2020年3月27日 下午 10:01

$X, Y$

$f_{XY}(x, y) = Prob(\text{"}X = x\text{"} 且 \text{"}Y = y\text{"})$

$\sum_{x=-\infty}^{\infty} f_{XY}(x, y) = f_Y(y)$ } marginal prob.

$\sum_{y=-\infty}^{\infty} f_{XY}(x, y) = f_X(x)$

$\frac{f_{XY}(x, y)}{f_Y(y)} = f_{X|Y}(x|y) = Prob(\text{"}X = x\text{"} \mid \text{"}Y = y\text{"})$

$\frac{f_{XY}(x, y)}{f_X(x)} = f_{Y|X}(y|x) = Prob(\text{"}Y = y\text{"} \mid \text{"}X = x\text{"})$

$f_{XY}(x, y)$, $0 \le x \le 2$, $0 \le y \le 2$

$f_X(x)$, $0 \le x \le 2$

$f_Y(y)$, $0 \le y \le 2$

$P(X = 0 \mid Y = 1)$

$= f_{X|Y}(\underset{0}{x} \mid \underset{1}{y})$

$= \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{f_{XY}(0, 1)}{f_Y(1)} = \frac{3/14}{3/7} = \frac{7}{14} = 0.5$

Table 3.1: Joint Probability Distribution for Example 3.14

| $f(x \mid y)$<br>$f(x,y)/f(y)$ | | $x$ = 0 | 1 | 2 | Row Totals |
|---|---|---|---|---|---|
| $y$ | 0 | $\frac{3}{28}/\frac{15}{28}$ | $\frac{9}{28}/.$ | $\frac{3}{28}/.$ | $\frac{15}{28}$ |
| | 1 | $\frac{3}{14}/\frac{3}{7}$ | $\frac{3}{14}/\frac{3}{7}$ | $0/\frac{3}{7}$ | $\frac{3}{7}$ |
| | 2 | $\frac{1}{28}/\frac{1}{28}$ | $0/.$ | $0/.$ | $\frac{1}{28}$ |
| Column Totals | | $\frac{5}{14}$ | $\frac{15}{28}$ | $\frac{3}{28}$ | 1 |

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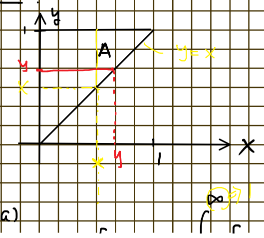
| $f(y \mid x)$<br>$f(x,y)/f(x)$ | | $x$ = 0 | 1 | 2 | Row Totals |
|---|---|---|---|---|---|
| $y$ | 0 | $\frac{3}{28}/\frac{5}{14}$ | $\frac{9}{28}/\frac{15}{28}$ | $\frac{3}{28}/\frac{3}{28}$ | $\frac{15}{28}$ |
| | 1 | $\frac{3}{14}/\frac{5}{14}$ | $\frac{3}{14}/\frac{15}{28}$ | 0 | $\frac{3}{7}$ |
| | 2 | $\frac{1}{28}/\frac{5}{14}$ | 0 | 0 | $\frac{1}{28}$ |
| Column Totals | | $\frac{5}{14}$ | $\frac{15}{28}$ | $\frac{3}{28}$ | 1 |

P.20 Exb 3.19




$h(y) = f_Y(y)$

$= \int_{x=0}^{y} \underbrace{f_{XY}(x, y)}_{10xy^2} \, dx$

$= 10y^2 \cdot \frac{1}{2} x^2 \Big]^{y}$

$f_{X|Y}(x|y)$

$= f_{XY}(x, y) / f_Y(y)$

$= 10xy^2 / 5y^4$

a)

a)

$$g(x) = f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_x^1 10xy^2\,dy$$

$$= 10x \cdot \frac{1}{3}y^3 \Big]_x^1$$

$$= 10x \cdot \frac{1}{3}(1^3 - x^3) = \frac{10}{3}x(1-x^3)$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = 3\frac{y^2}{1-x^3}, \quad 0<x<y<1$$

$$10xy^2$$

$$= 10y^2 \cdot \frac{1}{2}x^2 \Big]_{x=0}^{y}$$

$$= 5y^4$$

$$= 10xy^2 / 5y^4$$

$$= 2\, x/y^2, \quad 0<x<y<1$$

---

$X, Y$

$f_{XY}(x,y)$ : joint prob {mass / density} function

$f_X(x)$, $f_Y(y)$ } marginal prob function

$f_{X|Y}(x|y)$, $f_{Y|X}(y|x)$ } conditional prob func.

