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證明：

$$\frac{1}{2^{2k}}\Sigma_{i=0}^{2^k-1}\Sigma_{j=0}^{2^k-1}(i-j)^2=\frac{2^{2k}-1}{6} \tag{1}$$

化簡：

let $2^k=t$

$$\frac{1}{t^2}\Sigma_{i=0}^{t-1}\Sigma_{j=0}^{t-1}(i-j)^2 \tag{2}$$

$$\frac{1}{t^2}\Sigma_{i=0}^{t-1}\Sigma_{j=0}^{t-1}(i^2+j^2-2ij) \tag{3}$$

$$\frac{1}{t^2}\left(\left(t\Sigma_{i=0}^{t-1}i^2\right)+\left(t\Sigma_{j=0}^{t-1}j^2\right)+\Sigma_{i=0}^{t-1}\Sigma_{j=0}^{t-1}(-2ij)\right) \tag{4}$$

$$\frac{1}{t^2}\left((t)(\frac{1}{6})(t-1)(t)(2t-1)\times 2-2\times\Sigma_{i=0}^{2^t-1}\Sigma_{j=0}^{t-1}(ij)\right) \tag{5}$$

$$\frac{2}{t^2}\left((t)(\frac{1}{6})(t-1)(t)(2t-1)-\Sigma_{i=0}^{t-1}\Sigma_{j=0}^{t-1}(ij)\right) \tag{6}$$

$$\frac{2}{t^2}\left((t)(\frac{1}{6})(t-1)(t)(2t-1)-\frac{(t-1)(t)(t-1)(t)}{4}\right) \tag{7}$$

$$\frac{2\,\cancel{(t)}\,\cancel{(t)}\,(t-1)}{\cancel{t^2}}\left(\frac{2t-1}{6}-\frac{t-1}{4}\right) \tag{8}$$

$$(t-1)\left(\frac{2t-1}{3}-\frac{t-1}{2}\right) \tag{9}$$

$$(t-1)\times\frac{4t-2-3t+1}{6} \tag{10}$$

$$\frac{(t-1)(t-1)}{6} \tag{11}$$

$$\frac{t^2-1}{6} \tag{12}$$

代入$t=2^k$

$$\frac{2^{2k}-1}{6} \tag{14}$$

得證