Experiential Preference Model

The friend of my friend is?

Complex Social Systems: Modeling Agents, Learning and Games
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Abstract

In the diverse fields of social, behavioral, and economic sciences, a key area of study revolves around predicting how individual opinions will converge within a large group. The primary focus of this research is on understanding whether consensus can be reached and the duration required for this process. Consensus models, a subset of which includes opinion dynamics models, are crucial for evaluating whether a group of interacting individuals can agree on various choices, such as political voting, personal beliefs, or cultural traits. Opinion dynamics models specifically deal with the likelihood of reaching a unanimous opinion. These models are particularly relevant in large groups where moderate viewpoints, not strongly aligned with extreme political opinions, are common. Investigating the impact of moderates is essential for a deeper understanding of how political opinions evolve in large populations.

The Deffuant model is a key tool in this area, analyzing how consensus forms through random pairwise interactions that gradually shape the opinion distribution of a population.

In this study, we modify the traditional Deffuant model by introducing different matching schemes that more accurately reflect the influence of social networks, like friends and their connections. Our goal is to examine how these matching schemes affect the emergence of consensus in opinion dynamics within large populations.

We show that our model achieves opinion polarization faster and with greater certainty that the original model.

Contents

I. Introduction and Related Work

In the contemporary landscape of social dynamics, understanding the underpinnings of social influence is crucial. This importance is amplified in contexts like consensus formation, polarization, extremism, and the outcomes of various social and political initiatives. Such an understanding aids not only in comprehending present-day phenomena but also in the simulation and prediction of future trends, particularly in areas influenced by social media and cultural, political, or marketing trends. The modern realm of opinion dynamics situates itself within this rich tapestry of interdisciplinary insights, drawing from psychology, economics, physics, and computer science to provide a holistic view of human behavior and opinion formation.

After adopting models from biology e.g. the voter model (1970s [7]) –a discrete binary model– as well as finding inspiration in physics by adapting the Ising model (2000s [2]) –likewise a discrete but more complex model– technological progress enabled the developments and usage of more complex and nuanced approaches. This is especially true in the context of considering a continuous opinion space.

A significant contribution to the field of opinion dynamics came with models like the Deffuant model [4]. Developed in the early 2000s by Guillaume Deffuant and colleagues, the Deffuant model is a continuous opinion model under the umbrella of bounded confidence models. It posits that the extent of the influence on one individuals opinion is dependent on the difference between the opinion of the interaction individuals. This general expression can be formulated in different mathematical ways, so that the influence might be decreased up until to equal to zero, if the opinions are too far from each other. This feature of BCM's encapsulates the psychological effect of cognitive dissonance and the social behaviour that individuals are more prone to be in touch with like minded people. This range based on which the agents more or less influence each other is known as the confidence interval. The classic Deffuant model is characterized by pairwise interactions, where random encounters between individuals lead to adjustments in opinions. The core interaction rule states that if the difference in opinions is within the confidence bound, the opinions converge reciprocally and equivalentltowards an average.

This model has been compared to others like the Hegselmann-Krause model [8], which also uses continuous opinions but incorporates group influence, and the Axelrod model [1] which allows for multiple variables per agent. The strengths of the Deffuant model lie in its simplicity and adaptability to various scenarios, making it a valuable tool in the study of opinion dynamics. However, it is not without limitations. The classic Deffuant model assumes rational averaging of opinions (variations in averaging are examined in [6]), overlooks external factors and one sided interactions, like media influence, and relies on random interactions. It also does not account for gradual or delayed opinion changes over time, which can be crucial in understanding real-world opinion dynamics. Some of these limitations are the starting point of our study at hand, that tries to find the influence of alterations in the classic Deffuant model on its convergence types.

The Deffuant model, with its focus on bounded confidence and pairwise interactions, offers a framework to explore how opinions evolve and interact within a social context. This makes it a pivotal point of reference in our research project, where we aim to delve deeper into the intricacies of opinion dynamics and propose an alteration to the Deffuant model.

The report is organised as follows. We first provide the details on the classic Deffuant model. We then proceed to introduce two novelties we inserted to enrich the realism of the model. Finally, the experimental indicators we use to define the types of convergence. Finally we present the experimental analysis conducted, as well as the convergence results, concluding with an outlook for potential improvements and ideas to look into in future.

II. The Deffuant Model Framework

In this section, we aim to establish the fundamental mathematical principles underlying most bounded confidence models (BCMs).

The simplest form of BCM is exemplified by the bounded confidence model, initially introduced by Deffuant et al. [4]. This model assigns each agent not only an opinion but also a threshold, akin to a measure of uncertainty, beyond which it disregards divergent opinions. The classic Deffuant model (henceforth referred to as "classic model") is characterized by an iterative process that encompasses all agents within the model. During each iteration (henceforth mentioned as "time step"), every individual agent is matched with another randomly selected agent, and these matches engage in interactions with one another. This fundamental process of match-wise interaction lies at the heart of the Deffuant model, serving as the driving force behind the dynamic evolution of opinions within the system. A visualization of the Deffuant system algorithm is provided in figure 3.

The update of opinion x and threshold θ of Agent 1 and the opinion x' and threshold θ' of Agent 2 are updated according to following equations.

$$x := x + \mu \cdot k_\theta\left(x, x', \theta'\right) \cdot \left(x' - x\right) \tag{1}$$
$$x' := x' + \mu \cdot k_{\theta'}\left(x', x, \theta\right) \cdot \left(x - x'\right) \tag{2}$$

$$\theta := \theta + \mu \cdot k_\theta\left(x, x', \theta'\right) \cdot \left(\theta' - \theta\right) \tag{3}$$
$$\theta' := \theta' + \mu \cdot k_{\theta'}\left(x, x', \theta\right) \cdot \left(\theta - \theta'\right) \tag{4}$$

While μ represents the interaction intensity, a hyperparameter chosen at the experiment initialization, k_θ and k'_θ are the kernel functions of the respective agent. Different kernel functions can be chosen as we showcase in two examples. The simplest BCM uses the heaviside stepfunction as their kernel, which can be written as:

$$k_\theta\left(x, x', \theta'\right) = h_\theta\left(x, x'\right) = \begin{cases} 1 & \text{if } |\mathrm{x} - \mathrm{x}'| < \theta \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and is also visualized in Figure 1

However a Gaussian Bounded Confidence Model is also often used that can be expressed in following terms:

$$k_\theta\left(x, x', \theta'\right) = g_\theta\left(x, x'\right) = \exp\left(-\left(\frac{x - x'}{\theta}\right)^2\right) \tag{6}$$

The kernel functions k_θ for the Gaussian BCM is schematically plotted in fig. Figure 2a.

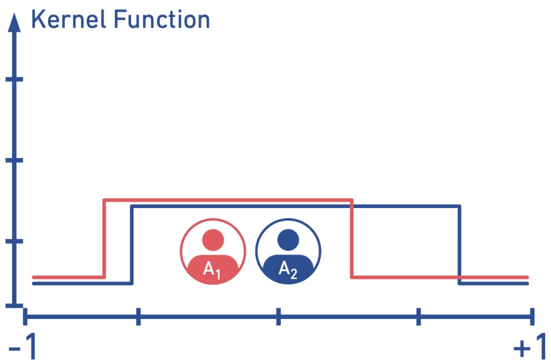
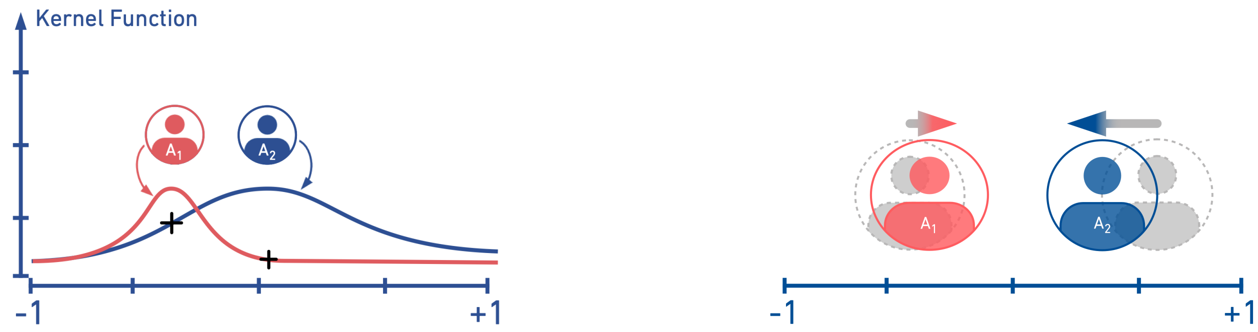


Figure 1: The Bounded Confidence Model (BCM), which uses the Heaviside kernel function h.



