INFERENCE IN LINEAR DYADIC DATA MODELS WITH NETWORK SPILLOVERS

Nathan Canen and Ko Sugiura

University of Houston

ABSTRACT. When using dyadic data (i.e., data indexed by pairs of units, such as trade flow data between two countries), researchers typically assume a linear model, estimate it using Ordinary Least Squares and conduct inference using "dyadic-robust" variance estimators. The latter assumes that dyads are uncorrelated if they do not share a common unit (e.g., if one country does not appear in both pairs of trade flow data). We show that this assumption does not hold in many empirical applications because indirect links may exist due to network connections, e.g., different country-pairs may have correlated trade outcomes due to sharing common trading partner links. Hence, as we prove, then show in Monte Carlo simulations, "dyadic-robust" estimators can be severely biased. We develop a consistent variance estimator appropriate for such contexts by leveraging results in network econometrics. Our estimator has good finite sample properties in numerical simulations. We then illustrate our message with an application to voting behavior by seating neighbors in the European Parliament.

KEY WORDS: Dyadic data, Networks, Inference, Cross-sectional dependence, Congressional Voting

1. Introduction

Dyadic data is categorized by the dependence between two sets of sampled units (dyads). For example, exports between the U.S. and Canada depends on both countries (and, plausibly, their characteristics), while arrangements between a seller and a buyer show similar dependence. This contrasts to classical data in the social sciences that only depend on a single unit of observation (e.g., the GDP of the U.S.).

The empirical relevance of dyadic data is showcased by its widespread use, which has increased over the past two decades (Graham (2020b) provides an extensive review). For example, applications are found in international trade (export-import outcomes across countries, Anderson and van Wincoop (2003)), macroeconomics (exchange rate determination, Lustig and Richmond (2020)), development (gift giving and insurance between two members in a village, Fafchamps and Gubert (2007)), political economy (correlation in voting behavior in Parliament across seating neighbors, Harmon, Fisman, and Kamenica (2019)), among many others. In all of these examples, applied researchers model the dependence between dyadic outcomes and observable characteristics using a linear model which they then estimate using Ordinary Least Squares (OLS). However, inference on such estimators for the linear parameters is more complex.

The main approach in applied work for the latter has been the use of the so-called "dyadic-robust" estimators (e.g., Cameron, Gelbach, and Miller (2011), Aronow, Samii, and Assenova (2015), Tabord-Meehan (2019), among others. This builds on the widely used assumption in dyadic data that the error term for dyad (i, j) and for dyad (j, k) can only be correlated if they share a unit (see Aronow, Samii, and Assenova (2015), Tabord-Meehan (2019) for a discussion, and Cameron and Miller (2014) for a review). Such spillovers could be rationalized as individual-level unobserved heterogeneity: e.g., an unmeasured export orientation of an economy and unmodeled tastes for consumption in the literature of international trade (see Graham (2020a) and references therein for detail). In the context of international trade, this implies that Canada-U.S. and France-U.S. trade may be correlated because the U.S. is present in both, but there is no correlation between Canada-U.S. exports and France-Germany ones (conditional on observable characteristics).

In this paper, we first show that such an assumption does not hold in many models using dyadic data where dyads may be indirectly connected along a network.[1] Figure 1 (A) presents a simple example in the context of trade. That is, it is completely possible that Canada-US and France-Germany trades might be correlated along unobservables because they have many

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[1]This is a concrete class of applied examples where the assumption fails. The possibility that cross-sectional dependence in dyadic data might be more extensive than assumed has been pointed out by Cameron and Miller (2014).

indirect connections (in the Figure, through Canada trading with China which then trades with France). This implies that the use of dyadic-robust variance estimators may be inappropriate for inference in linear models with dyadic data when spillovers exist along a network.

We make this point in two ways. First, we prove that existing "dyadic-robust" variance estimators are biased for the true asymptotic variance when there are network spillovers (i.e., when dyads may be correlated along a network even when they do not share a common unit). This result is verified in Monte Carlo simulations with standard set-ups, where we show that such variance estimators lead to a bias of approximately 14% and its associated confidence interval undercovers the true parameter by over 9 percentage points.[2] This performance worsens with the density of the underlying network (e.g., in political economy and international trade, networks are often dense, see Cameron and Miller (2014)) and with sample size, as these features increase the role of indirect spillovers. This issue may be prevalent in many applied examples: in the case of international trade (Cameron, Gelbach, and Miller (2011) and Cameron and Miller (2014)), dyads may be correlated even if they do not share a member country because there are many indirect connections among them. In political economy, those seating far apart from one another in Parliament may still have correlated behavior through the indirect links of whom they are seating next to sits with. In Macroeconomics, exchange rates may depend on other countries beyond the dyads.

To deal with these issues, we develop a consistent variance estimator that can account for such network spillovers even with dyadic data. This generalizes existing results for dyadic-robust variance estimators (e.g., Aronow, Samii, and Assenova (2015)) to explicitly account for the network structure of spillovers. To do so, we leverage recent results in inference in such cross-sectional data, such as Kojevnikov, Marmer, and Song (2021).[3] We prove that our proposed variance estimator is consistent for the true variance of the OLS estimator in linear models with dyadic data when the cross-sectional dependence follows an observed network.

Finally, we illustrate the extent to which neglecting network spillovers with dyadic data may bias inference results. Beyond the Monte Carlo simulations described above, we revisit the application in Harmon, Fisman, and Kamenica (2019) of voting in the European Parliament. The authors study whether random seating arrangements (based on naming conventions) induces neighboring politicians to agree with one another in policy votes. The outcome, whether politicians i and j vote the same way on a policy, is dyadic in nature. However, i and j's votes may be

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[2] These numbers are obtained from the simulation results in which the underlying network is modestly complex (the Barabśi and Albert's (1999) random network with parameter $v = 3$), the sample size is large ($N = 5000$) and the spillovers are significant ($\gamma = 0.8$). See Section 5 for details.

[3] We carefully review the existing literature on the econometrics of networks, including Vainora (2020), Kojevnikov, Marmer, and Song (2021), Leung (2022, 2021), and Leung and Moon (2021), in Section 2 after we introduce additional notation.

FIGURE 1. Hypothetical Example of Trade Network

(A) Trade network across countries

(B) Trade network across pairs of countries
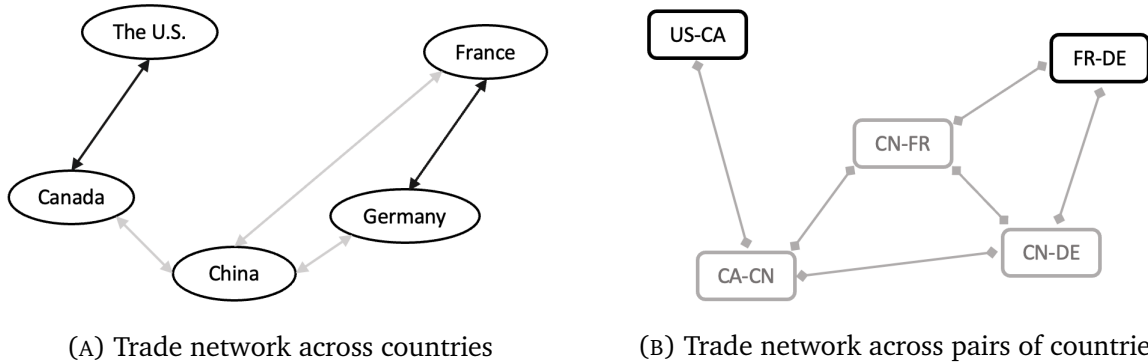
Note: Panel (A) shows a hypothetical example of trade networks across countries. Each double-headed arrow signifies the existence of bilateral trade flow between countries. (The absence of a double-headed arrow is interpreted as the nonexistence of any type of trade flow between the countries.) In this example, though the Canada-US and France-Germany trades (denoted by black arrows) do not have a country in common, they might still be correlated through two indirect links (denoted by grey arrows): the US-Japan-South Korea-France trade, and the Germany-China-Canada trade. Panel (B) illustrates the network across pairs of countries (i.e., dyads). Round squares stand for bilateral trade flows between countries and the adjacency of these trade flows are represented by turned square arrows.

correlated even if they are not neighbors: for instance, i and j may sit on either side of common neighbors k, m, who influence them both. This chain of influences is sufficient to induce strong correlation across non-dyads. We show that neglecting such higher order spillovers has large empirical consequences: their estimated variance using Aronow, Samii, and Assenova (2015) is roughly 22% smaller than using our consistent estimator accounting for such spillovers.

The paper is organized as follows. In the next section, we define the linear model and additional notation necessary for our results. Section 3 presents results on the consistency and asymptotic normality of the OLS estimator in this environment, necessary to prove the consistency of our variance estimator. The latter is our main result, proved in Section 4. Section 5 presents Monte Carlo simulation results comparing the finite sample performance of our estimator to White-robust estimators and dyadic-robust ones, while Section 6 presents the empirical application. Section 7 concludes. Throughout the paper, as appropriate, we compare our results to those in the literature. In Appendix, we provide all the proofs, additional Monte Carlo exercises and additional details on the empirical application.

2. Set-up

2.1. Sampling Scheme

Consider a population of individuals locating along a network that can be interpreted as consumers, firms, and other observation units depending on the context. Let each dyad between the units be indexed by $m \in \mathcal{M} \subseteq \mathbb{N}$ with $\mathcal{M}$ being an index set of the dyads. That is, the dyad for two units (e.g., firms, individuals) i and j is denoted as some m. Two dyads m and m' are said to be *adjacent* if they have an individual in common, and they are called *connected* if there are other pairs of adjacent dyads between them. In Figure 1 (A), for example, U.S.-Canada trade is adjacent to Canada-China, and connected with, though not adjacent to, France-Germany.

We emphasize that the adjacency relationship constitutes a network structure among dyads, and networks over individuals can thus be viewed as ones over dyads: for example, Figure 1 (B) provides a network over pairs of countries.[4] We define the geodesic distance between two connected dyads m and m' to be the smallest number of adjacent dyads between them. Note that adjacent dyads are a special case of connected dyads with the geodesic distance equal one.

We keep track of the adjacency of dyads through an array of binary variables $[g_{mm'}]_{m,m' \in \mathcal{M}}$. We restrict attention to environments whose networks structure among dyads satisfy the following standard conditions.

Assumption 2.1 (Adjacency of Dyads).

(i) For each $m \in \mathcal{M}$ and $m' \in \mathcal{M} \backslash m$,

$$g_{mm'} = \begin{cases} 1 & \text{if } m \text{ and } m' \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For all $m, m' \in \mathcal{M}$, $g_{mm'} = g_{m'm}$.

(iii) For all $m, m' \in \mathcal{M}$, $g_{mm'} = 1$ whenever $m = m'$.

Assumption 2.1 (i) is in line with the notational convention of the literature, in which the presence of a dyad (between dyads) is coded as one and its absence is coded as zero. This assumption also means that each dyad (between dyads) is equally weighted. Assumption 2.1 (ii) is again customary assumed in the literature to focus on symmetric cases. The results that follow can be extended to the asymmetric case at the cost of notational complexity. Assumption 2.1 (iii) allows for self-loops, easing notation. We expand on this point in Remark 2.1 below.

Suppose that each dyad m is endowed with a triplet of the dyad-specific variables $(y_m, x_m, \varepsilon_m)$, where $y_m \in \mathbb{R}$ is an observable outcome, $x_m \in \mathbb{R}^K$ is a K-dimensional vector of observable

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[4]This corresponds to thinking about the line graph of the original graph over individuals.

characteristics with $K \in \mathbb{N}$, and $\varepsilon_m \in \mathbb{R}$ is a random error term that is not observable to the econometrician. Furthermore, it is assumed that $[g_{mm'}]_{m,m' \in \mathcal{M}}$ is exogenously determined and conditionally independent of $\{\varepsilon_m\}_{m \in \mathcal{M}}$. This means that we only consider exogenous network formation.

The econometrician randomly draws $N \in \mathbb{N}$ individuals, and observes all the active (i.e., existing) dyads among the observed individuals. Let $\mathcal{M}_N$ be the index set of observed active dyads for the sample with $M := |\mathcal{M}_N| \in \mathbb{N}$. As soon as the sample of size N is drawn, the econometrician also observes the network structure across the active dyads, $[g_{mm'}]_{m,m' \in \mathcal{M}_N}$, and a collection of variables associated to the observed dyads, $\{(y_m, x_m)\}_{m \in \mathcal{M}_N}$.

It is worth noting that there are two different numbers of units at play: the number of sampling unit (i.e., individuals), N, and the number of dyads, M. The latter constitutes the basis of statistical analysis. The asymptotic theories of this paper are stated in the scenario that the number of sampling unit goes to infinity, i.e., large N asymptotics. To bridge these two units, we assume that the number of active dyads also goes to infinity as the number of sampling unit is taken to infinity. This is formalized by the following assumption.

Assumption 2.2 (Denseness)**.** $M = \mathcal{O}(N)$ whenever N is sufficiently large.

Assumption 2.2 requires the number of active dyads to grow at the same rate as the number of individuals (rate of N). This is similar in spirit to Assumption 2.3 of Tabord-Meehan (2019).

2.2. The Linear Model

2.2.1. **Setup & Identification.** The population model of interest takes the form of the linear network-regression model:

$$(1) \qquad y_m = \left(\sum_{m' \in \mathcal{M}} g_{mm'} x_{m'} \right)' \beta + \varepsilon_m \qquad \forall m \in \mathcal{M},$$

where

$$(2) \qquad Cov(\varepsilon_m, \varepsilon_{m'} \mid \mathbf{x}) = 0 \quad \text{unless } m \text{ and } m' \text{ are connected,}$$

and β is a $K \times 1$ vector of the regression coefficients and $\mathbf{x} := \{x_m\}_{m \in \mathcal{M}}$ stands for the collection of the dyad-specific characteristics. This formulation allows the dyad-specific outcome y_m to be affected by the characteristics of other dyads $x_{m'}$ through the network $[g_{m,m'}]_{m,m' \in \mathcal{M}_N}$. We note that equation (2) allows for there to be spillovers across the error terms even when dyads m and m' are not adjacent (i.e., when they are connected through indirect links).

Remark 2.1 (Variants of the Network-Regression Model)**.** Several well-known models are specific cases of the network-regression model (1). One example is when the outcome of dyad

$m = d(i,j)$ depends only on the characteristics of m (i.e., the characteristics of its units i and j) and is not affected by other dyads' or units' attributes:

$$y_m = x'_m \beta + \varepsilon_m \quad \forall m \in \mathcal{M}.$$

This specification has been of substantial empirical interest (See, e.g., Harmon, Fisman, and Kamenica (2019) and Lustig and Richmond (2020).). The model (1) can also be enriched by permitting different parameters specific to the own characteristics as well as a constant term:

$$y_m = \beta_0 + x'_m \beta_1 + \left(\sum_{m' \in \mathcal{M} \setminus m} g_{mm'} x_{m'} \right)' \beta_2 + \varepsilon_m \quad \forall m \in \mathcal{M}.$$

where $\beta_0 \in \mathbb{R}$ and $\beta_1, \beta_2 \in \mathbb{R}^K$ are the regression parameters. Finally, as in Kelejian and Prucha (1998) and Kelejian and Prucha (2010), one can allow the correlation across unobservables to be along a different network than those among observables.

Let us turn to dyadic models commonly studied in the econometrics literature. To do so, it is convenient to write $m = d(i,j)$ to denote that dyad m is the dyad related to units i and j.

Remark 2.2 (Dyadic-Regression Models). Aronow, Samii, and Assenova (2015) and Tabord-Meehan (2019) consider a linear dyadic-regression model of the form:

(3)
$$y_{d(i,j)} = x'_{d(i,j)} \beta + \varepsilon_{d(i,j)} \quad \forall i, j \in \mathbb{N},$$

where

(4)
$$Cov\left(\varepsilon_{d(i,j)}, \varepsilon_{d(k,l)} \mid x_{d(i,j)}, x_{d(k,l)} \right) \begin{cases} = 0 & \text{if} \quad \{i,j\} \cap \{k,l\} = \emptyset, \\ \neq 0 & \text{if} \quad \{i,j\} \cap \{k,l\} \neq \emptyset, \end{cases}$$

with $m = d(i,j)$ representing the dyad between i and j. The dyadic-regression model (3) can be embedded in our network-regression model (1) by: (i) dispensing the dependence of the dyad-specific outcome on other dyads' characteristics, and, most importantly, (ii) specifying the correlation structure (2) to be

$$Cov(\varepsilon_m, \varepsilon_{m'} \mid \mathbf{x}) \begin{cases} = 0 & \text{if } m \text{ and } m' \text{ are not adjacent,} \\ \neq 0 & \text{if } m \text{ and } m' \text{ are adjacent.} \end{cases}$$

It is clear from comparing the expression above to our assumption in (2) that our assumed covariance structure is more general than those proposed by the cited papers.

Towards identification of β in (1), we impose the following assumptions.

Assumption 2.3 (Identification Condition). For each $m \in \mathcal{M}$, the following conditions are satisfied:

(i) $E\left[|\varepsilon_m|^2\right]$ exits and is finite;

(ii) $E\left[\|x_m\|_2^2\right]$ exists and is finite;

(iii) $E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'} x_{m'}\right)\left(\sum_{m'\in\mathcal{M}} g_{mm'} x_{m'}\right)'\right]$ exists with finite elements and positive definite; and

(iv) $E\left[\varepsilon_m \mid \mathbf{x}\right] = 0$,

where $\|\cdot\|_2$ is meant to be the Euclidean norm.

Assumption 2.3 (i) and (ii) are standard and jointly imply the finite existence of the second moment of y_m, which in turn implies the finite existence of the cross moment of y_m and x_m. Provided that Assumption (ii) holds, Assumption 2.3 (iii) then requires that at most finitely many entries of $g_{mm'}$ are nonzero. For example, fulfilling (iii) rules out the the possibility of extremely dense network such as a complete network. Moreover, the positive definiteness of the matrix in this assumption ensures the existence of nondegenerate variations in the degree distribution[5] of the underlying network.[6] The fourth assumption states the strict exogeneity condition, a standard assumption in the context of the linear regression models.

Assumption 2.3 is used to prove identification of the linear parameters in equation (1):

Proposition 2.1 (Identification)**.** *Under Assumption 2.3, the regression parameter β in (1) is identified.*

Proof. See Appendix C.1. □

2.2.2. **Estimation.** Denote by G_M the collection of $\{g_{mm'}\}_{m,m'\in\mathcal{M}_N}$ arrayed in an $M\times M$ matrix, where the (m,m') entry is referenced by $g_{mm'}$. The latter represents whether dyad $m = d(i,j)$ and dyad $m' = d(i',j')$ share a common unit. To simplify notation, we drop the subscript M from G_M to write G in the subsequent discussion.

Stacking the dyad-specific observations, the model, in matrix notation, is given by

$$(5) \qquad\qquad\qquad y = GX\beta + \varepsilon,$$

where y is the $M\times 1$ vector of the observed dyad-specific outcomes, i.e., $y := [y_1,\ldots,y_M]'$; X denote the $M\times K$ matrix that records the observed dyad-specific characteristics, i.e., $X := [x_1,\ldots,x_M]'$; and ε represents the $M\times 1$ vector of unobserved dyad-specific error terms, i.e., $\varepsilon := [\varepsilon_1,\ldots,\varepsilon_M]'$.

Towards estimation, we assume the following rank condition.

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[5]The degree of a node in a network is the number of links at the node. The degree distribution describes relative frequencies of nodes with respect to the degrees.

[6]To see this, suppose, for example, that the underlying network is complete. In this case, the degree distribution degenerates at $M-1$ and every dyad-specific outcome is affected by characteristics of all dyads. If there is no variation in the number of connections, the effective regressor in the linear model would be the same for every dyad yielding multicollinearity.

Assumption 2.4 (Full Rank Condition). The matrix $X'G'GX$ is positive-definite with finite elements whenever M is sufficiently large.

