Fri: $z_i \sim \text{Bern}\left(\pi_i = \frac{n}{N}\right) \rightarrow \text{SRS}$

$$E(\bar{y}) \overset{SRS}{=} \bar{y}_u$$

Now $\boxed{Var(\bar{y}) \underset{SRS}{=} \left(1 - \frac{n}{N}\right)\frac{s^2}{n}}$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{N} z_i y_i$$

(math 240)

$$Var(\bar{y}) = Var\left(\frac{1}{n}\sum_{i=1}^{N} z_i y_i\right)$$

$$= \frac{1}{n^2}\left[\underbrace{\sum_{i=1}^{N} Var(z_i y_i)}_{a} + 2\underbrace{\sum_{i,j \text{ pairs}}\sum Cov(z_i, z_j) \times y_i y_j}_{b}\right]$$

z_i are not indep. RV under SRS

a $\sum_{i=1}^{N} Var(z_i y_i)$

$$V(z_i y_i) = y_i^2 \, Var(z_i) = y_i^2 \left(\frac{n}{N}\right)\left(1 - \frac{n}{N}\right)$$

a $\sum Var(z_i y_i) = \sum y_i^2 \left(\frac{n}{N}\right)\left(1 - \frac{n}{N}\right)$

$$= \left(\frac{n}{N}\right)\left(1 - \frac{n}{N}\right)\sum_{i=1}^{N} y_i^2$$

(b) $2 \sum\sum_{\text{all pairs}} Cov(z_i, z_j)\, y_i y_j$

$$Cov(z_i, z_j) = \underbrace{E(z_i z_j)} - E(z_i)E(z_j)$$

$$E(z_i) \downarrow \frac{n}{N} \qquad E(z_j) \downarrow \frac{n}{N}$$

$$z_i z_j = \begin{cases} 1, & z_i = z_j = 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(z_i z_j) = 1 \cdot \underbrace{P(z_i = 1, z_j = 1)}_{\text{both } i + j \text{ in SRS}} + 0 = \frac{n}{N}\frac{n-1}{N-1}$$

$$P(z_i = 1, z_j = 1) = \frac{\binom{N-2}{n-2}}{\binom{N}{n}} = \frac{n}{N} \cdot \frac{n-1}{N-1}$$

$$Cov(z_i, z_j) = \frac{n \cdot n-1}{N \cdot N-1} - \left(\frac{n}{N}\right)^2 = \text{.... algebra}$$

$$= \frac{n}{N} \times \frac{-1}{N-1} \times \left(1 - \frac{n}{N}\right)$$

* negative covariance

$$Var(\bar{y}) = \frac{1}{n^2}\left[a + b \right]$$

$$= \frac{1}{n^2}\left[\frac{n}{N}\left(1-\frac{n}{N}\right)\sum_{i=1}^{N} y_i^2 + 2 \sum_{\substack{\text{all} \\ \text{pairs}}}\sum y_i y_j \frac{n}{N}\left(\frac{-1}{N-1}\right)\left(1-\frac{n}{N}\right) \right]$$

= ... lots algebra! ...

$$= \left(1-\frac{n}{N}\right)\frac{S^2}{n} \quad *$$

$$S^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \bar{y}_u\right)^2$$

Optional!

*Key fact:

$$\sum_{i=1}^{N}(y_i-\bar{y}_u)^2 = \sum_{i=1}^{N}\left[y_i^2 - 2y_i\bar{y}_u + \bar{y}_u^2\right]$$

$$= \sum_{i=1}^{N} y_i^2 - 2\bar{y}_u \sum_{i=1}^{N} y_i \times \frac{N}{N} + N\bar{y}_u^2$$

$$= \sum_{i=1}^{N} y_i^2 - 2\bar{y}_u\bar{y}_u N + N\bar{y}_u^2$$

$$= \sum y_i^2 - 2\bar{y}_u^2 \cdot N + N\bar{y}_u^2$$

$$= \boxed{\sum_{i=1}^{N} y_i^2 - N\bar{y}_u^2}$$

$$\mathrm{Var}(\bar{y}) = \frac{1}{n^2}\left[\frac{n}{N}\left(1-\frac{n}{N}\right)\sum_{i=1}^{N} y_i^2 + 2\sum_{\text{all pairs}}\sum y_i y_j \frac{n}{N}\left(\frac{-1}{N-1}\right)\left(1-\frac{n}{N}\right)\right]$$

$$= \frac{1}{n}\left(1-\frac{n}{N}\right)\left[\frac{1}{N}\sum y_i^2 - \frac{1}{N(N-1)} \times 2\sum_{\text{all pairs}}\sum y_i y_j\right]$$

$$= \frac{1}{n}\left(1-\frac{n}{N}\right)\left[\frac{1}{N}\sum y_i^2 - \frac{1}{N(N-1)}\boxed{\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} y_i y_j}\right]$$

c $\displaystyle\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} y_i y_j = \sum_{i=1}^{N}\sum_{j=1}^{N} y_i y_j - \sum_{i=1}^{N} y_i^2$

$$\begin{array}{c|cccc} & y_1 & \cdots & & y_N \\ \hline y_1 & y_1y_1 & y_1y_2 & \cdots & y_1y_N \\ \vdots & & & & \\ y_N & y_Ny_1 & \cdots & & y_Ny_N \end{array}$$

$$= \sum_{i=1}^{N} y_i \times\frac{N}{N} \sum_{j=1}^{N} y_j \times\frac{N}{N} - \sum_{i=1}^{N} y_i^2$$

$$= N\bar{y}_u \times N\bar{y}_u - \sum_{i=1}^{N} y_i^2$$

$$= N^2\bar{y}_u^2 - \sum_{i=1}^{N} y_i^2$$

$Var(\bar{y}) =$

$$= \frac{1}{n}\left(1-\frac{n}{N}\right)\left[\frac{1}{N}\sum y_i^2 - \frac{1}{N(N-1)}\boxed{\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} y_i y_j}\right]$$

$$= \frac{1}{n}\left(1-\frac{n}{N}\right)\left[\frac{1}{N}\sum y_i^2 \times \frac{N-1}{N-1} - \frac{1}{N(N-1)}\left(N^2\bar{y}_u^2 - \sum_{i=1}^{N} y_i^2\right)\right]$$

$$= \frac{1}{n}\left(1-\frac{n}{N}\right)\left[\frac{\cancel{N}\sum y_i^2 - \cancel{\sum y_i^2} - N^{\cancel{2}}\bar{y}_u^2 + \cancel{\sum y_i^2}}{\cancel{N}(N-1)}\right]$$

$$= \frac{1}{n}\left(1-\frac{n}{N}\right)\frac{1}{N-1}\left[\sum y_i^2 - N\bar{y}_u^2\right]$$

$$= \frac{1}{n}\left(1-\frac{n}{N}\right)\underbrace{\frac{1}{N-1}\left[\sum_{i=1}^{N}\left(y_i-\bar{y}_u\right)^2\right]}_{S^2 \,=\, \text{pop variance}}$$

$$= \left(1-\frac{n}{N}\right)\frac{S^2}{n}$$

✓