Reciprocity in Directed Hypergraphs: Measures, Findings, and Generators (Online Appendix)

1 Proof of the Theorems

In this section together with the formal expressions of our proposals, we provide proofs of the theorems suggested in main paper.

1.1 Preliminaries of the Proofs

Prior to the proofs, we give general forms and equations of our proposal. Then, we introduce several important characteristics of *Jensen-Shannon Divergence(JSD)* [1], which play key roles during proofs. After, we examine how these concepts can be applied to our measure.

Proposed reciprocity measure **HYPERREC** for hyperarc $r(e_i|C_i)$ and hypergraph $r(G)$ are defined as

$$r(e_i|C_i) := \max_{C_i \subseteq E} \left(\frac{1}{|C_i|}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}}\right) \tag{1}$$

$$r(G) := \frac{1}{|E|} \sum_{i=1}^{|E|} r(e_i|C_i) \tag{2}$$

Where $\mathcal{L}(p_h, p_h^*)$ denotes a Jensen-Shannon Divergence [1] between transition probability distribution and optimal transition probability distribution. Here, transition probability of a target arc e_i, who has a single candidate arc $\{e_i'\}$ is defined as

$$p_h(v) = \begin{cases} \frac{1}{|H_i'|} & \text{if } v \in H_i' \\ 0 & \text{otherwise} \end{cases}$$

This is a case where $v_h \in \{H_i \cap T_i'\}$. For the nodes of $v_h \in H_i \setminus T_i'$, $p_h(v)$ only gets value of 1 at a sunken node $v = v_{sunk}$. For the target arc e_j who has an arbitrary number of candidate arcs $C_j = \{e_{j1}, \cdots e_{jk}\}$, transition probability is formally defined as

$$p_h(v) = \begin{cases} p_{h,1}(v) & \text{if } v_h \in \bigcup_{e_s \in C_j} T_s \\ p_{h,2}(v) & \text{otherwise} \end{cases}$$

$$p_{h,1}(v) = \frac{\sum_{e_s \in C_j} \left(\frac{\mathbf{1}[v_h \in T_s, v \in H_s]}{|H_s|}\right)}{\sum_{e_s \in C_j} (\mathbf{1}[v_h \in T_s])} \quad p_{h,2}(v) = \begin{cases} 1 & \text{if } v = v_{sunk} \\ 0 & \text{otherwise} \end{cases}$$

Throughout the proof, we use the formal expressions with a single candidate arc or two candidate arcs cases.

We now discuss the theoretical aspect of $JSD(P||Q)$. General formula of JSD for the discrete space is defined as

$$\mathcal{L}(P, Q) = \sum_{i=1}^{|V|} \ell(p_i, q_i) \tag{3}$$

$$\text{where} \quad \ell(p_i, q_i) = \frac{p_i}{2} \log \frac{2p_i}{p_i + q_i} + \frac{q_i}{2} \log \frac{2q_i}{p_i + q_i} \tag{4}$$

1. JSD is bounded as $0 \leq JSD(P||Q) \leq log2$ [1].

2. For the two discrete probability distributions P and Q where their non-zero probability domains do not overlap, (i.e., $p_i q_i = 0 \;\; \forall i = 1, ..., |V|$) overall JSD among two probability distribution is maximized as $\log 2$.

   Proof: Denote $\mathcal{X}_p$ as a domain where p_v has non-zero values and let $\mathcal{X}_q$ be the domain where q_v has non-zero values. As $\mathcal{X}_p$ and $\mathcal{X}_q$ is not overlapped, equation 3 is summarized as

$$\begin{aligned} &= \sum_{i \in \mathcal{X}_p} \frac{p_i}{2} log2 + \sum_{i \in \mathcal{X}_q} \frac{q_i}{2} log2 \\ &= \frac{log2}{2} \left(\sum_{i \in \mathcal{X}_p} p_i + \sum_{i \in \mathcal{X}_q} q_i \right) \\ &= \frac{log2}{2} * (1 + 1) = log2 \end{aligned}$$

3. For the two discrete probability distributions P and Q where there exist at least one overlapped non-zero probability domain between them, $JSD(P||Q) < log2$.

   Proof: Let $i = k$ be the domain which satisfies $p_k q_k \neq 0$. Then equation 3 can be summarized as

$$\mathcal{L}(p, q) = \sum_{i \in \mathcal{X}_p \setminus k} \frac{p_i}{2} log2 + \sum_{i \in \mathcal{X}_q \setminus k} \frac{q_i}{2} log2 + \left(\frac{p_k}{2} \log \frac{2p_k}{p_k + q_k} + \frac{q_k}{2} \log \frac{2q_k}{p_k + q_k} \right) \tag{5}$$

   In order to show the equation 5 is smaller than $log2$, it is required to show that

$$\left(\frac{p_k}{2} \log 2 + \frac{q_k}{2} \log 2 \right) - \left(\frac{p_k}{2} \log \frac{2p_k}{p_k + q_k} + \frac{q_k}{2} \log \frac{2q_k}{p_k + q_k} \right) > 0$$

$$\frac{p_k}{2} \log \left(1 + \frac{q_k}{p_k} \right) + \frac{q_k}{2} \log \left(1 + \frac{p_k}{q_k} \right) > 0 \because p_k, q_k > 0$$

   As log function has positive real value when its input is greater than 1, last inequality satisfies. Thus we can conclude that $JSD(P||Q) < log2$.

   We use above three statements as (A. 1), (A. 2), and (A. 3) to derive additional three statements, which are statement 6, 7 and 8.

   Extending these concepts to our interest, we can derive three cases where range or exact value of $r(e_i|C_i)$ can be simply induced. Note that first two conditions are not exclusive, but still they can be used without loss of generality.

$$\text{If } |H_i \cap \bigcup_{e_k \in C_i} T_k| = 0 \quad \text{then} \quad r(e_i|C_i) = 0. \tag{6}$$

$$\text{If } |T_i \cap \bigcup_{e_k \in C_i} H_k| = 0 \quad \text{then} \quad r(e_i|C_i) = 0. \tag{7}$$

$$\text{If } \sum_{e_k \in C_i} |H_i \cap T_k| \times |T_j \cap H_k| \geq 1 \quad \text{then} \quad r(e_i|C_i) > 0. \tag{8}$$

   Statement 6: For this case, as mentioned in the main paper, transition probability is heading toward sunken node v_{sunk}. On the other hand, optimal transition probability p^* is heading toward $v \in T_i$ where $v_{sunk} \notin T_i$. Thus, non-zero probability domain of transition probability and optimal transition probability is not overlapped, where probabilistic distance between them is maximized as $\mathcal{L}_{max}$ by (A. 2).

   This happens for all H_i, in sum,

$$\begin{aligned} r(e_i|C_i) &= \left(\frac{1}{|C_i|} \right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}_{max}}{|H_i| * \mathcal{L}_{max}} \right) \\ &= \left(\frac{1}{|C_i|} \right)^{\alpha} \times \left(1 - \frac{|H_i| * \mathcal{L}_{max}}{|H_i| * \mathcal{L}_{max}} \right) = \left(\frac{1}{|C_i|} \right)^{\alpha} \times 0 = 0 \end{aligned}$$

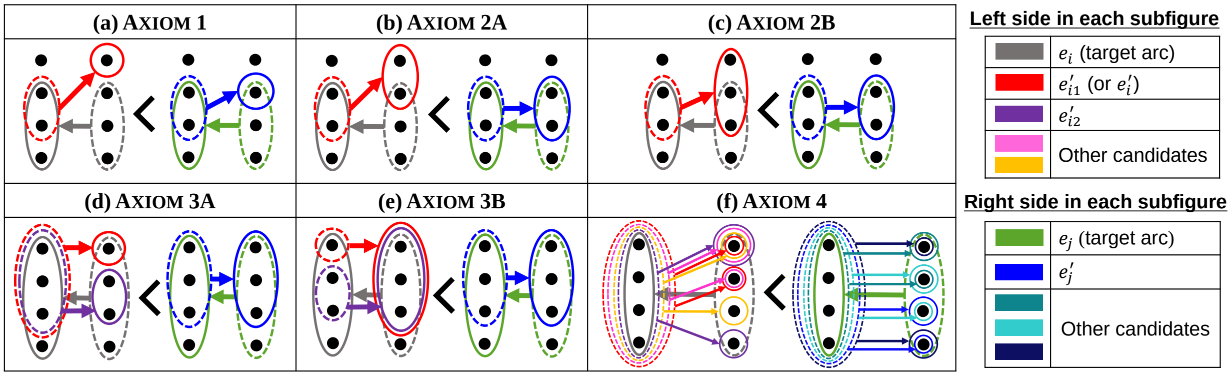


Figure 1: Examples for **AXIOMS** 1-4. In each subfigure, the reciprocity of the arc e_i on the left side should be smaller than that of the arc e_j on the right side.

