Homotopy (Wheeled) Props

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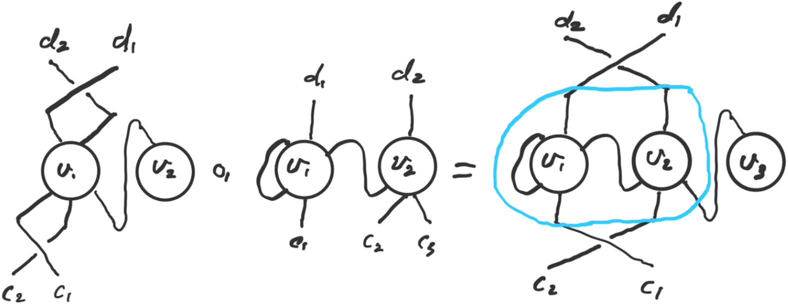
Operads Governing Operadic Structures

Let W be the set of strict isomorphism classes of $\mathcal{C}$-coloured, ordered, directed, wheeled graphs.



The Operad Governing Wheeled Props

Proposition

There exists an operad $\mathbb{W}$, whose algebras are wheeled props.

- W assembles into a coloured operad in Set.
- For a symmetric monoidal cat $\mathcal{E}$ the free object functor $F : Set \to \mathcal{E}, FX = \coprod_{x\in X} I$ is symmetric monoidal.
- This functor enriches W into $\mathbb{W}$, a coloured operad in $\mathcal{E}$.
- Algebras over $\mathbb{W}$ are wheeled props in $\mathcal{E}$.

Koszul Operads

A algebraic operad P is Koszul if, and only if, it has a quadratic model P_∞.

- model: $M \xrightarrow{\text{epimorphism}} P$, inducing iso of homology
- quadratic: $M = (F(E), d)$ and $d(E) \to F(E)^2$
- quadratic models $\subseteq$ minimal models

The point: If algebras over $\mathbb{W}$ are wheeled props, then algebras over $\mathbb{W}_\infty$ are homotopy associative wheeled props.

Koszul Operads Governing Operadic Structures

The Koszul machine has been used to produce homotopy weakened versions of most operadic structures,

- Operads: [Van der Laan, 2003]
- Modular operads: [Ward, 2019]
- Operadic structures living on connected graphs (e.g. dioperads, properads, wheeled properads,...):
  - [Kaufmann and Ward, 2021] with cubical Feynman categories
  - [Batanin and Markl, 2021] with partial operads in operadic categories

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These techniques don't readily extend to disconnected operadic structures.

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Construction

We can construct Koszul groupoid coloured operads $\mathbb{W}$ and $\mathbb{P}$ whose algebras govern wheeled props, and props respectively.

Today

1. Present a new biased definition of a wheeled prop.
2. Induce the definition of $\mathbb{W}$, which is groupoid coloured and quadratic.
3. Outline why $\mathbb{W}$ is Koszul
   - Extend Groebner bases for operads to **groupoid coloured** operads
4. Use the Koszul machine to define homotopy wheeled props
5. Explore some applications

All steps outlined also work for Props (modulo suffering).

Let $\mathcal{C}$ be a non-empty set of colours and $\underline{c}, \underline{d}$ be sequences from this set.

Definition

Let $\mathcal{P}(\mathcal{C})$ be the category whose

- objects are sequences of colours $\underline{c} = (c_1, ..., c_n)$ from $\mathcal{C}$
- and morphisms are permutations acting from left, i.e. $\sigma \in Hom(\underline{c}, \sigma\underline{c})$

- Then construct $\mathcal{P}(\mathcal{C})^{op} \times \mathcal{P}(\mathcal{C})$.