This assumption is an algebraic requirement and guarantees the invertibility of $X'G'GX$. The intuition and interpretation carry over from Assumption 2.3 (iii). Under Assumption 2.4, the Ordinary Least Squares (OLS) estimator of β is given by

$$\hat{\beta} = \left(X'G'GX\right)^{-1} X'G'y$$

(6)
$$= \left(\sum_{j \in \mathcal{M}_N} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N} g_{jk} g_{jl} x_k x_l'\right)^{-1} \left(\sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N} g_{mj} x_j y_m\right).$$

If the off-diagonal elements of the adjacency matrix are zero, i.e., $g_{mm'} = 0$ if $m \neq m'$, then the estimator (6) simplifies to the OLS estimator for the standard linear regression model without network connections.

From this expression, we can write

(7)
$$\hat{\beta} - \beta = \left(\sum_{j \in \mathcal{M}_N} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N} g_{jk} g_{jl} x_k x_l'\right)^{-1} \left(\sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N} g_{mj} x_j \varepsilon_m\right).$$

In light of Assumption 2.3 (iv), it is straightforward to verify that $\hat{\beta}$ is unbiased for β. However, a consistency result is by no means trivial due to the dependence along the network which induces a complex cross-sectional dependence which forbids the use of i.i.d. results. This is clarified in Section 2.3.

2.2.3. **Inference.** Inference about β is based on a normal approximation of the distribution of $\hat{\beta}$ around β. In practice, the hypothesis testing is conducted using the expression:

(8)
$$\left(\widehat{Var}(\hat{\beta})\right)^{-\frac{1}{2}} (\hat{\beta} - \beta),$$

where $\widehat{Var}(\hat{\beta})$ is a consistent estimator of the asymptotic variance of $\hat{\beta}$. Our main result is providing such an appropriate estimator, which takes the form:

(9)
$$\widehat{Var}(\hat{\beta}) := \left(\sum_{j \in \mathcal{M}_N} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N} g_{jk} g_{jl} x_k x_l'\right)^{-1} \left\{\sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N} \kappa_{m,m'} h_{m,m'} \left(\sum_{k \in \mathcal{M}_N} g_{mk} x_k \hat{\varepsilon}_m\right) \left(\sum_{l \in \mathcal{M}_N} g_{m'l} x_l \hat{\varepsilon}_{m'}\right)'\right\} \left(\sum_{j \in \mathcal{M}_N} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N} g_{jk} g_{jl} x_k x_l'\right)^{-1},$$

where $\kappa_{m,m'}$ is a kernel function that will be defined formally in Section 4; $h_{m,m'}$ represents an indicator function that takes one if dyads m and m' are connected and zero otherwise; and $\hat{\varepsilon}_m := y_m - \left(\sum_{m' \in \mathcal{M}_N} g_{mm'} x_{m'}\right)' \hat{\beta}$. This paper derives conditions under which $\widehat{Var}(\hat{\beta})$ is consistent for the asymptotic variance of $\hat{\beta}$. Before doing so, let us compare the variance estimator (9) and a dyadic-robust variance estimator.

Example 2.1 (Dyadic-Robust Variance Estimator). An increasing number of applied researches, such as Harmon, Fisman, and Kamenica (2019) and Lustig and Richmond (2020), estimate a version of the model (5):

$$(10) \qquad\qquad y = X\beta + \varepsilon,$$

where

$$(11) \qquad Cov(\varepsilon_m, \varepsilon_{m'} \mid X) \begin{cases} = 0 & \text{if } m \text{ and } m' \text{ are not connected,} \\ \neq 0 & \text{if } m \text{ and } m' \text{ are connected.} \end{cases}$$

To conduct inference, these works employ dyadic-robust variance estimators proposed by Aronow, Samii, and Assenova (2015) and Tabord-Meehan (2019) which are given by:

$$(12) \qquad \widehat{Var}(\hat\beta) := \left(\sum_{m \in \mathcal{M}_N} x_m x_m' \right)^{-1} \left(\sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N} \mathbb{1}_{m,m'} \hat\varepsilon_m \hat\varepsilon_{m'} x_m x_{m'}' \right) \left(\sum_{m \in \mathcal{M}_N} x_m x_m' \right)^{-1},$$

where $\mathbb{1}_{m,m'}$ equals one if dyads m and m' are adjacent and zero otherwise. On the other hand, our estimator (9) applied to this model yields

$$(13) \qquad \widehat{Var}(\hat\beta) := \left(\sum_{m \in \mathcal{M}_N} x_m x_m' \right)^{-1} \left(\sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N} h_{m,m'} \hat\varepsilon_m \hat\varepsilon_{m'} x_m x_{m'}' \right) \left(\sum_{m \in \mathcal{M}_N} x_m x_m' \right)^{-1},$$

where we assume, for the sake of simplicity, that $\kappa_{m,m'} = 1$ for all $m, m' \in \mathcal{M}_N$. Note that the use of the dyadic-robust variance estimator sets cases in which two dyads are not adjacent but connected to zero. However, our estimator accounting for network spillovers accomodates the correlation across both adjacent and connected dyads. This suggests that the dyadic-robust variance estimator may be inconsistent when there is cross-sectional dependence along a network: i.e., when non-adjacent dyads can still affect the correlation structure and outcomes of dyad m.[7] This conjecture is formally proven in Corollary 4.1 and illustrated in Monte Carlo simulations in Section 5.

In order to validate the test statistic (8) for inference, this paper establishes a consistency and asymptotic normality results of $\hat\beta$ and develops a consistent network-robust estimator for the asymptotic variance.

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[7]Standard clustering estimators (e.g., at the individual level) may be inappropriate as each agent in a network has a complex structure of connections. If the network model features positive spillovers, then the dyadic-robust variance estimator will probably underestimate the true variance, leading to conservative hypothesis testing. Meanwhile, it is likely to overstate the true variance when there are negative spillovers. We expand on this point in our numerical simulations.

2.3. Network Dependent Processes

Let $Y_{M,m}$ be a random vector[8] defined as

$$(14) \qquad Y_{M,m} := \left(\frac{1}{M} \sum_{j \in \mathcal{M}_N} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N} g_{jk} g_{jl} x_k x_l' \right)^{-1} \sum_{j \in \mathcal{M}_N} g_{mj} x_j \varepsilon_m,$$

and denote $\mathcal{C}_M := \{x_m\}_{m \in \mathcal{M}_N}$.[9] From equation (6) we can write

$$(15) \qquad \hat{\beta} - \beta = \frac{1}{M} \sum_{m \in \mathcal{M}_N} Y_{M,m}.$$

Our interest lies in proving the asymptotic properties of $\hat{\beta}$ taking advantage of the expression (15). However, the presence of the ε_m in $Y_{M,m}$, which is allowed to be correlated along the network G, renders our approach nonstandard. The canonical results about independently and identically distributed ($i.i.d.$) random variables are by no means applicable due to the complex form of dependence inherent to ε_m. The theory of dyadic data merely accounts for a special case where the correlations between connected dyads except adjacent dyads are muted. Moreover, the asymptotic theory of spatially-correlated and time-series data builds on the assumption that the index set is a metrizable space,[10] an assumption that may not necessarily be true in the network context. In this light, the right hand side of (15) cannot simply be embedded in other widely-studied variants of dependent random vectors.

In this paper, we embrace the dependence of the $Y_{M,m}$'s along the network G by exploiting the concept of network dependent random variables recently developed by Kojevnikov, Marmer, and Song (2021), in which the correlations of the random variables given a common shock are explicitly dictated by their distance on the network. Compared to the other existing frameworks, the network-dependent random variables circumvent the aforementioned limitations, so that the network-regression model (1) and (2) become amenable to formal statistical inference.

To formally define our notion of network dependence, we first introduce notation intentionally set close to those of Kojevnikov, Marmer, and Song (2021). By construction, $Y_{M,m}$ resides in the K-dimensional Euclidean space $\mathbb{R}^K$. First, let $\rho_M(m, m')$ denote the geodesic distance

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[8]Here we make explicit the double index of the random vector so that it conforms with the triangular array structure which we will exploit in theoretical derivations.

[9]For the case of stochastic networks, it is replaced by $\mathcal{C}_M := \{(x_m, \{g_{mj}\}_{j \in \mathcal{M}_N \setminus \{m\}})\}_{m \in \mathcal{M}_N}$.

[10]Typically the literature assumes a Euclidean space. See Conley (1999), Ibragimov and Müller (2010) and references therein.

between dyads m and m', and A and B be sets of dyads whose cardinalities are a and b, respectively. Define the distance between these two sets as

$$(16) \qquad \rho_M(A, B) := \min_{m \in A} \min_{m' \in B} \rho_M(m, m').$$

We denote by $\mathcal{P}_M(a, b; s)$ the collection of pairs of A and B with distance s. Next we consider a collection of bounded Lipschitz functions $f : \mathbb{R}^{K \times c} \to \mathbb{R}$ and denote it by $\mathcal{L}_{K,c}$. Let $\mathcal{L}_K$ be a collection that gathers $\mathcal{L}_{K,c}$ for all $c \in \mathbb{N}$.

Now we are finally in a position to formally define the notion of conditional ψ-dependence.

Definition 2.1 (Conditional ψ-Dependence given $\{\mathcal{C}_M\}$: Kojevnikov, Marmer, and Song (2021), Definition 2.2). A triangular array $\{Y_{M,m} \in \mathbb{R}^K : M \geq 1, m \in \{1, \ldots, M\}\}$ is called conditionally ψ-dependent given $\{\mathcal{C}_M\}$, if for each $M \in \mathbb{N}$, there exist a $\mathcal{C}_M$-measurable sequence $\theta_M := \{\theta_{M,s}\}_{s \geq 0}$ with $\theta_{M,0} = 1$, and a collection of nonrandom function $(\psi_{a,b})_{a,b \in \mathbb{N}}$ where $\psi_{a,b} : \mathcal{L}_{K,a} \times \mathcal{L}_{K,b} \to [0.\infty)$, such that for all $(A, B) \in \mathcal{P}_M(a, b; s)$ with $s > 0$ and all $f \in \mathcal{L}_{K,a}$ and $g \in \mathcal{L}_{K,b}$,

$$\left| Cov\left(f(Y_{M,A}), g(Y_{M,B}) \,\middle|\, \mathcal{C}_M \right) \right| \leq \psi_{a,b}(f, g)\theta_{M,s} \qquad a.s.$$

$\{\theta_{M,s}\}$ are the dependence coefficients of $\{Y_{M,m}\}$. Intuitively, $\psi_{a,b}(f, g)$ dictates the magnitude of covariation of $Y_{M,A}$ and $Y_{M,B}$ as well as those of f and g. Meanwhile the $\theta_{M,s}$ captures the strength of covariation net of the effect of scaling, analogous to correlation coefficients. As the minimum distance between A and B, s, grows, the dependence between $Y_{M,A}$ and $Y_{M,B}$, $\{\theta_{M,s}\}$, tends to zero.[11] We maintain Assumption 2.1 of Kojevnikov, Marmer, and Song (2021) on the upper bounds of both $\psi_{a,b}(f, g)$ and $\theta_{M,s}$ throughout the paper.[12]

Remark 2.3 (Vainora (2020)). Vainora (2020) develops a network version of a stationary concept, what he calls weakly $\mathcal{C}$-stationary. Weak $\mathcal{C}$-stationary, in our context, requires covariances between two random variables to be equal as long as the geodesic distance is the same. Definition 2.1, on the other hand, only refers to the upper bound of covariances with respect to the geodesic distance. Vainora (2020) further establishes a Law of Large Numbers and Central Limit Theorem for random variables whose dependence structure is tied to the network, with applications to linear regression models. But the consistency of the standard errors, the central object of our interest, is left unexplored.

Remark 2.4 (Leung and Moon (2021)). Leung and Moon (2021) derive a Central Limit Theorem (CLT) of an estimator involving a complex form of dependence along the network. As they are primarily interested in network formation models, which could be rationalized by strategic games of network formation, their estimator is applicable only for a discrete-choice-type

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[11]Note that $\{\theta_{M,s}\}$ are random because Definition 2.1 is a conditional statement.
[12]Details of the mathematical environment are delegated to Appendix A.

regression model. Our framework complements their work for the case of linear regression models with a continuous outcome.

Remark 2.5 (Leung (2021)). A notable literature which shares motivations with us is Leung (2021), who exploits the asymptotic theory of Kojevnikov, Marmer, and Song (2021) to perform robust inference under network dependence. His approach is different from ours in that he derives conditions under which the use of cluster-robust estimators for inference is justified. He finds a set of conditions that ensures the network is asymptotically segregated into disjoint subnetworks that are not connected with each other, and then makes use of the existing cluster-robust estimators. Our focus is on the role of network spillovers within the particular class of linear models with dyadic data, where the use of cluster-robust estimators still has the potential drawback raised in Example 2.1.

Remark 2.6 (Leung (2022)). Leung (2022) considers robust causal inference under network interference by applying the results of Kojevnikov, Marmer, and Song (2021). Although he analyzes a linear regression model akin to (1), his work differs from ours in two important ways. First, the source of randomness in his model is the uncertainty in assignments (*design-based uncertainty*), thereby featuring no random shocks, from which the stochastic nature of our model stems. Second, his analysis presumes an environment where the population is finite and focuses on characterizing the bias of the variance estimator in a finite population model.

3. Preliminary Results

This section establishes the asymptotic properties tailored for our network dependent random variables.

3.1. Notations

As will become transparent shortly, asymptotic theories for $\hat{\beta}$ rest on tradeoffs between the correlation of the network-dependent random vectors (i.e., the dependence coefficients) and the denseness of the underlying network. To measure the denseness, we first define two concepts of neighborhoods:

$$\mathcal{M}_N(m;s) := \left\{ m' \in \mathcal{M}_N : d_M(m,m') \le s \right\},$$
$$\mathcal{M}_N^{\partial}(m;s) := \left\{ m' \in \mathcal{M}_N : d_M(m,m') = s \right\}.$$

The first set $\mathcal{M}_N(m;s)$ collects all the m's neighbors whose distance from m is no more than s, whilst the other set $\mathcal{M}_N^{\partial}(m;s)$ registers all the m's neighbors whose distance from m is exactly s. In what follows, we refer to $\mathcal{M}_N(m;s)$ as the neighborhood of m within distance s, and call $\mathcal{M}_N^{\partial}(m;s)$ the neighborhood shell of m at distance s.

Next, we define two types of density measures of a network: for $k > 0$,

$$\Delta_M(s, r; k) := \frac{1}{M} \sum_{m \in \mathcal{M}_N} \max_{m' \in \mathcal{M}_N^\partial(m;s)} |\mathcal{M}_N(m; r) \backslash \mathcal{M}_N(m'; s-1)|^k,$$

$$\delta_M^\partial(s; k) := \frac{1}{M} \sum_{m \in \mathcal{M}_N} |\mathcal{M}_N^\partial(m; s)|^k,$$

where it is assumed that $\mathcal{M}_N(m'; -1) = \emptyset$. The measure $\Delta_M(s, r; k)$ gauges the denseness of a network in terms of the average size of a version of the neighborhood. The other measure $\delta_M^\partial(s; k)$ expresses the denseness of a network as the average size of the neighborhood shell. Moreover, we consider a composite of these two density measures:

$$c_M(s, m; k) := \inf_{\alpha > 1} \left(\Delta_M(s, r; k\alpha) \right)^{\frac{1}{\alpha}} \left(\delta_M^\partial \left(s; \frac{\alpha}{\alpha - 1} \right) \right)^{\frac{\alpha - 1}{\alpha}}.$$

Notice that any restriction on $c_M(s, m; k)$ indirectly regulates the limiting behaviors of $\Delta_M(s, r; k)$ and $\delta_M^\partial(s; k)$. In fact, Kojevnikov, Marmer, and Song (2021) show that controlling the asymptotic behavior of $c_M(s, m; k)$ is sufficient for the Law of Large Numbers (LLN) and Central Limit Theorem (CLT) of the network dependent random variables (Condition ND).

Lastly, conditions in this section are stated with respect to each element of $Y_{M,m}$ for notational simplicity. We refer to the u-th entry of the network-dependent random vector $Y_{M,m}$ as $Y_{M,m}^u$ for $u \in \{1, \ldots, K\}$.

3.2. Limit Theorems for Network Dependent Processes

3.2.1. **Consistency.** To prove that $\hat{\beta}$ is consistent for β requires two additional conditions.

Assumption 3.1 (Conditional Finite Moment of ε_m). There exists $\eta > 0$ such that

$$\sup_{N \geq 1} \max_{m \in \mathcal{M}_N} E\left[|\varepsilon_m|^{1+\eta} \,\big|\, \mathcal{C}_M \right] < \infty.$$

Assumption 3.2. $\frac{1}{M} \sum_{s \geq 1} \delta_M^\partial(s; 1) \theta_{M,s} \xrightarrow{a.s.} 0$ as $M \to \infty$.

Assumption 3.1 requires that the errors are not too large once conditioned on common shocks. This assumption is generally stronger than Assumption 2.3 (i) as it requires the finiteness of a conditional moment. Under Assumption 2.4, this assumption implies Assumption 3.1 of Kojevnikov, Marmer, and Song (2021) for each $Y_{M,m}^u$ with $u \in \{1, \ldots, K\}$.

Assumption 3.2 controls the tradeoff between the denseness of the underlying network and the covariability of the random vectors. If the network becomes dense, then the dependence of the associated random variables has to decay much faster. This is typically assumed in the literature of spatial econometrics upon which the empirical analyses of the aforementioned examples (i.e., international trade, legislative voting, and exchange rates) are based, because it embodies

the idea that spillovers decay as they propagate farther (a "fading" memory). See, e.g., Kelejian and Prucha (2010). For instance, Acemoglu, García-Jimeno, and Robinson (2015) assume that network spillovers are zero if agents are sufficiently distantly connected on a geographical network.

Theorem 3.1 (Consistency of $\hat{\beta}$). *Suppose that Assumptions 2.2, 2.3, 2.4, 3.1 and 3.2 hold. Then,*

$$\hat{\beta} \xrightarrow{p} \beta$$

as $N \to \infty$.

Proof. See Appendix C.2. □

Theorem 3.1 establishes the consistency of $\hat{\beta}$ under the scenario where the number of sampling units N goes to infinity. When Assumption 2.2 is dropped, the result continues to hold in terms of the number of active dyads M.

3.2.2. **Asymptotic Normality.** Before we can prove the consistency of our variance estimator, we must derive the asymptotic distribution of $\hat{\beta}$. To do so, we study the asymptotic distribution of the following sum of network dependent random vectors

$$S_M := \sum_{m \in \mathcal{M}_N} Y_{M,m},$$

which is present in $\hat{\beta}$ in equation (15). Let S_M^u be the u-th entry of S_M for $u \in \{1, \ldots, K\}$. We consider two types of variances: namely, unconditional and conditional variances. Denote the unconditional variance of S_M^u by

$$\tau_M^2 := Var\left(S_M^u\right),$$

and the conditional variance of S_M^u given $\mathcal{C}_M$ by

$$\sigma_M^2 := Var\left(S_M^u \mid \mathcal{C}_M\right).^{[13]}$$

Since S_M^u is not a sum of independent variables, τ_M^2 and σ_M^2 cannot be expressed as a sum of variances. We thus need to explicitly take into account covariance between the random variables $\{Y_{M,m}^u\}_{m \in \mathcal{M}_N}$. We study the CLT for the normalized sum of $Y_{M,m}^u$ given by $\frac{S_M^u}{\tau_M}$.

As usual, the Central Limit Theorem for this normalized sum requires us to strengthen conditions (3.1) and (3.2) to control the variability of higher moments of our random variables. This is done in Assumptions A.2 and A.3 in Appendix. These assumptions are written in terms of conditional expectations, whereas we are interested in the unconditional distribution of $\frac{S_M}{\tau_M}$.

