Relating Diagonals of the Permutahedra

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Joint work with Bérénice Delcroix-Oger, Matthieu Josuat-Vergès,
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The University of Melbourne

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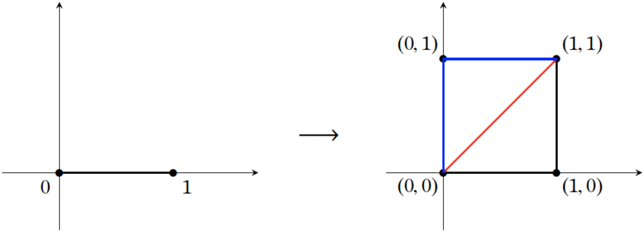
Cellular diagonals

Let P be a polytope in $\mathbb{R}^n$. In general, the set theoretic diagonal

$$\begin{array}{rcccc} \Delta & : & P & \to & P \times P \\ & & x & \mapsto & (x, x) \end{array}$$

is *not* cellular.



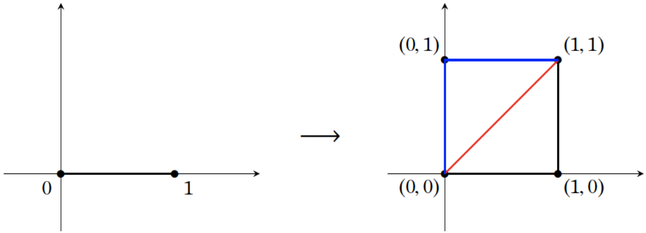
Slide: G. Laplante-Anfossi

Cellular diagonals

Definition

A *cellular diagonal* of a polytope P is a continuous map $P \to P \times P$ such that

1. its image is a union of dim P-faces of $P \times P$ (i.e. it is *cellular*),
2. it agrees with the thin diagonal on the vertices of P, and
3. it is homotopic to the thin diagonal, relative to the image of the vertices.



Slide: G. Laplante-Anfossi

Cellular diagonals

Example

- Simplices: Alexander–Whitney map (1935-38).
- Cubes: J.-P. Serre's thesis (1951).
- Associahedron:
  - Saneblidze–Umble (2004),
  - Markl–Shnider (2006),
  - Masuda–Tonks–Thomas–Vallette (2021).
- Permutahedron:
  - Saneblidze–Umble (2004),
  - Laplante-Anfossi (2022).

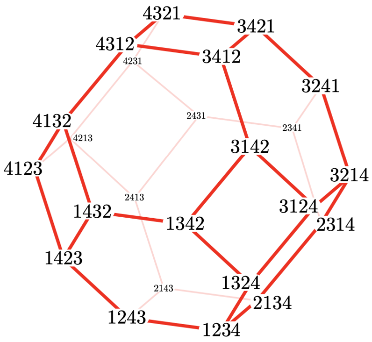
Slide: G. Laplante-Anfossi

The Permutahedra

Definition

The $(n-1)$-dimensional permutahedra P_n is the convex hull of the points

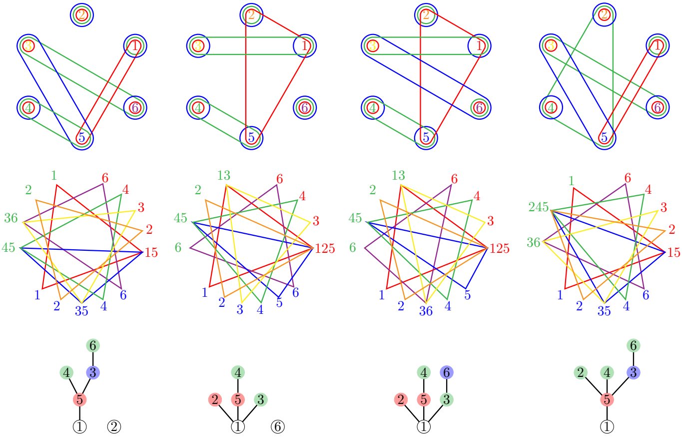
$$(\sigma(1), ..., \sigma(n)) \in \mathbb{R}^n, \sigma \in \mathbb{S}_n$$



Our Main Results

General enumeration results for **cellular** diagonals of the **permutahedra**

- (a) Using hyperplane arrangements and a theorem of Zaslavsky.
- (b) More explicit bijective formulae via Rainbow Trees/Forests



More general theory can be specialised to enumerate the diagonal!