Definition

For a symmetric monoidal cat $\mathcal{E}$, a **bimodule** is $P : \mathcal{P}(\mathcal{C})^{op} \times \mathcal{P}(\mathcal{C}) \to \mathcal{E}$

- for all pairs of profiles $\binom{\underline{d}}{\underline{c}}$ have an object $P\binom{\underline{d}}{\underline{c}} \in \mathcal{E}$.
- for all pairs of permutations $\binom{\sigma}{\tau}$ have isomorphisms $P\binom{\underline{d}}{\underline{c}} \xrightarrow{\binom{\sigma}{\tau}} P\binom{\sigma\underline{d}}{\underline{c}\tau}$

Example: $W = \bigcup_{\binom{\underline{d}}{\underline{c}}} W\binom{\underline{d}}{\underline{c}}$ is a bimodule.

Definition (Yau and Johnson Def 11.33)

A $\mathcal{C}$-coloured **wheeled prop** over $\mathcal{E}$ consists of

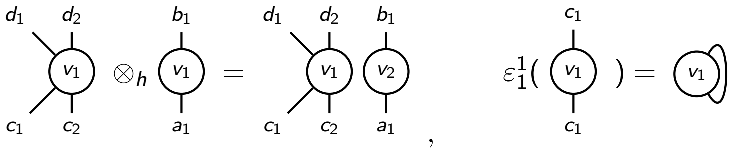
1. a bimodule $P : \mathcal{P}(\mathcal{C})^{op} \times \mathcal{P}(\mathcal{C}) \to \mathcal{E}$
2. a horizontal composition

$$P\binom{\underline{d}}{\underline{c}} \otimes P\binom{\underline{b}}{\underline{a}} \xrightarrow{\otimes_h} P\binom{\underline{d}, \underline{b}}{\underline{c}, \underline{a}}$$

3. a contraction

$$P\binom{\underline{d}}{\underline{c}} \xrightarrow{\varepsilon_j^i} P\binom{\underline{d} \setminus d_i}{\underline{c} \setminus c_j}$$

4. some units and axioms



Wheeled Props

Definition

An **alternate** $\mathcal{C}$-coloured wheeled prop consists of

1. the same bimodule P.
2. an extended horizontal composition

$$P\binom{\underline{d}}{\underline{c}} \otimes P\binom{\underline{b}}{\underline{a}} \xrightarrow{\otimes_h, \binom{\sigma}{\tau}} P\binom{\sigma(\underline{d}, \underline{b})}{(\underline{c}, \underline{a})\tau}$$

   where $(\otimes_h, \binom{\sigma}{\tau})(\alpha, \beta) := \otimes_h(\alpha, \beta) \cdot \binom{\sigma}{\tau}$

3. an extended contraction

$$P\binom{\underline{d}}{\underline{c}} \xrightarrow{(\varepsilon^i_j, \binom{\sigma}{\tau})} P\binom{\sigma(\underline{d} \setminus d_i)}{(\underline{c} \setminus c_j)\tau}$$

   where $(\varepsilon^i_j, \binom{\sigma}{\tau})(\alpha) := \varepsilon^i_j(\alpha) \cdot \binom{\sigma}{\tau}$

What does this accomplish?

Operations are right compatible

$$(\otimes_h, \binom{\sigma}{\tau})\binom{\sigma'}{\tau'} = (\otimes_h, \binom{\sigma'\sigma}{\tau\tau'})$$
$$(\varepsilon^i_j, \binom{\sigma}{\tau})\binom{\sigma'}{\tau'} = (\varepsilon^i_j, \binom{\sigma'\sigma}{\tau\tau'})$$

left compatible,

$$(\binom{\sigma_1}{\tau_1}, \binom{\sigma_2}{\tau_2})(\otimes_h, \binom{\sigma}{\tau}) = (\otimes_h, \binom{\sigma(\sigma_1 \times \sigma_2)}{(\tau_1 \times \tau_2)\tau})$$
$$\binom{\sigma'}{\tau'}(\varepsilon^{\sigma'^{-1}(i)}_{\tau'(j)}, \binom{\sigma}{\tau}) = (\varepsilon^i_j, \binom{\sigma(\sigma'^{(i)})}{(\tau'^{(j)})\tau})$$

and $\otimes_h$ has a well defined action of $\mathbb{S}_2$ on it.