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[13] We choose not to index these two variances by u for notational simplicity, as its omission still maintains expositional clarity.

The following assumption bridges the conditional variance σ_M and the unconditional variance τ_M.

Assumption 3.3 (Growth Rates of Variances). There exists a sequence of (possibly random) positive numbers, $\{\pi_{N,M}\}_{N>0}$, such that

$$\frac{\sigma_M^2}{\pi_{N,M}\tau_M^2} \xrightarrow{a.s.} 1 \qquad \text{as} \quad N \to \infty.$$

In Assumption 3.3, the relative growth rate of the conditional and unconditional variances are regulated by $\pi_{N,M}$. $\pi_{N,M}$ reflects the effects of the randomness conveyed in $\mathcal{C}_M$. Moreover, this assumption can be thought of as restricting the relationship between the growth rates of N and M. In the remainder of this paper, we assume that $\{\pi_{N,M}\}_{N>0}$ is known to the econometrician.

Under these assumptions, we obtain the asymptotic distribution of $\hat{\beta}$:

Theorem 3.2 (Asymptotic Normality of Sum of the Network-Dependent Random Variables). *Suppose that Assumptions 2.2, A.2, A.3 and 3.3. Then,*

$$(17) \qquad \sqrt{N}\left(\hat{\beta} - \beta\right) \xrightarrow{d} \mathcal{N}(0, AVar(\hat{\beta})) \qquad as \quad N \to \infty,$$

where $AVar(\hat{\beta})$ is given by:

(18)

$$AVar(\hat{\beta}) := \lim_{N\to\infty} N\pi_{N,M} E\left[\left(\sum_{j\in\mathcal{M}_N}\sum_{k\in\mathcal{M}_N}\sum_{l\in\mathcal{M}_N} g_{jk}g_{jl}x_k x_l'\right)^{-1}\left(\sum_{m\in\mathcal{M}_N}\sum_{k\in\mathcal{M}_N} g_{mk}x_k\varepsilon_m\right)\left(\sum_{m'\in\mathcal{M}_N}\sum_{l\in\mathcal{M}_N} g_{m'l}x_l\varepsilon_{m'}\right)'\left(\sum_{j\in\mathcal{M}_N}\sum_{k\in\mathcal{M}_N}\sum_{l\in\mathcal{M}_N} g_{jk}g_{jl}x_k x_l'\right)^{-1}\right],$$

which is assumed to have finite elements and be positive semidefinite, where $\pi_{N,M}$ is defined in Assumption 3.3.

Proof. See Appendix C.3. $\square$

Theorem 3.2 demonstrates the asymptotic normality of $\hat{\beta}$ in terms of N, the number of sampling units. If Assumption 2.2 is omitted, the theorem holds as long as M, the number of active dyads, goes to infinity. When $\pi_{N,M} = 1$, Theorem 3.2 corresponds to Lemma 1 of Aronow, Samii, and Assenova (2015) and also yields the same result as Proposition 3.1 of Tabord-Meehan (2019).

Remark 3.1. The requirement on the behavior of $AVar(\hat{\beta})$ mirrors Assumptions 2.4, 2.5 and 2.6 of Tabord-Meehan (2019): the expression (18) boils down to his Assumption 2.4, if $\pi_{N,M} = \frac{M}{N}$; it reduces to his Assumption 2.5, if $\pi_{N,M} = \frac{M}{N^2}$; and it coincides with Assumption 2.6, if $\pi_{N,M} = \frac{M}{N^{r+1}}$ with $r \in [0,1]$. Moreover, if $\pi_{N,M} = 1$, the asymptotic variance (18) appears in Lemma 1 of Aronow, Samii, and Assenova (2015).

4. Consistent Estimation of the Variance of β Under Network Spillovers

This section studies consistent estimation of $AVar(\hat{\beta})$. In light of Assumption 2.3 (iv), $Y_{M,m}$ are centered, i.e., $E\left[Y_{M,m}\right] = 0$ for each $m \in \mathcal{M}_N$. Define $Var(\hat{\beta})$ as

$$(19) \qquad Var(\hat{\beta}) := \frac{N \pi_{N,M}}{M^2} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N^{\partial}(m;s)} E\left[Y_{M,m} Y'_{M,m'}\right].$$

Clearly, $AVar(\hat{\beta}) = \lim_{N \to \infty} Var(\hat{\beta})$. The estimator developed in this section is a type of a kernel estimator. Define a kernel function $\omega : \bar{\mathbb{R}} \to [-1,1]$ such that $\omega(0) = 1$, $\omega(z) = 0$ whenever $|z| > 1$, and $\omega(z) = \omega(-z)$ for all $z \in \bar{\mathbb{R}}$. We denote by b_M the bandwidth, or the lag truncation. The feasible variance estimator of our interest (9) can be written as

$$(20) \qquad \widehat{Var}(\hat{\beta}) = \frac{1}{M^2} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N^{\partial}(m;s)} \omega_M(s) \hat{Y}_{M,m} \hat{Y}'_{M,m'},$$

with

$$\omega_M(s) := \omega\left(\frac{s}{b_M}\right) \qquad \forall s \geq 0,$$

and

$$\hat{Y}_{M,m} := \left(\frac{1}{M} \sum_{j \in \mathcal{M}_N} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N} g_{jk} g_{jl} x_k x'_l\right)^{-1} \sum_{j \in \mathcal{M}_N} g_{mj} x_j \hat{\varepsilon}_m,$$

where $\hat{\varepsilon}_m := y_m - \left(\sum_{m' \in \mathcal{M}_N} g_{mm'} x_{m'}\right)' \hat{\beta}$.

Assumption 4.1 (Kojevnikov, Marmer, and Song (2021), Assumption 4.1). There exists $p > 4$ such that

 (i) $\sup_{N \geq 1} \max_{m \in \mathcal{M}_N} E\left[|\varepsilon_m|^p \mid \mathcal{C}_M\right] < \infty$ a.s.;

 (ii) $\lim_{M \to \infty} \sum_{s \geq 1} |\omega_M(s) - 1| \delta_M^{\partial}(s) \theta_{M,s}^{1 - \frac{2}{p}} = 0$ a.s.; and

 (iii) $\lim_{M \to \infty} \frac{1}{M} \sum_{s \geq 0} c_M(s, b_M; 2) \theta_{M,s}^{1 - \frac{4}{p}} = 0$ a.s.

Assumption 4.1 (i) is essentially the same as Assumption A.2. Part (ii) posits a tradeoff between the kernel function, the denseness of a network and the dependence coefficients. Specifically, the kernel function ω_M is required to converge to one sufficiently fast. Kojevnikov, Marmer, and Song (2021) demonstrate primitive conditions under which this requirement is fulfilled (Proposition 4.2). Assumption (iii) indicates a condition between the bandwidth, the denseness and the dependence coefficients. Kojevnikov, Marmer, and Song (2021) provide a preferred choice of the bandwidth that satisfies this condition.

To prove our main result, we need three additional regularity assumptions.

Assumption 4.2. For all $N \geq 1$ and for each $m \in \mathcal{M}_N$, $\sum_{m' \in \mathcal{M}_N} g_{mm'} x_{m'}$ has uniformly bounded support.

Assumption 4.2 is reminiscent of Assumption 3.1 of Tabord-Meehan (2019), and is employed to guarantee the uniform boundedness of arbitrary moments. Unlike Tabord-Meehan (2019), the uniformly bounded support for x_m alone does not suffice to ensure the existence of moments of $\sum_{m' \in \mathcal{M}_N} g_{mm'} x_{m'}$ as it entails the $g_{mm'}$ terms, which may possibly diverge. Hence, in general, Assumption 4.2 further requires the boundedness of $\sum_{m' \in \mathcal{M}_M} g_{mm'}$.

Remark 4.1. As pointed out in Tabord-Meehan (2019), the bounded support assumption can be relaxed at the cost of imposing another condition pertain to higher-order moments. For instance, the boundedness of the 16th order moment of $\sum_{m' \in \mathcal{M}_N} g_{mm'} x_{m'}$ is sufficient for the subsequent results.

Assumption 4.3.

   (i) There exists $q \in (0,3]$ such that $\lim_{M \to \infty} \frac{1}{M^q} \sum_{s \geq 0} \delta_M^\partial(s;1) = 0$;

   (ii) There exists $q \in (0,5]$ such that $\lim_{M \to \infty} \frac{1}{M^q} \sum_{s \geq 0} c_M(s, b_M; 2) = 0$.

Both requirements of Assumption 4.3 restrict the denseness of the network, and rule out the situation where the network becomes progressively dense as the number of dyads increases.

Assumption 4.4. $\sup_{N \geq 1} \frac{N \pi_{N,M}}{M} < \infty$.

In light of Assumption 2.2, this condition amounts to requiring the boundedness of $\pi_{N,M}$. The following theorem is the main theoretical contribution of this paper.

Theorem 4.1 (Consistency of the Network-HAC Variance Estimator). *Suppose that Assumptions 4.1, 4.2, 4.3 and 4.4 hold. Then,*

$$\left\| N \pi_{N,M} \widehat{Var}(\hat{\beta}) - Var(\hat{\beta}) \right\|_F \xrightarrow{p} 0 \qquad as \quad N \to \infty,$$

where $\| \cdot \|_F$ indicates the Frobenius norm.[14]

Proof. See Appendix C.4. □

Theorem 4.1 establishes the consistency of our proposed variance estimator accounting for network spillovers across dyads in the sense of the Frobenius norm. By the equivalence of

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[14]The Frobenius norm of a real matrix $A := (a_{ij})_{i,j}$, dented by $\|A\|_F$, is given by

$$\|A\|_F := \sqrt{\text{tr}(A^T A)}$$

$$= \sqrt{\sum_i \sum_j a_{ij}^2}.$$

matrix norms, the consistency remains true in the sense of the spectral norm and the nuclear norm. It is immediate to see that this theorem yields results consonant with a number of important preceding results, though conditions are not necessarily nested. For example, if $\pi_{N,M} = 1$, Theorem 4.1 boils down to Proposition 1 of Aronow, Samii, and Assenova (2015). Under the condition that $\pi_{N,M} = \frac{M}{N}$, the theorem states the same result as Proposition 3.3 of Tabord-Meehan (2019), while the results of Proposition 3.4 and 3.5 can be replicated when $\pi_{N,M} = \frac{M}{N^2}$ and $\pi_{N,M} = \frac{M}{N^{r+1}}$, respectively. Moreover, it follows from this theorem that the dyadic-robust variance estimator (12) is inconsistent for the true variance when the underlying network involves a nonnegligible degree of far-away correlations.

Corollary 4.1 (Inconsistency of Dyadic-Robust Variance Estimators in the Presence of Network Spillovers Across Dyads). *Suppose that the assumptions required in Theorem 4.1 hold. Assume, in addition, that*

$$(21) \qquad \inf_{N \geq 1} \frac{1}{M} \sum_{s \geq 2} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N^\partial(m;s)} \left\| E\left[\varepsilon_m \varepsilon_{m'} x_m x_{m'}' \right] \right\|_F > 0.$$

Then, the dyadic-robust estimator (12) applied to the network-regression model (10) is inconsistent.

Proof. See Appendix C.4. □

The added assumption (21) in Corollary 4.1 pertains to both the network configuration of active dyads and the regression variables. It represents a setting where the spillovers from far-away neighbors is not negligible even when N is large. If, on the other hand, far-away neighbors in G only have a negligible effect on the cross-sectional dependence of our model, the dyadic-robust variance estimator remains a good approximation for the asymptotic variance of linear dyadic data models with network spillovers across dyads. This insight is further investigated through numerical simulations in Section 5.

5. Monte Carlo Simulation

This section explores the finite sample properties of the asymptotic inference of the proposed variance estimator by means of Monte Carlo simulations.

5.1. Simulation Design

We compare three types of variance estimators across different specifications and network configurations. We use the Eicker-Huber-White estimator as a benchmark, the dyadic-robust estimator of Tabord-Meehan (2019) as a comparison accounting for the dyadic nature of the data, and our proposed estimator which is robust to network spillovers across dyads.

We first generate networks on which random variables are assigned. We follow Canen, Schwartz, and Song (2020) among others by employing two models of random graph formations. They are referred to as Specifications 1 and 2. Specification 1 uses the Barabśi and Albert's (1999) model of preferential attachment, with the fixed number of edges $\nu \in \{1, 2, 3\}$ being established by each new node.[15] Specification 2 is based on the Erdös-Renyi random graph (Erdös and Rényi, 1959, 1960) with probability $p := \frac{\lambda}{N}$ for N denoting the number of nodes and $\lambda \in \{1, 2, 3\}$ being a parameter that governs the probability relative to the node size.

For each of the randomly generated networks, we consider a set of agents (denoted by i, j, etc) populated along the network. We can construct the network of active dyads from this simulated network among individual agents: there exists an edge between two active dyads only if they share a common agent. The simulation data is generated from the following simple network-linear regression:

$$y_m = x'_m \beta + \varepsilon_m,$$

with $m := d(i, j)$ representing the dyad between agent i and j.

The dyad-specific regressor x_m is defined as $x_m := |x_i - x_j|$, where both x_i and x_j are drawn independently from $\mathcal{N}(0, 1)$. The regression coefficient is fixed to be $\beta = 1$ across specifications.

The dyad-specific error term ε_m is constructed to exhibit non-zero correlation with $\varepsilon_{m'}$ as long as dyads m and m' are connected (i.e., in the network terminology, there exists a path in the simulated network), while the strength of the correlation is assumed to decay as they are more distant. To that end, we draw $\varepsilon_m := \sum_{m'} \gamma_{m,m'} \eta_{m,m'}$, where $\gamma_{m,m'}$ equals γ^s if the distance between m and m' is s, and 0 otherwise, for $\gamma \in [0, 1]$[16] and $s \in \{1, \ldots, S\}$ with S being the maximum geodesic distance that the spillover propagates. Each $\eta_{m,m'}$ is drawn i.i.d. from $\mathcal{N}(0, 1)$. If $\gamma = 1$, then spillover effects are the same no matter how far the agents are apart, i.e., the spillover effects do not decay. If $\gamma = 0$, there are no spillover effects, so the dyadic-robust estimator should be consistent. The case of $S = 2$ corresponds to a situation where up to friends of friends may matter for spillovers.

We consider three specifications for each type of network. In the main text, we set $S = 2$ and $\gamma = 0.8$. The results for $S = 2$ with $\gamma = 0.2$ are given in Appendix B.3, and the ones for $S = 1$ with $\gamma = 0.8$ are in Appendix B.4. For simplicity, the kernel function is set to be one. The nominal size is set at $\alpha = 0.05$ throughout the 5000 iterations of the Monte Carlo simulation.

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[15] In generating the Barabási-Albert random graphs, we follow Canen, Schwartz, and Song (2020) by choosing the seed to be the Erdös-Renyi random graph with the number of nodes equal the smallest integer above $5\sqrt{N}$, where N denotes the number of nodes.

[16] In this simulation, we focus on cases of positive spillovers. Cases of negative spillovers can be analyzed in an analogous manner.

5.2. Degree Characteristics

Table 1 reports the degree characteristics of the networks when viewed as networks over the active edges.[17] The table provides the average degree, the maximum degree, and the number of active edges (i.e., dyads). By construction, the maximum degree and the average degree increase monotonically as we increase the parameters in both specifications. The number of active edges increases with the sample size regardless of the specification. This reflects the fact that each node tends to have more far-away neighbors as the network become denser.

TABLE 1. The Average and Maximum Degrees of Networks among Edges in the Simulations

N		Specification 1			Specification 2		
		$v = 1$	$v = 2$	$v = 3$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
500	d_{act}	401	788	1175	238	484	740
	d_{max}	32	45	70	4	9	14
	d_{ave}	3.6858	6.0063	9.1881	0.9580	2.0248	3.0027
1000	d_{act}	859	1699	2540	498	981	1501
	d_{max}	35	55	76	5	8	10
	d_{ave}	3.9581	6.7810	9.2047	1.0341	1.9888	2.9594
5000	d_{act}	4663	9305	13952	2476	5008	7557
	d_{max}	74	161	210	7	12	15
	d_{ave}	5.2989	9.5159	12.8521	1.0137	2.0228	3.0341

Note: Observation units in this table are active edges (dyads), which departs from the convention. Active edges are edges that are at work in the original network over the nodes. The number of active edges is denoted by d_{act}. The maximum degree, d_{max}, expresses the maximum number of edges that are adjacent to an edges, and the average degree, d_{ave}, is the average number of edges adjacent to each edge of the network.

5.3. Results

In Table 2 we present the coverage probability for β and the average length of the confidence interval across simulations. To do so, we compute the t-statistic using the OLS estimator for β and different variance estimators under a Normal distribution approximation. We also follow Aronow, Samii, and Assenova (2015) in calculating the empirical standard error[18] and plotting it in the boxplots of the variance estimates together with the variance estimates in Figure 3.[19]

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[17]The degree characteristics of the original networks are summarized in Table 4 in Appendix B.1.

[18]The empirical standard error is obtained as a standard error of the parameter estimates.

[19]It is well known that estimates of variance-covariance matrix may be negative semidefinite when the sample size is very small. This occurs in 4 out of 5000 simulations when $N = 500$. Rather than dropping such observations,

The results for the empirical coverage probabilities depend on two dimensions: the sample size (N) and the denseness of the underlying network (parametrized by v and λ). The coverage probability for each estimator improves with the sample size. However, when spillovers are high ($\gamma = 0.8$), only our proposed network-robust estimator has coverage close to 95%. Meanwhile, both the Eicker-Huber-White and the dyadic-robust variance estimators perform poorly as the underlying network becomes denser, no matter which specification of the network is involved. For example, in Specification 1 with $v = 3$ and the largest sample size ($N = 5000$), the confidence intervals based on the Eicker-Huber-White and the dyadic-robust variance estimators do not cover the true parameter 615 and 455 times out of 5000 simulations (12.3% and 9.1%), respectively. This is because as the network gets denser, each agent tends to have more direct or indirect links to other agents. This leads to larger higher-order correlations, which is neither accounted by the Eicker-Huber-White nor by the dyadic-robust variance estimator. On the other hand, the network-robust estimator is designed to capture such higher-order correlations and, thus, its coverage remains stable across network configurations. A similar conclusion is drawn from the average length of the confidence intervals. The confidence intervals for the Eicker-Huber-White and dyadic-robust variance estimators are typically shorter than those for our proposed estimator when γ is large and $S = 2$. This is because the former undercover the true parameter (in the presence of positive spillovers), confirming our theoretical results. The difference is often around 10-20% of the length of the network-robust estimator.

However, as the magnitude of spillovers decreases (i.e. γ tends to 0), then higher-order spillovers are less pronounced, and the biases from using Eicker-Huber-White and dyadic-robust variance estimators disappear. This is shown in Table 6 of Appendix B when $S = 2$ and $\gamma = 0.2$. When $S = 1$, then the dyadic-robust variance estimator coincides with our proposed estimator (i.e., there are no spillovers from non-adjacent links). This is shown in Table 7.

The finite-sample properties of the three variance estimators are further illustrated in Figure 3, where the horizontal axes represent the sample size and the vertical axes indicate the standard error of the regression coefficient. The boxplots show the 25th and 75th percentiles across simulations, as well as the median, with the whiskers indicating the bounds that are not considered as outliers. The whisker length is set to cover ± 2.7 times the standard deviation of the standard-error estimates. The light-, medium- and dark-grey boxplots describe the distribution of the Eicker-Huber-White, dyadic-robust and our proposed network-robust variance estimates across simulations, respectively. The diamonds indicate the empirical standard errors of the estimates of the regression coefficients, what Aronow, Samii, and Assenova (2015)

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we follow Cameron, Gelbach, and Miller (2011) and augment the eigenvalues of the matrix by adding a small constant, say 0.005, thereby obtaining a new variance estimate that is more conservative.

