ComS574_HW1_Problem1

February 16, 2020

#

Coms 574

#

HW 1

#

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1 Problem 1:

1.1 (a)

[221]:
```
path = '';

%matplotlib inline
import numpy as np
from sklearn import linear_model
from sklearn.preprocessing import PolynomialFeatures
import matplotlib.pyplot as plt
import math
import random
import warnings

#ignore by message
warnings.filterwarnings("ignore", message="Polyfit may be poorly conditioned")
```

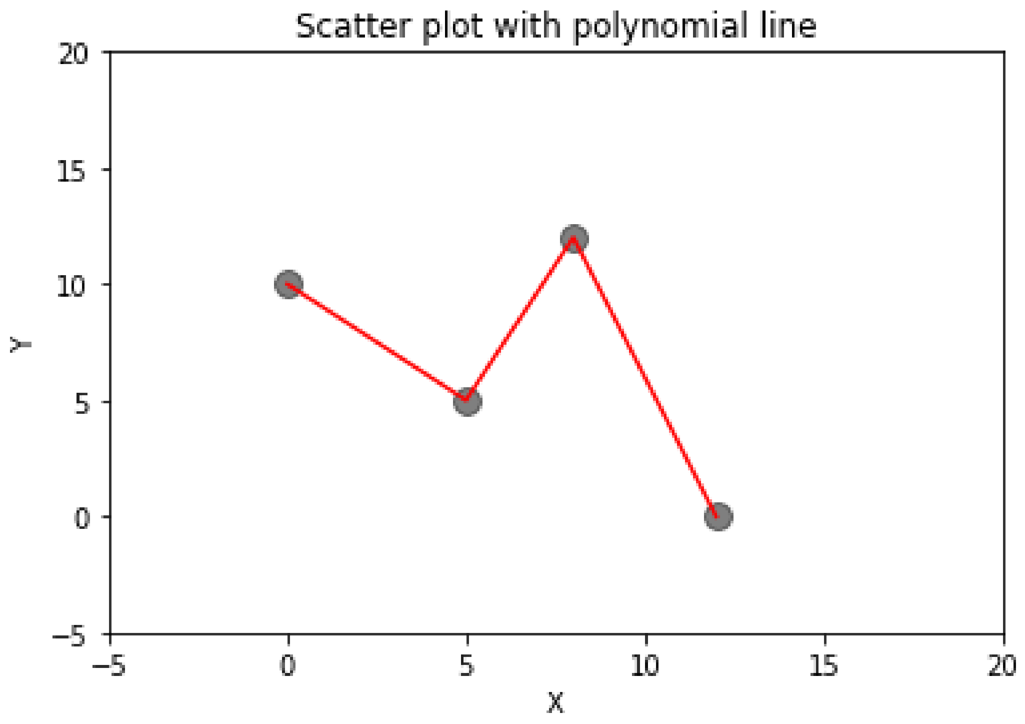
[222]:
```
import warnings
warnings.filterwarnings("ignore", message="divide by zero encountered in log")
warnings.filterwarnings("ignore", message="invalid value encountered in␣
↪multiply")
# warnings.filterwarnings("ignore", message="Polyfit may be poorly conditioned")


X = np.array([0, 5, 8, 12]).reshape(-1,1)
Y = np.array([10, 5, 12, 0])
lm = linear_model.LinearRegression()
```

```
X = PolynomialFeatures(degree=3).fit_transform(X)
model = lm.fit(X, Y)
print("Y^ = %.4f + (%.4f * X) + (%.4f * X^2) + (%.4f * X^3)"
      %(model.coef_[0], model.coef_[1], model.coef_[2], model.coef_[3]))
```

```
Y^ = 0.0000 + (-7.0119 * X) + (1.6935 * X^2) + (-0.0982 * X^3)
```

1.2 (b)

[223]:
```
# Xnew = np.arange(-5, 20).reshape(-1,1)
# pred = model.predict(PolynomialFeatures(degree=3).fit_transform(Xnew))
pred = model.predict(X)
plt.scatter(X[:,1], Y, alpha=0.5, facecolors='black', edgecolors='black', s=100)
plt.plot(X[:,1], pred, color = 'red')
plt.title('Scatter plot with polynomial line')
plt.xlabel('X')
plt.ylabel('Y')
plt.ylim(-5,20)
plt.xlim(-5,20)
plt.show()
```

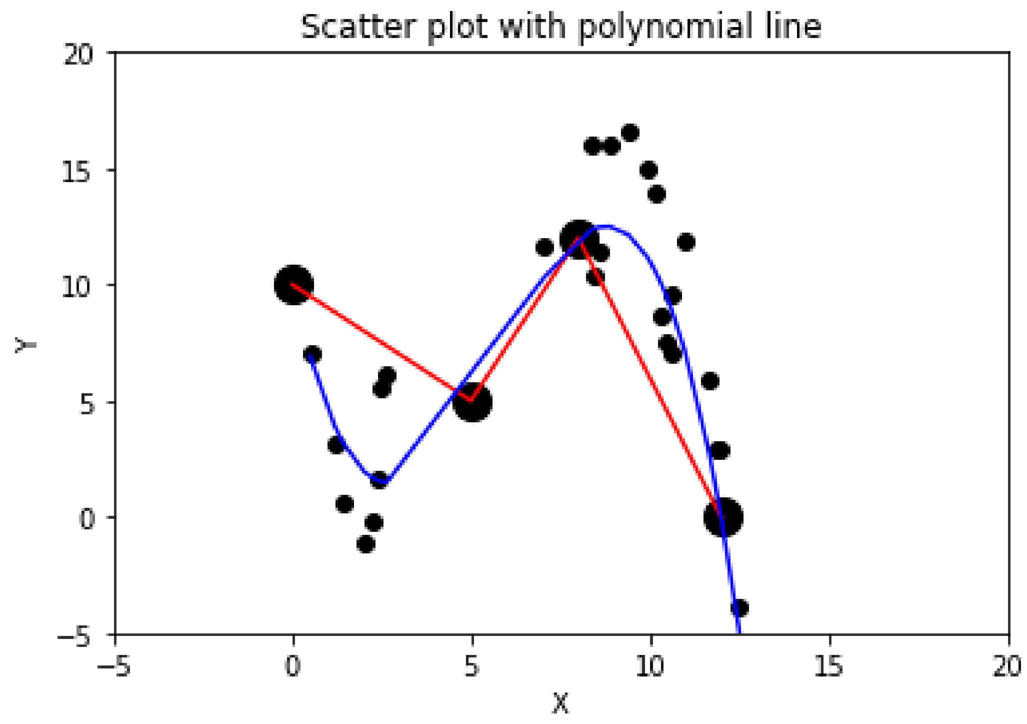


1.3 (c)

Loss function defines how well the model fits the data. MSE loss function minimizes average squared error that is average squared deviation of data from the regression line. Different loss function minimizes different objective function. For example, if we consider MAE (mean absolute error), we may get different coefficient that is different polynomial.

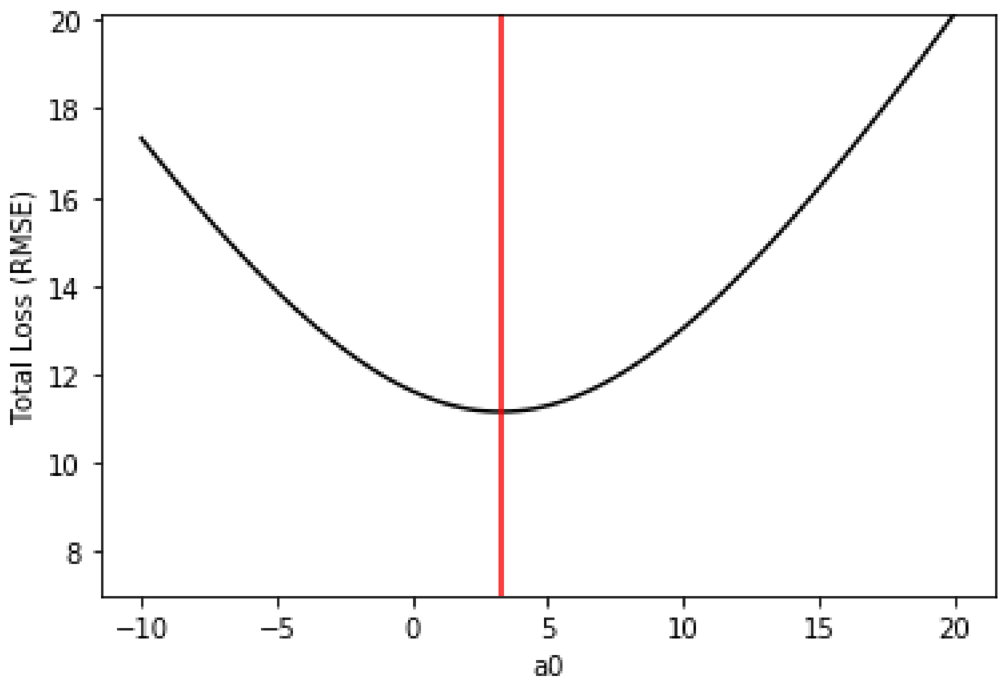
1.4 (d)

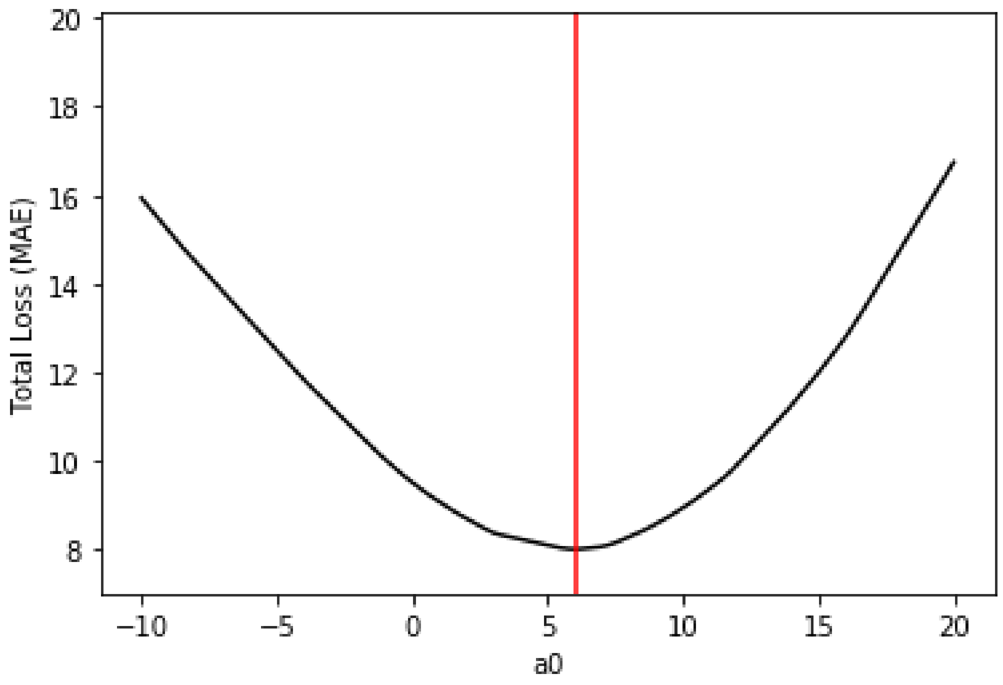
[224]:
```
np.random.seed(1234657)
nsample = 30
xnew = np.random.uniform(0, 15, nsample).reshape(-1,1)
error = np.random.normal(0, math.sqrt(10), nsample)
ynewhat = model.predict(PolynomialFeatures(degree = 3).fit_transform(xnew))
ynew = np.add(ynewhat, error)
# keydict = dict(zip(xnew.reshape(30), ynewhat))
plt.scatter(X[:,1], Y,  color = 'black', s=200)
plt.plot(X[:,1], pred, color = 'red')
plt.plot(sorted(xnew), list(zip(*sorted(zip(xnew.
 ↪reshape(nsample),ynewhat))))[1], color = 'blue')
plt.scatter(xnew, ynew, color = 'black')
plt.title('Scatter plot with polynomial line')
plt.xlabel('X')
plt.ylabel('Y')
plt.ylim(-5,20)
plt.xlim(-5,20)
plt.show()
```

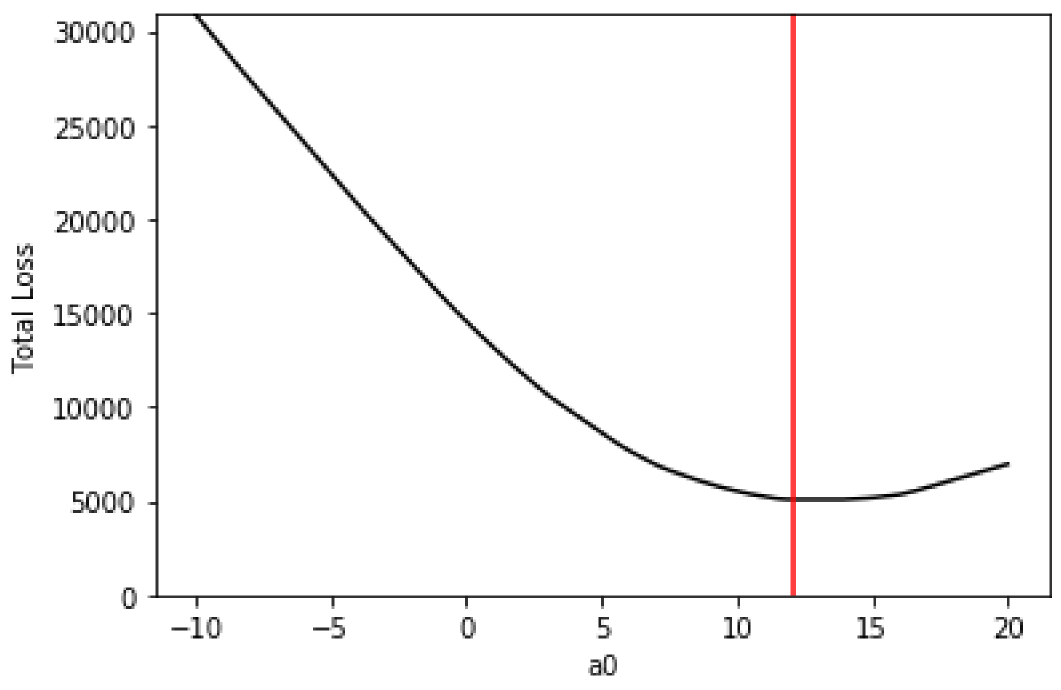


1.5 (e)

```
[225]: na0 = 100
a0 = np.linspace(-10, 20, na0)
def sqloss(y, yhat):
    return(np.mean((y-yhat)**2))

def absloss(y, yhat):
    return(np.mean(np.absolute(y-yhat)))



def specialloss(y, yhat, x):
    n = np.size(y)
    nyhat = np.size(yhat)
    nx = np.size(x)
    if nyhat == 1:
        yhat = np.repeat(yhat, n)
    if n == np.size(yhat):
        err = y - yhat
        err[err<0] = err[err<0]*(-1/5)
        err[err>=0] = err[err>=0]*10
        loss = np.sum(err / (abs(x-5)+0.01))
        return(loss)
    else:
        print("dimension of y or yhat is not correct")
        return()


loss = np.zeros(shape = (na0, 3))
loss = []
for i in range(0, na0):
    loss.append([sqloss(ynew, a0[i]), absloss(ynew, a0[i]), specialloss(ynew,␣
 ↪a0[i], xnew)])

loss = np.array(loss)
plt.plot(a0, np.sqrt(loss[:,0].astype(np.float64)),  color = 'black')
plt.plot([a0[np.argmin(loss[:,0])], a0[np.argmin(loss[:,0])]], [0, 30],  color␣
 ↪= 'red')
# plt.plot(a0, loss[:,2],  color = 'green')
plt.ylabel('Total Loss (RMSE)')
plt.xlabel('a0')
plt.ylim(7,np.max([np.sqrt(np.max(loss[:,0])), np.max(loss[:,1])]))
plt.show()
plt.plot(a0, loss[:,1],  color = 'black')
plt.plot([a0[np.argmin(loss[:,1])], a0[np.argmin(loss[:,1])]], [0, 30],  color␣
 ↪= 'red')
plt.ylabel('Total Loss (MAE)')
plt.xlabel('a0')
plt.ylim(7,np.max([np.sqrt(np.max(loss[:,0])), np.max(loss[:,1])]))
plt.show()
plt.plot(a0, loss[:,2],  color = 'black')
plt.plot([a0[np.argmin(loss[:,2])], a0[np.argmin(loss[:,2])]], [0, 50000], ␣
 ↪color = 'red')
plt.ylabel('Total Loss')
plt.xlabel('a0')
plt.ylim(0,np.max(loss[:,2]))
plt.show()
```


Total Loss (RMSE)
a0


Total Loss (MAE)
a0



1.6 (f)

Note: Red lines in the above plots indicate the minimized value for the loss function. For the training loss, I have used RMSE to have the same unite as MAE, so that we can compare.

