VIII. Mathematical Analysis

Using the MDP (Markov Decision Process) and minimal spanning tree, train rescheduling can be mathematically modeled using a number of mathematical concepts and methods, including:

1. **Graph theory:** To determine the shortest route between nodes, the minimal spanning tree approach makes use of graph theory.

2. **Linear algebra:** In most cases, matrix multiplication, eigenvectors, and eigenvalues are used to solve the matrices that describe the MDP.

3. **Theory of probabilities:** The MDP is a stochastic process that takes into account the probabilities of various states and actions. These probabilities are computed using probability theory, and the best course of action is determined.

4. **Game theory:** In some circumstances, it is possible to depict the train rescheduling issue as a contest between the railway operator and the passengers. The best solution is determined by using game theory to evaluate the strategic interactions between the various parties. The Multi-agent model uses numerous agents which strongly depend on the behaviour of agents, some external events and also on the present state.

As the railway system is very prone to disasters, the uncertainty of the situation makes the environment more vulnerable. In this paper, this scenario is modeled as Markov decision process (MDP) with its states (W) and transitions (υ).

$\Longrightarrow$ **Set of world state** (W)

W represents the set of agent's state(s) in railway network under disturbance. Train agents can sense three kind of states; if S_i is assumed to be a station, where disaster happens, then T_j is either on track connecting S_i or at a platform of S_i or at a platform of station $S_{i'}$, connected to S_i. i.e. $W = \{(L_{jil} = 1), (P_{jik} = 1), (P_{j'i'k} = 1)\}$

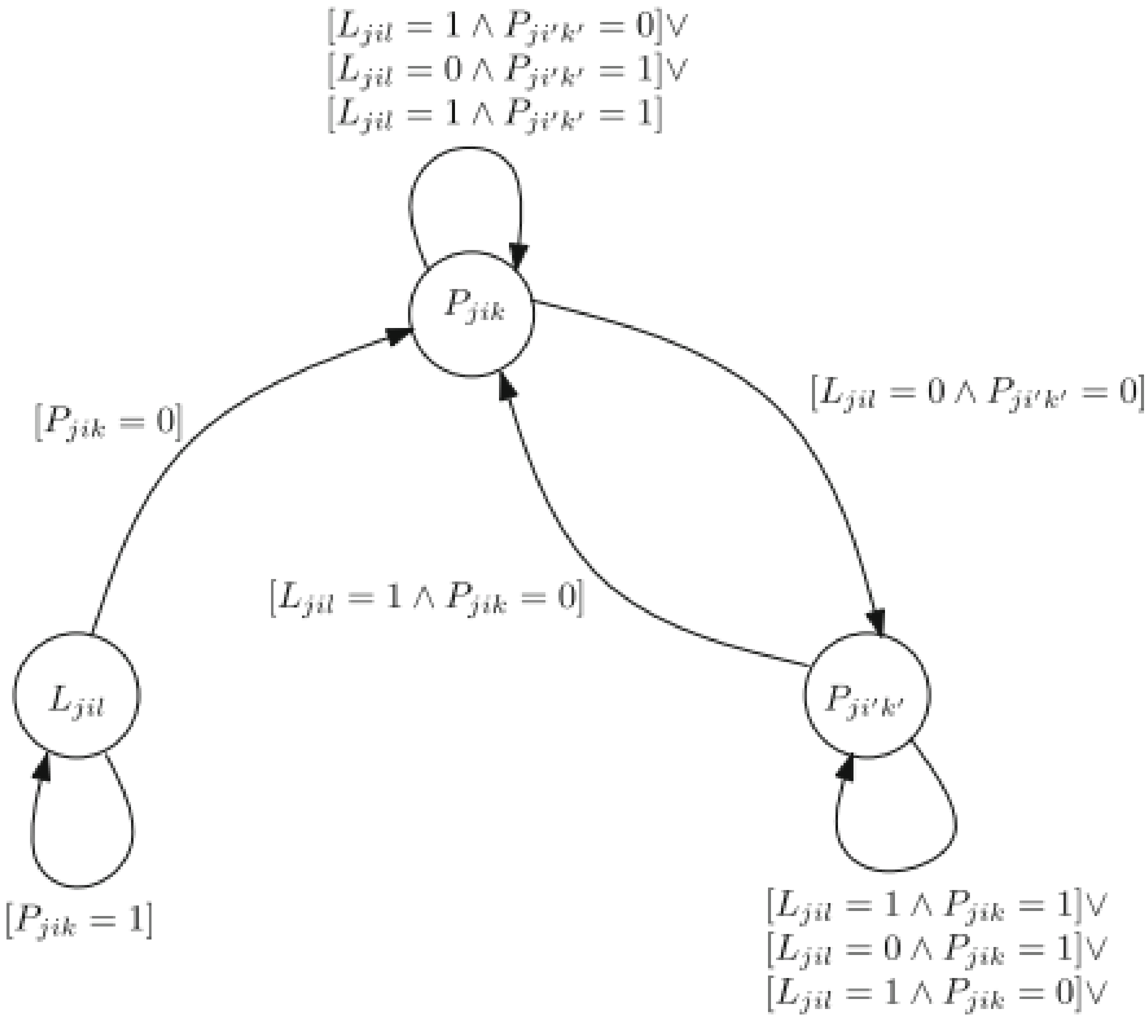
$\Longrightarrow$ **Transition function** (Ψ)