Our Main Results

There exists an isomorphism Θ which decomposes each face $A_1|\ldots|A_k$ of the permutahedron $P_{|A_1|+\cdots+|A_k|-1}$ as a product $P_{|A_1|-1} \times \cdots \times P_{|A_k|-1}$.

Definition

A diagonal of the permutahedra $\triangle$ is *operadic* if for every face $A_1|\ldots|A_k$ of the permutahedron $P_{|A_1|+\cdots+|A_k|-1}$, the map Θ induces a topological cellular isomorphism

$$\triangle(A_1) \times \ldots \times \triangle(A_k) \cong \triangle(A_1|\ldots|A_k) \ .$$

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There are exactly two operadic diagonals of the permutahedra, which respect the weak Bruhat order on the vertices:

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They are isomorphic cellularly, and at the level of face lattices.

The Goal for Today

Definition (Saneblidze–Umble, 2004)

The SU diagonal is given by the formula,

$$\triangle^{\mathrm{SU}}([n]) = \bigcup_{(\sigma,\tau)} \bigcup_{\mathbf{M},\mathbf{N}} R_{\mathbf{M}}(\sigma) \times L_{\mathbf{N}}(\tau)$$

where the unions are taken over all strong complementary partitions (σ, τ) of $[n]$, and over all admissible sequences of shifts $\mathbf{M}, \mathbf{N}$.

Definition (Laplante-Anfossi, 2022)

The LA diagonal is given by $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, which satisfy

$$\sum_{i \in I} v_i > \sum_{i \in J} v_j \ , \ \forall (I, J) \in \mathrm{LA}(n)$$

The Diagonals

Let $O(n) := \{(I, J) \mid I, J \subset [n], |I| = |J|, I \cap J = \emptyset\}$

Definition

We define $\mathrm{LA}(n)$ and $\mathrm{SU}(n)$ as subsets of $O(n)$,

- $\mathrm{LA}(n) := \{(I, J) \in O(n) \mid \ \min(I \cup J) = \min I\}$, and by
- $\mathrm{SU}(n) := \{(I, J) \in O(n) \mid \ \max(I \cup J) = \max J\}$.

Example

Underlined in LA, and overlined in SU,

$$O(2) = \{\underline{\overline{(1,2)}}, (2,1)\}$$

$$O(3) \ni \underline{\overline{(1,3)}}, \underline{\overline{(2,3)}}, (2,1), (3,2)$$

$$O(4) \ni \underline{\overline{(1,2)}}, (3,2), \underline{(14,23)}, \overline{(23,14)}, \underline{\overline{(13,24)}}$$

Geometric Formulae

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Definition

The 'SU Geometric diagonal' is given by $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, satisfying

$$\sum_{i \in I} v_i > \sum_{i \in J} v_j \ , \ \forall (I, J) \in \mathrm{SU}(n)$$

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Theorem (BDO,MJV,GLA,VP,KS)

This geometric definition of $\triangle^{\mathrm{SU}}$ recovers the original definition of $\triangle^{\mathrm{SU}}$.

A Geometric Formula

Definition (Laplante-Anfossi, 2022)

The LA diagonal is given by $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, satisfying

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Theorem (Laplante-Anfossi, 2022)

For a pair (σ, τ) of ordered partitions of $[n]$, we have

$$\begin{aligned}
(\sigma, \tau) \in \triangle^{\mathrm{LA}} \quad &\iff \quad \forall (I, J) \in \mathrm{LA}(\sigma, \tau), \exists k \in [n], \left|\sigma_{[k]} \cap I\right| > \left|\sigma_{[k]} \cap J\right| \ \textit{or} \\
&\qquad\quad \exists l \in [n], \left|\tau_{[l]} \cap I\right| < \left|\tau_{[l]} \cap J\right| \\
&\iff \quad \forall (I, J) \in \mathrm{LA}(n), \exists k \in [n], \left|\sigma_{[k]} \cap I\right| > \left|\sigma_{[k]} \cap J\right| \ \textit{or} \\
&\qquad\quad \exists l \in [n], \left|\tau_{[l]} \cap I\right| < \left|\tau_{[l]} \cap J\right| \ .
\end{aligned}$$