It is difficult to say whether these loss functions over-estimating or under estimating y values. It depends on objective of using loss function. For example, if data contains large number of outliers, then MSE loss function overestimate y values. Because MSE minimizes data around the mean and mean affects by the outliers. In such cases MAE would be a good choice for the estimation. The third loss function puts more weights on the positive error compare to negative error. So, this type of loos function is good if our objective is to estimate or predict large values of the data set.

It is difficult to say whether the loss function put more emphasis or less emphasis on points close to $x = 5$. It depends on mean and median of y values. MSE loss function put more emphasis around the conditional mean of y and MAE loss function puts more emphasis around the conditional median of y.

1.7 (g)

[226]:
```
ymean = np.mean(ynew)
ymedian = np.median(ynew)
print("Mean and Median of Y are %.3f, %.3f" %(ymean, ymedian))
```

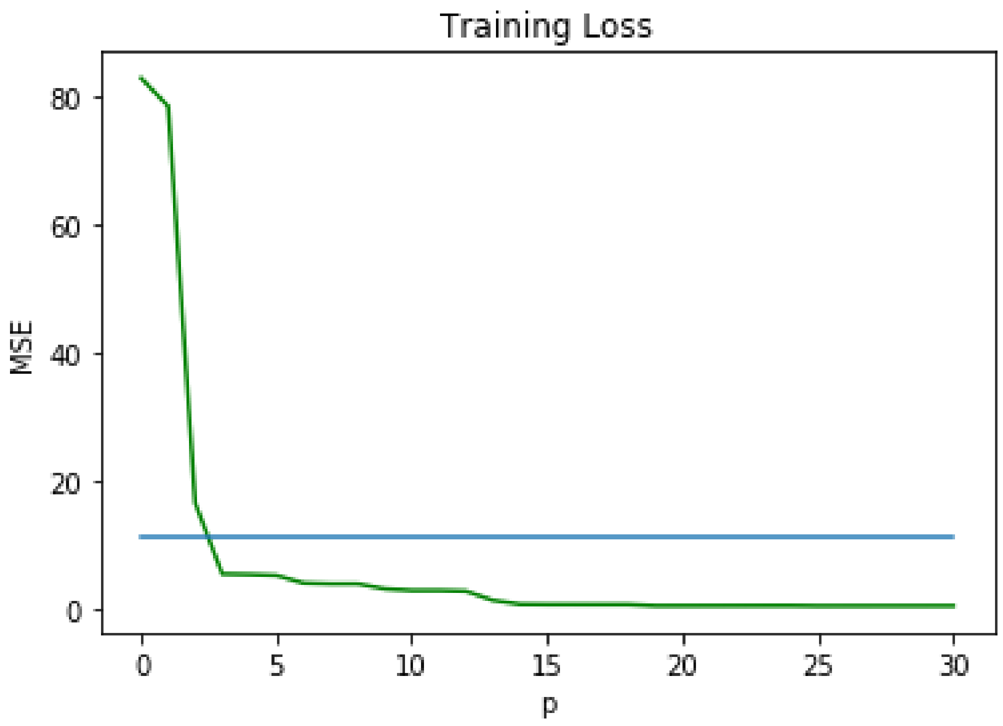
```
Mean and Median of Y are 3.258, 6.013
```

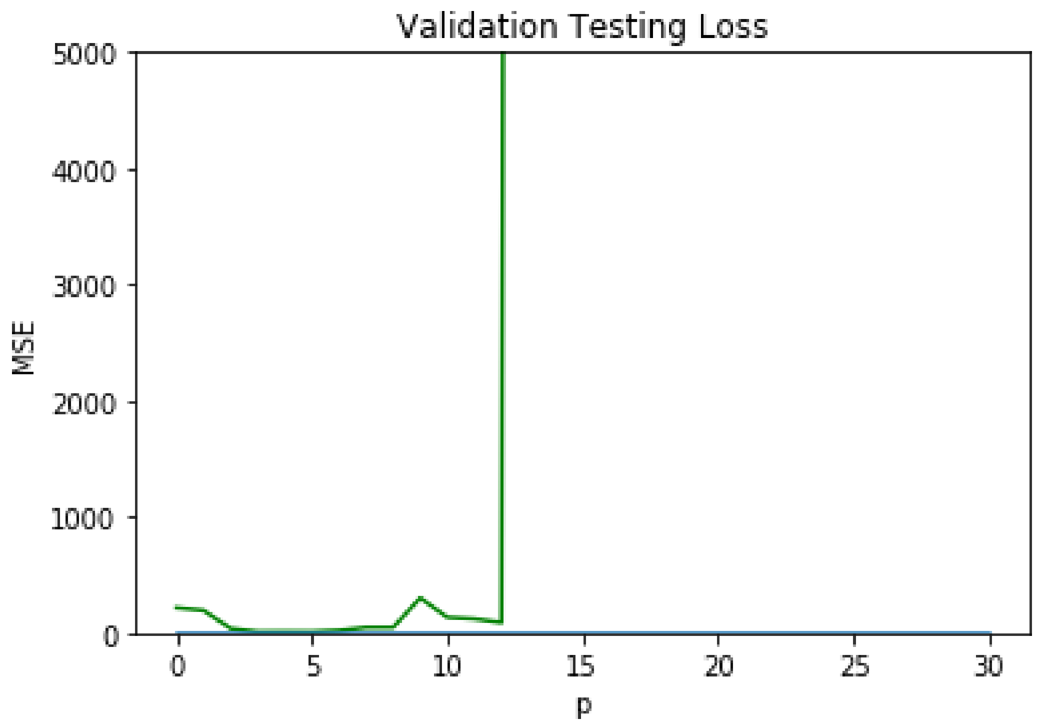
MSE minimizes around the mean of y and MAE minimizes around the median of y. MAE loss function penalize less on the extreme values. However, MSE loss function penalize more on the extreme values.

1.8 (h)

1.8.1 (i), (ii) and (iv)

[227]:
```
nn = 20
p = 30

xtrain = xnew[0:nn]
xtest = xnew[nn:nsample]
ytrain = ynew[0:nn]
ytest = ynew[nn:nsample]

xx1 = PolynomialFeatures(degree = 3).fit_transform(xtrain)
predtrain = model.predict(xx1)
groud_mse_tr = sqloss(ytrain, predtrain)

xx2 = PolynomialFeatures(degree = 3).fit_transform(xtest)
predtest = model.predict(xx2)
groud_mse_ts = np.mean((ytest - predtest)**2)


prange = range(0,p+1)
sqlossval_tr = np.zeros(p+1)
sqlossval_ts = np.zeros(p+1)




for j in prange:
    model_1 = np.polyfit(xtrain.reshape(nn), ytrain, j)
    ytrainhat_tr = np.polyval(model_1, xtrain.reshape(nn))
    ytrainhat_ts = np.polyval(model_1, xtest.reshape(10))
    sqlossval_tr[j] = sqloss(ytrain,  ytrainhat_tr)
    sqlossval_ts[j] = sqloss(ytest, ytrainhat_ts)

plt.plot(prange, sqlossval_tr,  color = 'green')
plt.plot([0,p], [groud_mse_tr, groud_mse_tr])
plt.title('Training Loss')
plt.ylabel('MSE')
plt.xlabel('p')
# plt.ylim(0,5)
plt.show()

plt.plot(prange, sqlossval_ts,  color = 'green')
plt.plot([0,p], [groud_mse_ts, groud_mse_ts])
plt.title('Validation Testing Loss')
plt.ylabel('MSE')
plt.xlabel('p')
plt.ylim(0, 5000)
plt.show()
```





1.8.2 (iii) and (v)

Note that validation loss plot axis has been truncated to have a better look.

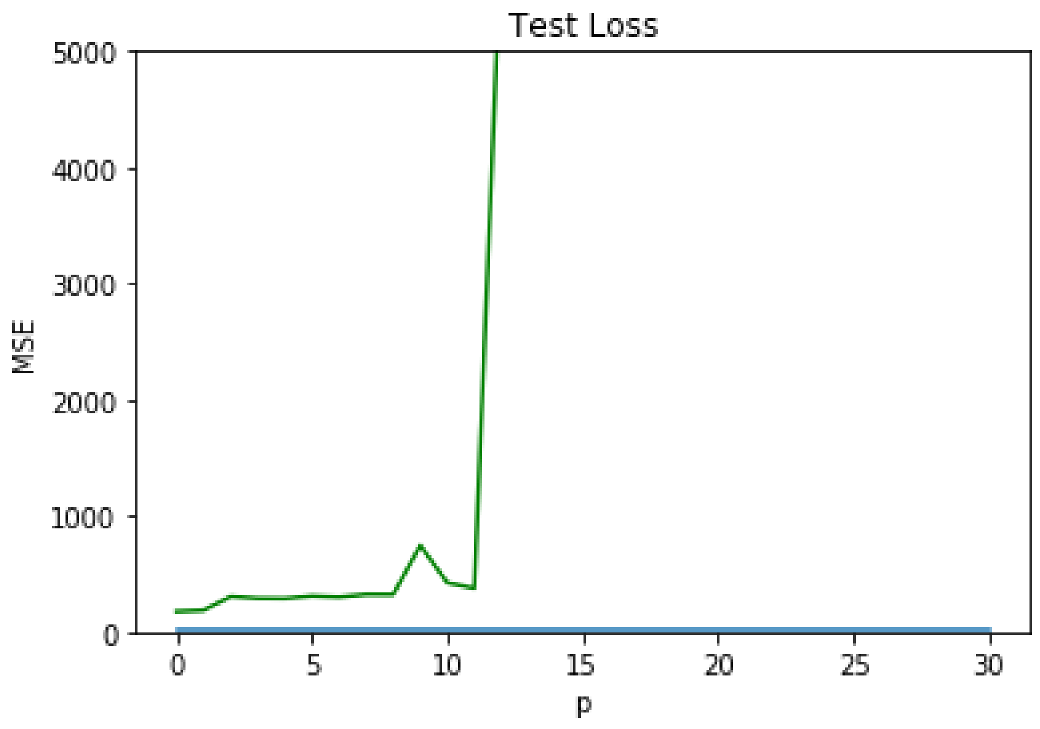
From the training loss curve, training loss decreases as the order of the polynomial increases and it goes to 0 after 19th order polynomial because we have used only 20 observations. There is a steep downward slope until 3rd order polynomial after it has a shallow slope because we have generated data from 3rd order polynomial.

For the validation loss curve, I have truncated y-axis to have a better look of the plot. From the plot, minimum testing loss is at 3rd order polynomial. Plot look like U shape. That is, test loss always higher for all polynomials except 3rd order. The difference between the training loss and validation loss curve is that training loss always decreases as the order of polynomial increases, but it is not same for validation loss.

1.8.3 (vi)

[228]:
```
np.random.seed(9876)
nnew_test = 1000
nn = 20
xnew_test = np.random.uniform(0, 15, nnew_test).reshape(-1,1)
error1 = np.random.normal(0, math.sqrt(10), nnew_test)
ynewhat_test = model.predict(PolynomialFeatures(degree = 3).
 ↪fit_transform(xnew_test))
```

```
ynew_test = np.add(ynewhat_test, error1)

prange = range(0,p+1)
sqlossval_tr = np.zeros(p+1)
sqlossval_test = np.zeros(p+1)

for j in prange:
    model_1 = np.polyfit(xtrain.reshape(nn), ytrain, j)
    ytrainhat_tr = np.polyval(model_1, xtrain.reshape(nn))
    ytrainhat_ts = np.polyval(model_1, xnew_test)
    sqlossval_tr[j] = sqloss(ytrain,  ytrainhat_tr)
    sqlossval_test[j] = sqloss(ynew_test, ytrainhat_ts)


xx2 = PolynomialFeatures(degree = 3).fit_transform(xnew_test)
predt_test = model.predict(xx2)
groud_mse_test = sqloss(ynew_test, predt_test)

plt.plot(prange, sqlossval_test,  color = 'green')
plt.plot([0,p], [groud_mse_test, groud_mse_test])
plt.title('Test Loss')
plt.ylabel('MSE')
plt.xlabel('p')
plt.ylim(0, 5000)
plt.show()
```



1.8.4 (vii)

Note that test loss plot axis has been truncated to have a better look.

Though the testing loss plot and validation loss plot look similar but closer look shows different results. For the test loss minimum loss found for polynomial order 0. However, for the validation loss, minimum loss found for polynomial order 3. Also, average testing loss is higher than training and validation loss. It indicates that even if we find a good model based on training and validation loss function does not guarantee to have same total loss on the test data.