The state transition function is denoted as $\Psi(\omega, C, \omega')$. In our proposed approach, each action C maps to constraint(s) of DCOP to satisfy to reach from state ω to the next state ω', where $\omega, \omega' \in W$.

Every node represents a state of the railway environment and each arc represents an action which is indeed a constraint. The transition from one state to another state happens if and only if the corresponding agent satisfies the specific constraint(s).

- In the figure below,

Ljil $= 1 \rightarrow$ Train j $\rightarrow$ present at station $i \rightarrow$ at Lth trek
Pjik $= 1 \rightarrow$ no platforms available
Pjik $= 0 \rightarrow$ platforms free
Pji$'k' = 1 \rightarrow$ platforms at next station not free


$[L_{jil} = 1 \wedge P_{ji'k'} = 0] \vee$
$[L_{jil} = 0 \wedge P_{ji'k'} = 1] \vee$
$[L_{jil} = 1 \wedge P_{ji'k'} = 1]$
P_{jik}
$[P_{jik} = 0]$
$[L_{jil} = 0 \wedge P_{ji'k'} = 0]$
$[L_{jil} = 1 \wedge P_{jik} = 0]$
L_{jil}
$P_{ji'k'}$
$[P_{jik} = 1]$
$[L_{jil} = 1 \wedge P_{jik} = 1] \vee$
$[L_{jil} = 0 \wedge P_{jik} = 1] \vee$
$[L_{jil} = 1 \wedge P_{jik} = 0] \vee$

Disaster handling and rescheduling approach

According to real-time scenario, in case of platform blockage or track blockage, disaster handling and rescheduling model of railway system refers three situations:

(1) Delay or stop at the station or on track (Retiming).

(2) Change in departure sequence of trains at the station depending on priority of trains (Reordering).

(3) Reschedule to alternative path (Rerouting).

$\Longrightarrow$ **Case 1: Partial node deletion from the graph** G

$Astation$S$_i$ faces problem due to disaster and the train T_j is on track l, approaching to the station S_i, that is

$$L_{jil} = 1$$

- **Case 1.1**

If S_i has a free platform at the time, when T_j reaches to S_i, then the system can allow T_j to reach S_i, if and only if the priority of the incoming train T_j has the highest priority amongst all trains T and the resource (Re) required for any other high priority train $T_{j''}$ does not hamper the resource requirement of T_j :

$$P_{jik}|_{x_{ji}^{AT}} = 1, \text{ if and only if } \left[\text{Prio}\left(T_j\right) > \text{Prio}\left(T_{j'}\right)_{j \neq j', j \in [1,m]}\right]$$
$$\wedge \left[\text{Re}\left(T_{j''}\right)|^{\tau^B}_{\text{Prio}\left(T_{j''}\right) > \text{Prio}(T_j)} \neq \text{Re}\left(T_j\right)|^{\tau^B}\right]$$

. Route of Tj after departure from Si at time $xDTji$ is [Rou$(Tj)|t \leq xDTji+$ Rou$(Tj)|xAT$ji $> x$ DT ji], that is

$$\text{if Rou}\left(T_j\right)|_{x_{ji}^{AT} > x_{ji}DT} = \bigwedge_{i'>i}^{n-1} \left(P_{ji'k} L_{ji'l}\right) \cup P_{jnk} = 0$$