(a) The crosses indicate the kernel values of one agent in the kernel function of the other agent

(b) The change of opinion of A_1 and A_2 as they mutually interact.

Figure 2: Opinion changes in the classic Deffuant model

The two above illustrations in fig. 2 explain how opinion changes can be influenced by the form and position of the kernel function and the respective opinion. In this case the two gaussian kernel functions are not equally sharp, indicating different tolerance thresholds. Put into context, this means that agent A_1 is less tolerant towards the opinion of agent A_2 and hence will be influenced less through their interaction.

The algorithm of the classic Deffuant model, shown in fig. 3 operates through a series of systematic steps to simulate opinion dynamics in a population of agents. The process begins with an initialization phase, where the agents' opinions and tolerances are –often in a uniformly random way– established within a graph structure. It is important to note that the terms "tolerance threshold" and "bounded confidence parameter" that are used in various literature, all refer to the same variable of θ (henceforth referred to as the "tolerance threshold"). Each agent is situated within this graph, which represents the social network or connections among individuals.

Following initialization, the algorithm enters a loop that repeats for each agent in the system. In this loop, an agent is paired with a neighbor selected uniformly at random from the network. Once paired, the agent interacts with this chosen neighbor. The interaction consists of two potential changes: an adjustment of the agent's opinion and an alteration of their tolerance levels. These adjustments are based on the specific rules of the Deffuant model, which typically involve convergence towards a common opinion if the difference between the agents' opinions is within their tolerance levels.

The process continues iteratively, with agents continuously matching and interacting with random neighbors, thereby evolving the overall opinion distribution of the population. This loop is governed by a termination condition, which checks whether certain criteria have been met to stop the simulation. These criteria can be related to the convergence of opinions, the number of iterations, or the stability of the system.

If the termination condition is not satisfied, the algorithm loops back, and agents continue to interact. If the condition is met, the simulation concludes, marking the end of the process. At this point, the final distribution of opinions can be analyzed to understand the outcomes of the simulation, such as the formation of consensus or the persistence of diverse opinion clusters within the population, later reffered to as "convergence types".

A summary of the provided description is given in Figure 3

III. Our Model

The purpose of the enhanced model is to incorporate the innate tendency of humans to prefer to interact with people with whom they have had a pleasant prior interactions, into the classic Deffuant model.

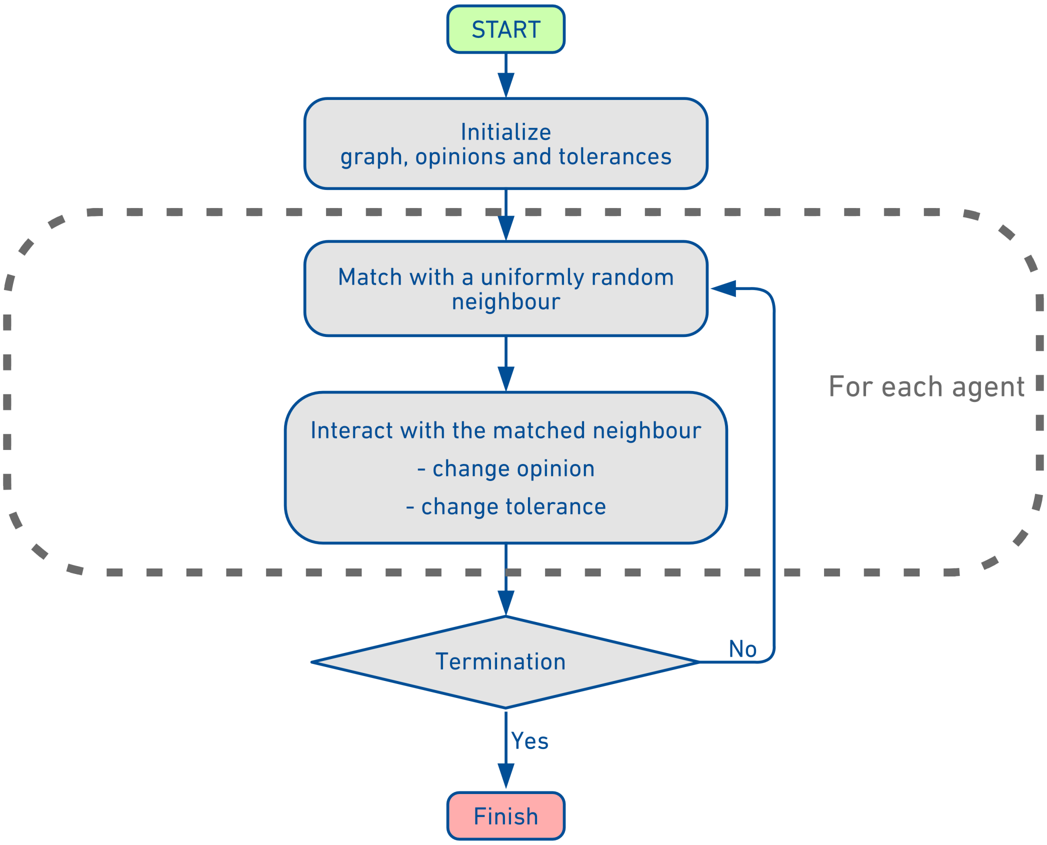


Figure 3: The simulation algorithm described in the classic Deffuant model

For that purpose, one novelty and one alteration were inserted into the classic model. The former is *link strengths*, while the latter is a slightly more complex, yet intuitive, *scheme to determine the matches formed at each time step.* [1].

A diagrammatical overview of our alterations to the classic model is present in Figure 4

A. Link strengths

The link strength integer map captures the memories and experience agent A_2 has accumulated about their interactions with other agents A_2 of the system up until time step t [2]. The more positive the link strength of agent A_1 towards A_2 is, the more constructive A_1 perceived their interactions with A_2. Similarly, the more negative the link strength, the more times A_1 has disagreed with A_2. A link strength of zero indicates a neutral experience with them. It should be stressed that a neutral experience is not identical to an absence of experience.

A natural question that arises from the previous definition is how to properly define a pleasant matching and an unpleasant matching. To this end, a threshold value is introduced, which is compared to the kernel value the matched agent A_2 has achieved in the kernel function of agent A_1:

[1]Without any loss of generality, for easiness of comprehension, for the rest of the report we shall assume that an agent A_1 is matched with an agent A_2

[2]For the more mathematically inclined, this is expressed as $l : \{\text{AGENTS} \times \text{AGENTS}\} \to \mathbb{Z} \cup \varnothing$, where $\varnothing$ indicates the abscense of a connection between two agents.

