BUFFON'S NEEDLE PROBLEM

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Abstract. In this paper, we will study Buffon's needle problem. This consists of trying to find the probability that a needle dropped onto a ruled surface will intersect one of these lines. We will solve this problem for small needles in two different ways: Barbier's method using linearity of expectations and a calculus approach. Then we will consider a large needle solution via calculus. Finally, we will look at a Monte Carlo method of approaximating π using the small needle solution.

1. Introduction

In its original form, Georges Louis Leclerc, the Comte de Buffon's famous 1777 geometric probability problem was based on a gambling game, in which a loaf of French bread was tossed onto a wide-board floor. Bets were made based on whether or not the loaf of bread would intersect one of the cracks between the floorboards. Buffon wanted to know what ratio of the length of bread in relation to the width of the floorboards would make the game fair, by giving a fifty-fifty chance of a crossing [1, p. 51]. In a more precise and replicable form, Buffon reimagined this game with needles and an equidistant parallel ruled surface. There are two solutions. One for small needles, for which the needle length is less than or equal to the distance between the ruled lines. The other for long needles, for which the needle length is greater than or equal to the distance between the ruled lines. Throughout this paper, l will refer to the length of the needle and d will equal the distance between the ruled lines.

2. Barbier's Solution for Small Needles

Allow $E(l)$ to equal the expected number of intersections for a needle of length l. Then, for any needle of length l, the expected number of points where it crosses a line on a single toss can be written as

$E(l) = 1p_1 + 2p_2 + 3p_3 + ... \geq 0$, where p_i is the probability of precisely i crossings.

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Additionally, the probability of at least one crossing can thus be defined as:

$$P = p_1 + p_2 + p_3 + ...$$

Small needles cannot intersect more than one line. Therefore, in the small needle $l \leq d$ case, $p_2 = p_3 = p_4 = ... = 0$. This is because, for the sake of simplicity of calculations, both the needle and the ruled lines are mathematically idealized to have zero width. Therefore, a needle will virtually never fall such that an endpoint lands on a line. This means that there is zero probability of having a needle fall exactly on a single line so that all points on the needle, including both endpoints, fall on the line and give an infinite number of intersection points. Furthermore, a small needle of length equal to the distance between the ruled lines ($l = d$) will not fall perfectly perpendicular in such a way that the one endpoint hits one line and the other endpoint hits an adjacent line. As a result,

$$E(l) = p_1 \ \& \ P = p_1, \text{ and consequently,}$$

$$P = p_1 = E(l).$$

More generally, the theorem for the expected value function is as follows:

Theorem 1. [1, p. 226] *For any random variable X, the expected value of X is*

$$E(X) = \sum_{i=1}^{n} p(x_i)x_i.$$

Here i runs through each of the n outcomes, for which $p(x_i)$ is the probability that X takes on a value of x_i. From this, the linearity of expectations theorem can be proven [2, p. 174-177].

Theorem 2. [2, p. 112] *The expected value for the sum of two discrete random variables X & Y is as follows:*

$$E(X + Y) = E(X) + E(Y).$$

Proof.

$$\begin{aligned} E(X + Y) &= \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)(x_i + y_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} (p(x_i, y_j)x_i + p(x_i, y_j)y_j) \end{aligned}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) x_i + \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) y_j$$

$$= \sum_{i=1}^{n} p(x_i) x_i + \sum_{j=1}^{m} p(y_j) y_j$$

$$= E(X) + E(Y)$$

□

Basically, linearity of expectations states that the expected value of a sum is equal to the sum of the individual expected values. This means that $E(l)$ can be calculated by breaking the needle up into smaller pieces (tiny needles) and adding together the individual expected number of intersections for each of these pieces:

$$\text{For a needle of length } l = \sum_{i=1}^{n} x_i = x_1 + x_2 + ... + x_n,$$

$$E(l) = \sum_{i=1}^{n} E(x_i) = E(x_1) + E(x_2) + ... + E(x_n).$$

Of course, these smaller, tiny needle pieces can be rearranged into polygonal needles, for which the sum of their expected number of crossings gives the expected number of crossings for this new, nontraditional needle. Consequently, the expected number of crossings for a traditional straight line needle of length l is the same for a polygonal needle of length l. All that matters here is that their smaller, tiny needle pieces add up to the same total length. Surprisingly, the shape these tiny needles form is irrelevant.