1.8.5 (viii)

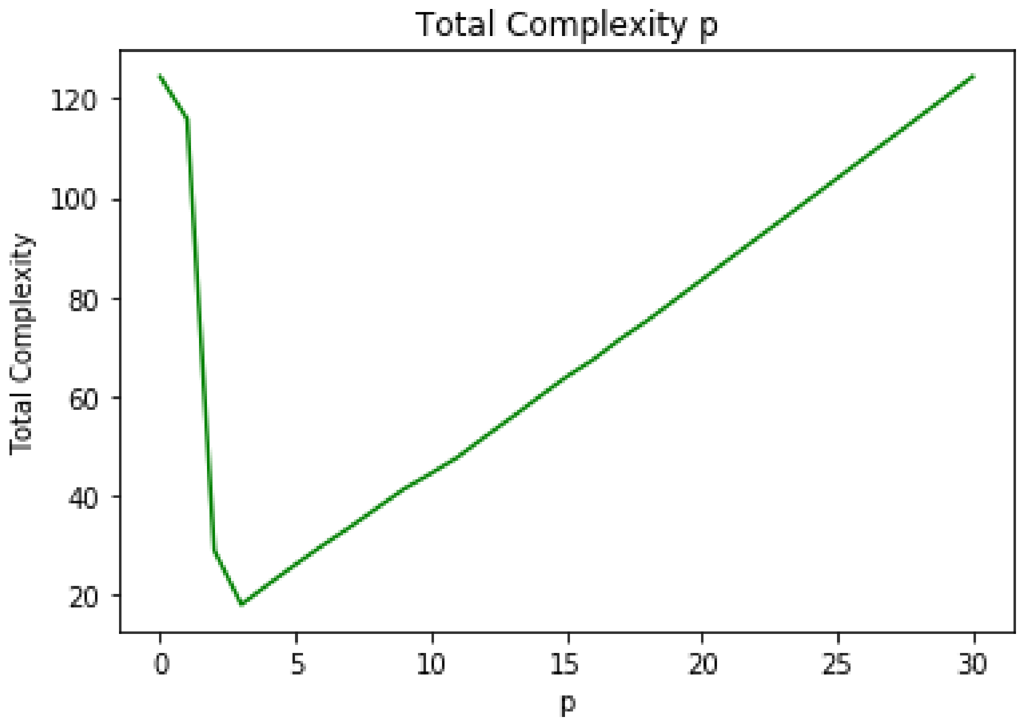
λ Value:

$$TC\left(p=0\right)=TC\left(p=30\right)$$

$$MSE\left(0\right)+\lambda*0=MSE\left(30\right)+\lambda*30$$

$$\lambda=\frac{MSE\left(0\right)-MSE\left(30\right)}{30}$$

[229]:
```
nn= 30
prange = range(0,p+1)
mse_loss_tr = np.zeros(p+1)




for j in prange:
    model_1 = np.polyfit(xnew.reshape(nn), ynew, j)
    ytrainhat_tr = np.polyval(model_1, xnew.reshape(nn))
    mse_loss_tr[j] = sqloss(ynew,  ytrainhat_tr)

lambda_v = (mse_loss_tr[0] - mse_loss_tr[p])/p
comx_loss_tr = mse_loss_tr + np.array(np.arange(0, p+1, 1)) * lambda_v

plt.plot(prange, comx_loss_tr,  color = 'green')
plt.title('Total Complexity p')
plt.ylabel('Total Complexity')
plt.xlabel('p')
# plt.ylim(0,5)
plt.show()
```

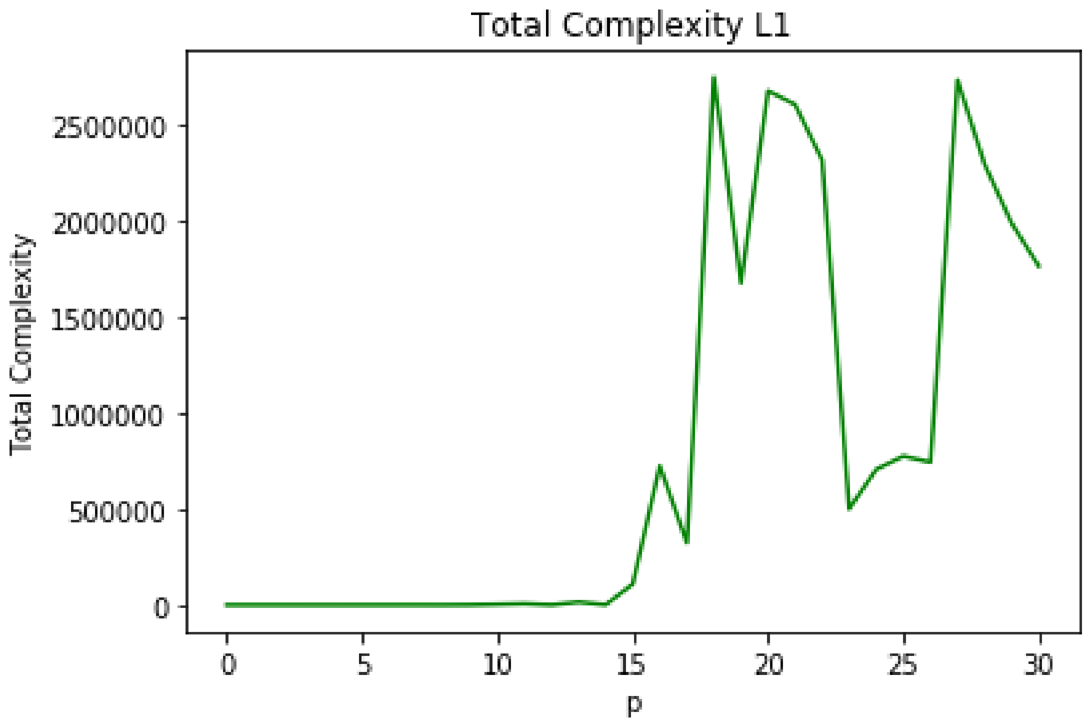


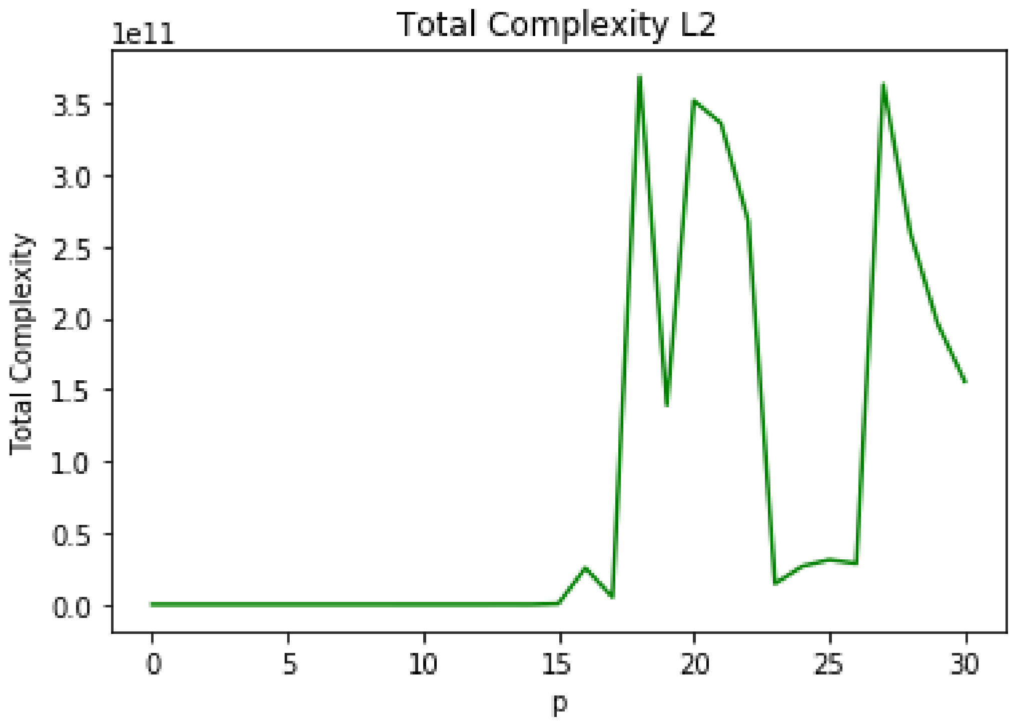
Based on this total complexity loss function, the minimum loss found for polynomial of order 3 that is same as the ground truth model. We also found the same for validation MSE. However, training MSE shows that more complex model has lower MSE.

1.8.6 ix.

[230]:
```
p = 30
nn=30
prange = range(0,p+1)
l1loss = np.zeros(p+1)
l2loss = np.zeros(p+1)

for j in prange:
    model_1 = np.polyfit(xnew.reshape(nn), ynew, j)
    ytrainhat_tr = np.polyval(model_1, xnew.reshape(nn))
    l1loss[j] = sqloss(ynew,  ytrainhat_tr) + lambda_v * np.sum(abs(model_1[0:
 ↪j]))
    l2loss[j] = sqloss(ynew,  ytrainhat_tr) + lambda_v * np.sum(model_1[0:j]**2)


plt.plot(prange, l1loss,  color = 'green')
plt.title('Total Complexity L1')
plt.ylabel('Total Complexity')
plt.xlabel('p')
```

```
# plt.ylim(0,5000)
plt.show()

plt.plot(prange, l2loss,  color = 'green')
plt.title('Total Complexity L2')
plt.ylabel('Total Complexity')
plt.xlabel('p')
# plt.ylim(0,5000)
plt.show()

print("Using L1 and L2 penalty, minimum loss found for polynomial order %d and␣
 ↪%d" %(np.argmin(l1loss), np.argmin(l2loss)))
```





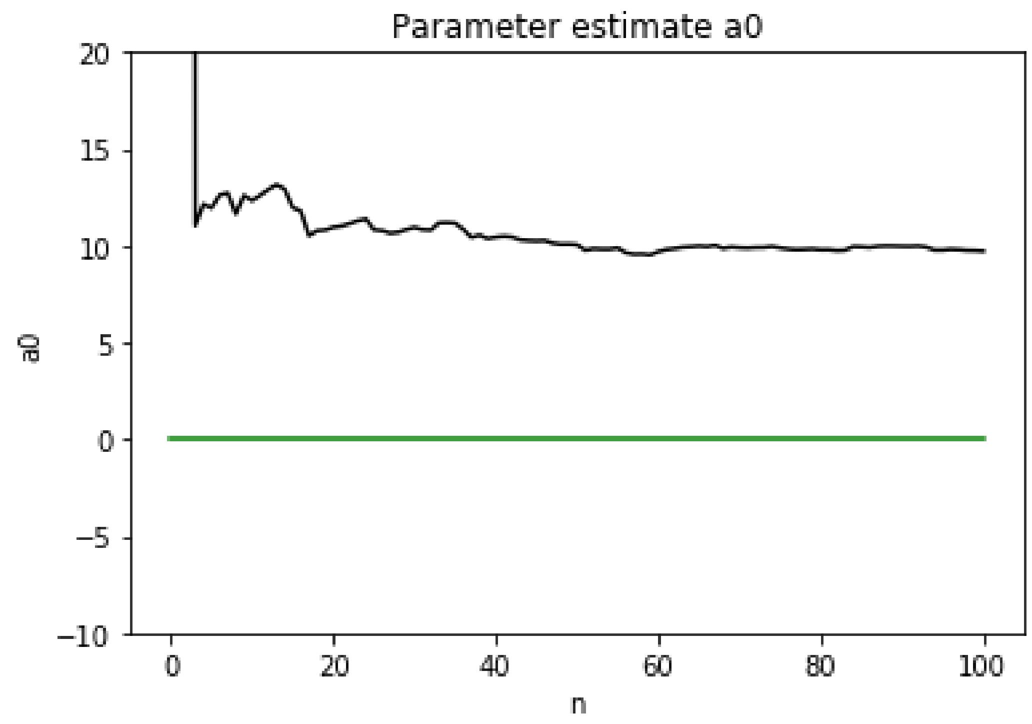
```
Using L1 and L2 penalty, minimum loss found for polynomial order 3 and 1
```

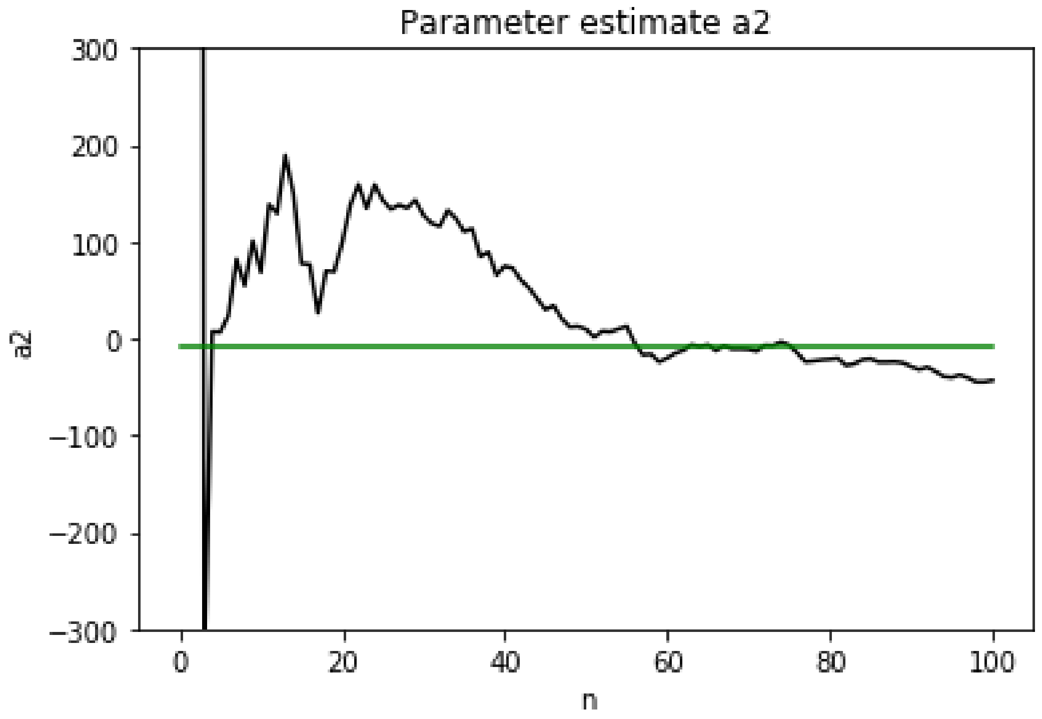
Note: intercept value for every order polynomial did not include in the penalty.

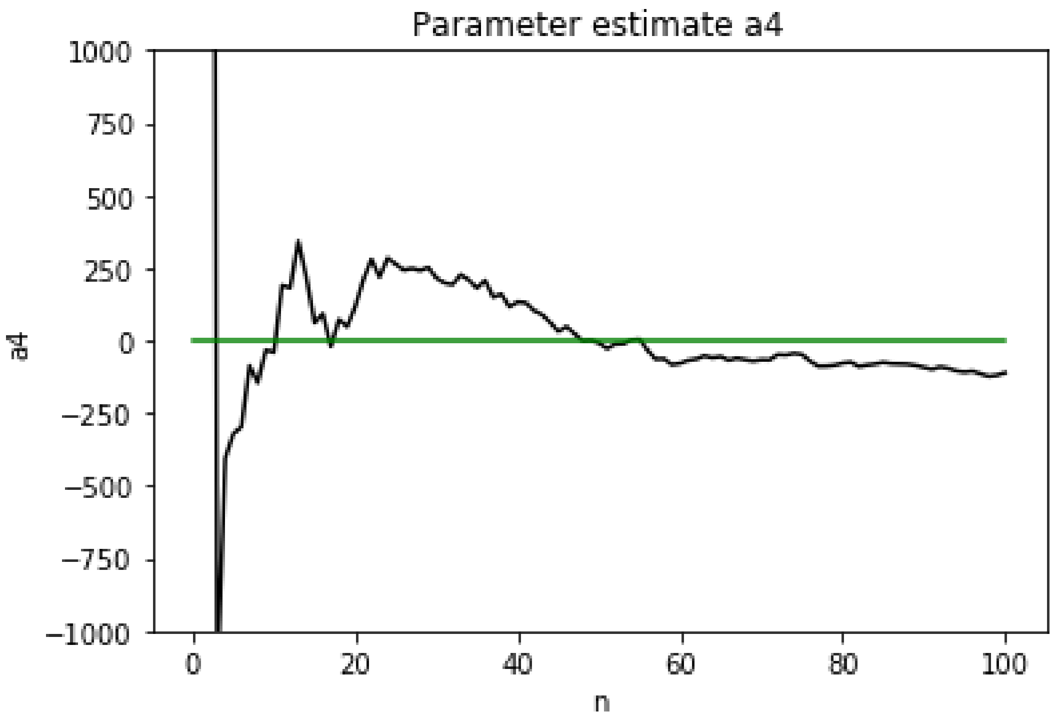
As the model complexity increases, these methods add more penalty in the total loss. The best model found for L1 penalty is the 3rd order polynomial which is our ground truth model and for L2 penalty 1st order polynomial is the best. It is also found similar pattern from the validation loss. However, based on test loss, 0th order polynomial found the best. It is important to mention that because of penalty factor training loss does not always go down.