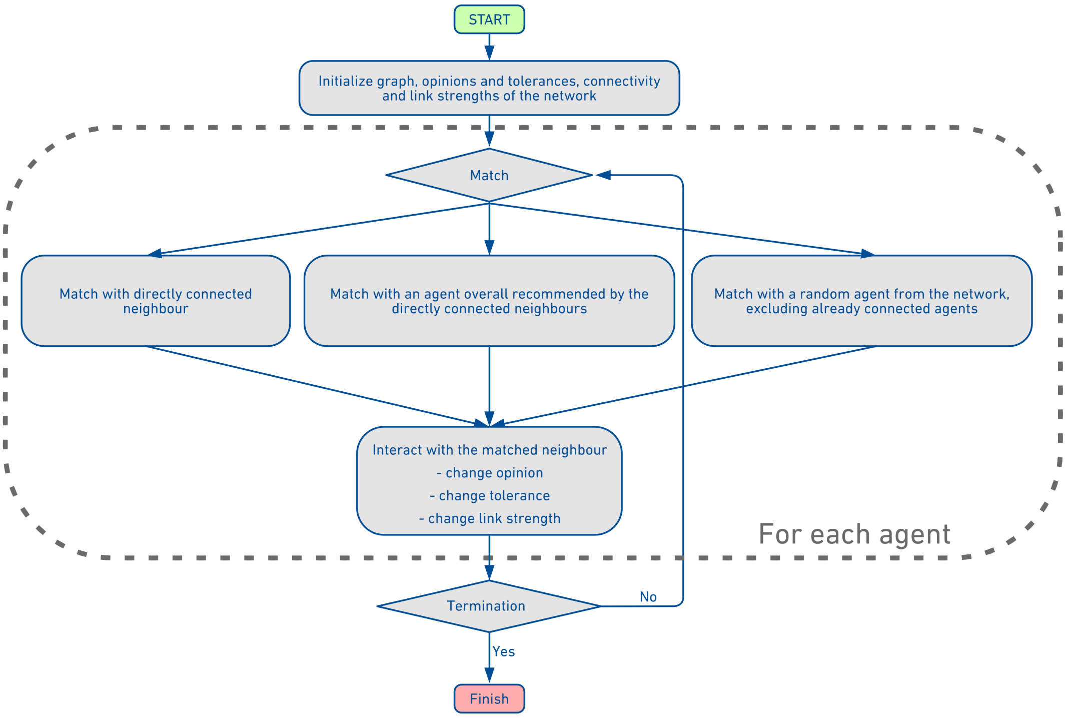


Figure 4: Our proposed variation to the Deffuant model

$$k_{\theta_1}(x_1, x_2, \theta_2) > \text{threshold} \Rightarrow \text{ pleasant matching} \Rightarrow \qquad l^{t+1}(A_1, A_2) = l^t(A_1, A_2) + 1 \tag{7}$$

$$k_{\theta_1}(x_1, x_2, \theta_2) \leq \text{threshold} \Rightarrow \text{ unpleasant matching} \Rightarrow \qquad l^{t+1}(A_1, A_2) = l^t(A_1, A_2) - 1 \tag{8}$$

Notice that the change of a link strength is independent of any potential change of opinion. This is consistent with the the psychological effect that the threshold value is tasked with modeling: in the case where an agent A_1 is somewhat persuaded by their interlocutor A_2 –thus observing a change of opinion towards the latter's opinion– but the two agents had fundamentally different opinions then the interaction itself was stressful for A_1, thus leaving a negative footprint in A_1's memory.
Of course, if the two agents agree on a fundamental level, then the impression left would be positive, thus increasing their willingness to interact with the same agent again.

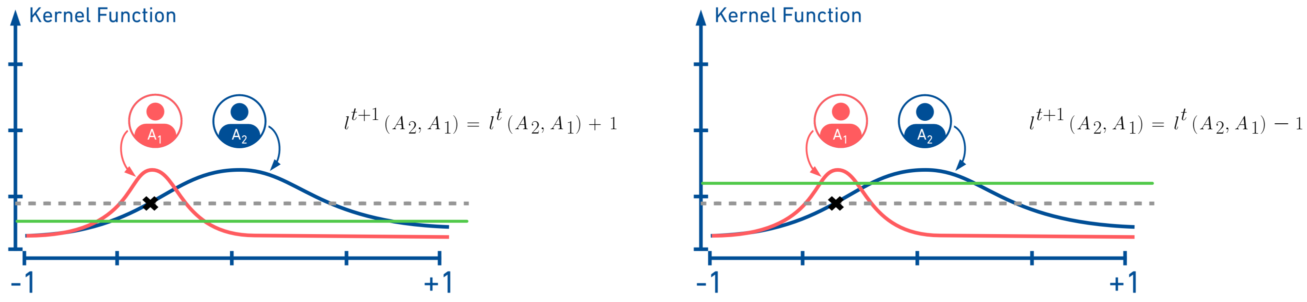
A visualization of the described mechanism is provided in Figure 5.

B. *Experiential Matching Scheme*

The novel matching scheme is what we consider to be our main contribution, as well as the driving force behind the model's behaviour. This new technique is an interpretation of what often occurs in practice with regards to socializing. At each time step, each agent follows a hierarchical order of preference dictated by their matching strategy [3] to select which (other) agent to match with. In this work, three cases are examined.

1) An agent chooses to interact with entities they already have an affinity towards (family, friends, politicians, favourite content creators etc.). Henceforth, this is referred to as the "*friend case*".
2) Next in order of preference, an agent chooses to match with entities which are recommended by the aforementioned (i.e. the already familiar) entities. Henceforth, this is referred to as the "*recommendation case*".

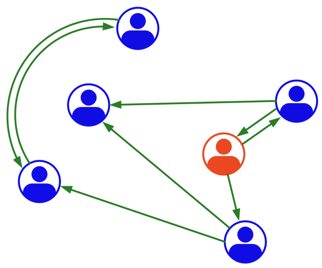
[3]Different matching strategies (e.g. evolving strategies) might be an interesting extension to this work. For simplicity, in this work the matching strategy consists of sampling one of the cases following a provided probability distribution at each iteration. This will be re-mentioned in the "Experiments" section of the report.



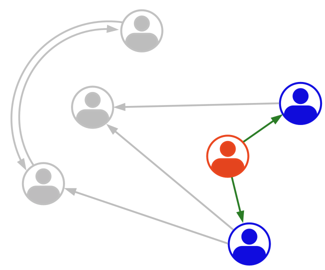
(a) The kernel value surpasses the threshold: A_2 had a positive interaction with A_1

(b) The kernel value does not surpass the threshold value: A_2 had a negative interaction with A_1

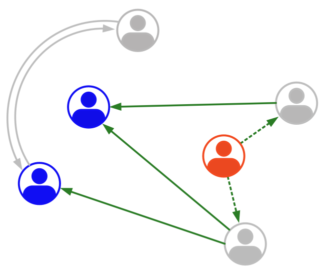
Figure 5: The link strength change mechanism



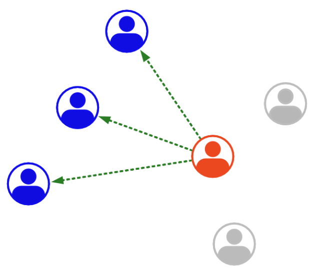
(a) The toy graph to be used. The matching possibilities of the red agent is examined for the subsequent subfigures.



(b) The red agent is matched with a directly neighbouring agent.



(c) The red agent is matched with a recommended agent.



(d) The red agent is matched with a (uniformly) random unconnected agent.

Figure 6: The possible options allowed in the proposed matching schemes demonstrated with a single agent of a toy-problem graph.

3) Lastly, it may happen that an agent randomly encounters an entity completely unaffiliated from their current social circle (e.g. random meetings on the street, parties, clubs, –seemingly– random recommendations on content creation platforms etc.). Henceforth, this is referred to as the "*random case*".

These discrete possible scenaria are visualized in Figure 6.

Once the matching scheme is selected, the next step for the agent is to define probability distribution function over all agents and then sample one of them. [4]

The probability distribution functions (PDFs) an agent i forms for each of our cases are:

1)
$$p_{\text{friend}_i}(k) = \frac{\max\{0,\, l(i,k)\}}{\sum_{j\in\mathcal{N}_i} \max\{0,\, l(i,j)\}} \tag{9}$$

where $\mathcal{N}_i$ refers to the neighbourhood (i.e. directly connected agents) of agent i.

