Suppose you woke up tomorrow and found

$$(x+y)+z = x+(y+z) \quad \text{had transformed into} \quad (x+y)+z \cong x+(y+z).$$

Just one of many sensible consequences is

$$w+((x+y)+z) \cong w+(x+(y+z)). \tag{†}$$

Computing these necessary equivalences, then equivalences between those equivalences, and so on, quickly becomes tiresome. This is where operadic structures such as operads and props step in. They readily serve as bookkeeping devices for higher homotopies such as †, and have found use in areas such as algebraic geometry, deformation theory, graph complexes, mathematical physics, and topology.

My research interests lie in the study of (∞ or homotopy) operadic structures and their algebras. I have a taste for the discrete, and am drawn to techniques, models and algebras of a more combinatorial flavour, such as those using graphs or polytopes. I'm particularly interested in general machinery for producing and studying ∞-operadic structures, such as Gröbner bases, Koszul duality and homotopy coherent nerves. Individually powerful, they collectively provide a factory for, the production of homotopy operadic structures, and their translation between the algebraic and topological worlds.

1. Recent Results

1.1. **Koszul Operads Governing Props, and Wheeled Props [Sto23].** Operads model the composition of functions with many inputs and a single output via trees. Specific operads such as the associative operad As (row one of Table 1), admit a presentation in terms of generators and relations. Algebras over As are associative algebras, as the sole binary generator $\wedge$ is concretely realised as a binary operation $\mu : V \otimes V \to V$, and the relation forces μ to be associative,

$$\mu(\mu(x,y),z) = \mu(x,\mu(y,z)).$$

More complicated operadic structures, such as props (and properads), model the composition of functions with many inputs and many outputs through directed (and respectively connected) graphs. As illustrated in Table 1, there exist properads whose algebras are bialgebras (and also Lie-bialgebras), and there exist props whose algebras are Hopf algebras. Note that every operad is a properad, and every properad is a prop, but the converse direction need not hold. For instance, disconnected graphs are needed to model the antipodal relations of the Hopf algebra in row three of Table 1.

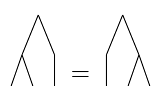
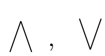
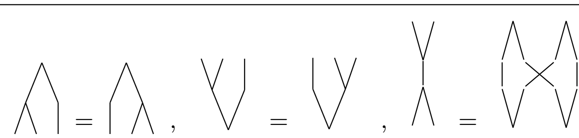
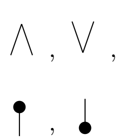
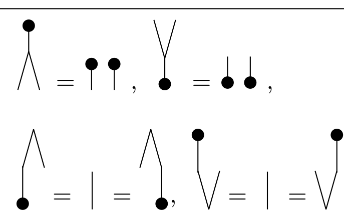
Family	An Example Element of Family		Algebras Over Example
	Generators	Relations	
Operads			Associative Algebras
Properads			Bialgebras
Props		All the above, 	Hopf Algebras

TABLE 1. Examples of Operadic Structures and their Algebras.

We say that a specific operad governs props (or other operadic structures), if an algebra over the operad is a prop. These operads are surprisingly useful, as if you can prove the operad admits the property of being Koszul, then you may infer many homotopical results about the operadic structure it governs (see for instance Corollary 4). This technique has been applied to many existing operadic structures, and the most general existing result, established by two groups of authors, was the following.

Theorem 0 ([BM21], [KW21])**.** *The operads governing connected operadic structures are Koszul.*

However, perhaps the most fundamental operadic structures were not covered by their result, props and wheeled props. This project addresses this gap in literature with three main findings.

Theorem 1 ([Sto23])**.** *The operads governing props and wheeled props are Koszul.*

Secondly, we reveal that the prior polytope based techniques of [BM21] and [KW21] could not have been used to obtain this result, via constructing explicit obstructions.

Theorem 2 ([Sto23])**.** *There exist sub-complexes of the minimal models of the operads governing props and wheeled props which are **not** isomorphic, as lattices, to the face poset of convex polytopes.*

Finally, as their methods were not sufficient to prove the desired result, we were forced to develop a more general toolkit. We extended the technique of Groebner bases for operads ([DK10], [KK21]) to groupoid coloured operads, with the following.

Theorem 3 ([Sto23])**.** *Let P be a groupoid coloured operad such that the associated coloured shuffle operad $(P^f)_*$ admits a quadratic Groebner basis, then P is Koszul.*

This powerful tool lets us prove that an operad has the complicated homological property of being Koszul, by proving there exists a conceptually simpler, confluent terminating rewrite system (see Fig. 1). Thus, we can apply this technique to not only prove Theorem 1, but also to recover Theorem 0.

