162. Using the method of Laplace transform, solve $y'' - 3y' + 2y = 4t + e^{3t}$

Given $y(0) = 1$, $y'(0) = -1$

[CUSAT 06 Nov 09] [CUSAT 06 Nov 11] [CUSAT 12 Nov 13] [KER 13 Apr 14]
[CUSAT 06 Nov 14] [CUSAT 12 Nov 14]

Ans:

The given differential equation is $y'' - 3y' + 2y = 4t + e^{3t}$

Taking Laplace transform on both sides, we get

$$L\{y''\} - 3L\{y'\} + 2L\{y\} = L\{4t + e^{3t}\}$$

$$\Rightarrow [s^2 L\{y\} - s\,y(0) - y'(0)] - 3[s\,L\{y\} - y(0)] + 2L\{y\} = L\{4t + e^{3t}\}$$

$$\Rightarrow [s^2 L\{y\} - s(1) + 1] - 3[s\,L\{y\} - 1] + 2L\{y\} = \frac{4}{s^2} + \frac{1}{s-3}$$

$$\Rightarrow (s^2 - 3s + 2)L\{y\} - s + 1 + 3 = \frac{4}{s^2} + \frac{1}{s-3}$$

$$\Rightarrow (s^2 - 3s + 2)L\{y\} = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$$

$$\Rightarrow s^2(s-3)(s^2 - 3s + 2)L\{y\} = 4(s-3) + s^2 + s^3(s-3) - 4s^2(s-3)$$

$$= s^4 - 7s^3 + 13s^2 + 4s - 12$$

$$\Rightarrow L\{y\} = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s^2 - 3s + 2)} = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s-1)(s-2)}$$

$$\Rightarrow y = L^{-1}\left[\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)}\right]$$

Let $\dfrac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)} = \dfrac{A}{s} + \dfrac{B}{s^2} - \dfrac{C}{s-1} + \dfrac{D}{s-2} + \dfrac{E}{s-3}$

$$\Rightarrow s^4 - 7s^3 + 13s^2 + 4s - 12 = A\,s(s-1)(s-2)(s-3) + B(s-1)(s-2)(s-3)$$
$$+ C\,s^2(s-2)(s-3) + D\,s^2(s-1)(s-3) + E\,s^2(s-1)(s-2)$$

Put $s = 0 \Rightarrow -12 = -6A \Rightarrow B = 2$

Put $s = 1 \Rightarrow -1 = 2C \Rightarrow C = -1/2$

Put $s = 2 \Rightarrow 8 = -4D \Rightarrow D = -2$

Put $s = 3 \Rightarrow 9 = 18E \Rightarrow E = 1/2$

Equating the terms containing "s^4", we get $1 = A + C + D + E \Rightarrow A = 3$

$$\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)} = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{2}\frac{1}{s-1} - \frac{2}{s-2} + \frac{1}{2}\frac{1}{s-3}$$