Tutorial on Stochastic Programming & Benders Decomposition

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Linear Programming [1]

- The primal form:

$$\min \quad \sum_i c_i x_i \tag{1.1}$$

$$\text{s.t.} \quad \sum_i a_{j,i} x_i = b_j \quad \forall\ j = 1, \ldots, m; \tag{1.2}$$

$$x_i \geq 0 \quad \forall\ i = 1 \ldots, n. \tag{1.3}$$

Graphic Representation

- Polyhedron;
- In two dimensions, i.e., $\boldsymbol{x} = (x_1, x_2)^T$:

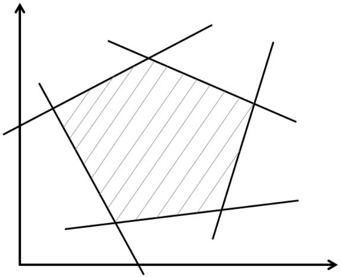

Figure: A polyhedron constructed using linear equations.

Dual

- "Rotate the primal 90° to the left";
- The primal form:

$$\min \quad \sum_i c_i x_i \tag{1.4}$$

$$\text{s.t.} \quad \sum_i a_{j,i} x_i = b_j \quad \forall\, j = 1, \ldots, m; \tag{1.5}$$

$$x_i \geq 0 \quad \forall\, i = 1 \ldots, n. \tag{1.6}$$

- The dual form:

$$\max \quad \sum_j b_j y_j \tag{1.7}$$

$$\text{s.t.} \quad \sum_j a_{j,i} y_j \leq c_i \quad \forall\, i = 1, \ldots, n; \tag{1.8}$$

$$y_j \text{ unrestricted} \quad \forall\, j = 1 \ldots, m. \tag{1.9}$$

Farkas' Lemma

- "Either the program is feasible, or infeasible";
- Exactly one of the following two holds:
  - There exists $x \succeq 0$, such that $\sum_i a_{j,i} x_i = b_j, \forall\, j = 1, \ldots, m$;
  - There exists y, such that $\sum_j a_{j,i} y_j \geq 0, \forall\, i = 1, \ldots, n$ and $\sum_j b_j y_j < 0$.

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Example: Production

- Produce two products: A and B;
- Revenue:
  - A: 40;
  - B: 50;
- Required material:
  - A: 1 α and 3 β;
  - B: 1 α, 4 β and 1 γ;
- Material cost:
  - α: 10;
  - β: 1;
  - γ: 1.

Example: Production (Cont.)

- Demand:
  - A: 20;
  - B: 60;
- Salvage cost:
  - α: 0;
  - β: 0.1;
  - γ: 0.1.
- How do we produce A and B, and how much material to purchase?

Example: LP

- Variables:
  - x: material purchase;
  - y: production;
  - z: remaining material.

- LP:

$$
\begin{aligned}
\min \quad & 10x_\alpha + 1x_\beta + 1x_\gamma - 40y_A - 50y_B - 0z_\alpha - 0.1z_\beta - 0.1z_\gamma && (1.10)\\
\text{s.t.} \quad & z_\alpha = x_\alpha - 1 \cdot y_A - 1 \cdot y_B; && (1.11)\\
& z_\beta = x_\beta - 3 \cdot y_A - 4 \cdot y_B; && (1.12)\\
& z_\gamma = x_\gamma - 0 \cdot y_A - 1 \cdot y_B; && (1.13)\\
& x \geq 0, z \geq 0; && (1.14)\\
& 0 \leq y_A \leq 20, 0 \leq y_B \leq 60. && (1.15)
\end{aligned}
$$

Example: LP (Cont.)