Statement 7: Similar to the statement 6, non-zero probability domain of transition probability and optimal probability is not overlapped since $|T_i \cap \bigcup_{e_k \in C_i} H_k| = 0$. Again, probabilistic distance between them is maximized as $\mathcal{L}_{max}$.

This happens for all H_i, in sum,

$$\begin{aligned} r(e_i|C_i) &= \left(\frac{1}{|C_i|}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}_{max}}{|H_i| * \mathcal{L}_{max}}\right) \\ &= \left(\frac{1}{|C_i|}\right)^{\alpha} \times \left(1 - \frac{|H_i| * \mathcal{L}_{max}}{|H_i| * \mathcal{L}_{max}}\right) = \left(\frac{1}{|C_i|}\right)^{\alpha} \times 0 = 0 \end{aligned}$$

Statement 8: According to the statement, there exist at least one candidate arc e_k whose tail set overlap to target arc's head set $|T_k \cap H_i| \geq 1$, and head set also overlap to target arc's tail set $|H_k \cap T_i| \geq 1$. Thus, for $v_h \in H_i \cap T_k$, p_h and p^*_h have common non-zero probability domain, which implies $\boldsymbol{\mathcal{L}(p_h, p^*_h) < log2}$ by (A. 3). In sum, we can derive following inequality (**see inequality sign between two middle terms**).

$$\begin{aligned} r(e_i|C_i) &= \left(\frac{1}{|C_i|}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p^*_h)}{|H_i| * \mathcal{L}_{max}}\right) \\ &> \left(\frac{1}{|C_i|}\right)^{\alpha} \times \left(1 - \frac{|H_i| * \mathcal{L}_{max}}{|H_i| * \mathcal{L}_{max}}\right) = \left(\frac{1}{|C_i|}\right)^{\alpha} \times 0 = 0 \end{aligned}$$

1.2 Proof of the Theorem 1.

For the **AXIOM** 1, we simply show that former's reciprocity gets zero $r(e_i|C_i) = 0$, while latter's reciprocity get's a positive value $r(e_j|C_j) > 0$. For the other **AXIOMS** , we first show the relationship between **AXIOMS** and the probabilistic distance (Step A). After, we show that less reciprocal case has higher probabilistic distance between observed transition probability and optimal transition probability for the certain head-set nodes of target arc (Step B). Overview of the **AXIOMS** is illustrated in the Figure 1.

1.2.1 Proof of Axiom 1

[**AXIOM1**]: *Existence of Inverse Overlap*

Assume $C_i = \{e'_i\}$ and $C_j = \{e'_j\}$. Then

$$\min(|H_i \cap T'_i|, |T_i \cap H'_i|) = 0 \text{ and } \min(|H_j \cap T'_j|, |T_j \cap H'_j|) \geq 1$$

Then, following inequality holds.

$$r(e_i|C_i) < r(e_j|C_j)$$

PROOF: We first show $r(e_i|C_i) = 0$. Term $\min(|H_i \cap T_i'|, |T_i \cap H_i'|) = 0$ implies at least one of $H_i \cap T_i'$ or $T_i \cap H_i'$ is an empty set. By the statement 6 and statement 7, we can derive reciprocity for this case is equal to zero $r(e_i|C_i) = 0$.

Now we show that $r(e_j|C_j) > 0$. As $\min(|H_j \cap T_j'|, |T_j \cap H_j'|) \geq 1$, we can guarantee that both $H_j \cap T_j'$ and $T_j \cap H_j'$ are non-empty set. By the statement 8, we prove the inequality $r(e_j|C_j) > 0$.

In conclusion as $r(e_i|C_i) = 0$ and $r(e_j|C_j) > 0$, we can derive $r(e_i|C_i) < r(e_j|C_j)$.

Note that from AXIOM 2-4, we assume two target arcs e_i and e_j have equal size $|\boldsymbol{H_i}| = |\boldsymbol{H_j}|$ and $|\boldsymbol{T_i}| = |\boldsymbol{T_j}|$.

1.2.2 Proof of Axiom 2

[AXIOM2A]: *Degree of Inverse Overlap: More Overlap*

Let $C_i = \{e_i'\}$ and $C_j = \{e_j'\}$ where

$$|H_i'| = |H_j'| \text{ and } |T_i'| = |T_j'|$$
$$\text{(I)} \quad 0 \leq |H_i' \cap T_i| < |H_j' \cap T_j| \text{ and } 0 \leq |T_i' \cap H_i| \leq |T_j' \cap H_j| \quad \text{or}$$
$$\text{(II)} \quad 0 \leq |H_i' \cap T_i| \leq |H_j' \cap T_j| \text{ and } 0 \leq |T_i' \cap H_i| < |T_j' \cap H_j|$$

Then, following inequality holds.

$$r(e_i|C_i) < r(e_j|C_j)$$

PROOF: As $|C_i| = |C_j| = 1$, cardinality penalty terms on both side can be discarded.

$$r(e_i|C_j) = \left(\frac{1}{|C_i|}\right)^{\alpha} \left(1 - \frac{\sum_{v_i \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}}\right) = 1 - \frac{\sum_{v_i \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}} \quad \text{Same for } r(e_j|C_j) \text{ with subscript } j.$$

<u>**Step A:**</u> Here, we are trying to compare $r(e_i|C_i) < r(e_j|C_j)$, which is equivalent to

$$\frac{\sum_{v_j \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}} > \frac{\sum_{v_j \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| \cdot \mathcal{L}_{max}}$$

As $|H_i| = |H_j|$, denominator of both terms are identical. By removing them, inequality can be summarized as

$$\equiv \sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*) > \sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*) \tag{9}$$

Each head set can be divided into two parts ; $H_k \setminus T_k'$ and $H_k \cap T_k'$ $\forall k = i, j$.

For $H_k \setminus T_k'$, as described in statement 6, probabilistic distance gets maximized to $\mathcal{L}_{max} = \log 2$.

For $H_k \cap T_k'$, we know that $T_k \cap H_k'$. For both $\forall k = i, j$, $\mathcal{L}(p_h, p_h^*) < \log 2$ satisfies for $\forall v_h \in H_k \cap T_k'$ by the statement 8. One more notable fact is that as there is a single candidate arc in both cases, $\mathcal{L}(p_h, p_h^*)$ are always same at $v_h \in H_k \cap T_k'$. Here, let $p_{h,i}$ be the transition probability regarding target arc e_i and its candidate set C_i. Together with it, as $|H_i \cap T_i'| < |H_j \cap T_j'|$ where $|H_i| = |H_j|$, we summarize main inequality 9 as follows.

$$\begin{aligned}
&\equiv \sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*) > \sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*) \\
&\equiv |H_i \setminus T_i'| * \log 2 + |H_i \cap T_i'| * \mathcal{L}(p_{h,i}, p_{h,i}^*) > |H_j \setminus T_j'| * \log 2 + |H_j \cap T_j'| * \mathcal{L}(p_{h,j}, p_{h,j}^*)
\end{aligned}$$

Further inequality can be developed differently according to the condition (whether it is (I) or (II)).

