

Two-point statistics without bins: A continuous-function generalization of the correlation function estimator for large-scale structure

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ABSTRACT

The two-point correlation function (2pcf) is the most important statistic in structure formation, used to measure the clustering of density field tracers (e.g. galaxies). Current estimators of the 2pcf, including the standard Landy-Szalay (LS) estimator, evaluate the 2pcf in hard-edged bins of separation between objects, which is inappropriate for the science context and results in a loss of information and a poor trade-off between bias and variance. We present a new estimator for the 2pcf, *the Continuous-Function Estimator*, which generalizes LS to a continuous representation and obviates binning in separation or any other pair property. Our estimator replaces the binned pair counts with a linear superposition of basis functions; it outputs the best-fit linear combination of basis functions to describe the 2pcf. It is closely related to the estimator used in linear least-squares fitting. The choice of basis can take into account the expected form of the 2pcf, as well as its dependence on properties other than separation. We show that the Continuous-Function Estimator can estimate the clustering of artificial data in representations that provide more accuracy with fewer basis functions than LS. The Continuous-Function Estimator achieves lower bias and lower variance than LS. We also demonstrate that the estimator can be used to directly estimate the Baryon Acoustic Oscillation scale. Critically, these will permit reductions in the number of mock catalogs required for covariance estimation, currently the limiting step in 2pcf measurements. We discuss applications and limitations of the Continuous-Function Estimator for present and future studies of large-scale structure, including determining the dependence of clustering on galaxy properties and potentially unifying real-space and Fourier-space approaches to clustering measurements.

KSF says: say in abstract that we'll need far fewer mocks; ok to be repetitive

Keywords: Astrostatistics techniques (1886), Baryon acoustic oscillations (138), Cosmology (343), Two-point correlation function (1951), Large-scale structure of the universe (902), Redshift surveys (1378)

1. INTRODUCTION

The large-scale structure (LSS) of the Universe is critical to our understanding of fundamental cosmology. It encodes information about the physics of the early Universe and the subsequent expansion history. In particular, LSS measures the Baryon Acoustic Oscillation (BAO) scale, which results from density fluctuations in the baryon-photon fluid. The distance traveled by these density waves before recombination imprints a feature on the statistical description of the LSS, which can be used to determine the characteristic BAO length scale (Eisenstein & Hu 1997). The LSS also contains the signature of redshift-space distortions caused by the peculiar velocities of galaxies, which are used to measure the growth rate of structure (Kaiser 2014). Additionally, the LSS can be used to constrain galaxy formation in conjunction with models of galaxy bias (e.g. Hamilton 1988). With current observations, the LSS is well-described by a cold dark matter model with a cosmological constant, the standard Λ CDM model. Upcoming galaxy surveys will observe larger volumes with improved measurements, allowing us to test Λ CDM to even higher precision.

The most important statistic for characterizing the LSS is the two-point correlation function (2pcf). The 2pcf measures the excess frequency at which any two galaxies are separated by a given distance, compared to a uniform distribution; effectively, it characterizes the strength of clustering at a given spatial scale. In calculating the 2pcf, the nontrivial survey boundaries of the surveys prevent us from directly summing pair counts. To account for the survey boundaries as well as regions corrupted by issues such as bright foreground stars, a set of random points are Poisson-distributed within the acceptable survey window. The pairwise correlations of these unclustered points are used to normalize out the survey window when estimating the 2pcf of the clustered data.

The 2pcf is traditionally estimated in bins of radial separation. This binning introduces inherent limitations. First, the choice of bins requires a trade-off between bias and variance: fewer bins may bias the result, while more bins increases the variance of measurement. Finite-width bins also result in a loss of information about the property in which one is binning. As we work towards extreme precision in large-scale structure, maximizing the information we extract with our analyses will become increasingly important. Finally, the error on the covariance matrix scales with the number of bins; a larger number of bins reduces bias, but means we must estimate a very large covariance matrix to achieve the required precision. This is currently the limiting step in LSS analyses, requiring many mock catalogs tailored to the survey, and

covariance matrix computation will further limit cosmological constraints as survey size increases.

More generally, binning adds arbitrary boundaries between continuous data; results should not depend on bin choice, yet they sometimes do. [Lanzuisi et al. \(2017\)](#) noted that the choice of binning axis impacts the detected correlation between the luminosity of active galactic nuclei and their host galaxies; [Grimmett et al. \(2020\)](#) devised a method to investigate this correlation in a continuous manner using a hierarchical Bayesian model, eliminating the need for binning. [Bailoni et al. \(2016\)](#) explored the dependence of clustering analyses on the number of redshift bins, finding a non-negligible difference in cosmological parameter uncertainties. The implications for BAO analyses were explored by [Percival et al. \(2014\)](#), who found that there is an optimal bin width given the analysis method. This balances the increasing statistical error with small bins and the offset in the derived BAO peak location with large bins; the effects are small but non-negligible. It is clear that, when analyzing smooth quantities such as LSS statistics, binning is sinning.

Estimators of the 2pcf have been studied extensively (e.g. [Peebles & Hauser 1974](#); [Davis & Peebles 1983](#); [Hamilton 1993](#)). The current standard estimator was proposed by [Landy & Szalay \(1993\)](#), hereafter LS. It is based on summing all data pairs DD with a given separation and using data-random pairs DR and random pairs RR to correct for the survey boundary. The LS estimator of the correlation function $\hat{\xi}_k$ for the k^{th} bin in separation r is

$$\hat{\xi}_k = \frac{DD_k - 2DR_k + RR_k}{RR_k}. \quad (1)$$

Compared with other estimators based on simple combinations of DD , DR and RR , LS has been shown to have the lowest bias and variance ([Kerscher et al. 2000](#)). Estimators of the 2pcf must also take into account the imperfect nature of the survey, including systematic effects, the target completeness, and fiber collisions. To account for these, each galaxy pair is sometimes assigned a weight, and pair counts are replaced by the sum of pair weights.

Variations on the random catalog pair count method have been proposed in recent years. [Demina et al. \(2016\)](#) replaced the DR and RR terms with an integral over the probability map, reducing computation time and increasing precision. An estimator proposed by [Vargas-Magaña et al. \(2013\)](#) iterates over sets of mock catalogs to find an optimal linear combination of data and random pair counts, reducing the bias and variance. An alternative estimator, the marked correlation function (e.g. [White & Padmanabhan 2009](#)), avoids the use of a random catalog altogether: it considers the ratio between the 2pcf and a weighted correlation function in which weights are assigned based on galaxy properties, such as the local density. These estimators have all taken probabilistic approaches; others have taken a likelihood approach. [Baxter & Rozo \(2013\)](#) introduced a maximum likelihood estimator for the 2pcf, which achieves

lower variance compared to the LS estimator, enabling finer binning and requiring a smaller random catalog for the same precision.

These estimators present improvements to LS, but they are still limited to estimates in separation bins. Some require additional computational costs or layers of complexity, so the standard formulation of LS continues to be the default estimator used in most analyses.

In this paper, we present a new estimator for the correlation function, the Continuous-Function Estimator, which generalizes the LS estimator to produce a continuous estimation of the 2pcf. The Continuous-Function Estimator projects the galaxy pairs onto a set of continuous basis functions and computes the best-fit linear combination of these functions. The basis representation can depend on the pair separation as well as other desired properties, and can also utilize the known form of the 2pcf. For top-hat basis functions, the Continuous-Function Estimator exactly reduces to the LS estimator. This estimator removes the need for binning and allows for the 2pcf to be represented by fewer basis functions, requiring fewer mock catalogs to compute the covariance matrix. It is particularly well-suited to the analysis of LSS features such as the BAO peak; we find that we can accurately locate the peak with fewer components.

This paper is organized as follows. In Section 2, we motivate our estimator and explain its formulation. We demonstrate its application on a simulated data set, including a toy BAO analysis, in Section 3. We discuss the implications and other possible applications in Section 4.

2. MOTIVATION AND FORMULATION

In this paper, we use the following notation. We write vectors in bold and lowercase, e.g. \mathbf{v} ; tensors in bold and uppercase, e.g. \mathbf{T} ; and unstructured data blobs in sans serif, e.g. \mathbf{G} . A hat above a symbol, e.g. $\hat{\xi}$, indicates an estimate of the value. *KSF says: where should this notation note go?*

2.1. Standard Two-Point Correlation Function Estimation

The standard approach to estimating the two-point correlation function involves counting pairs of tracers within a survey volume as a function of separation scale. Let's assume we have a data catalog with N_D objects within a sky volume. We also require a random catalog with N_R objects distributed uniformly throughout the same volume. We can define a set of separation bins which we will use to estimate the 2pcf at various scales. We are then ready to sum in each bin the relevant pairs of objects within and across our catalogs. In standard notation, these pair counts are written as DD , DR , and RR , as in Equation 1. To clarify that these are in fact vectors, with length K where K is the number of bins, we use the symbol \mathbf{v} ; then, for example, the data-data pair counts DD become \mathbf{v}_{DD} . We can then write the LS estimator as

$$\hat{\xi} = \frac{\mathbf{v}_{DD} - 2\mathbf{v}_{DR} + \mathbf{v}_{RR}}{\mathbf{v}_{RR}}. \quad (2)$$

The pair counts are defined explicitly as [KSF](#) says: should i define something like $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ so i don't have to keep writing it out? or better to be explicit throughout?

$$[\mathbf{v}_{\text{DD}}]_k \equiv \frac{2}{N_{\text{D}}(N_{\text{D}} - 1)} \sum_n \sum_{n'} i(g_k < |\mathbf{r}_n - \mathbf{r}_{n'}| < h_k) \quad (3)$$

$$[\mathbf{v}_{\text{DR}}]_k \equiv \frac{1}{N_{\text{D}}N_{\text{R}}} \sum_n \sum_m i(g_k < |\mathbf{r}_n - \mathbf{r}_m| < h_k) \quad (4)$$

$$[\mathbf{v}_{\text{RR}}]_k \equiv \frac{2}{N_{\text{R}}(N_{\text{R}} - 1)} \sum_m \sum_{m'} i(g_k < |\mathbf{r}_m - \mathbf{r}_{m'}| < h_k), \quad (5)$$

where $[\mathbf{v}]_k$ is the pair counts in bin k (which has bin edges g_k and h_k), i is an indicator function that returns 1 if the condition is true and otherwise returns 0, \mathbf{r} is the tracer position, the n and n' indices index data positions, and the m and m' indices index random catalog positions. We assume here that the sums are over unique pairs, and that for auto-correlations they exclude self-pairs; the normalization prefactors then account for the total number of possible pairs, explaining the difference between the auto- and cross-correlation factors. The tracer position can be in real or redshift space, or broken down into the transverse and line-of-sight directions in the anisotropic correlation function; in this paper we consider the isotropic real-space 2pcf for simplicity, but the estimators detailed here apply equally well to these alternative configurations. The estimator is also easily applicable to cross-correlations of two data sets.