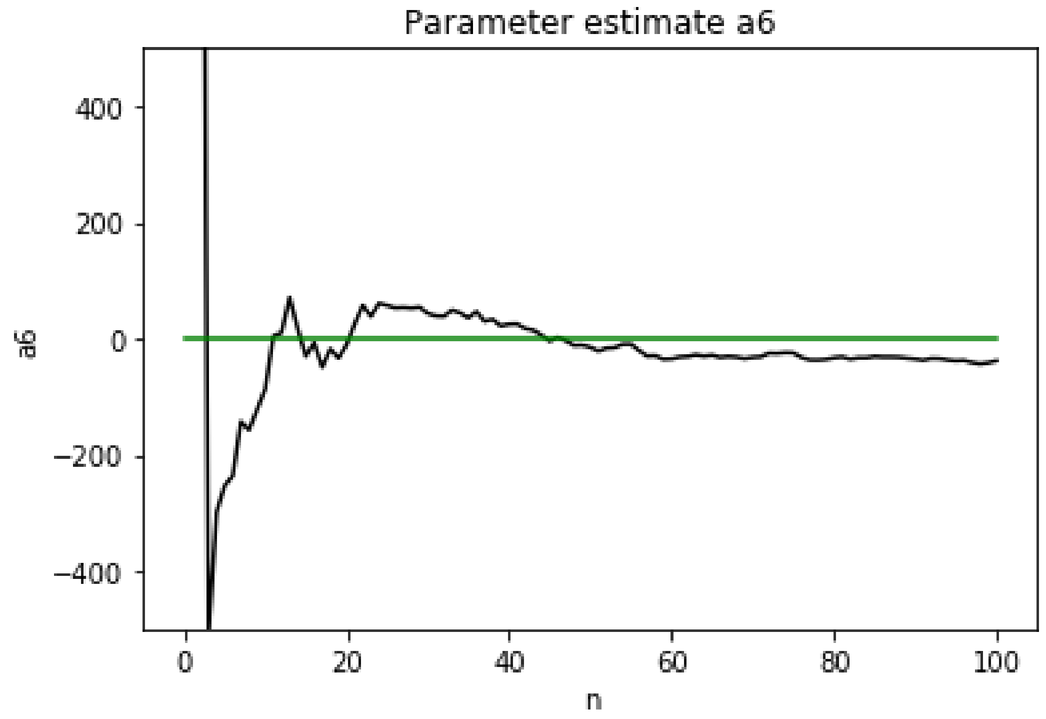
1.9 (i)

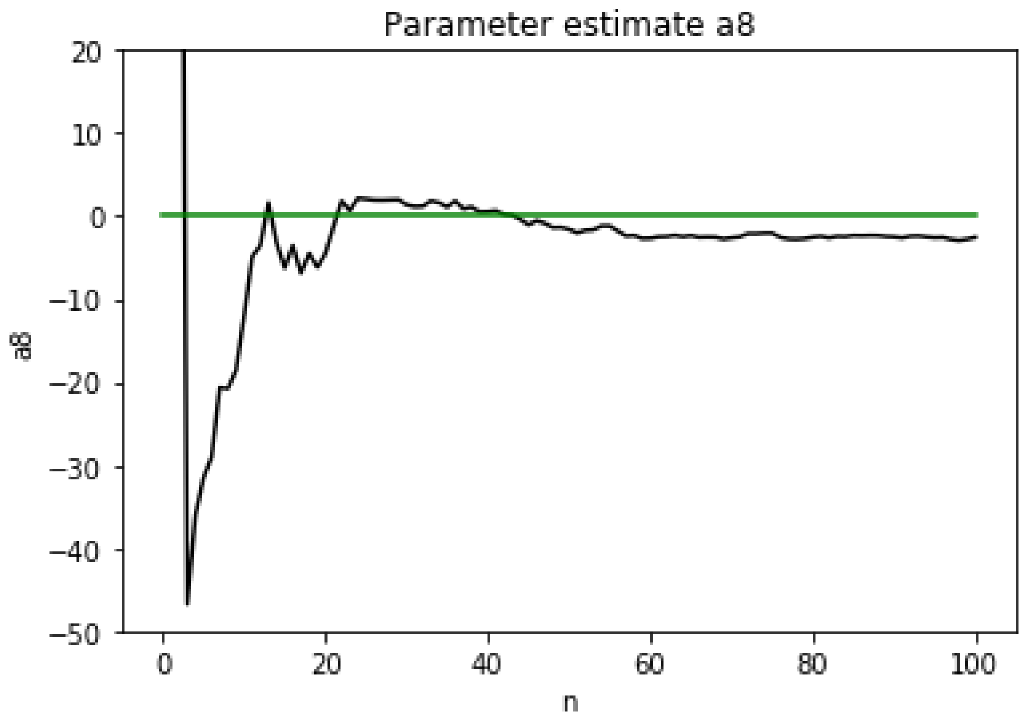
[231]:
```
np.random.seed(9876)
nnew_test = 5000
xnew_large = np.random.uniform(0, 15, nnew_test).reshape(-1,1)
error1 = np.random.normal(0, math.sqrt(10), nnew_test)
ynewhat_large = model.predict(PolynomialFeatures(degree = 3).
 ↪fit_transform(xnew_large))
ynew_large = np.add(ynewhat_large, error1)

prange = range(1,101)
prem = []
for i in prange:
    model_1 = np.polyfit(xnew_large[0:(i*50)].reshape(i*50), ynew_large[0:
↪(i*50)], 30)
    prem.append(model_1[np.array([30, 28, 26, 24, 22])])

prem = np.array(prem)
prem_ground = model.coef_

plt.plot(prange, prem[:,0],  color = 'black')
plt.plot([0,100], [prem_ground[0], prem_ground[0]], color = 'green')
plt.title('Parameter estimate a0')
plt.ylabel('a0')
plt.xlabel('n')
plt.ylim(-10,20)
plt.show()

plt.plot(prange, prem[:,1],  color = 'black')
plt.plot([0,100], [prem_ground[1], prem_ground[1]], color = 'green')
plt.title('Parameter estimate a2')
plt.ylabel('a2')
plt.xlabel('n')
plt.ylim(-300,300)
plt.show()


plt.plot(prange, prem[:,2],  color = 'black')
plt.plot([0,100], [0,0], color = 'green')
plt.title('Parameter estimate a4')
plt.ylabel('a4')
plt.xlabel('n')
plt.ylim(-1000,1000)
plt.show()

plt.plot(prange, prem[:,3],  color = 'black')
plt.plot([0,100], [0,0], color = 'green')
plt.title('Parameter estimate a6')
plt.ylabel('a6')
plt.xlabel('n')
plt.ylim(-500,500)
plt.show()

plt.plot(prange, prem[:,4],  color = 'black')
plt.plot([0,100], [0,0], color = 'green')
plt.title('Parameter estimate a8')
plt.ylabel('a8')
plt.xlabel('n')
plt.ylim(-50,20)
plt.show()
```


Parameter estimate a0
a0
n
20
15
10
5
0
−5
−10
0
20
40
60
80
100


Parameter estimate a2
a2
n
300
200
100
0
−100
−200
−300
0
20
40
60
80
100


Parameter estimate a4
1000
750
500
250
0
−250
−500
−750
−1000
a4
0
20
40
60
80
100
n


Parameter estimate a6
400
200
0
−200
−400
a6
0
20
40
60
80
100
n



It is clear from these plots that for large sample these parameter estimates are consistent. For small samples there are very high volatility of the parameter estimates. For this data set, as the sample size increases, parameter estimates become close to true estimate. It is important to mention that we have estimated 30th order model using data generated from 3rd order model as a result we are getting some difference from the true values (green line) that may contribute to the overfitting. It also indicate that as the model complexity increases, we need more data to have a consistent estimate of the parameters.

```
[ ]:
```

CS574 HW1

Kanak Choudhury

2/13/2020

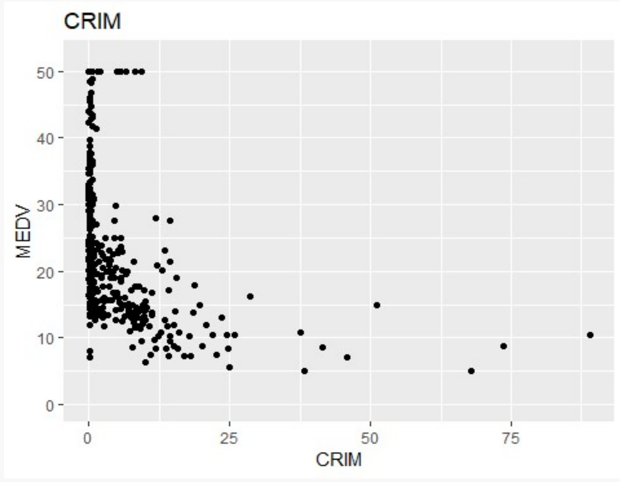
```r
library(glmnet)

library(ggplot2)
```

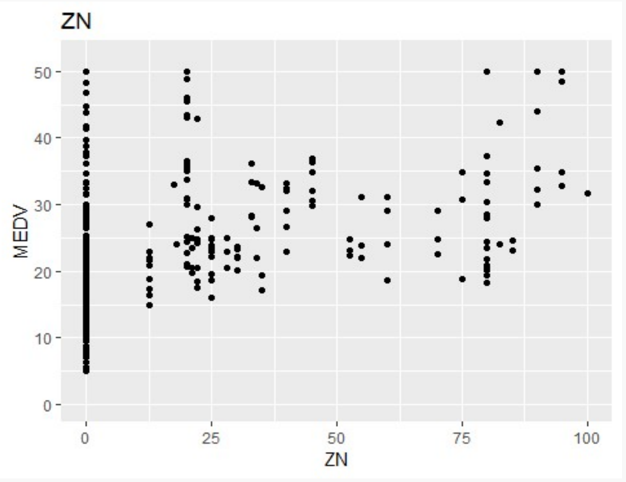
Problem 2

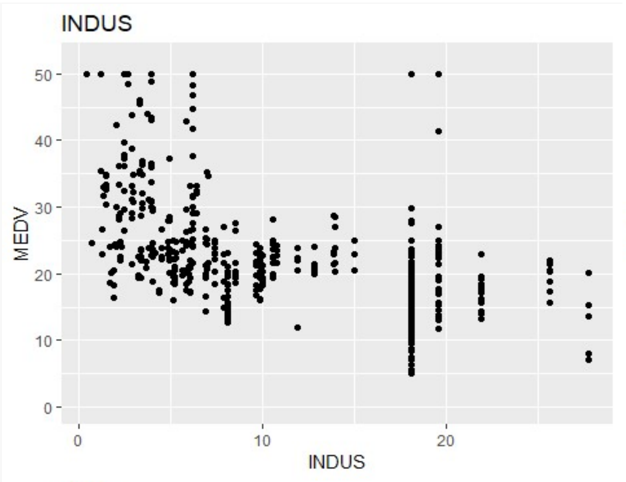
```r
path = "D:\\ISU\\COMS 574 - Introduction to Machine Learning\\HW\\HW1\\"

dt = read.csv(paste(path, "housingdata.csv", sep = ""), header = T)
varname = names(dt)
```

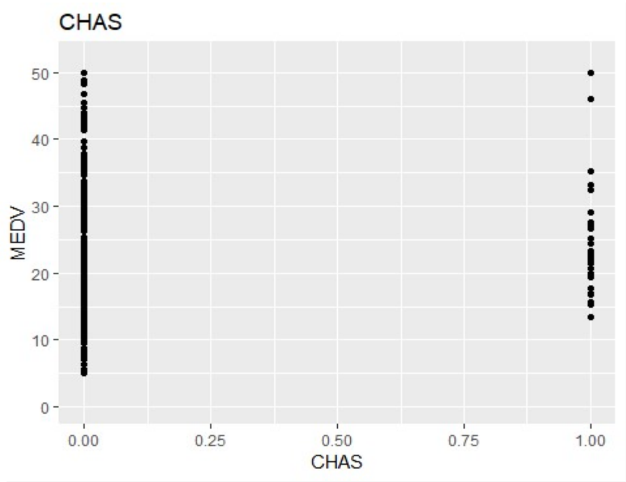
(a)

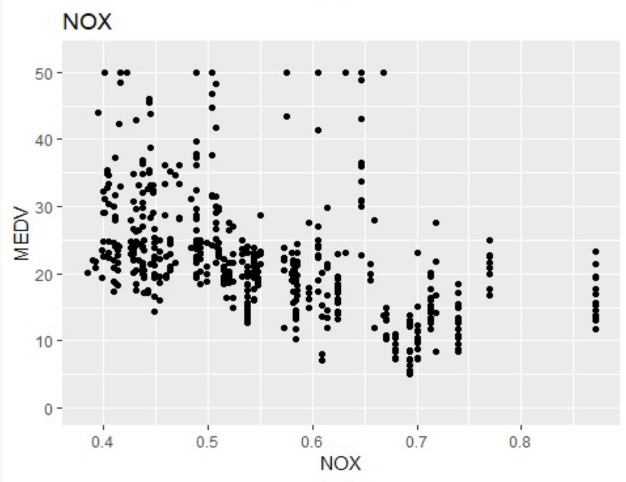
```r
for (i in c(1:13)){
 print(ggplot(dt, aes_string(x=varname[i], y=varname[14])) +
         ylim(0,max(dt$MEDV+2))+
         ggtitle(varname[i]) +
         geom_point())
}
```

