Homotopy probs and other G-Operadic Structures[1]

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supervised by
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[1]Extending "Koszul Operads Governing Props and Wheeled Props" arXiv:2308.08718

G-Operadic

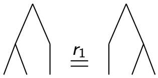
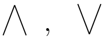
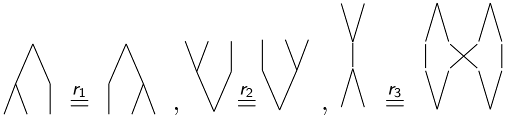
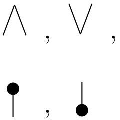
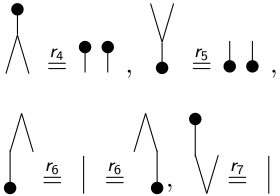
Idea: G-operadic structures model the composition of functions via graphs, where the edges of the graph are modelled by some group G.

Fancier Graphs →

Fancier Edges ↓

G	Operadic Family		
	Operad	Properad	Prop
Non-Symm.	...	...	...
Symmetric	...	...	...
Braided	...	...	...
Ribbon	...	...	...

Operadic Structures: Generalising the Graphs

Example Operadic	Its Presentation	
	Generators	Relations
The Operad Governing Ass. Algebras		
The Properad Governing Bialgebras		
The Prop Governing Hopf Algebras		r_1, r_2, r_3 and, 

Trees $\subset$ Connected Graphs $\subset$ Graphs

G-Props: Varying G

Definition

- Symmetric props are symmetric monoidal categories,

in each case generated by a single object.

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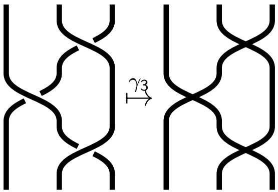
These are all instances of a more general definition called a G-prop.

$$G = (G_n, \gamma_n : G_n \to \mathbb{S}_n)_{n \in \mathbb{N}}$$



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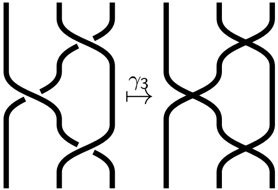

Claim: G-props can also be defined as algebras over operads!

Koszul Operads

Let $\mathbb{P}$ by the operad, whose algebras are symmetric props in $\mathit{Vect}_{\mathbb{K}}$.

Theorem (Stoeckl [2023])

The operad $\mathbb{P}$ is Koszul.

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The operad $\mathbb{P}$ is Koszul.

A algebraic operad P is Koszul if, and only if, it has a quadratic model P_∞.

- model: $P_\infty \xrightarrow{f} P$, f is an epimorphism and quasi-isomorphism.
- quadratic: $P_\infty = (F(E), d)$ and $d(E) \to F(E)^2$
- quadratic $\implies$ minimal

Why might we care?

Corollary

Algebras over $\mathbb{P}_\infty$ are homotopy-props.

Koszul Operads Governing Operadic Structures

Theorem

The operads governing all mainstream symmetric (and non-symmetric) operadic structures are all Koszul.

This is the culmination of many years of work by many authors.

- Non-symmetric Operads: [Van der Laan, 2003]
- Modular operads: [Ward, 2022]
- Operadic structures living on connected graphs:
  - [Kaufmann and Ward, 2023] with cubical Feynman categories.
  - [Batanin and Markl, 2023] with partial operads in operadic categories
- All operadic structures: [Stoeckl, 2023]

The Key Ideas in Extending the Theory

1. Non-symmetric Operads.
2. Symmetric connected operadic structures.
3. Disconnected operadic structures.

$1 \to 2$: Hide symmetric action in action of a groupoid (or equiv.).

$2 \to 3$: Use Gröbner bases methods instead of polytopes.

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Remainder of this talk:

- Describe these two ideas.
- Outline why they also work for G-operadic structures.

$\to$ Understand the operad governing props.

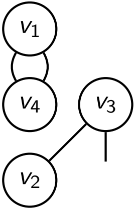
The Operad Governing Props

Generalised Graphs $\approx$ Vertices $+$ Flags/Half-Edges

Proposition (Specialisation of Lemma 14.2 Yau and Johnson [2015])

Let $Gr^{\uparrow}$ be the set of strict isomorphism classes of directed, vertex labelled generalised graphs, with no directed cycles, and partition it by profiles.

$$Gr^{\uparrow}\left(\substack{\binom{n}{m} \\ \binom{n_1}{m_1}, \dots, \binom{n_k}{m_k}}\right) := \{\gamma \in Gr^{\uparrow} : \gamma \textit{ profile } \binom{n}{m},\ v_i \textit{ profile } \binom{n_i}{m_i}\}$$



$$\in Gr^{\uparrow}\left(\substack{\binom{0}{1} \\ \binom{0}{2}, \binom{1}{0}, \binom{0}{2}, \binom{2}{0}}\right),$$

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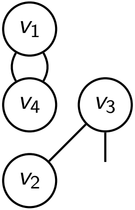
Then $Gr^{\uparrow}$ with $\circ_i$ given by graph substitution is a coloured operad in Set, whose algebras are symmetric props in Set.