The LS estimator is known to be optimal (i.e. it is unbiased and has minimum variance) under a particular set of conditions: in the limit of unclustered data, for a data volume much larger than the scales of interest, and an infinitely large random catalog. In practice the latter two limits are sufficiently satisfied, but the data we are interested in are clustered. [Vargas-Magaña et al. \(2013\)](#) show that for clustered data, the LS estimator has lower variance than other estimators, but does not reach the Poisson noise limit. When applied to clustered data, LS does show a bias on very large scales ($>130 h^{-1} \text{Mpc}$), but the bias is significantly smaller than that of most other estimators ([Kerscher 1999](#), [Vargas-Magaña et al. 2013](#)). LS is also less sensitive to the number of random points than other estimators ([Kerscher et al. 2000](#)). While LS has been sufficient for past analyses, its persisting bias and suboptimal variance under imperfect conditions mean that improvement is possible, and will be necessary for realistic large-scale structure measurements on modern datasets.

2.2. Least Squares Fitting

Estimating clustering is closely related to least-squares fitting. We are essentially trying to find the best representation of spatial data in the space of two-point radial separation. Recall that the linear least-squares fit to a set of data is

$$\mathbf{x} = [\mathbf{A}^{\text{T}} \mathbf{C}^{-1} \mathbf{A}]^{-1} [\mathbf{A}^{\text{T}} \mathbf{C}^{-1} \mathbf{y}] \quad (6)$$

where \mathbf{x} is the vector of best-fit parameters, \mathbf{A} is a design matrix with zeroth and first order terms (and possibly higher order) functions of fitting features, \mathbf{C} is the covariance matrix, and \mathbf{y} is a column vector of \mathbf{y} data to be fit. The second bracketed term projects the data onto the features; the first bracketed term renormalizes this quantity. [KSF says: renormalizes what? this phrase was from a Hogg comment, want to make sure it's clear here.](#) In the case of the 2pcf, the observed data is the pair counts at a given separation, and the weights are provided by the pair counts of the random catalog. Indeed, this is reminiscent of the so-called natural estimator of the 2pcf, $\xi = \mathbf{v}_{\text{DD}}/\mathbf{v}_{\text{RR}} - 1$ (e.g. [Kerscher et al. 2000](#)).

From this connection, we can infer the form of the estimator. [KSF says: What else to say in this section?? How to expand on what's here?](#)

2.3. The Continuous-Function Estimator

Inspired by least-squares fitting, we generalize the LS estimator defined above in Equations 3-5. We generalize the indicator function i to any function \mathbf{f} , which returns a vector of length K where K is the number of basis functions. We further generalize the arguments of the function to any properties of the galaxies, rather than just the separation between pairs; we call \mathbf{G} the data payload for a single galaxy. This gives us, instead of pair counts, a vector of amplitudes of the basis functions, \mathbf{v} . [KSF says: it's a bit confusing to call these amplitudes \(though they very much are\) because our \$\mathbf{a}\$ we're calling amplitudes; can we use an additional descriptor for the latter? correlation amplitudes?](#) These vectors are defined as

$$\mathbf{v}_{\text{DD}} \equiv \frac{2}{N_{\text{D}}(N_{\text{D}} - 1)} \sum_n \sum_{n'} \mathbf{f}(\mathbf{G}_n, \mathbf{G}_{n'}) \quad (7)$$

$$\mathbf{v}_{\text{DR}} \equiv \frac{1}{N_{\text{D}} N_{\text{R}}} \sum_n \sum_m \mathbf{f}(\mathbf{G}_n, \mathbf{G}_m) \quad (8)$$

$$\mathbf{v}_{\text{RR}} \equiv \frac{2}{N_{\text{R}}(N_{\text{R}} - 1)} \sum_m \sum_{m'} \mathbf{f}(\mathbf{G}_m, \mathbf{G}_{m'}) \quad (9)$$

$$\mathbf{T}_{\text{RR}} \equiv \frac{2}{N_{\text{R}}(N_{\text{R}} - 1)} \sum_m \sum_{m'} \mathbf{f}(\mathbf{G}_m, \mathbf{G}_{m'}) \cdot \mathbf{f}^{\text{T}}(\mathbf{G}_m, \mathbf{G}_{m'}) \quad (10)$$

where the summation notation and prefactors are the same as explained for the standard LS estimator.

We can now define the Continuous-Function Estimator as

$$\hat{\xi}(\mathbf{G}_l, \mathbf{G}_{l'}) \equiv \mathbf{a}^{\text{T}} \cdot \mathbf{f}(\mathbf{G}_l, \mathbf{G}_{l'}) \quad (11)$$

where \mathbf{a} is a K -vector of the computed *amplitudes* of the basis functions

$$\mathbf{a} \equiv \mathbf{T}_{\text{RR}}^{-1} \cdot (\mathbf{v}_{\text{DD}} - 2\mathbf{v}_{\text{DR}} + \mathbf{v}_{\text{RR}}) \quad (12)$$

and \mathbf{G}_l and $\mathbf{G}_{l'}$ contain the data values at which to evaluate $\hat{\xi}$. We emphasize that these are not real datapoints, but instead allow us to evaluate the 2pcf at any set

of parameters. In the standard case, G_l and $G_{l'}$ would effectively be an imaginary pair of galaxies that has a separate r at which we want to evaluate ξ , and we would compute $\hat{\xi}$ for such a pair at every separation we are interested in. With our general formulation, we could choose basis functions that depend on other galaxy properties, to investigate the effect of these on the 2pcf; then, we would also choose each G_l and $G_{l'}$ pair to have values of properties at which we want to evaluate $\hat{\xi}$. In the rest of this paper, however, we will only take into account the separation between pairs, and we will write $\hat{\xi}(r)$.

The Continuous-Function Estimator can be straightforwardly generalized to cross-correlations between two datasets. In this case, we consider datasets D_1 and D_2 , and associated random catalogs R_1 and R_2 . We then have cross-correlations rather than auto-correlations for the data-data and random-random terms, and two different data-random terms, crossing each dataset with the opposite random catalog. The data-data term becomes

$$\mathbf{v}_{D_1 D_2} \equiv \frac{1}{N_{D_1} N_{D_2}} \sum_{n_1} \sum_{n_2} \mathbf{f}(G_{n_1}, G_{n_2}) \quad (13)$$

where n_1 and n_2 index the data points in each catalog, and the normalization factor is now simply the product of catalog sizes as we are no longer concerned with double-counting. The other terms ($\mathbf{v}_{D_1 R_2}$, $\mathbf{v}_{D_2 R_1}$, $\mathbf{v}_{R_1 R_2}$, $\mathbf{T}_{R_1 R_2}$) generalize as one would expect. The amplitudes then becomes

$$\mathbf{a} \equiv \mathbf{T}_{R_1 R_2}^{-1} \cdot (\mathbf{v}_{D_1 D_2} - \mathbf{v}_{D_1 R_2} - \mathbf{v}_{D_2 R_1} + \mathbf{v}_{R_1 R_2}) \quad (14)$$

and we use this to compute the estimator as in Equation 11.

If we consider only the pair separation, and make a proper choice of \mathbf{f} , the Continuous-Function Estimator reduces to the Landy-Szalay estimator. Explicitly, from our full galaxy pair data G_n and $G_{n'}$, we can use only their separation, $|\mathbf{r}_n - \mathbf{r}_{n'}|$. We can then define a set of K basis functions \mathbf{f} as

$$\mathbf{f}_k(G_n, G_{n'}) = i(g_k < |\mathbf{r}_n - \mathbf{r}_{n'}| < h_k). \quad (15)$$

This is the common top-hat (or rectangular) function; the index k denotes a particular bin in separation, and here also indexes the basis functions, as each top-hat is a separate basis function. In this case the \mathbf{v}_{DD} , \mathbf{v}_{DR} and \mathbf{v}_{RR} vectors simply become binned pair counts, with bin edges g_k and h_k as before. The \mathbf{T}_{RR} tensor becomes diagonal, with its diagonal elements equal to the \mathbf{v}_{RR} vector elements. Then the evaluation of the amplitudes \mathbf{a} and the correlation function estimate $\hat{\xi}$ results in the equivalent of the LS estimator—just displayed in a continuous form.

We call this generalized 2pcf estimator the Continuous-Function Estimator. It replaces the binned pair counts of LS with any set of basis functions; the linear superposition of these basis functions is our estimate of the 2pcf. Essentially, the Continuous-Function Estimator outputs the best-fit linear combination of basis functions to describe the 2pcf. In this sense, it is deeply related to the linear least-squares

fitting described above. With our formulation, we no longer need to first bin our data and then fit a function; rather, the estimator directly projects the data (the pair counts) onto the desired function. This function can be nearly anything; the only limitation is it must be able to be written as a linear combination of basis functions.