$X, Y$ independent statistically

① $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

② $f_{X|Y}(x|y) = f_X(x) \iff \dfrac{f_{XY}(x,y)}{f_Y(y)} = f_X(x)$

③ $f_{Y|X}(y|x) = f_Y(y) \iff \dfrac{f_{XY}(x,y)}{f_X(x)} = f_Y(y)$

---

$X, Y$ independent ?

$$f_{XY}(x,y) \overset{?}{=} f_X(x) \cdot f_Y(y)$$

Table 3.1: Joint Probability Distribution for Example 3.14

| | $f(x,y)$ | $x=0$ | $x=1$ | $x=2$ | Row Totals |
|---|---|---|---|---|---|
| $y$ | 0 | $\frac{3}{28}$ | $\frac{9}{28}$ | $\frac{3}{28}$ | $\frac{15}{28}$ |
| | 1 | $\frac{3}{14}$ | $\frac{3}{14}$ | 0 | $\frac{3}{7}$ |
| | 2 | $\frac{1}{28}$ | 0 | 0 | $\frac{1}{28}$ |
| Column Totals | | $\frac{5}{14}$ | $\frac{15}{28}$ | $\frac{3}{28}$ | 1 |

$f(0,0)$ $f_X(0)$ $f_Y(0)$

$$\frac{3}{28} \neq \frac{15}{28} \cdot \frac{5}{14}$$

Not independent

---

$X_1, X_2, \cdots\cdots X_n$

independent

$\Leftrightarrow$

$$f_{X_1X_2\cdots X_n}(x_1,x_2,\cdots x_n)$$

$$= f_{X_1}(x_1)\cdot f_{X_2}(x_2)\cdots\cdots f_{X_n}(x_n)$$

$$f_{X_1}(x_1) = \sum_{x_2=-\infty}^{\infty}\cdots\sum_{x_{n-1}=-\infty}^{\infty}\sum_{x_n=-\infty}^{\infty} f_{X_1X_2\cdots X_n}(x_1,x_2\cdots x_n)$$

$\vdots$

$$f_{X_n}(x_n) = \sum_{x_1=-\infty}^{\infty}\sum\cdots\sum_{x_{n-1}=-\infty}^{\infty} f_{X_1X_2\cdots X_n}(x_1,x_2\cdots x_n)$$

---

$X_1, X_2, X_3$

$$\begin{cases} f_{X_1}(x_1) = \begin{cases} 0 & , x_1 \le 0 \\ e^{-x_1} & , x_1 > 0 \end{cases} \\ f_{X_2}(x_2) = \begin{cases} e^{-x_2} & , x_2 > 0 \\ 0 \end{cases} \\ f_{X_3}(x_3) = \begin{cases} e^{-x_3} & , x_3 > 0 \\ 0 \end{cases} \end{cases}$$

---

$$P(\text{"}X_1<2\text{"}, \text{"}1<X_2<3\text{"}, \text{"}X_3>2\text{"}) = P(\text{"}X_1<2\text{"}) \cdot P(\text{"}1<X_2<3\text{"}) \cdot P(\text{"}X_3>2\text{"})$$

$$= \int_{x=0}^{2} f_{X_1}(x)\,dx = e^{-x}(-1)\Big|_{x=0}^{2} = \underbrace{e^{-2}(-1) - e^{0}(-1)}_{1-e^{-2}}$$

$e^{-x}$

$$\int_{x=1}^{3} f_{X_2}(x)\,dx \quad e^{-1}-e^{-3}$$

$$\int^{\infty} f_{X_3}(x)\,dx$$

$X_3 < 2$ ) ! $\cdot P($ " $X_3 > 2$ " $)$

$x = 1$

$\int_2^{\infty} f_{X_3}(x)\,dx$

$e^{-1} - e^{-3}$

$1 - e^{-2}$

$e^{-2} - e^{-\infty}$

$(1 - e^{-2})(e^{-1} - e^{-3})(e^{-2} - 0)$

---

$X$ $x_1, x_2, \cdots x_n$

$f_X(x)$

mean, average, expected value

$$E\{X\} = \sum_{x=-\infty}^{\infty} x \cdot f_X(x)$$

$$= \int_{x=-\infty}^{\infty} x \cdot f_X(x)\,dx$$

---

$\boxed{\begin{matrix} 4 & g \\ 3 & d \end{matrix}} \overset{3}{\Longrightarrow} ???$

$X$ "good"