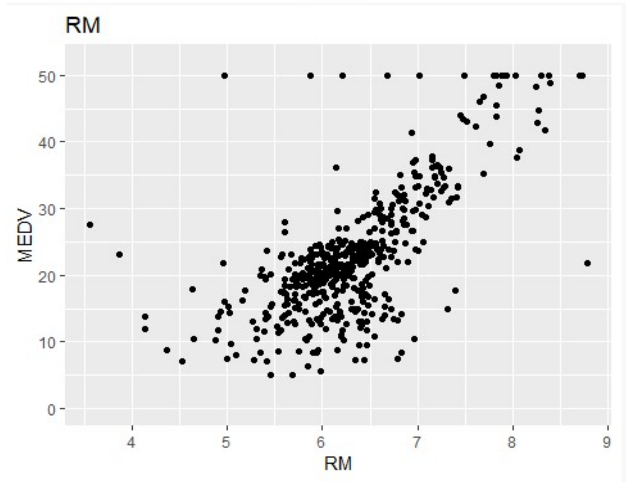


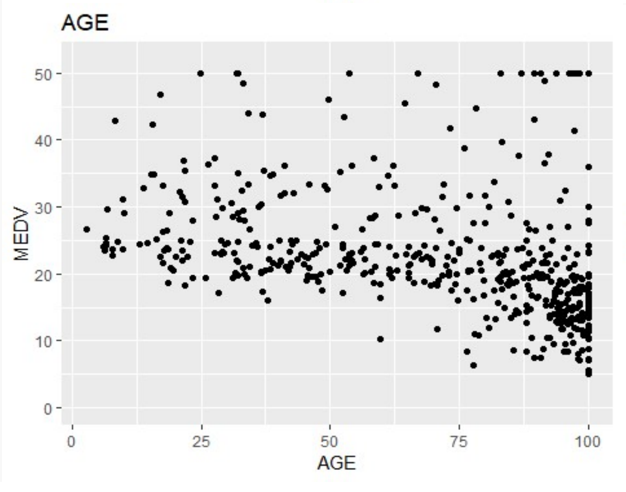


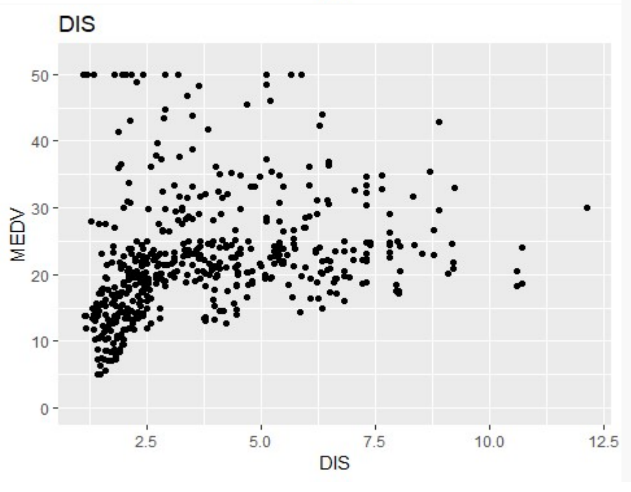

INDUS
MEDV
0
10
20
30
40
50
0
10
20
INDUS

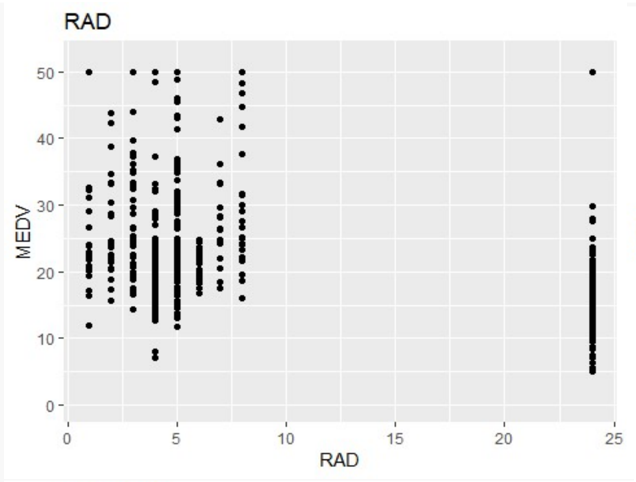

CHAS
MEDV
0
10
20
30
40
50
0.00
0.25
0.50
0.75
1.00
CHAS

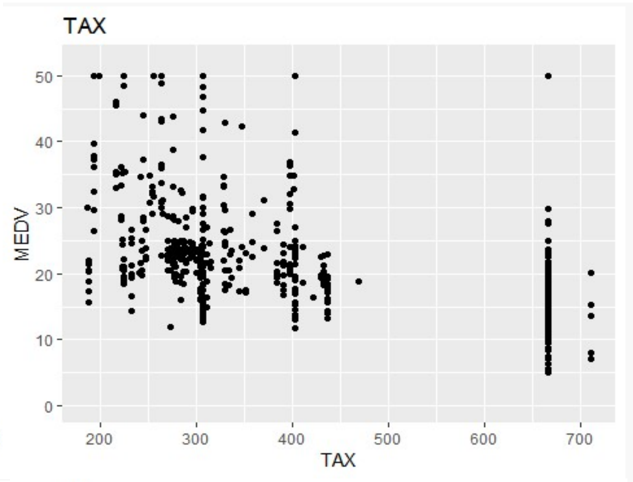

NOX
MEDV
0
10
20
30
40
50
0.4
0.5
0.6
0.7
0.8
NOX

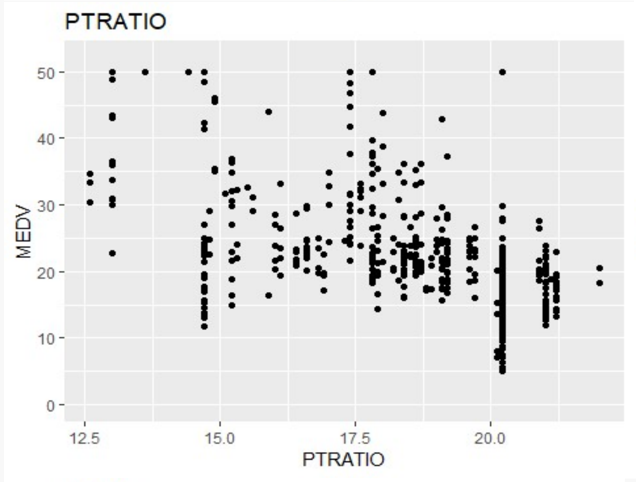

RM
MEDV
0
10
20
30
40
50
4
5
6
7
8
9
RM

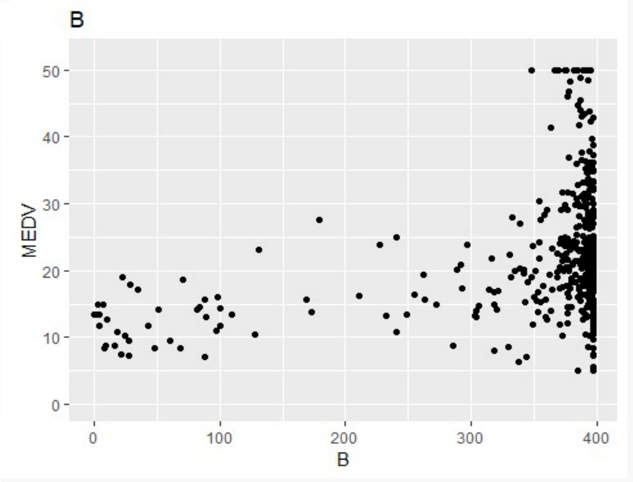

AGE
MEDV
0
10
20
30
40
50
0
25
50
75
100
AGE

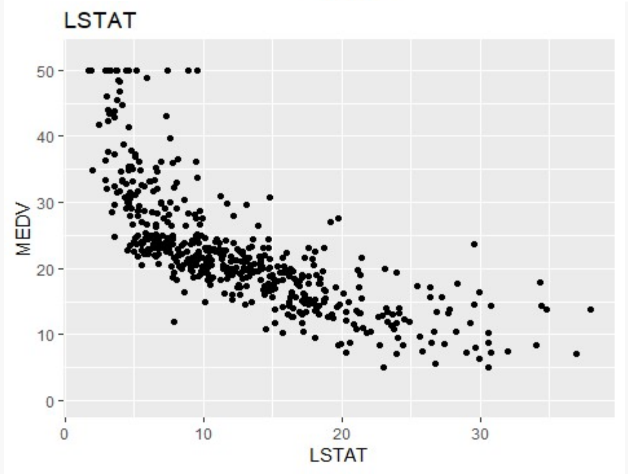

DIS
MEDV
0
10
20
30
40
50
2.5
5.0
7.5
10.0
12.5
DIS











Based on scatter plot, LSTAT (% lower status of the population) (negative relation) and RM (average number of rooms per dwelling) (positive relation) have moderate (highest among the features) linear association (-0.74 and 0.69) with MEDV (Median value of owner-occupied homes in \$1000's). On the other hand, B ($1000(Bk-0.63)^2$ where Bk is the proportion of black residents by town) has the least relation with the response variable.

Though, CHAS is a binary variable, it seems that there is a high relation with response variable. It is clear from the scatter plot that there is nonlinear or quadratic relation between DIS (weighted distances to five Boston employment centres) and MEDV. All other variables have some relation with response variable. However, with respect to high order space (interaction with other variables or higher order) there might have some strong relation with the response variable.

(b)

```
dt_train = dt[1:400,]
dt_test = dt[401:dim(dt)[1],]
varn = c("AGE", "INDUS", "NOX", "RM", "TAX")

reg_res = list()
tr_mse = list()
k = 1
for (i in c(0:length(varn))){
  comb = combn(varn, i)
  for (j in c(1:dim(comb)[2])){
    if (dim(comb)[1]==0){
      formu = "MEDV ~ 1"
      comb = matrix(c("Intercept"), nrow = 1, ncol = 1)
    } else{
      formu = paste("MEDV ~ 1", paste((comb[,j]), collapse = "+"), sep = "+")
    }

    reg_res[[k]] = lm(as.formula(formu), data = dt_train)
    tr_mse[[k]] = list(i = i, var = paste((comb[,j]), collapse = ","),
                       tr_mse = mean(reg_res[[k]]$residuals^2),
                       vel_mse = mean((predict(reg_res[[k]], newdata = dt_tes
t)-dt_test$MEDV)^2))
    k = k+1
  }

}

res = do.call(rbind.data.frame, tr_mse)
names(res) = c("subset", "variables", "tr_mse", "ts_mse")

trset = NULL
for (i in c(0:length(varn))){
  aa = which(res$tr_mse == res[res$subset == i,][which.min(res[res$subset ==
i,3]),3])
  trset = rbind(trset, res[aa,])
  res1 = reg_res[[aa]]
  len1 = length(res1$coefficients)
  if (i==0){
    print(paste("For subset model ", i, ":   y^hat = ", round(res1$coefficien
ts[1], 3) ,
                "   with MSE = ", round(res[aa,3],3)))
  } else {

    print(paste("For subset model ", i, ":   y^hat = ", round(res1$coefficien
ts[1], 3), " + " ,
              paste(round(res1$coefficients[2:len1], 3),
                    names(res1$coefficients)[2:len1], sep = " * ", collapse =
" + "),
        "   with training MSE = ", round(res[aa, 3],3)))
  }
}

print(trset)

aa = which.min(res$tr_mse)
res1 = reg_res[[aa]]
len1 = length(res1$coefficients)
print(paste("For subset model ", res[aa,]$subset, ":   y^hat = ", round(res1$
coefficients[1], 3),
            " + " , paste(round(res1$coefficients[2:len1], 3),
                  names(res1$coefficients)[2:len1], sep = " * ", collapse = "
+ "),
      "   with training MSE = ", round(res[aa, 3],3), " and testing MSE = ", ro
und(res[aa, 4],3)))

aa = which.min(res$ts_mse)
res1 = reg_res[[aa]]
len1 = length(res1$coefficients)
print(paste("For subset model ", res[aa,]$subset, ":   y^hat = ", round(res1$
coefficients[1], 3),
            " + " ,
            paste(round(res1$coefficients[2:len1], 3),
                  names(res1$coefficients)[2:len1], sep = " * ", collapse = "
+ "),
      "   with training MSE = ", round(res[aa, 3],3), " and testing MSE = ", ro
und(res[aa, 4],3)))
```

```
## [1] "For subset model  0 :   y^hat =  24.334    with MSE =  83.807"
## [1] "For subset model  1 :   y^hat =  -35.261  +  9.406 * RM    with train
ing MSE =  36.303"
## [1] "For subset model  2 :   y^hat =  -31.076  +  -0.034 * AGE + 9.09 * RM
with training MSE =  35.381"
## [1] "For subset model  3 :   y^hat =  -29.222  +  -0.028 * AGE + 8.944 * R
M + -0.004 * TAX    with training MSE =  35.193"
## [1] "For subset model  4 :   y^hat =  -30.586  +  -0.036 * AGE + 3.747 * N
OX + 8.991 * RM + -0.005 * TAX    with training MSE =  35.125"
```

```
## [1] "For subset model  5 :   y^hat =  -30.628  +  -0.034 * AGE + -0.031 *
INDUS + 4.573 * NOX + 8.947 * RM + -0.005 * TAX    with training MSE =  35.11
"
```

```
##    subset            variables   tr_mse    ts_mse
## 2       0            Intercept 83.80701 102.22659
## 5       1                   RM 36.30331  79.62583
## 9       2               AGE,RM 35.38101  67.97003
## 22      3           AGE,RM,TAX 35.19273  57.58572
## 30      4       AGE,NOX,RM,TAX 35.12503  57.85750
## 32      5 AGE,INDUS,NOX,RM,TAX 35.11013  57.66880
```

```
## [1] "For subset model  5 :   y^hat =  -30.628  +  -0.034 * AGE + -0.031 *
INDUS + 4.573 * NOX + 8.947 * RM + -0.005 * TAX    with training MSE =  35.11
and testing MSE =  57.669"
```

```
## [1] "For subset model  3 :   y^hat =  35.763  +  -0.04 * AGE + -7.286 * NO
X + -0.014 * TAX    with training MSE =  73.333  and testing MSE =  30.002"
```

The best fitting linear model for every subset of AGE, INDUS, NOX, RM, TAX using the first n = 400 samples based on training MSE is the model with all five features. However, based on test MSE, the based model is the model with variables AGE, NOX and TAX.

(c)

(i)

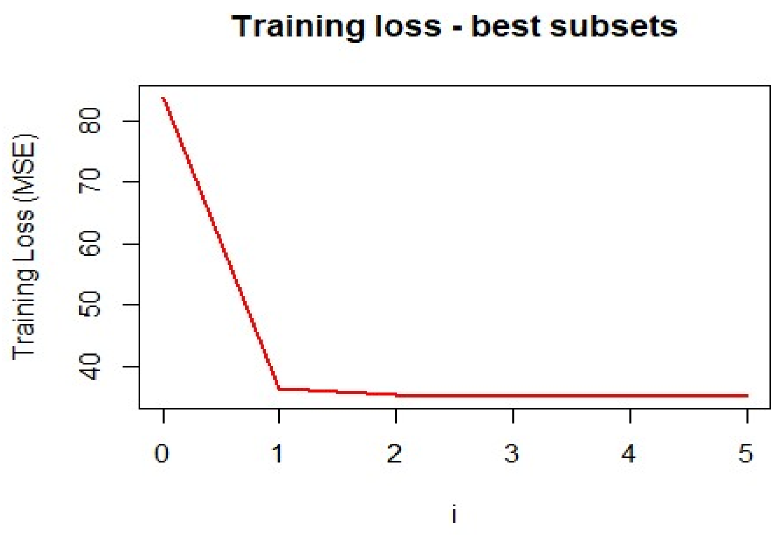
```
valset = NULL
for (i in c(0:length(varn))){
  aa = which(res$ts_mse == res[res$subset == i,][which.min(res[res$subset ==
i,4]),4])
  valset = rbind(valset, res[aa,])
  res1 = reg_res[[aa]]
  len1 = length(res1$coefficients)
  if (i==0){
    print(paste("For subset model ", i, ":   y^hat = ", round(res1$coefficien
ts[1], 3) ,
                "   with MSE = ", round(res[aa,4],3)))
  } else {
    print(paste("For subset model ", i, ":   y^hat = ", round(res1$coefficien
ts[1], 3), " + " ,
                paste(round(res1$coefficients[2:len1], 3),
                      names(res1$coefficients)[2:len1], sep = " * ", collapse
= " + "),
                "   with validation MSE = ", round(res[aa,4],3)))
  }
}

## [1] "For subset model  0 :   y^hat =  24.334    with MSE =  102.227"
## [1] "For subset model  1 :   y^hat =  32.103  +  -0.022 * TAX    with vali
dation MSE =  31.616"
## [1] "For subset model  2 :   y^hat =  33.753  +  -0.056 * AGE + -0.017 * T
AX    with validation MSE =  30.207"
## [1] "For subset model  3 :   y^hat =  35.763  +  -0.04 * AGE + -7.286 * NO
X + -0.014 * TAX    with validation MSE =  30.002"
## [1] "For subset model  4 :   y^hat =  31.131  +  -0.026 * AGE + -0.388 * I
NDUS + 3.887 * NOX + -0.01 * TAX    with validation MSE =  31.431"
## [1] "For subset model  5 :   y^hat =  -30.628  +  -0.034 * AGE + -0.031 *
INDUS + 4.573 * NOX + 8.947 * RM + -0.005 * TAX    with validation MSE =  57.
669"

print(valset)

##    subset            variables   tr_mse    ts_mse
## 2       0            Intercept 83.80701 102.22659
## 6       1                  TAX 75.69373  31.61631
## 10      2              AGE,TAX 73.59263  30.20671
## 21      3          AGE,NOX,TAX 73.33258  30.00196
## 28      4    AGE,INDUS,NOX,TAX 70.78130  31.43145
## 32      5 AGE,INDUS,NOX,RM,TAX 35.11013  57.66880
```

The best subset was for each *i* is given in the above table.

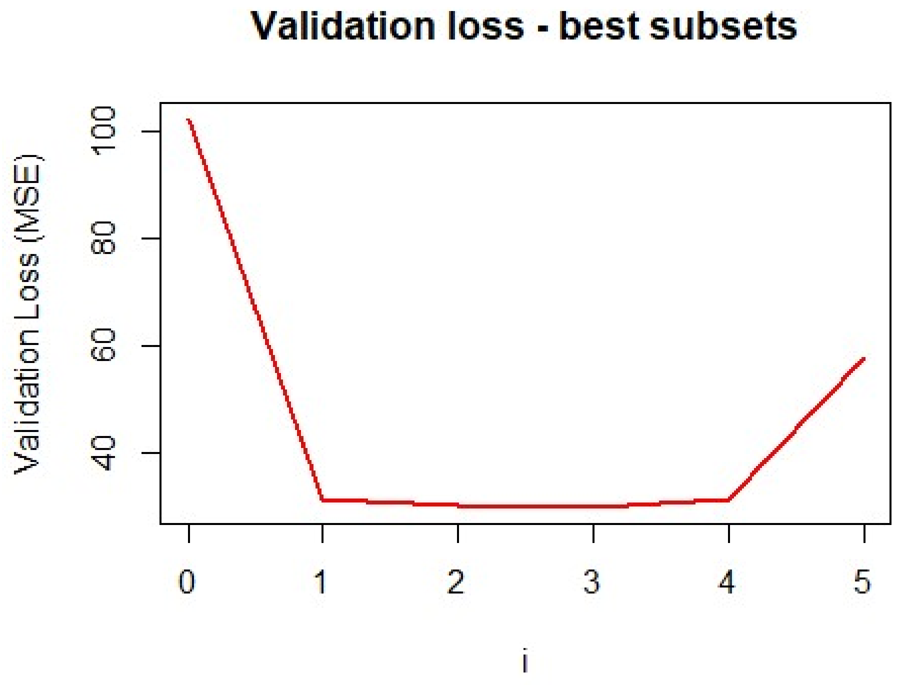
These six models are completely nested with largest order model. If we consider zero for the appropriate parameter(s), we can get any lower order model.