For the case of (I), intersection of target arc's head set and candidate arc's tail set is larger in e_j than e_i. As a result, inequality is re-written as follows.

$$\begin{aligned}
&\equiv |H_i \setminus T_i'| * \log 2 + |H_i \cap T_i'| * \mathcal{L}(p_{h,i}, p_{h,i}^*) > |H_j \setminus T_j'| * \log 2 + |H_j \cap T_j'| * \mathcal{L}(p_{h,i}, p_{h,i}^*) \\
&\geq |H_j \setminus T_j'| * \log 2 + |H_j \cap T_j'| * \mathcal{L}(p_{h,j}, p_{h,j}^*)
\end{aligned}$$

For the case of (II), intersection of target arc's head set and candidate arc's tail set can be identical, while target arc's tail set and candidate arc's head set intersection is greater in e_j than e_i. Then, inequality is re-written as follows.

$$
\begin{aligned}
&\equiv |H_i \setminus T_i'| * \log 2 + |H_i \cap T_i'| * \mathcal{L}(p_{h,i}, p_{h,i}^*) \geq |H_j \setminus T_j'| * \log 2 + |H_j \cap T_j'| * \mathcal{L}(p_{h,i}, p_{h,i}^*) \\
&> |H_j \setminus T_j'| * \log 2 + |H_j \cap T_j'| * \mathcal{L}(p_{h,j}, p_{h,j}^*)
\end{aligned}
$$

Here, one can clearly notice that if the following inequality holds, then both inequalities can be satisfied. This is following inequality is identical to the last inequality of (II), and more rigid inequality than (II).

$$
\begin{aligned}
&\equiv |H_j \setminus T_j'| * \log 2 + |H_j \cap T_j'| * \mathcal{L}(p_{h,i}, p_{h,i}^*) > |H_j \setminus T_j'| * \log 2 + |H_j \cap T_j'| * \mathcal{L}(p_{h,j}, p_{h,j}^*) \\
&\equiv |H_j \cap T_j'| * \mathcal{L}(p_{h,i}, p_{h,i}^*) > |H_j \cap T_j'| * \mathcal{L}(p_{h,j}, p_{h,j}^*) \\
&\equiv \mathcal{L}(p_{h,i}, p_{h,i}^*) > \mathcal{L}(p_{h,j}, p_{h,j}^*)
\end{aligned}
$$

Step B: We only need to show that probabilistic distance between transition probability and optimal transition probability is greater in e_i than e_j. Denote the intersection as $F_1 = |T_i \cap H_i'| \ < \ F_2 = |T_j \cap H_j'|$. We can decompose domain $v \in V$ into four parts as

$$
v \in T_k \cap H_k' \quad v \in T_k \setminus H_k' \quad v \in H_k' \setminus T_k \quad v \in V \setminus \{H_k' \cup T_k\} \quad \forall k = i, j
$$

For the last term, both transition probability and optimal of it have value of zero $p_h(v) = p_h^*(v) = 0$, that result in no penalty. We only need to consider first three terms for comparison. Here, probabilistic distance can be explicitly written as

$$
\begin{aligned}
\mathcal{L}(p_{h,i}, p_{h,i}^*) &= |F_1| * \ell\left(\frac{1}{T}, \frac{1}{A}\right) + (A - |F_1|) * \frac{1}{2A}\log 2 + (T - |F_1|) * \frac{1}{2T}\log 2 \\
\mathcal{L}(p_{h,j}, p_{h,j}^*) &= |F_2| * \ell\left(\frac{1}{T}, \frac{1}{A}\right) + (A - |F_2|) * \frac{1}{2A}\log 2 + (T - |F_2|) * \frac{1}{2T}\log 2
\end{aligned}
$$

where ℓ denotes JSD's functional form explained in the equation 4. Let $A = |S_i'| = |S_j'|$ and $T = |T_i| = |T_j|$. By showing $\mathcal{L}(p_{h,i}, p_{h,i}^*) - \mathcal{L}(p_{h,j}, p_{h,j}^*) > 0$, proof can be done. Formally,

$$
\begin{aligned}
&\equiv \mathcal{L}(p_{h,i}, p_{h,i}^*) - \mathcal{L}(p_{h,j}, p_{h,j}^*) > 0 \\
&= (|F_1| - |F_2|) * \ell\left(\frac{1}{T}, \frac{1}{A}\right) + (|F_2| - |F_1|) * \left(\frac{1}{2A} + \frac{1}{2T}\right)\log 2 > 0 \\
&\equiv (|F_2| - |F_1|) * \left(\frac{1}{2A}\log 2 + \frac{1}{2T}\log 2 - \ell\left(\frac{1}{T}, \frac{1}{A}\right)\right) > 0 \\
&\equiv \left(\frac{1}{2A}\log 2 - \frac{1}{2A}\log\frac{2T}{A+T} + \frac{1}{2T}\log 2 - \frac{1}{2T}\log\frac{2A}{A+T}\right) > 0 \quad \because (|F_2| - |F_1|) > 0 \\
&\equiv \frac{1}{2A}\log\frac{A+T}{T} + \frac{1}{2T}\log\frac{A+T}{A} > 0
\end{aligned}
$$

As $x > 1 \Rightarrow \log x > 0$, inequality holds. proof is done.

[AXIOM2B]: *Degree of Inverse Overlap: Small difference*

Let $C_i = \{e_i'\}$ and $C_j = \{e_j'\}$ where

$$
|H_i'| > |H_j'| \text{ and } |T_i'| = |T_j'| \quad 0 < |H_i' \cap T_i| = |H_j' \cap T_j| \text{ and } 0 < |T_i' \cap H_i| = |T_j' \cap H_j|
$$

Then, following inequality holds.

$$
r(e_i|C_i) < r(e_j|C_j)
$$

PROOF: As $|C_i| = |C_j| = 1$, cardinality penalty term is being erased as 1 for both terms.

Step A: Overall inequality can be re-written as

$$r(e_i|C_i) < r(e_j|C_j) \equiv 1 - \frac{\sum_{v_i \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}} < 1 - \frac{\sum_{v_j \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| \cdot \mathcal{L}_{max}}$$

Similar to the previous proof, $|H_i| = |H_j|$, and $\mathcal{L}(p_h, p_h^*)$ among different $v_h \in H_k \cap T_k'$ are all identical. In addition, here, $|H_i \cap T_i'| = |H_j \cap T_j'|$, $|T_i'| = |T_j'|$, number of target arc's head set nodes v_h that satisfy $\mathcal{L}(p_h, p_h^*) < \log 2$ are identical in both terms. Thus,

$$\begin{aligned}
&\equiv r(e_i|C_i) < r(e_j|C_j) \equiv \frac{\sum_{v_j \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| \cdot \mathcal{L}_{max}} < \frac{\sum_{v_i \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}} \\
&\equiv |H_j \cap T_j'| * \mathcal{L}(p_{h,j}, p_{h,j}^*) < |H_i \cap T_i'| * \mathcal{L}(p_{h,i}, p_{h,i}^*) \\
&\equiv \mathcal{L}(p_{h,j}, p_{h,j}^*) < \mathcal{L}(p_{h,i}, p_{h,i}^*)
\end{aligned}$$

Step B: We only need to show that probabilistic distance between transition probability and optimal transition probability is greater in e_i than e_j. Let $A = |H_i'| \ > \ B = |H_j'|$. We can decompose domain $v \in V$ into four parts as

$$v \in T_k \cap H_k' \quad v \in T_k \setminus H_k' \quad v \in H_k' \setminus T_k \quad v \in V \setminus \{H_k' \cup T_k\} \quad \forall k = i, j$$

Here, JSD in second term and fourth term are identical for both cases. That is, we only need to compare the probabilistic distance which are related to the first and third parts of above four domains.

$$\begin{aligned}
&\equiv \mathcal{L}(p_{h,j}, p_{h,j}^*) < \mathcal{L}(p_{h,i}, p_{h,i}^*) \\
&\equiv F * \ell\left(\frac{1}{B}, \frac{1}{T}\right) + \frac{B-F}{2B} \log 2 < F * \ell\left(\frac{1}{A}, \frac{1}{T}\right) + \frac{A-F}{2A} \log 2
\end{aligned}$$

where $F = |H_i' \cap T_i| = |H_j' \cap T_j|$ and $T = |T_i| = |T_j|$. Note that $\boldsymbol{A > B}$. Inequality can be summarized as

$$\begin{aligned}
&\equiv \mathcal{L}(p_{h,i}, p_{h,i}^*) - \mathcal{L}(p_{h,j}, p_{h,j}^*) \\
&\equiv \frac{F}{2}\left(\frac{1}{B} - \frac{1}{A}\right) \log 2 > F * \left(\ell\left(\frac{1}{B}, \frac{1}{T}\right) - \ell\left(\frac{1}{A}, \frac{1}{T}\right)\right)
\end{aligned}$$

