Elementary Background for Kazhdan-Lusztig Polynomials

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1 Introduction

This semester and the summer preceding it I have been working with Professor Scott to learn the necessary mathematical background for and develop programs to compute certain cases of Kazdhan-Lusztig polynomials, which provide a new basis for Hecke algebras related to Weyl groups. Here I'll outline the major concepts, emphasizing first the mathematial basis and then moving into the algorithmic realization.

2 Root Systems and Weyl Groups

Define a **root system** as a finite set Φ of objects called **roots** such that $\Phi \subseteq \mathbb{R}^n$ and the space has a positive-definite, symmetric biliniear form $\langle -, - \rangle$. Each root system has a set of **fundamental roots** where each positive root is a sum of these fundamental roots.

For example, consider $\Phi = \mathbb{R}^n$ and define $\epsilon_i = (0, 0, \ldots, 1, \ldots, 0, 0)$ where 1 is the i^{th} coordinate. Then the roots are elements $\epsilon_i - \epsilon_j$ where $i \neq j$ and the fundamental roots are

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \text{ for } 1 \leq i \leq n-1 \tag{1}$$

Now let

$$s_i(x) = x - \frac{2\langle x, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i \tag{2}$$

be the reflection s_i associated with the fundamental root α_i. Geometrically, each of these corresponds to reflection across some hyperplane through the origin. Each root system has an associated **Weyl group** generated by these reflections. It is interesting to note that the Weyl Group associated with the root system $Phi = \mathbb{R}^n$ (discussed above) is isomorphic to the symmetric group S_n. All roots in the root system are obtainable from the fundamental roots by applying elements of the associated Weyl group.

Each string of these reflections (viewed as a product) has a corresponding **length**, defined as the smallest number of reflections required to write an equivalent string. Note that every finitely generated Weyl group W is also a **Coxeter group**, so the only relations are

$$(s_i)^2 = 1 \text{ and } (s_i s_j)^{m(i,j)} = 1 \tag{3}$$

where $m(i,j)$ is the order of the product in W. We can see that appending some reflection s_i to the end (or beginning, depending on convention) will cause the length of the string to increase or decrease by 1. With these relations, each (finitely generated) Weyl group has an element w_0 of unique longest length. The length of w_0 is equal to the cardinality of Φ; appending any fundamental reflection to w_0 will decrease its length by 1.

3 Affine Weyl Groups, Cosets, and Weights

We can create an **affine Weyl group** by introducing the non-fundamental reflection $s_0 = s_n$ as a generator. Note that strings in affine Weyl groups do not have a finite maximal length, so we will continue to use w_0 to denote the longest element of the associated non-affine Weyl group.

In general, we consider elements not just of an affine Weyl group W_0 but of **cosets** $W_0 w$ where W is the associated non-affine Weyl group. Consider some coset $W_0 w'$ for an element $w' \in W_0$. For a reflection $s \in W$, it's possible that $W_0 w' s = W_0 w'$, in which case we would consider the length of w' to be unchanged by appending s. We can see that, when considering cosets of strings, appending elements can now increase length by 1, decrease length by 1, or leave length unchanged.

The idea of length-preserving reflections is more fundamental when we represent elements of affine Weyl groups as weights, which ,for a specific element, is an equivalent notation to the string notation we've used thus far. A **weight** is an n-tuple whose values correspond to coefficients of certain fundamental weights ϖ_i which are defined differently for each root system. We're generally concerned with elements $w_0 w$ (in string notation). Define ρ as the half-sum of positive roots. In our work, this corresponds to the weight $\rho = (1, 1, \ldots, 1)$. We can convert the string $w_0 w$ to weight notation by applying

$$w_0 w. - 2\rho = w_0 w(-\rho) - \rho \tag{4}$$

We are only concerned with so-called **dominant weights**, for which each coefficient of the weight is a non-negative integer. A dominant weight $w_0 w' s$ corresponds exactly to the case in which appending s did *not* leave the length of $w_0 w'$ unchanged. In other words, a weight $w_0 w$ is dominant if and only if $l(w_0 w) = l(w_0) + l(w)$ where $l(x)$ is the length of x.