With this generalized two-point estimator, the basis functions need not have hard edges like the top-hat function. They can instead be smooth functions of the pair separation, or chosen to suit the science use case. Further, the bases can make use of other information about the tracers or the survey; they are extremely general. The estimator also has the property that it is invariant under affine transformations, as it should be so that the result does not depend on e.g. the magnitude of the bases; we show this in Appendix A.

We can also write down the form of the Continuous-Function Estimator when we are working with a periodic box and don't need to worry about the survey window. In this case, we can analytically compute the \mathbf{v}_{RR} term, as well as the \mathbf{T}_{RR} term, and use the natural form of the 2pcf estimator. The derivation and formulation of these terms are shown in Appendix B.

KSF says: Discuss the limit of infinitesimal bins? Do we know what this is? Hogg says: Yes, we should discuss this, and show how we are related to that. KSF says: I'm not sure what to say about this.

We implement this estimator based on the correlation function package `Corrfunc` by Sinha & Garrison (2019). `Corrfunc` is the state-of-the-art package for computing correlation functions and other clustering statistics; it is extremely fast and user-friendly, and is used in many published analyses. It is also modular and open-source, making it a natural choice as a base for our implementation. Our implementation of the Continuous-Function Estimator is also open-source and available at github.com/kstoreyf/Corrfunc.

3. EXPERIMENTS AND RESULTS

3.1. Lognormal Mock Catalogs

We demonstrate the application of the Continuous-Function Estimator on a set of artificial data. We generate lognormal mock catalogs (Coles & Jones 1991) using the `lognormal_galaxies` code by (Agrawal et al. 2017). We use an input power spectrum with the Planck cosmology, the same parameters used for the MultiDark-PATCHY simulations (Kitaura et al. 2016) made for the Baryon Oscillation Spectroscopic Survey (Boss, Dawson et al. 2013). This assumes a cold dark matter model with $\Omega_m = 0.307115$, $\Omega_b = 0.048206$, $\sigma_8 = 0.8288$, $n_s = 0.9611$, and $h = 0.6777$. Our fiducial test set is 1000 realizations of periodic cubes with size $(750 h^{-1} \text{ Mpc})^3$ and a galaxy number density of 2×10^{-4} . We choose to perform these tests on periodic boxes so that we may compute the random-random term analytically (see Appendix B), significantly cutting down on computation time. We note, though, that we expect our results to hold for catalogs with realistic survey windows and random-random terms

computed directly with the Continuous-Function Estimator. *KSF says: I am realizing that it's a bit funny that our estimator hinges on the tensor random term, yet we are computing that analytically in all these tests. I think it's very fair, bc the formulation of the analytic \mathbf{v}_{RR} & \mathbf{T}_{RR} terms is based on our formulation. I don't think this is a reason to change everything now; but maybe i should include the analytic formulations above, and refer to the appendix for the full derivation?*

3.2. Comparison of Standard Tophat Basis Functions

We first estimate the correlation function of our mocks using the the standard estimator. We choose 15 separation (r) bins in the range $36 < r < 156 h^{-1} \text{ Mpc}$, each with a width of $8 h^{-1} \text{ Mpc}$; this was found to be the optimal bin width by Percival et al. (2014), and is standard for two-point analyses. We apply the estimator to each of our 1000 mock catalogs. The mean of these estimated correlation functions is shown in Figure 1 (grey squares); the error bars show the standard deviation of the 1000 mocks in each bin. *KSF says: do i note the legend info here, or just in figure caption?* We also show the true input correlation function (thin black line), and the bottom panel shows the absolute error between the estimated and true correlation functions.

There remains an ambiguity in the r -value at which to plot the result of the standard estimator. The volume-weighted average is often used, or a weighted average depending on the pairs in the bin; this choice propagates to differences in comparing the estimate to models (though at the precision of current surveys these differences are not significant). Here we plot the standard estimator with the volume-weighted average.

We demonstrate the Continuous-Function Estimator with a tophat basis function as a check. We choose tophat functions with the same locations and widths as the bins used for the standard estimator; these are shown in the top panel of Figure 1. As this estimator computes the 2pcf in a continuous form, we plot the result as a continuous function at every r value (thin blue line). This results in a step function form for the correlation function, which is in fact what the typical estimator is computing. The values of the correlation function at each step exactly align with the result of the standard estimator. In fact, we emphasize that this step function is exactly what the standard estimator is estimating; we have just made explicit the fact that the each estimate is for the entire bin. When we look at the error with respect to the truth (bottom panel), the error blows up at the edges of each bin, where the continuous estimate deviates most significantly from the truth. This demonstrates that the standard tophat estimator is not a good representation of the true 2pcf.

3.3. Demonstration using Spline Basis Functions

A natural extension of tophat basis functions is the B-spline. B-splines of order n are piecewise polynomials of order $n - 1$; they constitute the basis functions for spline interpolation. *KSF says: do i need a citation here? would be a textbook probs...* They have the nice property that the functions and their derivatives can be continuous,

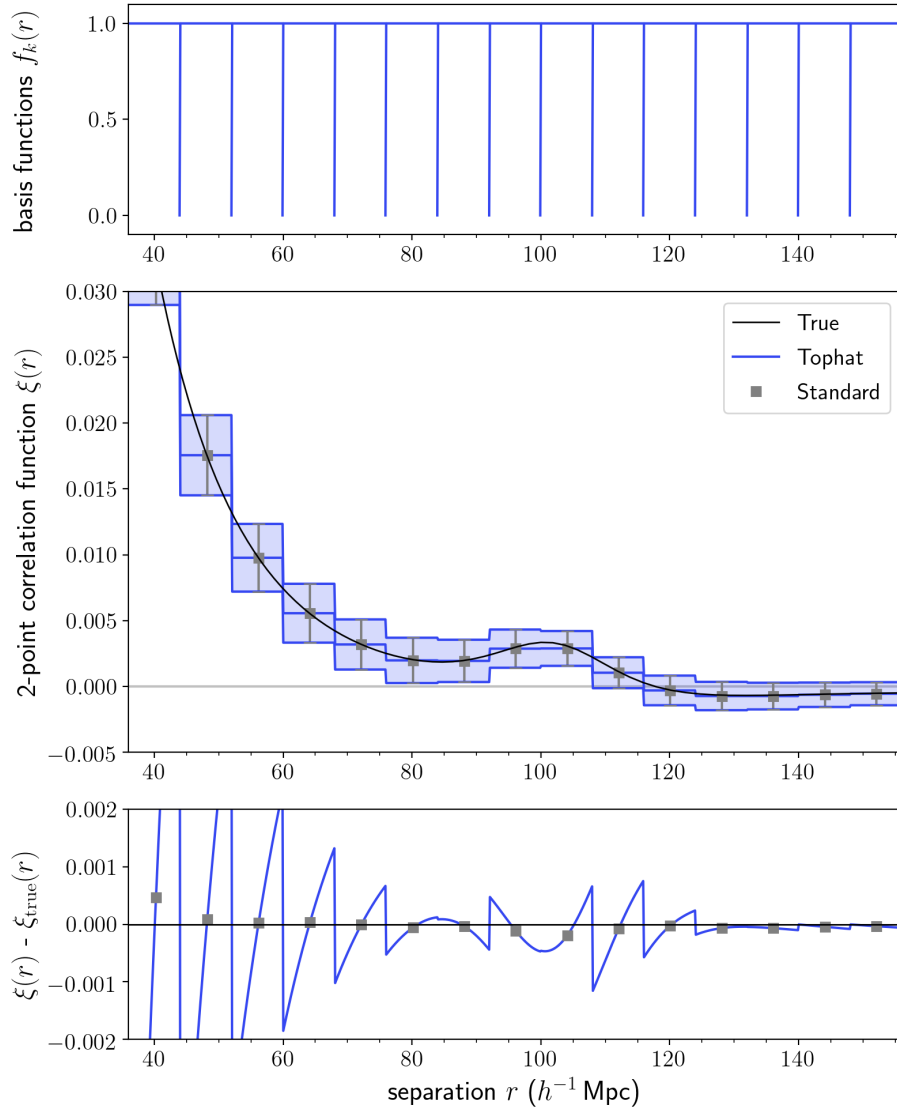


Figure 1. A comparison between the Continuous-Function Estimator with a tophat basis (thin blue lines) and the standard estimator (grey squares). The top panel shows the basis functions used for the tophat estimator. The middle panel shows the mean of the estimated correlation functions for 1000 mock catalogs, compared to the true input 2pcf (thin black line); the shaded region and errorbars are the standard deviation of the 2pcf estimate. The lower panel shows the absolute error between the estimate and true 2pcf. The Continuous-Function Estimator with a tophat basis is exactly equivalent to the standard estimator, but in a continuous form, emphasizing the fact that binning results in a poor representation of the true 2pcf.

depending on the order. Further, B-splines are well-localized, which provides a more direct comparison to the typical tophat basis (which is entirely localized). For this demonstration we use fourth-order B-splines, which constitute the set of basis functions for a cubic spline, as they are the lowest-order spline to have a continuous first derivative.

