# Can a Shallow Ice Approximation Model be Used to Model the Water Output of Alpine Glaciers?

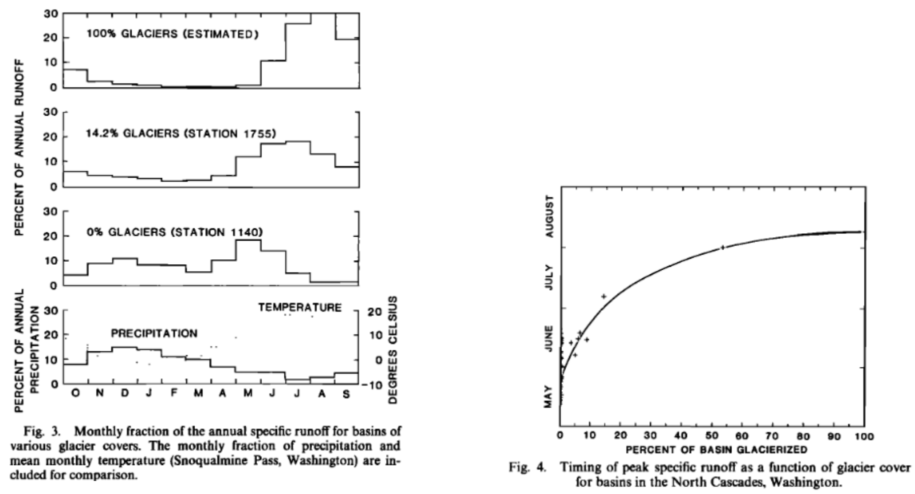
## Abstract

## 1 Introduction

### 1.1 Importance of glacial melting in mountain hydrology

Glacial melting plays a significant role in the hydrology of mountain catchment areas as shown in Fountain and Tangborn, 1985. Accurately capturing glacial dynamics with numerical models is important for understanding how water sources in high mountain catchments will evolve. The runoff produced by glaciers, especially later in the summer months, is a major contributor to stream flow. Without the runoff from the glaciers, the rivers would be even lower during the driest summer months as shown in Figure 1.

As glaciers melt and retreat, the change in mass takes the form of meltwater. This meltwater affects communities downriver in a variety of ways. For instance, many communities rely on these rivers as a water source for drinking and irrigation.




(a) Monthly fraction of the annual runoff for basins of various glacier cover.

(b) Timing of peak specific runoff as a function of glacier cover.

Figure 1: Figures 3 and 4 from Fountain and Tangborn (1985) showing how the percent of basin glacierized delays the peak runoff time of the basin.

## 1.2 Role of numerical modeling in understanding glacial runoff

One of the best ways to find out how glaciers will affect the stream flow of rivers in the glacier basin is by using computer models to approximate the water discharge of the glaciers. Scientists have been using computers to model glaciers for several decades, such as this paper by Iken A, 1981. As computational resources have grown, these models have grown in complexity and resolution leading to very computationally expensive models, meaning that they take a lot of computing power to run.

## 1.3 Challenges in computational modeling of glaciers

Many of these more advanced models can be quite complex to run due to there being a lot of different configurations and ways to run it. The goal of this paper is to write a simple Shallow Ice Approximation (SIA) - Mass Balance model that is easy to run and understand while still being reasonably generalizable to different small mountain glaciers. The advantage of using much more advanced models is that they often do a much better job at modeling the ice dynamics of glaciers. For some glaciers such as marine calving glaciers, accurate ice dynamics are crucial to accurately model them, as shown in Amaral et al., 2020. However on smaller mountain glaciers, modeling in 3-dimensions if often unnecessary and simpler models such as the SIA can be used. This paper will compare the results of a simple SIA - Mass Balance model to the results of OGGM, a much more advanced model using a simplified version of the Stokes Equations.

