Compatibility with Disk like Transfer Systems II

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Slides Part I

Slides Part II

Transfer Systems

Definition

Let $\mathcal{O}$ be a binary relation on $Sub(G)$ refining $\subset$. Then, $\mathcal{O}$ is said to be a G-transfer system if it is closed under

- conjugation,
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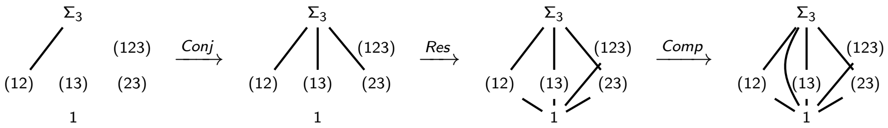
Theorem (A.2 of [Rub21])

Let R be a binary relation on $Sub(G)$ refining $\subset$. Let $T(R)$ denote the closure of R under

- *conjugation, then*
- *restriction, and then*
- *composition.*

Then $T(R)$ is the smallest G-transfer system containing R.

Example



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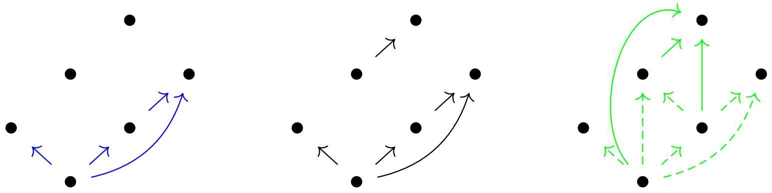
Disk like Transfer Systems

Definition

We say a transfer system $\mathcal{O}$ is **disk like** if it admits the presentation $\mathcal{O} = T(D_G)$, where D_G is a set of relations/transfers with target G.

- Left, a non-disk like $C_{p^2,q}$-transfer system.
- Mid, a non-disk like $C_{p^2,q}$-transfer system.
- Right, a disk like $C_{p^2,q}$-transfer system, its generators in solid green.



Definition

A transfer system $\mathcal{O}$ is **saturated** if it satisfies the 2 out of 3 property.

$$
\begin{tikzcd}
C & \\
& B \arrow[ul, dashed] \\
A \arrow[uu] \arrow[ur] \arrow[u, phantom, "\Rightarrow"] &
\end{tikzcd}
$$

- Left, a saturated $C_{p^2,q}$-transfer system.
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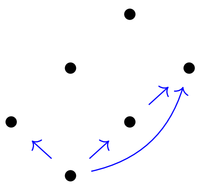
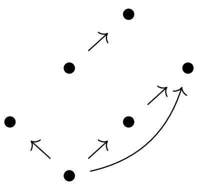
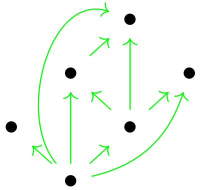




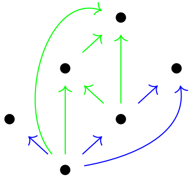
Compatible Transfer Systems

Definition ([Cha24, Definition 4.6])

Let $\mathcal{O}_a$ and $\mathcal{O}_m$ be a pair of G-transfer systems such that $\mathcal{O}_m \subseteq \mathcal{O}_a$. We say $(\mathcal{O}_a, \mathcal{O}_m)$ are **compatible** if we can complete all squares of the form

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, "e"] & & B \arrow[ul, dashed] \\
 & A \cap B \arrow[ul, "a"] \arrow[ur, "r"'] \arrow[uu, phantom, "\Rightarrow"] &
\end{tikzcd}$$

with $e, r \in \mathcal{O}_m$ and $a \in \mathcal{O}_a$.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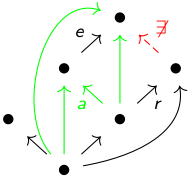

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Maximal Compatible Transfer Systems

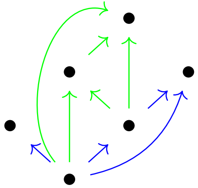
Proposition ([BH22])

If $(\mathcal{O}_a, \mathcal{O}_m)$ and $(\mathcal{O}_a, \mathcal{O}'_m)$ are both compatible, then $(\mathcal{O}_a, \mathcal{O}_m \vee \mathcal{O}'_m)$ is compatible.

