Sampling with unequal probabilities of selection (WOR)

Week 9 (6.4)

Stat 260, St. Clair

Horvitz-Thompson Estimator

- You take a (one-stage) random sample where:
  - t_i = "response" in PSU i
  - n = PSU sample size (or unique PSU sampled)
  - π_i = sample inclusion prob for PSU i
  - $w_i = 1/\pi_i$ = number of population units represented by PSU i
- The Horvitz-Thompson estimator is

$$\hat{t}_{HT} = \sum_{i=1}^{n} w_i t_i$$

Horvitz-Thompson Estimator

$$\hat{t}_{HT} = \sum_{i=1}^{n} w_i t_i$$

- For any design (with or without replacement), the H-T estimator is an unbiased estimator of population total t.

$$E(\hat{t}_{HT}) = t = \sum_{i=1}^{N} t_i$$

- Week 2 theory slides

Horvitz-Thompson Estimator

- All total estimates so far, except for ratio estimates, have been HT estimators
  - SRS: $w_i = N/n$
  - Stratified: $w_{hj} = N_h/n_h$
  - One-stage cluster: $w_{ij} = N/n$
- For these designs, the total estimator SE's can be derived from a general variance calculation
  - using the fact that **without replacement** designs leads to **dependence** among units being sampled

$$Var(\hat{t}_{HT}) = \sum_{i=1}^{N} \frac{1-\pi_i}{\pi_i} t_i^2 + 2 \underbrace{\sum_{i=1}^{N} \sum_{\substack{k=1 \\ i<k}}^{N} \frac{\pi_{ik} - \pi_i \pi_k}{\pi_i \pi_k} t_i t_k}_{\text{WOR}}$$

Σ over all unordered pairs $\{i,k\}$

Horvitz-Thompson Estimator

$$Var(\hat{t}_{HT}) = \sum_{i=1}^{N} \frac{1-\pi_i}{\pi_i} t_i^2 + 2 \sum_{\substack{i=1 \\ i<k}}^{N} \sum_{k=1}^{N} \frac{\pi_{ik} - \pi_i \pi_k}{\pi_i \pi_k} t_i t_k$$

- We need to compute the **joint** inclusion probability

$$\pi_{ik} = \pi_{ki} = P(\text{both } \underbrace{i, k}_{\text{not order}} \text{ included in the sample})$$

- What is π_{ik} for a SRS?

$\pi_{12} = P(\text{both } 1 \,\&\, 2 \text{ in SRS})$

sample that includes 1 & 2 $= \underbrace{\{1, 2, \ ?\ \}}_{n}$ n − 2 units to select

$$\pi_{12} = \frac{\#\text{ of samples with units } 1 \,\&\, 2}{\binom{N}{n}} = \frac{\binom{N-2}{n-2}}{\binom{N}{n}} = \frac{n(n-1)}{N(N-1)}$$

Horvitz-Thompson Estimator

- **SRS:** We measure $t_i = y_i$ for each unit and $\hat{t}_{HT} = N\bar{y}$.

$$\pi_i = \frac{\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{n}{N} \qquad \pi_{ik} = \frac{\binom{N-2}{n-2}}{\binom{N}{n}} = \frac{n(n-1)}{N(N-1)}$$

- **SRS:** variance is then

π_{ik}

$$\underline{Var(\hat{t}_{HT})} = \sum_{i=1}^{N} \frac{1-\frac{n}{N}}{\frac{n}{N}} y_i^2 + 2\sum_{\substack{i=1\\i<k}}^{N}\sum_{k=1}^{N} \frac{\frac{n(n-1)}{N(N-1)} - \frac{n}{N}\frac{n}{N}}{\frac{n}{N}\frac{n}{N}} y_i y_k$$

$$\ldots\text{. lots of algebra}\ldots\text{.}$$

$$= N^2\left(1-\frac{n}{N}\right)\frac{S^2}{n}$$

$\Rightarrow$ <u>ch. 2</u>

theory

Horvitz-Thompson Estimator

- All designs covered so far have used a SRS
  - Definition: each sample of size n is equally likely
  - Implication: each PSU is equally likely (equal π_i for all i)
- What if we don't use a SRS?
  - Take a random sample of PSU without replacement
  - Let inclusion probs π_i vary (unequal π_i)
- Our unbiased total estimate is still $\hat{t}_{HT}$
- Our variance is computed using π_i and π_{ik}!