A Combinatorial Interpretation

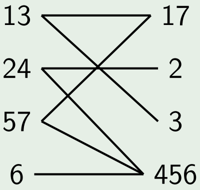
Definition

A n-partition tree is a pair (σ, τ) of set partitions of $[n]$ whose intersection graph is a bipartite tree.

Example

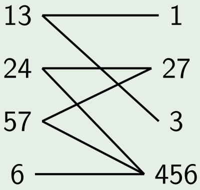
An example and counter example,

$13|24|57|6 \times 17|2|3|456$



✓

$13|24|57|6 \times 1|27|3|456$



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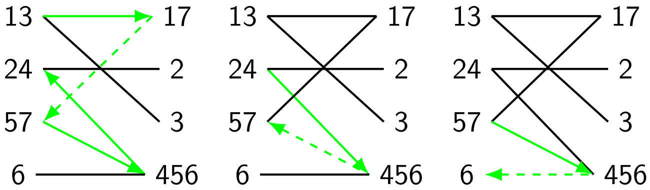
Proposition (BDO,MJV,GLA,VP,KS)

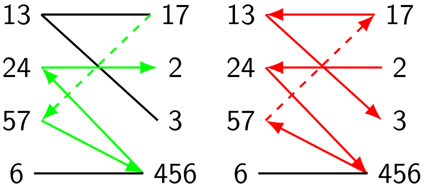
Let (σ, τ) be a pair of ordered partitions of $[n]$ forming an n-partition tree. If for all pairs of adjacent blocks, the directed path between them traverses

1. *the* **maximal** *path element* **right to left**, *then* $(\sigma, \tau) \in \triangle^{\mathrm{SU}}$.

σ is Good:



τ is Bad:

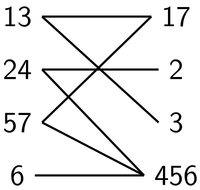


Re-orienting

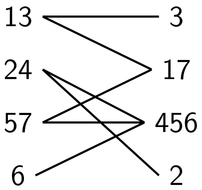
Proposition (BDO,MJV,GLA,VP,KS)

Every n-partition tree can be uniquely oriented into an element of $\triangle^{\mathrm{SU}}$.

$$13|24|57|6 \times 17|2|3|456 \qquad 13|24|57|6 \times 3|17|456|2$$



$\mapsto$



$\in \triangle^{\mathrm{SU}}$

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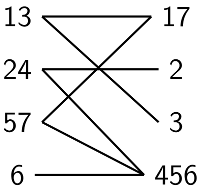

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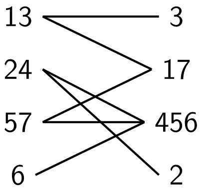
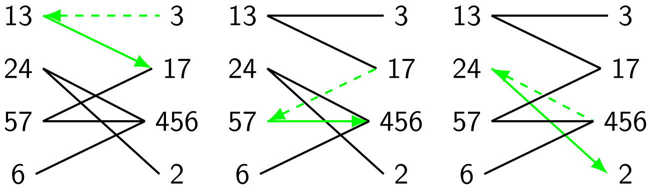


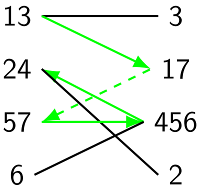
$\in \triangle^{\mathrm{SU}}$



$$(\sigma,\tau) \in \triangle^{\mathrm{SU}} \iff \forall (I,J) \in \mathrm{SU}(\sigma,\tau), \quad \begin{array}{l} \exists k \in [n], \left|\sigma_{[k]} \cap I\right| > \left|\sigma_{[k]} \cap J\right| \text{ or} \\ \exists l \in [n], \left|\tau_{[l]} \cap I\right| < \left|\tau_{[l]} \cap J\right| \end{array}$$