# 2 Literature Review

## 2.1 Prior research on SIA vs. Stokes models

As shown in Le Meur et al., 2004, there are significant differences in computational time between a SIA model and a Stokes model. When computing the free surface and associated velocity field, the SIA model took 1 minute of CPU time, whereas the Stokes model took 2 hours. This disparity grew even larger for 3D models. The authors show that there are some instances where SIA models do significantly worse than Stokes models, such as glaciers on steep slopes and glaciers in steep, narrow valleys because SIA models only approximate the Stokes equations. One of these approximations is to ignore horizontal stress gradients. This can cause a SIA model to deviate from a Stokes model significantly in predicted glacier flow and expansion. In one example, the resulting SIA model can have an upper free surface that is 15–20% greater than the Stokes model and velocities up to a factor of 2 greater (Le Meur et al., 2004). In the 2D model, the bed characteristics and slope become the limiting factor of the SIA model; Le Meur et al., 2004 note that the maximum velocity ratio of their SIA and Stokes models goes from 1.9 in a 3D model to 1.3 in a 2D model, which will differ depending on model configuration, but it tends to indicate that the horizontal stress gradients played a large part in this error. They found instances in which the SIA models performed well compared to Stokes models—particularly large flat glaciers with relatively free edges. One thing to note about this comparison study is that the authors are looking at the shape, area, and velocity profile of the glacier, whereas this study will focus on the water output (surface mass loss) of the glacier.

## 2.2 Using SIA models to model alpine glaciers

There are several papers, such as Le Muer et al., 2003 and Kessler et al., 2006, that use an SIA model for alpine glaciers. The consensus from those papers is that SIA models only work well on alpine glaciers with a low aspect ratio, defined as the thickness-to-extent ratio in Le Muer et al., 2004. The glacier used by this study will have a low aspect ratio and therefore a SIA model should work well to model it.

## 2.3 Using SIA models to model water runoff from glaciers

Additionally, there is precedent for using a SIA-Mass Balance model for modeling water runoff from glaciers (Naz et al., 2014). They used the SIA model to approximate the ice dynamics and a mass balance model to approximate the accumulation and ablation patterns on the glacier. As shown in their paper, the SIA model was able to accurately predict the glacier, and the coupled hydrological model was able to predict the stream flow accurately—only overestimating the July flow by an average of 13% and underestimating the August and September flow by an average of 2%.

# 3 Thesis Statement

How much do ice dynamics affect the model result when modeling small mountain glaciers for mass balance? I theorize that if using a simple 1-dimensional SIA-Mass Balance model on small mountain glaciers (with a low aspect ratio), the mass balance profile will have a much larger effect on the output of the model and its overall accuracy than the modeled ice dynamics. The results of the SIA-Mass Balance model and the OGGM model will be compared to the actual stream flow data to verify this hypothesis.

## 3.1 Study Site

### South Cascade Glacier, Washington State

In this study, I will focus on modeling the South Cascade Glacier in the North Cascades region of Washington State. South Cascade Glacier is roughly 1.68 square kilometers, has a mean elevation of roughly 1900 meters, (GLIMS) faces North, an average thickness of 99 meters, and a maximum thickness of 195 meters (GlaThiDa Consortium, 2020). The glacier is small, not overly steep, and has a low aspect ratio such that a SIA model should be able to accurately represent its ice dynamics. On the other hand, the glacier is large enough to exhibit some movement and produce a measurable amount of runoff throughout the year. It is also one of the USGS Benchmark glaciers which means there is a large amount of data on it available such as temperature, precipitation, mass balance, front variation, and thickness change. In 1992 a stream gauge was installed just below the glacier which allows one to calculate the runoff from the glacier.