Every other (non-unital) axiom is quadratic.

The Groupoid Coloured Approach

~~Let $\mathcal{C}$ be a non-empty set of colours and $\underline{c}, \underline{d}$ be sequences from this set.~~
Let $\mathbb{V}$ be a groupoid, and $\underline{c}, \underline{d}$ be sequences of objects from this category.

Definition

Let $\cancel{\mathcal{P}(\mathcal{C})}$ $\mathcal{W}(\mathbb{V})$ be the category whose

- ~~objects are sequences of colours $\underline{c}$ from $\mathcal{C}$~~
- objects are sequences of objects $\underline{c}$ from $\mathbb{V}$
- ~~and morphisms are permutations acting from left, i.e. $\sigma \in Hom(\underline{c}, \sigma\underline{c})$~~
- and morphisms are permutations with 'extensions' by morphisms of $\mathbb{V}$

Formally, $\mathcal{W}_k(\mathbb{V}) := \mathbb{V}^k \rtimes \Sigma_k$, and $\mathcal{W}(\mathbb{V}) := \coprod_{k \geq 0} \mathcal{W}_k(\mathbb{V})$.

- ~~A bimodule is $P : \mathcal{P}(\mathcal{C})^{op} \times \mathcal{P}(\mathcal{C}) \to \mathcal{E}$~~
- A **groupoid coloured bimodule** is $P : \mathcal{W}(\mathbb{V})^{op} \times \mathcal{W}(\mathbb{V}) \to \mathcal{E}$

If $\mathbb{V}$ is discrete, then the $\mathbb{V}$-bimodule is a bimodule.

Groupoid Coloured Operads

Definition

A **non-unital partial groupoid coloured operad** consists of

- a groupoid coloured module P
- partial compositions

$$P\binom{d}{\underline{c}} \otimes P\binom{c_i}{\underline{b}} \xrightarrow{\circ_i} P\binom{d}{\underline{c} \circ_i \underline{b}}$$

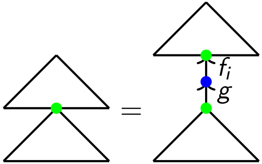
It satisfies the standard associativity axioms, and this extended equivariance axiom

$$\begin{tikzcd}
P\binom{d}{\underline{c}} \otimes P\binom{c_i}{\underline{b}} \arrow[r, "(\sigma,f,g)\otimes(\tau,f',g')"] \arrow[d, "\circ_i"] & P\binom{d'}{\underline{c}'} \otimes P\binom{c'_{\sigma(i)}}{\underline{b}'} \arrow[d, "\circ_{\sigma(i)}"] \\
P\binom{d}{\underline{c}\circ_i\underline{b}} \arrow[r, "(\sigma,f,g)\circ_i(\tau,f',g')"] & P\binom{d'}{\underline{c}'\circ_{\sigma(i)}\underline{b}'}
\end{tikzcd}$$

Understanding Equivariance

$$
\begin{tikzcd}
P\binom{d}{\underline{c}} \otimes P\binom{c_i}{\underline{b}} \arrow[r, "(id,id,(id,\dots,f_i,\dots,id))\otimes(id,g,\vec{id})"] \arrow[d, "\circ_i"] & P\binom{d}{c_1,\dots,c_i',\dots,c_{|\underline{c}|}} \otimes P\binom{c_i'}{\underline{b}} \arrow[d, "\circ_i"] \\
P\binom{d}{\underline{c}\circ_i\underline{b}} \arrow[r, "id=(id,id,\vec{id})"] & P\binom{d}{\underline{c}\circ_i\underline{b}}
\end{tikzcd}
$$

$$c_i \xrightarrow{g} c_i' \xrightarrow{f_i} c_i \in Aut(c_i)$$



i.e. we have an action of the groupoid on the internal edges of groupoid coloured operads, via the automorphism group of the colour.