2)
$$p_{\text{recommendation}_i}(k) = \frac{\max\left(0,\, \sum_{j\in\mathcal{N}_i} \max\{0,\, l(i,j)\} l(j,k)\right)}{\sum_{\kappa\in\mathcal{N}_j-\{i\}} \max\left(0,\, \sum_{j\in\mathcal{N}_i} \max\{0,\, l(i,j)\} l(j,\kappa)\right)} \tag{10}$$

3)
$$p_{\text{random}_i}(k) = \frac{1}{|\text{AGENTS} - (\mathcal{N}_i \cup i)|} \tag{11}$$

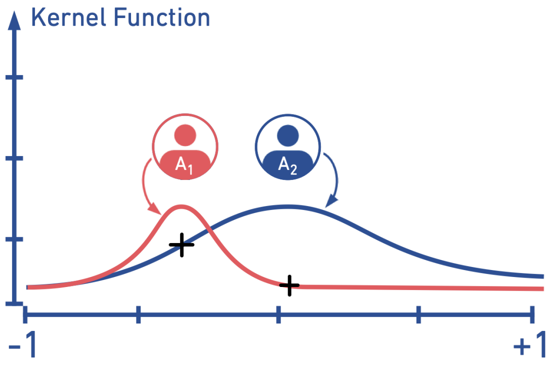
where AGENTS refers to the set of agents of the system and $|\cdot|$ is the cardinality operator (i.e. counts how many elements the input set has).

For the derivation, intuitive interpretation and special case handling of these PDFs, the reader is advised to read the corresponding appendix.

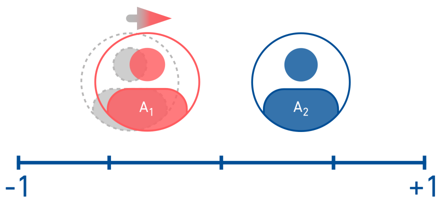
At this juncture, an assumption we made about our model must be revealed. Matchings in our model are not reciprocal, as the common case of the Deffuant model (visualized in Figure 2b). On the contrary, our model assumes unilateral matchings, consequently leading to unilateral changes of opinion, as is often the case in digital social networks and media. In those, an agent might be influenced by the content a different agent publishes, while the reverse is not necessary (and in practice, is often not possible).

So as for no confusion to arise, we provide a corrected visualization at Figure 7

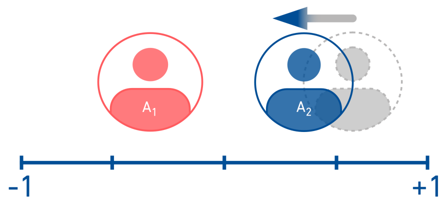
[4] Although our codebase allows for the selection of multiple agents, we considered that the computational overhead would not offer anything of substance to the final experimental results analysis.



(a) The crosses indicate the kernel value of one agent in the kernel function of the other agent



(b) Change of opinion in case A_1 interacts with A_2



(c) Change of opinion in case A_2 interacts with A_1

Figure 7: Our model's assumption about unilateral matching and – consequently– opinion changes

IV. Experiments Framework

A. Metrics

Adopted from an article by Deffuant [3], we characterize the convergence type of a simulation, by employing an indicator. This indicator is the generalized number of clusters, represented by n, a smooth count of opinion clusters. This count is calculated using the methodology outlined by Derrida and Flyvbjerg (1986 [5]). It considers the final state of the population that contains k clusters of opinion x_i, each with a proportion p_i of initially moderate individuals. For the proportion p_i this we count for every value of opinion the number of agents that has that opinion, and round the opinion values to 3 digits so that the agents that have close enough opinions are considered having identical opinions. The generalized number of clusters is then given by:

$$n = \frac{1}{\sum_{i=1}^{k} p_i^2} \tag{12}$$

This generalized number n of clusters deliver an exact count of clusters, where each cluster has a equal share of individuals, thereby clusters with small number of agents are weighted less.

The convergence type is determined by combining these two indicators, following these criteria:

- If $n < 1.25$, the convergence type is classified as "single extreme".
- If $n > 1.66$, the convergence type is "double extreme".
- If $1.25 < n < 1.66$, the convergence type is described as "intermediate between single and double extreme"
- For $n > 2.5$, the convergence is termed "moderate".

B. Implementation details

a) Matching Strategy: In subsection III-B there was a brief mention of matching strategies, which do not consist part of the proposed model and, as such, would not be an aspect to be researched in the present work. Despite that being the case, a matching strategy needed to be devised, so as to complete an experimentation framework. For the present work, a simple sampling strategy is followed: at each time step, each agent samples one of the matching cases ("*friend*", "*recommendation*" and "*random*") according to a constant PDF dictated by the user as part of the configuration of the experiment. Details for the involved parameters will be provided in the "Parameterization" subsection.

b) Exception handling during matching: p_{friend_i} and $p_{\text{recommendation}}$ presented in subsection III-B may possibly collapse to $\frac{0}{0}$ results, which is mathematically undefined behaviour (and quite clearly probability distribution functions cannot be formed by it). Depending on the conditions of the agent, the matching strategy PDF may be subjected to a metamorphosis. To counter this, the preprocessing scheme visualized in Table I was devised and applied as a check before each agent sampled a matching case.

Has friend(s)	Has good recommendation(s)	Is fully connected	Probabilities (p_f, p_r, p_{random})
			$(0, 0, 1)$
		✓	$(1, 0, 0)$
	✓		$(p_f + \frac{p_r}{2}, 0, p_{\text{random}} + \frac{p_r}{2})$
✓	✓		$(p_f, p_r, p_{\text{random}})$

Table I: Changes of the default probabilities in cases of extreme conditions. Checkmarks indicate that the condition mentioned on the top row is met for the agent which will proceed to sample a matching case. The probability triplet at the top row is the default probability distribution of the matching strategy.

C. Parameterization

Each of our experiments was configured using the following parameters

- **Number of agents** N. Due to time constraints, we limited our exploration $N = 100$ agents.
- **Time steps** T. The total number of iterations the components of the social system interact with one another before stopping the simulation. It was set to a constant of $T = 1,000$.
- **Interaction intensity** μ. One of the axes of the grid search was this parameter. We limited our search to the values $[0.1, 0.3, 0.5, 0.7]$ for all agents.
- **Tolerance threshold** θ. The other axis of the grid search was this parameter. We limited our search to the values $[0.1, 0.3, 0.5, 0.7]$ for all agents.
- **Expected number of initial edges** E. The directed graph representing the social network is initialized using the Erdős–Rényi–Gilbert model. This model is traditionally defined as $G(N, p)$, where N is the number of vertices in the graph and p is the probability of a (directed) edge of the fully connected graph being part of the generated graph.
  Since our initial vision was to conduct experiments with a varying number of nodes we decided to express p with respect to the number of edges that (on average) a single agent will have $p = \frac{E}{N}$. This aimed to save time for the user, who no longer has to compute the corresponding probability on their own. We conducted experiments using $E = [7, 13, 20]$ to simulate different scenaria of connectivity (which we dubbed "sparse", "moderate" and "dense" respectively). Link strengths are randomly generated by $\mathcal{U}[-10, 10]$.
- **Probability of sampling the *friend* case** p_f.
- **Probability of sampling the *recommendation* case** p_r. In the code and the figures, this has the name `pff`
  NOTE: p_f *and* p_r *uniquely determine* p_{random}*, as* $p_{random} = 1 - (p_f + p_r)$.

D. Implementation tools

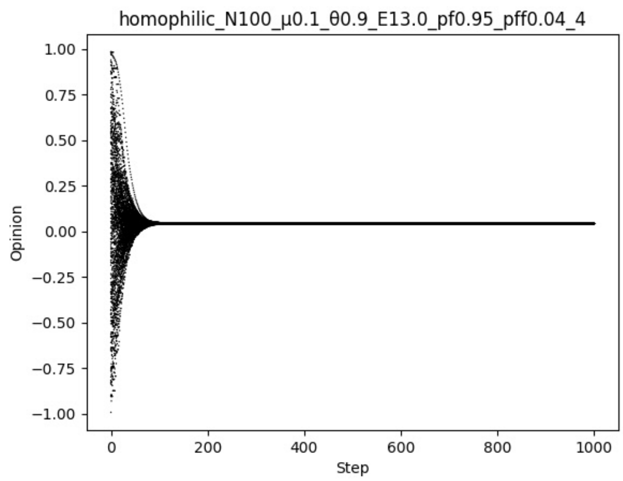
The models code was programmed in Python 3 utlizing the NumPy and SciPy libraries for the computational aspect –like (sparse) linear algebra random selections et al– and the NetworkX library for graph initialization, storage and manipulation. The experiments were conducted on the ETH Euler Cluster, which provides 20 CPU cores per user, which allowed for better throughput, since our codebase runs experiment configurations in parallel.