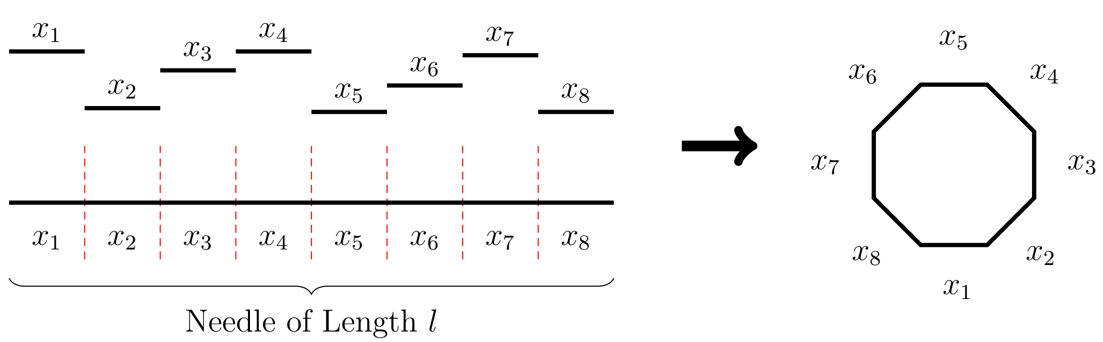

FIGURE 1. Traditional Needle Broken down and Rearranged as a Polygonal Needle

Furthermore, *Thm. 2* suggests that for a needle of integer length k,

$$E(k) = E(\underbrace{1+1+...+1}_{\text{k times}}) = kE(1).$$

However, this is even true for needles of rational length $k \in \mathbb{Q}$.

Proof. Suppose a and b are integers such that $a/b \in \mathbb{Q}$.

$$E(a/b) = \frac{b}{b}E(a/b) = \frac{1}{b}E(ba/b) = \frac{1}{b}E(a) = \frac{a}{b}E(1)$$

$\square$

Intuitively, the expected number of crossings function is monotone for $l \geq 0$, since the expected number of crossings should increase as l increases. As a result of this monotone property, it can be inferred that $E(l) = lE(1)$ for all $l \in \mathbb{R}$ [4, p. 2]. This is because limits exist at all points on monotone functions [5, p. 5]. With this is mind, the key theorem emerges.

Theorem 3. [2, p. 176] *For a polygonal needle of real length $l \in \mathbb{R}$, the probability of an intersection is:*

$$P = E(l) = l \cdot E(1).$$

Barbier's method for finding $E(1)$ is based on $Needle_C$, a circular "needle" with its diameter equal to the distance d between the lines of a horizontally ruled surface [2, p. 176-177]. Therefore l_C, the length of $Needle_C$, is equal to the needle's perimeter $\pi d = l_C$. Such a needle must always cross the lines at exactly two points. Either the same line penetrates the round needle, entering the circle at one point and then exiting again at another, or in a much less likely scenario, the needle can fall such that one ruled line is tangent at the uppermost point of the needle and a lower adjacent ruled line is tangent at the lowermost point of the needle. Fig. 2 shows this case where needle A is intersected by two different tangent lines, while needles B and C are intersected by the same line, as will most often happen.