The Groupoid Coloured Operad Governing Wheeled Props

The operad governing wheeled props is $\mathbb{W} = F_{\Sigma}(E)/\langle R \rangle$ where

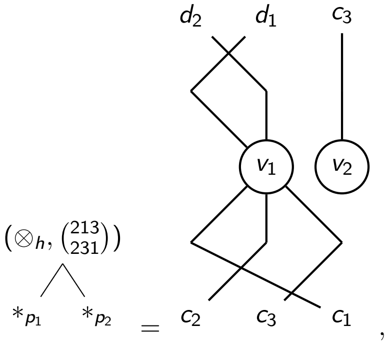
- the groupoid is $\mathcal{P}(\mathcal{C})^{op} \times \mathcal{P}(\mathcal{C})$
- the groupoid coloured module E is given by

$$E(\binom{\underline{d}}{\underline{c}};\binom{\underline{b}}{\underline{a}}) := \{(\varepsilon^i_j, \binom{\sigma}{\tau})(-) : d_i = c_j \wedge \binom{\sigma(\underline{d} \setminus \{d_i\})}{(\underline{c} \setminus \{c_j\})\tau} = \binom{\underline{b}}{\underline{a}}\}$$

$$E(\binom{\underline{d}}{\underline{c}}, \binom{\underline{b}}{\underline{a}};\binom{\underline{f}}{\underline{e}}) := \{(\otimes_h, \binom{\sigma}{\tau})(-,-) : \binom{\sigma(\underline{d},\underline{b})}{(\underline{c},\underline{a})\tau} = \binom{\underline{f}}{\underline{e}}\}$$

  where the action of the groupoid on these generators is given by right and left compatibility, and the action of $\mathbb{S}_2$ on $\otimes_h$.
- the relations R are given by the remaining quadratic axioms

If we have profiles $p_1 = \binom{d_1, d_2}{c_1, c_2, c_3}$, $p_2 = \binom{c_3}{\emptyset}$, $p = \binom{d_2, d_1, c_3}{c_2, c_3, c_1}$, $p_\gamma = \binom{d_2, d_1}{c_2, c_1}$ then $(\otimes_h, \binom{213}{231}) \in E\binom{p}{p_1, p_2}$ and $(\varepsilon_2^3, \binom{id}{id}) \in E\binom{p_\gamma}{p}$



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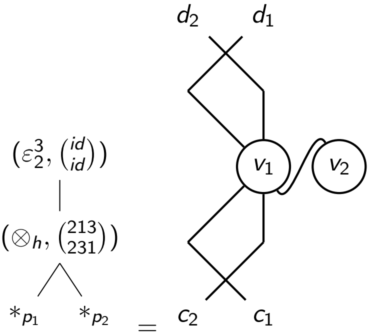
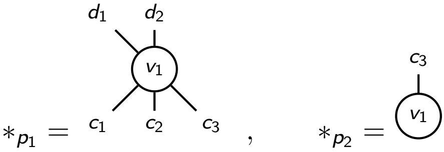
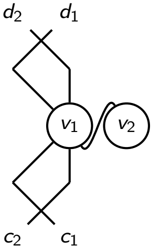


Where,



$$\begin{array}{c} (\varepsilon^3_2, \binom{id}{id}) \\ | \\ (\otimes_h, \binom{213}{231}) \\ \wedge \\ *_{p_1} \quad *_{p_2} \end{array} =$$

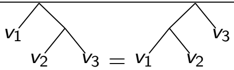
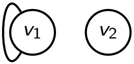
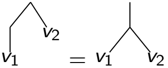
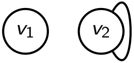
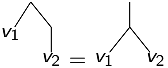
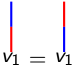


$$= \begin{array}{c} (\varepsilon^3_3, \binom{21}{21}) \\ | \\ (\otimes_h, \binom{id}{id}) \\ \wedge \\ *_{p_1} \quad *_{p_2} \end{array}$$