Example: Unequal inclusion probabilities

- **Supermarkets** estimate total sales t for $N = 4$ stores.

$$t_i = \text{total sales (thousands of dollars) at store } i$$

- **Design:** Random sample WOR with probability proportional to physical store size

- Here's our **population**:

Store	t_i	**Size** m^2
A	11	100
B	20	200
C	24	300
D	245	1000
total	$t = 300$	1600

what we want to estimate!

Example: Unequal inclusion probabilities

Design: Random sample WOR with probability proportional to physical store size → $n = 2$ stores

$$\psi_i = \text{probability of selecting store } i \text{ on your } \underline{\text{first draw}}$$

Store	t_i	**Size** m^2	ψ_i
A	11	100	$\psi_1 = \frac{100}{1600} = \frac{1}{16}$
B	20	200	$\psi_2 = 2/16$
C	24	300	$\psi_3 = 3/16$
D	245	1000	$\psi_4 = 10/16$
total	$t = 300$	1600	1

$$\psi_i = \frac{\text{size store } i}{\text{total of all store sizes}}$$

total of all store sizes → 1600

Example: Unequal inclusion probabilities

Design: Random sample WOR with probability proportional to physical store size

$$\psi_i = \text{probability of selecting store } i \text{ on your first draw}$$

- **Catch:** WOR, probabilities for the second draw are not equal to ψ_i
  - π_i is the probability that store i is one of the two stores sampled, and not equal to ψ_i

ψ_1 = prob. store 1 on first draw

$\neq \pi_1$ = prob. store 1 is picked in the sample of 2 stores.

Example: Unequal inclusion probabilities

- **Draw 1:** we sample store B

$$\psi_{i|B} = P(\text{draw 2 is } i \mid \text{draw 1 is } B)$$

Store	**Size** m^2	$\psi_{i\|B}$
A	100	1/14
B	---	0
C	300	3/14
D	1000	10/14
total	1400	1

$= \psi_{A|B} = \frac{100}{1400} = \frac{1}{14}$

$\rightarrow$ WOR

$= \psi_{C|B}$

$\psi_{D|B}$

Example: Unequal inclusion probabilities

- Use the **individual PSU** selection probs (draw to draw) to compute the **joint inclusion** prob for each pair
- π_{AB}: the probability that both A and B are included is

$$\pi_{AB} = P(A \text{ then } B) + P(B \text{ then } A)$$

$$= \psi_A \times \psi_{B|A} + \psi_B \, \psi_{A|B}$$

$$P(B \text{ on } 2^{nd} \mid A \text{ } 1^{st})$$

$$= \frac{1}{16} \times \frac{200}{1500} \times \frac{2}{16} \times \frac{1}{14} \approx .0173$$

Example: Unequal inclusion probabilities

- π_A: the probability that store A is included is the **sum of all probs of samples that contain that PSU**

$$\pi_A = \underline{\pi_{AB}} + \underline{\pi_{AC}} + \underline{\pi_{AD}}$$

$n = 2$

$$= .0173 + .0269 + .1458$$

$$= .19$$

Example: Unequal inclusion probabilities

- The matrix below gives joint probs π_{ik} in the body and single PSU probs in the margins

π_{AB}

	A	B	C	D	π_i
A	--	0.0173	0.0269	0.1458	0.1900
B	0.0173	--	0.0556	0.2976	0.3705
C	0.0269	0.0556	--	0.4567	0.5393
D	0.1458	0.2976	0.4567	--	0.9002
π_i	0.1900	0.3705	0.5393	0.9002	$n = 2$