V. Results Analysis

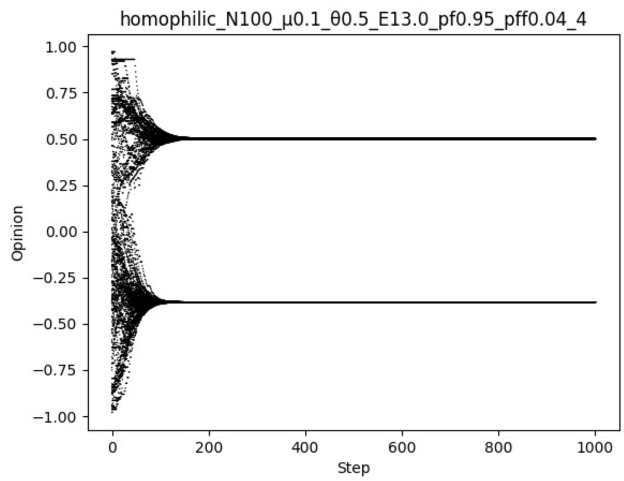
Our qualitative analysis examines the emerging convergence types of our experiential model, while also comparing it to the classic Deffuant model. The following section will discuss the results of our experiments and illustrate them in two different kind graphs. First a phase diagram showing the classification into convergence types with regard to the bounded confidence parameter θ and the interaction intensity μ. Second we will analyze the exact evolution of opinions in the system by plotting the time series of opinions.

The representative categories of convergences encountered are shown in the time series depicted in Figure 8. These consist of:

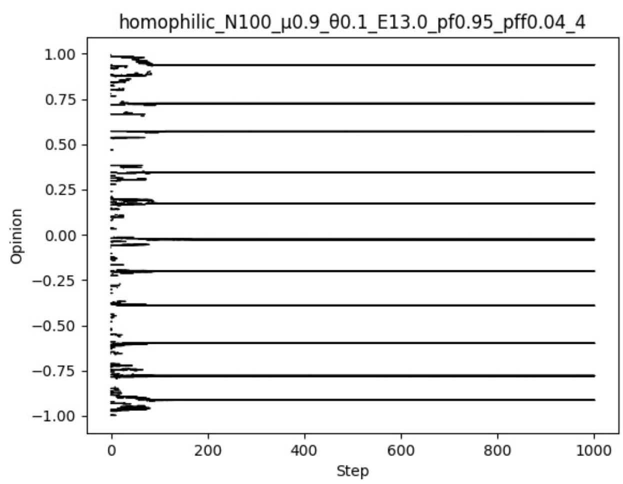
- Single Convergence Figure 8a: The population's opinion collapses to a single opinion value.
- Double Convergence Figure 8b: The population's opinion converge towards two discrete opinion poles.
- Moderate Convergence Figure 8c: The population's opinions converge into multiple small clusters.
- Intermediate Convergence Figure 8d: Similar to double convergence, but instead there is a small number of outliers which form their own opinion.



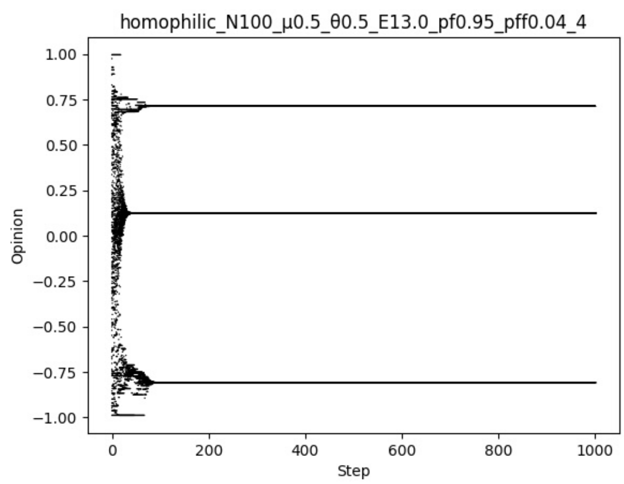
(a) Single opinion convergence outcome



(b) Double opinion convergence outcome



(c) Moderate opinion convergence outcome



(d) Intermediate opinion convergence outcome

Figure 8: Time evolution plot of opinions, showcasing the different convergence types

In addition we plot phase diagrams to explore the dependencies of the convergence outcomes of the parameters θ and μ. Due to the computing time of each time step the resolution is kept quite low. An example figure which provides information as to how to read the content of plotted phase diagram is provided in Figure 9.

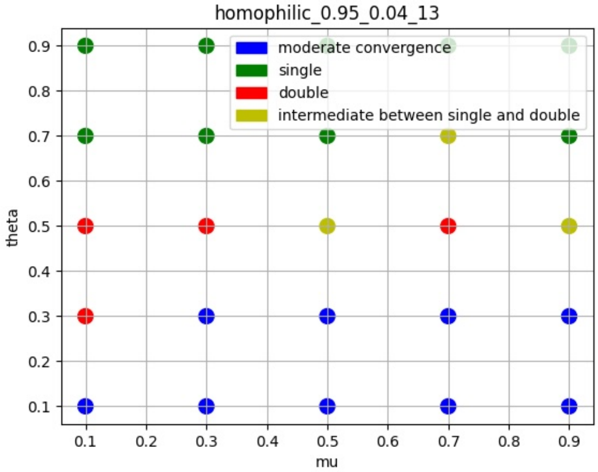


Figure 9: Phase diagram for our model (`homophilic`) showing the dependencies of the convergence outcomes of the parameters θ and μ. The chosen parameters are the probability to match with a friend $p_f = 0.95$, the probability to match with a recommendation $p_r = 0.04$ and the expected number of links per agent E. The dots represent the moderate convergence, the red ones represent the double convergence, the green dots represent the single opinion and the yellow dots represent the Intermediate opinion convergence outcomes.

With this one can observe a clear separation of zones with single on top, double in middle and moderate opinion convergence in the bottom. Nevertheless there are occurrences of intermediate convergences in special cases. All in all the significant importance of the bounded confidence parameter θ needs to be noted.

a) Influence of Experience: Experiments visualized in Figure 10 show that the classic (`backbone`) model does concludes with an unsatisfactory result, as moderate convergence arise in more configurations; there is no clear separation in the searching space where moderate convergence stops appearing.

The experiential (`homophilic`) models seems to converge much more often to extreme cases (either single convergence or double convergence). The authors attribute to the dynamic nature of the graph, which paired with the proposed matching scheme, encourage an early stage community formation, which ultimately leads to globally clearer extreme outcomes.

In contrast, the classic model, due to its uniformly random formulation, asymptotically encourages agents to make more compromises, thus forming smaller clusters. In addition, it is hypothesized that the static formulation of the classic model slows down the propagation of opinion to the rest of the network, thus the agent's ultimate is "destined" to be an average (compromise) of its neighbourhood.

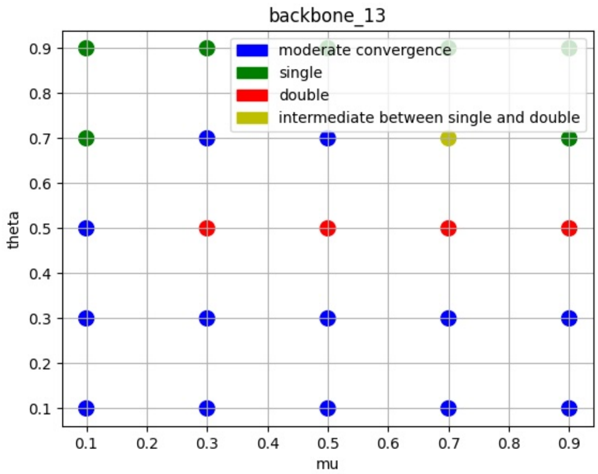
b) Effect of initial expected links per agent E: The experimental results corresponding to the effect of E are visualized in Figure 11. A natural deduction to make about this parameter is that its increase leads to the enlargement of subcommunities formed initially in the network. However, this effect is asymptotically mitigated, as more and more connections are formed as time passes.