To simplify the equation, we unfold $\ell(p, q)$ as follows.

$$\begin{aligned}
&\equiv \frac{F}{2}\left(\frac{1}{B} - \frac{1}{A}\right) \log 2 > F * \left(\ell\left(\frac{1}{B}, \frac{1}{T}\right) - \ell\left(\frac{1}{A}, \frac{1}{T}\right)\right) \\
&\equiv \frac{1}{2}\left(\frac{1}{B} - \frac{1}{A}\right) \log 2 > \frac{1}{2T} \log \frac{2B}{B+T} + \frac{1}{2B} \log \frac{2T}{B+T} - \frac{1}{2T} \log \frac{2A}{A+T} - \frac{1}{2A} \log \frac{2T}{A+T} \\
&\equiv \left(\frac{1}{B} - \frac{1}{A}\right) \log 2 > \frac{1}{T} \log \frac{2B}{B+T} + \frac{1}{B} \log \frac{2T}{B+T} - \frac{1}{T} \log \frac{2A}{A+T} - \frac{1}{A} \log \frac{2T}{A+T} \\
&\equiv 0 > \frac{1}{T} \log \frac{B}{B+T} + \frac{1}{B} \log \frac{T}{B+T} - \frac{1}{T} \log \frac{A}{A+T} - \frac{1}{A} \log \frac{T}{A+T} \qquad \because \text{pull 2 inside the log term out} \\
&\equiv 0 > \frac{1}{T} \log \frac{B(A+T)}{A(B+T)} - \frac{1}{B} \log \frac{B+T}{T} + \frac{1}{A} \log \frac{A+T}{T} \qquad \text{multiply } T \text{ in both terms} \\
&\equiv 0 > \log \frac{AB+BT}{AB+AT} + \frac{T}{A} \log(1 + \frac{A}{T}) - \frac{T}{B} \log(1 + \frac{B}{T})
\end{aligned}$$

We show the last inequality by splitting it to two terms ; $\log \frac{AB+BT}{AB+AT} < 0$ and $\frac{T}{A} \log(1 + \frac{A}{T}) - \frac{T}{B} \log(1 + \frac{B}{T}) < 0$. Here, first term is trivial as $A > B$. To show the second term which is

$$\frac{T}{A} \log(1 + \frac{A}{T}) - \frac{T}{B} \log(1 + \frac{B}{T}) < 0 \tag{10}$$

Note that last term's nominator and denominator has a form of $f(x) = \frac{1}{x}\log(1+x)$. By using the fact that $A > B$, if a function $f(x)$ is a decreasing function at $x > 0$, inequality 10 is satisfied. We have show that $f'(x) < 0 \quad x > 0$, that is

$$
\begin{aligned}
f'(x) &= -\frac{1}{x^2}\log(x+1) + \frac{1}{x(x+1)} < 0 \quad \text{multiply } x^2(x+1) \text{ in both terms} \\
f'(x) &= -(x+1)\log(x+1) + x < 0 \quad \text{where } f'(0) = 0 \\
f''(x) &= -1 - \log(x+1) + 1 < 0 \quad \forall x > 0
\end{aligned}
$$

As $f''(x) < 0$, and $f'(x) = 0$ we can derive $f'(x) < 0$ for $x > 0$. This implies a function $f(x)$ is a decreasing function. Thus we have shown inequality 10 satisfies, proof is done.

1.2.3 Proof of Axiom 3

[AXIOM3.A]: *Number of Candidates where Tail Sets are Fixed.*

Let $e'_k \subseteq_{(R)} e_k$ indicates $H'_k \subseteq T_k$ and $T'_k \subseteq H_k$.
Assume $C_i = \{e'_{i1}, e'_{i2}\}$ and $C_j = \{e'_j\}$ where

$$
\begin{gathered}
e'_{i1} \subseteq_{(R)} e_i \quad e'_{i2} \subseteq_{(R)} e_i \quad e'_j \subseteq_{(R)} e_j \quad T'_{i1} = T'_{i2} \quad |T'_{i1}| = |T_j| \\
H'_{i1} \cap H'_{i2} = \emptyset \quad \text{and} \quad |(H'_{i1} \cup H'_{i2}) \cap T_i| = |H'_j \cap T_j|
\end{gathered}
$$

Then, following inequality holds.

$$r(e_i|C_i) < r(e_j|C_j)$$

PROOF: Unlike previous cases where there exist only a single hyperarc in a candidate set, there are two arcs in a former case. Thus, cardinality penalty terms are not discarded. Two reciprocity terms $r(e_i|C_i)$ and $r(e_j|C_j)$ becomes

$$
\begin{aligned}
r(e_i|C_i) &= \left(\frac{1}{2}\right)^{\alpha} \times \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p^*_h)}{|H_i| * \mathcal{L}_{max}}\right) \\
r(e_j|C_j) &= \left(1 - \frac{\sum_{v_h \in H_j} \mathcal{L}((p_h, p^*_h))}{|H_j| \times \mathcal{L}_{max}}\right)
\end{aligned}
$$

Step A: From the inequality $r(e_i|C_i) < r(e_j|C_j)$, we can derive that

$$
\begin{aligned}
&\equiv r(e_i|C_i) < r(e_j|C_j) \\
&= \left(\frac{1}{2}\right)^{\alpha} \times \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p^*_h)}{|H_i| * \mathcal{L}_{max}}\right) < \left(1 - \frac{\sum_{v_h \in H_j} \mathcal{L}((p_h, p^*_h))}{|H_j| \times \mathcal{L}_{max}}\right) \\
&\equiv \left(\frac{1}{2}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p^*_h)}{|H_i| * \mathcal{L}_{max}}\right) < \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p^*_h)}{|H_i| * \mathcal{L}_{max}}\right) \le \left(1 - \frac{\sum_{v_h \in H_j} \mathcal{L}(p_h, p^*_h)}{|H_j| \times \mathcal{L}_{max}}\right)
\end{aligned}
$$

Last less or equal relation can be induced because $\alpha > 0$. In sum, proof can be summarized as

$$\frac{\sum_{v_h \in H_j} \mathcal{L}(p_h, p^*_h)}{|H_j| \times \mathcal{L}_{max}} \le \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p^*_h)}{|H_i| * \mathcal{L}_{max}} \quad \text{as } |H_i| = |H_j| \text{, this can be re-written as}$$

$$\sum_{v_h \in H_j} \mathcal{L}(p_h, p^*_h) \le \sum_{v_h \in H_j} \mathcal{L}(p_h, p^*_h) \tag{11}$$

For e_i, as $T'_{i1} = T'_{i2}$, each $p_h \; \forall v_h \in H_i$ has same distribution. For e_j, there is only one single candidate arc, still each $p_h \; \forall v_h \in H_i$ has same distribution. That is, inequality 11 can be summarized as

$$
\begin{aligned}
&\equiv \sum_{v_h \in H_j} \mathcal{L}(p_h, p^*_h) \le \sum_{v_h \in H_j} \mathcal{L}(p_h, p^*_h) \\
&\equiv |H_j \cap T'_j| * \mathcal{L}(p_{h,j}, p^*_{h,j}) \le |H_i \cap T'_{i1}| * \mathcal{L}(p_{h,i}, p^*_{h,i})
\end{aligned}
$$

Since $e'_{i1} \subseteq_{(R)} e_i, \quad e'_{i2} \subseteq_{(R)} e_i, \quad e'_j \subseteq_{(R)} e_j$ and $|T'_j| = |T'_{i1}|$, last inequality can be written by

$$\begin{aligned}
&\equiv |H_j \cap T'_j| * \mathcal{L}(p_{h,j}, p^*_{h,j}) \leq |H_i \cap T'_{i1}| * \mathcal{L}(p_{h,i}, p^*_{h,i}) \\
&\equiv |T'_j| * \mathcal{L}(p_{h,j}, p^*_{h,j}) \leq |T'_{i1}| * \mathcal{L}(p_{h,i}, p^*_{h,i}) \\
&\equiv \mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i})
\end{aligned}$$

Proof can be done by simply showing the last inequality, $\mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i})$.