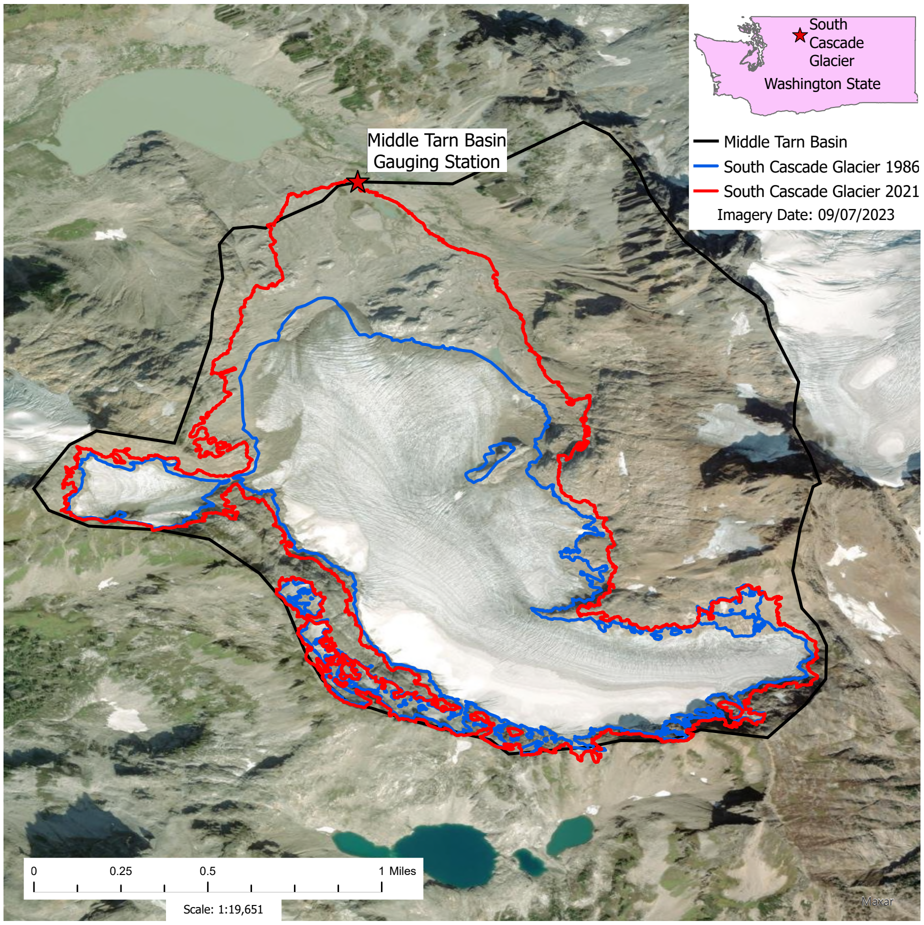



Figure 2: Map of the South Cascade Glacier in the North Cascades of Washington.

# 4 Methods

## 4.1 Model Development

### 4.1.1 Model Overview

The model is a one-dimensional Shallow Ice Approximation (SIA) model coupled with a temperature degree day mass balance model. The structure of the model is

1. SIA model for ice dynamics
2. Temperature degree day and precipitation mass balance model
3. Temperature degree day and precipitation snow model

| Symbol | Description |
|---|---|
| $Z_g$ | Glacier surface elevation above sea level (m) |
| ELA | Equilibrium line altitude (m) |
| $\gamma$ | Spinup mass balance equation gradient (m/year) |
| $Q$ | Ice flux ($m^2$/day) |
| $A$ | Flow rate factor ($5.87 * 10^{-19}Pa^{-3}day^{-1}$) |
| $n$ | Flow law exponent (3) |
| $p_{ice}$ | Density of ice ($917kg/m^3$) |
| $g$ | Acceleration due to gravity ($9.81m/s^2$) |
| $\frac{\partial z_s}{\partial x}$ | Slope of the glacier |
| $H$ | Ice thickness (m) |
| $b_s$ | Summer mass balance (m/day) |
| $T$ | Temperature (°C) |
| $M_{snow}$ | Snow melt factor (m/day/K) |
| $M_{ice}$ | Ice melt factor (m/day/K) |
| $b_w$ | Winter mass balance (m/day) |
| $p$ | Precipitation (m/day) |
| $\alpha$ | Precipitation conversion factor |
| $Accum_{lower}$ | Accumulation lower bound (m/day) |
| $y$ | Year |
| $Accum_{upper}$ | Accumulation upper bound (m/day) |
| $s$ | Snow depth (m) |
| $s_{melt-vol}$ | Snow melt volume ($m^3$) |
| $Area_{basin}$ | Basin area ($m^2$) |
| $Area_{glacier}$ | Glacier area ($m^2$) |
| $r_{vol}$ | Rain volume ($m^3$) |
| $g_{melt-vol}$ | Glacial melt volume ($m^3$) |

Table 1: Symbols Table

There are two main sections to the model run, spinup and run. The spinup section of the model runs for 500 years and aims to replicate the state of the glacier in 1984 when weather data becomes readily available. This section uses the simple mass balance equation below

$$(Z_g - \text{ELA}) * \gamma / 365.25 \tag{1}$$

This calculates the mass balance in meters per day. When the spinup hits the year 1900 the ELA is shifted up from 1880m to 1930m to simulate the retreat state of the glacier. The figure below shows the model in 1986 after 500 years of spinup and 2 years of run compared with the actual glacier in 1986 derived from a DEM (reference). The measured line has some gaps due to the DEM not being complete over the glacier area.