The variation of simulation results do not provide evidence to support that E has a decisive role in the outcome or what its effect is.

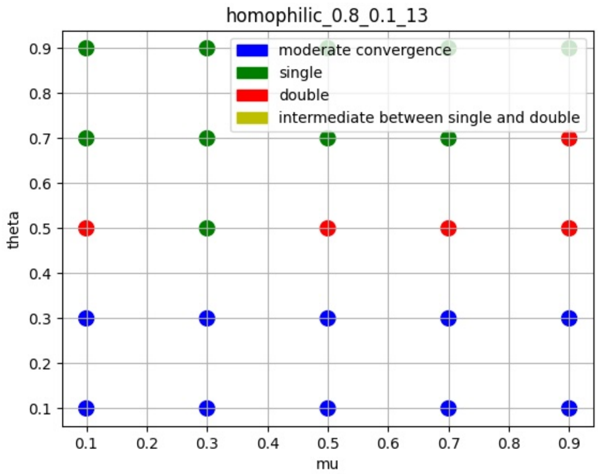
The plotted time series of the relevant experiment in Figure 13 do not provide any conclusive evidence as to the influence of E in the experiments. More experimentation is required.

c) Effect of p_f and p_r: The synergy of these two parameters –which also dictate the probability of choosing a random agent– affect the speed of opinion subgroups formation, as well as the intercommunication between different subgroups (starting with "graph bridges" as these are defined in graph theory, and then the socialization is propagated via the *recommendation* case), similarly to the parameter E.

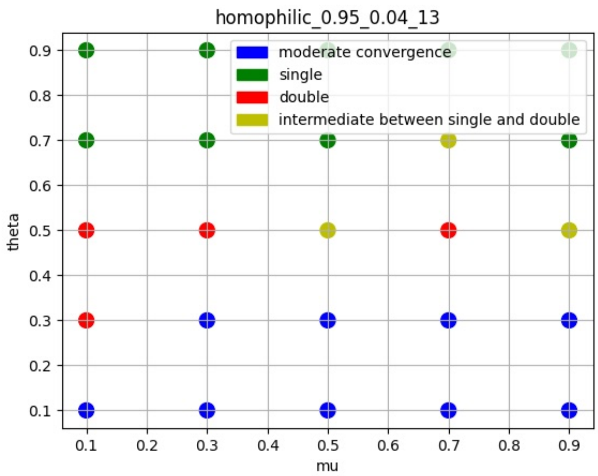
The visualized grid search with respect to this parameter (Figure 12) indicate that p_{random}, which expresses extroversion and open-mindedness, has a significant impact on the non-formation of extreme opinions. This correlation is more evident when peeking into the time series regarding these experiments at Figure 14.



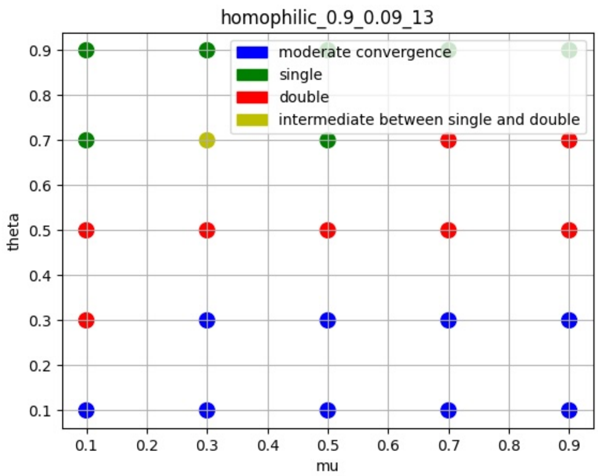
(a)



(b)



(c)



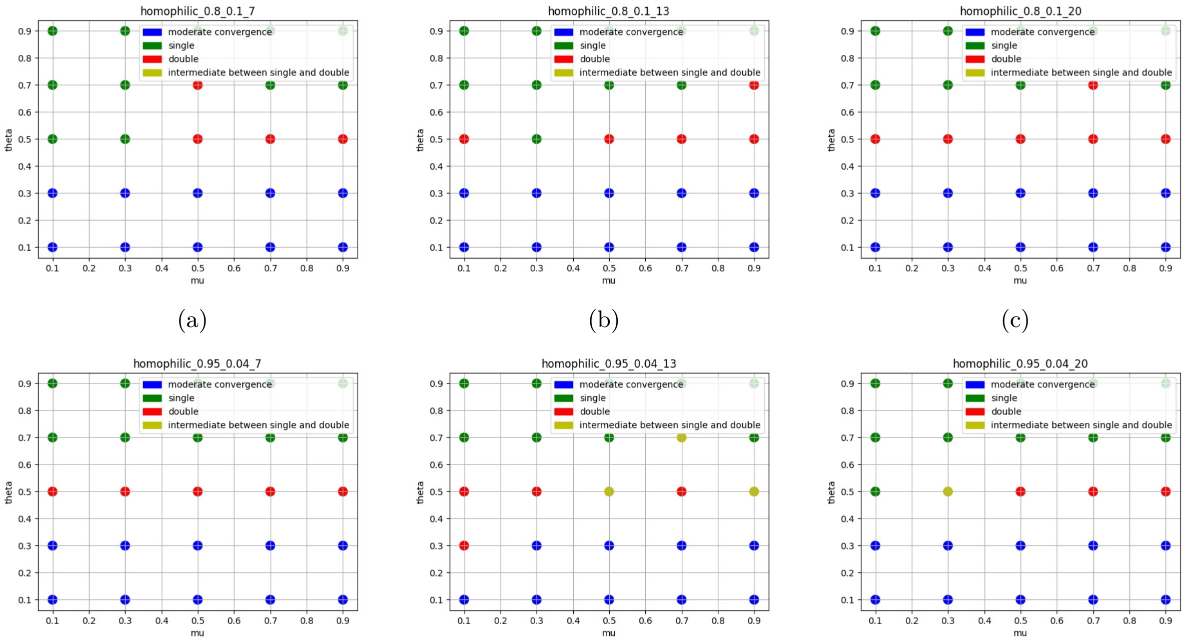
(d)

Figure 10: Classic VS Ours: Effect of experience

In the top row of the figure, it may be observed that the cases of moderate convergences decreases. This is not as visible, due to the existence of monads at the opinion extremes, due to the low value of tolerance θ. Though correlation was found, no clear trade-off p_f and p_r relation was able to be expressed.

It is interesting to observe that, although slight, it is noticeable that in the moderate convergence cases where the p_{random} is $10\times$ larger than in the others, the opinion subgroups form much faster (compare (a) and (d) with their right counterparts). On the contraty, in the double convergence cases, lower p_{random} seem to encourage fast cluster formations.

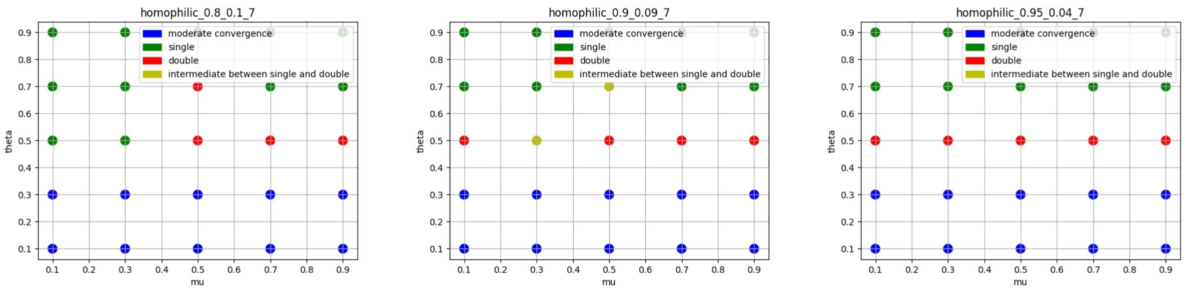
d) Effect of θ : In all aforementioned experiments, the factor that dominated the emergence of different convergence type was the tolerance threshold θ. This can be seen in most phase diagrams, where there seems to be a line drawn in the cases where $\theta < 0.5$, where moderate convergences appear, and where $\theta > 0.5$, where single or double convergences appear. This phenomenon is in agreement with the intuition behind the parameter: in low values, agents are unwilling to make compromises and consequently form multiple small clusters, while higher values represent a lack of confidence in their opinion, thus settling for an opinion that was even vastly different than their original one.



