Comparing One-stage cluster sampling to SRS

Week 6 (5.2)

Stat 260, St. Clair

When is a one-stage cluster sample more precise than SRS?

When does

$$SE(\hat{t}_{cluster}) \overset{???}{<} SE(\hat{t}_{SRS})$$

answer: It depends on the measurement's **Analysis of Variance** (ANOVA)

$$SST = \underbrace{SSB}_{\text{between clusters}} + \underbrace{SSW}_{\text{within clusters}}$$

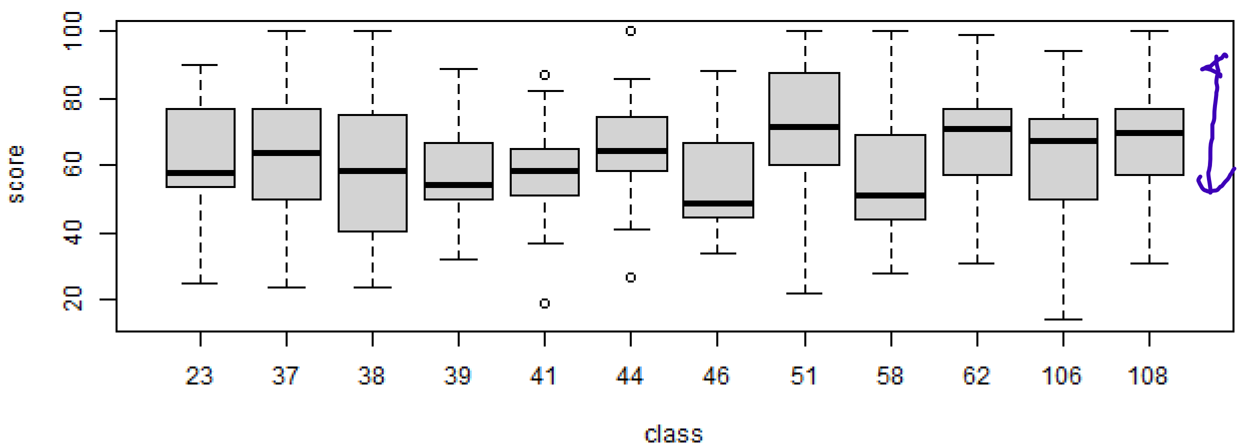
Lohr Examples 5.6: design effect

```
> svymean(~score, alg_design, deff = TRUE)
          mean      SE   DEff
score 62.5686  1.4916 2.245
> boxplot(score ~ class, data = algebra)
```

$\frac{\text{cluster}}{\text{SRS}} \approx 2.2$

SE for cluster est is bigger than SE for same sized SRS

same # of SSU (students)



y

clusters

Population ANOVA

Let y_{ij} be your measurement of unit j in cluster i

ANOVA breaks the **total** sum of squares of y into **between cluster** and **within cluster** variation:

$$SST = SSB + SSW$$

For now, **assume that cluster sizes are equal**

$$M_i = M \text{ for all clusters } i = 1, \ldots, N$$

Population ANOVA

Source	df	Sum of Squares	Mean Square
Between	$N-1$	$SSB = \sum_{i=1}^{N} M(\bar{y}_{i\mathcal{U}} - \bar{y}_{\mathcal{U}})^2$	$MSB = \dfrac{SSB}{N-1}$
Within	$N(M-1)$	$SSW = \sum_{i=1}^{N} (M-1)S_i^2$	$MSW = \dfrac{SSW}{N(M-1)}$
total	$NM-1$	$SSTot = \sum_{i=1}^{N} \sum_{j=1}^{M} (y_{ij} - \bar{y}_{\mathcal{U}})^2$	$S^2 = \dfrac{SSTot}{NM-1}$

Between: $\bar{y}_{\mathcal{U}}$ = overall mean per SSU (pop.)
$\bar{y}_{i\mathcal{U}}$ = cluster i mean per SSU (pop.)

Within: S_i^2 = pop. variance for cluster i

Variance: SRS

Equal cluster sizes: We've sampled nM **observation units** (SSU) out of $M_0 = NM$ possible units.

For a SRS of nM **observation units**, we can write the variance, SE^2, of $\hat{t}_{SRS}$ as

$$Var(\hat{t}_{SRS}) = (NM)^2 \left(1 - \frac{nM}{NM}\right) \frac{S^2}{nM}$$

where S is the SD of the measurements in the population.