Three factors were tuned to create the spinup glacier, the initial ELA, the shifted ELA in 1900 and gamma. These variables were tuned to match the glacier ice profile in 1958 and 1986 when DEM's are available (reference).

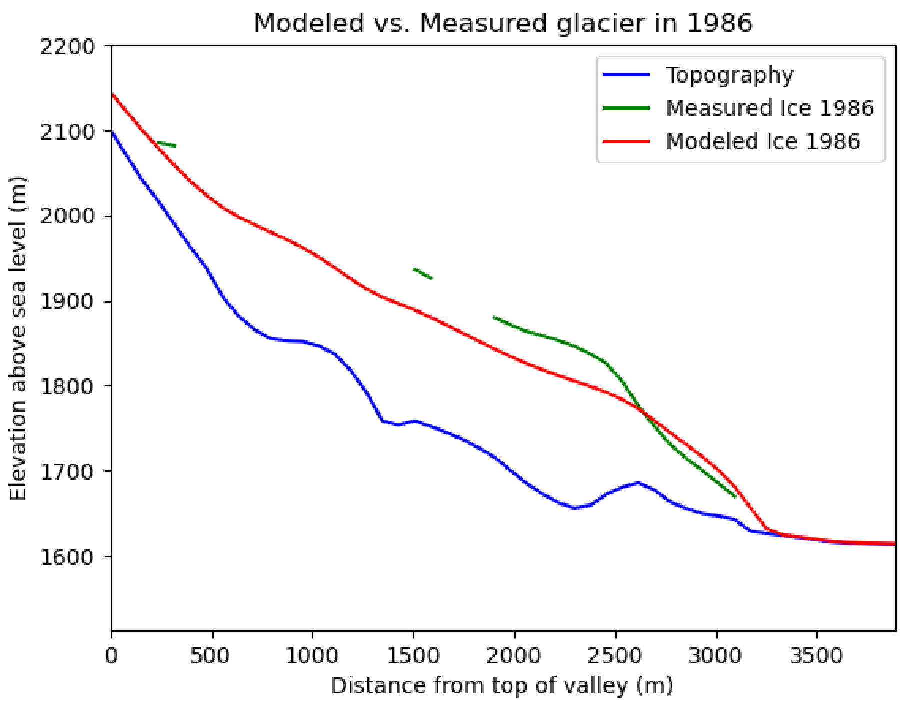



Figure 3: Spinup comparison of the model in 1986 after 500 years of spinup and 2 years of run compared with the actual glacier in 1986.

### 4.1.2 Ice Dynamics

**SIA Model** The SIA model is a one-dimensional model that uses the shallow ice approximation to approximate the ice dynamics of the glacier. This model calculates the one dimensional ice flux of the glacier using equation 2.

$$Q = \frac{2A}{n+2}(p_{ice}g|\frac{\partial z_s}{\partial x}|)^n \frac{H^5}{5} \tag{2}$$

**Assumptions** The SIA equations make several assumptions. First, the equations are 1-dimensional, so they neglect longitudinal stress, and ice only flows downhill. Second, the equations also assume that there is no basal sliding of the glacier. Third, the equations only use gravity as the driver of ice flow; they ignore other forces such as lateral and basal stress. Fourth, this set of equations assumes that the horizontal dimensions of the modeled glacier are much larger than the vertical dimensions.

The South Cascade Glacier was chosen because of some of these assumptions, it's horizontal dimensions are significant larger than its vertical dimensions (widths ranging from 400m-1200m and maximum ice thickness of 195m), and it is not a steep (average slope of 7.14 degrees along my centerline in 2021) or fast flowing glacier, allowing the SIA assumptions to hold.