(ii)

```
plot(trset$subset, trset$tr_mse, type = "l", xlab = "i", lwd = 2, col = 2,
     ylab = "Training Loss (MSE)")
title("Training loss - best subsets")
```



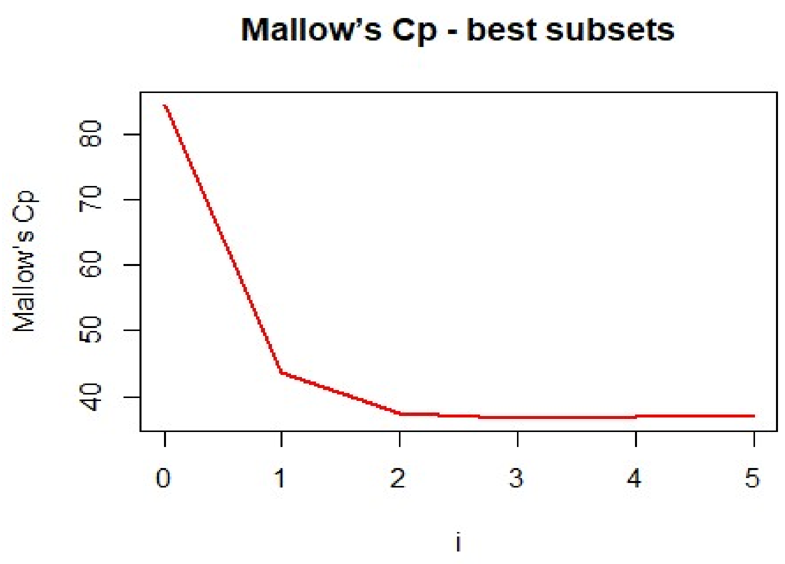
```r
plot(valset$subset, valset$ts_mse, type = "l", xlab = "i", lwd = 2, col = 2,
     ylab = "Validation Loss (MSE)")
title("Validation loss - best subsets")
```



As the number of features increases in the model, the training MSE decreases. However, based on the validation loss, it looks like "U" shaped. That indicates that more complex model is not always better for prediction.

(iii)

```
reg_res_cp = list()
tr_mse_cp = list()
k = 1
for (i in c(0:length(varn))){
  comb = combn(varn, i)
  for (j in c(1:dim(comb)[2])){
    if (dim(comb)[1]==0){
      formu = "MEDV ~ 1"
      comb = matrix(c("Intercept"), nrow = 1, ncol = 1)
    } else{
      formu = paste("MEDV ~ 1", paste((comb[,j]), collapse = "+"), sep = "+")
    }

    reg_res_cp[[k]] = lm(as.formula(formu), data = dt)
    trmse = mean(reg_res_cp[[k]]$residuals^2)
    tr_mse_cp[[k]] = list(i = i, var = paste((comb[,j]), collapse = ","),
                           tr_mse_cp = trmse)
    k = k+1
  }

}

res_cp = do.call(rbind.data.frame, tr_mse_cp)
names(res_cp) = c("subset", "variables", "tr_mse")
res_cp$cp = res_cp$tr_mse + 2*res_cp$subset*res_cp$tr_mse[nrow(res_cp)] / nro
w(dt)


trset_cp = NULL
for (i in c(0:length(varn))){
  aa = which(res_cp$cp == res_cp[res_cp$subset == i,][which.min(res_cp[res_cp
$subset == i,4]),4])
  trset_cp = rbind(trset_cp, res_cp[aa,])
  res_cp1 = reg_res_cp[[aa]]
  len1 = length(res_cp1$coefficients)
  if (i==0){
    print(paste("For subset model ", i, ":   y^hat = ", round(res_cp1$coeffic
ients[1], 3) ,
                "   with MSE = ", round(res_cp[aa,3],3)))
  } else {

    print(paste("For subset model ", i, ":   y^hat = ", round(res_cp1$coeffic
ients[1], 3), " + " ,
                paste(round(res_cp1$coefficients[2:len1], 3),
                      names(res_cp1$coefficients)[2:len1], sep = " * ", colla
pse = " + "),
                "   with Mallow’s Cp = ", round(res_cp[aa, 4],3)))
  }
}

aa = which.min(res_cp$cp)
res_cp1 = reg_res_cp[[aa]]
len1 = length(res_cp1$coefficients)
print(paste("For subset model ", res_cp$subset[aa], ":   y^hat = ", round(res
_cp1$coefficients[1], 3), " + " ,
            paste(round(res_cp1$coefficients[2:len1], 3),
                  names(res_cp1$coefficients)[2:len1], sep = " * ", collapse
= " + "),
            "   with Mallow’s Cp = ", round(res_cp[aa, 4],3)))

## [1] "For subset model  3 :   y^hat =  -18.955  +  -0.037 * AGE + 7.843 * R
M + -0.013 * TAX    with Mallow’s Cp =  36.73"

plot(trset_cp$subset, trset_cp$cp, type = "l", xlab = "i", lwd = 2, col = 2,
     ylab = "Mallow’s Cp")
title("Mallow’s Cp - best subsets")



## [1] "For subset model  0 :   y^hat =  22.533    with MSE =  84.42"
## [1] "For subset model  1 :   y^hat =  -34.671  +  9.102 * RM    with Mallo
w’s Cp =  43.744"
## [1] "For subset model  2 :   y^hat =  -21.233  +  7.993 * RM + -0.016 * TA
X    with Mallow’s Cp =  37.384"
## [1] "For subset model  3 :   y^hat =  -18.955  +  -0.037 * AGE + 7.843 * R
M + -0.013 * TAX    with Mallow’s Cp =  36.73"
## [1] "For subset model  4 :   y^hat =  -18.803  +  -0.035 * AGE + -0.022 *
INDUS + 7.808 * RM + -0.012 * TAX    with Mallow’s Cp =  36.866"
## [1] "For subset model  5 :   y^hat =  -18.497  +  -0.034 * AGE + -0.016 *
INDUS + -0.942 * NOX + 7.807 * RM + -0.012 * TAX    with Mallow’s Cp =  37.00
5"
```



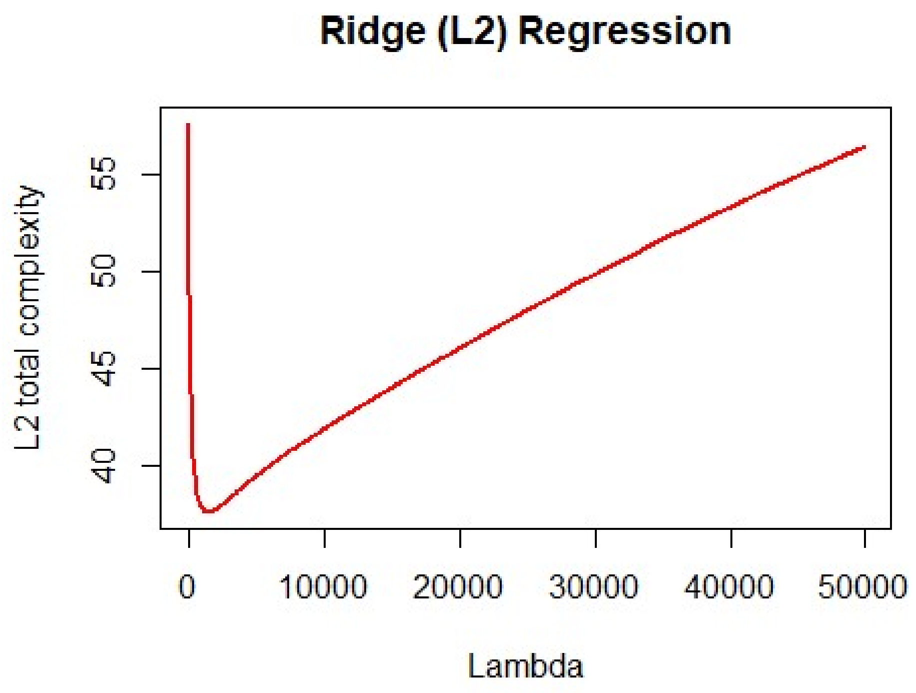
The best fitting linear model for every subset of AGE, INDUS, NOX, RM, TAX using all the samples based on C_p is the model with variables AGE, RM and TAX.

Plot using total complexity C_p and plot using validation set MSE show the same pattern. C_p gives more weight to the model with higher complexity. However, That does not mean C_p and validation set MSE behave the same way. It actually depends on data. For some data set, these two methods can provide the same result. However, if prediction is the main purpose of study, model section based on validation set total loss would be better.

(iv)

(i) - (iii)

```
dt_train = dt[1:400,]
dt_test = dt[401:dim(dt)[1],]
varn = c("AGE", "INDUS", "NOX", "RM", "TAX")

lambda = 0
l2_res = NULL

loop = TRUE

while (loop) {
  a = glmnet(x = as.matrix(dt_train[,varn]), y = dt_train[,"MEDV"], standardi
ze=FALSE,
             alpha = 0, lambda = lambda)
  pred = predict(a, as.matrix(dt_test[,varn]))
  l2 = mean((dt_test$MEDV - pred)^2) + lambda * sum(a$beta^2)
  l2_res = rbind(l2_res, c(a$lambda, l2))
  if (lambda > 1400 & lambda <1600){
    lambda = lambda + 0.05
  }else {
    lambda = lambda + 100
  }

  if(sum(a$beta^2) < 1e-5 | lambda > 50000) loop = FALSE
}

l2_res = as.data.frame(l2_res)
plot(l2_res$V1, l2_res$V2, type = "l", xlab = "Lambda", lwd = 2, col = 2,
     ylab = "L2 total complexity")
title("Ridge (L2) Regression")
```

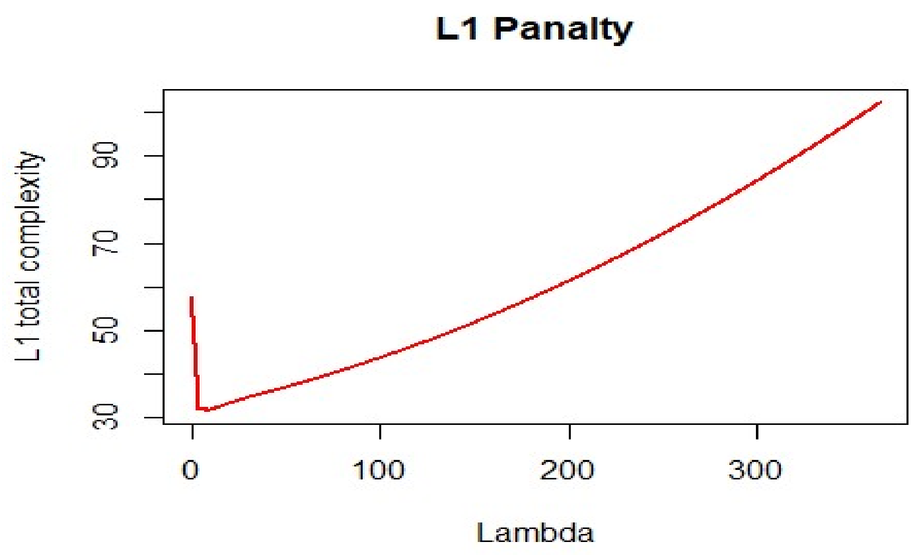


```
aa = which.min(l2_res$V2)
a = glmnet(x = as.matrix(dt_train[,varn]), y = dt_train[,"MEDV"], standardize
=FALSE,
             alpha = 0, lambda = l2_res[aa,1])
pred = predict(a, as.matrix(dt_test[,varn]))
l2 = mean((dt_test$MEDV - pred)^2)

print(paste("Model with lambda = ", round(l2_res[aa,1],3), ":   y^hat = ", ro
und(a$a0, 3),
            " + " ,
            paste(round(a$beta, 3),
                  row.names(a$beta), sep = " * ", collapse = " + "),
    "   with testing MSE = ", round(l2,3)))

## [1] "Model with lambda =  1503.3 :   y^hat =  33.047  +  -0.042 * AGE + -0
.045 * INDUS + 0 * NOX + 0.026 * RM + -0.017 * TAX    with testing MSE =  30.
591"

print(paste("Model with lambda = ", l2_res[aa,1], " has lowest validation MSE
, ", round(l2_res[aa,2],3)))


## [1] "Model with lambda =  1503.3  has lowest validation MSE,  37.628"
```

From this plot, we can see that model with $\lambda = 1503.3$ with AGE, INDUS, RM, TAX variables has the smallest validation set total complexity. However, using C_p and validation set MSE, we got smaller set model though both were different.

(iv)

```

lambda = 0
l1_res = NULL

loop = TRUE

while (loop) {
  a = glmnet(x = as.matrix(dt_train[,varn]), y = dt_train[,"MEDV"], standardi
ze=FALSE,
             alpha = 1, lambda = lambda)
  pred = predict(a, as.matrix(dt_test[,varn]))
  l1 = mean((dt_test$MEDV - pred)^2) + lambda * sum(abs(a$beta))
  l1_res = rbind(l1_res, c(a$lambda, l1))
  lambda = lambda + 0.05
  if(sum(abs(a$beta)) < 1e-8) loop = FALSE
}

l1_res = as.data.frame(l1_res)
plot(l1_res$V1, l1_res$V2, type = "l", xlab = "Lambda", lwd = 2, col = 2,
     ylab = "L1 total complexity")
title("L1 Panalty")
```



```
aa = which.min(l1_res$V2)
a = glmnet(x = as.matrix(dt_train[,varn]), y = dt_train[,"MEDV"], standardize
=FALSE,
                alpha = 1, lambda = l1_res[aa,1])
pred = predict(a, as.matrix(dt_test[,varn]))
mmse = mean((dt_test$MEDV - pred)^2)

print(paste("Model with lambda = ", round(l1_res[aa,1],3), ":   y^hat = ", ro
und(a$a0, 3),
              " + " ,
              paste(round(a$beta, 3),
                     row.names(a$beta), sep = " * ", collapse = " + "),
    "   with testing MSE = ", round(mmse,3)))

## [1] "Model with lambda =  8.6 :   y^hat =  33.229  +  -0.044 * AGE + -0.00
1 * INDUS + 0 * NOX + 0 * RM + -0.017 * TAX    with testing MSE =  31.106"

print(paste("Model with lambda = ", round(l1_res[aa,1], 3), " has lowest vali
dation MSE, ", round(mmse,3)))

## [1] "Model with lambda =  8.6  has lowest validation MSE,  31.106"
```

LASSO model also shows the same patter as Ridge regression. However, LASSO model selects smaller set variables that the Ridge regression. It includes only AGE, INDUS, TAX variables. It is important to mention that model selection based on different criteria selects different sets of variables in the respective best model.