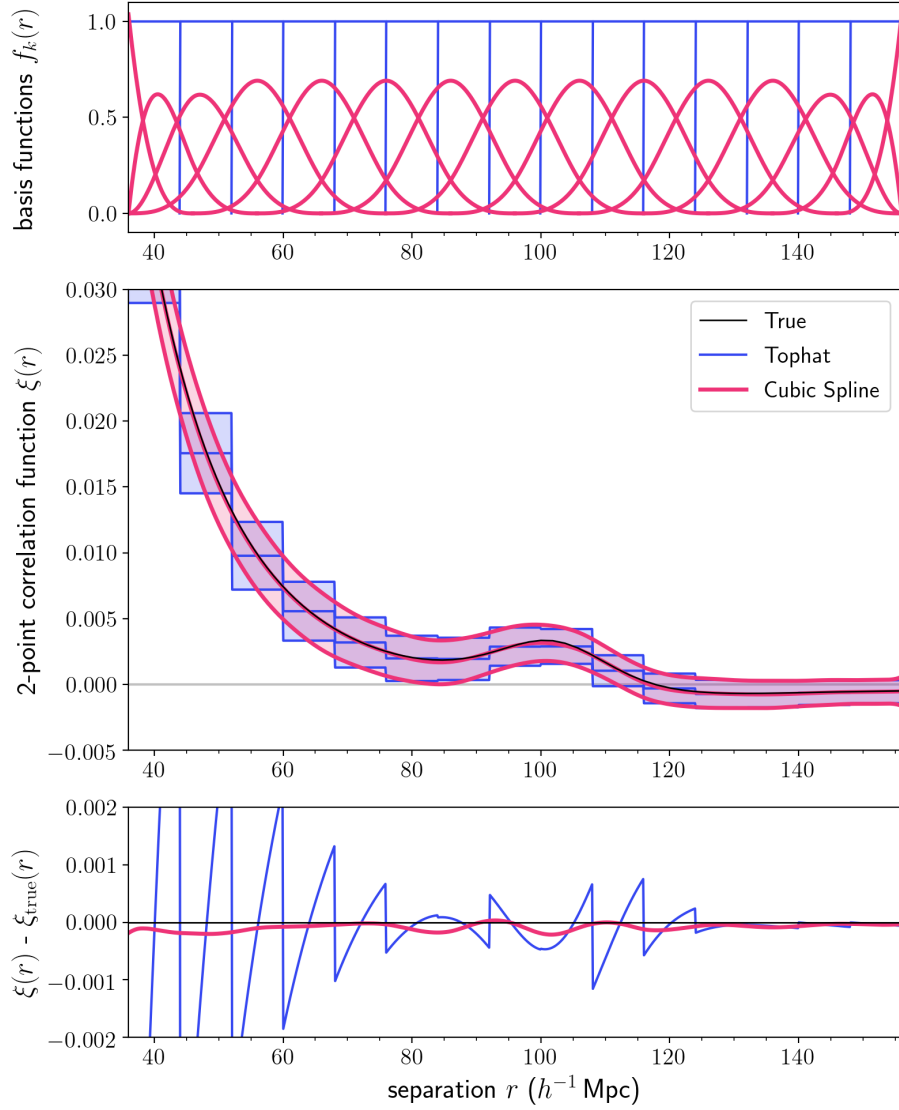


Figure 2. A comparison between the Continuous-Function Estimator with a cubic spline basis function (thick red) and a standard tophat basis (thin blue). The top panel shows the basis functions used for each measurement. The middle panel shows the mean of the estimated correlation functions for each of the 1000 mock catalogs compared to the true input 2pcf (thin black); the shaded region is the standard deviation. The lower panel shows the absolute error between the estimate and true 2pcf. It is clear that the spline basis function results in a correlation function that is a better representation of the true 2pcf in its shape and smoothness.

We compare the estimator with a cubic spline basis to the standard estimator, reformulated as continuous functions using a tophat basis; the results are shown in Figure 2. The basis functions are shown in the top panel of the figure. We use the same tophat basis as above (thin blue). For the cubic spline basis, we use the same r -range and number of basis functions, and knots chosen to evenly span the range (thick red). The cubic spline bases on the edge have different shapes such that they remain normalized; we note that generally, one should choose the basis functions such

that the range of the 2pcf that is of interest does not depend on the range of the basis functions.

The estimator using the cubic spline basis clearly produces a better fit to the true correlation function (thin black) in its shape and smoothness at every point across the scale range, compared to the estimator using the tophat basis. The bottom panel shows the error with respect to the truth; the cubic spline estimator is more generally more accurate, and straightforward to compare to the truth (or model) at every scale. On the other hand, in order to compare the binned correlation to a model, one must integrate the model over the bin range, though in practice the model is often just evaluated at the effective r of each bin. This comparison demonstrates that there exist other sets of basis functions that produce better representations of the data compared to the standard tophat/binned estimator. The choice of a high-order spline may be useful for cases in which one wants a mostly localized yet representative estimate of the 2pcf, or smooth derivatives. Generally, the choice of basis functions should be tailored to the scientific use case; in the next section we explore the case of a BAO analysis.

3.4. BAO Scale Estimation Test

The measurement of the baryon acoustic oscillation (BAO) scale provides an apt use case for our estimator. The BAO feature is a peak in clustering on large scales, ~ 150 Mpc ($\sim 100h^{-1}$ Mpc), making it less sensitive to small-scale astrophysical effects. It is one of the best tools for constraining cosmological models, in particular the distance-redshift relation (Kazin et al. 2010; Anderson et al. 2011, 2014; Alam et al. 2016). [KSF says: what else do I need to cite for BAO?](#)

We base our BAO analysis on the method of the BOSS DR10 and 11 analysis (Anderson et al. 2014). We estimate the spherically averaged 3-dimensional correlation function, $\hat{\xi}(r)$, where r is the separation between pairs. (BAO analyses are typically done in redshift space, estimating $\hat{\xi}(s)$, where s is the redshift-space separation between pairs, but here we are using a periodic box in which we know the true galaxy positions, so we just use the real-space distance r .) In order to extract information about the baryon acoustic feature from galaxy clustering, we must choose a fiducial cosmological model to convert redshifts to distances. If we choose an incorrect model, the scales in the power spectrum will be dilated, so the oscillation wavelength—and thus the BAO peak position—will be shifted. We can model this shift as a scale dilation parameter, α , which is a function of the relevant distance scales in the true and fiducial cosmologies, defined as

$$\alpha = \left(\frac{D_A(z)}{D_A^{\text{mod}}(z)} \right)^{2/3} \left(\frac{H^{\text{mod}}(z)}{H(z)} \right)^{1/3} \left(\frac{r_s^{\text{mod}}}{r_s} \right) \quad (16)$$

where D_A is the angular diameter distance, H is the Hubble parameter, z is the redshift, r_s is the sound horizon scale at the drag epoch, and the superscript “mod”

denotes the value for the chosen fiducial model (the non-superscripted parameters are the true values). Qualitatively, if the fit prefers $\alpha > 1$, this suggests the true position of the BAO peak is at a smaller scale than in the fiducial model, whereas if $\alpha < 1$, the peak is at a larger scale. With isotropic analyses, there is a degeneracy between D_A and H , so typically a combination of these values is reported; the degeneracy can be broken with anisotropic BAO analyses. Our estimator could straightforwardly perform an estimate of the anisotropic correlation function, but for demonstration purposes we perform an isotropic analysis here and focus on the recovered value of α .

In standard practice, the fitting function used to determine the value of α is [KSF](#) says: should ξ here should be bold? different than earlier because here it's written as a function of r - so for a given r , it returns a single value. but given a set of r values, it's a vector...

$$\xi^{\text{fit}}(r) = B^2 \xi^{\text{mod}}(\alpha r) + \frac{a_1}{r^2} + \frac{a_2}{r} + a_3 \quad (17)$$

where B is a constant that allows for a large-scale bias, and a_1 , a_2 , and a_3 are nuisance parameters to account for the broadband shape. A χ^2 fit is performed with five free parameters: α , B , a_1 , a_2 , and a_3 . [KSF](#) says: **TODO: KSF is figuring out how this actually done and flesh out this paragraph!** The resulting value for α is used to derive the actual values of the distance scales of interest. Typically, density-field reconstruction is performed before applying the estimator to correct for nonlinear growth around the BAO scale ([Eisenstein et al. 2007](#)); for our toy example, we omit this step.

The form of the standard fitting function is well-suited to our estimator, as it is a few-parameter model with a linear combination of terms. To use our estimator to estimate α , we add a term that includes the partial derivative of the model with respect to α . This allows us to have fixed basis functions, and for an initial choice of α_{guess} , determine the change in this value needed to improve the fit. Our fitting function is then

$$\xi^{\text{fit}}(r) = B^2 \xi^{\text{mod}}(\alpha_{\text{guess}} r) + C k_0 \frac{d\xi^{\text{mod}}(\alpha_{\text{guess}} r)}{d\alpha} + a_1 \frac{k_1}{r^2} + a_2 \frac{k_2}{r} + a_3 k_3, \quad (18)$$

where C is an additional coefficient that describes the contribution of the derivative term, and k_0 , k_1 , k_2 , and k_3 are constants that determine the initial amplitude of the basis functions. In this case, the free parameters are B^2 , C , a_1 , a_2 , and a_3 . Note that in theory the choice of k_i values shouldn't matter as the estimator is affine invariant (see [Appendix A](#)), but in practice reasonable choices are important for stability. The adopted k_i values are noted in [Appendix C](#)

To use the estimator for a BAO measurement, we input these five terms as the five basis functions of our estimator. The estimator outputs an amplitude vector \mathbf{a} as described in [Section 2.3](#), which describes the contribution of each basis function—precisely the values of the free parameters, scaled by k_i . From the value of C , we can determine our estimate of the scale dilation parameter, $\hat{\alpha}$, as $\hat{\alpha} = \alpha_{\text{guess}} + C k_0$,

based on the definition of finite derivatives. With this formulation, a value of $C = 0$ indicates that the current α_{guess} gives the best fit to the data (given the chosen cosmological model), while nonzero values give the magnitude and direction of the necessary change in the scale dilation parameter to optimally fit the data. In practice, we apply an iterative procedure to converge at our best estimate $\hat{\alpha}$; this procedure and other implementation details are described in Appendix C.