Now consider two n sided polygons. In relation to $Needle_C$, let one be an inscribed small polygon P_S and the other a circumscribed large polygon P_L, such that $Needle_C$ is bounded between these two $Needle_C$ polygon approximations. Let their lengths be defined as l_{SP} and l_{LP}, respectively. Now consider the expected number of intersections for each of these. Every line that intersects P_S must also intersect the

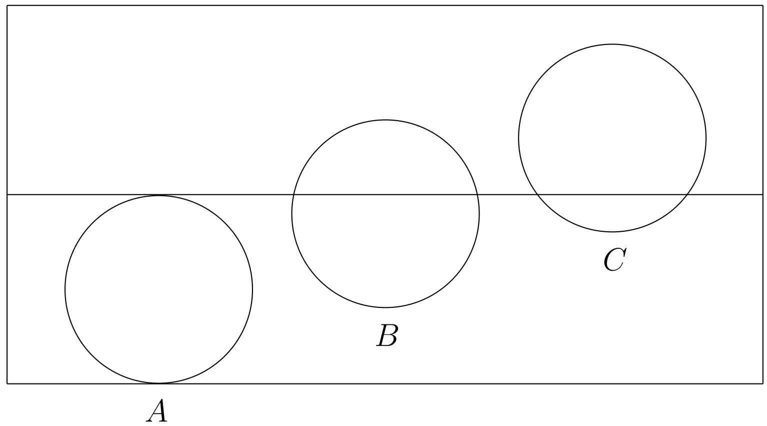

FIGURE 2. Circular "needles" dropped on a horizontally ruled surface

circumscribed $Needle_C$. However, it is possible for lines to intersect $Needle_C$ without intersecting P_S. Similarly, every line that insects $Needle_C$ must also intersect the circumscribed P_L, but it is possible for lines to intersect P_L without intersecting $Needle_C$. Thus intuitively,

$$E(l_{LP}) \geq E(\text{crossings of } Needle_C) \geq E(l_{SP}).$$

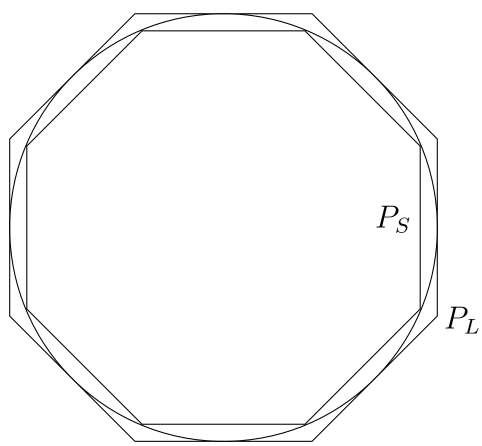

FIGURE 3. $Needle_C$ bounded by circumscribed P_L and inscribed P_S

Of course, by *Thm. 3* the expected number of intersections for each of these polygons can be rewritten as linear functions of their lengths multiplied by $E(1)$. Furthermore, as seen in *fig. 2*, the number of line crossings for $Needle_C$ is always equal to 2. Therefore, the previous inequality can be rewritten as:

$$l_{LP} \cdot E(1) \geq 2 \geq l_{SP} \cdot E(1) \tag{1}$$

Next suppose that the n number of sides of these polygons approaches infinity. In this case, their lengths will converge towards the perimeter of the circle $d\pi$:

$$\lim_{n\to\infty} l_{LP} = d\pi = \lim_{n\to\infty} l_{SP}$$

Therefore, plugging these values into *eq. 1* gives,

$$d\pi \cdot E(1) = 2 = d\pi \cdot E(1).$$

Finally, rearranging this equation gives,

$$E(1) = \frac{2}{d\pi}.$$

Now that the value of $E(1)$ has been found, the probability that a small needle of length $l \leq d$ will cross a line can be calculated:

$$P = E(l) = l \cdot E(1) = \frac{2l}{d\pi}. \tag{2}$$

3. Calculus Solution to Buffon's Problem for a Small Needle

Suppose a small needle of length l is dropped onto a horizontally ruled surface of equidistant d length spaces between the lines. Additionally, let y equal the vertical distance between the midpoint of the needle and the closest ruled line, such that $0 \leq y \leq \frac{d}{2}$. Next, suppose that the needle falls such that the angle made with the horizontal axis can be defined by $0 \leq \theta \leq \pi/2$. This special case is enough to find the probability of an intersection, since the other instances are merely the result of a different perspective, and thus share the same probability of an intersection.