Variance: One-stage cluster sample

Equal cluster sizes: Under this assumption the variance of $\hat{t}_{unb}$ is equal to

$$Var(\hat{t}_{unb}) = N^2 \left(1 - \frac{n}{N}\right) \frac{M \times MSB}{n}$$

$$Var(\hat{t}_{unb}) = N^2\left(1-\frac{n}{N}\right)\frac{S_t^2}{n} \rightarrow \text{variance of cluster totals}$$

$$S_t^2 = \frac{1}{N-1}\sum_{i=1}^{N}(t_i - \bar{t}_u)^2 = \frac{1}{N-1}\sum(M\bar{y}_{iu} - M\bar{y}_u)^2$$

$$t_i = \underbrace{\sum_{j=1}^{M} y_{ij} \times \frac{M}{M}}_{\bar{y}_{iu}} = M\bar{y}_{iu} \quad \Bigg\| \quad \bar{t}_u = \underbrace{\frac{\sum_i\sum_j y_{ij}}{N} \times \frac{NM}{NM}}_{\bar{y}_u} = M\bar{y}_u$$

$$S_t^2 = \frac{M^2}{N-1}\sum_{i=1}^{N}(\bar{y}_{iu} - \bar{y}_u)^2 = \underbrace{\frac{M \cdot SSB}{N-1}}_{MSB} = M \times MSB$$

Variance: SRS vs. Stratified sample

- **Equal cluster sizes:** Under this assumption, the design effect for a one-stage cluster sample total estimate is

$$DEff(\hat{\bar{y}}_{unb}) = DEff(\hat{t}_{unb}) = \frac{Var(\hat{t}_{unb})}{Var(\hat{t}_{SRS})} = \frac{MSB}{S^2}$$

Variance: SRS vs. Stratified sample

Cluster sampling is more precise than an equal sized SRS when

$$MSB < S^2$$

$$\Rightarrow \text{ between cluster variation is small}$$

$$\Rightarrow \text{ measurements are heterogenous within clusters}$$

small big

$$SST = \overset{\text{small}}{SSB} + \overset{\text{big}}{SSW}$$

$$SST \rightarrow S^2 \qquad SSB \rightarrow MSB$$

Opposite result from strat. sampling

→ strat. good when measurements within strata are homogeneous.

Measuring homogeneity within clusters

- **Intraclass correlation coefficient:** for equal sized clusters

$$ICC = 1 - \frac{M}{M-1}\frac{SSW}{SSTot} \quad \text{where } -\frac{1}{M-1} \leq ICC \leq 1$$

- **Adjusted R-squared:** can be used for unequal cluster sizes

$$R_a^2 = 1 - \frac{MSW}{S^2} \quad \text{where } 1 - \frac{NM-1}{N(M-1)} \leq R_a^2 \leq 1$$

- **For both:**
  - values near 1 indicate **homogeneous** (similar) responses **within** clusters $SSW \approx MSW \approx 0$
  - values near 0 indicate **heterogeneous** (dissimilar) responses **within** clusters

$SSW \approx SStot \quad (SSB \approx 0)$

Design effect revisted

Equal cluster sizes: Under this assumption the ~~variance~~ DEff of $\hat{t}_{unb}$ is equal to

$$\begin{aligned} DEff(\hat{t}_{unb}) &= \frac{MSB}{S^2} \\ &= \frac{MN-1}{M(N-1)}(1+(M-1)ICC) \\ &= 1 + \frac{N(M-1)}{N-1} R_a^2 \end{aligned}$$

Design effect revisted

What is the design effect if

- N is big
- $M = 11$
- $R_a^2 = 0.5$

$$\text{Deff(cluster)} = 1 + \frac{N(M-1)}{N-1} R_a^2 = 1 + \frac{N}{N-1}(10)\left(\frac{1}{2}\right)$$

N big $\approx 1 + 10\left(\frac{1}{2}\right) = \underline{\underline{6}} = \frac{\text{Var(cluster)}}{\text{Var(SRS)}}$

$\text{Var(SRS)} = \frac{\text{Var(cluster)}}{6}$ → $\underline{\underline{n}}$ = # clusters sampled

$n \times 6$ = # clusters needed to sample to have same SE as an SRS of $n \cdot 11$ observation units

equal SE: SRS $n \cdot 11$ units One-stage $n \cdot 6 \cdot 11$ units

Big picture

- One-stage cluster sampling is "good" for precision if SSU within clusters have very **heterogeneous** responses
    - true whether or not cluster sizes are equal
- But often SSU within clusters have very **homogeneous** responses
    - clusters contain "similar" observation units
    - clusters defined for **cost-saving** reasons, not for precision

Post-Hoc comparison

Q: How do we compute the design effect with a **sample of data** from any one-stage cluster sample?

1. (Any cluster sizes) Use sampling weights to estimate $Var(\hat{t}_{srs})$

- This is what the `survey` package when you use `deff=TRUE`

$S^2 \Rightarrow$ ratio estimate

"raw" data
SSU-level

Post-Hoc comparison

Q: How do we compute the design effect with a **sample of data** from any one-stage cluster sample?