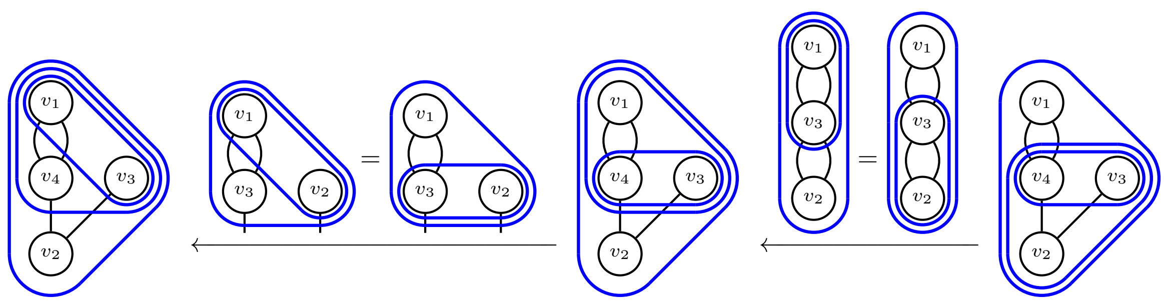


FIGURE 1. Proving that the operad governing props $\mathbb{P}$ is Koszul using Theorem 3 amounts to showing that every labelled directed acyclic graph has a unique minimal nesting, and every other nesting can be rewritten into it using quadratic relations. Here is one such minimal nesting, and two successive rewrites to it, where the relation corresponding to each rewrite is displayed above the arrow.

Theorem 1 and the Koszul machine provides several immediate consequences. We obtain explicit minimal models for operads governing props and wheeled props. Algebras over these minimal models are ∞-props, and ∞-wheeled props. Furthermore, we can apply the technique of homotopy transfer theory to any homotopy retract of a (wheeled) prop. In particular, as the homology of a (wheeled)-prop is an example of a homotopy retract, we may transfer to it the structure of an ∞-(wheeled) prop. These higher operations present in the homology are known as Massey products, and it is straightforward to show, in characteristic 0, that a (wheeled) prop is formal if, and only if, it has no Massey products.

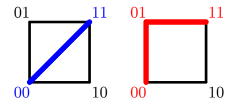
This non-trivial characterisation of formality casts new light on old results. In [ML65], Mac Lane provided an early calculation demonstrating that the prop governing (co)commutative Hopf algebras (the commutative and cocommutative variant of Table 1) has Massey products. Thus, we can now draw the following conclusion from Theorem 1.

Corollary 4 ([Sto23])**.** *The prop governing (co)commutative Hopf algebras is non-formal.*

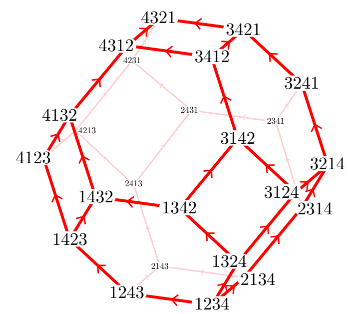
1.2. **Cellular Diagonals of the Permutahedra [DOLAPS23].**

A cellular diagonal of a polytope P is a cellular approximation of the thin diagonal,

$$\triangle : P \to P \times P, \quad x \mapsto (x, x).$$



The thin and a cellular diagonal of the interval $[0, 1]$



A projection of $\mathsf{Perm}(4)$ in $\mathbb{R}^3$.

The permutahedra $\mathsf{Perm}(n)$ is the polytope defined as the convex hull of all permutations in the nth symmetric group, considered as coordinates in $\mathbb{R}^n$. Cellular diagonals, henceforth referred to as diagonals, and coherent families of them called operadic diagonals, are of great interest in algebraic geometry and topology. For instance, they define universal tensor products of homotopy operads [LA22]. Furthermore, the combinatorics of the diagonals of the permutahedra are subtle, as they do not satisfy the magical formula of [MTTV21]. This joint project with Bérénice Delcroix-Oger, Guillaume Laplante-Anfossi and Vincent Pilaud sought to answer two related questions.

(1) Can we provide a complete enumeration of the faces of cellular diagonals of the permutahedra?
(2) What is the relation between the only known formulae for operadic diagonals of the permutahedra, the LA diagonal of [LA22] and the SU diagonal of [SU04]?

To solve the first problem, we realised that as the normal fan of the permutahedra is the braid fan, the diagonal of the permutahedra is dual to two generically translated copies of the braid arrangement. Thus, we enumerated the faces of ℓ generically translated copies of the braid arrangement, and then specialised to the case $\ell = 2$.