- Concisely:
  - q_i: revenue of i;
  - d_i: demand of i;
  - $m_{i,j}$: product i using material j;
  - c_j: cost of j;
  - s_j: salvage cost of j;
- LP:

$$\min \quad \sum_{j=\alpha}^{\gamma} c_j x_j - \sum_{i=A}^{B} q_i y_i - \sum_{j=\alpha}^{\gamma} s_j z_j \tag{1.16}$$

$$\text{s.t.} \quad z_j = x_j - \sum_{i=A}^{B} m_{i,j} y_i \quad \forall\, j = \alpha, \beta, \gamma; \tag{1.17}$$

$$x \geq 0, z \geq 0, 0 \leq y_i \leq d_i, \forall i. \tag{1.18}$$

Uncertain Future

- E.g., stochastic demand;
- Scenarios, $k = 1, 2, 3$:
  1. With pr. $p_1 = 0.3$, $d_A = 10$, $d_B = 30$;
  2. With pr. $p_2 = 0.5$, $d_A = 20$, $d_B = 60$;
  3. With pr. $p_3 = 0.2$, $d_A = 40$, $d_B = 80$;
- How should we process this information?

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- How should we process this information?

Three Types of Solutions

- Wait & see (WS):
  - Perfect information, knows the future;
  - Have three purchase & production plans according to each of the scenario;
- Here & now:
  - Know the future up to a distribution;
  - Have only one purchase plans, but three production plans according to each of the scenario;
  - Recourse policy (RP);
- Expected value (EV):
  - Know the future up to a distribution;
  - Have only one purchase & production plan, based on the average information.
- WS $\leq$ RP $\leq$ EV.

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Formulation: Extensive Form

- Adopting RP:

$$
\begin{aligned}
\min \quad & \sum_{j=\alpha}^{\gamma} c_j x_j + \sum_{k=1}^{3} p_k \cdot \Big(-\sum_{i=A}^{B} q_i y_i^k - \sum_{j=\alpha}^{\gamma} s_j z_j^k \Big) && (2.1) \\
\text{s.t.} \quad & z_j^k = x_j - \sum_{i=A}^{B} m_{i,j} y_i^k \quad \forall\, j = \alpha, \beta, \gamma, k = 1, 2, 3; && (2.2) \\
& x_j \geq 0 \quad \forall\, j = \alpha, \beta, \gamma; && (2.3) \\
& 0 \leq y_i^k \leq d_i \quad \forall\, i = A, B, k = 1, 2, 3; && (2.4) \\
& z_j \geq 0 \quad \forall\, j = \alpha, \beta, \gamma, k = 1, 2, 3. && (2.5)
\end{aligned}
$$

Generic Form [2]

- Variable in the present: $\boldsymbol{x}$;
- Variable in the future: $\boldsymbol{y}$;
- The generic extensive form:

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T\boldsymbol{x} + \mathbb{E}_\xi\left[\min \boldsymbol{q}^T(\xi)\boldsymbol{y}(\xi)\right] && (2.6)\\
\text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b}; && (2.7)\\
& T(\xi)\boldsymbol{x} + W\boldsymbol{y}(\xi) = \boldsymbol{h}(\xi); && (2.8)\\
& \boldsymbol{x} \succeq \boldsymbol{0}, \boldsymbol{y}(\xi) \succeq \boldsymbol{0}. && (2.9)
\end{aligned}
$$

Problem of Interest [3]

- Two variables, $\boldsymbol{x}$ and $\boldsymbol{y}$;

$$\begin{aligned} \max \quad & \boldsymbol{c}^T\boldsymbol{x} + f(\boldsymbol{y}) && (3.1)\\ \text{s.t.} \quad & A\boldsymbol{x} + F(\boldsymbol{y}) \preceq \boldsymbol{b}; && (3.2)\\ & \boldsymbol{x} \in R^p, \boldsymbol{y} \in S^q. && (3.3) \end{aligned}$$

- Equivalent form:

$$\begin{aligned} \max \quad & x_0 && (3.4)\\ \text{s.t.} \quad & x_0 - \boldsymbol{c}^T\boldsymbol{x} - f(\boldsymbol{y}) \leq 0; && (3.5)\\ & A\boldsymbol{x} + F(\boldsymbol{y}) \preceq \boldsymbol{b}; && (3.6)\\ & \boldsymbol{x} \in R^p, \boldsymbol{y} \in S^q. && (3.7) \end{aligned}$$

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Using Farkas' Lemma

- Consider fixed $\bar{\boldsymbol{y}}$ and arbitrary $\bar{x}_0$;
- Feasible region for $\boldsymbol{x}$ becomes

$$\begin{cases} -\boldsymbol{c}^T\boldsymbol{x} \leq -\bar{x}_0 + f(\bar{\boldsymbol{y}}); \\ A\boldsymbol{x} \preceq \boldsymbol{b} - F(\bar{\boldsymbol{y}}) \end{cases} \tag{3.8}$$

