A Koszul Operad Governing Wheeled Props

(and one for Props as well)

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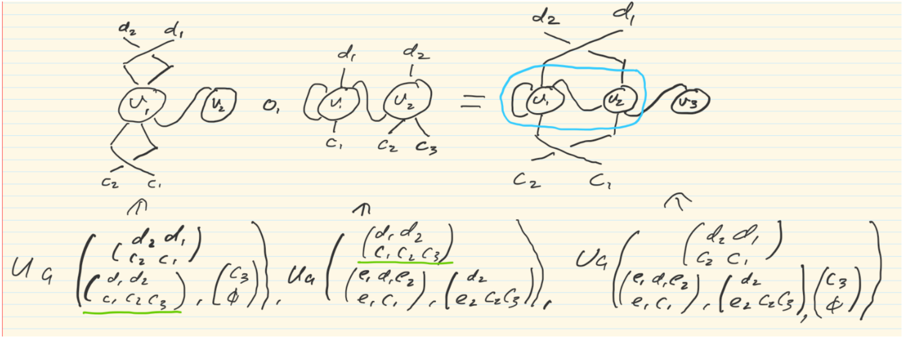
Let U_G be the set of strict isomorphism classes of coloured, ordered, directed, wheeled graphs.



- U_G is a coloured operad in Set
- For a symmetric monoidal cat $\mathcal{E}$ the free object functor $F : Set \to \mathcal{E}, FX = \coprod_{x \in X} I$ is symmetric monoidal.
- This functor enriches U_G into $\overline{U_G}$ a coloured operad in $\mathcal{E}$.
- Algebras over $\overline{U_G}$ are wheeled props in $\mathcal{E}$.
- Or we say the operad $\overline{U_G}$ governs wheeled props.

What is an algebra over an operad?

- Representation is to group as an algebra is to operad

Similar methods generate operads governing other operadic structures (operads, props, ...)

Koszul Operads Governing Operadic Structures

- Operads: [Van der Laan, 2003]
- Modular operads: [Ward, 2019]
- Operadic structures living on connected graphs (e.g. dioperads, properads, wheeled properads,...):
  - [Kaufmann and Ward, 2021] with cubical Feynman categories
  - [Batanin and Markl, 2021] with partial operads in operadic categories
- Props and wheeled props live on disconnected graphs...

Why might we care?

- Koszul operad governing x gives a model of an ∞-x

Theorem

There exists a Koszul groupoid coloured operad governing wheeled props.

Proof outline,

- Give a certain biased presentation of a wheeled prop.
- Leads to biased presentation of the operad governing wheeled props.
- Show isomorphic to a groupoid coloured operad which is quadratic.
- Show this operad is Koszul.

Let $\mathcal{C}$ be a non-empty set of colours and $\underline{c}, \underline{d}$ be sequences from this set.

Definition

Let $\mathcal{P}(\mathcal{C})$ be the category whose

- objects are sequences of colours $\underline{c}$ from $\mathcal{C}$
- and morphisms are permutations acting from left, i.e. $\sigma \in Hom(\underline{c}, \sigma\underline{c})$

- $\mathcal{P}(\mathcal{C})^{op} \times \mathcal{P}(\mathcal{C})$ is a groupoid.

Definition

For a symmetric monoidal cat $\mathcal{E}$, a **bimodule** is $P : \mathcal{P}(\mathcal{C})^{op} \times \mathcal{P}(\mathcal{C}) \to \mathcal{E}$

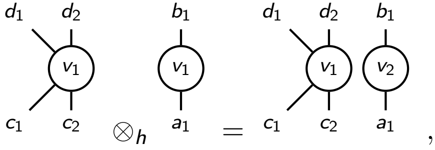
- for all pairs of profiles $\binom{\underline{d}}{\underline{c}}$ have an object $P\binom{\underline{d}}{\underline{c}} \in \mathcal{E}$.
- for all pairs of permutations $\binom{\sigma}{\tau}$ have isomorphisms $P\binom{\underline{d}}{\underline{c}} \xrightarrow{\binom{\sigma}{\tau}} P\binom{\sigma\underline{d}}{\underline{c}\tau}$

Example: $U_G = \bigcup_{\binom{\underline{d}}{\underline{c}}} U_G\binom{\underline{d}}{\underline{c}}$ is a bimodule.