Corollary

For a fixed transfer system $\mathcal{O}_a$,

- *there exists a maximal compatible transfer system $\mathcal{O}_m$, and*
- *all other compatible transfers systems are sub-transfer systems of $\mathcal{O}_m$.*



Why?

Work including [BH15, GW18, BBR21, BP21, Rub21, BMO24, Cha24], provides the correspondences

N_∞-operads	Transfer Systems
Additive Transfers	Disk Like Transfer Systems
Multiplicative Norms	Saturated Transfer Systems
Bi-incomplete Transfers and Norms	Compatible Transfer Systems

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For a fixed transfer system $\mathcal{O}_a$,

- *there exists a maximal compatible transfer system $\mathcal{O}_m$, and*
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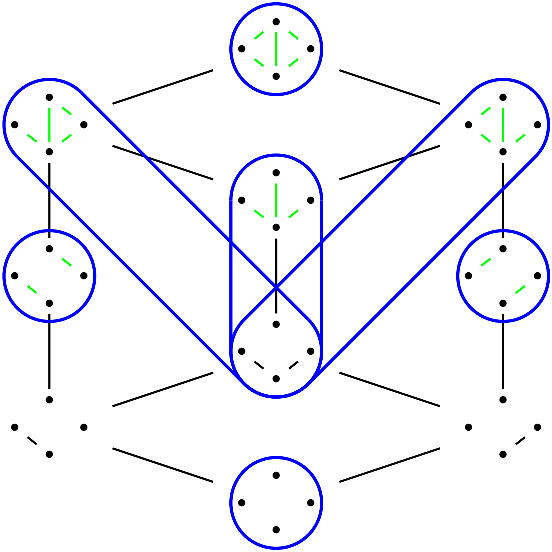
Thus identifying the maximal compatible transfer system identifies all bi-incomplete/compatible multiplicative norms for a fixed additive transfer.

Maximal Compatible Pairs of Disk like Transfers of $C_{p,q}$



How?

In Part 1 with David, we saw that

Proposition (DHKNSVNY)

The maximal compatible transfer $\mathcal{O}_m$ of $\mathcal{O}_a$ is always saturated.

Is $\mathcal{O}_m$ the 'maximal saturated sub-transfer system' of $\mathcal{O}_a$?

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Is $\mathcal{O}_m$ the 'maximal saturated sub-transfer system' of $\mathcal{O}_a$?

No!

1. There can exist multiple incomparable saturated transfer systems smaller than $\mathcal{O}_a$.
2. Saturated elements can exist in the open interval $(\mathcal{O}_m, \mathcal{O}_a)$.

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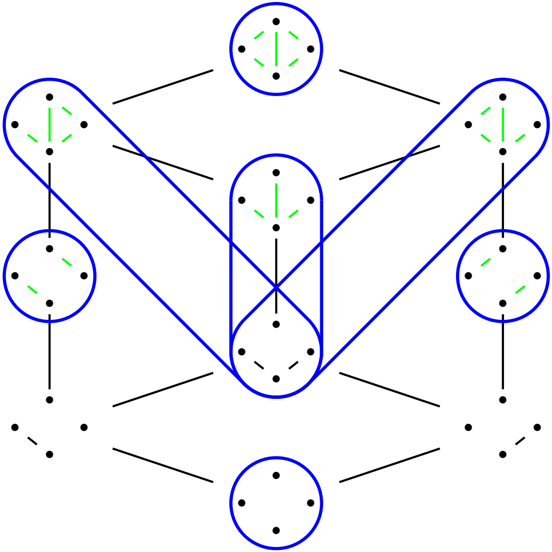