- Lagrangian multiplier (dual variable) μ_0, $\boldsymbol{\mu}$;
- Region (3.8) is feasible **if and only if**

$$\mu_0\bar{x}_0 - \mu_0 f(\bar{\boldsymbol{y}}) + \boldsymbol{\mu}^T F(\bar{\boldsymbol{y}}) \leq \boldsymbol{\mu}^T\boldsymbol{b}, \tag{3.9}$$

$$\forall\ (\mu_0, \boldsymbol{\mu}) \in C := \{(\mu_0, \boldsymbol{\mu}) : A^T\boldsymbol{\mu} - \mu_0\boldsymbol{c} \succeq \boldsymbol{0}, \boldsymbol{\mu} \succeq \boldsymbol{0}, \mu_0 \geq 0\}. \tag{3.10}$$

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Graphic Illustration

- Define

$$G := \bigcap_{(\mu_0,\boldsymbol{\mu})\in C} \Big\{(x_0,\boldsymbol{y}) : \mu_0 x_0 - \mu_0 f(\boldsymbol{y}) + \boldsymbol{\mu}^T F(\boldsymbol{y}) \le \boldsymbol{\mu}^T \boldsymbol{b}\Big\}; \tag{3.11}$$

- Finite halflines $h = 1, 2, \ldots, H$,

$$G := \bigcup_{h=1,2,\ldots,H} \Big\{(x_0,\boldsymbol{y}) : \mu_0^h x_0 - \mu_0^h f(\boldsymbol{y}) + \boldsymbol{\mu}^{h^T} F(\boldsymbol{y}) \le \boldsymbol{\mu}^{h^T} \boldsymbol{b}\Big\}. \tag{3.12}$$

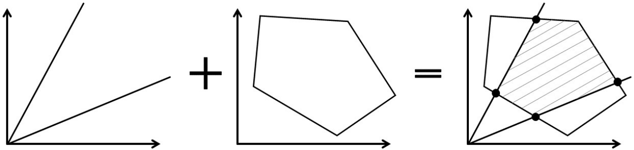

Figure: Intersecting a polyhedron with a cone.

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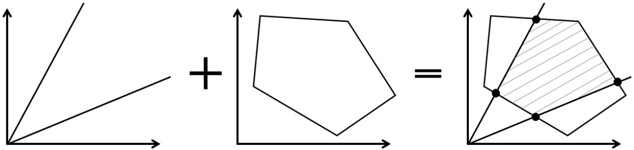

Figure: Intersecting a polyhedron with a cone.

Equivalence to The Original Problem

- The problem ($\diamondsuit$)

$$\max \quad x_0 \tag{3.13}$$
$$\text{s.t.} \quad x_0 \in \{x_0 : (x_0, \boldsymbol{y}) \in G\} \tag{3.14}$$

is equivalent to the problem ($\heartsuit$)

$$\max \quad x_0 \tag{3.15}$$
$$\text{s.t.} \quad x_0 - \boldsymbol{c}^T \boldsymbol{x} - f(\boldsymbol{y}) \leq 0; \tag{3.16}$$
$$A\boldsymbol{x} + F(\boldsymbol{y}) \preceq \boldsymbol{b}; \tag{3.17}$$
$$\boldsymbol{x} \in R^p, \boldsymbol{y} \in S^q. \tag{3.18}$$

Optimal Solutions

- Let $(x_0^*, \boldsymbol{y}^*)$ be an optimal solution to $(\diamondsuit)$;
- There must exist $\boldsymbol{x}^*$, such that $\boldsymbol{x}^*$ is an optimal solution to $(\heartsuit)$;
- We know

$$\begin{cases} x_0^* = \boldsymbol{c}^T \boldsymbol{x}^* + f(\boldsymbol{y}^*); \\ x_0^* \geq \boldsymbol{c}^T \boldsymbol{x} + f(\boldsymbol{y}^*) \quad \forall\ \boldsymbol{x}. \end{cases} \tag{3.19}$$