Definition (Yau and Johnson def 11.33)

A $\mathcal{C}$ coloured **wheeled prop** over $\mathcal{E}$ (a symmetric monoidal cat) consists of

1. a bimodule $P : \mathcal{P}(\mathcal{C})^{op} \times \mathcal{P}(\mathcal{C}) \to \mathcal{E}$
2. a horizontal composition
$$P\binom{\underline{d}}{\underline{c}} \otimes P\binom{\underline{b}}{\underline{a}} \xrightarrow{\otimes_h} P\binom{\underline{d}, \underline{b}}{\underline{c}, \underline{a}}$$
3. a contraction
$$P\binom{\underline{d}}{\underline{c}} \xrightarrow{\varepsilon^i_j} P\binom{\underline{d} \setminus d_i}{\underline{c} \setminus c_j}$$
4. some units and axioms



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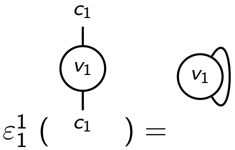


Wheeled Props

Definition

An **alternate** $\mathcal{C}$ coloured wheeled prop over $\mathcal{E}$ consists of

1. the same bimodule P.
2. an extended horizontal composition

$$P\binom{\underline{d}}{\underline{c}} \otimes P\binom{\underline{b}}{\underline{a}} \xrightarrow{\otimes_h, \binom{\sigma}{\tau}} P\begin{pmatrix}\sigma(\underline{d}, \underline{b}) \\ (\underline{c}, \underline{a})\tau\end{pmatrix}$$

   where $(\otimes_h, \binom{\sigma}{\tau})(\alpha, \beta) := \otimes_h(\alpha, \beta) \cdot \binom{\sigma}{\tau}$

3. an extended contraction

$$P\binom{\underline{d}}{\underline{c}} \xrightarrow{(\varepsilon^i_j, \binom{\sigma}{\tau})} P\begin{pmatrix}\sigma(\underline{d} \setminus d_i) \\ (\underline{c} \setminus c_j)\tau\end{pmatrix}$$

   where $(\varepsilon^i_j, \binom{\sigma}{\tau})(\alpha) := \varepsilon^i_j(\alpha) \cdot \binom{\sigma}{\tau}$

What does this accomplish?

Operations are right compatible

$$(\otimes_h, \begin{pmatrix}\sigma\\ \tau\end{pmatrix})\begin{pmatrix}\sigma'\\ \tau'\end{pmatrix} = (\otimes_h, \begin{pmatrix}\sigma'\sigma\\ \tau\tau'\end{pmatrix})$$
$$(\varepsilon^i_j, \begin{pmatrix}\sigma\\ \tau\end{pmatrix})\begin{pmatrix}\sigma'\\ \tau'\end{pmatrix} = (\varepsilon^i_j, \begin{pmatrix}\sigma'\sigma\\ \tau\tau'\end{pmatrix})$$

left compatible,

$$(\begin{pmatrix}\sigma_1\\ \tau_1\end{pmatrix}, \begin{pmatrix}\sigma_2\\ \tau_2\end{pmatrix})(\otimes_h, \begin{pmatrix}\sigma\\ \tau\end{pmatrix}) = (\otimes_h, \begin{pmatrix}\sigma(\sigma_1 \times \sigma_2)\\ (\tau_1 \times \tau_2)\tau\end{pmatrix})$$
$$\begin{pmatrix}\sigma'\\ \tau'\end{pmatrix}(\varepsilon^{\sigma'^{-1}(i)}_{\tau'(j)}, \begin{pmatrix}\sigma\\ \tau\end{pmatrix}) = (\varepsilon^i_j, \begin{pmatrix}\sigma(\sigma'^{(i)})\\ (\tau'^{(j)})\tau\end{pmatrix})$$

and $\otimes_h$ has a well defined action of $\mathbb{S}_2$ on it.

Every other (non-unital) axiom is quadratic.

Definition

For a groupoid $\mathbb{V}$, a $\mathbb{V}$ **coloured bimodule** P has the following data

- For any pair of profiles it has an object $P\binom{\underline{d}}{\underline{c}} \in \mathcal{E}$.
- Given permutations $(\tau;\sigma) \in \Sigma_{|\underline{c}|} \times \Sigma_{|\underline{d}|}$ and isomorphisms $g : \sigma\underline{d} \to \underline{d}' \in \mathbb{V}^{|\underline{d}|}$, $f : \underline{c}' \to \underline{c}\tau \in \mathbb{V}^{|\underline{c}|}$ there is an isomorphism

$$P\begin{pmatrix}\underline{d}\\ \underline{c}\end{pmatrix} \xrightarrow{(\binom{\sigma}{\tau},\binom{g}{f})} P\begin{pmatrix}\underline{d}'\\ \underline{c}'\end{pmatrix}$$

in $\mathcal{E}$ such $(\binom{id}{id},\binom{id}{id})$ is the identity and composition satisfies

$$(\begin{pmatrix}\sigma'\\ \tau'\end{pmatrix},\begin{pmatrix}g'\\ f'\end{pmatrix}) \circ (\begin{pmatrix}\sigma\\ \tau\end{pmatrix},\begin{pmatrix}g\\ f\end{pmatrix}) = (\begin{pmatrix}\sigma'\sigma\\ \tau\tau'\end{pmatrix},\begin{pmatrix}g''\\ f''\end{pmatrix})$$

where if $g' : \sigma'\underline{d}' \to \underline{d}''$, $f' : \underline{c}'' \to \underline{c}'\tau'$ then g'' and f'' are given by

$$g'' : \sigma'\sigma\underline{d} \to \sigma'\underline{d}' \to \underline{d}'', \qquad f'' : \underline{c}'' \to \underline{c}'\tau' \to \underline{c}\tau\tau'$$