TABLE 2. The empirical coverage probability and average length of confidence intervals for β at 95% nominal level: $S = 2$, $\gamma = 0.8$.
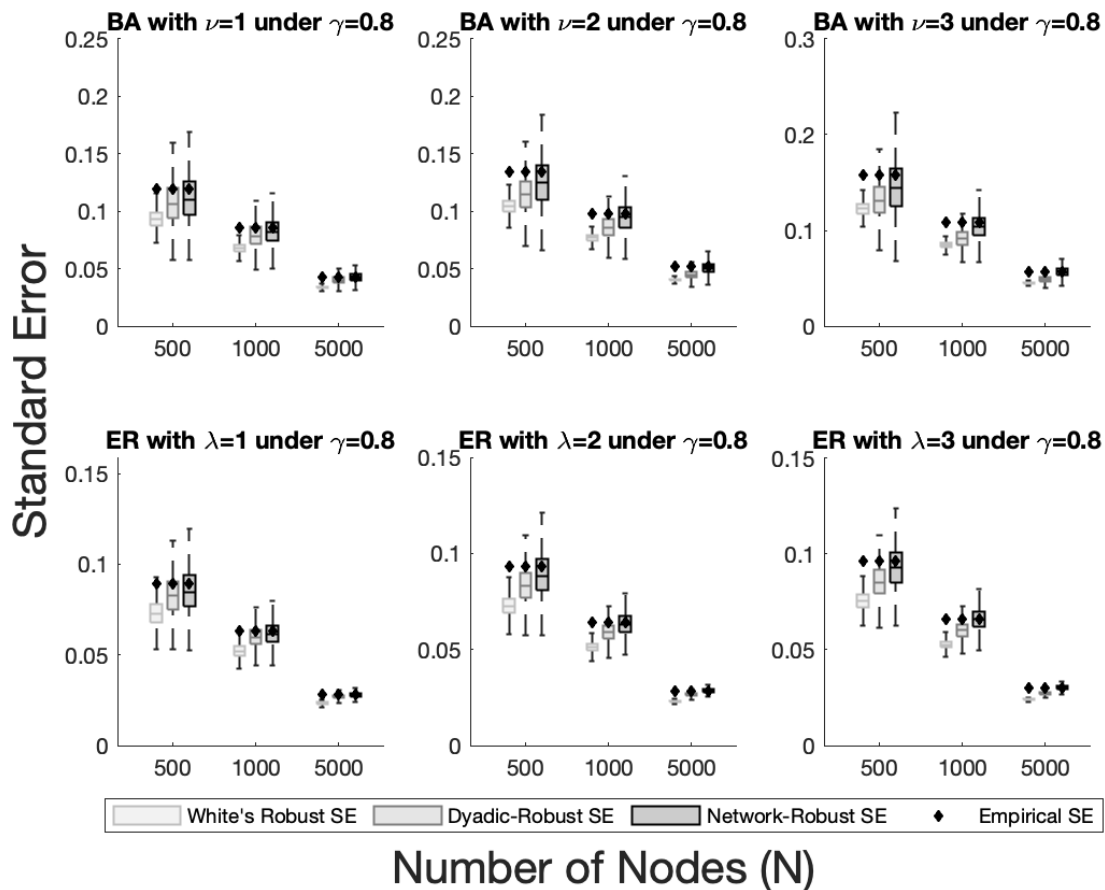
	N	Specification 1			Specification 2		
		$v = 1$	$v = 2$	$v = 3$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
		Coverage Probability					
Eicker-Huber-White	500	0.8774	0.8680	0.8712	0.8910	0.8700	0.8750
	1000	0.8798	0.8732	0.8728	0.8924	0.8810	0.8876
	5000	0.8792	0.8708	0.8770	0.8928	0.8890	0.8790
Dyadic-robust	500	0.9216	0.8976	0.8938	0.9320	0.9212	0.9172
	1000	0.9286	0.9134	0.9006	0.9366	0.9274	0.9242
	5000	0.9342	0.9118	0.9090	0.9388	0.9332	0.9224
Network-robust	500	0.9300	0.9166	0.9146	0.9370	0.9370	0.9410
	1000	0.9392	0.9338	0.9328	0.9456	0.9446	0.9476
	5000	0.9486	0.9440	0.9426	0.9468	0.9472	0.9480
		Average Length of the Confidence Intervals					
Eicker-Huber-White	500	0.3678	0.4093	0.4819	0.2874	0.2853	0.2962
	1000	0.2663	0.3024	0.3309	0.2048	0.2013	0.2068
	5000	0.1321	0.1585	0.1756	0.0916	0.0898	0.0938
Dyadic-robust	500	0.4260	0.4543	0.5204	0.3280	0.3285	0.3365
	1000	0.3124	0.3393	0.3611	0.2359	0.2323	0.2366
	5000	0.1580	0.1775	0.1920	0.1060	0.1043	0.1077
Network-robust	500	0.4413	0.4932	0.5681	0.3370	0.3487	0.3655
	1000	0.3263	0.3733	0.4079	0.2435	0.2482	0.2585
	5000	0.1666	0.1994	0.2215	0.1097	0.1117	0.1183

Note: The upper-half of the table displays the empirical coverage probability of the asymptotic confidence interval for β_1, and the lower-half showcases the average length of the estimated confidence intervals. As the sample size (N) increases, the empirical coverage probability for our estimator accounting for network spillovers approaches 0.95, the correct nominal level. However, that is not the case for alternative estimators.

call the true standard error. It is unsurprising that the empirical standard errors are the same across different variance estimators, as we use the same $\hat{\beta}$. The boxplots show that as the sample size increases, the variation of the network-robust variance estimator shrinks, reaching the empirical standard error (the diamonds). This is expected since this estimator is consistent for the true variance (Theorem 4.1). The estimates appear to vary little for moderate sample sizes (e.g., $N = 1000$). However, the other variance estimators (the light- and medium-grey boxplots) converge to lower values than the empirical standard errors (the diamonds), verifying their inconsistency in this environment with network spillovers, as shown by Corollary 4.1. As we make such spillovers very small (e.g., $\gamma = 0.2$ in Appendix B.3), all estimators have

similar performance. This highlights the role of condition ([21](#)): namely, the dyadic-robust variance estimator might perform satisfactorily well as long as higher-order correlations beyond immediate neighbors are negligible.

FIGURE 3. Boxplots of Standard Errors for Specifications 1 and 2 ($S = 2$, $\gamma = 0.8$)

Note: This figure shows boxplots describing the estimated standard errors and the empirical standard errors for various combinations of parameters under Specification 1 (Barabási-Albert networks) and Specification 2 (Erdös-Renyi networks). The horizontal axis shows the number of nodes and the vertical axis represents the the standard error of the coefficient. The shaded boxes represent the 25th, 50th and 75th percentiles of estimated standard errors with the whiskers indicating the most extreme values that are not considered as outliers. The light-grey box illustrates the Eicker-Huber-White standard error, the medium-grey one the dyadic-robust standard error and the dark-grey one the network-robust standard error. The diamonds stand for the empirical standard error, defined as the standard deviation of the estimates of the regression coefficient. The estimator is not covering the true standard error when the diamond is outside of the shaded area.

6. Empirical Illustration: Legislative Voting in the European Parliament

We now turn to an empirical application to demonstrate the performance of our variance estimator with real-world data. In so doing, we revisit the work of Harmon, Fisman, and Kamenica (2019) on whether legislators who sit next to each other in Parliament tend to vote more alike on policy proposals.

In their original article, the authors study the existence of peer effects in legislative voting. They focus on the European Parliament, whose Members (MEPs) are voted in through elections in each European Union (EU) member country every five years. The Parliament convenes once or twice a month, in either Brussels or Strasbourg, to debate and vote on a series of proposals.

Once elected to the European Parliament (EP), these MEPs are organized into European Political Groups (EPGs), which aggregate similar ideological members/parties across countries. As Harmon, Fisman, and Kamenica (2019) describe, these EPGs function as parties for many of the traditional party-functions in other legislatures, including coordination on policy and policy votes. Most importantly, MEPs sit within their EPG groups in the chamber. However, within each EPG group, non-party leaders traditionally sit in alphabetical order by last name. See Figure 4 for an example. Hence, to the extent that last names are exogenous (i.e., politicians are not selected or choose last names anticipating seating arrangements in the European Parliament), seating arrangements are used as a quasi-natural experiment: it is sometimes by chance that one legislator sits next to another. And across different periods, as politicians get replaced, they find themselves sitting with new neighbors.

6.1. Data

We adopt the same dataset used in Harmon, Fisman, and Kamenica (2019), which collects the MEP-level data on votes cast in the EP. The dataset records what each MEP voted for (Yes or No), where she is seated, and a number of individual characteristics (e.g. country, age, education, gender, tenure, etc). Its sample period are the plenary sessions held in both Brussels and Strasbourg between October 2006 and November 2010. This corresponds to the sixth and seventh terms of the European Parliament.

For our empirical illustration, we restrict the sample to the policies voted in Strasbourg during the seventh term and we focus on the seating pattern between July 14th - July 16th, 2009 (which involved 116 different proposals being voted on). The resulting sample has 2,431,261 observations, which are split into 422 politicians forming 26,099 pairs (i.e., dyads) of MEPs[20]

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[20]There are 334 pairs of adjacent dyads and 591 pairs of connected dyads.

FIGURE 4. Seating Plan at the European Parliament: Strasbourg, September 14, 2009
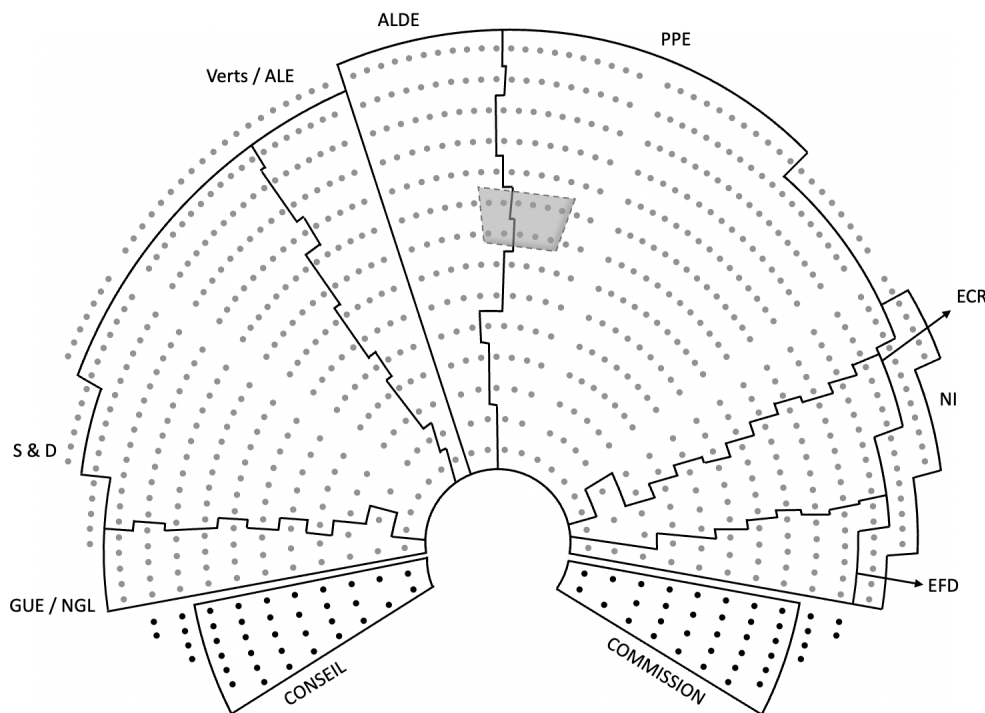
(A) Zoomed-Out View

	ALDE		PPE				
Row 10 $\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
Row 9 $\cdots$	Seat 36: ALFANO	Seat 37: ALVARO	Seat 38: CAVADA	Seat 39: COELHO	Seat 40: COLLINO	Seat 41: COMI	$\cdots$
Row 8 $\cdots$	Seat 32: OJULAND	Seat 33: OVIIR	Seat 34: BACH	Seat 35: BALDASSARRE	Seat 36: BALZ		$\cdots$
Row 7 $\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

(B) Zoomed-In View

Note: Panel (A) illustrates a zoomed-out view of a seating plan for the European parliament in Strasbourg on September 14, 2009. Grey circles are individual MEPs, while black circles embody members of conseil and commission. The associated party (EPG) is denoted at the top. Panel (B) provides a zoomed-in view elaborating on the part of Panel (A) marked by the dotted trapezoid shaded in grey. Alafano and Alvaro are treated as adjacent because they are sitting next to each other and belong to the same political party, i.e., ALE. Similarly, Ojuland and Oviir are considered to be adjacent. On the other hand, following the original authors, Alvaro and Cavada are not regarded as adjacent though they are seated together because they belong to different political parties, i.e., ALE and PPE, respectively. In terms of dyad-level adjacency, Cavada-Coelho and Coelho-Collino are adjacent dyads as they share Coelho, whereas Cavada-Coelho and Collino-Comi are not adjacent, but they are still connected as they have indirect paths to one another along the dyadic network.

over 116 proposals. Further information on the construction of our sample is detailed in Appendix D. These restrictions keep the main set-up in Harmon, Fisman, and Kamenica (2019), while allowing us to evaluate variance estimators with smaller sample sizes.

6.2. Empirical Setup

The empirical analysis in Harmon, Fisman, and Kamenica (2019) studies whether MEPs who are seated together are more likely to vote similarly. To make it consistent with our notation above, let $m = d(i, j)$ denote the dyad between legislator i and j, with t being used to index the proposal. Let $Agree_{d(i,j),t}$ be an indicator that takes one if MEP i and j cast the same vote on proposal t, and zero otherwise. Likewise, $SeatNeighbors_{d(i,j),t}$ is a binary variable that equals one if MEP i and j are seated next to each other when the vote for proposal t is taken place, and zero otherwise. We view the seating arrangement as a network among the MEPs: i.e., two MEPs who are seated next to each other within the same political group are treated as adjacent. (See the note below Figure 4.)[21]

We follow Harmon, Fisman, and Kamenica (2019) in assuming that such seating arrangements are exogenously determined, i.e., the adjacency relation G is exogenously formed. Their main specification is a linear model:

$$(22) \qquad Agree_{d(i,j),t} = \beta_0 + \beta_1 SeatNeighbors_{d(i,j),t} + \varepsilon_{d(i,j),t}.$$

The authors originally conducted inference using the estimator in Aronow, Samii, and Assenova (2015), assuming that dyads $m = d(i, j)$ cannot be correlated with $m' = d(k, l)$ unless they share a common member (i.e., there is no correlation across errors if i, j and k, l do not include a common unit). We will now compare this approach to using the variance estimator introduced in Section 4, which allows error terms to exhibit arbitrary correlations as long as they are connected on the network represented by the adjacency relation of seating arrangements in Parliament.

Inspired by Harmon, Fisman, and Kamenica (2019), we consider three specifications: (I) a simple linear regression model as given in (22); (II) the model (22) augmented with a flexible set of other demographic variables;[22] and (III) the model (22) with both a flexible set of other demographic variables and day-specific fixed effects.[23] When fixed effects are present in their

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[21]This accommodates the row-by-EP-by-EPG clustering implemented by the authors. This is already an extension to Harmon, Fisman, and Kamenica (2019) which assumed that only seating neighbors would have correlated ε. One could extend this further by allowing connections across parties.

[22]Following Harmon, Fisman, and Kamenica (2019), we include indicators whether country of origins, quality of education, freshman status and gender, respectively, are the same, as well as differences in ages and tenures. See Table 3 for details.

[23]The day-specific fixed effects are meant to control for unmeasured factors specific to a single day, possibly correlated to the covariates.

original estimation, we estimate a within-difference model via OLS. Since our focus is to study how the standard errors are different when the far-away neighbors are considered, we do not pursue instrumenting the possible endogeneity in sitting together by means of name adjacency.

6.3. Results

The main results of our empirical analysis are summarized in Table 3. Panel A displays the parameter estimates with the standard errors obtained from our proposed variance estimator from equation (9). The full model with all covariates, Specification (III), shows that our point-estimates are consistent in magnitude with the original estimates of Harmon, Fisman, and Kamenica (2019) (column 7 of Table 4), as they are close to 0.006 (their original results) and stable across specifications. Hence, changes to point-estimates are not due to sample selection.[24] The positive coefficient for *SeatNeighbors* indicates that the MEPs sitting together tend to vote more similarly than those sitting apart, providing evidence in favor of their original hypothesis. The coefficients on the covariates are also quantitatively and qualitatively similar to those in their original paper. For instance, our estimates for *SameCountry* are 0.056, while their estimates are around 0.051, suggesting politicians from the same country are more likely to vote similarly on policies.

Panel B shows the standard errors for the regression coefficient of *SeatNeighbors* using different variance estimators. Building on the last section, the panel compares three different types of heteroskedasticity robust estimators: namely, Eicker-Huber-White, the dyadic-robust estimator (used in their original work), and our proposed network-robust estimator. We note that, due to the smaller sample size, the standard errors in our exercise are larger than the original authors'. Hence, we do not wish to draw conclusions from our inference relative to theirs.

As foreshadowed in the Monte Carlo simulations, the Eicker-Huber-White estimates are the smallest, followed by the dyadic-robust estimates, which, in turn, are smaller than the network-robust estimates. In fact, for Specification (III), Eicker-Huber-White estimates are roughly 73% smaller than using the estimator accounting for network spillovers across dyads, while dyadic-robust ones are 22% smaller. This fact entails two implications. First, our finding provides empirical evidence in support of the existence of *indirect* positive spillovers among the MEPs: even distant connections may indirectly generate correlated behavior among politicians i and j. Second, the use of alternative estimators not accounting for such spillovers undercover the true parameter and may generate biased hypothesis testing about the regression coefficient of

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[24]Note that our dependent variable is equal to one if two MEPs vote the same and zero otherwise, while Harmon, Fisman, and Kamenica (2019) code it as one if MEPs vote differently. Hence, to compare our estimates with their paper, the signs on the estimates of $SeatNeighbors$ should be flipped around.

SeatNeighbors. The difference in estimates appears quantitatively meaningful in this empirical example.

TABLE 3. Spillovers in Legislative Voting – Main Analysis

	Specification (I)	Specification (II)	Specification (III)
Panel A: Parameter estimates			
Seat neighbors	0.0069	0.0060	0.0060
	(0.0095)	(0.0104)	(0.0112)
Same country		0.0561	0.0562
		(0.0008)	(0.0008)
Same quality education		0.0030	0.0028
		(0.0007)	(0.0007)
Same freshman status		-0.0070	-0.0070
		(0.0008)	(0.0008)
Same gender		0.0004	0.0004
		(0.0007)	(0.0006)
Age difference		0.0007	0.0004
		(0.0004)	(0.0004)
Tenure difference		-0.0149	-0.0149
		(0.0006)	(0.0006)
Day-level FE	No	No	Yes
Panel B: Standard errors for Seat neighbors			
Eicker-Huber-White	0.0031	0.0030	0.0030
Dyadic-robust	0.0075	0.0082	0.0087
Network-robust	0.0095	0.0104	0.0112

Note: Adjacency of MEPs is defined at the level of a row-by-EP-by-EPG. (See the note below Figure 4.) Independent variables are self-explained: *Seat neighbors* is an indicator variable denoting whether both MEPs sit together; *Same country* represents an indicator for whether both MEPs are from the same country; *Same quality education* is an indicator showing whether both MEPs have the same quality of education background, measured by if both have the degree from top 500 universities; *Same freshman status* encodes if both MEPs are freshman; *Age difference* is the difference in the MEPs' ages; and *Tenure difference* measures the difference in the MEPs' tenures.