Step B: In order to show $\mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i})$, we should clarify the transition probability of e_i, since it is different from the previous arcs with single candidate arc. That is

$$p_{h,i}(v) = \begin{cases} \frac{1}{2|H'_{i1}|} & \text{if } v \in H'_{i1} \\ \frac{1}{2|H'_{i2}|} & \text{if } v \in H'_{i2} \\ 0 & \text{otherwise} \end{cases}$$

Since $e'_{i1} \subseteq_{(R)} e_i, \quad e'_{i2} \subseteq_{(R)} e_i, \quad e'_j \subseteq_{(R)} e_j$, probabilistic domain can be separated as

$$\begin{aligned}
&v \in H'_{i1} \quad v \in H'_{i2} \quad v \in T_i \setminus \{H'_{i1} \cup H'_{i2}\} \quad v \in V \setminus T_i \quad \text{for } e_i \\
&v \in H'_j \quad v \in T_j \setminus H'_j \quad v \in V \setminus T_j \quad \text{for } e_j
\end{aligned}$$

As $|T_i| = |T_j|$, last term is identical for both terms. By using this information, we can formally express $\mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i})$ by below inequality. Since $|H_{i1} \cup H_{i2}| = |H_j|$ where $|H_{i1} \cap H_{i2}| = 0$, we derive $|H_{i1}| + |H_{i2}| = |H_j|$. By using it, we simplify the size of each set as follows ; $|H_j| = A, |H_{i1}| = B, |H_{i2}| = A - B$, and $|T_i| = |T_j| = T$.

$$\begin{aligned}
&\equiv \mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i}) \\
&\equiv A * \ell\left(\frac{1}{T}, \frac{1}{A}\right) + \frac{T-A}{2T}\log 2 \leq B * \ell\left(\frac{1}{T}, \frac{1}{2B}\right) + (A-B) * \ell\left(\frac{1}{T}, \frac{1}{2(A-B)}\right) + \frac{T-B-(A-B)}{2T}\log 2 \\
&\equiv A * \ell\left(\frac{1}{T}, \frac{1}{A}\right) \leq B * \ell\left(\frac{1}{T}, \frac{1}{2B}\right) + (A-B) * \ell\left(\frac{1}{T}, \frac{1}{2(A-B)}\right) \\
&\equiv \frac{A}{2T}\log\frac{2A}{A+T} + \frac{A}{2A}\log\frac{2T}{A+T} \quad \text{unfolding } \ell(p,q) \\
&\leq \frac{B}{2T}\log\frac{4B}{2B+T} + \frac{B}{4B}\log\frac{2T}{2B+T} + \frac{(A-B)}{2T}\log\frac{4(A-B)}{2(A-B)+T} + \frac{(A-B)}{4(A-B)}\log\frac{2T}{2(A-B)+T}
\end{aligned}$$

If we pull out $\log 2$ terms of both sides

$$\left(\frac{A}{2T} + \frac{A}{2A}\right)\log 2 \qquad \left(\frac{B}{2T} + \frac{B}{4B} + \frac{A-B}{2T} + \frac{B-A}{4(B-A)}\right)\log 2$$

where LHS and RHS have identical terms. By erasing them, original inequality can be re written as

$$\begin{aligned}
&\equiv \frac{A}{2T}\log\frac{A}{A+T} + \frac{A}{2A}\log\frac{T}{A+T} \\
&\leq \frac{B}{2T}\log\frac{2B}{2B+T} + \frac{B}{4B}\log\frac{T}{2B+T} + \frac{(A-B)}{2T}\log\frac{2(A-B)}{2(A-B)+T} + \frac{(A-B)}{4(A-B)}\log\frac{T}{2(A-B)+T}
\end{aligned}$$

Instead of directly comparing every term, we partially compare inequality as

$$\frac{A}{2T}\log\frac{A}{A+T} \leq \frac{B}{2T}\log\frac{2B}{2B+T} + \frac{(A-B)}{2T}\log\frac{2(A-B)}{2(A-B)+T} \tag{12}$$

$$\frac{1}{2}\log\frac{T}{A+T} \leq \frac{1}{4}\log\frac{T}{2B+T} + \frac{1}{4}\log\frac{T}{2(A-B)+T} \tag{13}$$

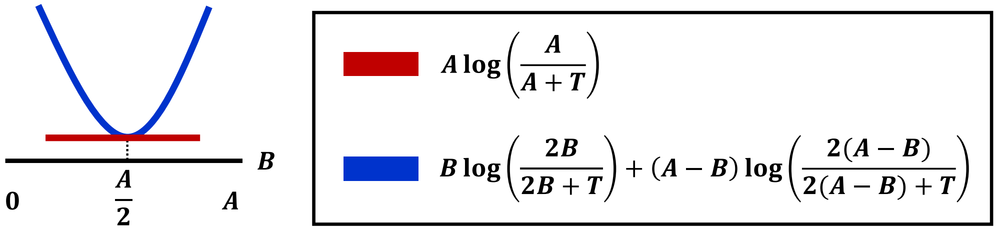


Figure 2: Result of the inequality 12

We first show inequality 12. By multiplying $2T$ on both side, we have

$$\begin{aligned}
&\frac{A}{2T}\log\frac{A}{A+T} \le \frac{B}{2T}\log\frac{2B}{2B+T} + \frac{(A-B)}{2T}\log\frac{2(A-B)}{2(A-B)+T} \\
&\equiv A\log\frac{A}{A+T} \le B\log\frac{2B}{2B+T} + (A-B)\log\frac{2(A-B)}{2(A-B)+T}
\end{aligned}$$

Here, we prove this inequality by using the functional form of $f(B) = B\log\frac{2B}{2B+T} + (A-B)\log\frac{2(A-B)}{2(A-B)+T}$ where B lies in $0 < B < A$.

$$\begin{aligned}
f(B) &= B[\log(2B) - \log(2B+T)] + (A-B)[\log(2(A-B)) - \log(2(A-B)+T)] \\
\frac{\partial f(B)}{\partial B} &= \log(2B) - \log(2B+T) + B\left(\frac{2}{2B} - \frac{2}{2B+T}\right) \\
&\quad - \log(2(A-B)) + \log(2(A-B)+T) - (A-B)\left(\frac{2}{2(A-B)} - \frac{2}{2(A-B)+T}\right)
\end{aligned}$$

Cancel out several identical terms. We have

$$\begin{aligned}
f'(B) &= \log(2B) - \log(2B+T) - \frac{2B}{2B+T} - \log(2(A-B)) + \log(2(A-B)+T) + \frac{2(A-B)}{2(A-B)+T} \\
&\equiv \log\frac{2B}{2B+T} - \frac{2B}{2B+T} - \log\frac{2(A-B)}{2(A-B)+T} + \frac{2(A-B)}{2(A-B)+T}
\end{aligned}$$

Note that functional form of last equation is $f'(B) = \log x - x - (\log y - y)$. Note that $x = \frac{2B}{2B+T}$ and $y = \frac{2(A-B)}{2(A-B)+T}$ who lie in $0 < x, y < 1$. Here, we derive functional from of $f(B)$ by using the following facts.

- If we plug in $B = \frac{A}{2}$, we have $f'(B) = 0$.
- $\log x - x$ is an increasing function for $0 < x < 1$.
- If $0 < B < \frac{A}{2}$, then $(\log y) - y > (\log x) - x$, which implies $f'(B) < 0$.
- If $\frac{A}{2} < B < A$, then $(\log x) - x > (\log y) - y$, which implies $f'(B) > 0$.

