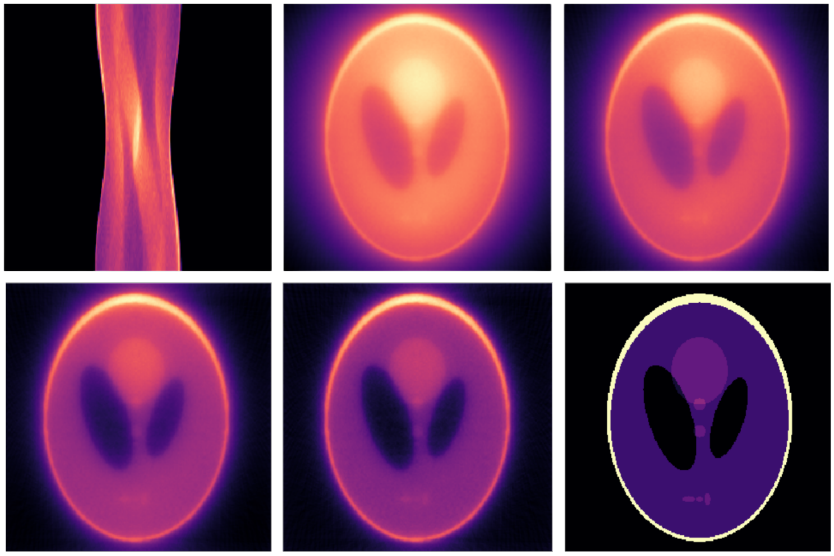
Image Reconstruction with Cimmino's Algorithm

Task M4.T1D - Project

Computed Tomography (CT) Scan and Reconstruction of the Shepp-Logan Phantom using Metal/C++



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Please see the GitHub repository for the complete code as I am only able to upload one code file to OnTrack. I have added my tutor Daniel and the unit chair Maksym to the repository as collaborators. This project is based off the SIT292 HD report I wrote this trimester, with permission from Maksym. The report is included in the repository for reference and focuses on the theoretical analysis of Cimmino's algorithm and proving its convergence as $k \to \infty$ and the characterisation of the limit point.

1 Introduction

In this project, we explore the implementation of Cimmino's algorithm for image reconstruction problems, such as those encountered in computed tomography (CT). The algorithm is particularly useful for solving large-scale and sparse linear systems that arise in these applications due to its ability to handle inconsistent systems and its parallelisable nature.

We do not aim to provide a perfectly reconstructed scan or an in-depth insight into CT reconstruction techniques, but rather to demonstrate the idea of using Cimmino's algorithm in a parallel computing environment to achieve a reasonable approximation of the original image. The main program uses Metal for GPU parallelisation, and we also provide sequential and OpenMP implementations for comparison. The focus of this project is on the Metal program, therefore, the other implementations are kept simple and straightforward.

2 Project Structure

The project is structured as follows:

- **Metal Implementation:** The main program is written in C++ and Metal Shading Language (MSL) with some Objective-C++ for interfacing with Metal. This implementation leverages the parallel processing capabilities of the GPU to efficiently perform the computations required by Cimmino's algorithm. The source files are located in the `metal-src` directory and the header files in the `metal-include` directory. The metal kernels are in the `metal-shaders` directory. I have split the Metal compute and render logic into two separate classes (`MTLComputeEngine` and `MTLRenderEngine`) for better organisation and modularity.
- **Sequential Implementation:** A simple C++ program that implements Cimmino's algorithm in a single-threaded manner for baseline performance comparison. This is in `Other-Implementations/Sequential/sequential.cpp`.
- **OpenMP Implementation:** A C++ program that uses OpenMP to parallelise the computation across multiple CPU cores, providing a middle ground between the sequential and Metal implementations. This is in `Other-Implementations/OpenMP/openmp.cpp`.
- **Python Scripts:** Python scripts are used to generate the projection matrix and phantom image using the Astra Toolbox, as well as to visualise the results outside of the Metal application. These can be found in the `metal-data` directory.