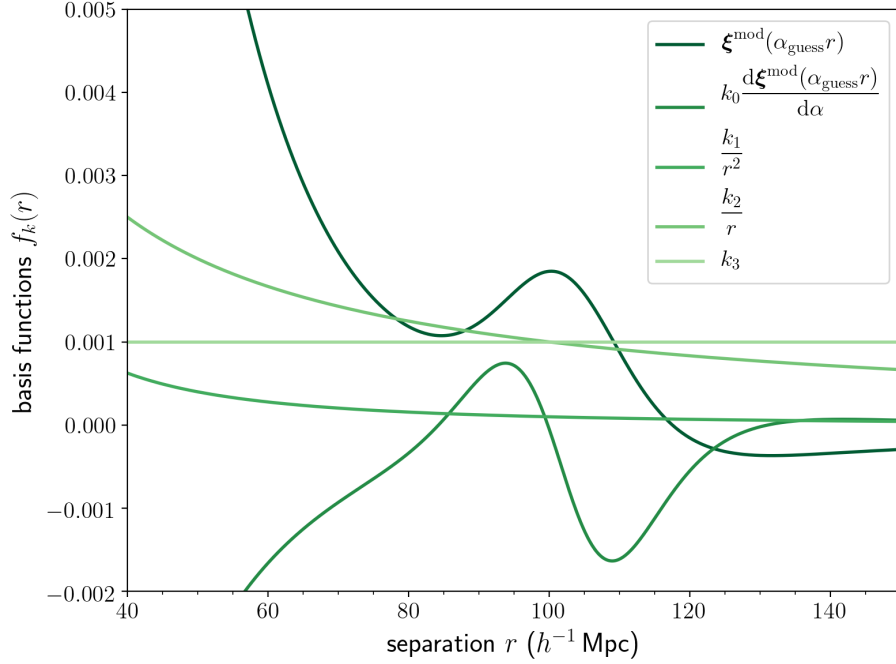


Figure 3. The set of basis functions used to fit for the BAO scale using our estimator. The $\xi^{\text{mod}}(\alpha_{\text{guess}} r)$ term (darkest green) is the correlation function computed using fiducial model. The $\frac{d\xi^{\text{mod}}(\alpha_{\text{guess}} r)}{d\alpha}$ term (second-to-darkest green) is the derivative of this model with respect to α , which allows for the direct estimation of this scale dilation parameter. *KSF says: what do about this inline fraction?* The other three terms (lighter greens) are nuisance parameters to fit the broadband shape. Here we show the basis functions with arbitrary scaling. *KSF says: as these are shown scaled slightly differently than the k_i values adopted for the fitting, should i still show the k s in the legend? or explain this instead of just saying “arbitrary scaling”? KSF says: should i leave “guess” in the legend labels, or remove for simplicity?*

We demonstrate this method using our set of lognormal mock catalogs. We construct a recovery test following that in Hinton et al. (2019). We assume the fiducial cosmological model used in Beutler et al. (2017): $\Omega_{\text{m}} = 0.31$, $h = 0.676$, $\Omega_{\text{b}} = 0.04814$, $n_s = 0.97$. As we know the cosmology used for our mock catalogs, we can compute the true value of the scale dilation parameter, $\alpha_{\text{true}} = 0.9987$. (Here our choice of fiducial model happened to be close to the true model, so our α_{true} is very close to 1; this is typical, as our cosmological model is fairly well-constrained.) With this fiducial model, we can construct the basis functions for our estimator; these are shown (with $\alpha = 1$ and arbitrary scaling) in Figure 3.

We apply our iterative estimation procedure to each of the 1000 mocks; the mean of the resulting estimates for the correlation function is shown in Figure 3 (thick green). We show the BAO basis functions in the top panel, as in Figure 3. We compare the Continuous-Function Estimator using the BAO bases with that using tophat bases (thin blue), as in the previous sections. The correlation function estimated with the BAO bases clearly produces a more representative estimate at all scales. The estimate is also smoother than that produced using the cubic spline basis functions (Figure 2). More importantly, it is scientifically motivated: the estimator directly gives us the relative contributions of the terms of the BAO fitting functions. Further, the BAO-based estimate requires only five components, while the tophat basis requires 15 components (or bins, in the standard approach) in the same scale range. This is critical for the efficient computation of a precise covariance matrix, as the errors depend on the number of components used for the estimate, as described in Section 4.4. The Continuous-Function Estimator with the BAO basis functions could reduce the number of mocks needed to achieve the same precision by a factor of a few to an order of magnitude; as these expensive cosmological simulations are currently the limiting step in two-point analyses, this could be highly impactful.

We note that these basis functions are significantly different than the tophat or B-spline bases previously explored, mainly because they are not localized. This means that data at all scales could contribute to all basis functions. It is then critical to ensure that the final parameter estimate does not rely on the range of scales chosen. We have confirmed that in this application, the result is robust to the chosen range as long as the scales cover the range where the feature of interest sits. *KSF says: Need to check this again & update*

KSF says: I will flesh out this paragraph once I perform the traditional method! The Continuous-Function Estimator with the BAO bases gives us an estimate for α for each of the 1000 mock catalogs. These are shown in comparison with the values estimated using the traditional fitting approach in Figure ZZZ. *KSF says: update* The methods produce similar values of α ... The mean estimate of the recovered scale dilation parameter is $\alpha = XXX \pm YYY$, very close to the true value of $\alpha = 0.9987$. The mean estimate using the traditional approach is $\alpha = XXX \pm YYY$. Our estimate is more accurate and precise ...

4. DISCUSSION

4.1. Relationship to Existing Estimators

The Continuous-Function Estimator has properties similar to existing estimators, including kernel density estimators and the marked correlation function. With the proper choice of basis functions, the Continuous-Function Estimator can in fact produce both of these estimators; it is more general than either of them.

Kernel density estimation (KDE) is a class of methods for estimating a probability density function from a set of data. KDE methods essentially smooth the data with

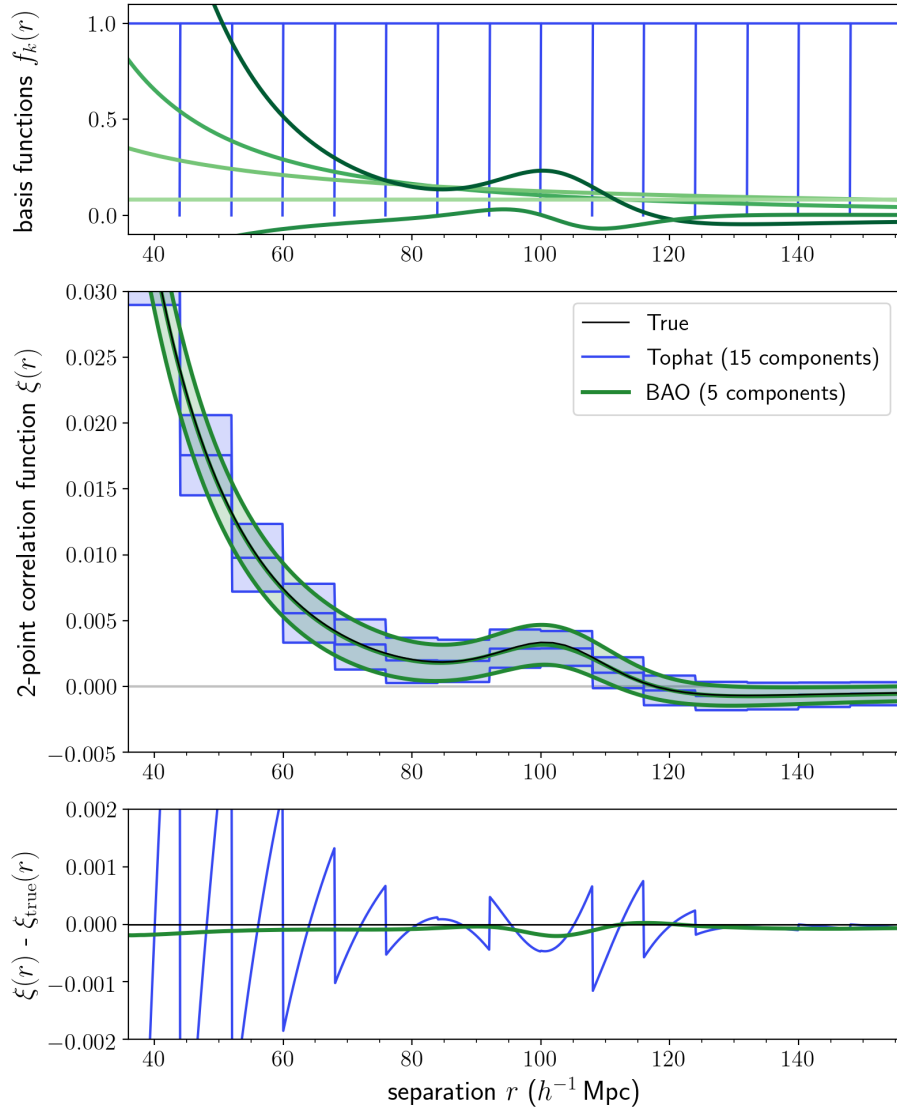


Figure 4. Estimation of the correlation function using our estimator with basis functions based on the BAO fitting function (orange dot-dashed). The line is the mean of the final estimate from the iteration procedure for 1000 mocks, and the shaded region is the 1σ variation. We also show the standard Landy-Szalay estimator, displayed as a tophat function (blue), as well as the true input correlation function (black). The basis functions are the same as those shown in Figure 3, with arbitrary scaling.

a given kernel, often a Gaussian. *KSF says: should i cite the original KDE papers from the 50s and 60s? or not necessary bc not the focus?* This is useful when we want to reconstruct a distribution without making many assumptions about the data, as is required in parametric methods. KDEs have found use in many areas of astrophysics, for example to measure the 21cm power spectrum with reduced foreground contamination (Trott et al. 2019), and to estimate luminosity functions with superior performance compared to binned methods (Yuan et al. 2020). *KSF says: there are others, should i list more citations? without describing?* Hatfield et al. (2016) uses

a KDE approach to estimate the angular correlation function, in order to address the issues of information loss and arbitrary bin choice inherent to binning; they optimize for the kernel choice, and find a correlation function consistent with that of the binned method. [KSF says: I couldn't find any other papers that use KDEs on correlation functions, but I found this one by chance. more to say here? the paper doesn't draw strong conclusions from this.](#)