$x$ : 0, 1, 2, 3

$f_X(x)$ : $p_0$, $p_1$, $p_2$, $p_3$

$\{ 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 \} \Rightarrow E(x)$

$\mu$

" $X = x$ " prob

$$= \frac{\binom{4}{x} \cdot \binom{3}{3-x}}{\binom{7}{3}} = f_X(x)$$

$$E(x) = \sum_{x=0}^{3} x \cdot f_X(x)$$

```
import numpy as np
from   scipy.special import comb

x= np.arange(0, 4)  # x = [0,1,2,3]

μ= ( comb(4, x)
    *comb(3, 3-x)
    /comb(7, 3)
    * x
    ).sum()

μ   # = 1.7142857142857144
```

```
import numpy as np
from   scipy.special import comb

x= np.arange(0, 4)  # x = [0,1,2,3]

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    *comb(3, 3-x)
    /comb(7, 3)
    * x
    ).sum()

μ   # = 1.7142857142857144
```

---

① $p = .7 \longrightarrow x_1 = 1000$
$q = .3$

② $p = .4 \longrightarrow x_2 = 1500$
$q = .6$

---

$X_1 + X_2 = X$

| | | | | |
|---|---|---|---|---|
| 0, 1000 | 0, 1500 | 0 = 0 + 0 | .3 * .6 | = .18 |
| | | 1000 = 1000 + 0 | .7 * .6 | .42 |
| | | 1500 = 0 + 1500 | .3 * .4 | .12 |

$2500 = 1000 + 1500 \quad .7 * .4 \quad .28$

$1.00$

$$E(X) = \sum_{\forall x} x \cdot P(x)$$

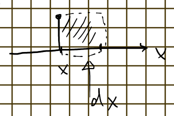
$$= 0 \cdot .18 + 1000 * .42 + 1500 * .12 + 2500 * .28 = 1300 \ \#$$

---

$X$ : (hours)

$$f_X(x) = \begin{cases} \frac{20000}{x^3}, & x > 100 \\ 0 \end{cases}$$

$$E(X) = \int_{x=-\infty}^{\infty} x \cdot f(x)\, dx$$




$$= \int_{100}^{\infty} x \cdot \frac{20000}{x^3}\, dx \qquad 20000\, x^{-2}\, dx \qquad d(x^{-1})(-1)$$

$$= -20000 \int_{100}^{\infty} d(x^{-1}) = 20000 \cdot 100^{-1} = 200 \text{ (hour)} \ \#$$

$$x^{-1} \Big|_{100}^{\infty} \qquad (\infty^{-1} - 100^{-1})$$

---

P.29, Th2.1

$$X, \ f_X(x) \leftarrow E(X) = \sum_{\forall x} x \cdot f_X(x)$$

$aX+b$, $aX^2+bX+c$ ← $g(\cdot)$ difficult ??

$$Y = g(X), \ f_Y(y) \leftarrow E(Y) \cancel{=} \sum_{\forall y} y \cdot f_Y(y)$$

$$E(Y) = E(g(X))$$

$$= \sum_{\forall x} g(x) \cdot f_X(x)$$

---

1/12+1/12+1/4+1/4+1/6+1/6 = 1

4*1/12+5*1/12+6*1/4+7*1/4+8*1/6+9*1/6=6.8333

| | | | | | | |
|---|---|---|---|---|---|---|
| x | 4 | 5 | 6 | 7 | 8 | 9 |
| P(x) | 1/12 | 1/12 | 1/4 | 1/4 | 1/6 | 1/6 |
| g(x)=2x-1 | 7 | 9 | 11 | 13 | 15 | 17 |

E(g(X))= 7*1/12+9*1/12+11*1/4+13*1/4+15*1/6+17*1/6=12.6667

6.8333*2-1=12.6666

---

$X : f_X(x) = \begin{cases} \frac{1}{3}x^2, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$

$X \to \boxed{g(\cdot)} \to g(X) = 4 \cdot X + 3$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$x=-1$, $x=2$; $g(x) = (4x+3)$, $f_X(x) = \frac{1}{3}x^2$

$$\left(\frac{4}{3}x^3 + 1 \cdot x^2\right) dx$$

$$\frac{4}{3} \underbrace{x^3 dx}_{= d(x^4)\cdot\frac{1}{4}} + 1 \cdot \underbrace{x^2 dx}_{= d(x^3)\cdot\frac{1}{3}}$$

$$= \frac{4}{3} \cdot \frac{1}{4} x^4 \Big]_{-1}^{+2} + \frac{1}{3} \cdot x^3 \Big]_{-1}^{+2}$$