In sum, we can derive that

$$f'(B) = \begin{cases} < 0 & \text{if } 0 < B < \frac{A}{2} \\ = 0 & \text{if } B = \frac{A}{2} \\ > 0 & \text{if } \frac{A}{2} < B < A \end{cases}$$

Thus, we can infer that $f(B)$ is a convex function at $0 < B < A$. Value of $f(B = \frac{A}{2}) = \frac{A}{2}\log\frac{A}{A+T} + \frac{A}{2}\log\frac{A}{A+T} = A\log\frac{A}{A+T}$. Rewind that our original goal is to show

$$\equiv A\log\frac{A}{A+T} \le B\log\frac{2B}{2B+T} + (A-B)\log\frac{2(A-B)}{2(A-B)+T}$$

That is, RHS term of inequality is a convex function which has its minimum value at $B = A/2$, and its minimum value is equal to the LHS term. Thus we can guarantee the inequality 12 is satisfied. Overview of this result is illustrated in the Figure 2.

Now we show the inequality 13.

$$\begin{aligned}
&\equiv \frac{1}{2}\log\frac{T}{A+T} \le \frac{1}{4}\log\frac{T}{2B+T} + \frac{1}{4}\log\frac{T}{2(A-B)+T} \\
&\equiv \frac{1}{4}\log\frac{T}{A+T} + \frac{1}{4}\log\frac{T}{A+T} \le \frac{1}{4}\log\frac{T}{2B+T} + \frac{1}{4}\log\frac{T}{2(A-B)+T} \\
&\equiv \frac{1}{4}\log\frac{T^2}{(A+T)(A+T)} \le \frac{1}{4}\log\frac{T^2}{(2B+T)(2(A-B)+T)}
\end{aligned}$$

This can be shown by comparing denominator of both terms, we can rewrite last inequality by

$$\begin{aligned}
&\equiv (2B+T)(2(A-B)+T) \le (A+T)(A+T) \\
&= 4B(A-B) + 2AT + T^2 \le A^2 + 2AT + T^2 \\
&\equiv A^2 - 4AB + 4B^2 = (A-2B)^2 \ge 0
\end{aligned}$$

Where equality holds at $B = A/2$. As both inequality 12 and 13 hold, where equality occur at $B = A/2$ in both terms, $\mathcal{L}(p_{h,j}, p^*_{h,j}) \le \mathcal{L}(p_{h,i}, p^*_{h,i})$ is satisfied, and proof is done.

[AXIOM3.B]: *Number of Candidates where Head Sets are Fixed.*

Let $e'_k \subseteq_{(R)} e_k$ indicates $H'_k \subseteq T_k$ and $T'_k \subseteq H_k$.
Assume $C_i = \{e'_{i1}, e'_{i2}\}$ and $C_j = \{e'_j\}$ where

$$e'_{i1} \subseteq_{(R)} e_i \quad e'_{i2} \subseteq_{(R)} e_i \quad e'_j \subseteq_{(R)} e_j \quad H'_{i1} = H'_{i2} \quad |H'_{i1}| = |H_j|$$
$$T'_{i1} \cap T'_{i2} = \emptyset \quad |(T'_{i1} \cup T'_{i2}) \cap H_i| = |T'_j \cap H_j|$$

Then, following inequality holds.

$$r(e_i|C_i) < r(e_j|C_j)$$

PROOF: Settings are all same with the proof of **AXIOM3.A** until inequality 11.

Step A: Following the proof **AXIOM3.A** of two inequalities are equivalent.

$$r(e_i|C_i) < r(e_j|C_j) \equiv \sum_{v_h \in H_j} \mathcal{L}(p_h, p^*_h) \le \sum_{v_h \in H_j} \mathcal{L}(p_h, p^*_h)$$

Here, note that as $H'_{i1} = H'_{i2}$, two candidate arcs of e_i have same transition probability. Furthermore, as $T'_{i1} \cap T'_{i2} = \emptyset$, transition probability for every $v_h \in H_i \cap \{T'_{i1} \cup T'_{i2}\}$ are identical. Thus, let transition probability of e_i in this case as $p_i(v)$, which is

$$p_i(v) = p_{h,i}(v) \quad \forall v_h \in H_i \cap \{T'_{i1} \cup T'_{i2}\}$$

As $|H_i \cap \{T'_{i1} \cup T'_{i2}\}| = |H_j \cap T'_j|$, above inequality can be summarized as

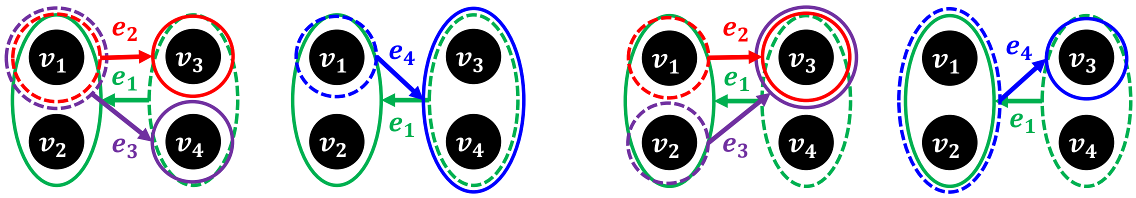
$$\mathcal{L}(p_{h,i}, p^*_{h,i}) \le \mathcal{L}(p_{h,j}, p^*_{h,j})$$

Step B: Note that $e'_{i1} \subseteq_{(R)} e_i \quad e'_{i2} \subseteq_{(R)} e_i \quad e'_j \subseteq_{(R)} e_j$ and $|H'_{i1}| = |H'_j|$, probabilistic distance at e_i and e_j are identical.

$$\mathcal{L}(p_{h,i}, p^*_{h,i}) = \mathcal{L}(p_{h,j}, p^*_{h,j})$$

Although their probabilistic distance are identical, penalty term $\left(\frac{1}{|C_i|}\right)^\alpha = \left(\frac{1}{2}\right)^\alpha < 1 \quad \forall \alpha > 0$ satisfies the inequality $r(e_i|C_i) < r(e_j|C_j)$. Proof is done.

Failure of Baselines at Axiom 3



(a) Axiom3.A (b) Axiom3.B

Figure 3: Counter examples for baselines. Left hypergraph of each subfigure is G_1 and right hypergraph of each subfigure is G_2.

B1. Pearcy et al. [2]: They use clique expansion which bi-cliques hypergraph to the weighted digraph (see related work section of main paper). We provide simple example of the **AXIOMS** where hypergraphs become indistinguishable after clique expansion (see Figure 3). Note that as [2] does not propose the arc level reciprocity, we compare the cases of $r(G_1) = (V_1 = \{v_1, v_2, v_3, v_4\}, E_1 = \{e_1, e_2, e_3\})$ and $r(G_2) = (V_2 = \{v_1, v_2, v_3, v_4\}, E_2 = \{e_1, e_4\})$. We show the counter example by proving $r(G_1) = r(G_2)$.

Here, for Figure 3 (a), resulting clique expansion $\bar{A}_1$ and $\bar{A}_2$ becomes

$$\bar{A}_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad \bar{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

One can clearly notice that resulting clique expanded adjacency matrices are identical. Here, perfect reciprocal hypergraphs $\bar{A}'_1$ and $\bar{A}'_2$ are also the same.

$$\bar{A}'_1 = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \quad \bar{A}'_2 = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

Thus, resulting reciprocity $r(G_1) = \frac{tr(\bar{A}_1^2)}{tr(\bar{A}_1'^2)}$ and $r(G_2) = \frac{tr(\bar{A}_2^2)}{tr(\bar{A}_2'^2)}$ are surely identical. That is, **AXIOM3.A** which should be $r(G_1) < r(G_2)$ has been violated.

For Figure 3 (b), resulting clique expansion $\bar{A}_1$ and $\bar{A}_2$ becomes

$$\bar{A}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad \bar{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Similar to the previous case, resulting clique expanded adjacency matrices are identical. Furthermore, perfect reciprocal hypergraphs $\bar{A}'_1$ and $\bar{A}'_2$ also become identical.

$$\bar{A}'_1 = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad \bar{A}'_2 = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Similar to the **AXIOM3.A**, resulting reciprocity $r(G_1) = \frac{tr(\bar{A}_1^2)}{tr(\bar{A}_1'^2)}$ and $r(G_2) = \frac{tr(\bar{A}_2^2)}{tr(\bar{A}_2'^2)}$ are identical. Thus, **AXIOM3.B** which should be $r(G_1) < r(G_2)$ has been violated.