#### 4.1.3 Model Setup

The model relies on a variety of input data in order to run. Besides the temperature and precipitation data described below the model also needs a bed topography, yearly glacier area, and total basin area to run. In order to tune the input parameters the model needs mass balance data, and ice thickness data along the bed centerline to tune the glacier spinup. In order to run the model with the temperature and precipitation data the model needs seven input parameters: ice melt factor, snow melt factor, lapse rate, start accumulation factor, end accumulation factor, avalance percentage and precipitation conversion factor. The melt factors were tuned so the model summer mass balance matched the measured summer mass balance. The accumulation factors were tuned so the model winter mass balance matched the measured winter mass balance. The lapse rate was empiraclly calculated using data from the Diablo Dam weather station and a weather station at 1830 meters next to the South Cascade glacier. The avalanche percentage was tuned to minimize the average snow depth over the model run. The precipitation conversion factor was obtained from (reference).

#### 4.1.4 Mass Balance Model

The mass balance of the glacier is calculated using temperature and precipitation data from the Diablo Dam weather station at 272m. The temperature at the glacier is calculated by using a month-specific lapse rate. This month-specific lapse rate was empiraclly calculated using data from the Diablo Dam weather station and the South Cascade Glacier weather station at 1830m from 2010-2018. The precipitation at the glacier is calculated by multiplying the precipitation at the Diablo Dam weather station by the precipitation conversion factor of 1.58 (reference). The ablation of the glacier is calculated by using a combination of an ice melt factor and a snow melt factor. Above the ELA the ablation is calculated by the equation

$$b_s = T * M_{snow} \tag{3}$$

Below the ELA the ablation is calculated by the equation

$$b_s = T * (M_{snow} + ((ELA - Z - g)/(ELA - min(Z_g))) * (M_{ice} - M_{snow})) \tag{4}$$

The result of this equation is the snow melt factor being used at the ELA and a linear increase in the melt factor until it hits the ice melt factor at the base of the glacier. The accumulation of the glacier is calculated using a similar linear equation that increases with time.

$$b_w = p*\alpha*(Accum_{lower}+((y-1984)/(2024-1984))*(Accum_{lower}-Accum_{upper})) \tag{5}$$

This resuls in the accumulation increasing with time until it hits the end accumulation at 2024.

#### 4.1.5 Snow and Rain Model

The snow melt model uses precipitation and temperature data to melt and accumulate snow. This model uses 14 elevation bins and keeps track of the snow depth in each bin. Each elevation bin has a corresponding area that represents the area of the basin in the bin's elevation range. The equation below is used to calculate the change in snow depth per timestep

$$s+ = \begin{cases} p * \alpha & \text{if } T \leq 0, \\ -min((s * T), s) & \text{if } T > 0 \end{cases} \tag{6}$$

The snow melt is constrained so that there cannot be more melt than there is snow. The rain is simply modeled by $p * \alpha$ for positive temperatures.

The total volume of snow is calculated by the equation

$$s_{melt-vol} = (s * T) * (Area_{basin} - Area_{glacier}) \tag{7}$$

This give us the total volume of snow melting off the glacier. The glacial melt is calculated elsewhere. The volume of rain is calculated by

$$r_{vol} = p * \alpha * Area_{basin} \tag{8}$$

This calculates the rain for the whole basin, assuming that any rain that falls off the glacier runs off immediately.

#### 4.1.6 Glacial Melt Model

The glacial melt model uses the mass balance of the glacier to calculate how much volume the glacier is losing. The volume of runoff from the glacier per timestep is calculated by the equation

$$g_{melt-vol} = b_s * Area_{glacier} \tag{9}$$

### 4.2 Data Used for Model

The temperature and precipitation data used for the model is from the Diablo Dam weather station at 272m. The data is available from 1984-2024 and missing 298 days of temperature measurements and 292 days of precipitation measurements. The missing temperature data was interpolated using the interp function from the numpy python library, the missing precipitation data was assumed to be 0. The glacier area data used in the model is from the USGS (reference). The basin area data was calculated using a DEM from the USGS and the basin outline shown in figure 1.

### 4.2.1 Model Calibration

The spinup initial ela, ela shift and gamma were manually optimized to match the glacier extent in 1984.

The ice and snowmelt factors were calibrated using summer mass balance data available from the USGS from 1984-2024. I used the minimize function using the Nelder-Mead method from the scipy library to minimize the mean squared error between the model and the data. The accumulation factors were calculated using the same methodology for the winter mass balance data available from the USGS from 1984-2024.