- **Data and Log Files:** The `metal-data` directory contains the projection matrix and phantom files as well as the reconstructed image files. The `metal-logs` directory contains log files generated during the execution of the Metal program for debugging and performance analysis.
- **CMake Build System:** The project uses CMake for building the C++ and Metal code. It includes all the necessary frameworks and headers and automatically compiles the Metal shaders. The CMake configuration file is located in the root directory.
- **Instructions:** A `README.md` file is provided in the root directory with instructions on how to build and run the project.

3 Cimmino's Algorithm Overview

To give a high-level overview, Cimmino's algorithm is a row-iterative algorithm that simultaneously reflects the current approximation point (estimate) across all hyperplanes defined by the equations of the linear system. The subsequent approximation is then computed as the weighted average of these reflections. The result is a convergent sequence of approximations that approaches either a solution of a consistent system or the weighted least-squares solution in the case of an inconsistent system [1]. Mathematically, Cimmino's algorithm is guaranteed to converge for any intiial approximation x^0. This is rigourously proven in the accompanying SIT292 HD report.

Given an $m \times n$ matrix A with rows A_i, a vector $b \in \mathbb{R}^m$ and non-negative weights ω_i for $i = 1, 2, \ldots, m$, the algorithm can be expressed as follows:

$$x^{k+1} = \sum_{i=1}^{m} \frac{\omega_i}{\omega} y^{(k,i)} = x^k + 2 \sum_{i=1}^{m} \frac{\omega_i}{\omega} \frac{b_i - A_i^T x^k}{\|A_i\|^2} A_i \quad \text{for } k = 0, 1, 2, \ldots \tag{1}$$

Here, x^k is the current approximation, $y^{(k,i)}$ is the reflection of x^k across the hyperplane defined by the i-th equation, ω_i are non-negative weights assigned to each equation, and $\omega = \sum_{i=1}^{m} \omega_i$ [1].

In simpler terms, the algorithm can be expressed in matrix form as:

$$x^{k+1} = x^k + \frac{2}{\omega} A^T D^T D (b - A x^k) \quad \text{for } k = 0, 1, 2, \ldots \tag{2}$$

where D is a diagonal matrix with entries $D_{ii} = \sqrt{\omega_i} / \|A_i\|$.

$$D = \begin{bmatrix} \frac{\sqrt{\omega_i}}{\|a_1\|} & 0 & \ldots & 0 \\ 0 & \frac{\sqrt{\omega_2}}{\|a_2\|} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{\sqrt{\omega_m}}{\|a_m\|} \end{bmatrix} \tag{3}$$

For simplicity, we choose particular weights $\omega_i = \|a_i\|^2$ for $i = 1, 2, \ldots, m$. This choice simplifies the diagonal matrix D to the identity matrix I, leading to a more

straightforward update rule:

$$x^{k+1} = x^k + \frac{2}{\omega} A^T (b - Ax^k) \quad \text{for } k = 0, 1, 2, \ldots \tag{4}$$

To explicitly measure the reconstruction quality and check for convergence, we compute the relative error norm E between the current approximation and the phantom P (original image) every 50 iterations. The relative error norm is defined as:

$$E = \frac{\|x^k - P\|_2^2}{\|P\|_2^2} \tag{5}$$

The convergence criteria is set to a relative error norm of less than 10^{-2} or until the maximum number of iterations is reached, as specified by the user. Mathematically,

$$E < 10^{-2} \quad \text{or} \quad k = \text{max\_iterations}$$

4 Implementation Details

We intend to perform a simulation of a CT scan by setting the geometry parameters of our 'scanner', generating a projection matrix that models the scanner geometry, performing a scan of a phantom to obtain a sinogram (a vector of measurements), and then using Cimmino's algorithm to reconstruct the original image from the sinogram.

For this image reconstruction problem, we use the well-known Shepp-Logan phantom as our test image. The Shepp-Logan phantom is a commonly used synthetic image modeling the cross-section of a human head and is widely used in the field of computed tomography (CT) for testing and evaluating reconstruction algorithms [2]. Our objective is to reconstruct this phantom image from its projections (sinogram) using Cimmino's algorithm. The projections are obtained by simulating the passage of X-rays through the phantom at various angles [3].