$2^4 - (-1)^4 = 15$, $5$; $2^3 - (-1)^3 = 9$

$5$, $3$

$$= 8$$

---

$X, Y : f_{XY}(x, y)$

$X, Y \to \boxed{g(x, y)} \Downarrow$

$$G = g(X, Y)$$

$$E(G) = E(g(X, Y))$$

$$= \sum_y \sum_x g(x, y) \cdot f_{XY}(x, y)$$

$$\int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f(x, y)\, dx\, dy$$

---

$f_{XY}(x, y)$

$X, Y \to \boxed{g(\cdot)} \Downarrow$

$X, Y$

$\Downarrow$

$\boxed{g(\cdot)}$

$\Downarrow$

$G = X \cdot Y$

$$E(G) = E(X \cdot Y)$$
$$= \sum \sum x \cdot y \cdot f_{XY}(x, y)$$

**範例 2.28**

令 $X$ 和 $Y$ 爲隨機變數，其聯合機率分布顯示於表 2.1。求 $g(X,Y) = XY$ 的期望值。爲了方便起見，將該表再次列印於此。

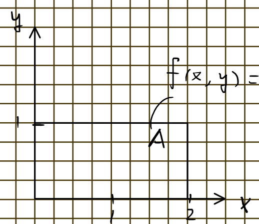
| | $f(x,y)$ | $x$ = 0 | 1 | 2 | 列加總 |
|---|---|---|---|---|---|
| $y$ | 0 | 3/28 | 9/28 | 3/28 | 15/28 |
| | 1 | 3/14 | 3/14 | 0 | 3/7 |
| | 2 | 1/28 | 0 | 0 | 1/28 |
| 行加總 | | 5/14 | 15/28 | 3/28 | 1 |

| x*y | 0 | 1 | 2 | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | |
| 1 | 0 | 1 | 2 | |
| 2 | 0 | 2 | 4 | |

| x*y *f(x,y) | 0 | 1 | 2 | |
|---|---|---|---|---|
| 0 | 0*3/28 | 0*9/28 | 0*3/28 | |
| 1 | 0*3/14 | 1*3/14 | 2*0 | |
| 2 | 0*1/28 | 2*0 | 4*0 | 1*3/14 |

---

$f(x,y) = x(1+3y^2)/4$




$G = \frac{Y}{X}$ , $E(G) = E\left(\frac{Y}{X}\right)$

$$= \int_{y=0}^{1}\int_{x=0}^{2} \frac{y}{\cancel{x}} \cdot \cancel{x}(1+3y^2)/4 \, dx\, dy$$

$y + 3y^{3}$

$d(y^2)\frac{1}{2} + 3d(y^4)\frac{1}{4} = \frac{1}{2}y^2 + \frac{3}{4}y^4 \Big]_0^1$

$= \frac{1}{2} \cdot 1 + \frac{3}{4} = \frac{5}{4}$

$= \int_{x=0}^{2} \frac{5}{4} dx / 4 = \frac{5}{4} \cdot 2 / 4 = 5/8$

```
import sympy as sp

x,y= sp.symbols('x, y')

sp.integrate(
    y *( 1 + 3*y**2) /4,
    (x, 0, 2),
    (y, 0, 1)
    )
```

$\frac{5}{8}$

---

$X$ . $f_X(\cdot)$

$E(X) = \mu = \sum_{\forall x} x \cdot f_X(x)$

$Var(X) = \sigma^2 = \sum (x-\mu)^2 f_X(\cdot)$

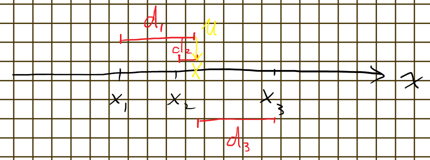
$E(X) = \mu = \sum_{\forall x} x \cdot f_X(x)$

$Var(X) = \sigma^2 = \boxed{\sum_{\forall x} (x-\mu)^2 f_X(x)}$

$Std(X) = \sigma = \sqrt{\sigma^2}$

$g(X) = (X-\mu)^2$

$E(g(X)) = E((X-\mu)^2)$




$d$ : deviation
distance

$\left.\begin{matrix}\\ \end{matrix}\right\} D \equiv X - \mu$

$+, +$
$-\ -\ -$

$E(D) = 0$

$|d|$

$D = |X-\mu|$

$$E(D) = \sum_{\forall x} |x-\mu| \cdot f_X(x)$$
$$= \sum_{x \geq \mu} (x-\mu) f_X(x)$$
$$+ \sum_{x < \mu} [-(x-\mu)] f_X(x)$$

---

$D = |X-\mu|$

$$V = D^2 = |X-\mu|^2$$
$$= (X-\mu)^2$$
$$= X^2 - 2X\mu + \mu^2$$

$$E(V) = E(D^2) = E((X-\mu)^2)$$

平方

取平均　任意点　中心點

$$\underline{Var(X)} = E(V)$$
$$= E((X-\mu)^2)$$

天　　$m^2$

$\sqrt{Var(X)}$

$\sigma$ : standard deviation