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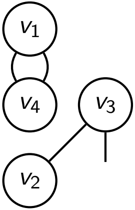
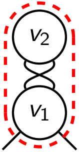


$\circ_3$



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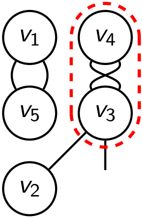
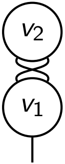


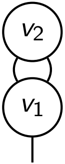
The Operad Governing Props

Proposition (Specialisation of Lemma 14.4 Yau and Johnson [2015])

The operad $Gr^{\uparrow}$ governing symmetric props in Set, can be enriched into the operad $\mathbb{P}$ governing symmetric props in $Vect_{\mathbb{K}}$.

$$\mathbb{P}\binom{\binom{n}{m}}{\binom{n_1}{m_1},\dots,\binom{n_k}{m_k}} \cong \mathbb{K}\left\langle Gr^{\uparrow}\binom{\binom{n}{m}}{\binom{n_1}{m_1},\dots,\binom{n_k}{m_k}}\right\rangle,$$

k_1 

$+k_2$ 

$\in \mathbb{P}\binom{\binom{0}{1}}{\binom{2}{1},\binom{0}{2}}$

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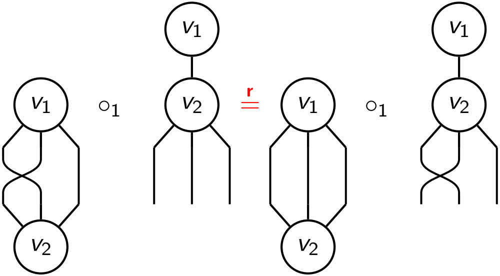

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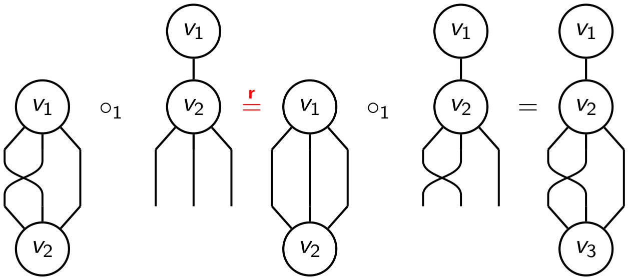

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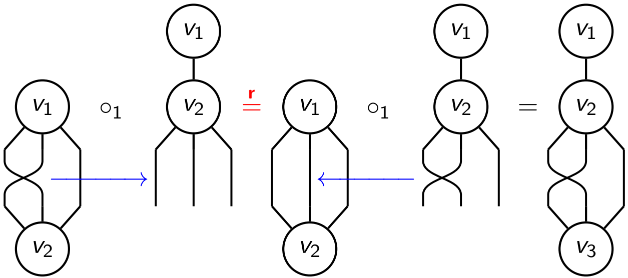


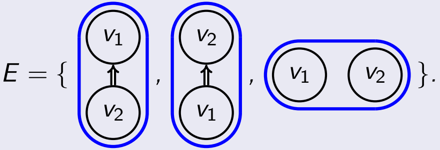
Key Idea: Hide symmetric structure of graph in action of a groupoid

- Reconsider $\mathbb{P}$ as a groupoid coloured operad.

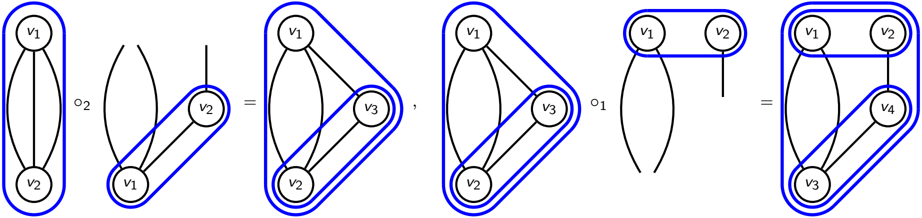
The Free Full Nesting Operad

Proposition

Let E be the set of graphs in $Gr^{\uparrow}$ with two vertices, partitioned as follows.



The free operad $F(E)$ is equivalent to 'fully nested graphs' with $\circ_i$ given by nested graph substitution.