- To find $\boldsymbol{x}^*$, solve

$$\max \quad \boldsymbol{c}^T \boldsymbol{x} \tag{3.20}$$

$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b} - F(\boldsymbol{y}^*); \tag{3.21}$$

$$\boldsymbol{x} \succeq \boldsymbol{0}. \tag{3.22}$$

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Decomposition

- Master problem:

$$\begin{aligned} \max \quad & x_0 && (3.23) \\ \text{s.t.} \quad & x_0 \in \{x_0 : (x_0, \boldsymbol{y}) \in G\}. && (3.24) \end{aligned}$$

- Subproblem:

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} && (3.25) \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} - F(\boldsymbol{y}^*); && (3.26) \\ & \boldsymbol{x} \succeq \boldsymbol{0}. && (3.27) \end{aligned}$$

Stochastic: Two-stage

- Master problem:

$$\min \quad \boldsymbol{c}^T\boldsymbol{x} + \mathcal{Q}(\boldsymbol{x}) \tag{3.28}$$
$$\text{s.t.} \quad A\boldsymbol{x} = \boldsymbol{b}; \tag{3.29}$$
$$\boldsymbol{x} \succeq \boldsymbol{0}, \tag{3.30}$$

where $\mathcal{Q}(\boldsymbol{x}) := \mathbb{E}_\xi[Q(\boldsymbol{x},\xi)]$;

- Subproblem:

$$Q(\boldsymbol{x},\xi) := \min \quad \boldsymbol{q}^T(\xi)\boldsymbol{y}(\xi) \tag{3.31}$$
$$\text{s.t.} \quad T(\xi)\boldsymbol{x} + W\boldsymbol{y}(\xi) = \boldsymbol{h}(\xi); \tag{3.32}$$
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Recall: The Production Problem

- Extensive form:

$$\min \quad \sum_{j=\alpha}^{\gamma} c_j x_j + \sum_{k=1}^{3} p_k \cdot \left(-\sum_{i=A}^{B} q_i y_i^k - \sum_{j=\alpha}^{\gamma} s_j z_j^k \right) \tag{3.34}$$

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$$x_j \geq 0 \quad \forall\, j = \alpha, \beta, \gamma; \tag{3.36}$$

$$0 \leq y_i^k \leq d_i^k \quad \forall\, i = A, B, k = 1, 2, 3; \tag{3.37}$$

$$z_j \geq 0 \quad \forall\, j = \alpha, \beta, \gamma, k = 1, 2, 3. \tag{3.38}$$

Reformulate The Production Problem

- Master problem:

$$\min \quad \sum_{j=\alpha}^{\gamma} c_j x_j + \sum_{k=1}^{3} p_k \cdot Q(\boldsymbol{x}, k) \tag{3.39}$$

$$\text{s.t.} \quad x_j \geq 0 \quad \forall\, j = \alpha, \beta, \gamma; \tag{3.40}$$

- Subproblems, $\forall\ k = 1, 2, 3$:

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INTUITION

- Recall the master problem:

$$\max \quad x_0 \tag{4.1}$$
$$\text{s.t.} \quad x_0 \in \{x_0 : (x_0, \boldsymbol{y}) \in G\}; \tag{4.2}$$

- How can we construct G?
- We know that there are finitely halflines $h = 1, 2, \ldots, H$,

$$G := \bigcup_{h=1,2,\ldots,H} \left\{(x_0, \boldsymbol{y}) : \mu_0^h x_0 - \mu_0^h f(\boldsymbol{y}) + \boldsymbol{\mu}^{h^T} F(\boldsymbol{y}) \le \boldsymbol{\mu}^{h^T} \boldsymbol{b}\right\}. \tag{4.3}$$

- Construct 'G' iteratively.

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Feasibility Concern

- Say we have a solution $\bar{\boldsymbol{x}}$;
- Will the subproblem

$$Q(\boldsymbol{x}, \xi) := \min \quad \boldsymbol{q}^T(\xi)\boldsymbol{y}(\xi) \tag{4.4}$$
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always be feasible?

- Not necessarily.