The avalanche percentage was optimized using the same methods as the mass balance variables, but instead of a mean squared error being minimized, the mean of the snow depth over the period 1984-2024 was minimized.

### 4.2.2 Model Comparison

**Running OGGM Model for the Same Glacier** The OGGM model was run using the run_with_hydro task from the oggm library. This model run used CMIP5 historical temperature and precipitation data to model the hydrology of the glacier from 1984-2019. Using the output of this model I was able to calculate the total runoff from the glacier using the melt_on_glacier_monthly, snowmelt_on_glacier_monthly and liq_precip_on_glacier_monthly variables. The MSE of the OGGM model and my model is 5.31% and the MSE of the OGGM model and the measured runoff data is 3.93%.

**Validation Using Real-World Streamflow Data** The calculated runoff of the SIA-Mass Balance model was validated using measured streamflow data from 1992-2007. This data was measured using a stream gauge located just below the glacier. The data is in units of mm per day averaged over the basin area ($4.46 km^2$). I converted this to cubic meters per day by divinding each value by 1000 and multiplying by the basin area in square meters. None of the input parameters to the model were specifically tuned to match the streamflow data. The melt factors were tuned to match the yearly summer mass balance, the accumulation factors were tuned to match the yearly winter mass balance, the avalance percentage was tuned to minimize the average snow depth over the model run, the lapse rates were empirically calculated using data from the Diablo Dam weater station and a weather station at 1830 meters next to the South Cascade glacier, and the precipitation conversion factor was obtained from (reference). The model was able to calculate the runoff from the whole basin (snowmelt, glacier melt and rainfall) from 1984-2024 with a mean squared error of 1.41%.

# 5 Expected Results

## 5.1 Accuracy of SIA

The SIA-Mass Balance proved to be quite accurate with a mean squared error of 1.42% for the available data from 1992-2007, consisting of 2418 data points. Below is a graph of the real-world measured runoff data and my model calculated runoff

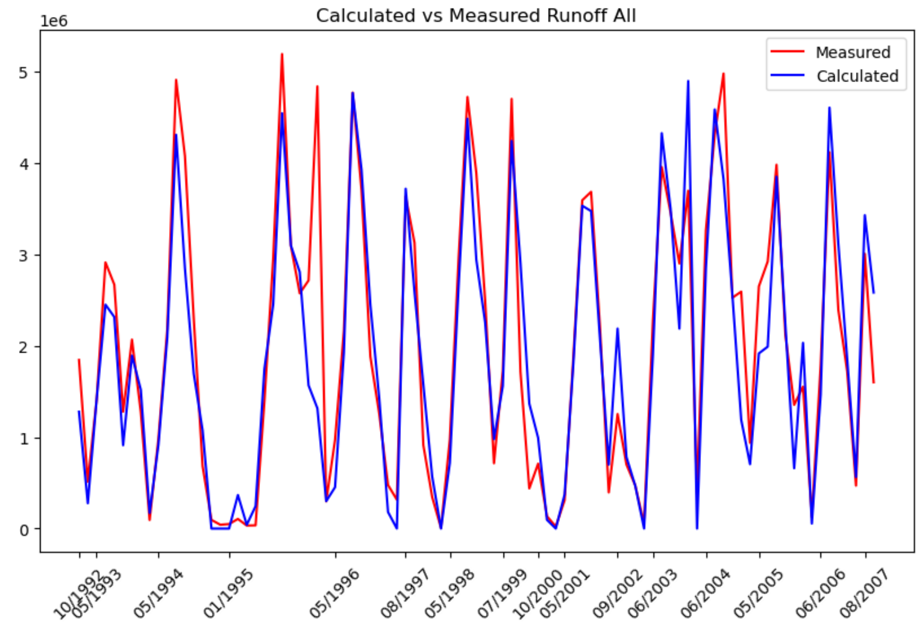



Figure 4: Measured vs. Calculated runoff data from 1992-2007

## 5.2 Accuracy of OGGM

The OGGM Model proved to be slightly less accurate when compared to the real-world measured runoff data. It was also run on slightly different temperature and precipitation data compared to my SIA-Mass Balance model. It used the CMIP5 (confirm this) climate modeled data which goes back to 1970 (check this). It had a mean squared error of 3.22% for all of the runoff data from 1992-2007.