$= \pi_A$

π_{BA}

- Note that

theory

$$\sum_{i=1}^{N} \pi_i = n$$

Example: Unequal inclusion probabilities

- Suppose you sampled stores C and D

$\to \frac{1}{\pi_i}$

Store	Size m^2	π_i	w_i	t_i
C	300	0.5393	1.854	24
D	1000	0.9002	1.111	245

- **Estimated total:**

$$\hat{t}_{HT} = \sum_{\text{sampled}} w_i t_i = (1.854)(24) + (1.111)(245)$$

$$= \$316.66$$

Example: Unequal inclusion probabilities

true (theoretical) variance of $\hat{t}_{HT}$
→ depends on all 4 stores

- **Variance** of $\hat{t}_{HT}$ is

$$Var(\hat{t}_{HT}) = \left(\frac{1-0.1900}{0.1900} 11^2 + \cdots + \frac{1-0.9002}{0.9002} 245^2 \right)$$

$$+ 2 \left(\frac{0.0173-(0.1900)(0.3705)}{(0.1900)(0.3705)} (\underset{A}{11})(\underset{B}{20}) + \right.$$

$$\left. \cdots + \frac{0.4567-(0.5393)(0.9002)}{(0.5393)(0.9002)} (\underset{C}{24})(\underset{D}{245}) \right) = 4383.6$$

- The estimated total sales is \$316.67 thousand with a SE of \$66.2 thousand.

- How does this compare to a SRS of $n = 2$ stores?

$$Var(\hat{t}_{HT}) = \sum_{i=1}^{4} \frac{1-\pi_i}{\pi_i} t_i^2 + 2 \underbrace{\sum\sum}_{\text{all pairs of store}} \frac{\pi_{ik} - \pi_i \pi_k}{\pi_i \pi_k} t_i t_k$$

Example: Unequal inclusion probabilities

- Suppose stores C and D were selected from an SRS.
- **SRS Estimated total:** \$538 thousand

$$\hat{t}_{SRS} = N\bar{y} = 4\frac{24+245}{2} = 538$$

`

- **SRS Variance** of $\hat{t}_{SRS}$ is

ch.2 S^2

$$Var(\hat{t}_{HT}) = 4^2(1-\frac{2}{4})\frac{12874}{2} = 51496$$

where $S^2 = \dfrac{1}{N-1}\sum_{i=1}^{N}(t_i - \bar{t}_{\mathcal{U}}) = 12874$

```
> pop <- c(11,20,24,245)
> var(pop)
[1] 12874
```

Example: Unequal inclusion probabilities

- **Probability proportional to size:** The estimated total sales is \$316.67 thousand with a SE of \$66.2 thousand.
- **SRS:** The estimated total sales is \$538 thousand with a SE of \$226.9 thousand.
- One important reason for selecting PSU with unequal probabilities:
  - can reduce SE (compared to SRS) when selection probability π_i is positively associated with the response t_i
  - called **probability proportional to size (pps)** sampling
  - most samples will contain large t_i making variation in $\hat{t}_{pps} = \hat{t}_{HT}$ less than when a small t_i is just as likely as a large

Store: probs. prop. to store size.

→ size & t_i (sales) are positively correlated

Example: Unequal inclusion probabilities

- One important reason for **not** selecting PSU with unequal probabilities:
  - if some PSU have very small π_i, then they have very high weight w_i
  - can cause imprecise **estimates of** $Var(\hat{t}_{HT})$ because of these high weights

$$\frac{1}{\pi_i} = w_i$$

very small

very big

Estimating HT variance

$$Var(\hat{t}_{HT}) = \sum_{i=1}^{N} \frac{1-\pi_i}{\pi_i} t_i^2 + 2 \sum_{\substack{i=1 \\ i<k}}^{N} \sum_{k=1}^{N} \frac{\pi_{ik} - \pi_i \pi_k}{\pi_i \pi_k} t_i t_k$$