If $\mathbb{V}$ is a discrete groupoid (of colours) then obtain a coloured bimodule.

Definition

A **non-unital partial groupoid coloured operad** consists of

- a groupoid coloured module P
- partial compositions

$$P\binom{d}{\underline{c}} \otimes P\binom{c_i}{\underline{b}} \xrightarrow{\circ_i} P\binom{d}{\underline{c} \circ_i \underline{b}}$$

It satisfies the standard associativity axioms, and this extended equivariance axiom

$$\begin{tikzcd}
P\binom{d}{\underline{c}} \otimes P\binom{c_i}{\underline{b}} \arrow[r, "(\sigma,f,g)\otimes(\tau,f',g')"] \arrow[d, "\circ_i"] & P\binom{d'}{\underline{c}'} \otimes P\binom{c'_{\sigma(i)}}{\underline{b}'} \arrow[d, "\circ_{\sigma(i)}"] \\
P\binom{d}{\underline{c}\circ_i\underline{b}} \arrow[r, "(\sigma,f,g)\circ_i(\tau,f',g')"] & P\binom{d'}{\underline{c}'\circ_{\sigma(i)}\underline{b}'}
\end{tikzcd}$$

$$\begin{array}{ccc}
P\binom{d}{\underline{c}} \otimes P\binom{c_i}{\underline{b}} & \xrightarrow{(\sigma,f,g)\otimes(\tau,f',g')} & P\binom{d'}{\underline{c'}} \otimes P\binom{c'_{\sigma(i)}}{\underline{b'}} \\
\Big\downarrow \circ_i & & \Big\downarrow \circ_{\sigma(i)} \\
P\binom{d}{\underline{c}\circ_i\underline{b}} & \xrightarrow{(\sigma,f,g)\circ_i(\tau,f',g')} & P\binom{d'}{\underline{c'}\circ_{\sigma(i)}\underline{b'}}
\end{array}$$

$$\begin{array}{ccc}
P\binom{d}{\underline{c}} \otimes P\binom{c_i}{\underline{b}} & \xrightarrow{(id,id,(id,\ldots,f_i,\ldots,id))\otimes(id,g,\vec{id})} & P\binom{d}{c_1,\ldots,c'_i,\ldots,c_{|\underline{c}|}} \otimes P\binom{c'_i}{\underline{b}} \\
\Big\downarrow \circ_i & & \Big\downarrow \circ_i \\
P\binom{d}{\underline{c}\circ_i\underline{b}} & \xrightarrow{id=(id,id,\vec{id})} & P\binom{d}{\underline{c}\circ_i\underline{b}}
\end{array}$$

$$c_i \xrightarrow{g} c'_i \xrightarrow{f_i} c_i \in Aut(c_i)$$

i.e. we have an action of the groupoid on the internal edges of groupoid coloured operads, via the automorphism group of the colour.

The Groupoid Coloured Operad Governing Wheeled Props

The operad governing wheeled props in Vect is isomorphic to the groupoid coloured operad $\mathbb{W} = F_\Sigma(E)/\langle G \rangle$ where

- the groupoid is $\mathcal{P}(\mathcal{C})^{op} \times \mathcal{P}(\mathcal{C})$
- the groupoid coloured module E is given by

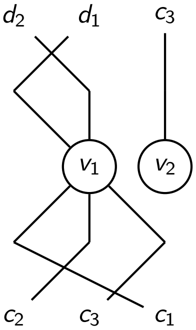
$$(\varepsilon^i_j, \binom{\sigma}{\tau}) = U_G(\binom{\underline{d}}{\underline{c}}; \binom{\sigma(\underline{d} \setminus \{d_i\})}{(\underline{c} \setminus \{c_j\})\tau})$$

$$(\otimes_h, \binom{\sigma}{\tau}) = U_G(\binom{\underline{d}}{\underline{c}}, \binom{\underline{b}}{\underline{a}}; \binom{\sigma(\underline{d}, \underline{b})}{(\underline{c}, \underline{a})\tau})$$

  where the action of the groupoid on these generators is given by right and left compatibility, and the action of $\mathbb{S}_2$ on $\otimes_h$.
- the relations G are given by the remaining quadratic axioms