Feasibility Concern

- Say we have a solution $\bar{\boldsymbol{x}}$;
- Will the subproblem

$$\begin{aligned} Q(\boldsymbol{x},\xi) := \min \quad & \boldsymbol{q}^T(\xi)\boldsymbol{y}(\xi) && (4.4)\\ \text{s.t.} \quad & T(\xi)\boldsymbol{x} + W\boldsymbol{y}(\xi) = \boldsymbol{h}(\xi); && (4.5)\\ & \boldsymbol{y}(\xi) \succeq 0. && (4.6) \end{aligned}$$

  always be feasible?
- Not necessarily.

Feasibility Concern (Cont.)

- **Complete recourse**: if the subproblems are always feasible no matter $\bar{\boldsymbol{x}}$;
- Otherwise, $Q(\boldsymbol{x}, \xi)$ can be infeasible;

Feasibility Concern (cont.)

- **Complete recourse**: if the subproblems are always feasible no matter $\bar{\boldsymbol{x}}$;
- Otherwise, $Q(\boldsymbol{x}, \xi)$ can be infeasible;
- Consider the dual of $Q(\boldsymbol{x}, \xi)$;
- Primal infeasible, dual ________;

Feasibility Concern (Cont.)

- **Complete recourse**: if the subproblems are always feasible no matter $\bar{\boldsymbol{x}}$;
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- Primal infeasible, dual ________;
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- Infeasible: "the stochastic program is not well-formulated";

Feasibility Concern (Cont.)

- **Complete recourse**: if the subproblems are always feasible no matter $\bar{\boldsymbol{x}}$;
- Otherwise, $Q(\boldsymbol{x}, \xi)$ can be infeasible;
- Consider the dual of $Q(\boldsymbol{x}, \xi)$;
- Primal infeasible, dual ________;
- Unbounded & infeasible;
- Infeasible: "the stochastic program is not well-formulated";
- Unbounded: the halfline associated with vector $\bar{\boldsymbol{\mu}}$ goes to infinity.

Feasibility Concern (Cont.)

- Action: tell the master problem to cuts $\bar{\boldsymbol{x}}$;
- Dual of subproblem, with variable $\boldsymbol{\mu}$:

$$\max \quad \boldsymbol{\mu}^T\left(h(\xi) - T(\xi)\bar{\boldsymbol{x}}\right) \tag{4.7}$$

$$\text{s.t.} \quad W^T\boldsymbol{\mu} \preceq \boldsymbol{q}(\xi); \tag{4.8}$$

- Solve to find the unbounded ray $\bar{\boldsymbol{\delta}}$;
- Add a feasibility cut

$$\bar{\boldsymbol{\delta}}^T(h(\xi) - T(\xi)\boldsymbol{x}) \leq 0 \tag{4.9}$$

to the mater problem.

Optimality Concern

- Say we have a solution $\bar{\boldsymbol{x}}$;
- Will there be another $\boldsymbol{x}'$,

$$\mathcal{Q}(\boldsymbol{x}') \leq \mathcal{Q}(\bar{\boldsymbol{x}})? \tag{4.10}$$

- Probably.

Optimality Concern

- Say we have a solution $\bar{\boldsymbol{x}}$;
- Will there be another $\boldsymbol{x}'$,

$$\mathcal{Q}(\boldsymbol{x}') \leq \mathcal{Q}(\bar{\boldsymbol{x}})? \tag{4.10}$$

- Probably.

Optimality Concern (Cont.)

- How do we find better solutions (improving halflines h)?
- Let $\theta = \mathcal{Q}(\boldsymbol{x})$;
- Write the master problem as

$$\min \quad \boldsymbol{c}^T \boldsymbol{x} + \theta \tag{4.11}$$

$$\text{s.t.} \quad A\boldsymbol{x} = \boldsymbol{b}; \tag{4.12}$$

$$\boldsymbol{x} \succeq 0, \tag{4.13}$$

- Adding the information of the current solution using the dual of subproblems as a cut:

$$\sum_k p_k \left(\bar{\boldsymbol{\mu}}_k^T (h(\xi^k) - T(\xi^k)\boldsymbol{x}) \right) \le \theta. \tag{4.14}$$

Optimality Concern (Cont.)