Phantom (P): A phantom is a standard test image comprising of various shapes and intensities, used in medical imaging to model the human body [4]. The Shepp-Logan phantom is our ground truth image that we aim to reconstruct and measure our reconstruction quality against. The 256x256 Shepp-Logan phantom is generated in Python using ASTRA-Toolbox [5].

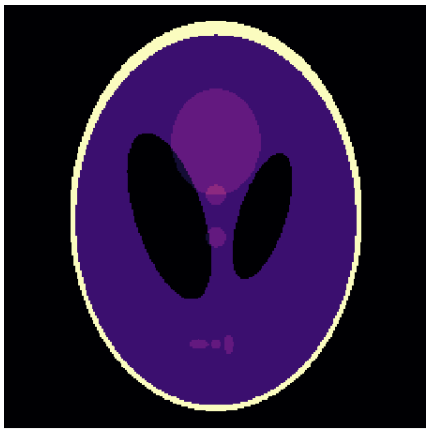


Figure 1: Shepp-Logan Phantom (256x256)

Projection Matrix (A): A matrix that transforms the image space into the projection space based on the scanner geometry [6]. This matrix represents the system of equations we need to solve to reconstruct the image. In other words, it is A in the linear system $Ax = b$. The projection matrix is generated using ASTRA-Toolbox in Python, which provides efficient methods for creating projection matrices based on specified scanner geometries [7]. In other words, it is complex mathematics that I am not confident in implementing myself. The projection matrix is then saved to a binary file as a sparse matrix in the Compressed Sparse Row (CSR) format to optimise memory usage and read into the Metal application, as well as the other implementations.

Sinogram (b): A vector of measured projections obtained by simulating the CT scan of the phantom [8]. This can be thought of as b in the linear system $Ax = b$. The sinogram is computed by projecting it using the projection matrix A, i.e. $b = AP$, where P is the vectorised phantom image. This is computed in a Metal kernel function with the following thread configuration: Grid Size: (65536, 1), threadgroup Size: (1024, 1), number of threadgroups: 64.

As seen below, the sinogram does not look like the original phantom, but it contains all the information needed to reconstruct it with the projection matrix.

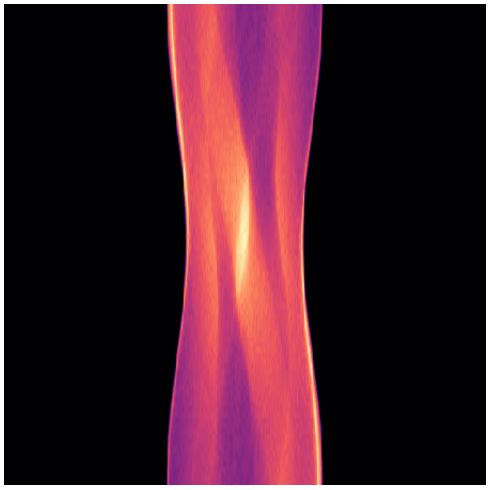


Figure 2: Sinogram (90 angles, 725 detectors)

We also use Metal kernel functions to normalise the sinogram texture values to a [0,1] range for better visualisation in the render window after computation. For this, a three-step process is used:

> **Step 1:** Find the maximum value in the sinogram. We use a SIMD parallel partial reduction algorithm, inspired by an example from the MSL specification [9], to find the maximum value in the sinogram. For the SIMD `findMaxPerThreadgroupKernel`, a local threadgroup memory is used to store the intermediate maximum values found by each thread in the threadgroup. Each thread computes the maximum value for its assigned elements and stores it in the local memory. After all threads have completed their computations, a single thread (thread 0) iterates through the local memory to find the overall maximum value for the entire threadgroup. The thread configuration for this kernel is as follows: Grid size: (725, 90, 1), threadgroup size: (16, 16, 1), number of threadgroups:

(46, 6, 1), total thread groups: 276.

Step 2: Find the maximum value across all threadgroups. This is computed on the CPU after reading back the maximum values from each threadgroup. This is done in a simple for loop iterating through the array of maximum values from each threadgroup to find the overall maximum value. Since we only have 276 threadgroups, this is a quick operation and does not benefit from parallelisation in this context.