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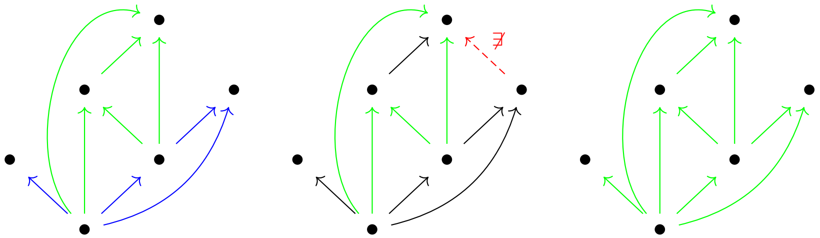

Computing $\mathcal{O}_m$

Worst case need to check

$$\begin{tikzcd} & C & \\ A \arrow[ur, "e"] & & B \arrow[ul, dashed] \\ & A \cap B \arrow[ul, "a"] \arrow[ur, "r"'] \arrow[uu, phantom, "\Nearrow"] & \end{tikzcd}$$

for all $e, r \in \mathcal{O}_m$ and $a \in \mathcal{O}_a$.

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Lemma (DHKNSVNY)

If $\mathcal{O}_a = Comp(Rest(Conj(B_a)))$ and $\mathcal{O}_m = Comp(Rest(Conj(B_m)))$. Then, $(\mathcal{O}_a, \mathcal{O}_m)$ are compatible, if, and only if, $\mathcal{O}_m \subseteq \mathcal{O}_a$ and

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for all $e, f \in Rest(Conj(B_m))$ and $a \in Rest(Conj(B_a))$.

Also: It is possible to not conjugate one of the sets of generators!

Computing $\mathcal{O}_m$ Via Its Complement

Proposition (DHKNSVNY)

The complement of the maximal compatible transfer system of $\mathcal{O}_a$ satisfies

$$\mathcal{O}_m^c := \mathcal{O}_a \setminus \mathcal{O}_m = \{e \in \mathcal{O}_a : \exists r, r' \in Res(e), a \in \mathcal{O}_a, \nexists a' \in \mathcal{O}_a \text{ such that } \begin{tikzcd} & D \arrow[dr, phantom, "\supseteq"] & & \\ A \arrow[ur, "e"] \arrow[dr, phantom, "\supseteq"] & & B & \\ & A \cap B \arrow[ur, "r"] & & C \arrow[ul, dashed, red, "\nexists a'"'] \\ & & A \cap B \cap C \arrow[ul, "a"] \arrow[uu, "c"] \arrow[ur, "r'"'] & \end{tikzcd} \}.$$

Idea:

- Delete the top left factors of all saturation failures in $\mathcal{O}_a$.
- Every occurrence of the pattern above deletes e, r and c.

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Proof sketch:

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Proof sketch:

- $e \in \mathcal{O}_m$ if, and only if, $(\mathcal{O}_a, T(e))$ is compatible.

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Proof sketch:

- $e \in \mathcal{O}_m$ if, and only if, $(\mathcal{O}_a, T(e))$ is compatible.
- Then check the compatibility of $(\mathcal{O}_a, T(e))$ with prior lemma.

Case: $\mathcal{O}_m^c = \emptyset$

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\end{tikzcd} \}.$$

Proposition (DHKNSVNY)

A transfer system $\mathcal{O}_a$ is self compatible, if, and only if, it is saturated.

Proof:

- If $\mathcal{O}_a$ is saturated then $\mathcal{O}_m^c = \emptyset$.
- Every saturation failure of $\mathcal{O}_a$ is in $\mathcal{O}_m^c = \emptyset$, thus $\mathcal{O}_a$ is saturated.