Specifically, kernel density estimators take the contribution of each data point to be a kernel function centered on that value, and sum these to determine the full distribution. In contrast, the Continuous-Function Estimator projects each data point onto fixed basis functions, which are distinct from the typical understanding of kernels. As such, our estimator is not smearing out the data, as KDEs do; it is using the data to directly infer the contribution of each basis function. This preserves the information in the data to the degree given by the chosen set of basis functions, which can in fact enhance features rather than smooth them. That said, the formulation of the Continuous-Function Estimator is general enough that it can perform a kernel density estimate of the correlation function, by choosing $f(r)$ to be a kernel centered on r . However, our uses here of the estimator are fundamentally different from KDE methods, as they use fixed basis functions that can take advantage of the science use case and preserve maximal information from the data. [KSF says: this last sentence is a bit repetitive / are there more differences im missing?](#)

Another method that shares similarities with the Continuous-Function Estimator is the marked correlation function (MCF, [Beisbart & Kerscher 2000](#); [Sheth 2005](#)). This estimator weights the two-point statistic by “marks,” which are typically properties of the tracers. The MCF is useful for studying the connection between galaxies and their spatial clustering. [Skibba et al. \(2006\)](#) used it to determine that luminosity-dependent clustering is a straightforward consequence of mass dependence. [Armijo et al. \(2018\)](#) applied the MCF to test a class of modified gravity theories by marking with local density, demonstrating that there is additional information in the environmental dependence of clustering. The MCF has also been shown to break the degeneracy between halo occupation distribution parameters and the cosmological parameter σ_8 ([White & Padmanabhan 2009](#)).

The Continuous-Function Estimator can straightforwardly incorporate the idea of marks by choosing the basis functions to be functions of the desired properties of the tracer in addition to pair separation. Combined with the choice of tophat basis functions and proper normalization, this would be equivalent to the MCF. However, the Continuous-Function Estimator can generalize this concept even further. Rather than still producing a two-point function that is only a function of separation, weighted by the marks, our estimator can elevate the marking properties to another continuous axis. That is, it can estimate a multi-dimensional correlation function as a function of both separation and the given property. This provides a more flexible way to look at the dependence of the 2pcf on the property, and has similar applicability to break-

ing parameter degeneracies. We elaborate on the use cases for incorporating further tracer information into the choice of bases functions in Section 4.5.

4.2. Beyond the Landy-Szalay Estimator

While we have formulated our estimator as a generalization of LS, as this is the standard used in 2pcf analyses and has optimal properties under certain conditions, we can also reformulate it for other estimators. Our formulation currently requires a normalization term (i.e. denominator) based on the random-random counts; for LS we replace this with our \mathbf{T}_{RR} term (Equation 10). This is also the case for the Peebles & Hauser (1974) (natural) estimator and the Hewett (1982) estimator:

$$\hat{\xi}_{\text{PH}} = \frac{\mathbf{v}_{\text{DD}} - \mathbf{v}_{\text{RR}}}{\mathbf{v}_{\text{RR}}} \rightarrow \mathbf{T}_{\text{RR}}^{-1} \cdot (\mathbf{v}_{\text{DD}} - \mathbf{v}_{\text{RR}}) \quad (19)$$

$$\hat{\xi}_{\text{Hew}} = \frac{\mathbf{v}_{\text{DD}} - \mathbf{v}_{\text{DR}}}{\mathbf{v}_{\text{RR}}} \rightarrow \mathbf{T}_{\text{RR}}^{-1} \cdot (\mathbf{v}_{\text{DD}} - \mathbf{v}_{\text{DR}}). \quad (20)$$

We can also straightforwardly generalize estimators which have a data-random cross-correlation as the normalization term, such as the Davis & Peebles (1983) estimator, KSF says: yikes the DP subscript is confusingly similar to D for data, not Davis... is it fine?

$$\hat{\xi}_{\text{DP}} = \frac{\mathbf{v}_{\text{DD}} - \mathbf{v}_{\text{DR}}}{\mathbf{v}_{\text{DR}}} \rightarrow \mathbf{T}_{\text{DR}}^{-1} \cdot (\mathbf{v}_{\text{DD}} - \mathbf{v}_{\text{DR}}) \quad (21)$$

where we define

$$\mathbf{T}_{\text{DR}} = \frac{2}{N_{\text{D}} N_{\text{R}}} \sum_n \sum_m \mathbf{f}(\mathbf{G}_n, \mathbf{G}_m) \cdot \mathbf{f}^{\text{T}}(\mathbf{G}_n, \mathbf{G}_m). \quad (22)$$

This formulation could be extended to nearly any linear combination of pair counts. The estimator of Vargas-Magaña et al. (2013), for instance, selects the optimal combination of pair counts; our estimators could be combined to create an even more generalized estimator. KSF says: should i mention the fact the other possibilities for the normalization term? e.g. some of the terms in V-M include DD in the denom, and there's an estimator with DRsquared in the denom. but i think these are straightforward to reformulate in our generalization? We note that the continuous form of these estimators can be extended to cross-correlations in a straightforward way as expected.

4.3. Implementation and Computational Performance

We implement the Continuous-Function Estimator within the correlation function package Corrfunc KSF says: cite, texttt. The package is written in C with python bindings and utilities. For every computation of the pair separation, we additionally pass the separation and any additional tracer information to the user-defined basis function. We output the traditional pair counts as well as our component vector and, if desired, the component tensor.

The computational scaling for our estimator is by definition the same as the traditional method, as pair-finding remains the limiting factor. However, because the

Continuous-Function Estimator must evaluate the set of basis functions for each pair of galaxies, it can take significantly longer. For simple basis functions like splines, this will only marginally decrease performance. For more complicated functions, the Continuous-Function Estimator may incur significant extra computational expense. Basis functions can also be input on a grid (of separation or any other property) and then interpolated; the performance is then similar for all functions, depending on how the interpolation is done, but interpolating each function for each pair does somewhat decrease the performance. Though the performance at the time of estimation may be slower than the traditional estimator, the choice of basis may significantly save computational time in other areas, such as reducing the number of mock catalogs required for covariance matrix estimation; see Section 4.4.

We detail a number of implementation choices here. Our formulation of the Continuous-Function Estimator requires the inverse of the random-random tensor \mathbf{T}_{RR} to compute the amplitudes (Equation 12). However, we don't compute this inverse directly, as can be unstable, and is not in fact the end result we are interested in: we want the dot product between $\mathbf{T}_{\text{RR}}^{-1}$ and the numerator \mathbf{v} of the estimator. For this reason, we use the “solve” operation which computes the solution \mathbf{a} of the well-determined matrix equation $\mathbf{T}_{\text{RR}} \mathbf{a} = \mathbf{v}$.

KSF says: What else should go in this section?

4.4. Effect on Covariance Matrix Estimation

We have shown that the Continuous-Function Estimator results in 2pcf estimates that are just as accurate with fewer components. This is critical when estimating the covariance matrix, which is necessary for parameter inference. The covariance matrix is difficult to compute analytically, though there is promising progress on this front (e.g. Wadekar et al. 2020). For major analyses, it is usually estimated by evaluating the 2pcf on a large number of mock catalogs and computing the covariance between the bins (e.g. Reid et al. 2010; Anderson et al. 2014). The unbiased estimator for the sample covariance matrix is (e.g. Anderson 2003)

$$\hat{C}_{ij}^{\text{ML}} = \frac{1}{N_{\text{mocks}} - 1} \sum_{q=1}^{N_{\text{mocks}}} \left([\xi_q]_i - \bar{\xi}_i \right) \left([\xi_q]_j - \bar{\xi}_j \right)^{\text{T}}, \quad (23)$$

where q denotes the index of the mock, i and j denote the index of the bin or component, ξ denotes the estimate in that bin for that mock, and $\bar{\xi}$ denotes the mean value of the estimate in that bin across the mocks, where we have omitted the hat for clarity. KSF says: i removed the hats here because they didn't play nicely w the average bar; thoughts? To get an unbiased estimate of the inverse covariance matrix, we require a correction factor, as the inverse of an unbiased estimator is not necessarily unbiased. The unbiased estimator for the sample inverse covariance matrix can be shown to be (Hartlap et al. 2007)

$$\hat{C}^{-1} = \frac{N_{\text{mocks}} - N_{\text{bins}} - 2}{N_{\text{mocks}} - 1} \left(\hat{C}^{\text{ML}} \right)^{-1}. \quad (24)$$

The variance in the elements of this estimator then have a dependence on N_{mocks} and N_{bins} . This propagates to the derived cosmological parameters, resulting in an overestimation of the error bars (Hartlap et al. 2007; Dodelson & Schneider 2013 Percival et al. 2014; Taylor & Joachimi 2014). Assuming that $N_{\text{mocks}} \gg N_{\text{bins}}$ (and both much larger than the number of parameters to be estimated), and that the measurements are Gaussian distributed, the error bars are inflated by a factor of $(1 + N_{\text{bins}}/N_{\text{mocks}})$ (i.e., the true constraints are tighter than the derived ones). This factor becomes critical at the precision of cosmological parameter estimation (Percival et al. 2014).

Typically, this is dealt with by generating a very large number of mocks. For the Baryon Oscillation Spectroscopic Survey (BOSS, Dawson et al. 2013) DR9 analysis, 600 mocks were needed and the two-point correlation function used 41 bins (Sánchez et al. 2012) (though they also perform a restricted 15-bin analysis over the BAO peak scales). For the BOSS DR14 fiducial 2pcf results, 1000 mocks and 18 bins were used Ata et al. (2017). Future surveys will have more costly requirements on mock catalogs, with larger simulations necessary to cover the larger survey volumes. KSF says: I went down some rabbit holes trying to find the numbers for DES and other surveys, it's not as clear. Is it ok as is or should I keep digging? KSF says: it turns out that at least for DR14, some approximate method was used, so they weren't full cosmo sims and didn't take that long. should i try to figure out which analyses still used/use full sims to cite? any leads here?