In sum, B1. Pearcy et al. [2] satisfy neither **AXIOM3.A** nor **AXIOM3.B**.

B2. Ratio of Covered Pairs: From now on, we are able to compare hyperarc level reciprocity. Here, we show that this metric results in $r(e_1|C_1 = \{e_2, e_3\}) = r(e_1|C_1 = \{e_4\})$, which is again a violation of **AXIOM3**.

For Figure 3 (a), each pair interaction can be defined as

$$R(e_1) = \{\langle v_1, v_3\rangle, \langle v_1, v_4\rangle, \langle v_2, v_3\rangle, \langle v_2, v_4\rangle\}.$$
$$R(e_2) = \{\langle v_3, v_1\rangle\} \quad R(e_3) = \{\langle v_4, v_1\rangle\} \quad R(e_4) = \{\langle v_3, v_1\rangle, \langle v_4, v_1\rangle\}$$
$$R^{-1}(e_2) = \{\langle v_1, v_3\rangle\} \quad R^{-1}(e_3) = \{\langle v_1, v_4\rangle\} \quad R^{-1}(e_4) = \{\langle v_1, v_3\rangle, \langle v_1, v_4\rangle\}$$

Thus, each reciprocity is defined as

$$r(e_1|C_1 = \{e_2, e_3\}) = \frac{2}{4} = 0.5$$
$$r(e_1|C_1 = \{e_4\}) = \frac{2}{4} = 0.5$$

This gives identical result on both example, which is a violation of the **AXIOM3.A.** Similarly, for Figure 3 (b), interaction pairs are defined as

$$R(e_1) = \{\langle v_1, v_3\rangle, \langle v_1, v_4\rangle, \langle v_2, v_3\rangle, \langle v_2, v_4\rangle\}.$$
$$R(e_2) = \{\langle v_3, v_1\rangle\} \quad R(e_3) = \{\langle v_3, v_2\rangle\} \quad R(e_4) = \{\langle v_3, v_1\rangle, \langle v_3, v_2\rangle\}$$
$$R^{-1}(e_2) = \{\langle v_1, v_3\rangle\} \quad R^{-1}(e_3) = \{\langle v_2, v_3\rangle\} \quad R^{-1}(e_4) = \{\langle v_1, v_3\rangle, \langle v_2, v_3\rangle\}$$

Similarly, each reciprocity is defined as

$$r(e_1|C_1 = \{e_2, e_3\}) = \frac{2}{4} = 0.5$$
$$r(e_1|C_1 = \{e_4\}) = \frac{2}{4} = 0.5$$

Here again identical reciprocity values on both example, which is a violation of the **AXIOM3.B.**. We conclude that **B2. Ratio of Covered Pairs** also violates **AXIOM3.B.**.

B5. HYPERREC w/o Size Penalty: While proving general **HYPERREC**, original inequality $r(e_i|C_i) < r(e_j|C_j)$ has been relaxed to $\mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i})$ because of the size penalty term $\left(\frac{1}{|C_i|}\right)^{\alpha}$ (refer to **Step A** of **AXIOM3.A.** proof). Thus, if we remove this size penalty term, equality within $\leq$ cannot guarantee the original reciprocity.

In sum, if **HYPERREC** without size penalty term $(1/|C_i|)^{\alpha}$ cannot satisfy the **AXIOM3**.

1.2.4 Proof of Axiom 4

[AXIOM4]: *Bias in the Candidate Arcs*

For two target arcs e_i and e_j,

$$|C_i| = |C_j| = |T_i| = |T_j| \quad |T'_i| = |T'_j|$$
$$T'_i = H_i \ , \ H'_i \subset T_i \ , \ |H'_i| = 2, \forall e'_i \in C_i$$
$$T'_j = H_j \ , \ H'_j \subset T_j \ , \ |H'_j| = 2, \forall e'_j \in C_j$$

$$\exists u, v \in T_i \quad \text{where} \quad |\{u \in e'_i \mid e'_i \in C_i\}| \neq |\{v \in e'_i \mid e'_i \in C_i\}| \tag{14}$$
$$\forall u, v \in T_j \quad \text{where} \quad |\{u \in e'_j \mid e'_j \in C_j\}| = |\{v \in e'_j \mid e'_j \in C_j\}| \tag{15}$$

Then, following inequality holds.

$$r(e_i|C_i) < r(e_j|C_j)$$

PROOF: This **AXIOM** implies if there exist a bias in the candidate arcs, (roughly speaking, candidate arcs are concentrated on specific nodes only) then it is less reciprocal than the cases where candidate arcs are uniformly pointing target arc's tail set nodes.

Step A: General form can be written as

$$\equiv r(e_i|C_i) < r(e_j|C_j)$$
$$= \left(\frac{1}{|C_i|}\right)^{\alpha}\left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right) < \left(\frac{1}{|C_j|}\right)^{\alpha}\left(1 - \frac{\sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| * \mathcal{L}_{max}}\right)$$

As $|C_i| = |C_j|,\ \ |H_i| = |H_i|,\ \ T_i' = H_i,$ and $T_j' = H_j$, above inequality can be summarized as

$$\equiv r(e_i|C_i) < r(e_j|C_j)$$
$$\equiv \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right) < \left(1 - \frac{\sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| * \mathcal{L}_{max}}\right)$$
$$\equiv \sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*) < \sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*)$$
$$\equiv \mathcal{L}(p_{h,j}, p_{h,j}^*) < \mathcal{L}(p_{h,i}, p_{h,i}^*)$$

Step B: Here, we prove the **AXIOM** by showing that given e_j is an optimally reciprocal case on the **AXIOM**'s setting, and other cases are inevitably less reciprocal than it. For the target arc $e_k\ \forall k = i, j$, corresponding candidate arcs are having head set of size 2, and given candidate set has cardinality of $|C_k| = |T_k|$. In this setting, equation 15 (about e_j) implies that every node in target arc's tail set $v \in T_i$ is included in candidate arcs' head set twice. Thus,

- There are $|T_j|$ number of candidate arcs, whose heat set sizes are all 2.
- Every tail set node of target arc is being involved in candidate arc's head set twice. $|T_j| * 2$
- Every head set candidate arcs is a subset of target arc's tail set.

From the above three statements, transition probability can be induced as

$$p_{h,j} = \begin{cases} \frac{1}{2|T_j|} + \frac{1}{2|T_j|} = \frac{1}{|T_j|} & \text{if } v \in T_j \\ 0 & \text{otherwise} \end{cases}$$

Note that this is identical to the optimal transition probability.

Consider the case of e_i. Here, as there exist at least one inequality between the number of inclusion of each target arc's tail set node to the candidate arcs' head set, transition probability cannot be uniform as in the case of e_j.

Let's assume $v_{j1}' \in T_i$ belongs to the candidate arcs' head set $K \neq 2$ times. Then, transition probability at v_{j1}' is defined as $p(v_{j1}') = \frac{1}{2|T_i|} * K \neq \frac{1}{|T_i|}$. This result implies transition probability of e_i is not optimal at all. Thus, we can guarantee that

$$\mathcal{L}(p_{h,j}, p_{h,j}^*) < \mathcal{L}(p_{h,i}, p_{h,i}^*) \quad \forall v_{h,i} \in T_i' \ \ \forall v_{h,j} \in T_j'$$

As above inequality holds, proof is done.

Failure of Axiom 4

B2. Ratio of Covered Pairs: If every pair of target arc has been covered, then reciprocity for such case is always equal to 1. That is, how many times each node has been pointed is indistinguishable with this metric. In such case, reciprocity for both cases become $r(e_i|C_i) = r(e_j|C_j)$, which violates the **AXIOM4**.

B3. Penalized Ratio of Covered Pairs: Although size penalty term has been added to **B2**, cardinality of candidate sets are identical in this term. Thus, this baseline also suffers same problem with **B2**, $r(e_i|C_i) = r(e_j|C_j)$, which violates the **AXIOM4**.