Example: Computing $\mathcal{O}_m$ in C_{p^2,q^2} using the complement

$$\mathcal{O}_m^c := \mathcal{O}_a \setminus \mathcal{O}_m = \left\{ e \in \mathcal{O}_a : \exists r, r' \in Res(e), a \in \mathcal{O}_a,\ \nexists a' \in \mathcal{O}_a \text{ such that }
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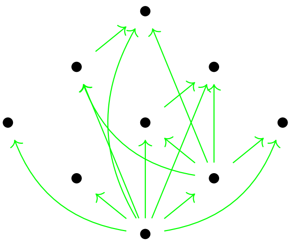
\arrow[from=1-2, to=2-4, phantom, "\supseteq"]
\end{tikzcd}
\right\}.$$



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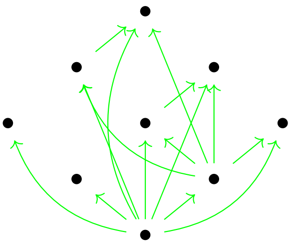

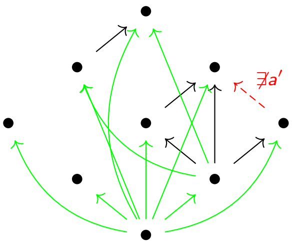

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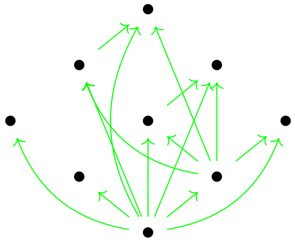

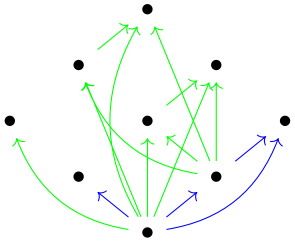

Some Questions and Future Directions

- How can we use the disk like assumption to aid in computing $\mathcal{O}_m$?
- Can we use subset bounds on $\mathcal{O}_m$ for faster computation?
- Can we compute all maximal compatible pairs for all disk like transfer systems of a fixed group G in a relatively efficient manner?
  - i.e. maybe it is 'hard' to compute $\mathcal{O}_m$ for arbitrary $\mathcal{O}_a$,
  - but we can induct to all $(\mathcal{O}_a, \mathcal{O}_m)$ from computing $(T(H \to G), \mathcal{O}_m)$?

[BBR21] Scott Balchin, David Barnes, and Constanze Roitzheim. N_∞-operads and associahedra. *Pacific J. Math.*, 315(2):285–304, 2021.

[BH15] Andrew J. Blumberg and Michael A. Hill. Operadic multiplications in equivariant spectra, norms, and transfers. *Adv. Math.*, 285:658–708, 2015.

[BH22] Andrew J. Blumberg and Michael A. Hill. Bi-incomplete Tambara functors. In *Equivariant topology and derived algebra*, volume 474 of *London Math. Soc. Lecture Note Ser.*, pages 276–313. Cambridge Univ. Press, Cambridge, 2022.

[BMO24] Scott Balchin, Ethan MacBrough, and Kyle Ormsby. Composition closed premodel structures and the Kreweras lattice. *European J. Combin.*, 116:Paper No. 103879, 22, 2024.

[BP21] Peter Bonventre and Luís A. Pereira. Genuine equivariant operads. *Adv. Math.*, 381:Paper No. 107502, 133, 2021.

Mentioned Sources II

[Cha24] David Chan. Bi-incomplete Tambara functors as $\mathcal{O}$-commutative monoids. *Tunis. J. Math.*, 6(1):1–47, 2024.

[GW18] Javier J. Gutiérrez and David White. Encoding equivariant commutativity via operads. *Algebr. Geom. Topol.*, 18(5):2919–2962, 2018.

[Rub21] Jonathan Rubin. Detecting Steiner and linear isometries operads. *Glasg. Math. J.*, 63(2):307–342, 2021.