An alternative to increasing N_{mocks} is decreasing N_{bins} to achieve the same error on precision. In the standard method, this is shown to *increase* the statistical error, albeit only slightly (Percival et al. 2014). A substantial increase in bin width would prevent capturing information in finer clustering features; even the relatively broad BAO peak requires a bin size on the order of its width of $\sim 10h^{-1}$ Mpc. In fact, in the standard method more bins would typically be desirable, but the number is limited by the available number of mocks for covariance matrix computation.

We have shown that we can use the Continuous-Function Estimator to estimate the 2pcf using fewer components and without sacrificing accuracy. This means that we can safely reduce N_{bins} , or in our case, the number K of components (or basis functions). The covariance matrix will be the covariance between these components. To then achieve the same precision on the error on the cosmological parameters, a lower value of N_{mocks} becomes possible. KSF says: I think the result of discussions was that there wasn't a good way of showing this without propagating all the way to cosmological parameters. Would love a figure showing lower covariance errors but not sure how without full propagation This will significantly reduce requirements on mock catalog construction, which will be particularly important for upcoming large surveys.

Another issue with the covariance of the standard estimator is that the uncertainty is highly correlated across bins. Thus the diagonal terms of the covariance matrix are

poor representations of the true error on each bin. The errors can be decorrelated by choosing a new estimator that is a linear combination of the original 2pcf bins. [Hamilton & Tegmark \(2000\)](#) proposed a transformation using the symmetric square root of the Fisher matrix, and this was shown in [Anderson et al. \(2014\)](#) to significantly suppress the off-diagonal elements of the covariance matrix. While this decorrelated covariance matrix is not used in the fitting in that analyses, it is useful for visualizing the uncertainty of the 2pcf estimates. The Continuous-Function Estimator could also be used to obtain a decorrelated covariance matrix. One could perform an initial estimation with standard bins or basis functions, and then apply a transformation to decorrelation them. These decorrelated bins could then be passed to the estimator as basis functions, and the analysis run again, in order to obtain a direct estimation that produces representative diagonal errors. This would be particularly important for unlocalized basis functions such as the BAO basis functions, which will have highly correlated errors.

4.5. Further Applications

The formulation of the Continuous-Function Estimator opens up many possibilities for extracting information from the correlation function. The most straightforward applications are standard basis functions or linearizeable astrophysical models, as we have shown here. Other applications for the direct estimation of cosmological parameters could include the growth rate of cosmic structure f ([Satpathy et al. 2016](#); [Reid et al. 2018](#)) and primordial non-Gaussianity in the local density field f_{NL}^{local} ([Karagiannis et al. 2014](#)). *KSF says: mention idea of doing full cosmo model analysis by taking derivs wrt cosmological params? cool but less connected to citeable papers perhaps. or do you know any relevant papers to look at?*

We can take our estimator a step further by choosing basis functions that depend not only on the separation between tracer pairs, but also on the properties of the tracers themselves. One such application is the redshift dependence of the Alcock-Paczynski effect ([Alcock & Paczynski 1979](#)), which can be used to constrain the matter density Ω_m and the dark energy equation of state parameter w ([Li et al. 2016](#)). The basis functions f in this case would take the form

$$f_k(\mathbf{G}_n, \mathbf{G}_{n'}) = f_k(|\mathbf{r}_n - \mathbf{r}_{n'}|, z_n, z_{n'}), \quad (25)$$

where z is the redshift of tracer n or n' . Another potential use case is the luminosity and color dependence of galaxy clustering, which can be used to understand the relationship between galaxy formation and the LSS (e.g. [Zehavi et al. 2011](#)). *KSF says: do i need to cite more here?* This could be extended to other galaxy properties; the Continuous-Function Estimator provides the flexibility to explore a high-dimensional parameter space, while binned methods become quickly limited by number statistics as one tries to include more parameters.

Beyond these standard use cases, the estimator gives us the opportunity to investigate more subtle or exotic signals which are anomalous with respect to our conven-

tional models. Anomalies could appear as inhomogeneities or anisotropies in the data. For example, [Mukherjee & Wandelt \(2018\)](#) investigated whether there is a directional dependence in estimated cosmological parameters across the sky, by performing analyses on patches of the Cosmic Microwave Background. Another possibility is anisotropy in the cosmic acceleration, which could leave signatures in measurements made using various phenomena including baryon acoustic oscillations ([Faltenbacher et al. 2012](#)) and Type Ia supernovae ([Colin et al. 2019](#)). With our estimator, we could introduce a dependence on location or direction into our basis functions, and constrain the potential deviation from homogeneity or isotropy. The Continuous-Function Estimator would allow for a more precise estimate of this dependence as it doesn't require any sort of patches or spatial binning, instead estimating a multi-dimensional continuous 2pcf. While these effects would be highly degenerate with systematics, our estimator combined with robust systematics mitigation opens investigation channels into the possibility of new physics.

Finally, our estimator can be directly related to a power spectrum analysis. We could choose a Fourier basis as our set of continuous functions. This would allow us to directly project the data onto Fourier modes. This represents a step towards unifying the correlation function and the power spectrum. [KSF says: need to elaborate here; pointers on what to say appreciated!](#)

5. SUMMARY

[KSF says: TODO: write short summary. probably. if we decide it is useful and not too repetitive.](#)

KSF was supported by the NASA FINESST grant [grant number] during the completion of this work. The authors thank Jeremy Tinker and Michael Blanton for helpful discussions, Roman Scoccimarro for insightful conversation, and the members of the Flatiron Astronomical Data Group for useful feedback. KSF would like to acknowledge significant code feedback and support from Manodeep Sinha, as well as Lehman Garrison. KSF thanks Drew Jamieson, Chris Lovell for helpful discussion... All of the code used in this paper is available open-source at github.com/kstoreyf/Corrfunc and github.com/kstoreyf/continuous-estimator.

APPENDIX

A. AFFINE INVARIANCE

The estimate of the 2pcf with the Continuous-Function Estimator should not depend on the scaling of the chosen basis functions. Thus we expect the Continuous-Function Estimator to be invariant under affine transformations of the basis functions, meaning transformations that preserve collinearity and distance ratios; the following demonstrates this affine invariance.

We represent the affine transformation by a transformation matrix \mathbf{M} that modifies the basis functions \mathbf{f} , such that

$$\mathbf{f}' \leftarrow \mathbf{M} \mathbf{f} \quad (\text{A1})$$

where the prime indicates our affine-transformed basis. *KSF says: im also using primes for indices, looks a bit confusing; another notation for this?* Then in the primed basis, the pair counts become

$$\mathbf{v}'_{\text{DD}} = \sum_n \sum_{n'} \mathbf{f}'_{nn'} = \sum_{nn'} \mathbf{M} \mathbf{f}_{nn'} = \mathbf{M} \mathbf{v}_{\text{DD}} \quad (\text{A2})$$

$$\mathbf{v}'_{\text{DR}} = \sum_n \sum_m \mathbf{f}'_{nm} = \sum_{nm} \mathbf{M} \mathbf{f}_{nm} = \mathbf{M} \mathbf{v}_{\text{DR}} \quad (\text{A3})$$

$$\mathbf{v}'_{\text{RR}} = \sum_m \sum_{m'} \mathbf{f}'_{mm'} = \sum_{mm'} \mathbf{M} \mathbf{f}_{mm'} = \mathbf{M} \mathbf{v}_{\text{RR}} \quad (\text{A4})$$

where we use the shorthand $\mathbf{f}_{ij} = \mathbf{f}(\mathbf{G}_i, \mathbf{G}_j)$ and we have omitted the normalization factors for clarity. In the last step, we have factored \mathbf{M} out of the summation and written the primed vectors in terms of the unprimed vectors.

For the random-random tensor we have

$$\mathbf{T}'_{\text{RR}} = \sum_m \sum_{m'} (\mathbf{M} \mathbf{f}_{mm'}) \cdot (\mathbf{M} \mathbf{f}_{mm'})^\top \quad (\text{A5})$$

$$= \mathbf{M} \left[\sum_m \sum_{m'} \mathbf{f}_{mm'} \cdot \mathbf{f}_{mm'}^\top \right] \mathbf{M}^\top \quad (\text{A6})$$

$$= \mathbf{M} \mathbf{T}_{\text{RR}} \mathbf{M}^\top \quad (\text{A7})$$

Then the amplitudes in the primed basis become

$$\mathbf{a}' = \mathbf{T}_{\text{RR}}'^{-1} \cdot (\mathbf{v}'_{\text{DD}} - 2 \mathbf{v}'_{\text{DR}} + \mathbf{v}'_{\text{RR}}) \quad (\text{A8})$$

$$\mathbf{a}' = [\mathbf{M} \mathbf{T}_{\text{RR}} \mathbf{M}^\top]^{-1} \cdot [\mathbf{M} \mathbf{v}_{\text{DD}} - 2 \mathbf{M} \mathbf{v}_{\text{DR}} + \mathbf{M} \mathbf{v}_{\text{RR}}] \quad (\text{A9})$$

$$= (\mathbf{M}^\top)^{-1} \mathbf{T}_{\text{RR}}^{-1} \mathbf{M}^{-1} \cdot \mathbf{M} [\mathbf{v}_{\text{DD}} - 2 \mathbf{v}_{\text{DR}} + \mathbf{v}_{\text{RR}}] \quad (\text{A10})$$

$$= (\mathbf{M}^\top)^{-1} \mathbf{T}_{\text{RR}}^{-1} \cdot [\mathbf{v}_{\text{DD}} - 2 \mathbf{v}_{\text{DR}} + \mathbf{v}_{\text{RR}}] \quad (\text{A11})$$

$$= (\mathbf{M}^\top)^{-1} \mathbf{a} \quad (\text{A12})$$

and the estimator $\hat{\xi}'$ in the primed basis, using the shorthand $\hat{\xi}_{ij} = \hat{\xi}(\mathbf{G}_i, \mathbf{G}_j)$, is

$$\hat{\xi}'_{ll'} = \mathbf{a}'^\top \cdot \mathbf{f}_{ll'} \quad (\text{A13})$$

$$\hat{\xi}'_{ll'} = [(\mathbf{M}^\top)^{-1} \mathbf{a}]^\top \cdot (\mathbf{M} \mathbf{f}_{ll'}) \quad (\text{A14})$$

$$= \mathbf{a}^\top [(\mathbf{M}^{-1})^\top]^\top \cdot (\mathbf{M} \mathbf{f}_{ll'}) \quad (\text{A15})$$

$$= \mathbf{a}^\top \mathbf{M}^{-1} \cdot \mathbf{M} \mathbf{f}_{ll'} \quad (\text{A16})$$

$$= \mathbf{a}^\top \cdot \mathbf{f}_{ll'} \quad (\text{A17})$$

$$= \hat{\xi}_{ll'}. \quad (\text{A18})$$

Thus after an affine transformation of the basis function, the resulting estimator is equivalent to the estimator in the original basis.