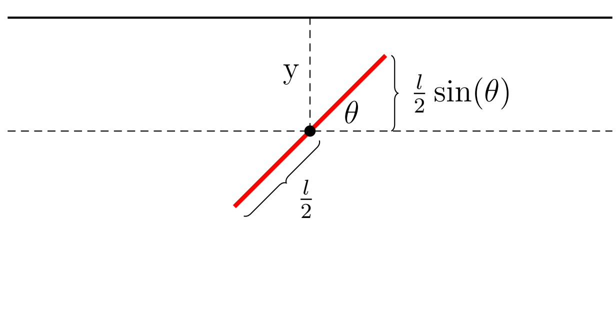

Figure 4. Small Dropped Needle

The key distances are thus y and $\frac{l}{2}\sin\theta$, where the latter term defines the vertical extension of the needle in both directions (up and down) from its midpoint. Therefore,

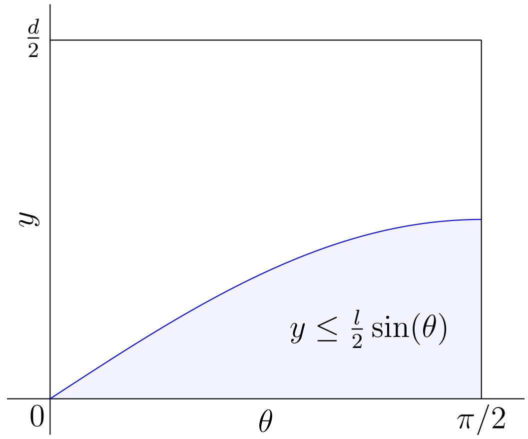

FIGURE 5. Areas for $l = 1$ & $d = 2$

the needle will only intersect a ruled line if $y \le \frac{l}{2}\sin\theta$. The key to solving this problem is to think of it graphically. Since $0 \le y \le \frac{d}{2}$ and $0 \le \theta \le \pi/2$, then all of the possible combinations of y and θ outcomes must lie in a rectangle of length $\pi/2$, width $d/2$, and area $d\pi/4$. Furthermore, the portion of this rectangle that consists of y and θ values that satisfy $y \le \frac{l}{2}\sin\theta$ represents the area of the outcomes that result in an intersection. These areas can be seen in *fig. 5*, for which the $y \le \frac{l}{2}\sin\theta$ function has been drawn according to a needle of length $l = 1$ and the rectangle dimensions are based on a distance between the ruled lines $d = 2$.

The probability of a small needle intersection can thus be calculated by dividing the $y \le \frac{l}{2}\sin\theta$ area of intersections by the total area of the rectangle $d\pi/4$, as shown in *eq. 11*. The integral calculates the $y \le \frac{l}{2}\sin\theta$ area, while the $4/d\pi$ in front of it is just the reciprocal of the total area of the rectangle. As expected, this matches the answer given by Barbier.

$$
\begin{aligned}
P &= \frac{4}{d\pi}\int_{\theta=0}^{\pi/2}\int_{y=0}^{\frac{l}{2}\sin\theta} dy d\theta \\
&= \frac{4}{d\pi}\int_{\theta=0}^{\pi/2}\frac{l}{2}\sin\theta d\theta \\
&= \frac{4}{d\pi}\frac{l}{2}\Big| -\cos(\theta)\Big|_0^{\pi/2} \\
&= \frac{2l}{d\pi}[-0-(-1)] \\
&= \frac{2l}{d\pi}
\end{aligned}
$$

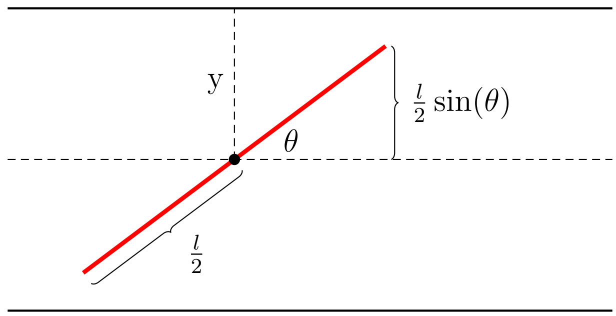

FIGURE 6. Large Dropped Needle

4. Calculus Solution to Buffon's Problem for a Large Needle

Now suppose that a large needle of length $l \geq d$ is dropped onto a horizontally ruled surface. As in the case of the small needle, let y equal the vertical distance between the midpoint of the needle and the closest ruled line, such that $0 \leq y \leq \frac{d}{2}$. Also, suppose that the needle falls such that the angle θ that it makes with the horizontal axis can be defined by $0 \leq \theta \leq \pi/2$.