- To estimate variance, treat the summations as population totals
    - estimate the total with a HT-estimator!
    - weight sampled values by w_i's

theory

Estimating HT variance

- There are three commonly used estimates of $Var(\hat{t}_{HT})$
- **Horvitz-Thompson (HT)**: unbiased and often the default software version (as in `survey`), but can be negative for samples with small inclusion probs

$$\hat{V}_{HT}(\hat{t}_{HT}) = \sum_{i=1}^{n} \frac{1-\pi_i}{\pi_i^2} t_i^2 + 2 \sum_{i} \sum_{\substack{k \\ i<k}} \frac{\pi_{ik} - \pi_i \pi_k}{\pi_{ik}} \frac{t_i}{\pi_i} \frac{t_k}{\pi_k},$$

sample units

pairs

- **Sen-Yates-Grundy (SYG):** unbiased and more stable than HT version

$$\hat{V}_{SYG}(\hat{t}_{HT}) = \sum_{i} \sum_{\substack{k \\ i<k}} \frac{\pi_i \pi_k - \pi_{ik}}{\pi_{ik}} \left(\frac{t_i}{\pi_i} - \frac{t_k}{\pi_k} \right)^2$$

sampled units

Estimating HT variance

- **With Replacement:** is a biased estimate that overestimates the variance, but it doesn't require joint inclusion probs!

$$\hat{V}_{WR}(\hat{t}_{HT}) = \frac{n}{n-1}\sum_{i=1}^{n}\left(\frac{t_i}{\pi_i} - \frac{\hat{t}_{HT}}{n}\right)^2$$

$\hat{V}_{HT}(\hat{t}_{HT}) \longrightarrow$ stores C+D sample $\pi_C = .5393$ $\pi_D = .9002$

$t_C = 24$ $t_D = 245$ $\pi_{CD} = .4567$

$$\hat{V}_{HT}(\hat{t}_{HT}) = \frac{1-.5393}{.5392^2}(24)^2 + \frac{1-.9002}{.9002^2}(245)^2$$

$$+ 2 \times \left(\frac{.4567 - (.5393)(.9002)}{.4567}\right)\frac{24}{.5393} \times \frac{245}{.9002}$$

≈ 6778

Example: Estimating HT variance

$\to \pi_{ik}$

Sample $\mathcal{S}$	$P(\mathcal{S})$	$\hat{t}_{HT}$	$\hat{V}_{HT}(\hat{t}_{HT})$	$\hat{V}_{SYG}(\hat{t}_{HT})$
A,B	0.01726	111.87	-14,691.5	47.1
A,C	0.02692	102.39	-10,832.1	502.8
A,D	0.14583	330.06	4,659.3	7,939.8
B,C	0.05563	98.48	-9,705.1	232.7
B,D	0.29762	326.15	5,682.8	5,744.1
C,D	0.45673	316.67	6,782.8	3,259.8

- If we happen to sample two small stores, our HT estimate of variance is negative!
- But both are unbiased estimators of the true variance.

$E[\hat{V}_{HT}] = (-14691.5)(.01726) + \text{other 5} \ldots = V_{HT} = 4383$

↳ all samples

$E[\hat{V}_{SYG}] = 47.1(.01726) + \ldots = V_{HT} = 4383$

What about estimating population mean?

$$\sum_{all\ elements} w_i$$

→ observation unit

- Summing sampling weights over all **elements** sampled will give
  - actual population size (of elements) when weights are equal for all elements and number of elements per PSU is constant
  - an unbiased estimated population size (of elements)
- The Horvitz-Thompson estimate of population mean (per element) is

$$\hat{\bar{y}}_{HT} = \frac{\hat{t}_{HT}}{\sum_{all\ elements} w_i} = \frac{\sum w_i t_i}{\sum w_i}$$

- The `survey` package uses this when you run `svymean`
  - gives $\hat{t}_{HT}$ when you run `svytotal`