7. Conclusion

Dyadic data have widespread applications in social sciences. Researchers typically assume that dyads are uncorrelated if they do not share a common unit, an assumption that is leveraged in inference. We showed that this is not the case in many models whereby strategic interactions occur on a network: while data may be dyadic, the cross-sectional dependence

may be much more complex and spillover beyond pairwise interactions. For instance, trade between countries may depend on trade between auxiliary partners, even those beyond the dyads being considered. In political economy, whether a politician votes with a colleague may depend on all intermediate seating neighbors beyond one's own immediate ones. We verified this using both theoretical results and Monte Carlo simulations. A new consistent variance estimator for parameters in a linear model with dyadic data were derived, which has correct asymptotic coverage and good finite sample properties. This estimator generalizes existing results (e.g. Aronow, Samii, and Assenova (2015)) even in non-i.i.d. settings. It is also simple to implement and provided stable inference in an application to concordance in legislative voting across pairs of members using real-world data. We found that, when accounting for network spillovers across dyads, standard errors grew by at least 20% compared to the original specification. One of the limitations of our approach is the assumption that $\pi_{N,M}$ is known to the econometrician *a priori*. Extension to the case where $\pi_{N,M}$ is unknown is left for future work.

To conclude, we would like to clarify that our goal in this exercise is neither to criticize dyadic-robust variance estimators, which are a fundamental part of the empiricist's toolkit, nor to suggest our approach should always be used. Rather, we wish to draw attention that researchers should fully specify the cross-sectional dependence in their model. If it is a purely dyadic environment or one where such spillovers might be negligible, then previous approaches suffice and will be equal, or possibly more efficient, than the one proposed here. However, as we have discussed above, many existing applications seemingly apply the latter method even if it is seemingly inappropriate to their setting. Hence, we recommend researchers to continue to fully specify their model, including full specification of their covariance structure, thereby clarifying what type of inference procedure is most appropriate for their environment.

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Appendix A. Mathematical Setup

This section lays out the mathematical setup of our model in more detail, heavily drawing from Kojevnikov, Marmer, and Song (2021).

We first define a collection of pairs of sets of dyads. For any positive integers a, b and s, define

$$\mathcal{P}_M(a,b;s) := \{(A,B) : A,B \subset \mathcal{M}_N, |A| = a, |B| = b, \rho_M(A,B) \geq s\},$$

where

(23)
$$\rho_M(A,B) := \min_{m \in A} \min_{m' \in B} \rho_M(m,m'),$$

with $\rho_M(m,m')$ denoting the geodesic distance between dyads m and m'. In words, the set $\mathcal{P}_M(a,b;s)$ collects all two distinct sets of active dyads whose sizes are a and b and that have no dyads in common.

Next we consider a collection of bounded Lipschitz functions. Define

$$\mathcal{L}_K := \left\{\mathcal{L}_{K,c} : c \in \mathbb{N}\right\},$$

where

$$\mathcal{L}_{K,c} := \left\{f : \mathbb{R}^{K \times c} \to \mathbb{R} : \|f\|_\infty < \infty, \mathrm{Lip}(f) < \infty\right\},$$

with $\|\cdot\|_\infty$ representing the supremum norm and $\mathrm{Lip}(f)$ being the Lipschitz constant.[25] In words, the set $\mathcal{L}_{K,c}$ collects all the bounded Lipschitz functions on $\mathbb{R}^{K \times c}$ and the set $\mathcal{L}_K$ moreover gathers such sets with respect to $c \in \mathbb{N}$.

Lastly, we write

$$Y_{M,A} := (Y_{M,m})_{m \in A},$$

and $Y_{M,B}$ is analogously defined. Let $\{\mathcal{C}_M\}_{M \geq 1}$ denote a sequence of σ-algebras and be suppressed as $\{\mathcal{C}_M\}$.

The network dependent random variables are characterized by the upper bound of their covariances.

Definition A.1 (Conditional ψ-Dependence given $\{\mathcal{C}_M\}$: Kojevnikov, Marmer, and Song (2021), Definition 2.2). A triangular array $\{Y_{M,m} \in \mathbb{R}^K : M \geq 1, m \in \{1,\dots,M\}\}$ is called conditionally ψ-dependent given $\{\mathcal{C}_M\}$, if for each $M \in \mathbb{N}$, there exist a $\mathcal{C}_M$-measurable sequence $\theta_M := \{\theta_{M,s}\}_{s \geq 0}$ with $\theta_{M,0} = 1$, and a collection of nonrandom function $(\psi_{a,b})_{a,b \in \mathbb{N}}$ where $\psi_{a,b} : \mathcal{L}_{K,a} \times$

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[25]It is immediate to see that $\mathbb{R}$ is a normed space with respect to the Euclidean norm, while the $\mathbb{R}^{K \times c}$ can be equipped with the norm $\rho_c(x,y) := \sum_{\ell=1}^{c} \|x_\ell - y_\ell\|$ where $x,y \in \mathbb{R}^{K \times c}$ and $\|z\| := (z'z)^{\frac{1}{2}}$, thereby the Lipschitz constant is defined as $\mathrm{Lip}(f) := \min\{w \in \mathbb{R} : |f(x) - f(y)| \leq w\rho_c(x,y) \ \forall x,y \in \mathbb{R}^{K \times c}\}$.

$\mathcal{L}_{K,b} \to [0.\infty)$, such that for all $(A, B) \in \mathcal{P}_M(a, b; s)$ with $s > 0$ and all $f \in \mathcal{L}_{K,a}$ and $g \in \mathcal{L}_{K,b}$,

$$\left| Cov\left(f(Y_{M,A}), g(Y_{M,B}) \,\big|\, \mathcal{C}_M \right) \right| \leq \psi_{a,b}(f, g) \theta_{M,s} \quad a.s.$$

Intuitively, this definition states that the upper bound must be decomposed into two components, one of which is deterministic and depends on nonlinear Lipschitz functions f and g, and the other of which is stochastic and depends only on the distance of the random variables on the underlying network. The former, nonrandom component reflects the scaling of the random variables as well as that of the Lipschitz transformations, while the latter random part stands for the covariability of the two random variables. We follow Kojevnikov, Marmer, and Song (2021) in assuming boundedness for these two components.

Assumption A.1 (Kojevnikov, Marmer, and Song (2021), Assumption 2.1)**.** The triangular array $\{Y_{M,m} \in \mathbb{R}^K : M \geq 1, m \in \{1, \dots, M\}\}$ is conditionally ψ-dependent given $\{\mathcal{C}\}$ with the dependence coefficients $\{\theta_{M,s}\}$ satisfying the following conditions:

(a) There exists a constant $C > 0$ such that

$$\psi_{a,b}(f, g) \leq C \times ab \left(\|f\|_\infty + \mathrm{Lip}(f) \right) \left(\|g\|_\infty + \mathrm{Lip}(g) \right);$$

(b) $\sup_{M \geq 1} \max_{s \geq 1} \theta_{M,s} < \infty$ a.s.

Assumption A.1 is maintained throughout the paper and employed to show asymptotic properties of our estimators such as the consistency and asymptotic normality and the consistency of the network-robust variance estimator for dyadic data.

To prove Theorem 3.2, we make use of the following assumptions which strengthen their counterparts in the proof of consistency.

Assumption A.2 (Conditional Finite Moment of ε_m)**.** There exists $p > 4$ such that

$$\sup_{N \geq 1} \max_{m \in \mathcal{M}_N} E\left[|\varepsilon_m|^p \,|\, \mathcal{C}_M \right] < \infty \quad a.s.$$

Assumption A.2 allows for the same interpretation as Assumption 3.1, i.e., the random error term ε_m cannot be too large, conditional on a common component. This assumption, however, is more stringent than the previous one because it now requires the finiteness of a higher moment of ε_m. When coupled with Assumption 2.4, Assumption A.2 implies that Assumption 3.3. of Kojevnikov, Marmer, and Song (2021) for each $Y_{M,m}^u$ with $u \in \{1, \dots, K\}$.

Assumption A.3 (Kojevnikov, Marmer, and Song (2021), Assumption 3.4)**.** There exists a positive sequence $r_M \to \infty$ such that for $k = 1, 2,$

$$\frac{M^2 \theta_{M,r_M}^{1-1/p}}{\sigma_M} \xrightarrow{a.s.} 0,$$

$$\frac{M}{\sigma_M^{2+k}} \sum_{s \geq 0} c_n(s, r_M; k) \theta_{M,s}^{1-\frac{2+k}{p}} \xrightarrow{a.s.} 0,$$

as $M \to \infty$, where $p > 4$ is the same as the one that appears in Assumption A.2.

Similar to Assumption 3.2, this assumption binds the covariance of the random variables, the dependence reflected in the dependence coefficients, and the underlying network. The first part of this assumption requires that σ_M grows at least as fast as M does. Kojevnikov, Marmer, and Song (2021) claim that it is satisfied if the long-run variance $\frac{\sigma_M^2}{M}$ is bounded away from c^2 for all $M \geq 1$ for some constant c independent of M, i.e., $\sigma_M \geq c\sqrt{M}$. The second part can be interpreted in an analogous manner to Assumption 3.2. That is, given σ_M growing at least at the same rate of M, the composite of the density of the network and the magnitude of the correlations of the random variables must decay fast enough.

Appendix B. Additional Monte Carlo Simulation Results

B.1. Summary Statistics

Table 4 shows summary statistics (i.e., the average and maximum degrees) of the networks across nodes that are used in our simulation study. By construction, the maximum degree and the average degree monotonically increase in the parameters for both specifications.

TABLE 4. The Average and Maximum Degrees of Networks among Nodes in the Simulations

N		Specification 1			Specification 2		
		$\nu = 1$	$\nu = 2$	$\nu = 3$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
500	d_{max}	23	40	41	5	7	8
	d_{ave}	0.8020	1.5780	2.3540	0.4760	0.9680	1.4800
1000	d_{max}	26	36	47	4	7	8
	d_{ave}	0.8590	1.7000	2.5410	0.4980	0.9810	1.5010
5000	d_{max}	53	125	130	6	9	10
	d_{ave}	0.9326	1.8618	2.7910	0.4952	1.0016	1.5114

Notes: Observation units in this table are nodes (individuals) as usual in the literature. The maximum degree, d_{max}, means the maximum number of nodes that are adjacent to a node, and the average degree, d_{ave}, is the average number of nodes adjacent to each node of the network.

B.2. $S = 2$ and $\gamma = 0.8$

Table 5 describes the the standard deviation of the estimated regression coefficients (what Aronow, Samii, and Assenova (2015) calls the true standard errors) and the means of the estimated standard errors for each variance estimator. The round brackets indicate the biases of each estimate relative to the true standard error in percentage (%). For instance, the Eicker-Huber-White variance estimator and the dyadic-robust variance estimator, when applied to Specification 1 with $\nu = 3$, underestimate the true standard error by 21.45% and 14.14%, respectively.

TABLE 5. Means and Biases of the Standard Errors: $N = 5000$, $S = 2$, $\gamma = 0.8$.

	Specification 1			Specification 2		
	$v = 1$	$v = 2$	$v = 3$	$m = 1$	$m = 2$	$m = 3$
True	0.0430	0.0518	0.0570	0.0285	0.0283	0.0302
Eicker-Huber-White	0.0337	0.0404	0.0448	0.0234	0.0229	0.0239
	(-21.61)	(-21.92)	(-21.45)	(-17.91)	(-19.14)	(-20.68)
Dyadic-robust	0.0403	0.0453	0.0490	0.0270	0.0266	0.0275
	(-6.19)	(-12.54)	(-14.14)	(-5.01)	(-6.12)	(-8.93)
Network-robust	0.0425	0.0509	0.0565	0.0280	0.0285	0.0302
	(-1.09)	(-1.78)	(-0.92)	(-1.70)	(0.58)	(0.09)

Note: This table shows the standard deviation of the estimated regression coefficients (the true standard error) and the mean of the estimated standard errors for each variance estimator with the round brackets indicating the biases relative to the true standard error in percentage (%). To facilitate the comparison, the biases are rounded off to the second decimal places.

B.3. $S = 2$ and $\gamma = 0.2$

Table 6 presents the empirical coverage probability and average length of confidence intervals for β at 5% nominal size when $S = 2$ and $\gamma = 0.2$. The associated boxplots are given in Figure 6. Since the magnitude of spillovers is now much smaller than the case of $\gamma = 0.8$, there are only minor differences in performance between the network-robust variance estimator and the other two existing methods (namely, Eicker-Huber-White and dyadic-robust variance estimators). In terms of convergence, the comparable performance of the dyadic-robust-variance estimator is evident in Figure 6.

TABLE 6. The empirical coverage probability and average length of confidence intervals for β at 95% nominal level: $S = 2$, $\gamma = 0.2$.

	N	Specification 1			Specification 2		
		$\nu = 1$	$\nu = 2$	$\nu = 3$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
		Coverage Probability					
Eicker-Huber-White	500	0.9286	0.9186	0.9108	0.9434	0.9292	0.9308
	1000	0.9320	0.9158	0.9148	0.9338	0.9322	0.9322
	5000	0.9200	0.9110	0.9124	0.9434	0.9382	0.9308
Dyadic-robust	500	0.9342	0.9350	0.9336	0.9454	0.9368	0.9422
	1000	0.9454	0.9376	0.9458	0.9398	0.9422	0.9486
	5000	0.9446	0.9448	0.9432	0.9486	0.9490	0.9472
Network-robust	500	0.9284	0.9246	0.9162	0.9428	0.9360	0.9384
	1000	0.9414	0.9294	0.9370	0.9392	0.9410	0.9456
	5000	0.9454	0.9476	0.9418	0.9494	0.9492	0.9470
		Average Length of the Confidence Intervals					
Eicker-Huber-White	500	0.1578	0.1214	0.1092	0.1860	0.1360	0.1141
	1000	0.1088	0.0846	0.0743	0.1290	0.0955	0.0799
	5000	0.0486	0.0388	0.0346	0.0579	0.0423	0.0357
Dyadic-robust	500	0.1648	0.1316	0.1213	0.1890	0.1410	0.1205
	1000	0.1158	0.0931	0.0833	0.1319	0.0994	0.0848
	5000	0.0532	0.0439	0.0398	0.0594	0.0443	0.0381
Network-robust	500	0.1637	0.1291	0.1174	0.1885	0.1404	0.1196
	1000	0.1154	0.0922	0.0825	0.1318	0.0993	0.0848
	5000	0.0533	0.0440	0.0401	0.0594	0.0444	0.0382

Note: The upper-half of the table displays the empirical coverage probability of the asymptotic confidence interval for β, and the lower-half showcases the average length of the estimated confidence intervals. One computational issue that plagues the Monte Carlo simulation is the potential lack of positive-semi-definiteness of the estimated variance-covariance matrix. In general, this problem prevails only when the sample size (N) is small. In our case, when $N = 500$, four variance estimates out of five thousands take negative values. We deal with this issue by first applying the eigenvalue decomposition to the estimated variance-covariance matrix and then augmenting the diagonal matrix of eigenvalues by a small constant, followed by pre- and post-multiplications by the matrix of eigenvectors to obtain the updated estimate for the variance-covariance matrix. As the sample size (N) increases, the empirical coverage probability approaches 0.95, the nominal level. This convergence is accompanied by the shrinking average length of confidence intervals.

FIGURE 6. Boxplots of Standard Errors for Specifications 1 and 2 ($S = 2$, $\gamma = 0.2$)
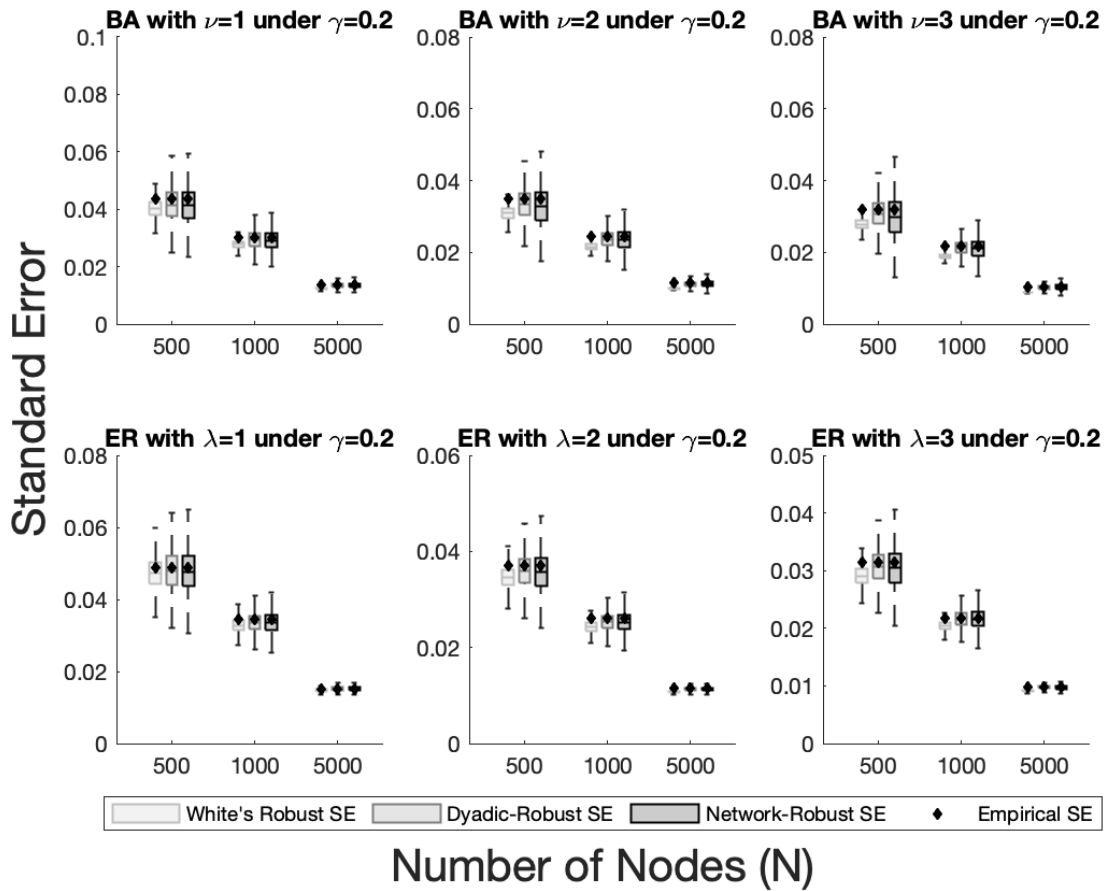
Note: This picture shows boxplots describing the estimated standard errors and the empirical standard errors for various combinations of parameters under Specification 1 (Barabási-Albert networks) and Specification 2 (Erdös-Renyi networks). The horizontal axis shows the number of nodes and the vertical axis represents the the standard error of the coefficient. The shaded boxes represent the 25th, 50th and 75th percentiles of estimated standard errors with the whiskers indicating the most extreme values that are not considered as outliers. The light-grey box illustrates the Eicker-Huber-White standard error, the medium-grey one the dyadic-robust standard error and the dark-grey one the network-robust standard error. The diamonds stand for the empirical standard error, defined as the standard deviation of the estimates of the regression coefficient. This figure showcases the boxplots for the case when $\gamma = 0.2$.

B.4. $S = 1$

For comparison purposes, this subsection explores the results for $S = 1$. If $S = 1$, there are no higher-order correlations beyond direct (adjacent) neighbors. Then, the network-robust variance estimator ought to coincide with the dyadic-robust variance estimator by definition,

for any γ, as pointed out in Section 2.1 and Example 2.1. This is verified below for the case of $\gamma = 0.8$. Table 7 shows the simulation results.

TABLE 7. The empirical coverage probability and average length of confidence intervals for β at 95% nominal level: $S = 1$, $\gamma = 0.8$.