Step 3: Normalise the sinogram using the maximum value found in the previous steps. The `normaliseKernel` divides each element in the sinogram by the maximum value to scale it to the [0,1] range. The thread configuration for this kernel is as follows: Grid size: (725, 90, 1), threadgroup size: (32, 32, 1), number of threadgroups: (23, 3, 1), total thread groups: 69.

Normalisation of the sinogram is not necessary for the computation and is only done for better visualisation in the render window as well as to demonstrate other Metal/GPU computation capabilities. Without normalisation, we are not able to properly visualise the sinogram as an image since the values are not in the [0,1] range. Below is the non-normalised sinogram for comparison.

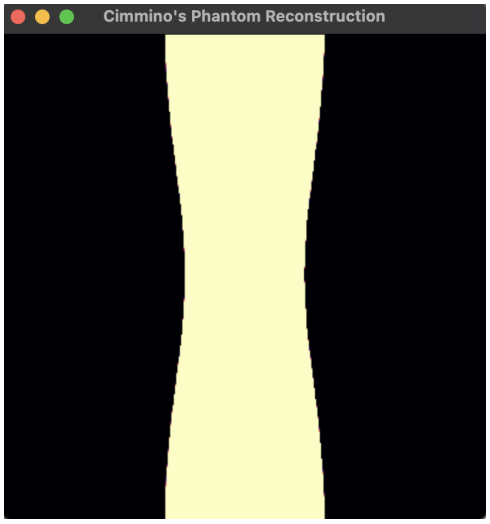


Figure 3: Non-normalised Sinogram (90 angles, 725 detectors)

Scanner Geometry: The configuration of the CT scanner, including the number of angles, the number of detectors, the detector spacing and total angle degree [10]. This defines how the projection matrix is constructed and influences the quality of the reconstruction. In this project, we set the scanner geometry parameters as follows:

Number of angles: 90 (i.e., projections taken every 2 degrees over 180 degrees)

Number of detectors: $\lceil 2 * \sqrt{2} * 256 \rceil = 725$ (to ensure full coverage of the image width) **systemmatrix**

Detector spacing: 1 unit

Total angle degree: 180 degrees

It's important to note, that the more angles and detectors we have, the better the reconstruction quality likely will be, but this also increases the computational load and memory requirements. For example, using the parameters we have chosen (90 angles and 725 detectors based on a 256x256 image) which are rather modest, the projection matrix becomes quite large ($256 \cdot 256 \times 90 \cdot 725 = 65536 \times 65250$), with 18737864 non-zero elements and ≈ 4.28 billion total elements if stored as a dense matrix. This is why we opted to use sparse matrix representation and parallel processing to handle the computations efficiently.

4.1 Cimmino's Algorithm Implementation

The core of the project is the implementation of Cimmino's algorithm in a parallel computing environment using Metal. The algorithm iteratively updates the image estimate by reflecting it across the hyperplanes defined by each equation in the linear system and averaging these reflections. The algorithm implementation is decomposed into three steps/kernels:

Reconstruction Kernel: Each thread is responsible for one row of the projection matrix. It computes the dot product of the current estimate and the corresponding row of the projection matrix, subtracts this from the corresponding measurement in the sinogram to obtain the residual, and then computes that ray's contribution to the reflection. These results are stored in an update buffer. The thread configuration for this kernel is as follows: Grid Size: (65536, 1), threadgroup Size: (1024, 1), number of threadgroups: 64.

Update Kernel: This kernel reads the update buffer and adds the contributions from all rays to the current estimate, effectively averaging the reflections to produce the new estimate. In this kernel, the relative update norm is also partially computed - specifcally the difference between the approximation and the phantom and the phantom norm. The thread configuration for this kernel is as follows: Grid Size: (65536, 1), threadgroup Size: (1024, 1), number of threadgroups: 64.