Theorem 5 ([DOLAPS23])**.** *The Möbius polynomial of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ is given by*

$$\boldsymbol{\mu}_{\mathcal{B}_n^\ell}(x, y) = x^{n-1-\ell n} y^{n-1-\ell n} \sum_{\boldsymbol{F} \le \boldsymbol{G}} \prod_{i \in [\ell]} x^{\#F_i} y^{\#G_i} \prod_{p \in G_i} (-1)^{\#F_i[p]-1} (\#F_i[p] - 1)!\,,$$

where $\boldsymbol{F} \le \boldsymbol{G}$ ranges over all intervals of the (ℓ, n)-partition forest poset, and $F_i[p]$ denotes the restriction of the partition F_i to the part p of G_i.

This formula admitted many elegant specialisations, providing formulae for the number of regions and bounded regions in terms of the Fuss-Catalan numbers. Furthermore, these simpler cases often had alternate combinatorial interpretations. For instance, we found that the faces of the diagonal of $\mathsf{Perm}(n)$, were in bijection with bipartite forests. In particular, there are the $2(n+1)^{n-2}$ outer faces, and they are in bijection with bipartite trees (see the 8 outer faces of Fig. 4).

This bijection to bipartite trees was the key combinatorial insight needed to relate the LA and SU diagonals. We found, in a non-trivial proof, that a new geometric definition recovered that of [SU04]. Moreover, the new geometric characterisation revealed that these are the only two operadic diagonals.

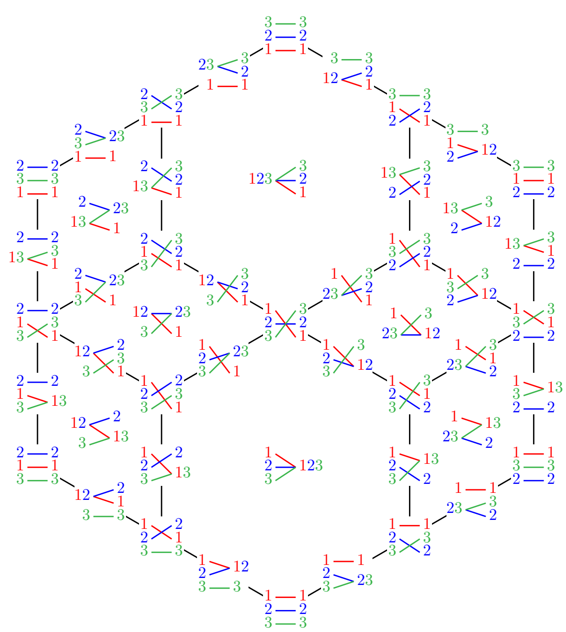


FIGURE 4. The faces of a diagonal of $\mathsf{Perm}(3)$ and their corresponding ordered bipartite forests.

Theorem 6 ([DOLAPS23])**.** *There are exactly four operadic geometric diagonals of the permutahedra, the geometric* LA *and* SU *diagonals and their opposites, and only the first two respect the weak order on permutations. Moreover, their cellular images are isomorphic as posets.*

This surprising result induces consequences in homotopical algebra, such as the following.

Theorem 7 ([DOLAPS23])**.** *There are exactly two geometric universal tensor products of homotopy operads, and they are related by an ∞-isotopy.*

2. Current Projects

2.1. **Homotopy G-Operadic Structures.** From Theorems 0 and 1, we now know that the operads governing all (mainstream) symmetric-operadic structures are Koszul. This brings into sights a useful extension, G-operadic structures. Informally, G-operadic structures model the composition of functions whose inputs and outputs are connected by some G-action. For instance, if G is the symmetric group, we obtain the symmetric operadic structures discussed in Section 1.1. Alternatively, if G is something more exotic like the braid group, then we can visualise the inputs and outputs of our braided-operadic structures as being connected by braided wires. Structures such as braided-props, otherwise known as probs, have drawn recent interest in the work of Kapranov on perverse sheaves [KS21].

A key technical insight used to prove that the operads governing symmetric-operadic structures were Koszul, is that one can obtain a quadratic presentation of these operads by hiding certain (equivariance) axioms in a groupoid colouring. This insight also applies to G-operadic structures. Thus, we may construct a quadratic groupoid coloured operad P governing each G-operadic structure. To prove P is Koszul, we would like to apply Theorem 3. It is straightforward to show that the associated coloured shuffle operad $(P^f)_*$ admits a quadratic Groebner basis, however this only currently implies P is Koszul when its groupoid has finite automorphisms. Thus, two possible paths are being considered.

- We directly extend the theory to groupoids with infinite automorphisms.
- We switch to using non-homogenous coloured operads (quadratic and unary relations). With existing theory, it is straightforward to show these more complicated presentations are Koszul.

2.2. **The Homotopy Coherent Nerves of (Wheeled) Props.** This joint project with Philip Hackney and Marcy Robertson aims to build a bridge between algebraic and topological notions of ∞-operadic structures. An ∞-operad, is an operad whose operations are only defined up to some notion of coherent homotopy. There are many models of these structures in literature such as: the algebraic model given by the Koszul machine *shOperad*; the dendroidal/graphical model *dSet*; the simplicial model *sOperad*; and the topological model *dSpace*. Furthermore, the relations between these models of ∞-operads are well understood through various homotopy coherent nerves.