- The internal edge of the left tree monomial is $p = \binom{d_2, d_1, c_3}{c_2, c_3, c_1}$.
- $id = p \xrightarrow{\binom{213}{231}^{-1}} p' \xrightarrow{\binom{213}{231}} p \in Aut(p)$ where $p' = \binom{d_1, d_2, c_3}{c_1, c_2, c_3}$

$$\begin{array}{c} (\varepsilon^3_2, \binom{id}{id}) \\ | \\ (\otimes_h, \binom{213}{231}) \\ \wedge \\ *_{p_1} \quad *_{p_2} \end{array} \overset{eq.\ axiom}{=} \begin{array}{c} \binom{213}{231} \cdot (\varepsilon^3_2, \binom{id}{id}) \\ | \\ (\otimes_h, \binom{213}{231}) \cdot \binom{213}{231}^{-1} \\ \wedge \\ *_{p_1} \quad *_{p_2} \end{array} \overset{\text{left+right actions}}{=} \begin{array}{c} (\varepsilon^3_3, \binom{21}{21}) \\ | \\ (\otimes_h, \binom{id}{id}) \\ \wedge \\ *_{p_1} \quad *_{p_2} \end{array}$$

The Quadratic Relations of $\mathbb{W} = F_\Sigma(E)/\langle R \rangle$

Graphs	Relations
	
	
	
 ,	

Showing $\mathbb{W}$ is Koszul

We can extend Groebner bases for operads to Groupoid coloured operads!

Theorem

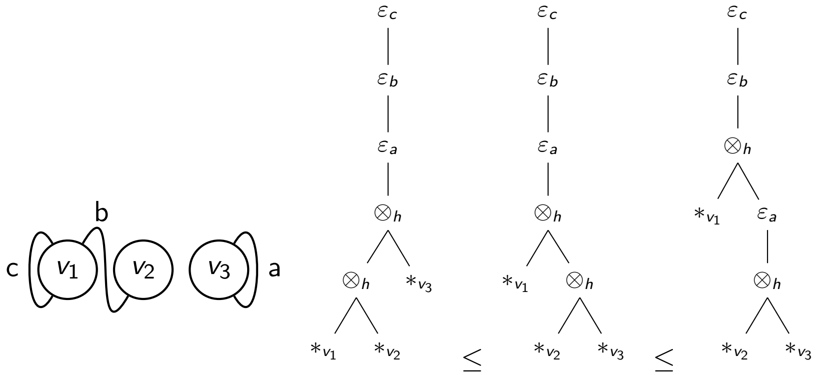
Let $\mathcal{O} = F_\Sigma(\mathcal{X})/\langle\mathcal{G}\rangle$ be a $\mathbb{V}$-coloured operad, where $Aut(v)$ is finite for all $v \in ob(\mathbb{V})$. If $(\mathcal{O}^f)_*$ admits a quadratic Groebner basis then $\mathcal{O}$ is Koszul.

$(\mathbb{W}^f)_*$ admits a quadratic Groebner basis $\implies$

Theorem

The groupoid coloured operad $\mathbb{W}$ governing wheeled props is Koszul.

What does it mean to have a quadratic Groebner basis?



- Every graph has a unique minimal tree monomial forming it.
- Every non-minimal tree monomial can be rewritten to the minimal tree via the relations of the operad.

Homotopy (Wheeled) Props

Given a groupoid coloured operad O, and cooperad C, let

- $O^{¡}$ be the Koszul dual cooperad of O, and
- $\Omega(C)$ be the cobar construction applied to the cooperad C.

As $\mathbb{W}$ and $\mathbb{P}$ are Koszul,

- $\mathbb{W}_\infty := \Omega(\mathbb{W}^{¡})$ is a (quadratic) minimal model for $\mathbb{W}$.
- $\mathbb{P}_\infty := \Omega(\mathbb{P}^{¡})$ is a (quadratic) minimal model for $\mathbb{P}$.

As such we define

- A homotopy wheeled prop is a algebra over $\mathbb{W}_\infty$.
- A homotopy prop is a algebra over $\mathbb{P}_\infty$.