- **Case 1.1.2**

If case 1.1.1 is invalid, then stop T_j at S_i until recovery is done or any other alternative path becomes free. Therefore, T_j occupies one of the platforms at S_i, that is

$\longrightarrow$ **Analysis of PN2:**

Tables 5 and 6 presents the description of the places $P = \{P1, P2, P3, P4, P5, P6, P7\}$ and transitions $Tr = \{\text{Tr}\,1, \text{Tr}\,2, \text{Tr}\,3, \text{Tr}\,4, \text{Tr}\,5, \text{Tr}\,6, \text{Tr}\,7, \text{Tr}\,8\}$ and the initial marking is $M_0 = [1, 1, 0, 0, 0, 0, 0]$.

$\longrightarrow$ **Reachability graph analysis:**

Reachability graph analysis is the simplest method to analyse the behaviour of a Petri-Net. It decides whether the system is bounded and live or not. From our resultant tree in Fig. 4 it can be proved that: (a) the reachability set $R\left(M_0\right)$ is finite, (b) maximum number of tokens that a place can have is 2 , so our $PN2$ is 2-bounded, (c) all transitions can be fired, so there are no dead transitions.

$\Longrightarrow$ **State Matrix :**

The statement behaviour of the petri-net model can be measured using the Algebraic Analysis called the incidence matrix.

Incidence matrix represents the connections between places and transitions, the number of total transitions can be represented through the incidence matrix.

No $\rightarrow$ Initial Marking
N $\rightarrow$ Reachable Marking

It can be done as : $\boldsymbol{M}_0 + [\boldsymbol{A}] \times X_\sigma = \boldsymbol{M}$

$\Longrightarrow$ **Incidence Matrix :**

The order of the places in the matrix is as follows :

$P = \{P1, P2, P3, P4, P5, P6, P7\}$ denoted by rows and

$\mathrm{Tr} = \{\mathrm{Tr}\,1, \mathrm{Tr}\,2, \mathrm{Tr}\,3, \mathrm{Tr}\,4, \mathrm{Tr}\,5, \mathrm{Tr}\,6, \mathrm{Tr}\,7, \mathrm{Tr}\,8\}$ denoted by columns

X_σ is an m-dimensional vector with its j th entry denoting the number of times transition t_j occurs in σ :

$$\boldsymbol{A} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

Thus, if we view a marking $\boldsymbol{M}_0$ as a k-dimensional column vector in which the i th component is $M_0(p_i)$, each column of $[\boldsymbol{A}]$ is then a k-dimensional vector such that

$$\boldsymbol{M}_0 \xrightarrow{\sigma} \boldsymbol{M}$$

In our system, marking $\boldsymbol{M} = [1, 1, 0, 0, 0, 0, 0]$ is reachable from initial marking $\boldsymbol{M}_0 = [1, 1, 0, 0, 0, 0, 0]$ through the firing sequence $\sigma_1 = \text{Tr}\,2, \text{Tr}\,3, \text{Tr}\,4, \text{Tr}\,6, \text{Tr}\,7$.

$$M_0 \xrightarrow{Tr2} M_1 \xrightarrow{Tr3} M_2 \xrightarrow{Tr4} M_3 \xrightarrow{Tr6} M_4 \xrightarrow{Tr7} M_5 (= \text{'old }')$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \textit{Similarly we can reach th}$$

$M_0 \xrightarrow{Tr2} M_1 \,(= \text{ 'old }')\,.$
$M_0 \xrightarrow{Tr2} M_1 \xrightarrow{Tr3} M_2 \xrightarrow{Tr5} M_3 \xrightarrow{Tr8} M_2 \xrightarrow{Tr4} M_4 \xrightarrow{Tr6}$
$M_5 \xrightarrow{Tr7} M_6 (= \text{ oold }) :$

Currently multiple trains are on various platforms which follow a sequence accordingto

τ^B. Here, τ^B is related to the disaster recovery time t_R of that station. The track is free but more than one

$L_{jil} = 0$

$\sum_{j=1} P_{jik} \leq (p-1)$

As the station S_i faces disaster, a particular k th platform can not be used until the recovery time has elapsed. If there is any incoming train T_j within buffer period τ^B, the system allows T_j to reach S_i if a platform is available, i.e. $P_{jik} = 0$. The system also checks for the priority of T_j to reorder the departure schedule of all trains from S_i introducing delay $\delta_{ji} to T_j$, if needed.