	N	Specification 1			Specification 2		
		$v = 1$	$v = 2$	$v = 3$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
		Coverage Probability					
Eicker-Huber-White	500	0.8804	0.8676	0.8734	0.8906	0.8768	0.8692
	1000	0.8678	0.8810	0.8710	0.8984	0.8864	0.8856
	5000	0.8752	0.8652	0.8742	0.8996	0.8910	0.8778
Dyadic-robust	500	0.9292	0.9304	0.9384	0.9366	0.9416	0.9368
	1000	0.9364	0.9426	0.9432	0.9428	0.9454	0.9484
	5000	0.9474	0.9414	0.9498	0.9452	0.9518	0.9506
Network-robust	500	0.9292	0.9304	0.9384	0.9366	0.9416	0.9368
	1000	0.9364	0.9426	0.9432	0.9428	0.9454	0.9484
	5000	0.9474	0.9414	0.9498	0.9452	0.9518	0.9506
		Average Length of the Confidence Intervals					
Eicker-Huber-White	500	0.3282	0.2901	0.2881	0.2664	0.2377	0.2235
	1000	0.2321	0.2088	0.1964	0.1887	0.1665	0.1564
	5000	0.1131	0.1042	0.0980	0.0844	0.0742	0.0704
Dyadic-robust	500	0.3934	0.3591	0.3603	0.3104	0.2888	0.2776
	1000	0.2853	0.2625	0.2500	0.2227	0.2037	0.1950
	5000	0.1428	0.1330	0.1259	0.0998	0.0913	0.0882
Network-robust	500	0.3934	0.3591	0.3603	0.3104	0.2888	0.2776
	1000	0.2853	0.2625	0.2500	0.2227	0.2037	0.1950
	5000	0.1428	0.1330	0.1259	0.0998	0.0913	0.0882

Note: The upper-half of the table displays the empirical coverage probability of the asymptotic confidence interval for β, and the lower-half showcases the average length of the estimated confidence intervals. As the sample size (N) increases, the empirical coverage probability approaches 0.95, the nominal level. This convergence is accompanied by the shrinking average length of confidence intervals.

Appendix C. Proof of Main Theorems

C.1. Identification of β

Proof of Proposition 2.1. Premultiply the model (1) by $\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)$ to obtain

$$\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)y_m = \left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)'\beta + \left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)\varepsilon_m$$

Taking the expectation with respect to $(x_m, y_m, \varepsilon_m)$,

$$E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)y_m\right] = E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)'\right]\beta + E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)\varepsilon_m\right].$$

Since it holds by Assumption 2.3 (iv) that

$$E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)\varepsilon_m\right] = E\left[E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)\varepsilon_m\,\middle|\,\mathbf{x}\right]\right]$$

$$= E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)\underbrace{E\left[\varepsilon_m\,|\,\mathbf{x}\right]}_{0}\right]$$

$$= 0,$$

then

$$E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)y_m\right] = E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)'\right]\beta.$$

Next, Assumption 2.3 (iii) ensures existence of the inverse of the expectation term in the right hand side, so that we have

$$\beta = E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)'\right]^{-1} E\left[\left(\sum_{m'\in\mathcal{M}} g_{mm'}x_{m'}\right)y_m\right].$$

This proves the identification of the regression parameter β. □

C.2. Consistency of $\hat{\beta}$

Proof of Theorem 3.1. From (7), (14) and (15), we can write

$$\hat{\beta} - \beta = \left(\sum_{j \in \mathcal{M}_N} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N} g_{jk} g_{jl} x_k x_l' \right)^{-1} \left(\sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N} g_{mj} x_j \varepsilon_m \right)$$

$$= \frac{1}{M} \sum_{m \in \mathcal{M}_N} Y_{M,m},$$

and then

$$\hat{\beta}_u - \beta_u = \frac{1}{M} \sum_{m \in \mathcal{M}_N} Y_{M,m}^u,$$

where $\hat{\beta}_u$ and β_u, respectively, denote the u-th entry of $\hat{\beta}$ and β and thus are one-dimensional. In view of Assumption 2.3 (iv), it holds that for each $m \in \mathcal{M}_N$

$$E\left[Y_{M,m} \mid \mathcal{C}_M \right] = E\left[\left(\frac{1}{M} \sum_{j \in \mathcal{M}_N} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N} g_{jk} g_{jl} x_k x_l' \right)^{-1} \sum_{j \in \mathcal{M}_N} g_{mj} x_j \varepsilon_m \; \middle| \; \mathcal{C}_M \right]$$

$$= \left(\frac{1}{M} \sum_{j \in \mathcal{M}_N} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N} g_{jk} g_{jl} x_k x_l' \right)^{-1} \sum_{j \in \mathcal{M}_N} g_{mj} x_j E\left[\varepsilon_m \mid \mathcal{C}_M \right]$$

$$= 0,$$

and thus

$$E\left[Y_{M,m}^u \; \middle| \; \mathcal{C}_M \right] = 0.$$

By Theorem 3.1 of Kojevnikov, Marmer, and Song (2021),

$$\left\| \frac{1}{M} \sum_{m \in \mathcal{M}_N} \left(Y_{M,m}^u - \underbrace{E\left[Y_{M,m}^u \; \middle| \; \mathcal{C}_M \right]}_{0} \right) \right\|_{\mathcal{C}_M, 1} \xrightarrow{a.s.} 0$$

which implies $\left\| \dfrac{1}{M} \sum_{m \in \mathcal{M}_N} Y_{M,m}^u \right\|_{\mathcal{C}_M, 1} \xrightarrow{a.s.} 0$,

so that

$$E\left[|\hat{\beta}_u - \beta_u| \right] = E\left[E\left[|\hat{\beta}_u - \beta_u| \; \middle| \; \mathcal{C}_M \right] \right]$$

$$= E\left[\|\hat{\beta}_u - \beta_u\|_{\mathcal{C}_M, 1} \right]$$

$$= E\left[\left\|\frac{1}{M}\sum_{m\in\mathcal{M}_N}Y^u_{M,m}\right\|_{\mathcal{C}_M,1}\right]$$

$$\to 0,$$

where the last implication is a consequence of the Dominated Convergence Theorem. Since it holds by the Markov inequality that for any $c > 0$

$$\Pr(\left|\hat{\beta}_u - \beta_u\right| > c) \le \frac{E\left[\left|\hat{\beta}_u - \beta_u\right|\right]}{c},$$

it then follows that

$$\Pr(\left|\hat{\beta}_u - \beta_u\right| > c) \to 0,$$

as $N \to \infty$. Hence we have

$$\hat{\beta}_u \xrightarrow{p} \beta_u \qquad as \quad N \to \infty.$$

Finally, we can invoke the Cramér-Wold device to obtain

$$\hat{\beta} \xrightarrow{p} \beta \qquad as \quad N \to \infty,$$

as desired. □

C.3. Asymptotic Normality of $\hat{\beta}$

Proof of Proposition 3.2. First of all, we prove

$$\frac{S^u_M}{\sigma_M} \xrightarrow{d} \mathcal{N}(0,1),$$

as $N \to \infty$. Denote $\tilde{S}^u_M := \frac{S^u_M}{\sigma_M}$. Under A.2 and A.3, it holds by Theorem 3.2 of Kojevnikov, Marmer, and Song (2021) that for any $\epsilon > 0$, there exists $M_0 > 0$ such that for each $M > M_0$ and for each $x \in \mathbb{R}$,

$$\sup_{t\in\mathbb{R}}\left|\Pr(\tilde{S}^u_M \le t \mid X = x) - \Phi(t)\right| < \epsilon.$$

Then, by the law of total probability, we have

$$\left|\Pr(\tilde{S}^u_M \le t) - \Phi(t)\right| = \left|\int \Pr(\tilde{S}^u_M \le t \mid X = x)dF_X(x) - \Phi(t)\right|$$

$$= \left|\int \Pr(\tilde{S}^u_M \le t \mid X = x) - \Phi(t)dF_X(x)\right|$$

$$\leq \int \left| \Pr(\tilde{S}_M^u \leq t \mid X = x) - \Phi(t) \right| dF_X(x)$$

$$\leq \int \sup_{t \in \mathbb{R}} \left| \Pr(\tilde{S}_M^u \leq t \mid X = x) - \Phi(t) \right| dF_X(x),$$

where $F_X(\cdot)$ denotes the probability distribution function of X. Now pick arbitrarily $\epsilon > 0$. Then there exists $M_0 > 0$ such that for each $M > M_0$

$$\left| \Pr(\tilde{S}_M^u \leq t) - \Phi(t) \right| \leq \int \underbrace{\sup_{t \in \mathbb{R}} \left| \Pr(\tilde{S}_M^u \leq t \mid X = x) - \Phi(t) \right| dF_X(x)}_{\epsilon}$$

$$\leq \epsilon,$$

Since the right hand side does not depend on t, we then have that for each $M > M_0$,

$$\sup_{t \in \mathbb{R}} \left| \Pr(\tilde{S}_M^u \leq t) - \Phi(t) \right| \leq \epsilon,$$

which implies, when coupled with Assumption 2.2, that

$$\sup_{t \in \mathbb{R}} \left| \Pr(\tilde{S}_M^u \leq t) - \Phi(t) \right| \to 0 \qquad as \quad N \to \infty,$$

Then we have

$$\frac{S_M^u}{\sigma_M} \xrightarrow{d} \mathcal{N}(0,1) \qquad as \quad N \to \infty.$$

Next this can be combined with Assumption 3.3 by using the Slutsky theorem, yielding that

$$\frac{S_M^u}{\tau_M \sqrt{\pi_{N,M}}} \xrightarrow{d} \mathcal{N}(0,1) \qquad as \quad N \to \infty.$$

Moreover, applying the Cramér-Wold device gives

$$\frac{\tau_M^{-1}}{\sqrt{\pi_{N,M}}} S_M \xrightarrow{d} \mathcal{N}(0, I_K) \qquad as \quad N \to \infty,$$

where I_K is the $K \times K$ identity matrix and τ_M is understood as the variance-covariance matrix.[26] Now notice that by the definition of S_M we have

$$\hat{\beta} - \beta = \frac{1}{M} S_M,$$

so that we obtain

$$\sqrt{N} \left(\hat{\beta} - \beta \right) \xrightarrow{d} \mathcal{N}(0, AVar(\hat{\beta})) \qquad as \quad N \to \infty,$$

---

[26] To economize on notation, we use the same notation τ_M to denote the case of one-dimensional parameter and the case of multiple-dimensional parameters as we believe this causes little confusion.

where

$$AVar(\hat{\beta}) := \lim_{N\to\infty} \frac{N\pi_{N,M}}{M^2} E\left[\left(\sum_{j\in\mathcal{M}_N}\sum_{k\in\mathcal{M}_N}\sum_{l\in\mathcal{M}_N} g_{jk}g_{jl}x_k x_l'\right)^{-1}\left(\sum_{m\in\mathcal{M}_N}\sum_{k\in\mathcal{M}_N} g_{mk}x_k\varepsilon_m\right)\left(\sum_{m'\in\mathcal{M}_N}\sum_{l\in\mathcal{M}_N} g_{m'l}x_l\varepsilon_{m'}\right)'\left(\sum_{j\in\mathcal{M}_N}\sum_{k\in\mathcal{M}_N}\sum_{l\in\mathcal{M}_N} g_{jk}g_{jl}x_k x_l'\right)^{-1}\right],$$

which is assumed to be well-defined.

$\square$

C.4. Consistency of $\widehat{Var}(\hat{\beta})$

Proof of Theorem 4.1. Denote the variance of $\frac{S_M}{\sqrt{M}}$ as $V_{N,M} := Var\left(\frac{S_M}{\sqrt{M}}\right)$. It can readily be shown that $V_{N,M}$ takes the form of

$$V_{N,M} = \sum_{s\geq 0}\Omega_{N,M}(s),$$

where

$$\Omega_{N,M}(s) := \frac{1}{M}\sum_{m\in\mathcal{M}_N}\sum_{j\in\mathcal{M}_N^\partial(m;s)} E\left[Y_{M,m}Y_{M,j}'\right].$$

We define the kernel HAC estimator of $V_{N,M}$ as

$$\hat{V}_{N,M} := \sum_{s\geq 0}\omega_M(s)\hat{\Omega}_{N,M}(s),$$

where

$$\omega_M(s) := \omega\left(\frac{s}{b_M}\right)$$

$$\hat{\Omega}_{N,M}(s) := \frac{1}{M}\sum_{m\in\mathcal{M}_N}\sum_{j\in\mathcal{M}_N^\partial(m;s)} \hat{Y}_{M,m}\hat{Y}_{M,j}'.$$

Moreover, we define a empirical analogy of $V_{N,M}$, though infeasible, by

$$\tilde{V}_{N,M} := \sum_{s\geq 0}\omega_M(s)\tilde{\Omega}_{N,M}(s),$$

where

$$\tilde{\Omega}_{N,M}(s) := \frac{1}{M}\sum_{m\in\mathcal{M}_N}\sum_{j\in\mathcal{M}_N^\partial(m;s)} Y_{M,m}Y_{M,j}'.$$

Additionally, we denote a conditional version of $V_{N,M}$ by $V_{N,M}^c := Var(\frac{S_M}{\sqrt{M}} \mid \mathcal{C}_M)$, i.e.,

$$V_{N,M}^c = \sum_{s\geq 0}\omega_M(s)\Omega_{N,M}^c(s),$$

where

$$\Omega_{N,M}^c(s) := \frac{1}{M} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} E\left[Y_{M,m} Y_{M,j}' \,\Big|\, \mathcal{C}_M \right].$$

Notice that since $E\left[Y_{M,m} \,\big|\, \mathcal{C}_M \right] = 0$, it follows from the law of total variance that,

$$V_{N,M} = E\left[V_{N,M}^c \right].$$

Notice moreover that it holds that

$$Var(\hat{\beta}) = \frac{N \pi_{N,M}}{M} V_{N,M},$$

and

$$\widehat{Var}(\hat{\beta}) = \frac{N \pi_{N,M}}{M} \hat{V}_{N,M}.$$

Since

$$\begin{aligned}
\|\widehat{Var}(\hat{\beta}) - Var(\hat{\beta})\|_F &= \left\| \frac{N \pi_{N,M}}{M} \hat{V}_{N,M} - \frac{N \pi_{N,M}}{M} V_{N,M} \right\|_F \\
&= \left\| \frac{N \pi_{N,M}}{M} \left(\hat{V}_{N,M} - V_{N,M} \right) \right\|_F \\
&= \frac{N \pi_{N,M}}{M} \left\| \left(\hat{V}_{N,M} - V_{N,M} \right) \right\|_F,
\end{aligned}$$

and $\frac{N \pi_{N,M}}{M}$ is bounded due to Assumption 4.4, it suffices to show that

$$\|\hat{V}_{N,M} - V_{N,M}\|_F \xrightarrow{p} 0,$$

which, in light of the technique of "add and subtract," is equivalent to

$$\|\hat{V}_{N,M} - \tilde{V}_{N,M} + \tilde{V}_{N,M} - V_{N,M}\|_F \xrightarrow{p} 0.$$

Since by the triangular inequality,

$$\begin{aligned}
0 &\le \|\hat{V}_{N,M} - \tilde{V}_{N,M} + \tilde{V}_{N,M} - V_{N,M}\|_F \\
&\le \|\hat{V}_{N,M} - \tilde{V}_{N,M}\|_F + \|\tilde{V}_{N,M} - V_{N,M}\|_F,
\end{aligned}$$

we wish to prove that

(i) $\|\tilde{V}_{N,M} - V_{N,M}\|_F \xrightarrow{p} 0$;
(ii) $\|\hat{V}_{N,M} - \tilde{V}_{N,M}\|_F \xrightarrow{p} 0$,

as $N \to \infty$.

(i) $\|\tilde{V}_{N,M} - V_{N,M}\|_F \xrightarrow{p} 0$:

To economize on notation, we reference the (i,j) entry of $V_{N,M}$ by $V_{i,j}$. Analogous notations are applied to $\tilde{V}_{N,M}$ and $V_{N,M}^c$. The proof proceeds in multiple steps:

(1) $E\left[|\tilde{V}_{i,j} - V_{i,j}|\right] \to 0$;

(2) $|\tilde{V}_{i,j} - V_{i,j}| \xrightarrow{p} 0$,

as $N \to \infty$.

(1) $E\left[|\tilde{V}_{i,j} - V_{i,j}|\right] \to 0$:

We consider two cases:

(a) the case when $\tilde{V}_{i,j} - V_{i,j} \geq 0$;

(b) the case when $\tilde{V}_{i,j} - V_{i,j} < 0$,

separately.

(a) the case when $\tilde{V}_{i,j} - V_{i,j} \geq 0$:

If $\tilde{V}_{i,j} - V_{i,j} \geq 0$, it holds that

$$
\begin{aligned}
E\left[|\tilde{V}_{i,j} - V_{i,j}|\right] &= E\left[\tilde{V}_{i,j} - V_{i,j}\right] \\
&= E\left[\tilde{V}_{i,j} - E\left[V_{i,j}^c\right]\right] \\
&= E\left[\tilde{V}_{i,j} - V_{i,j}^c\right] \\
&\leq \left|E\left[\tilde{V}_{i,j} - V_{i,j}^c\right]\right| \\
&\leq E\left[\left|\tilde{V}_{i,j} - V_{i,j}^c\right|\right] \\
&= E\left[\underbrace{E\left[\left|\tilde{V}_{i,j} - V_{i,j}^c\right|\,\Big|\,\mathcal{C}_M\right]}_{\xrightarrow{a.s.} 0}\right] \\
&\to 0,
\end{aligned}
$$

where the last implication is the result of Proposition 4.1 of Kojevnikov, Marmer, and Song (2021)[27] and the Dominated Convergence Theorem.