We note that this requires \mathbf{M} be invertible. However, any two equivalent bases must be related by the inverse of a transformation matrix, so this requirement is already satisfied. *KSF says: I don't understand what i wrote here, what was i getting at?*

KSF says: what more to say about this?

B. COMPUTING THE RANDOM-RANDOM TERMS ANALYTICALLY

The autocorrelation of the random catalog is meant to approximate the window function. When we have a periodic cube, we can compute this \mathbf{v}_{RR} term analytically in the standard approach to correlation function estimation. Here we derive this, and then derive the equivalent for our continuous-basis \mathbf{v}_{RR} and \mathbf{T}_{RR} terms.

Our goal is to estimate the number of pairs in a periodic cubic volume filled uniformly with tracers, $\mathbf{v}_{\text{RR}}^{\text{ana}}$. We first consider an annulus indexed by k around a single galaxy, with radial edges g_k and h_k . This annulus has a volume V_k . Taking the box to have an average number density \bar{n} , the number of galaxies expected in the annulus is $N_k = V_k \bar{n}$, and thus our selected galaxy contributes N_k pairs to the count. *KSF says: do i put mathspaces between multiplications when there's already a parenthesis?* We do this for each of the $N_{\text{D}} - 1$ other galaxies, and after including a factor of $\frac{1}{2}$ accounts for the fact that this double-counts pairs, we find a total pair count of $[\mathbf{v}_{\text{RR}}^{\text{ana}}]_k = \frac{1}{2}(N_{\text{D}} - 1)N_k = \frac{1}{2}(N_{\text{D}} - 1)V_k \bar{n}$. For a cubic volume, $\bar{n} = N_{\text{D}}/L^3$, so our final pair count for the annulus is

$$[\mathbf{v}_{\text{RR}}^{\text{ana}}]_k = \frac{1}{2} \frac{N_{\text{D}}}{L^3} (N_{\text{D}} - 1) V_k. \quad (\text{B19})$$

We next need to compute V_k ; for hard-edged radial bins, we can compute V_k simply as the difference between spherical volumes. We can represent this more generally as an integral,

$$V_k = \int_{g_k}^{h_k} dV = 4\pi \int_{g_k}^{h_k} r^2 dr \quad (\text{B20})$$

where we assume spherical symmetry. We can easily generalize this to any basis function $\mathbf{f}_k(r)$ that is only a function of r ,

$$V_k = 4\pi \int_{g_k}^{h_k} \mathbf{f}_k(r) r^2 dr \quad (\text{B21})$$

where k is now the index of the basis functions. We can see that this reduces to Equation B20 when $\mathbf{f}(r)$ is the tophat function (returning 1 or 0 depending on whether or not r falls between g_k and h_k).

Combining the above equations gives us our full generalized analytic random-random vector $\mathbf{v}_{\text{RR}}^{\text{ana}}$, which has elements

$$[\mathbf{v}_{\text{RR}}^{\text{ana}}]_k = \frac{1}{2} \frac{N_{\text{D}}}{L^3} (N_{\text{D}} - 1) 4\pi \int_{r_{\text{min}}}^{r_{\text{max}}} \mathbf{f}_k(r) r^2 dr \quad (\text{B22})$$

where we are now integrating over all values of r we are interested in from some r_{\min} to r_{\max} . (For non-localized basis functions, the fully correct thing would be to integrate from $-\infty$ to ∞ , though some bounds must be chosen in practice.)

Based on the definition of \mathbf{T}_{RR} in Equation 10 as the outer product of the basis function vector and its transpose, we can see that the elements of the analytic random-random tensor $\mathbf{T}_{\text{RR}}^{\text{ana}}$ can be written as

$$[\mathbf{T}_{\text{RR}}^{\text{ana}}]_{kk'} = \frac{1}{2} \frac{N_{\text{D}}}{L^3} (N_{\text{D}} - 1) 4\pi \int_{r_{\min}}^{r_{\max}} \mathbf{f}_k(r) \mathbf{f}_{k'}(r) r^2 dr \quad (\text{B23})$$

This could be further generalized to account for basis functions that take other properties as input.

When considering a periodic box, the natural estimator is no longer biased, so we can also avoid computing the cross-correlation term \mathbf{v}_{DR} and calculate the amplitudes as

$$\mathbf{a}_{\text{ana}} = [\mathbf{T}_{\text{RR}}^{\text{ana}}]^{-1} \cdot \mathbf{v}_{\text{DD}}. \quad (\text{B24})$$

Looking back, it might have seemed strange that we use N_{D} in calculating the analytical term $\mathbf{v}_{\text{RR}}^{\text{ana}}$, but we now see that this normalization prefactor cancels out with that of the \mathbf{v}_{DD} term. Finally, we use these amplitudes \mathbf{a}_{ana} to compute the correlation function $\hat{\xi}_{\text{ana}}$ as before in Equation 11.

This analytic form for the continuous estimator could be extended to basis functions that depend on other tracer properties in addition to pair separation. In this case, one would have to integrate over these axes as well, but the idea is the same.

C. IMPLEMENTATION OF ESTIMATION WITH BAO BASIS FUNCTIONS

C.1. Iterative Procedure

The Continuous-Function Estimator can be used to measure the baryon acoustic oscillation (BAO) scale by choosing the basis functions to terms of a BAO fitting function, as described in Section 3.4. For this application, we need to choose a fiducial cosmology for our bases, which will be offset from the true cosmology. This offset can be encoded by a scale dilation parameter α , which contains the information about the BAO scale; see Equation 16. As our fitting function requires a fiducial model and an initial guess of this parameter, α_{guess} , and then determines the change needed, an iterative procedure is needed to converge to the best-fit value.

We start with assuming that we have chosen our fiducial model to match our true cosmology (we in all likelihood have not, but it's not a bad initial guess), giving us an initial $\alpha_{\text{guess}} = 1.0$. We then apply the Continuous-Function Estimator to perform the measurement, and obtain the amplitude C for the derivative term in our model as in Equation 18. This gives us our estimate $\hat{\alpha}$ of the scale dilation parameter from this initial model; for the i th iteration, we have

$$\hat{\alpha}_i = \alpha_{\text{guess},i} + C_i k_0 \quad (\text{C25})$$

where k_0 is the chosen scaling parameter for the derivative basis function as in Equation 18.

We choose the convergence criterion to be when the fractional change in $\hat{\alpha}$ between subsequent iterations falls below a threshold, c_{thresh} KSF says: name for this variable?,

$$\left| \frac{\hat{\alpha}_i - \hat{\alpha}_{i-1}}{\hat{\alpha}_i} \right| < c_{\text{thresh}}. \quad (\text{C26})$$

For our application we choose $c_{\text{thresh}} = 0.00001$.

To achieve convergence, we need to be careful in choosing our next $\alpha_{\text{guess},i}$. If it is far from the best estimate, C_i will be large, and our resulting estimate $\hat{\alpha}_i$ will be inaccurate. We thus include a damping parameter η between 0 and 1 to improve our convergence. Our next guess is then KSF says: what should the assignment symbol be here?

$$\alpha_{\text{guess},i+1} \leftarrow \alpha_{\text{guess},i} + \eta C_i k_0. \quad (\text{C27})$$

The choice of η is important for stability and speed of convergence; too large a value can lead to a back-and-forth cycle in which the result hops between two values and never converges, and too small a value would make convergence take a very long time. In our application, we start with $\eta = 0.5$. We check if our estimate is jumping over the true value by checking if the error changes sign; if it does, we reduce η by a factor of 0.75.

C.2. Implementation Details

We implement the partial derivative in the fitting function as a finite difference between model with the our chosen value of α_{guess} , and the model with a value shifted by a small $\Delta\alpha$, KSF says: what should the assignment symbol be here?

$$\frac{d\xi^{\text{mod}}(\alpha s)}{d\alpha} \leftarrow \frac{\xi^{\text{mod}}(\alpha_{\text{guess}} s) - \xi^{\text{mod}}((\alpha_{\text{guess}} + \Delta\alpha) s)}{\Delta\alpha}. \quad (\text{C28})$$

In our implemenation we take $\Delta\alpha = 0.001$; we check that our results are insensitive to this choice.

We choose the amplitudes of the basis functions k to set them at similar scales, providing improved stability in convergence. We find that the values $k_0 = 0.1$, $k_1 = 10.0$, $k_2 = 0.1$, and $k_3 = 0.001$ provide the fastest convergence, though the results are insensitive to choices near these values.

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