4 Classification of Root Systems

Irreducible root systems can be classified by the relations between their fundamental roots as type A_n, B_n, C_n, or D_n for any $n \geq 2$ as well as F_4, G_2, or E_n for $3 \leq n \leq 8$. Root systems of order $n = 0$ or 1 are also (trivially) defined as type A. As an example, the root system constructed above (for which the associated Weyl group is isomorphic to S_n) is of type A_n.

Each root system can be visualized via a Coxeter diagram or, more commonly, a **Dynkin diagram**, which classifies a root system up to isomorphism (e.g. $D_3 \cong A_3$). In Dynkin diagrams, each fundamental root is represented by a node. The nodes are connected based on the following conventions:

- If $(s_i s_j)^2 = 1$, leave the nodes corresponding to α_i and α_j unconnected.
- If $(s_i s_j)^3 = 1$, connect the nodes corresponding to α_i and α_j with a single line segment.
- If $(s_i s_j)^4 = 1$, connect the nodes corresponding to α_i and α_j with a double line segment.

- If $(s_i s_j)^6 = 1$, connect the nodes corresponding to α_i and α_j with a triple line segment.

Root systems can have fundamental roots of two lengths, which are referred to as long and short roots. In the case of a double or triple line segment connecting roots of differing length, draw an arrow pointing from the long to the short root.

It is interesting to note that the Dynkin diagrams for types B_n and C_n are distinguished only by the directions of the arrows. Our work so far has dealt mostly with root systems of type A_n, D_n, and E_n, which have fundamental roots of only one length. This greatly reduces the complexity of calculations within the root systems.

The placement of nodes in general follows the convention established by Bourbaki. Convention for the addition of an α_0 node (i.e. the move to an affine root system) varies; our choice is often referred to as the "twisted" affine case.

Coxeter diagrams generally have only single lines which are labelled according to the relations denoted. There is no directionality.

Root systems can also be classified (up to isomorphism) by their **Cartan matrix**, defined as the matrix for which $\alpha_{i,j} = \langle \alpha_i, \alpha_j \rangle$. These matrices (and their inverses) are especially important in the computational aspect of our work, as they encode the relations defining root systems in a machine-readable format. All operations involving appending elements of the Weyl group to a string or converting between root (string) and weight notation involve Cartan matrix calculations.

5 Kazhdan-Lusztig Polynomials

Kazhdan-Lusztig polynomials are indexed by two elements of a Coxeter group (in our case a Weyl group) and give a basis for the Hecke algebra of this group. They are defined recursively with respect to appending reflections s_i to the indexing strings, which requires every polynomial to be stored upon completion in order to make further computations. This causes memory problems for moderately high-order root systems and, combined with the general complexity of the computations, means that Kazhdan-Lusztig polynomials are known only for select root systems.

6 Algorithm and Implementation

The algorithm used was inherited from work by a previous student of Professor Scott, who used it to compute Kazhdan-Lusztig polynomials for the Weyl groups associated with type A_n for $2 \leq n \leq 8$. It decreases memory requirements by taking advantage of repetitions in values of polynomials for various indices and storing pointers to these values rather than always storing the full polynomials for each index.

The algorithm's implementation is fairly general because it takes an input file which states the lengths of and relations between all dominant weights (up to a specified maximum weight) in the root system. Most of my work, then, has been with the program which produces these input files. The program was designed explicitly for type A but has now been generalized to run for any root system having roots of only one length (so e.g. it works for type D but not for type B).

References

[1] Humphreys, James E. *Reflection Groups and Coxeter Groups*. Cambridge University, 1990. Print.

[2] Bourbaki, Nicolas. *Groupes et Algèbres de Lie*. Hermann, 1968. Print.