(b) the case when $\tilde{V}_{i,j} - V_{i,j} < 0$:

If $\tilde{V}_{i,j} - V_{i,j} < 0$, it holds that

$$
\begin{aligned}
E\left[|\tilde{V}_{i,j} - V_{i,j}|\right] &= E\left[-(\tilde{V}_{i,j} - V_{i,j})\right] \\
&= -E\left[\tilde{V}_{i,j} - V_{i,j}\right] \\
&= -E\left[\tilde{V}_{i,j} - E\left[V_{i,j}^c\right]\right]
\end{aligned}
$$

---

[27]Notice that the definitions of $V_{N,M}$, $\hat{V}_{N,M}$ and $\tilde{V}_{N,M}$ are slightly different from those used in Proposition 4.1 of Kojevnikov, Marmer, and Song (2021).

$$= -E\left[\tilde{V}_{i,j} - V_{i,j}^c\right]$$

$$\leq \left|E\left[\tilde{V}_{i,j} - V_{i,j}^c\right]\right|$$

$$\leq E\left[\left|\tilde{V}_{i,j} - V_{i,j}^c\right|\right]$$

$$= E\left[\underbrace{E\left[\left|\tilde{V}_{i,j} - V_{i,j}^c\right|\,\Big|\,\mathcal{C}_M\right]}_{\xrightarrow{a.s.} 0}\right]$$

$$\to 0,$$

where the last implication is a consequence of Proposition 4.1 of Kojevnikov, Marmer, and Song (2021) and the Dominated Convergence Theorem. Hence in either case, we have

$$E\left[|\tilde{V}_{i,j} - V_{i,j}|\right] \to 0.$$

(2) $|\tilde{V}_{i,j} - V_{i,j}| \xrightarrow{p} 0$:

By the Markov inequality, for any positive constant $c > 0$,

$$Pr\left(|\tilde{V}_{i,j} - V_{i,j}| > c\right) \leq \frac{1}{c}\underbrace{E\left[|\tilde{V}_{i,j} - V_{i,j}|\right]}_{\to 0}$$

$$\to 0,$$

where the last implication is the result of part (1). Hence it holds that

$$|\tilde{V}_{i,j} - V_{i,j}| \xrightarrow{p} 0,$$

which in turn implies

$$\|\tilde{V}_{N,M} - V_{N,M}\|_F \xrightarrow{p} 0.$$

(ii) $\left\|\hat{V}_{N,M} - \tilde{V}_{N,M}\right\|_F \xrightarrow{p} 0$:

The proof mimics the standard proof of consistency of a heteroskedasticity-robust standard error. To lighten the notation, we denote

$$D_m := \sum_{k \in \mathcal{M}_N} g_{mk} x_k,$$

in which case we can write

$$Y_{M,m} = \left(\sum_{k \in \mathcal{M}_N} D_k D_k'\right)^{-1} D_m \varepsilon_m,$$

and

$$\hat{Y}_{M,m} = \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} D_m \hat{\varepsilon}_m.$$

First, applying the properties of the Frobenius norm,[28] we have

$$
\begin{aligned}
\left\| \hat{V}_{N,M} - \tilde{V}_{N,M} \right\|_F &= \left\| \sum_{s \geq 0} \omega_M(s) \frac{1}{M} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} D_m \hat{\varepsilon}_m \hat{\varepsilon}_j D_j' \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right. \\
&\quad \left. - \sum_{s \geq 0} \omega_M(s) \frac{1}{M} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} D_m \varepsilon_m \varepsilon_j D_j' \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F \\
&= \left\| \sum_{s \geq 0} \omega_M(s) \frac{1}{M} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} D_m \left(\hat{\varepsilon}_m \hat{\varepsilon}_j - \varepsilon_m \varepsilon_j \right) D_j' \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F \\
&\leq \left\| \frac{1}{M} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} D_m \left(\hat{\varepsilon}_m \hat{\varepsilon}_j - \varepsilon_m \varepsilon_j \right) D_j' \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F \\
&= \left\| \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \left\{ \frac{1}{M} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \left(\hat{\varepsilon}_m \hat{\varepsilon}_j - \varepsilon_m \varepsilon_j \right) D_m D_j' \right\} \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F \\
&= \left\| \left(\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \left\{ \frac{1}{M^3} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \left(\hat{\varepsilon}_m \hat{\varepsilon}_j - \varepsilon_m \varepsilon_j \right) D_m D_j' \right\} \left(\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F \\
&\leq \left\| \left(\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F \left\| \frac{1}{M^3} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \left(\hat{\varepsilon}_m \hat{\varepsilon}_j - \varepsilon_m \varepsilon_j \right) D_m D_j' \right\|_F \left\| \left(\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F.
\end{aligned}
$$

In what follows we aim to show that the "bread" part ($\left\| \left(\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F$) remains finite, and that the "meat" part ($\left\| \frac{1}{M^3} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \left(\hat{\varepsilon}_m \hat{\varepsilon}_j - \varepsilon_m \varepsilon_j \right) D_m D_j' \right\|_F$) converges to zero in probability. We examine these by showing the convergence in mean-square, a strategy employed in Aronow, Samii, and Assenova (2015) and Tabord-Meehan (2019). First, in light of Continuous Mapping Theorem, the proof of the bread part amounts to showing the convergence in probability of $\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k'$. We prove this by considering an element-by-element convergence. Let $D_{k,i}$ denote the i-th element of D_m. Then the (i, j) entry of $\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k'$ is

---

[28]Recall that

$$\|AB\|_F \leq \|A\|_F \|B\|_F,$$

where A and B are matrices whose product is well-defined. Also, It should be noted that by Assumption 4.2, there exists a nonnegative finite constant $C > 0$ such that

$$C = \sup_{N \geq 1} \left\| \left(\sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F.$$

given by $\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_{k,i} D_{k,j}$. Consider its expectation,

$$E\left[\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_{k,i} D_{k,j}\right] = \frac{1}{M} \sum_{k \in \mathcal{M}_N} E\left[D_{k,i} D_{k,j}\right].$$

From Assumption 4.2, there exists a nonnegative finite constant $C_{0,1}$ such that

$$C_{0,1} = \sup_{N \geq 1} \max_{m \in \mathcal{M}_N} E\left[D_{m,i} D_{m,j}\right].$$

Hence $E\left[\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_{k,i} D_{k,j}\right]$ exists and finite. Next, the variance can be expressed as a sum of covariance:

$$Var\left(\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_{k,i} D_{k,j}\right) = \frac{1}{M^2} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N} Cov\left(D_{m,i} D_{m,j}, D_{m',i} D_{m',j}\right)$$

$$= \frac{1}{M^2} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N(m;s)} Cov\left(D_{m,i} D_{m,j}, D_{m',i} D_{m',j}\right).$$

Again from Assumption 4.2, there exists a nonnegative finite constant $C_{0,2}$ such that

$$C_{0,2} = \sup_{N \geq 1} \max_{m,m' \in \mathcal{M}_N} Cov\left(D_{m,i} D_{m,j}, D_{m',i} D_{m',j}\right).$$

Hence

$$Var\left(\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_{k,i} D_{k,j}\right) \leq \frac{1}{M^2} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N(m;s)} C_{0,2}$$

$$= \frac{C_{0,2}}{M^2} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N(m;s)} 1$$

$$= \frac{C_{0,2}}{M^2} \sum_{s \geq 0} \delta_M^\partial(s; 1)$$

$$= C_{0,2} \underbrace{\frac{1}{M^2} \sum_{s \geq 0} \delta_M^\partial(s; 1)}_{\to 0}$$

$$\to 0,$$

where the last implication is due to Assumption 4.3. Therefore, we obtain

$$\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_{k,i} D_{k,j} \xrightarrow{p} E\left[\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_{k,i} D_{k,j}\right].$$

By repeating the argument above, we arrive at

$$\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \xrightarrow{p} E\left[\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \right].$$

Furthermore, applying Continuous Mapping Theorem yields

$$\left(\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \xrightarrow{p} \left(E\left[\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \right] \right)^{-1},$$

obtaining the result.

   Next the proof of the meat part consists of multiple steps. To begin with, observe that by definition, $\hat{\varepsilon}_m$ can be written as

$$\begin{aligned}
\hat{\varepsilon}_m &= \underbrace{y_m}_{D_m'\beta + \varepsilon_m} - D_m'\hat{\beta} \\
&= D_m'\beta + \varepsilon_m - D_m'\hat{\beta} \\
&= \varepsilon_m - D_m'(\hat{\beta} - \beta),
\end{aligned}$$

and then

$$\begin{aligned}
\hat{\varepsilon}_m \hat{\varepsilon}_j' &= \left\{ \varepsilon_m - D_m'(\hat{\beta} - \beta) \right\} \left\{ \varepsilon_j - D_j'(\hat{\beta} - \beta) \right\}' \\
&= \varepsilon_m \varepsilon_j' - \varepsilon_m (\hat{\beta} - \beta)'D_j - D_m'(\hat{\beta} - \beta)\varepsilon_j' + D_m'(\hat{\beta} - \beta)(\hat{\beta} - \beta)'D_j.
\end{aligned}$$

Hence

$$\hat{\varepsilon}_m \hat{\varepsilon}_j' - \varepsilon_m \varepsilon_j' = -\varepsilon_m (\hat{\beta} - \beta)'D_j - D_m'(\hat{\beta} - \beta)\varepsilon_j' + D_m'(\hat{\beta} - \beta)(\hat{\beta} - \beta)'D_j,$$

so that by the triangular inequality and the Cauchy-Schwartz inequality,

$$\begin{aligned}
\left| \hat{\varepsilon}_m \hat{\varepsilon}_j' - \varepsilon_m \varepsilon_j' \right| &= \left| -\varepsilon_m (\hat{\beta} - \beta)'D_j - D_m'(\hat{\beta} - \beta)\varepsilon_j' + D_m'(\hat{\beta} - \beta)(\hat{\beta} - \beta)'D_j \right| \\
&\leq \underbrace{\left| \varepsilon_m (\hat{\beta} - \beta)'D_j \right|}_{\leq |\varepsilon_m||(\hat{\beta} - \beta)'D_j|} + \underbrace{\left| D_m'(\hat{\beta} - \beta)\varepsilon_j' \right|}_{\leq |D_m'(\hat{\beta} - \beta)||\varepsilon_j|} + \underbrace{\left| D_m'(\hat{\beta} - \beta)(\hat{\beta} - \beta)'D_j \right|}_{|D_m'(\hat{\beta} - \beta)||(\hat{\beta} - \beta)'D_j|} \\
&\leq |\varepsilon_m| \underbrace{\left| (\hat{\beta} - \beta)'D_j \right|}_{\|\hat{\beta} - \beta\|_2 \|D_j\|_2} + \underbrace{\left| D_m'(\hat{\beta} - \beta) \right|}_{\leq \|D_m\|_2 \|\hat{\beta} - \beta\|_2} |\varepsilon_j| + \underbrace{\left| D_m'(\hat{\beta} - \beta) \right|}_{\leq \|D_m\|_2 \|\hat{\beta} - \beta\|_2} \underbrace{\left| (\hat{\beta} - \beta)'D_j \right|}_{\leq \|\hat{\beta} - \beta\|_2 \|D_j\|_2} \\
&\leq \left\| \hat{\beta} - \beta \right\|_2 \left\| D_j \right\|_2 |\varepsilon_m| + \left\| \hat{\beta} - \beta \right\|_2 \|D_m\|_2 |\varepsilon_j| + \left\| \hat{\beta} - \beta \right\|_2^2 \|D_m\|_2 \left\| D_j \right\|_2.
\end{aligned}$$

Hence the meat part can be bounded as

$$\left\| \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \left(\hat{\varepsilon}_m\hat{\varepsilon}_j - \varepsilon_m\varepsilon_j\right) D_m D_j' \right\|_F$$

$$\leq \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} |\hat{\varepsilon}_m\hat{\varepsilon}_j - \varepsilon_m\varepsilon_j| \underbrace{\|D_m D_j'\|_F}_{\leq \|D_m\|_2\|D_j\|_2}$$

$$\leq \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \left(\left\|\hat{\beta}-\beta\right\|_2 \|D_j\|_2 |\varepsilon_m| + \left\|\hat{\beta}-\beta\right\|_2 \|D_m\|_2 |\varepsilon_j| + \left\|\hat{\beta}-\beta\right\|_2^2 \|D_m\|_2 \|D_j\|_2 \right) \|D_m\|_2\|D_j\|_2$$

$$= \left\|\hat{\beta}-\beta\right\|_2 \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \|D_m\|_2\|D_j\|_2^2|\varepsilon_m| + \left\|\hat{\beta}-\beta\right\|_2 \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \|D_m\|_2^2\|D_j\|_2|\varepsilon_j|$$

$$+ \left\|\hat{\beta}-\beta\right\|_2^2 \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \|D_m\|_2^2\|D_j\|_2^2.$$

Denote

$$R_{N,1} := \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \|D_m\|_2\|D_j\|_2^2|\varepsilon_m|$$

$$R_{N,2} := \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \|D_m\|_2^2\|D_j\|_2|\varepsilon_j|$$

$$R_{N,3} := \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \|D_m\|_2^2\|D_j\|_2^2.$$

Now, since by Theorem 3.1,

$$\|\hat{\beta}-\beta\|_2 \xrightarrow{P} 0,$$

and the application of Continuous Mapping Theorem yields

$$\|\hat{\beta}-\beta\|_2^2 \xrightarrow{P} 0,$$

it thus suffices to prove that each of $R_{N,1}$, $R_{N,2}$ and $R_{N,3}$ converges in probability to a finite number.

First let us study the expectation of $R_{N,1}$. By applying the Cauchy-Schwartz inequality repeatedly, it holds

$$E\left[R_{N,1}\right] = E\left[\frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \|D_m\|_2\|D_j\|_2^2|\varepsilon_m|\right]$$

$$= \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \underbrace{E\left[\|D_m\|_2\|D_j\|_2^2|\varepsilon_m|\right]}_{\leq \left(E\left[\|D_m\|_2^2\|D_j\|_2^4\right]\right)^{\frac{1}{2}} \left(E[|\varepsilon_m|^2]\right)^{\frac{1}{2}}}$$

$$\leq \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^\partial(m;s)} \left(\underbrace{E\left[\|D_m\|_2^2\|D_j\|_2^4\right]}_{\leq\left(E\left[\|D_m\|_2^4\right]\right)^{\frac{1}{2}}\left(E\left[\|D_j\|_2^8\right]\right)^{\frac{1}{2}}} \right)^{\frac{1}{2}} \left(\underbrace{E\left[|\varepsilon_m|^2\right]}_{=E[E[|\varepsilon_m|^2|\mathcal{C}_M]]} \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^\partial(m;s)} \left(\left(E\left[\|D_m\|_2^4\right]\right)^{\frac{1}{2}} \left(E\left[\|D_j\|_2^8\right]\right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left(E\left[E\left[|\varepsilon_m|^2 \mid \mathcal{C}_M\right]\right] \right)^{\frac{1}{2}}.$$

Here, in light of Assumption 4.1, there exists a nonnegative finite number $C_1 > 0$ such that

$$C_1 = \sup_{N\geq 1} \max_{m\in\mathcal{M}_N} E\left[|\varepsilon_m| \mid \mathcal{C}_M\right],$$

and moreover by Assumption 4.2, there exists a nonnegative finite number $C_2 > 0$ such that

$$C_2 = \sup_{N\geq 1} \max_{m\in\mathcal{M}_N} E\left[\|D_m\|_2^8\right].$$

With a slight abuse of notation, we have

$$E\left[R_{N,1}\right] \leq \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^\partial(m;s)} C_1 C_2$$

$$= \frac{C_1 C_2}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^\partial(m;s)} 1$$

$$= \frac{C_1 C_2}{M^3} \sum_{s\geq 0} \delta_M^\partial(s;1)$$

$$= C_1 C_2 \underbrace{\frac{1}{M^3} \sum_{s\geq 0} \delta_M^\partial(s;1)}_{\to 0}$$

$$\to 0,$$

where the last implication is because of Assumption 4.3. Next let us study the variance of $R_{N,1}$. It suffices to show that

$$E\left[R_{N,1}^2\right] \to 0,$$

By the Cauchy-Schwartz inequality, it holds that

$$E\left[R_{N,1}^2\right] = E\left[\frac{1}{M^6} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^\partial(m;s)} \sum_{t\geq 0} \sum_{k\in\mathcal{M}_N} \sum_{l\in\mathcal{M}_N^\partial(m;s)} \|D_m\|_2\|D_j\|_2^2\|D_k\|_2\|D_l\|_2^2|\varepsilon_m||\varepsilon_k| \right]$$

$$= \frac{1}{M^6} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^\partial(m;s)} \sum_{t\geq 0} \sum_{k\in\mathcal{M}_N} \sum_{l\in\mathcal{M}_N^\partial(m;s)} E\left[\|D_m\|_2\|D_j\|_2^2\|D_k\|_2\|D_l\|_2^2|\varepsilon_m||\varepsilon_k|\right]$$

$$\leq \frac{1}{M^6} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \sum_{t\geq 0} \sum_{k\in\mathcal{M}_N} \sum_{l\in\mathcal{M}_N^{\partial}(m;s)} \left(E\left[\|D_m\|_2^2\|D_j\|_2^4\|D_k\|_2^2\|D_l\|_2^4\right]\right)^{\frac{1}{2}} \left(E\left[|\varepsilon_m|^2|\varepsilon_k|^2\right]\right)^{\frac{1}{2}}.$$

Here, by Assumption 4.2 and the Cauchy-Schwartz inequality, there exists a nonnegative finite constant $C_3 > 0$ such that

$$C_3 = \sup_{N\leq 1} \max_{m,j,k,l\in\mathcal{M}_N} E\left[\|D_m\|_2^2\|D_j\|_2^4\|D_k\|_2^2\|D_l\|_2^4\right].$$

Then, with a slight abuse of notation in writing $C_3^{\frac{1}{2}}$ as C_3, we have

$$E\left[R_{N,1}^2\right] \leq \frac{1}{M^6} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \sum_{t\geq 0} \sum_{k\in\mathcal{M}_N} \sum_{l\in\mathcal{M}_N^{\partial}(m;s)} \underbrace{\left(E\left[\|D_m\|_2^2\|D_j\|_2^4\|D_k\|_2^2\|D_l\|_2^4\right]\right)^{\frac{1}{2}}}_{\leq C_3^{\frac{1}{2}}} \left(E\left[|\varepsilon_m|^2|\varepsilon_k|^2\right]\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{M^6} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \sum_{t\geq 0} \sum_{k\in\mathcal{M}_N} \sum_{l\in\mathcal{M}_N^{\partial}(m;s)} C_3^{\frac{1}{2}} \left(E\left[|\varepsilon_m|^2|\varepsilon_k|^2\right]\right)^{\frac{1}{2}}$$

$$= \frac{C_3}{M^6} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^{\partial}(m;s)} \sum_{t\geq 0} \sum_{k\in\mathcal{M}_N} \sum_{l\in\mathcal{M}_N^{\partial}(m;s)} \left(E\left[|\varepsilon_m|^2|\varepsilon_k|^2\right]\right)^{\frac{1}{2}}$$

$$= \frac{C_3}{M^6} \sum_{s\geq 0} \sum_{(m,j,k,l)\in H_M(s;b_M)} \left(E\left[|\varepsilon_m|^2|\varepsilon_k|^2\right]\right)^{\frac{1}{2}}$$

$$= \frac{C_3}{M^6} \sum_{s\geq 0} \sum_{(m,j,k,l)\in H_M(s;b_M)} \left(E\left[E\left[|\varepsilon_m|^2|\varepsilon_k|^2 \mid \mathcal{C}_M\right]\right]\right)^{\frac{1}{2}},$$

Corollary A.2 of Kojevnikov, Marmer, and Song (2021) shows that there exists a nonnegative finite constant C_4 such that

$$E\left[|\varepsilon_m|^2|\varepsilon_k|^2 \mid \mathcal{C}_M\right] \leq C_4 \bar{\theta} \theta_{M,s}^{1-\frac{4}{p}},$$

where

$$\bar{\theta} := \sup_{M\geq 1} \max_{s\geq 1} \theta_{M,s}.$$

This implies that,

$$E\left[R_{N,1}^2\right] = \frac{C_3}{M^6} \sum_{s\geq 0} \sum_{(m,j,k,l)\in H_M(s;b_M)} \left(E\left[\underbrace{E\left[|\varepsilon_m|^2|\varepsilon_k|^2 \mid \mathcal{C}_M\right]}_{\leq C_4(\mu\vee 1)^2\bar{\theta}\theta_{M,s}^{1-\frac{4}{p}}}\right]\right)^{\frac{1}{2}}$$

$$\leq \frac{C_3}{M^6} \sum_{s\geq 0} \sum_{(m,j,k,l)\in H_M(s;b_M)} \left(E\left[C_4(\mu\vee 1)^2\bar{\theta}\theta_{M,s}^{1-\frac{4}{p}}\right]\right)^{\frac{1}{2}}$$

$$\leq \frac{C_3 C_4}{M^6} \sum_{s\geq 0} \sum_{(m,j,k,l)\in H_M(s;b_M)} \left(E\left[\theta_{M,s}^{1-\frac{4}{p}}\right]\right)^{\frac{1}{2}}$$

$$\leq \frac{C_3 C_4}{M^6} \sum_{s\geq 0} \left(E\left[\theta_{M,s}^{1-\frac{4}{p}} \right] \right)^{\frac{1}{2}} \sum_{(m,j,k,l)\in H_M(s;b_M)} 1$$

$$\leq \frac{C_3 C_4}{M^6} \sum_{s\geq 0} \left(E\left[\theta_{M,s}^{1-\frac{4}{p}} \right] \right)^{\frac{1}{2}} \underbrace{|H_M(s;b_M)|}_{\leq 4Mc_M(s,b_M;2)}$$

$$\leq \frac{C_3 C_4}{M^6} \sum_{s\geq 0} \left(E\left[\theta_{M,s}^{1-\frac{4}{p}} \right] \right)^{\frac{1}{2}} 4Mc_M(s,b_M;2)$$

$$= \frac{4C_3 C_4}{M^5} \sum_{s\geq 0} \underbrace{\left(E\left[\theta_{M,s}^{1-\frac{4}{p}} \right] \right)^{\frac{1}{2}}}_{<\infty} c_M(s,b_M;2)$$

$$= \frac{4C_3 C_4}{M^5} \sum_{s\geq 0} c_M(s,b_M;2)$$

$$= 4C_3 C_4 \underbrace{\frac{1}{M^5} \sum_{s\geq 0} c_M(s,b_M;2)}_{\to 0}$$

$$\to 0,$$

where we apply Assumption 4.3 for the last implication. Hence we have shown that

$$R_{N,1} \xrightarrow{p} 0.$$

The proof of $R_{N,2}$ is analogous. It remains to show that $R_{N,3}$ converges to zero in probability. Let us first study the expectation of $R_{N,3}$.