Relative Error Norm/Convergence Check: Every 50 iterations, the relative error norm E is calculated. The difference between each element of the current estimate and the phantom is squared and summed up in a Metal kernel with the following thread configuration: Grid Size: (65536, 1), threadgroup Size: (1024, 1), number of threadgroups: 64. The phantom norm is precomputed on the CPU since it does not change during the iterations. The relative error norm is then computed on the CPU and checked against the convergence criteria.

5 Results

We have three implementations of Cimmino's algorithm for comparison: a sequential C++ implementation, an OpenMP parallelised C++ implementation, and the main Metal implementation. All programs are run on a Macbook Air with an Apple M3 chip (8-core CPU, 10-core GPU, 8GB unified memory).

All programs are tested with the same parameters: 90 angles, 725 detectors and 256x256 image size and use the same projection matrix generated with ASTRA-Toolbox. The

sinogram produced in the Metal application is saved to a .txt file and loaded into other applications for direct comparison. We are specifcally looking at the computation time of the reconstruction algorithm itself as detailed in Section 4.1. The sequential and OpenMP implementations mimic the same steps as the Metal implementation, but without the GPU acceleration.

We are only comparing the reconstruction time (the reconstruction loop including reconstruction, update and convergence check) to have a fair comparison between the different programs. The time taken to generate the sinogram, load the projection matrix and render the images is not included in this comparison. Furthermore, while we should consider the total time taken for the Metal implementation (including data transfer to/from the GPU), even in our case, this time is neglible ($\approx$ 1ms).

Let's have a look at the execution times for 10 - 2000 iterations. The times are averaged over multiple runs. We have included the final relative error norm E for all iteration counts and programs to give an analytical measure of the reconstruction quality and ensure each program is working correctly and consistently. We expect that each implementation will produce the same relative error norm for a given number of iterations.

All results are compiled and tabulated in Python (`plots.ipynb`) from the log files generated by each program.

5.1 Relative Error Norms and Execution Times

The key thing we observe here is the consistent relative error norms across all implementations, indicating that they are all functioning correctly and producing the same results. This is important for validating the correctness of the parallel implementations against the sequential baseline - often parallel implementsations can be deceivingly fast simply because they are not producing the correct results and visually inspecting the reconstructed images may not always reveal subtle differences. We can confirm that for the same number of iterations, all implementations yield the same relative error norm, which is a good indication that they are all functioning consistently.

Iterations	Seq. (ms)	OMP (ms)	Metal (ms)	Relative Error Norm		
				Seq.	OMP	Metal
1	126.711	71.500	13.193	0.996	0.996	0.996
10	1231.700	732.940	67.532	0.965	0.965	0.965
100	20751.050	7805.349	546.398	0.808	0.808	0.808
500	129114.500	39806.083	2400.600	0.661	0.661	0.661
1000	226751.500	88153.525	4824.020	0.576	0.576	0.576

Figure 4: Relative Error Norms for Different Implementations up to 1000 Iterations

The relative error norm is quite high, but this is expected given the limited number of angles and detectors used in the scan as well as the lack of preconditioning, regularisation and other advanced techniques that are often employed in practical CT reconstruction scenarios [3].

Increasing the number of angles and detectors would likely improve the reconstruction quality but also increase the computational load, memory requirements and execution times. However, a decrease in the relative error norm is observed with an increasing number of iterations, indicating that the algorithm is converging towards a better approximation of the original image.

Secondly, we can see huge performance improvements using GPU parallelisation with Metal. This is further reflected in the following speedup table.

Iterations	Seq. (ms)	OMP (ms)	Metal (ms)	Speedup		
				OMP vs Seq	Metal vs Seq	Metal vs OMP
1	126.711	71.500	13.193	1.772	9.604	5.419
10	2154.880	735.737	67.532	2.929	31.909	10.895
100	20751.050	7805.349	532.843	2.659	38.944	14.648
500	129114.500	39806.083	2368.883	3.244	54.504	16.804
1000	226751.500	88153.525	4824.020	2.572	47.005	18.274

Figure 5: Execution Times for Different Implementations up to 1000 Iterations

The OpenMP program, though simple, still offers a decent speedup ($\approx$ 3x) over the sequential version. However, for larger-scale image reconstruction problems this is not sufficient. We are only performing 2000 iterations and are yet to achieve a good reconstruction in terms of the relative error norm. Therefore, the Metal implementation is the most impressive, achieving a speedup of over 50x compared to the sequential version and an 18x speedup compared to the OpenMP version for 1000 iterations.