$$\text{i.e. } \forall j, \text{ if } \operatorname{Prio}(T_{j'}) > \operatorname{Prio}(T_j)$$

$$x_{ji}^{DT} = o_{ji}^{DT} + \delta_{ji}$$

Otherwise, if all the resources are available for T_j and it has the highest priority among all the trains currently waiting at S_i, the scheduled departure of T_j is the original departure time as per the original railway timetable, that is

$$x_{ji}^{DT} = o_{ji}^{DT}$$

$$\textit{if and only if } \operatorname{Prio}(T_j) > \operatorname{Prio}(T_{j'})_{j \neq j', j \in [1,m]}$$

Analysis of PN3

$P = \{P1, P2, P3, P4, P5, P6\}$ and transitions $Tr = \{\mathrm{Tr}\,1, \mathrm{Tr}\,2, \mathrm{Tr}\,3, \mathrm{Tr}\,4, \mathrm{Tr}\,5\}$ and the initial marking is $\boldsymbol{M}_0 = [2, 3, 0, 0, 0, 0]$.

The order of the places in the incidence matrix $\boldsymbol{A}$ is $P =$

IX. $\{P1, P2, P3, P4, P5, P6\}$, denoted by rows and the order of the transitions is $\mathrm{Tr} = \{\mathrm{Tr}\,1, \mathrm{Tr}\,2, \mathrm{Tr}\,3, \mathrm{Tr}\,4, \mathrm{Tr}\,5\}$, denoted by columns:

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Here, marking $\boldsymbol{M} = [1, 2, 0, 1, 1, 0]$ is reachable from initial marking $\boldsymbol{M}_0 = [2, 3, 0, 0, 0, 0]$ through the firing sequence $\sigma_1 = \mathrm{Tr}\,1, \mathrm{Tr}\,2, \mathrm{Tr}\,3, \mathrm{Tr}\,4. M_0 \xrightarrow{Tr1} M_1 \xrightarrow{Tr2} M_2 \xrightarrow{Tr3} M_3 \xrightarrow{Tr4} M_4$

$$\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Train T_j is neither waiting at the station S_i, where disaster happened, i.e. $P_{jik} = 0$ nor on the connecting track, i.e. $L_{jil} = 0$. However, T_j reaches the station S_i within τ^B.

Train T_j is at station $S_{i''}$, where $S_{i''}$ is in neighbourhood of S_i

that is

$$P_{ji''k} = 1, \quad S_{i''} \in S \backslash S_i \text{ and } i \in [1, n]$$

If any platform is available at the next station and the connecting track is also free, the system checks for the priority of the train T_j. T_j maintains its original schedule if and only if it has the highest priority while reaching S_i, i.e., if

$$\begin{aligned}&(P_{jik'} = 0) \wedge (L_{jil} = 0) \wedge \Big(\operatorname{Prio}(T_j)\big|_{t=x_{ji}^{AT}} \\ &> \operatorname{Prio}(T_{j'})\big|_{j \neq j', j \in [1,m]} \Big)\end{aligned}$$

then

$$x_{ji}^{AT} = o_{ji}^{AT}$$

$$L_{jil} = 1 \text{ and } P_{jik'}\big|_{t=x_{ji}^{AT}} = 0$$

$$P_{jik'}\big|_{t=x_{ji}^{AT}} = 1 \text{ and } L_{jil} = 0$$

Here, damaged platform is $k.k' = \{1, 2, \ldots, p\} \backslash \{k\}$

Case 3.2 Train T_j is at $S_{i'}$, i.e. $P_{ji'k} = 1$, where $i, i' \in [1, n]$ and $i \neq i'$

There are multiple tracks between two stations S_i and $S_{i'.}$. i.e. $1 < l \leq 4$. If the track l breaks down due to disaster, it is assumed that track l is not free, that is

$$L_{ji'l} = 1, 1 \ \leq \ l < 4$$

Then, the trains which are scheduled to use that track face problem. In that case, first $S_{i'}$ checks for other available tracks, one of which can be allotted to T_j, provided T_j has the highest priority satisfying all the constraints and there is no resource conflict within τ^B.

$$[L_{ji'l'} = 0] \wedge \Big[\mathrm{Prio}\,(T_j) |\, t = xjj'^{DT} > \mathrm{Prio}\,(T_{j'}) |\, j \neq j',, j \in [1, m] \Big] \wedge \Big[\mathrm{Re}\,(T_j) |_{xji^{AT}} \neq \mathrm{Re}\,(T_{j'}) |\, xji^{AT} \Big]$$

Figure 6 represents Petri-Net model for the scenario of Case 3

Analysis of PN4: Tables 11 and 12 presents the description of the places $P = \{P1, P2, P3, P4, P5, P6\}$ and transitions $Tr = \{\mathrm{Tr}\,1, \mathrm{Tr}\,2, \mathrm{Tr}\,3\}$ and the initial marking is $\boldsymbol{M}_0 = [2, 1, 0, 0,\ 1, 0]$.