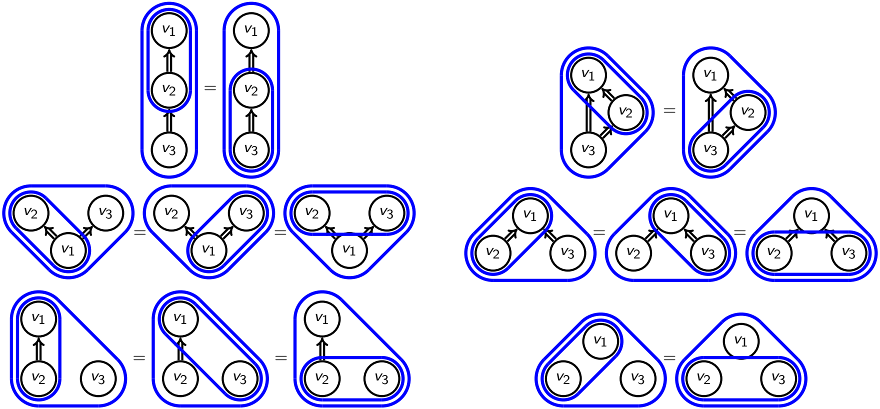


A Quadratic Presentation of the Operad Governing Props

Proposition (Reformulation of Stoeckl [2023])

The groupoid coloured operad $\mathbb{P}$ admits a quadratic presentation $\mathbb{P} \cong F(E)/\langle R\rangle$.

The quadratic relations R are,



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- All G-operadic structures only differ in their equivariance axioms.
  - Symmetric equivariance axioms hidden in action of a groupoid.
  - G-equivariance axioms in action of a groupoid.

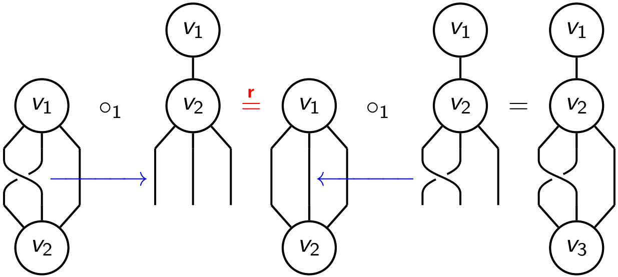
Why does this work for all G-props?

- All G-operadic structures only differ in their equivariance axioms.
  - Symmetric equivariance axioms hidden in action of a groupoid.
  - G-equivariance axioms in action of a groupoid.

- Varying G, the operads governing G-props have 'same' presentation,
  - with different groupoid colouring!

An Informal Illustration for Braided Props

Governing operad $\mathbb{P}_{\mathcal{B}}$ can be described in terms of **braided** graphs.



$$\mathbb{P}_{\mathcal{B}} \cong F(E_{\mathcal{B}})/\langle R_{\mathcal{B}} \rangle$$

where

- $E_{\mathcal{B}}$ is the set of **braided** graphs with two vertices.
- $F(E_{\mathcal{B}})$ is the set of fully nested **braided** graphs.
- $R_{\mathcal{B}}$ is R re-interpreted with nested **braided** graphs.

Koszulness of Operads Governing G-Operadic Structures

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Theorem (Stoeckl [2023])

Let $\mathcal{O} = F_\Sigma(\mathcal{X})/\langle \mathcal{G} \rangle$ be a $\mathbb{V}$-coloured operad, where $Aut(v)$ is finite for all $v \in ob(\mathbb{V})$. If $(\mathcal{O}^f)_*$ admits a quadratic Groebner basis then $\mathcal{O}$ is Koszul.

Proposition

If G has a solvable word problem, then the operad O governing each G-operadic structure satisfies: $(\mathcal{O}^f)_*$ has a quadratic Groebner basis.

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Proposition

If G has a solvable word problem, then the operad O governing each G-operadic structure satisfies: $(\mathcal{O}^f)_*$ has a quadratic Groebner basis.

Recall $G = (G_n, \gamma_n : G_n \to \mathbb{S}_n)_{n\in\mathbb{N}}$.

Corollary

The groupoid coloured operads governing all G-operadic structures are Koszul, if G_n is a finite group for all $n \in \mathbb{N}$.

What I'm Thinking On

- The operads governing all G-operadic structures have quadratic Groebner bases.
- Using this result, what is the nicest possible way to produce associated Koszul operads?
  - Can seek to extend groupoid coloured theory.
  - Can use coloured operads with quadratic-unary presentations.
    - Mirroring Dehling and Vallette [2021].

Later,

- Currently working with Marcy Robertson and Phil Hackney on homotopy coherent nerves between various models of symmetric ∞-(wheeled)-prop(erad)s.
- These nerves could also extend nicely in the G-case.

Mentioned Sources

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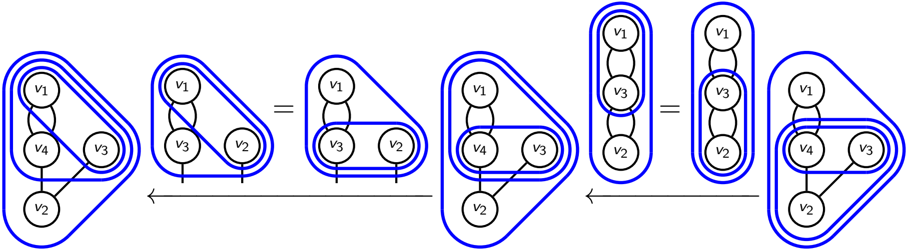
What does it mean to have a quadratic Groebner basis?



- Every graph has a unique minimal shuffle tree monomial forming it.
- Every non-minimal tree monomial can be rewritten to the minimal tree via the relations of the operad.