$$E\left[R_{N,3} \right] = E\left[\frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^\partial(m;s)} \|D_m\|_2^2 \|D_j\|_2^2 \right]$$

$$= \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^\partial(m;s)} E\left[\|D_m\|_2^2 \|D_j\|_2^2 \right]$$

In view of Assumption 4.2, there exists a nonnegative finite number $C_5 > 0$ such that

$$C_5 = \sup_{N\geq 1} \max_{m\in\mathcal{M}_N} E\left[\|D_m\|_2^2 \|D_j\|_2^2 \right].$$

Hence

$$E\left[R_{N,3} \right] = \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^\partial(m;s)} \underbrace{E\left[\|D_m\|_2^2 \|D_j\|_2^2 \right]}_{\leq C_5}$$

$$\leq \frac{1}{M^3} \sum_{s\geq 0} \sum_{m\in\mathcal{M}_N} \sum_{j\in\mathcal{M}_N^\partial(m;s)} C_5$$

$$= \frac{C_5}{M^3} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} 1$$

$$= \frac{C_5}{M^3} \sum_{s \geq 0} \delta_M^\partial(s; 1)$$

$$= C_5 \underbrace{\frac{1}{M^3} \sum_{s \geq 0} \delta_M^\partial(s; 1)}_{\to 0}$$

$$\to 0,$$

where we apply Assumption 4.3 in the last implication. Next let us consider the variance of $R_{M,3}$.

$$E\left[R_{N,3}^2\right] = E\left[\frac{1}{M^6} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \sum_{t \geq 0} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N^\partial(m;s)} \|D_m\|_2^2 \|D_j\|_2^2 \|D_k\|_2^2 \|D_l\|_2^2\right]$$

$$= \frac{1}{M^6} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \sum_{t \geq 0} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N^\partial(m;s)} E\left[\|D_m\|_2^2 \|D_j\|_2^2 \|D_k\|_2^2 \|D_l\|_2^2\right].$$

Once again, Assumption 4.2, in conjunction with the Cauchy-Schwartz inequality, implies that there exists a nonnegative finite number $C_6 > 0$ such that

$$C_6 = \sup_{N \geq 1} \max_{m,j,k,l \in \mathcal{M}_N} E\left[\|D_m\|_2^2 \|D_j\|_2^2 \|D_k\|_2^2 \|D_l\|_2^2\right].$$

We now use the following lemma from Kojevnikov, Marmer, and Song (2021), p.903:

Lemma C.1. *Define*

$$H_M(s, r) := \left\{(m, j, k, l) \in \mathcal{M}_N^4 : j \in \mathcal{M}_N(m; r), \ l \in \mathcal{M}_N(k; r), \ d_M(\{m, j\}, \{k, l\}) = s\right\}.$$

Then

$$|H_M(s, r)| \leq 4M c_M(s, r; 2).$$

Then by Lemma C.1,

$$E\left[R_{N,3}^2\right] = \frac{1}{M^6} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \sum_{t \geq 0} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N^\partial(m;s)} \underbrace{E\left[\|D_m\|_2^2 \|D_j\|_2^2 \|D_k\|_2^2 \|D_l\|_2^2\right]}_{\leq C_6}$$

$$\leq \frac{1}{M^6} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \sum_{t \geq 0} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N^\partial(m;s)} C_6$$

$$= \frac{C_6}{M^6} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^\partial(m;s)} \sum_{t \geq 0} \sum_{k \in \mathcal{M}_N} \sum_{l \in \mathcal{M}_N^\partial(m;s)} 1$$

$$
= \frac{C_6}{M^6} \sum_{s \geq 0} \sum_{(m,j,k,l) \in H_M(s;b_M)} 1
$$

$$
= \frac{C_6}{M^6} \sum_{s \geq 0} \underbrace{|H_M(s;b_M)|}_{\leq 4M c_M(s,b_M;2)}
$$

$$
= \frac{C_6}{M^6} \sum_{s \geq 0} 4M c_M(s,b_M;2)
$$

$$
= \frac{4C_6}{M^5} \sum_{s \geq 0} c_M(s,b_M;2)
$$

$$
= 4C_6 \underbrace{\frac{1}{M^5} \sum_{s \geq 0} c_M(s,b_M;2)}_{\to 0}
$$

$$
\to 0,
$$

where the last implication is a consequence of Assumption 4.3 (ii). Therefore we have shown that

$$
\left\| \frac{1}{M^3} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^{\partial}(m;s)} (\hat{\varepsilon}_m \hat{\varepsilon}_j - \varepsilon_m \varepsilon_j) D_m D_j' \right\|_F = \|\hat{\beta} - \beta\|_2 \underbrace{\frac{1}{M^3} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^{\partial}(m;s)} \|D_m\|_2 \|D_j\|_2^2 |\varepsilon_m|}_{R_{M,1}}
$$

$$
+ \|\hat{\beta} - \beta\|_2 \underbrace{\frac{1}{M^3} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^{\partial}(m;s)} \|D_m\|_2^2 \|D_j\|_2 |\varepsilon_j|}_{R_{M,2}}
$$

$$
+ \|\hat{\beta} - \beta\|_2^2 \underbrace{\frac{1}{M^3} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^{\partial}(m;s)} \|D_m\|_2^2 \|D_j\|_2^2}_{R_{M,3}}
$$

$$
= \underbrace{\|\hat{\beta} - \beta\|_2}_{\xrightarrow{p} 0} \underbrace{R_{M,1}}_{\xrightarrow{p} 0} + \underbrace{\|\hat{\beta} - \beta\|_2}_{\xrightarrow{p} 0} \underbrace{R_{M,2}}_{\xrightarrow{p} 0} + \underbrace{\|\hat{\beta} - \beta\|_2^2}_{\xrightarrow{p} 0} \underbrace{R_{M,3}}_{\xrightarrow{p} 0}
$$

$$
\xrightarrow{p} 0,
$$

which proves the convergence of the meat part to zero.

Hence, we have established

$$
\|\hat{V}_{N,M} - \tilde{V}_{N,M}\|_F \leq \underbrace{\left\| \left(\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F}_{< \infty} \underbrace{\left\| \frac{1}{M^3} \sum_{s \geq 0} \sum_{m \in \mathcal{M}_N} \sum_{j \in \mathcal{M}_N^{\partial}(m;s)} (\hat{\varepsilon}_m \hat{\varepsilon}_j - \varepsilon_m \varepsilon_j) D_m D_j' \right\|_F}_{\xrightarrow{p} 0} \underbrace{\left\| \left(\frac{1}{M} \sum_{k \in \mathcal{M}_N} D_k D_k' \right)^{-1} \right\|_F}_{< \infty}
$$

$$
\xrightarrow{p} 0,
$$

completing part (ii) of the proof. Combining part (i) and (ii), we arrive at

$$\|\hat{V}_{N,M} - V_{N,M}\|_F = \|\hat{V}_{N,M} - \tilde{V}_{N,M} + \tilde{V}_{N,M} - V_{N,M}\|_F$$

$$\leq \underbrace{\|\hat{V}_{N,M} - \tilde{V}_{N,M}\|_F}_{\xrightarrow{p} 0} + \underbrace{\|\tilde{V}_{N,M} - V_{N,M}\|_F}_{\xrightarrow{p} 0}$$

$$\xrightarrow{p} 0,$$

as desired. □

Proof of Corollary 4.1. For simplicity we denote

$$\hat{V}_{N,M}^{Dyad} := \left(\sum_{m \in \mathcal{M}_N} x_m x_m' \right)^{-1} \left(\sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N} \mathbb{1}_{m,m'} \hat{\varepsilon}_m \hat{\varepsilon}_{m'} x_m x_{m'}' \right) \left(\sum_{m \in \mathcal{M}_N} x_m x_m' \right)^{-1},$$

and

$$\hat{V}_{N,M}^{Network} := \left(\sum_{m \in \mathcal{M}_N} x_m x_m' \right)^{-1} \left(\sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N} h_{m,m'} \hat{\varepsilon}_m \hat{\varepsilon}_{m'} x_m x_{m'}' \right) \left(\sum_{m \in \mathcal{M}_N} x_m x_m' \right)^{-1}.$$

Define moreover $\widetilde{Var}(\hat{\beta})$ to be the same variance as (19), but is now applied to the network-regression model (10) and (11). By the triangular inequality,

$$\left\| N \pi_{N,M} \hat{V}_{N,M}^{Network} - \hat{V}_{N,M}^{Dyad} \right\|_F = \left\| N \pi_{N,M} \hat{V}_{N,M}^{Network} - \widetilde{Var}(\hat{\beta}) + \widetilde{Var}(\hat{\beta}) - N \pi_{N,M} \hat{V}_{N,M}^{Dyad} \right\|_F$$

$$\leq \left\| N \pi_{N,M} \hat{V}_{N,M}^{Network} - \widetilde{Var}(\hat{\beta}) \right\|_F + \left\| \widetilde{Var}(\hat{\beta}) - N \pi_{N,M} \hat{V}_{N,M}^{Dyad} \right\|_F.$$

Since Theorem 4.1 implies $\left\| N \pi_{N,M} \hat{V}_{N,M}^{Network} - \widetilde{Var}(\hat{\beta}) \right\|_F \xrightarrow{p} 0$, then in the limit we are left with

(24) $$\left\| \hat{V}_{N,M}^{Network} - \hat{V}_{N,M}^{Dyad} \right\|_F \leq \left\| \widetilde{Var}(\hat{\beta}) - N \pi_{N,M} \hat{V}_{N,M}^{Dyad} \right\|_F.$$

Now we prove the statement by way of contradiction. Assume for the sake of contradiction that the dyadic-robust variance estimator $\hat{V}_{N,M}^{Dyad}$ is consistent, i.e., $\left\| \widetilde{Var}(\hat{\beta}) - N \pi_{N,M} \hat{V}_{N,M}^{Dyad} \right\|_F \xrightarrow{p} 0$. This, combined with the inequality (24), implies $\left\| N \pi_{N,M} \hat{V}_{N,M}^{Network} - N \pi_{N,M} \hat{V}_{N,M}^{Dyad} \right\|_F \xrightarrow{p} 0$. Now, observe that

$$\left\| N \pi_{N,M} \hat{V}_{N,M}^{Network} - N \pi_{N,M} \hat{V}_{N,M}^{Dyad} \right\|_F$$

$$
= \left\| N\pi_{N,M}\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\left(\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N}h_{m,m'}\hat{\varepsilon}_m\hat{\varepsilon}_{m'}x_mx_{m'}'\right)\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\right.
$$
$$
\left. -N\pi_{N,M}\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\left(\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N}\mathbb{1}_{m,m'}\hat{\varepsilon}_m\hat{\varepsilon}_{m'}x_mx_{m'}'\right)\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\right\|_F
$$

$$
= \left\| N\pi_{N,M}\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\left(\sum_{s\geq0}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}h_{m,m'}\hat{\varepsilon}_m\hat{\varepsilon}_{m'}x_mx_{m'}'\right)\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\right.
$$
$$
\left. -N\pi_{N,M}\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\left(\sum_{s=\{0,1\}}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}\mathbb{1}_{m,m'}\hat{\varepsilon}_m\hat{\varepsilon}_{m'}x_mx_{m'}'\right)\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\right\|_F
$$

$$
= \left\| N\pi_{N,M}\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\left(\sum_{s=\{0,1\}}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}\underbrace{(h_{m,m'}-\mathbb{1}_{m,m'})}_{0}\hat{\varepsilon}_m\hat{\varepsilon}_{m'}x_mx_{m'}'\right)\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\right.
$$
$$
\left. +N\pi_{N,M}\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\left(\sum_{s\geq2}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}\left(\underbrace{h_{m,m'}}_{1}-\underbrace{\mathbb{1}_{m,m'}}_{0}\right)\hat{\varepsilon}_m\hat{\varepsilon}_{m'}x_mx_{m'}'\right)\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\right\|_F
$$

$$
= \left\| N\pi_{N,M}\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\left(\sum_{s\geq2}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}\hat{\varepsilon}_m\hat{\varepsilon}_{m'}x_mx_{m'}'\right)\left(\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\right\|_F
$$

$$
= \frac{N\pi_{N,M}}{M}\left\| \left(\frac{1}{M}\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\left(\frac{1}{M}\sum_{s\geq2}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}\hat{\varepsilon}_m\hat{\varepsilon}_{m'}x_mx_{m'}'\right)\left(\frac{1}{M}\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}\right\|_F,
$$

where we note that $\frac{N\pi_{N,M}}{M}<\infty$ by Assumption 4.4. We prove that the inside the Frobenius norm does not converge in probability to zero. First it can be shown that the "bread" part $\left(\frac{1}{M}\sum_{m\in\mathcal{M}_N}x_mx_m'\right)^{-1}$ converges to $\left(\frac{1}{M}\sum_{m\in\mathcal{M}_N}E\left[x_mx_m'\right]\right)^{-1}$ by an argument analogous to the one in proof of Theorem 4.1. Next plugging the definition of $\hat{\varepsilon}$ into the middle part, we have

$$
\frac{1}{M}\sum_{s\geq2}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}\hat{\varepsilon}_m\hat{\varepsilon}_{m'}x_mx_{m'}'
$$
$$
= \frac{1}{M}\sum_{s\geq2}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}\left\{\varepsilon_m+x_m'(\beta-\hat{\beta})\right\}\left\{\varepsilon_{m'}+x_{m'}'(\beta-\hat{\beta})\right\}x_mx_{m'}'
$$
$$
= \frac{1}{M}\sum_{s\geq2}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}\varepsilon_m\varepsilon_{m'}x_mx_{m'}'+\varepsilon_m(\beta-\hat{\beta})x_{m'}x_mx_{m'}'+x_m'(\beta-\hat{\beta})\varepsilon_{m'}x_mx_{m'}'+x_m'(\beta-\hat{\beta})(\beta-\hat{\beta})'x_{m'}x_mx_{m'}'
$$
$$
= \frac{1}{M}\sum_{s\geq2}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}\varepsilon_m\varepsilon_{m'}x_mx_{m'}'+\frac{1}{M}\sum_{s\geq2}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}\varepsilon_m(\beta-\hat{\beta})x_{m'}x_mx_{m'}'
$$
$$
+ \frac{1}{M}\sum_{s\geq2}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}x_m'(\beta-\hat{\beta})\varepsilon_{m'}x_mx_{m'}'+\frac{1}{M}\sum_{s\geq2}\sum_{m\in\mathcal{M}_N}\sum_{m'\in\mathcal{M}_N^\partial(m;s)}x_m'(\beta-\hat{\beta})(\beta-\hat{\beta})'x_{m'}x_mx_{m'}'.
$$

Denote

$$Q_{N,1} := \frac{1}{M} \sum_{s \geq 2} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N^\partial(m;s)} \varepsilon_m \varepsilon_{m'} x_m x_{m'}'$$

$$Q_{N,2} := \frac{1}{M} \sum_{s \geq 2} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N^\partial(m;s)} \varepsilon_m (\beta - \hat{\beta}) x_{m'} x_m x_{m'}'$$

$$Q_{N,3} := \frac{1}{M} \sum_{s \geq 2} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N^\partial(m;s)} x_m'(\beta - \hat{\beta}) \varepsilon_{m'} x_m x_{m'}'$$

$$Q_{N,4} := \frac{1}{M} \sum_{s \geq 2} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N^\partial(m;s)} x_m'(\beta - \hat{\beta})(\beta - \hat{\beta})' x_{m'} x_m x_{m'}'.$$

Since by Theorem 3.1, it can be seen that $Q_{M,2}$, $Q_{M,3}$ and $Q_{M,3}$ either converge to zero or diverge as N goes to infinity. When it comes to $Q_{M,1}$, observe that

$$E\left[Q_{N,1}\right] = E\left[\frac{1}{M} \sum_{s \geq 2} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N^\partial(m;s)} \varepsilon_m \varepsilon_{m'} x_m x_{m'}'\right]$$

$$= \frac{1}{M} \sum_{s \geq 2} \sum_{m \in \mathcal{M}_N} \sum_{m' \in \mathcal{M}_N^\partial(m;s)} E\left[\varepsilon_m \varepsilon_{m'} x_m x_{m'}'\right],$$

which never converges to zero due to the hypothesis (21) of this corollary. In either case, the middle part does not converge in probability to zero, meaning that $\left\|N\pi_{N,M}\hat{V}_{N,M}^{Network} - N\pi_{N,M}\hat{V}_{N,M}^{Dyad}\right\|_F \xrightarrow{P} 0$ is not true. This, however, contradicts the implication of the assumption that the dyadic-robust variance estimator is consistent. Hence, by means of contradiction, we conclude that the dyadic-robust variance estimator is not consistent, which completes the proof. $\square$

Appendix D. Data Construction for the Empirical Illustration

Data construction for our empirical exercise in Section 6 proceeds in multiple steps:

Step 1: Our subsample consists of the location of interest, i.e., Strasbourg, for the period of interest, i.e., Term 7. We select a further subset of the extracted data by seating arrangement (i.e., we focus on Pattern 1 for the present analysis - see Table 8).

Step 2: Since our analysis is concerned with voting concordance, we follow the original authors and drop entries with missing data or "abstain" in the variable "vote."[29]

Step 3: The resulting data still contains individuals belonging to "Identity, Tradition and Sovereignty (ITS)," one of the European Political Groups that dissolved in November 7, during the sixth term. We drop such MEPs from our analysis.

Step 4: The selected data is used to form the dyadic-data registering the pair-of-MEPs-specific information. When pairing two MEPs, we follow Harmon, Fisman, and Kamenica (2019) in focusing on those pairs of MEPs, both of whom are

  (i) in the same EPG;

 (ii) from an alphabetically-seated EPG; and

(iii) non-leaders at the time of voting.

Our dyadic-data consists of two types of variables: binary variables and numerical variables. The dyad-level binary (i.e., indicator) variables are defined to be one if the individual-level binary variables are the same, and zero otherwise. The dyadic-specific numerical variables in our analysis are the differences between the individual-level numerical variables, such as age and tenure. When calculating the differences in ages and tenures, we take the absolute values as we do not consider directional dyads, and we then rescale them into ten-year units. See the note for Table 3 for details.

Department of Economics, University of Houston

Email address: `ncanen@uh.edu, ksugiura@uh.edu`

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[29]This amounts to assuming that those observations are missing completely at random (MCAR).

TABLE 8. Patterns of Seating Arrangements: Strasbourg, Term 7

Pattern	Date			Number of Proposals
1	7/14/2009	~	7/16/2009	116
2	8/18/2009	~	8/21/2009	72
3	9/23/2009	~	9/25/2009	114
4	10/13/2009	~	10/16/2009	40
5	11/19/2009	~	12/11/2009	94
6	1/5/2010	~	1/8/2010	79
7	3/17/2010	~	3/19/2010	45
8	4/14/2010	~	4/16/2010	120
9	5/5/2010	~	5/7/2010	79
10	7/7/2010	~	7/9/2010	34
11	7/21/2010	~	7/22/2010	50
12	8/18/2010	~	8/20/2010	118

Note: This table presents patterns of seating arrangements with the corresponding dates and the number of total observations for each pattern. Since voting may be taken place for multiple proposals within the same day, the total number proposals tends to be higher than that of days in a single pattern. For example, the first line indicates that 116 proposals were discussed and votes were cast over the three days (from the 14th of July, 2009 to the 16th of July, 2009).