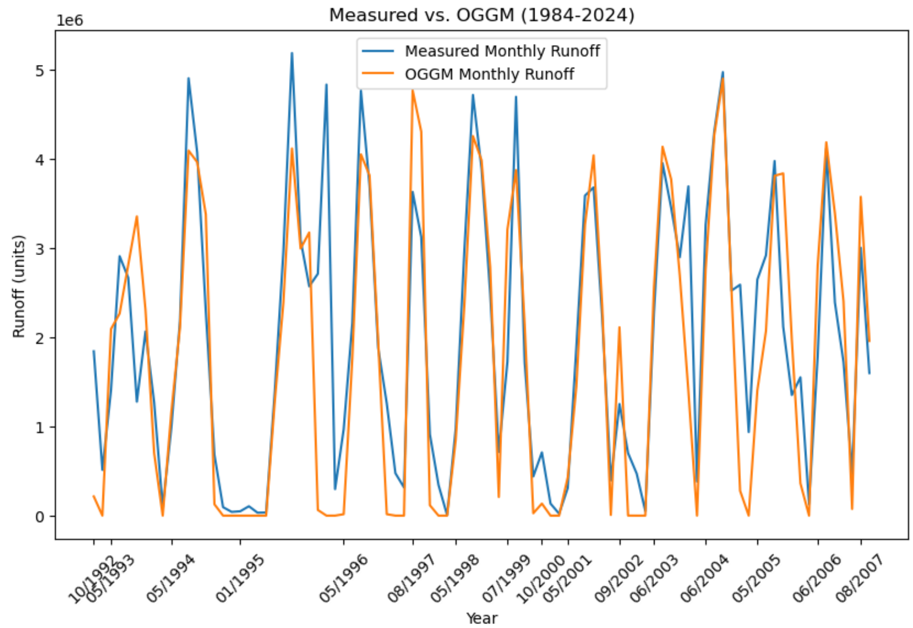



Figure 5: Measured vs. Calculated runoff data from 1992-2007

### 5.3 Comparison of Accuracy

The accuracy of the simpler SIA-Mass Balance model is quite impressive compared to the much more advanced OGGM model. The SIA-Mass Balance model's accuracy is roughly two times better than OGGM which is quite impressive. The OGGM model was run on modeled climate data instead of real-world measured data which most likely contributed to its worse accuracy.

## 6 Implications of Research

### 6.1 Importance of Simplified Ice Dynamics in Numerical Glacier Modeling

These results show that complicated and computationally intensive mass balance, melt and ice dynamics are not required to accurately model the runoff from small mountain glaciers. This means that we can use much simpler and less computationally intensive models such as the ones used in this paper to model the runoff from mountain glaciers over a much larger area. The modeling techniques used in this paper could easily be scaled to a much larger region if mass balance is available to tune the input parameters. I would hypothesize that the input parameters used in my model of the South Cascade Glacier could be used for a much larger model over a similar region as almost all of the input parameters are climate dependet, but this is a topic for another paper.

### 6.2 Applications

The work of this paper shows that complex models are not always required to accurately model the runoff from small mountain glaciers. This can have applications in everything from regional glacier modeling to water resource managment as glaciers are a significant source of water for many communities around the world.

The result of this research project will tell us how important it is to accurately model the ice dynamics of small mountain glaciers when predicting mass balance changes. If the hypothesis is correct, then researchers will be able to run less computationally expensive models to achieve a similar output as a much more sophisticated and computationally expensive model such as the Stokes model. As referenced in Verbunt et al., 2003, glacier models are often just one component of hydrology or other types of models. If we can make certain assumptions, like using an SIA-Mass Balance model when modeling a small mountain glacier as part of a larger hydrology model, that glacier component will run much faster. As a result, the whole model will run faster which means researchers can run this model over a larger area or run it many more times. This could lead to more accurate water predictions for water management teams and subsequently better water management techniques.

## 7 Discussion

### 7.1 What Worked

### 7.2 What Didn't Work and Why

### 7.3 What Can Be Improved

## 8 Conclusion

### 8.1 Summary of Results

### 8.2 Conclusion of Model Accuracy

## 9 References