(a) (b) (c)

(d) (e) (f)

Figure 11: Networking: Effect of E

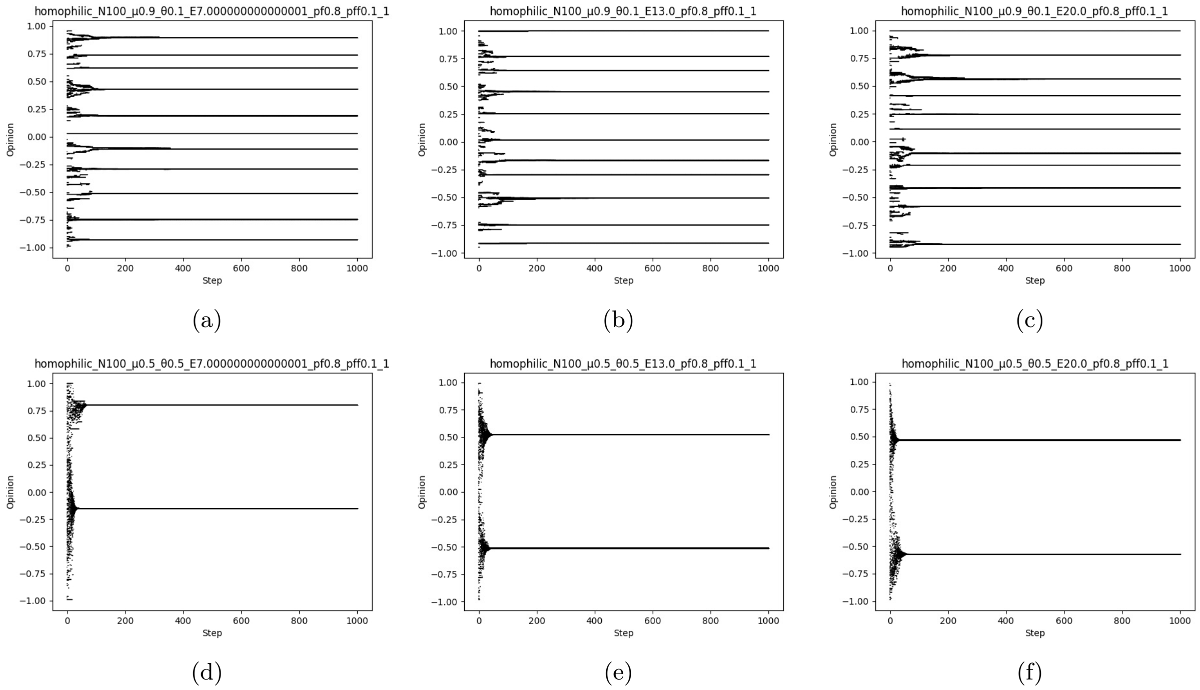


(a) (b) (c)

Figure 12: Extroversion, Cliqueness or Devotion: Effect of p_f and p_r.

VI. Individual Contributions

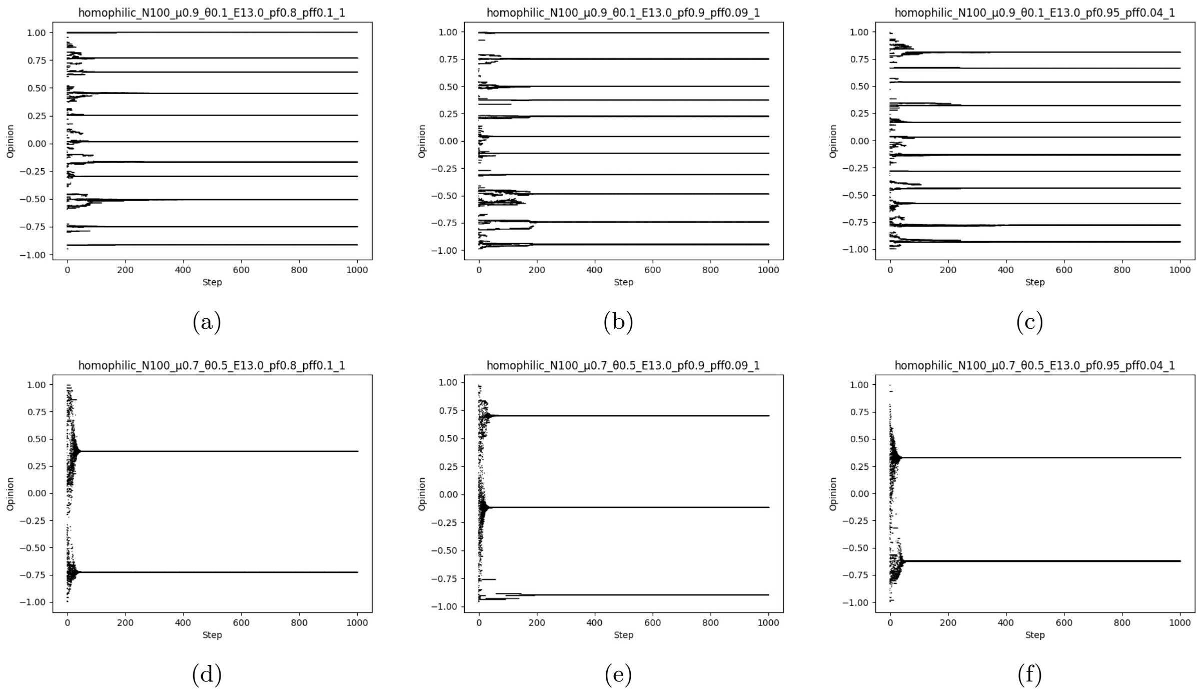
- **Gabor Hollbeck**: Literature review, Report ("Introduction and Related Work", "The Deffuant Model Framework", Callaesthetic figures (3, 2, 8, 2a, 1, 5, 6, 4, 7)
- **Adrien Lanne**: Experimental analysis and visualizations, Metrics design, Experiments design, Euler cluster setup, Data and result collection, Report ("Abstract", "Results Analysis")
- **Konstantinos Stavratis**: Literature Review, Mathematical Modeling, Implementation of mathematical models, Code architecture, Implementation testing, Euler cluster setup, Experiments design, Data and result collection, Project coordination, Report ("Abstract", "The Deffuant Model Framework", "Our Model", "Experiments Framework", "Results Analysis", "Appendix A")



(a) (b) (c)

(d) (e) (f)

Figure 13: Networking time series: Effect of E



(a) (b) (c)

(d) (e) (f)

Figure 14: Exroversion, Cliqueness and Devotion time series

Appendix A
Matching Scheme Probability Distribution Functions

The purpose of this appendix is to help the reader understand the resulting PDFs presented in III-B in an informal didactic manner. This will be achieved by stepping over a clear description of the desired goal, expressing it in the mathematical language at each step, until the final results are reached. Our wish is that the reader will see the incremental building process and appreciate its simplicity.

We are first aware that the Deffuant model describes a procedure in which *related* agents form pairs and then adjust their opinions. We must therefore search the literature to find a fitting structure. Thankfully, there is no need for a thorough search, as long as the reader is aware of graph theory.

Our most fundamental block is understanding what a graph is.