If we have profiles $p_1 = \binom{d_1, d_2}{c_1, c_2, c_3}$, $p_2 = \binom{c_3}{\emptyset}$, $p = \binom{d_2, d_1, c_3}{c_2, c_3, c_1}$, $p_\gamma = \binom{d_2, d_1}{c_2, c_1}$ then $(\otimes_h, \binom{213}{231}) \in E\binom{p}{p_1, p_2}$ and $(\varepsilon_2^3, \binom{id}{id}) \in E\binom{p_\gamma}{p}$

$$\begin{array}{c} (\otimes_h, \binom{213}{231}) \\ \diagup \quad \diagdown \\ *_{p_1} \qquad *_{p_2} \end{array} =$$



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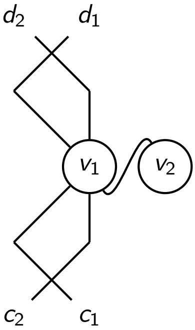
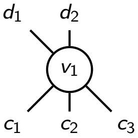
$$\begin{array}{c} (\varepsilon_2^3, \binom{id}{id}) \\ | \\ (\otimes_h, \binom{213}{231}) \\ \diagup \quad \diagdown \\ *_{p_1} \qquad *_{p_2} \end{array} =$$

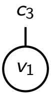


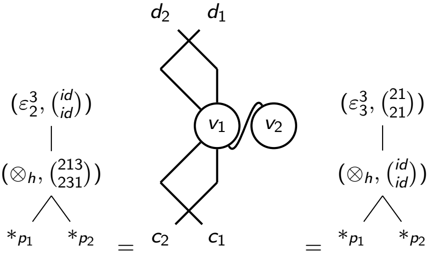
Where,

$*_{p_1} =$

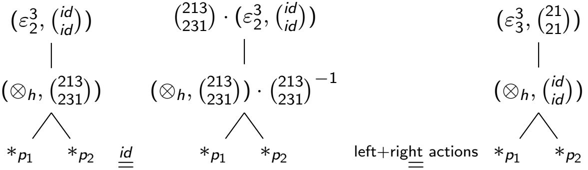


, $*_{p_2} =$

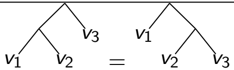
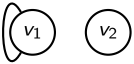
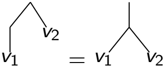
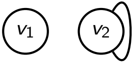
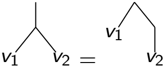
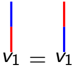




- The internal edge of the left tree monomial is $p = \binom{d_2, d_1, c_3}{c_2, c_3, c_1}$.
- $id = p \xrightarrow{\binom{213}{231}^{-1}} p' \xrightarrow{\binom{213}{231}} p \in Aut(p)$ where $p' = \binom{d_1, d_2, c_3}{c_1, c_2, c_3}$



The Quadratic Relations of $\mathbb{W} = F_{\Sigma}(E)/\langle G \rangle$

Graphs	Relations
	
	
	
 ,	

Showing $\mathbb{W}$ is Koszul

Theorem

Let $\mathcal{O} = F_\Sigma(\mathcal{X})/\langle \mathcal{G} \rangle$ be a $\mathbb{V}$ coloured operad, where $Aut(v)$ is finite for all $v \in ob(\mathbb{V})$. If $\mathcal{O}^f$ admits a quadratic Groebner basis then $\mathcal{O}$ is Koszul.

Straightforward extension of existing work on Groebner basis for operads.

Some Consequences

$\mathbb{W}^f$ admits a quadratic Groebner basis $\implies$ the operad $\mathbb{W}$ is Koszul.

- this gives a model of an infinity wheeled prop
- which enables homotopy transfer theory for wheeled props

By similar methods we can construct a Koszul operad $\mathbb{P}$ governing props.

Mentioned Sources

M. Batanin and M. Markl. Koszul duality for operadic categories. *arXiv preprint arXiv:2105.05198*, 2021.

V. Dotsenko and A. Khoroshkin. Gröbner bases for operads. *Duke Mathematical Journal*, 153(2):363–396, 2010.

R. M. Kaufmann and B. C. Ward. Koszul feynman categories. *arXiv preprint arXiv:2108.09251*, 2021.

P. Van der Laan. Coloured koszul duality and strongly homotopy operads. *arXiv preprint math/0312147*, 2003.

B. C. Ward. Massey products for graph homology. *arXiv preprint arXiv:1903.12055*, 2019.

D. Yau and M. W. Johnson. *A foundation for PROPs, algebras, and modules*, volume 203. American Mathematical Soc., 2015.