2. Equal cluster sizes: Estimate population sum of square values from **sample** mean square values msw and msb:

$$\widehat{SSW} = N(M-1)msw \quad \widehat{SSB} = (N-1)msb$$

The estimated design effect is

$$\widehat{DEff}(\hat{t}_{unb}) = \frac{\widehat{MSB}}{\hat{S}^2} = \frac{msb}{(\widehat{SSW} + \widehat{SSB})/(NM-1)}$$

$$\hat{S}^2 = \frac{\widehat{SStotal}}{NM-1} = \frac{\widehat{SSW} + \widehat{SSB}}{NM-1}$$

Estimating ICC and R_a^2

$$\widehat{SSW} = N(M-1)msw, \quad \widehat{SSB} = (N-1)msb, \quad \widehat{SST} = \widehat{SSB} + \widehat{SSW}$$

- Estimated ICC is

$$ICC = 1 - \frac{M}{M-1}\frac{\widehat{SSW}}{\widehat{SST}}$$

- Estimated R_a^2 is

$$\hat{R}_a^2 = 1 - \frac{msw}{\hat{S}^2}$$

Example - GPA

$N = 100, n = 5, M_i = 4, M_0 = 400$

	Suite 1	Suite 2	Suite 3	Suite 4	Suite 5
1	3.08	2.36	2.00	3.00	2.68
2	2.60	3.04	2.56	2.88	1.92
3	3.44	3.28	2.52	3.44	3.28
4	3.04	2.68	1.88	3.64	3.20
total	12.16	11.36	8.96	12.96	11.08

```
> dorm <- read.csv("http://math.carleton.edu/kstclair/data/Dorm_Clust
> dplyr::glimpse(dorm)
Rows: 20
Columns: 2
$ gpa  <dbl> 3.08, 2.60, 3.44, 3.04, 2.36, 3.04, 3.28, 2.68, 2.00, 2.
$ room <int> 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5,
```

ter.csv")

y_{ij}

cluster id

Example - GPA

$N = 100, n = 5, M_i = 4, M_0 = 400$

What is the design effect, ICC and R_a^2 for estimating mean GPA?

y ~ cluster

```
> dorm_lm <- lm(gpa ~ factor(room), data = dorm)
> anova(dorm_lm)
Analysis of Variance Table

Response: gpa
             Df Sum Sq Mean Sq F value  Pr(>F)
factor(room)  4 2.2557 0.56392  3.0476 0.05039 .
Residuals    15 2.7756 0.18504
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

because room is a numeric variable in the data

between cluster = factor(room)

with = Residuals

msb = .564 msw = .185

pop. SS

$SSB = (N-1)msb = (100-1)(.564) = 55.83$

$SSW = N(M-1)msw = 100(4-1)(.185) = 55.51$

$$\hat{s}^2 = \frac{\widehat{SSW} + \widehat{SSB}}{NM-1} = \frac{55.51 + 55.83}{100(4)-1} \approx .279$$

$$DEff = \frac{msb}{\hat{s}^2} = \frac{.564}{.279} \approx 2.02$$

$$R_a^2 = 1 - \frac{msw}{\hat{s}^2} = 1 - \frac{.185}{.279} \approx .34$$

$$ICC = 1 - \frac{M}{M-1}\frac{\widehat{SSW}}{\widehat{SST}} = 1 - \frac{4}{4-1}\frac{55.51}{55.51+55.83}$$

$$\approx .34$$

$$\boxed{\text{survey package } DEff \approx 2.12}$$

Lohr Examples 5.6: design effect of 2.245

What if cluster sizes are not equal?

```
> alg_lm <- lm(score ~ factor(class), data = algebra)
> anova(alg_lm)
Analysis of Variance Table

Response: score
               Df Sum Sq Mean Sq F value  Pr(>F)
factor(class)  11   7086  644.14  2.1184 0.01915 *
Residuals     287  87270  304.08
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
> msb <- 644.14
> msw <- 304.08
> msb/var(algebra$score)    # rough DEff guess
[1] 2.03437
```

bias est. of σ^2!

Lohr Examples 5.6: design effect of 2.245

What if cluster sizes are not equal?

```
> summary(alg_lm)$adj.r.squared # rough R^2_a guess
[1] 0.03964506
> library(dplyr)
> algebra %>%
+   group_by(class) %>%
+   summarize(Mi = n()) %>%  # gets Mi values by class
+   summarize(mean(Mi))      # mean Mi per class
# A tibble: 1 x 1
  `mean(Mi)`
       <dbl>
1       24.9
> 1 +  187*(25-1)*.04/(187-1)  # rough DeFF guess based on R^2_a
[1] 1.965161
```

≈ .04

mean # students/class