(v)

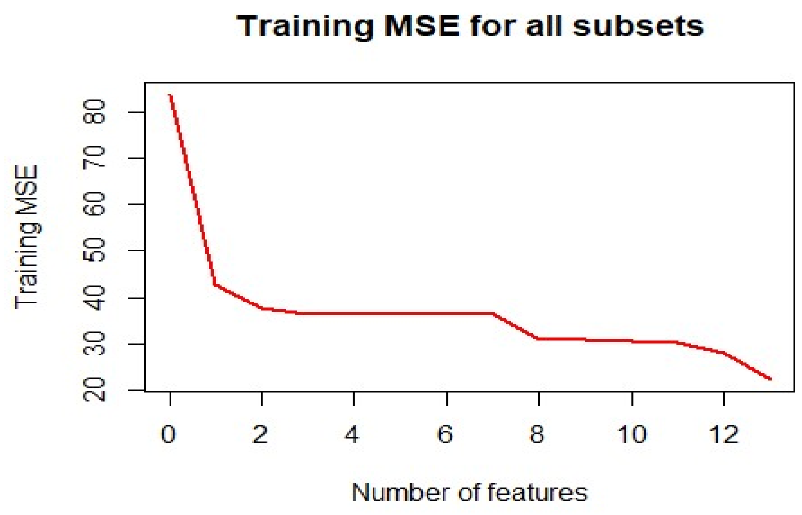
It is important to normalize all features when using L1 or L2 penalty for model estimation. Because, estimated parameters have a same unit as the corresponding variables and when we use these estimated parameters as the penalty factor, it will have high influence if the variable represents with very high values. As a result, prediction based on unnormalize features might be misleading.

(d)

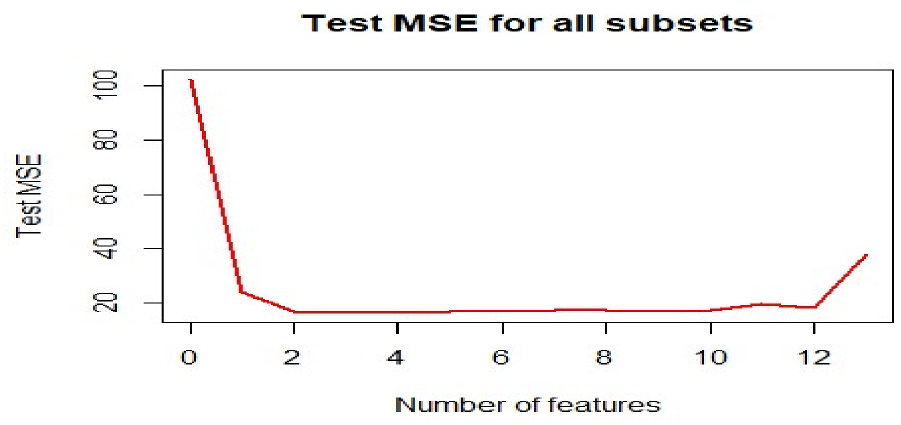
(i)

```
dt_train = dt[1:400,]
dt_test = dt[401:dim(dt)[1],]
varn = c("CRIM",     "ZN",      "INDUS",    "CHAS",     "NOX",      "RM",
         "AGE",      "DIS",     "RAD",      "TAX",      "PTRATIO",  "B" ,       "L
STAT")


varinclude = c()
varexlude = varn
fi_res = list()
fi_reg = list()
k=2

formu = "MEDV ~ 1"
tem_reg = lm(as.formula(formu), data = dt_train)
temp_res = list(0, "intercept",
                    mean(tem_reg$residuals^2),
                    mean((predict(tem_reg, newdata = dt_test)-dt_test$MEDV)^
2),
                    formu)


fi_reg[[1]] = tem_reg
fi_res[[1]] = c(temp_res)


while (!is.null(varexlude) & length(varexlude)>0) {
  tem_reg = list()
  temp_res = list()
```

```r
  for (i in c(1:length(varexlude))){

    if (is.null(varinclude)){
      formu = paste("MEDV ~ 1", varexlude[i], sep = "+")
    } else {
      formu = paste("MEDV ~ 1", paste(varinclude, collapse = "+"), varexlude[
i], sep = "+")
    }

    tem_reg[[i]] = lm(as.formula(formu), data = dt_train)
    temp_res[[i]] = list(length(varinclude)+1, varexlude[i],
                 mean(tem_reg[[i]]$residuals^2),
                 mean((predict(tem_reg[[i]], newdata = dt_test)-dt_test$MEDV)
^2),
                 formu)

  }

  temp_res = do.call(rbind.data.frame, temp_res)
  names(temp_res) = c("nvar", "var_include", "tr_mse", "ts_mse", "formula")
  temp_res$var_include = as.character(temp_res$var_include)
  temp_res$formula = as.character(temp_res$formula)
  varinclude = c(varinclude, as.character(temp_res[which.min(temp_res$ts_mse)
, 2]))
  varexlude = varexlude[!(varexlude %in% varinclude)]

  fi_reg[[k]] = tem_reg[[which.min(temp_res$ts_mse)]]
  fi_res[[k]] = c(temp_res[which.min(temp_res$ts_mse),])

  k=k+1
}

fi_res = do.call(rbind.data.frame, fi_res)
names(fi_res) = c("nvar", "var_include", "tr_mse", "ts_mse", "formula")
forward_res = fi_res

plot(fi_res$nvar, fi_res$tr_mse, type = "l", xlab = "Number of features", lwd
= 2, col = 2,
     ylab = "Training MSE")
title("Training MSE for all subsets")
```



```
plot(fi_res$nvar, fi_res$ts_mse, type = "l", xlab = "Number of features", lwd
= 2, col = 2,
     ylab = "Test MSE")
title("Test MSE for all subsets")
```



```
print(fi_res)

print(fi_res[which.min(fi_res$ts_mse),])

aa = which.min(fi_res$ts_mse)
res_131 = fi_reg[[aa]]
len1 = length(res_131$coefficients)
print(paste("Forward Best Subset Model :", fi_res[aa,]$formula, ":   y^hat =
", round(res_131$coefficients[1], 3), " + " ,
            paste(round(res_131$coefficients[2:len1], 3),
                  names(res_131$coefficients)[2:len1], sep = " * ", collapse
= " + "),
            "   with training MSE = ", round(fi_res[aa,3],3),
            " and validation MSE = ", round(fi_res[aa,4],3)))
```

```
##    nvar var_include   tr_mse    ts_mse
## 2     0   intercept 83.80701 102.22659
## 21    1       LSTAT 42.62158  23.92312
## 3     2     PTRATIO 37.65518  16.76650
## 4     3        CHAS 36.43344  16.55229
## 5     4          ZN 36.42852  16.61204
## 6     5       INDUS 36.41458  16.87186
## 7     6         TAX 36.38863  17.15157
## 8     7         NOX 36.37764  17.36296
## 9     8         DIS 30.84108  17.31292
## 10    9        CRIM 30.83273  16.81341
## 11   10         AGE 30.51697  17.53081
## 12   11           B 30.28257  19.59739
## 13   12         RAD 28.17532  18.49887
## 14   13          RM 22.30523  37.89378
##                                                                    formula
## 2                                                                 MEDV ~ 1
## 21                                                          MEDV ~ 1+LSTAT
## 3                                                   MEDV ~ 1+LSTAT+PTRATIO
## 4                                              MEDV ~ 1+LSTAT+PTRATIO+CHAS
## 5                                           MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN
## 6                                     MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS
## 7                                 MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX
## 8                             MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX
## 9                         MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS
## 10                   MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS+CRIM
## 11               MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS+CRIM+AGE
## 12             MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS+CRIM+AGE+B
## 13         MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS+CRIM+AGE+B+RAD
## 14 MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS+CRIM+AGE+B+RAD+RM
```

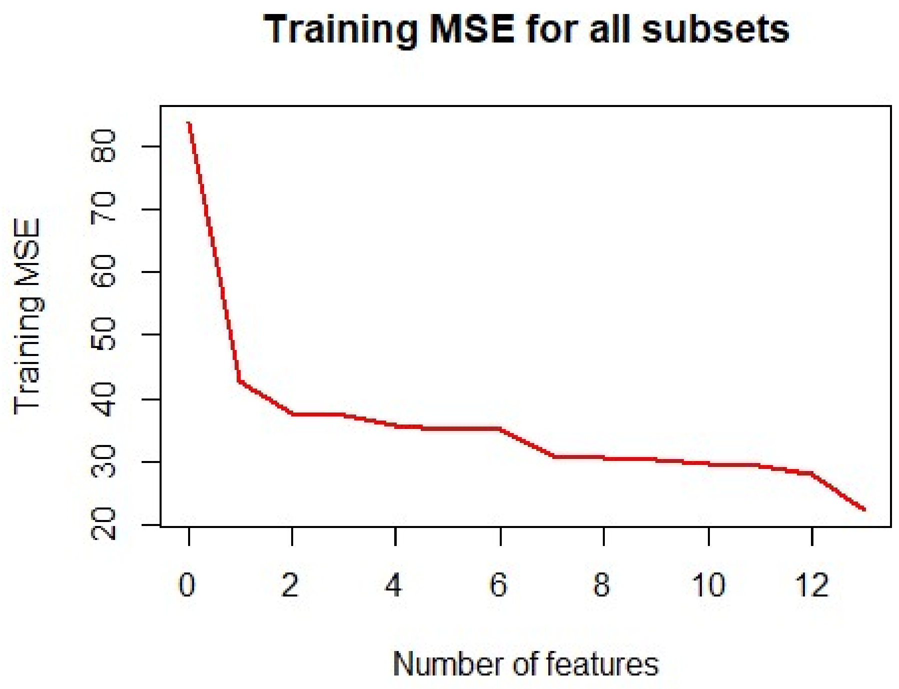
```
##   nvar var_include   tr_mse   ts_mse                     formula
## 4    3        CHAS 36.43344 16.55229 MEDV ~ 1+LSTAT+PTRATIO+CHAS

## [1] "Forward Best Subset Model : MEDV ~ 1+LSTAT+PTRATIO+CHAS :   y^hat = 
51.786  +  -0.853 * LSTAT + -1.011 * PTRATIO + 3.923 * CHAS    with training 
MSE =  36.433  and validation MSE =  16.552"
```

Using forward search, based on validation MSE, the best model includes three variables (LSTAT, PTRATIO, CHAS).

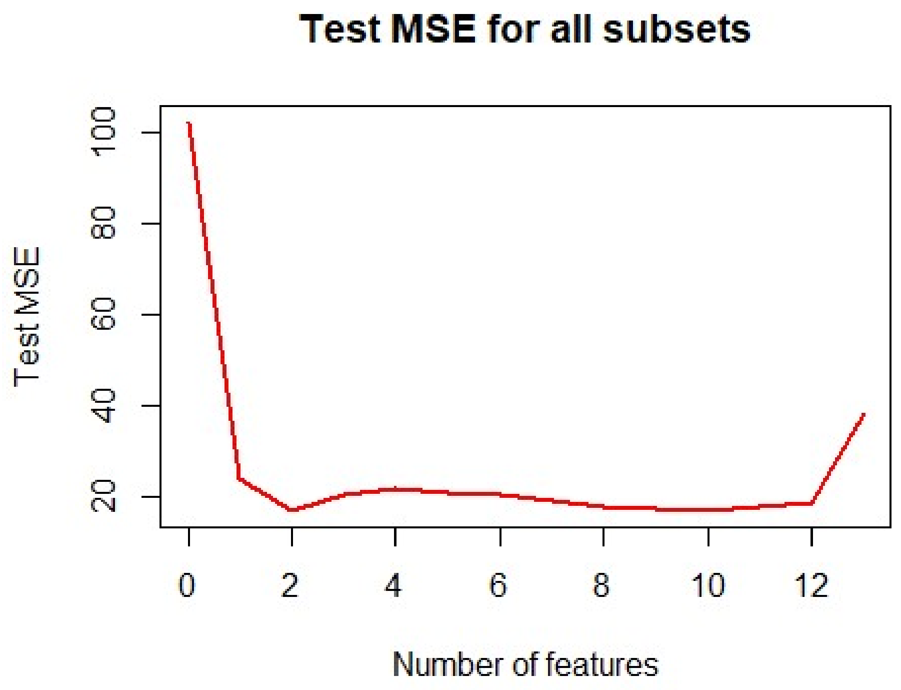
(d)

(ii)

```
dt_train = dt[1:400,]
dt_test = dt[401:dim(dt)[1],]
varn = c("CRIM",    "ZN",      "INDUS",   "CHAS",    "NOX",     "RM",
         "AGE",     "DIS",     "RAD",     "TAX",     "PTRATIO", "B" ,      "L
STAT")

varinclude = varn
varexlude = c()
fi_res = list()
fi_reg = list()
k=2

formu = paste("MEDV ~ 1", paste(varinclude, collapse = "+"), sep = "+")
tem_reg = lm(as.formula(formu), data = dt_train)
temp_res = list(13, "None",
                mean(tem_reg$residuals^2),
                mean((predict(tem_reg, newdata = dt_test)-dt_test$MEDV)^2),
                formu)

fi_reg[[1]] = tem_reg
fi_res[[1]] = c(temp_res)

while (!is.null(varinclude) & length(varinclude)>0) {
  tem_reg = list()
  temp_res = list()

  for (i in c(1:length(varinclude))){

    if (is.null(varinclude) | length(varinclude)==1){
      formu = "MEDV ~ 1"
    } else {
      formu = paste("MEDV ~ 1", paste(varinclude[-i], collapse = "+"), sep =
"+")
    }

    tem_reg[[i]] = lm(as.formula(formu), data = dt_train)
    temp_res[[i]] = list(length(varinclude)-1, varinclude[i],
                         mean(tem_reg[[i]]$residuals^2),
                         mean((predict(tem_reg[[i]], newdata = dt_test)-

                               dt_test$MEDV)^2),  formu)
  }
  temp_res = do.call(rbind.data.frame, temp_res)
  names(temp_res) = c("nvar", "var_exclude", "tr_mse", "ts_mse", "formula")
  temp_res$var_exclude = as.character(temp_res$var_exclude)
  temp_res$formula = as.character(temp_res$formula)
  varexlude = c(varexlude, as.character(temp_res[which.min(temp_res$ts_mse),
2]))
  varinclude = varinclude[!(varinclude %in% varexlude)]

  fi_reg[[k]] = tem_reg[[which.min(temp_res$ts_mse)]]
  fi_res[[k]] = c(temp_res[which.min(temp_res$ts_mse),])

  k=k+1
}

fi_res = do.call(rbind.data.frame, fi_res)
names(fi_res) = c("nvar", "var_include", "tr_mse", "ts_mse", "formula")
backward_res = fi_res

plot(fi_res$nvar, fi_res$tr_mse, type = "l", xlab = "Number of features", lwd
= 2, col = 2,
     ylab = "Training MSE")
title("Training MSE for all subsets")
```



```
plot(fi_res$nvar, fi_res$ts_mse, type = "l", xlab = "Number of features", lwd
= 2, col = 2,
     ylab = "Test MSE")
title("Test MSE for all subsets")
```



```
print(fi_res)

print(fi_res[which.min(fi_res$ts_mse),])

aa = which.min(fi_res$ts_mse)
res_131 = fi_reg[[aa]]
len1 = length(res_131$coefficients)
print(paste("For subset model ", fi_res[aa,]$formula, ":   y^hat = ", round(r
es_131$coefficients[1], 3), " + " ,
            paste(round(res_131$coefficients[2:len1], 3),
                  names(res_131$coefficients)[2:len1], sep = " * ", collapse
= " + "),
            "   with training MSE = ", round(fi_res[aa,3],3),
            " and validation MSE = ", round(fi_res[aa,4],3)))



##    nvar var_include   tr_mse    ts_mse
## 2    13        None 22.30523  37.89378
## 21   12          RM 28.17532  18.49887
## 3    11         TAX 29.25133  17.51549
## 4    10         AGE 29.58767  16.93445
## 5     9          ZN 30.43122  17.26423
## 6     8        CRIM 30.72554  17.80309
## 7     7       INDUS 30.90940  18.75269
## 8     6         DIS 34.93004  20.21552
## 9     5         NOX 34.94184  20.86509
## 10    4        CHAS 35.82823  21.82590
## 11    3         RAD 37.32106  20.31659
## 12    2           B 37.65518  16.76650
## 13    1     PTRATIO 42.62158  23.92312
## 14    0       LSTAT 83.80701 102.22659
##                                                                formula
## 2  MEDV ~ 1+CRIM+ZN+INDUS+CHAS+NOX+RM+AGE+DIS+RAD+TAX+PTRATIO+B+LSTAT
## 21    MEDV ~ 1+CRIM+ZN+INDUS+CHAS+NOX+AGE+DIS+RAD+TAX+PTRATIO+B+LSTAT
## 3         MEDV ~ 1+CRIM+ZN+INDUS+CHAS+NOX+AGE+DIS+RAD+PTRATIO+B+LSTAT
## 4             MEDV ~ 1+CRIM+ZN+INDUS+CHAS+NOX+DIS+RAD+PTRATIO+B+LSTAT
## 5                MEDV ~ 1+CRIM+INDUS+CHAS+NOX+DIS+RAD+PTRATIO+B+LSTAT
## 6                     MEDV ~ 1+INDUS+CHAS+NOX+DIS+RAD+PTRATIO+B+LSTAT
## 7                           MEDV ~ 1+CHAS+NOX+DIS+RAD+PTRATIO+B+LSTAT
## 8                               MEDV ~ 1+CHAS+NOX+RAD+PTRATIO+B+LSTAT
## 9                                   MEDV ~ 1+CHAS+RAD+PTRATIO+B+LSTAT
## 10                                       MEDV ~ 1+RAD+PTRATIO+B+LSTAT
## 11                                           MEDV ~ 1+PTRATIO+B+LSTAT
## 12                                             MEDV ~ 1+PTRATIO+LSTAT
## 13                                                     MEDV ~ 1+LSTAT
## 14                                                           MEDV ~ 1
```

```
##    nvar var_include   tr_mse  ts_mse                formula
## 12    2           B 37.65518 16.7665 MEDV ~ 1+PTRATIO+LSTAT
```

```
## [1] "For subset model  MEDV ~ 1+PTRATIO+LSTAT :   y^hat =  52.79  +  -1.05
* PTRATIO + -0.85 * LSTAT    with training MSE =  37.655  and validation MSE
=  16.766"
```

Using backward search, based on validation MSE, the best model includes only two variables (LSTAT, PTRATIO).