The difference in the large needle $l \geq d$ calculation is that for large θ values,

$$\frac{lsin\theta}{2} \geq \frac{d}{2} \geq \text{ all } y \text{ values.}$$

Basically, if a large needle falls upright enough, then its vertical component $\frac{lsin\theta}{2}$ will always be at least as large as the largest y value: $d/2$. Consequently, these θ values will always produce an intersection, regardless of the y value. Therefore, the key to calculating the probability of an intersection is to break the problem into two integrals: one for cases where $\frac{lsin\theta}{2} < d/2$ and the other cases where $\frac{lsin\theta}{2} \geq d/2$:

$$P = \underbrace{\int_{\theta=0}^{\arcsin(d/l)} \int_{y=0}^{\frac{l}{2}\sin\theta} \frac{4}{d\pi} dy d\theta}_{\frac{l}{2}\sin\theta < \frac{d}{2}} + \underbrace{\int_{\theta=\arcsin(d/l)}^{\pi/2} \int_{y=0}^{d/2} \frac{4}{d\pi} dy d\theta}_{\frac{l}{2}\sin\theta \geq \frac{d}{2}} \tag{3}$$

$$= \frac{4}{d\pi} \int_{\theta=0}^{\arcsin(d/l)} \frac{l}{2}\sin\theta d\theta + \frac{2}{\pi} \int_{\theta=\arcsin(d/l)}^{\pi/2} d\theta$$

$$= \frac{2l}{d\pi} \Big| -\cos(\theta) \Big|_{0}^{\arcsin(d/l)} + \frac{2}{\pi} \Big|\theta\Big|_{\arcsin(d/l)}^{\pi/2}$$

$$= \frac{2l}{d\pi} \left[-\frac{\sqrt{l^2-d^2}}{|l|} - (-1) \right] + \frac{2}{\pi} \left[\pi/2 - \arcsin(d/l) \right]$$

$$= \frac{2l}{d\pi}\left[1 - \frac{\sqrt{l^2 - d^2}}{l}\right] + \left[1 - \frac{2}{\pi}(\arcsin(d/l))\right]$$

The first term on the right hand side of the equation is for the case where the needle falls at a small angle where it is not guaranteed to cross a line for all y values. Notice that for this integral, θ is bounded between 0 and $\arcsin(d/l)$. This upper limit was chosen so that

$$\frac{l}{2}\sin(\sin^{-1}(d/l)) = d/2, \text{ the maximum value of } y.$$

Additionally, the second term on the right hand side of the equation is for the case where the needle falls at a large angle where it is guaranteed to cross a line for all y values. *Fig. 5* shows the rectangular area graph for a large needle of length $l = 2.5$ and a distance between the ruled lines $d = 2$. Notice how the graph of the $y \leq \frac{l}{2}\sin(\theta)$ function extends beyond $d/2$ for $\theta > \sin^{-1}(d/l)$. For these θ values, all of the y values are below the critical line, and thus are irrelevant in determining a crossing. Conversely, for $\theta < \sin^{-1}(d/l)$, there are still some y and θ pairings above the line, for which the needle will not have an intersection. The probability resulting from the blue shaded area is calculated in the first term on the right hand side of *eq. 3*, while the probability added in the second term comes from the green shaded area.