Homotopy Transfer Theory for Operads

A homotopy retract of chain complexes is of the form,

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (B, d_B) \quad , \quad id_A - ip = d_A h + h d_A \ , \quad i \text{ is quasi-iso}$$

e.g. the homology of a chain complex is a homotopy (deformation) retract

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0)$$

If A is an P-algebra, can we transfer the P-algebra structure to B?

- In general no!
- But we can transfer a P_∞-algebra structure to B

See for instance Loday and Vallette [2012].

HTT for Groupoid Coloured Operads

A homotopy retract of $\mathbb{V}$-modules is a family of homotopy retracts, indexed by $ob(\mathbb{V})$, such that h_v, i_v, p_v are $Aut(v)$-equivariant.

$$h_v \circlearrowright A(v) \underset{i_v}{\overset{p_v}{\rightleftarrows}} B(v)$$

Lemma (2.57 of Ward [2019])

The homology of $\mathbb{V}$-module, with finite $Aut(v)$ for all $v \in Ob(\mathbb{V})$, over a field of characteristic 0, is a homotopy (deformation) retract.

If A is an P-algebra, can we transfer the P-algebra structure to B?

- In general no!
- But we can transfer a P_∞-algebra structure to B

See Ward [2019].

Massey Products

HTT applied to homology gives what are known as Massey products.

Theorem (2.58 of Ward [2019])

Let P be a Koszul groupoid coloured operad and A a P_∞-algebra.

- *The homology of A, $H(A)$ has a P_∞-structure.*
- *This P_∞-structure extends the induced P-algebra structure.*

Formality

Definition (Section 11.46 Loday and Vallette [2012])

A dg algebra A over a Koszul groupoid coloured operad P is said to be **formal** if there exists a ∞-quasi-iso of dg P-algebras between it and its homology $H(A)$.

Massey products obstruct formality, furthermore

Proposition

Let P be a Koszul groupoid coloured operad and A a dg P-algebra. If the Massey products on $H(A)$ vanish, then A is formal.

A is formal if, and only if, the Massey products vanish.

(An extension of 11.4.10 Loday and Vallette, using results of Ward)

Massey Products for (Wheeled) Props

Corollary

The homology of a (wheeled) prop admits Massey products.

Corollary

A (wheeled) prop is formal if, and only if, its Massey products vanish.

An Early Instance of Homotopy Transfer Theory

Let $K(H_{fc})$ be the prop governing dg Hopf algebras with commutative products. Let $K(H_{fcc})$ be the prop governing dg Hopf algebras with commutative products and co-commutative co-products.

Theorem (25.1 of Mac Lane [1965])

If U is a $K(H_{fcc})$ algebra, then there is a PACT $P \supset K(H_{fc})$, which acts on the bar construction $B^{\cdot}(U)$ and on the reduced bar construction $\overline{B^{\cdot}}(U)$, and a map $\theta : P \to K(H_{fcc})$ of PACTs such that the induced homology map

$$\theta_* : H(P) \cong H(K(H_{fcc}))$$

is an isomorphism.

"This is a covert statement on the existence of higher homotopies."

Summary

1. New biased definitions for (wheeled) props.
2. $\mathbb{W}$ and $\mathbb{P}$ are Koszul.
3. Proven via extending Groebner bases to groupoid coloured operads.
4. This defines infinity (wheeled) props.
5. Homotopy transfer theory for (wheeled) props can be used to,
   - Obtain Massey products
   - Study formality
   - Study deformation theory

Other models?

- Many models for infinity operadic structures e.g. dendroidal, simplicial, equifibered, topological, Lurie-esque, ...
- Few for props,
  - Simplicial props (Hackney and Robertson)
  - Via symmetric monoidal infinity categories (Haugseng and Kock)
- None for wheeled props (?)

- Can we develop these other models for infinity (wheeled) props?
- Can we relate them via extensions of existing nerves?

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