$\mathrm{SU}(\sigma,\tau) = \{(I,J)$ encoded in paths between adj. blocks $\}$

Existential Statement $\cong$ Maximal path element traversed right to left



$(I,J) = (\{1,5\},\{4,7\})$

The Diagonal Via Shifts

Definition (Saneblidze–Umble, 2004)

The SU diagonal is given by the formula,

$$\triangle^{\mathrm{SU}}([n]) = \bigcup_{(\sigma,\tau)} \bigcup_{\mathbf{M},\mathbf{N}} R_{\mathbf{M}}(\sigma) \times L_{\mathbf{N}}(\tau)$$

where the unions are taken over all strong complementary partitions (σ, τ) of $[n]$, and over all admissible sequences of shifts $\mathbf{M}, \mathbf{N}$.

Strong Complementary Partitions

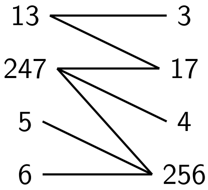
Definition

Given a permutation v, we define its strong complementary pair (σ, τ) by,

- σ is obtained by merging all decreasing sequences of v
- τ is obtained by merging all increasing sequences of v

$$13|247|5|6 \times 3|17|4|256$$

$3|1|7|4|2|5|6 \cong$



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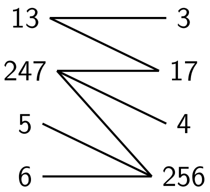


Proposition

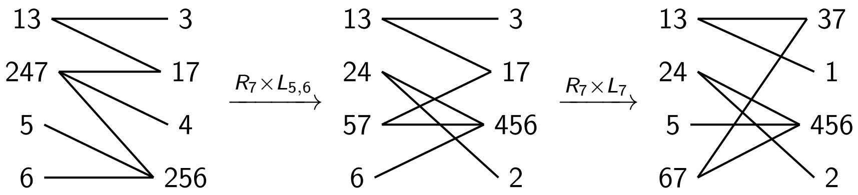
The maximal path elements of SCPs are always traversed right to left.

Shifts

Definition

Let $\sigma = \sigma_1|\ldots|\sigma_k$ be an ordered partition, and let $M_i \subsetneq \sigma_i$ be a non-empty subset of the block σ_i. We define the right/left shift operators

$$R_{M_i}(\sigma) := \sigma_1|\ldots|\sigma_i \smallsetminus M_i|\sigma_{i+1} \cup M_i|\ldots|\sigma_k$$
$$L_{M_i}(\sigma) := \sigma_1|\ldots|\sigma_{i-1} \cup M_i|\sigma_i \smallsetminus M_i|\ldots|\sigma_k\ .$$



Admissible Shifts

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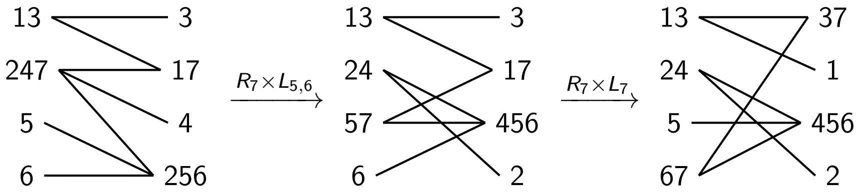
Let $\sigma = \sigma_1 | \ldots | \sigma_k$ be an ordered partition

- A right shift is admissible if $\min \sigma_i \notin M_i$, and $\min M_i > \max \sigma_{i+1}$.

Dually,

- A left shift is admissible if $\min \sigma_i \notin M_i$, and $\min M_i > \max \sigma_{i-1}$.



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- A sequence of right shifts $\mathbf{M} = (M_{i_1}, \dots, M_{i_p})$, is admissible if $i_1 < \dots < i_p < k$, and each sequential shift is admissible.