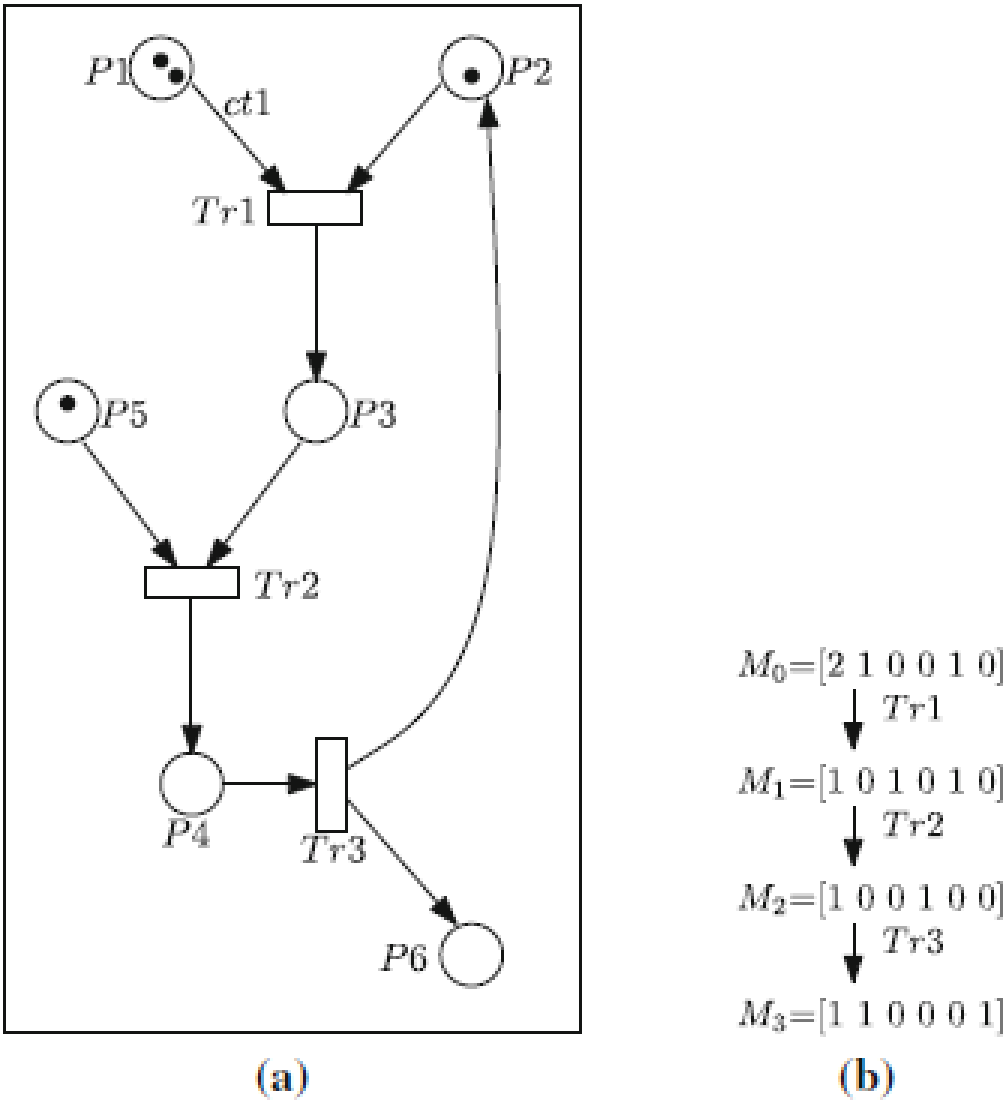


Fig. 6 Petri-Net $PN4$. **a** Petri-Net model $PN4$ of Case 3. **b** Reachability tree of $PN4$ for different firing sequences