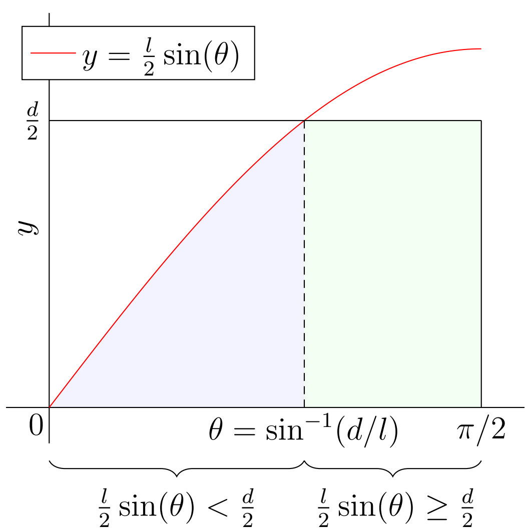

FIGURE 7. Areas for $l = 2.5$ & $d = 2$

Since both the small and large needle probability calculations work for needles where $l = d$, the probabilities calculated in *eqs. 2 and 3* should give the same answer for $l = d = k$:

$$\begin{aligned}
\frac{2k}{k\pi} &= \frac{2k}{k\pi}\left[1 - \frac{\sqrt{k^2 - k^2}}{k}\right] + \left[1 - \frac{2}{\pi}(\arcsin(k/k))\right] \\
\frac{2}{\pi} &= \frac{2}{\pi}\left[1 - 0\right] + \left[1 - \frac{2}{\pi}\frac{\pi}{2}\right] \\
\frac{2}{\pi} &= \frac{2}{\pi} + \left[1 - 1\right] \\
\frac{2}{\pi} &= \frac{2}{\pi}
\end{aligned}$$

As expected, they both give the same answer for the case where $l = d$.

5. Using Buffon's Small Needle Problem to perform a Monte Carlo Estimation of π

Since the probability that a small needle will cross a line is $P = \frac{2l}{d\pi}$, π can be solved by rearranging this equation:

$$\pi = \frac{2l}{dP} \tag{4}$$

This formula allows for a Monte Carlo method approximation of π. A Monte Carlo method is where a mathematical problem is solved through random, probabilistically distributed data. By performing the Buffon Needle game many times and recording the data, the P probability that a small needle will cross a line can be approximated experimentally as:

$$P = \frac{\textit{number of hits}}{\textit{number of tosses}}$$

Then by plugging in this P approximation along with the length of the needle l and the spacing between the lines d that were used in the estimation of P, *eq. 4* gives an approximation of π. Note that it gives an approximation of π because an approximation of P is being used. Thus, using a better estimate of P (based on more trials) should give a better estimate of π. Over time, many such experiments have been performed to estimate π as can be seen in *table 1.* Of these, Lazzarini's stands out as surprisingly accurate. Due to the odd number of casts, it is likely that he

Experimenter	Needle Length	Number of Casts	Number of Crossings	π estimation
Wolf, 1850	0.8	5000	2532	3.1596
Smith, 1855	0.6	3204	1218.5	3.1553
De Morgan, c. 1860	1.0	600	382.5	3.137
Fox, 1864	0.75	1030	489	3.1595
Lazzarini, 1901	0.83	3408	1808	3.1415929
Reina, 1925	0.5419	2520	869	3.1795

TABLE 1. Experimental Approximations of π [1, p. 51]

conducted his experiment with this outcome in mind. A famous approximation of π is $355/113 \approx 3.1415929$. With Lazzarini's needle length $l = 2.5cm$ and line spacing $d = 3cm$, It is likely that he set up the experiment so that he would eventually arrive at this approximation. With these figures and the desired π ratio of 355/113, Lazzarini's approximation looks like [3, p. 85-86]:

$$\hat{\pi} = \frac{5}{3}\frac{tosses}{hits} = \frac{355}{113}, \text{ when } \frac{tosses}{hits} = \frac{213k}{113k}, \text{ for some natural number } k \tag{5}$$

This means that Lazzarini would be able to get such an estimate by attempting 213 casts k times, until he achieved $113k$ hits. He supposedly conducted $4,000$ casts, and recorded this π estimation following the 3408^{th} toss, sighting it as a noteworthy point in the experiment. From his 4,000 casts, the probability of getting an outcome of $\frac{213k}{113k}$ for some natural number k is still quite unlikely, at about 30 percent [3, p. 86].

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