Gröbner Bases for Operads Cannot be Generalised to Wheeled Structures

A Simple Counter Example

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What we're going to discuss

1. ~~What~~ Why is a Gröbner basis for operads?
2. What do we need to construct a GB?
3. Generalisations
4. A counter example for wheeled structures

~~What~~ Why is a Gröbner basis for operads?

- Poincaré-Birkhoff-Witt (PBW) Operad [Hoffbeck 2007]
- Gröbner Basis (GB) for Operads [Dotsenko, Khoroshkin 2008]

(Quadratic GB $\iff$ PBW) $\implies$ Koszul

Algorithm for Constructing GB $\implies$ Algorithm for Proving Koszul

Shuffle Operads

Problem: Symmetric structures are yuck.

Solution: Work with a related non-symmetric structure $\to$ **shuffle** operads.

Store some information in the modules in new composition maps.

Definition (Shuffle Composition)

Let $\alpha \in \mathcal{O}(n)$ and $\beta \in \mathcal{O}(m)$ be elements of a symmetric (or non-symmetric) collection and $i \in \{1, \dots, n\}$ then $\alpha \circ_{i,\sigma} \beta$ is the operation

$$\alpha\big(x_1, \dots, x_{i-1}, \beta(x_i, x_{\sigma(i+1)}, \dots, x_{\sigma(i+m-1)}\big), x_{\sigma(i+m)}, \dots, x_{\sigma(n+m-1)}\big)$$

where σ is a shuffle permutation.

Shuffle Operads

Corollary (5.3.3.3. Bremner and Dotsenko [2016])

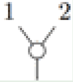
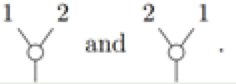
If G is a symmetric collection, and f is the forgetful functor from symmetric collections to non-symmetric collections then there is an isomorphism of non-symmetric shuffle operads.

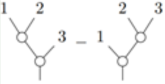
$$f(\mathcal{O}_\Sigma(G)) \simeq \mathcal{O}_{sh}(f(G))$$

Theorem (6.3.3.2 Bremner and Dotsenko [2016])

If $\mathcal{O}_\Sigma$ is a symmetric operad and the shuffle operad $f(\mathcal{O}_\Sigma)$ admits a quadratic GB then $\mathcal{O}_\Sigma$ is Koszul.

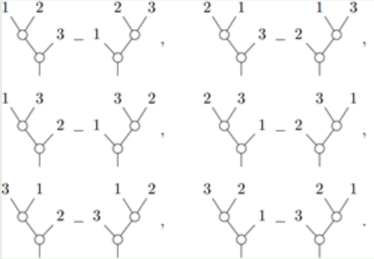
Example (A symmetric associative algebra $Ass = \mathcal{O}_\Sigma(U)/I$)

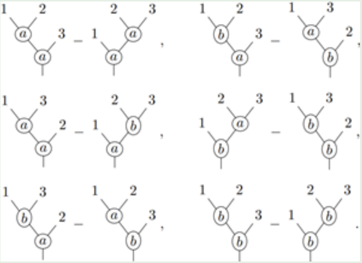
$U :=$  , $\Sigma_2 U =$ 

, $I :=$ 

Example (As a shuffle operad $f(Ass) \simeq \mathcal{O}_{sh}(U')/I'$)

$U' :=$  ,

$\Sigma_3 I =$ 

$I' :=$ 

What do we need to construct a GB?

Constructed by a generalisation of Buchberger's algorithm which needs

1. Divisibility (easy)
2. A total admissible order

Lemma (Dotsenko and Khoroshkin [2010])

There exists an algorithm for constructing the GB of an ideal of a freely generated non-symmetric shuffle operad.

Total Admissible Orders

Definition (Total Order)

A total order on a set is a binary relation which is

- Reflexive
- Transitive
- Anti-symmetric
- Total

Definition (Admissible Order)

A order $\leq$ is admissible for a non-symmetric shuffle operad $\mathcal{O}$ if $\forall \alpha, \alpha' \in \mathcal{O}(n)$ and $\beta, \beta' \in \mathcal{O}(m)$

- $n < m \implies \alpha < \beta$
- $\alpha \leq \alpha' \wedge \beta \leq \beta' \implies \alpha \circ_{i,\sigma} \beta \leq \alpha' \circ_{i,\sigma} \beta'$

Generalisations

Known

- GB for Operads: Dotsenko and Khoroshkin [2010]
- GB for Coloured Operads: Kharitonov and Khoroshkin [2021]
- GB for Dioperads: Khoroshkin [Pending]

Unknown

- GB for Properads
- GB for Props
- GB for Wheeled Operads/Properads/Props

A counter example for wheeled structures

Lemma

There does not exist a total admissible order for most wheeled structures (operads/properads/props).

Claim

If $\leq$ is an admissible order for the wheeled shuffle operad $\mathcal{O}$ and $\alpha, \beta, \alpha', \beta' \in \mathcal{O}\binom{1}{1}$ and $\alpha \leq \alpha'$, $\beta \leq \beta'$ then

$$\alpha \circ_1 \beta \leq \alpha' \circ_1 \beta'$$
$$\epsilon_{1,1}\alpha \leq \epsilon_{1,1}\alpha'$$

Notation

$$\alpha\beta := \alpha \circ_1 \beta, \qquad (\alpha)_c := \epsilon_{1,1}\alpha$$

A counter example for wheeled structures

Proof.

Let $\mathcal{O}$ be a non-symmetric wheeled operad with two distinct non-identity elements in $\mathcal{O}\binom{1}{1}$ say f and g. If $\leq$ is a total order than either

$$fg < gf \quad \vee \quad fg > gf$$

W.L.O.G we suppose that $fg < gf$. If $\leq$ is admissible then,

$$\begin{gathered} (fg)f < (gf)f \\ g(fg)f < g(gf)f \\ (g(fg)f)_c < (g(gf)f)_c \\ (fgfg)_c < (ffgg)_c \end{gathered} \qquad \textit{and} \qquad \begin{gathered} (fg)(gf) < (gf)(gf) \\ ((fg)(gf))_c < ((gf)(gf))_c \\ (ffgg)_c < (fgfg)_c \end{gathered}$$

A contradiction. $\square$

What I'm currently working on

- GB for properads/props?
- Have an appropriate notion of a shuffle properad/prop which has
    - divisibility
    - total admissible orders
- So have gröbner basis for properads/props
- Still need quadratic GB for shuffle properads/prop $\implies$ Koszul

Mentioned Sources

M. R. Bremner and V. Dotsenko. *Algebraic operads: an algorithmic companion*. CRC Press, 2016.

V. Dotsenko and A. Khoroshkin. Gröbner bases for operads. *Duke Mathematical Journal*, 153(2):363–396, 2010.

V. Kharitonov and A. Khoroshkin. Gröbner bases for coloured operads. *Annali di Matematica Pura ed Applicata (1923-)*, pages 1–39, 2021.