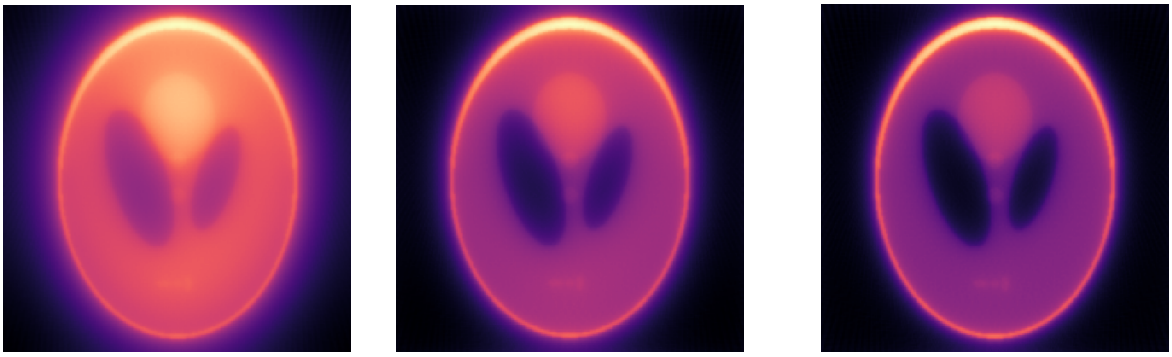
This demonstrates the power of parallel computing on GPUs for computationally intensive tasks like image reconstruction though, it's important to note that we would likely need significantly more iterations to achieve a high-quality reconstruction, especially since the error norm reduces more slowly with more iterations. Even with only 1000 iterations, the sequential and OpenMP programs become impractical, taking over 3.5 minutes and 1.4 minutes respectively.

Thus, this eseentially eliminates the use case of the sequential and OpenMP programs for larger scale image reconstruction.

6 Comparing the Reconstructed Images

Finally, let's compare the reconstructed images after each iteration count. Since we have already established that all implementations produce the same results (same error norm), we will only show the Metal reconstructions here. Recall that we begin with an initial guess of a zero image (all black) and iteratively improve it.

(a) $k = 0$, $E = 1$ (b) $k = 1$, $E = 0.996$ (c) $k = 10$, $E = 0.965$



(d) $k = 100$, $E = 0.808$ (e) $k = 500$, $E = 0.661$ (f) $k = 1000$, $E = 0.576$

Figure 6: Reconstructed Shepp-Logan phantoms using Cimmino's algorithm in C++/Metal for $k = 0, 1, 10, 100, 500, 1000$ iterations with relative error norms E.

Even with 1 iteration, we can already see some structure of the original image and the phantom is identifiable. As the number of iterations increases, the reconstruction quality improves, with more details becoming visible. However, even with 1000 iterations, the reconstruction is still quite rough and lacks fine details. Regardless, the results are quite impressive.

Out of interest and without comparing execution times to the sequential and OpenMP programs (since they would take too long), we also ran the Metal implementation for 5000 and 10000 iterations to see if the reconstruction quality improves further. The results are shown below:

(a) $k = 5000$, $E = 0.337$ (b) $k = 10000$, $E = 0.254$ (c) Original Phantom

Figure 7: Reconstructed Shepp-Logan phantoms for $k = 5000$ and 10000 iterations with relative error norms E. The original phantom is shown on the right for reference.

We can see some sharper lines and more details in the 5000 and 10000 iteration reconstructions, but of course, there is still a lack of fine details and clarity. This further emphasises the need for efficient computing for image reconstruction tasks, as achieving high-quality results may require a large number of iterations and more memory-intensive operations.

In practice, CT reconstructions often require many more iterations and more sophisticated techniques to achieve high-quality results. However, this project demonstrates the feasibility of using Cimmino's algorithm with GPU parallelism for image reconstruction tasks.

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