(d)

(iii)

```
print(forward_res)
```

```
##    nvar var_include   tr_mse    ts_mse
## 2     0   intercept 83.80701 102.22659
## 21    1       LSTAT 42.62158  23.92312
## 3     2     PTRATIO 37.65518  16.76650
## 4     3        CHAS 36.43344  16.55229
## 5     4          ZN 36.42852  16.61204
## 6     5       INDUS 36.41458  16.87186
## 7     6         TAX 36.38863  17.15157
## 8     7         NOX 36.37764  17.36296
## 9     8         DIS 30.84108  17.31292
## 10    9        CRIM 30.83273  16.81341
## 11   10         AGE 30.51697  17.53081
## 12   11           B 30.28257  19.59739
## 13   12         RAD 28.17532  18.49887
## 14   13          RM 22.30523  37.89378
##                                                               formula
## 2                                                            MEDV ~ 1
## 21                                                     MEDV ~ 1+LSTAT
## 3                                              MEDV ~ 1+LSTAT+PTRATIO
## 4                                         MEDV ~ 1+LSTAT+PTRATIO+CHAS
## 5                                      MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN
## 6                                MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS
## 7                            MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX
## 8                        MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX
## 9                    MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS
## 10              MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS+CRIM
## 11          MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS+CRIM+AGE
## 12        MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS+CRIM+AGE+B
## 13    MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS+CRIM+AGE+B+RAD
## 14 MEDV ~ 1+LSTAT+PTRATIO+CHAS+ZN+INDUS+TAX+NOX+DIS+CRIM+AGE+B+RAD+RM
```

```r
print(backward_res)
```

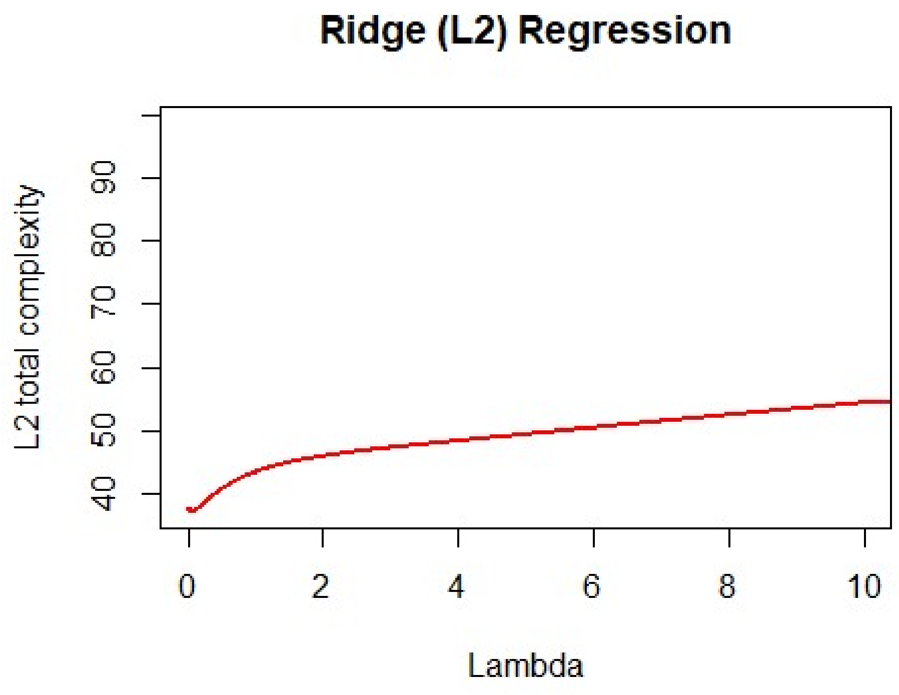
```
##    nvar var_include   tr_mse    ts_mse
## 2    13        None 22.30523  37.89378
## 21   12          RM 28.17532  18.49887
## 3    11         TAX 29.25133  17.51549
## 4    10         AGE 29.58767  16.93445
## 5     9          ZN 30.43122  17.26423
## 6     8        CRIM 30.72554  17.80309
## 7     7       INDUS 30.90940  18.75269
## 8     6         DIS 34.93004  20.21552
## 9     5         NOX 34.94184  20.86509
## 10    4        CHAS 35.82823  21.82590
## 11    3         RAD 37.32106  20.31659
## 12    2           B 37.65518  16.76650
## 13    1     PTRATIO 42.62158  23.92312
## 14    0       LSTAT 83.80701 102.22659
##                                                               formula
## 2  MEDV ~ 1+CRIM+ZN+INDUS+CHAS+NOX+RM+AGE+DIS+RAD+TAX+PTRATIO+B+LSTAT
## 21    MEDV ~ 1+CRIM+ZN+INDUS+CHAS+NOX+AGE+DIS+RAD+TAX+PTRATIO+B+LSTAT
## 3         MEDV ~ 1+CRIM+ZN+INDUS+CHAS+NOX+AGE+DIS+RAD+PTRATIO+B+LSTAT
## 4             MEDV ~ 1+CRIM+ZN+INDUS+CHAS+NOX+DIS+RAD+PTRATIO+B+LSTAT
## 5                MEDV ~ 1+CRIM+INDUS+CHAS+NOX+DIS+RAD+PTRATIO+B+LSTAT
## 6                     MEDV ~ 1+INDUS+CHAS+NOX+DIS+RAD+PTRATIO+B+LSTAT
## 7                                MEDV ~ 1+CHAS+NOX+DIS+RAD+PTRATIO+B+LSTAT
## 8                                    MEDV ~ 1+CHAS+NOX+RAD+PTRATIO+B+LSTAT
## 9                                        MEDV ~ 1+CHAS+RAD+PTRATIO+B+LSTAT
## 10                                            MEDV ~ 1+RAD+PTRATIO+B+LSTAT
## 11                                                MEDV ~ 1+PTRATIO+B+LSTAT
## 12                                                  MEDV ~ 1+PTRATIO+LSTAT
## 13                                                          MEDV ~ 1+LSTAT
## 14                                                                MEDV ~ 1
```

All lower order models are nested in the full model with all features. The best model found from backward search is nested within model found from forward search because forward model includes LSTAT, PTRATIO, CHAS variables and backward model includes only LSTAT, PTRATIO that can be found by considering CHAS coefficient equal zero.

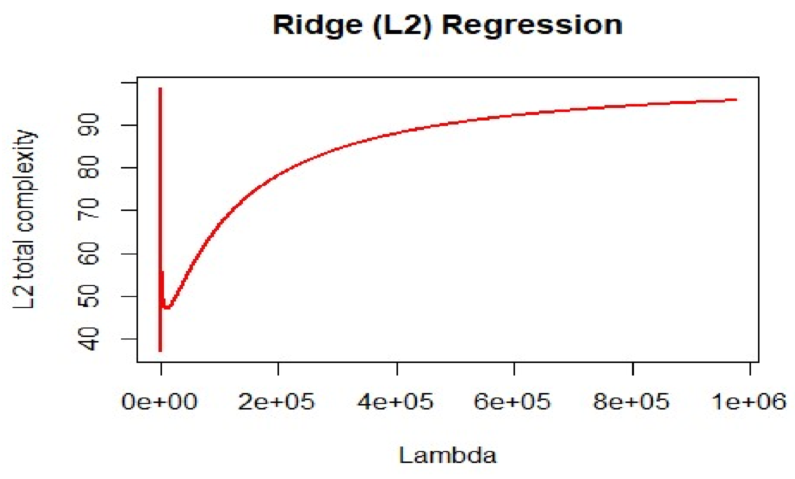
(d)

(iv)

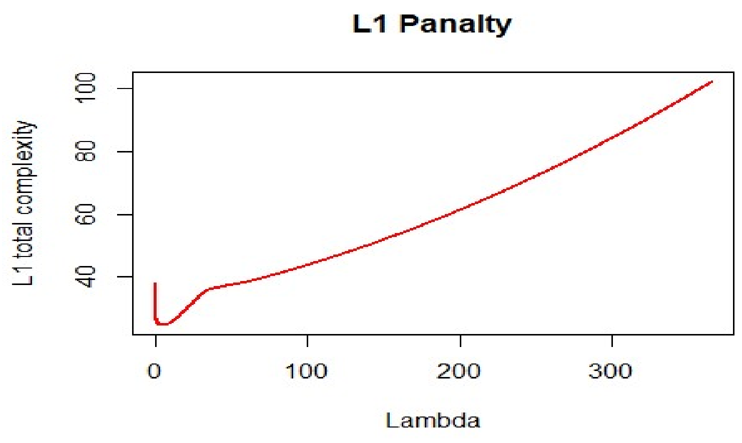
```
# Ridge

dt_train = dt[1:400,]
dt_test = dt[401:dim(dt)[1],]
varn = c("CRIM",    "ZN",       "INDUS",    "CHAS",     "NOX",      "RM",
         "AGE",     "DIS",      "RAD",      "TAX",      "PTRATIO", "B" ,       "L
STAT")

lambda = 0
l2_res = NULL

loop = TRUE

while (loop) {
  a = glmnet(x = as.matrix(dt_train[,varn]), y = dt_train[,"MEDV"], standardi
ze=FALSE,
             alpha = 0, lambda = lambda)
  pred = predict(a, as.matrix(dt_test[,varn]))
  l2 = mean((dt_test$MEDV - pred)^2) + lambda * sum(a$beta^2)
  l2_res = rbind(l2_res, c(a$lambda, l2))
  if (lambda <10){
    lambda = lambda + .005
  }else {
    lambda = lambda + 100
  }

  if(sum(a$beta^2) < 1e-5 | lambda > 1500000) loop = FALSE
}

l2_res = as.data.frame(l2_res)
plot(l2_res$V1, l2_res$V2, type = "l", xlab = "Lambda", lwd = 2, col = 2, xli
m = c(0, 10),
     ylab = "L2 total complexity")
title("Ridge (L2) Regression")
```



```
plot(l2_res$V1, l2_res$V2, type = "l", xlab = "Lambda", lwd = 2, col = 2,
     ylab = "L2 total complexity")
title("Ridge (L2) Regression")
```



```
aa = which.min(l2_res$V2)
a = glmnet(x = as.matrix(dt_train[,varn]), y = dt_train[,"MEDV"], standardize
=FALSE,
             alpha = 0, lambda = l2_res[aa,1])
pred = predict(a, as.matrix(dt_test[,varn]))
l2 = mean((dt_test$MEDV - pred)^2)

print(paste("Model with lambda = ", round(l2_res[aa,1],3), ":   y^hat = ", ro
und(a$a0, 3),
            " + " ,
            paste(round(a$beta, 3),
                  row.names(a$beta), sep = " * ", collapse = " + "),
    "   with testing MSE = ", round(l2,3)))

## [1] "Model with lambda =  0.065 :   y^hat =  21.105   +  -0.184 * CRIM + 0.
047 * ZN + 0.006 * INDUS + 1.548 * CHAS + -4.246 * NOX + 4.849 * RM + -0.005
* AGE + -1.159 * DIS + 0.46 * RAD + -0.017 * TAX + -0.696 * PTRATIO + 0.002 *
B + -0.539 * LSTAT    with testing MSE =  34.081"

print(paste("Model with lambda = ", l2_res[aa,1], " has lowest validation MSE
, ", round(l2_res[aa,2],3)))

## [1] "Model with lambda =  0.065  has lowest validation MSE,  37.091"

# LASSO

lambda = 0
l1_res = NULL

loop = TRUE

while (loop) {
  a = glmnet(x = as.matrix(dt_train[,varn]), y = dt_train[,"MEDV"], standardi
ze=FALSE,
             alpha = 1, lambda = lambda)
  pred = predict(a, as.matrix(dt_test[,varn]))
  l1 = mean((dt_test$MEDV - pred)^2) + lambda * sum(abs(a$beta))
  l1_res = rbind(l1_res, c(a$lambda, l1))
  lambda = lambda + 0.05
  if(sum(abs(a$beta)) < 1e-8) loop = FALSE
}

l1_res = as.data.frame(l1_res)
plot(l1_res$V1, l1_res$V2, type = "l", xlab = "Lambda", lwd = 2, col = 2,
     ylab = "L1 total complexity")
title("L1 Panalty")
```



```
aa = which.min(l1_res$V2)
a = glmnet(x = as.matrix(dt_train[,varn]), y = dt_train[,"MEDV"], standardize
=FALSE,
             alpha = 1, lambda = l1_res[aa,1])
pred = predict(a, as.matrix(dt_test[,varn]))
mmse = mean((dt_test$MEDV - pred)^2)

print(paste("Model with lambda = ", round(l1_res[aa,1],3), ":   y^hat = ", ro
und(a$a0, 3),
            " + " ,
            paste(round(a$beta, 5),
                  row.names(a$beta), sep = " * ", collapse = " + "),
    "   with testing MSE = ", round(mmse,3)))
```

```
## [1] "Model with lambda =  5.1 :   y^hat =  29.115  +  0 * CRIM + 0.04001 *
ZN + 0 * INDUS + 0 * CHAS + 0 * NOX + 0 * RM + 0.05182 * AGE + 0 * DIS + 0 *
RAD + -0.00581 * TAX + 0 * PTRATIO + 0.00771 * B + -0.85077 * LSTAT    with t
esting MSE =  19.898"
```

```
print(paste("Model with lambda = ", round(l1_res[aa,1], 3), " has lowest vali
dation MSE, ", round(mmse,3)))
```

```
## [1] "Model with lambda =  5.1  has lowest validation MSE,  19.898"
```

Using Ridge regression best model found for $\lambda = 0.065$ which includes all variables with validation $MSE = 37.091$. On the other hand, best LASSO model found fro $\lambda = 5.1$ with validation $MSE = 19.898$ which includes only ZN, AGE, TAX, B and LSTAT and it is mach lower that Ridge regression. However, forward search model includes only LSTAT, PTRATIO, CHAS variables with validation $MSE = 16.552$ and it is the lowest MSE among forward, backward, Ridge and LASSO models.