"*In discrete mathematics, and more specifically in graph theory, a graph is a structure amounting to a set of objects in which some pairs of the objects are in some sense "related". The objects correspond to mathematical abstractions called vertices (also called nodes or points) and each of the related pairs of vertices is called an edge (also called link or line). Typically, a graph is depicted in diagrammatic form as a set of dots or circles for the vertices, joined by lines or curves for the edges. Graphs are one of the objects of study in discrete mathematics.*"
∼Wikipedia

Now that we have established a structure that encompasses connections, we wish to find structures which store concrete information about the relations between the aforementioned nodes. One such structure is an "adjacency matrix"

"*In graph theory and computer science, an adjacency matrix is a square matrix used to represent a finite graph. The elements of the matrix indicate whether pairs of vertices are adjacent or not in the graph.* "
∼Wikipedia

Let us consider the graph depicted in Figure 15 .

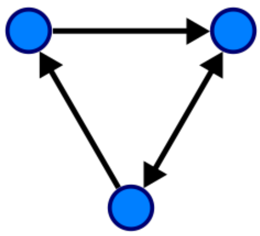


Figure 15: Toy directed graph. Image taken from Wikipedia.
Without loss of generality, we consider the left vertex to have an identifier of "1", the right "2" and the bottom one "3".

Its corresponding (simple) adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

with the first row representing which vertices vertex 1 is connected ("pointing") to, the second row representing which vertices vertex 2 is connected to and the third row representing which vertices vertex 3 is connected to.

By now, the reader might have pondered the following question: is it possible to extract the vertices with which a vertex is away two "hops" (edges) by only using the information inside the adjacency matrix? The answer to that is "yes"! And quite simply in fact.

The only thing needed is to multiply the adjacency matrix with itself.

For the sake of completeness, we provide the definitions of a dot product between two vectors and two matrices.

The dot product between two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ is defined as of a summation of scalar products:

$$\mathbf{a} \cdot \mathbf{b} := \sum_{i=1}^{n} a_i b_i$$

The (dot) product between two matrices A and B is defined as:

$$[AB]_{i,j} = \sum_k a_{i,k} \cdot b_{k,j}$$

By using the above with $B = A$ for our example we get:

$$[A^2]_{i,j} = \sum_k a_{i,k} \cdot a_{k,j} \iff A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

But why is that the case? At this stage, it is helpful to look the adjacency matrix from a different angle: the columns! A column of an adjacency matrix holds information about which vertices are connected to the vertex of the corresponding column.
In our example, the first column indicates that vertex 1 can be reached by the third vertex (only the third element of the column has a value of 1), the second column indicates that it can be reached by the vertices 1 and 3 (the first and third element of the column have a value of 1) and the third column indicates that vertex 3 may be reached by vertex 2 (the second element of the third column).

If we combine this information with the definition of matrix multiplication it is visible that what occurs is:

$\mathbf{r}_i \cdot \mathbf{c}_j \rightarrow$ number of different "two-hop" paths that the i-th vertex can take to reach the j-th vertex.

At this point, the attentive reader might have realized that this can be generalized to "n-hop" paths, which will be A^n. This is indeed true, however two-hops suffice for this work.

Let us return our attention to the proposed Deffuant model and try to associate the graph concepts that have been introduced so far in the appendix with the model.

- Vertices are agents
- Edges denote the existence of links
- The content of the adjacency matrix are the link strengths

As briefly mentioned in the core paper, link strengths are integer values which convey the level of pleasant past experience with other agents; positive values encode comfort, negative values encode trauma, and zero values denote neutrality.
Let us consider one vector representing the outgoing links (each having a link strength) of agent i $\mathbf{r}_i$ and one vector representing the ingoing links connected to agent j $\mathbf{c}_j$ and examine their dot product.

$$\mathbf{r}_i \cdot \mathbf{c}_j := \sum_{k=1}^{n} r_{i\to k} \cdot c_{k\to j}$$

Notice how a transitive-like relation $i \to k \to j$ emerges from the indices of the individual scalars. The critical point to focus on is the (mathematical) sign of each individual scalar product, as shown in Table II.

	$c_{k\to j} > 0$	$c_{k\to j} = 0$	$c_{k\to j} < 0$
$r_{i\to k} > 0$	$+$	0	$-$
$r_{i\to k} = 0$	0	0	0
$r_{i\to k} < 0$	$-$	0	$+$

Table II: Possible recommendation outcomes. A positive sign denotes that agent k would encourage agent i to meet agent j, while the negative sign means that k would advise against agent i meeting agent j.

Moreover, a very convenient property of products [5] is exploited in this model to simulate recommendations.

- The larger the value of $r_{i\to k}$, the more agent i takes to heart the advice of agent k.

[5] Essentially, the referred property are the partial derivatives of the scalar product.
$\frac{\partial(r_{i\to k} \cdot c_{k\to j})}{\partial c_{k\to j}} = r_{i\to k}$ and $\frac{\partial(r_{i\to k} \cdot c_{k\to j})}{\partial r_{i\to k}} = c_{k\to j}$

- The larger the absolute value of $c_{k\to j}$, the more determined agent k is about their impression of agent j. In the case of a positive value, agent k "sings praises" for j to i, while with a negative value agent k "propagandizes" agent i against j.

Another operation that we get for free with the adjacency matrix multiplication is the addition operator (a.k.a the sum). With this, agent i's final evaluation of agent j is a superposition of the feedback agent i has received from all neighbouring agents k.

With the provided information, the reader now hopefully sees why the adjacency matrix was chosen as a main operation for acquiring results for both the "*friend*" case and the "*recommendation*" case. To complete the equations presented in III-B, an important assumption for the *recommendation* mechanism must be revealed. We thought it natural that agents do not seek out the advice of agents with which they have negative link strengths (due to trauma and/or hostility associated with negative link strengths). This is how the subterms $\max\{0, l(i,k)\}$ $\max\{0, l(i,j)\}$ appear in p_{friend} and $p_{\text{recommendation}}$ respectively. Notice that in *recommendation* case, this effectively reduces the knowledge an agent holds about the graph; agent i is unaware of the link strengths of an agent j they dislike. Hence, i cannot detect possible other agents they might dislike (those friendly to j) or apply "the enemy of my enemy is my friend" motto to socialize with potentially friendly agents (those that j dislikes).

Last but not least, the denominators in the equations are the normalizing constant to form proper PDFs with a total probability of 1.

$$l(i,k) \xRightarrow[\text{friends only}]{\text{consider}} \max\{0,\, l(i,k)\} \xRightarrow[\text{constant}]{\text{normalizing}} \frac{\max\{0,\, l(i,k)\}}{\sum_{j\in\mathcal{N}_i} \max\{0,\, l(i,j)\}} =: p_{\text{friend}_i}(k)$$

$$\begin{aligned} l(i,j) &\xRightarrow[\text{friends only}]{\text{consider}} \\ \max\{0,\, l(i,j)\} &\xRightarrow[\text{of friend } j \text{ for } k]{\text{listen to advice}} \\ \max\{0,\, l(i,j)\} \cdot l(j,k) &\xRightarrow[\text{friends' opinions}]{\text{listen to all}} \\ \sum_{j\in\mathcal{N}_i} \max\{0,\, l(i,j)\} \cdot l(i,k) &\xRightarrow[\text{overall good recommendations}]{\text{consider meeting only}} \\ \max\left(0,\, \sum_{j\in\mathcal{N}_i} \max\{0,\, l(i,j)\} \cdot l(i,k)\right) &\xRightarrow[\text{constant}]{\text{normalizing}} \\ \frac{\max\left(0,\, \sum_{j\in\mathcal{N}_i} \max\{0,\, l(i,j)\} \cdot l(i,k)\right)}{\sum_{\kappa\in\mathcal{N}_j-\{i\}} \max\left(0,\, \sum_{j\in\mathcal{N}_i} \max\{0,\, l(i,j)\} \cdot l(i,\kappa)\right)} &=: p_{\text{recommendation}_i}(k) \end{aligned}$$

The PDF for the *random* case is just a discrete uniform PDF which takes into consideration only the agents *not* connected with the agent to be matched.

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