This project aims to subsume these various models, and nerves at the level of ∞-(wheeled) props. In illustration, the topmost nerves of Figure 5 subsume all the lower nerves via a sieve (or informally restriction), and a similar statement holds for the wheeled variants.

$$
\begin{tikzcd}
shProp \arrow[r, leftrightarrow, dotted] & Set^{\Gamma^{op}} \arrow[r, leftrightarrow, dotted] & sProp \\
shProperad \arrow[u, hook] \arrow[r, leftrightarrow, dotted] & Set^{\Gamma_c^{op}} \arrow[u, hook] \arrow[r, leftrightarrow, dotted] & sProperad \arrow[u, hook] \\
shOperad \arrow[u, hook] \arrow[r, leftrightarrow] & dSet \arrow[u, hook] \arrow[r, leftrightarrow] & sOperad \arrow[u, hook] \\
\mathcal{A}_\infty\text{-}Cat \arrow[u, hook] \arrow[r, leftrightarrow] & sSet \arrow[u, hook] \arrow[r, leftrightarrow] & sCat \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
Set^{\Gamma^{op}} \arrow[r, leftrightarrow, dotted] & \Gamma\text{-}Space \\
Set^{\Gamma_c^{op}} \arrow[u, hook] \arrow[r, leftrightarrow, dotted] & \Gamma_c\text{-}Space \arrow[u, hook] \\
dSet \arrow[u, hook] \arrow[r, leftrightarrow] & dSpace \arrow[u, hook] \\
sSet \arrow[u, hook] \arrow[r, leftrightarrow] & sSpace \arrow[u, hook]
\end{tikzcd}
$$

Figure 5. Various models of ∞-operadic structures, with inclusions and nerves between them. The known nerves are indicated by solids lines, and the new nerves by dotted.

At the level of operads and categories, the existing nerves are known to be Quillen equivalences. This is no longer generally true for the more complicated graphical categories of (wheeled)-prop(erads). Thus, the central goal of the project is,

Characterise when these new nerves are Quillen equivalences via an obstruction theory.

We already have one characterisation through the Σ-cofibrancy of the operads governing these different structures. Thus, applying and reinterpreting this graphically will yield the desired obstruction theory.

3. Future Work

3.1. **Koszul Duality for Props.** As of today, there is still no known Koszul duality theory for either props, or wheeled props. However, given we now have Koszul groupoid coloured operads $\mathbb{P}$ and $\mathbb{W}$ governing props, and wheeled props respectively, such a theory is within reach. In [Mil12], a Koszul duality theory for algebras over a Koszul (non-groupoid coloured) operad was introduced. To clarify their theory by example, the operad As whose algebras are associative algebras is known to be Koszul, and their construction recovers the classical notion of Koszul duality for associative algebras.

Similarly to Section 2.1, there are two possible paths for extending this theory to Props. One can either seek to extend existing theory to the groupoid coloured case, or one can work entirely within the known, using more complicated non-homogeneous quadratic coloured presentations. The second path is certainly viable, and is currently being employed by [DV21], in the pursuit of a Koszul duality theory for symmetric operads, where the symmetry is also relaxed up to coherent homotopy.

3.2. **Tensor Products for Homotopy Properads.** Although homotopy props do not admit a minimal model governed by polytopes, properads do. In particular, the appropriate minimal model for the operad governing properads is given by the poset associahedra of [Gal21]. This poset has been recently realised as a convex polytope in [MPP23] and [Sac23]. Why is this interesting? Well, in [LA22], explicit functorial tensor products of homotopy operads were constructed from the following two results,

(1) The appropriate minimal model for homotopy operads is given by nested planar trees.
(2) This nesting model admits an explicit realisation as a convex polytope.

Thus, we have the necessary ingredients to construct functorial tensor products of homotopy properads. Indeed, the existence of this tensor product would follow from the following technical steps.

- Take a definition of the minimal model for the operad governing properads P_∞. This can be produced by restricting the known minimal model for the operad governing props [Sto23].
- Apply the cellular chains functor $C_\bullet^\infty(P_\infty)$ to produce a (non-strictly coassociative) Hopf Operad, whose coproduct corresponds to the diagonal.
- Mirror Theorem 4.23 of [LA22]) to show that $C_\bullet^\infty(P_\infty) \cong P_\infty$ as dg-symmetric operads. This is comparable to showing the associahedron encodes the signed differential of the A_∞ operad.

The diagonal of $C_\bullet^\infty(P_\infty)$ encodes a tensor product of homotopy properads. Thus, a chosen tensor product can then be constructed through a choice of orientation vector, and the toolkit of [LA22].

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