- In the cut, $\bar{\boldsymbol{\mu}}_k$ is the solution from the dual of subproblems
- The part

$$\bar{\boldsymbol{\mu}}_k^T(h(\xi^k) - T(\xi^k)\boldsymbol{x}) \tag{4.15}$$

  approximates the objective of a subproblem using its dual.
- The variable θ approximate the value of all subproblems for the master problem.

The L-shaped Algorithm

Algorithm 1: The L-shaped Algorithm

```
while True do
    Solve the master problem and obtain solution x̄;
    Feasible ← True;
    for k = 1, 2, ..., K do
        Construct the dual of subproblem using x̄ and the ransom variable ξ_k;
        Solve the dual problem;
        if Subproblem Unbounded then
            Obtain unbounded ray δ̄;
            Add cut δ̄^T (h(ξ) − T(ξ)x) ≤ 0 to the master problem;
            Feasible ← False;
            break;
        end
        if Subproblem Optimal then
            Obtain the solution μ̄_k;
        end
    end
    if Feasible then
        v ← Σ_k p_k ( μ̄_k^T (h(ξ^k) − T(ξ^k)x̄) );
        if v > θ then
            Add cut Σ_k p_k ( μ̄_k^T (h(ξ^k) − T(ξ^k)x) ) ≤ θ to the master problem;
        else
            break;
        end
    end
end
```

Recall: The Production Problem, Two-stage

- Master problem:

$$\min \quad \sum_{j=\alpha}^{\gamma} c_j x_j + \sum_{k=1}^{3} p_k \cdot Q(\boldsymbol{x}, k) \tag{5.1}$$

$$\text{s.t.} \quad x_j \geq 0 \quad \forall\, j = \alpha, \beta, \gamma; \tag{5.2}$$

- Subproblems, $\forall\ k = 1, 2, 3$:

$$Q(\boldsymbol{x}, k) := \min \quad -\sum_{i=A}^{B} q_i y_i^k - \sum_{j=\alpha}^{\gamma} s_j z_j^k \tag{5.3}$$

$$\text{s.t.} \quad z_j^k = x_j - \sum_{i=A}^{B} m_{i,j} y_i^k \quad \forall\, j = \alpha, \beta, \gamma; \tag{5.4}$$

$$0 \leq y_i^k \leq d_i^k \quad \forall\ i = A, B; \tag{5.5}$$

$$z_j \geq 0 \quad \forall\ j = \alpha, \beta, \gamma. \tag{5.6}$$

Dualize The Subproblem

- Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be the dual variables associated with the constraints;
- Taking the dual, $\forall\ k = 1, 2, 3$:

$$\max \quad \sum_{j=\alpha}^{\gamma} x_j \lambda_j^k + \sum_{i=A}^{B} d_i^k \mu_i^k \tag{5.7}$$

$$\text{s.t.} \quad \sum_{j=\alpha}^{\gamma} m_{i,j} \lambda_j^k + \mu_i^k \leq -q_i \quad \forall\ i = A, B; \tag{5.8}$$

$$\lambda_j^k \leq -s_j \quad \forall\ j = \alpha, \beta, \gamma; \tag{5.9}$$

$$\mu_i^k \leq 0 \quad \forall\ i = A, B. \tag{5.10}$$

Feasibility Cuts

- Let $\lambda_j^{k'}$ and $\mu_i^{k'}$ be the unbounded rays associated with the variables;
- The feasibility cut:

$$\sum_{j=\alpha}^{\gamma} x_j \lambda_j^{k'} + \sum_{i=A}^{B} d_i^k \mu_i^{k'} \leq 0. \tag{5.11}$$

Optimality Cuts

- Let $\bar{\lambda}_j^k$ and $\bar{\mu}_i^k$ be the optimal solutions;
- The optimality cut:

$$\sum_{k=1}^{3} p_k \cdot \left(\sum_{j=\alpha}^{\gamma} x_j \bar{\lambda}_j^k + \sum_{i=A}^{B} d_i^k \bar{\mu}_i^k \right) \leq \theta. \tag{5.12}$$

Implementation

- Python + Gurobi.

Thank You!

Questions?

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Reference I

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