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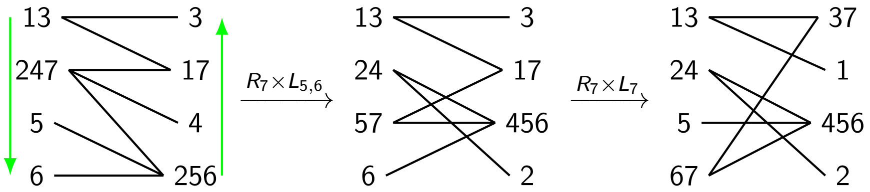

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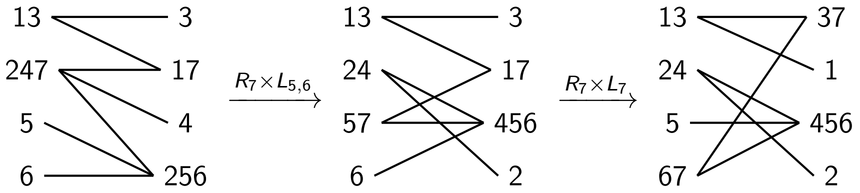


Shift $\triangle^{\mathrm{SU}} \subseteq$ Geometric $\triangle^{\mathrm{SU}}$.

We previously saw that,

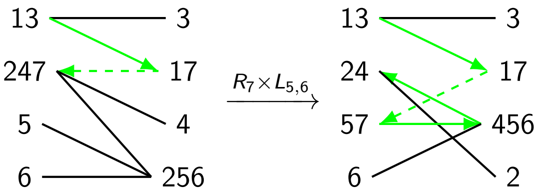
Proposition (BDO,MJV,GLA,VP,KS)

Let (σ, τ) be a pair of ordered partitions of $[n]$ forming an n-partition tree. If for all pairs of adjacent blocks, the directed path between them traverses

1. *the* **maximal** *path element* **right to left**, *then $(\sigma, \tau) \in$ Geo. $\triangle^{\mathrm{SU}}$.*

Show all elements of shift $\triangle^{\mathrm{SU}}$ also satisfy the path condition.

1. We know strong complementary partitions meet the path condition,
2. We show admissible sequences of shifts conserve the path condition,



Consequently, Shift $\triangle^{\mathrm{SU}} \subseteq$ Geometric $\triangle^{\mathrm{SU}}$.

Geometric $\triangle^{\mathrm{SU}} \subseteq$ Shift $\triangle^{\mathrm{SU}}$.

Conversely we need,

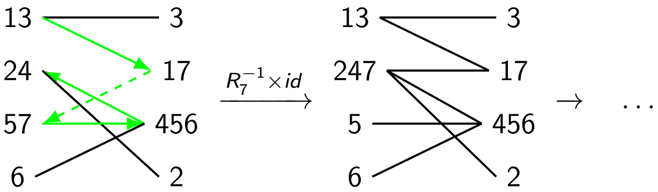
Lemma

Let (σ, τ) be a pair of ordered partitions of $[n]$ forming an n-partition tree. If for all pairs of adjacent blocks, the directed path between them traverses

1. *the* **maximal** *path element* **right to left**,

then it is either a strong complementary pair, or generated by shifts.

Idea: For anything that is not a strong complementary partition we can identify an inverse shift operator, e.g.



A Zoo of Formulae

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Consequently, have many different encodings of the LA and SU diagonals.

- Geometric formulae
- Min/max path formulae
- Shift formulae
- Cubical formulae
- Matrix formulae

Theorem (BDO,MJV,GLA,VP,KS)

There are exactly two operadic diagonals of the permutahedra, which respect the weak Bruhat order on the vertices:

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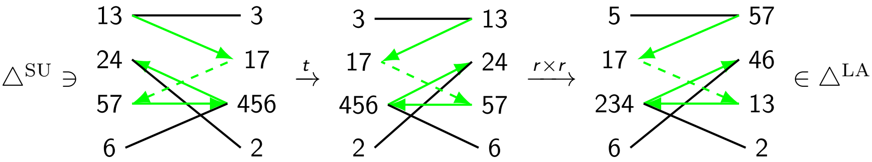

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1. *the* **maximal** *path element* **right to left**, *then* $(\sigma, \tau) \in \triangle^{\mathrm{SU}}$.
2. *the* **minimal** *path element* **left to right**, *then* $(\sigma, \tau) \in \triangle^{\mathrm{LA}}$.