1.2.5 Proof of Axiom 5

[AXIOM5]: *Upper and Lower Bounds of Hyperarc Reciprocity*

For every hyperarc $e_i \in E$ with $C_i \subseteq E$, following range should hold.

$$0 \leq r(e_i|C_i) \leq 1$$

Proof: We begin from the $\mathcal{L}(p, q)$.

$$\begin{aligned}
&\equiv 0 \leq \frac{\mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}} \leq 1 \quad \because 0 \leq \mathcal{L}(p, q) \leq \log 2 \\
&\equiv 0 \leq \sum_{v_h \in H_i} \frac{\mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}} \leq |H_i| \\
&\equiv 0 \leq \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}} \leq 1 \\
&\equiv 0 \leq 1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}} \leq 1 \\
&\equiv 0 \leq \left(\frac{1}{|C_i|}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right) \leq 1 \quad \because \alpha > 0
\end{aligned}$$

Proof is done.

<u>**Failure of Axiom 5**</u>

<u>**B4. HYPERREC w/o Normalization**</u> If we go similar flow without $|H_i|$ in the denominator, we get

$$\begin{aligned}
&\equiv 0 \leq \frac{\mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}} \leq 1 \quad \because 0 \leq \mathcal{L}(p, q) \leq \log 2 \\
&\equiv 0 \leq \sum_{v_h \in H_i} \frac{\mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}} \leq |H_i| \\
&\equiv 0 \leq |H_i| - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}} \leq |H_i| \\
&\equiv 0 \leq \left(\frac{1}{|C_i|}\right)^{\alpha} \left(|H_i| - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}}\right) \leq |H_i| \quad \because \alpha > 0
\end{aligned}$$

We can see the reciprocity lies between $0 \leq r(e_i|C_i) \leq |H_i|$, which is a violation of the **AXIOM5**.

This contains problem that since hyperarc reciprocity's range depends on the size of the target arc's head set, it is hard to compare different hyperarc with different head set size.

1.2.6 Proof of Axiom 6

[AXIOM6]: *Inclusion of Graph Reciprocity*

Digraph reciprocity is a special case of directed hypergraph reciprocity. That is, for $G = (V, E)$ if every hyperarc's head set and tail set size are equal to 1 $|H_i| = |T_i| = 1 \quad \forall e_i \in E$ (i.e., digraph), then following equality should hold.

$$r(G) = \frac{|E^{\leftrightarrow}|}{|E|} \quad \text{where } E^{\leftrightarrow} = \{e_i \mid \exists e_k = \langle H_k = T_i, T_k = H_i \rangle, (e_i, e_k \in E)\} \tag{16}$$

Note that this reciprocity is identical to the normal digraph reciprocity [3, 4].

Proof: Hypergraph level of **HyperRec** is defined as

$$r(G) = \frac{1}{|E|}\sum_{i=1}^{|E|} r(e_i \mid C_i)$$

As every hyperarc's head set size is equal to 1, each $r(e_i|C_i)$ can be written as

$$r(e_i|C_i) = \max_{C_i \subseteq E} \left(\frac{1}{|C_i|}\right)^{\alpha} (\mathcal{L}(p_h, p_h^*)) \quad \text{where } \{v_h\} = H_i$$

Here, optimal transition probability is defined as

$$p_h^*(v) = \begin{cases} 1 & \text{if } \{v\} = T_i \\ 0 & \text{otherwise} \end{cases}$$

One can clearly notice that only one arc is required for candidate set, since there is no partially covered arc (i.e., $T_i' \subset H_i, \quad H_i' \subset T_i$ does not exist), and we only need to decide whether an arc inversely-overlap or not, because max term will filter arcs which are not necessary.

For $C_i = \{e_k\}$ where $T_k \neq H_i$, by the statment 6, reciprocity for such hyperarc gets zero.

For $C_i = \{e_k\}$ where $T_k = H_i$, transition probability for the target arc e_i is defined as

$$p_h^*(v) = \begin{cases} 1 & \text{if } \{v\} = H_k \\ 0 & \text{otherwise} \end{cases}$$

Here, transition probability and optimal transition probability becomes identical if and only if $H_k = T_i$. Otherwise, as their non-zero probability domain do not overlap, reciprocity gets zero.

In sum, $r(e_i|C_i)$ is formally re-written as

$$r(e_i|C_i) = \begin{cases} 1 & \text{if } H_k = T_i \text{ and } T_k = H_i \\ 0 & \text{otherwise} \end{cases} \quad \text{where } C_i = \{e_k\}$$

Thus, $r(e_i|C_i)$ works as an indicator function which gives a value of 1 if there exist its inverse-pair, $\exists e_k \in E$ where $e_k = \langle H_k = T_i, T_k = H_i \rangle$ and zero elsewhere. Formally,

$$\begin{aligned} r(G) &= \frac{1}{|E|}\sum_{i=1}^{|E|} r(e_i|C_i) \\ &= \frac{1}{|E|}\sum_{i=1}^{|E|} \mathbb{1}((\exists e_k = \langle H_k = T_i, T_k = H_i \rangle), (e_k \in E)) \qquad = \frac{|E^{\leftrightarrow}|}{|E|} \end{aligned}$$

which is identical to the digraph reciprocity measure. Proof is done.

Failure of Axiom 6

B1. Pearcy et al. [2]: We provide simple counter example. See the following example shown in the Figure 4
Digraph reciprocity for Figure 4 can be computed as

$$E = \{e_1, e_2, e_3\}, \quad E^{\leftrightarrow} = \{e_1, e_2\} \quad \rightarrow \quad r(G) = \frac{2}{3}$$

In this case, clique expanded adjacency matrix of actual data and its optimal case are

$$\bar{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \bar{A}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

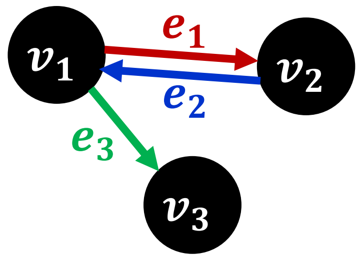


Figure 4: Counter example why certain baselines fail in **AXIOM 6**.

Where $tr(\bar{A}^2) = 2$ and $tr(\bar{A}'^2) = 4$, which result in $\frac{2}{4} = 0.5$. Since $\frac{2}{4} \neq \frac{2}{3}$, **B1. Pearcy et al. [2]** violates the **AXIOM 6**.

<u>B6. HYPERREC with All Arcs as Candidates:</u> As we put all the hyperarcs in C_i, this measure cannot achieve hyperarc reciprocity of 1 even in the case where there exist its perfectly reciprocal opponent.

In the Figure 4, although e_2 has its perfect reciprocal arc i.e., e_1, it takes e_3 also in consideration, which results in transition probability of $p_1 = [v_1, v_2, v_3, v_{sunk}] = [0, 0.5, 0.5, 0]$. Here, final reciprocity becomes $r(e_2|C_2 = \{e_1, e_3\}) = 0.7842 \neq 1$, which implies this metric cannot work as a proper indicator function whether its perfect opponent does exist. Thus, **AXIOM 6** cannot be achieved.

1.2.7 Proof of Axiom 7

[AXIOM7]: *Upper and Lower Bounds of Hypergraph Reciprocity*

The reciprocity of any hypergraph should be within a fixed range. Specifically, for any hypergraph G, $0 \leq r(G) \leq 1$ should hold.

<u>Proof:</u> Hypergraph level reciprocity of **HYPERREC** is defined as

$$r(G) = \frac{1}{|E|}\sum_{i=1}^{|E|} r(e_i|C_i)$$

Here, by the **AXIOM5**, we verify that arbitrary hyperarc reciprocity $r(e_i|C_i)$ lies in $[0, 1]$. From this fact, we can derive that

$$\begin{aligned}
&0 \leq r(e_i|C_i) \leq 1 \quad \forall i = \{1, \cdots, |E|\} \\
&0 \leq \sum_{i=1}^{|E|} r(e_i|C_i) \leq |E| \\
&0 \leq \frac{1}{|E|}\sum_{i=1}^{|E|} r(e_i|C_i) \leq 1
\end{aligned}$$

Proof is done.

1.2.8 Proof of Axiom 8

[AXIOM8]: *Surjection of Reciprocity*

The maximum reciprocity, which is 1 by the **AXIOM7**, should be reachable from any hypergraph by adding specific hyperarcs. That is, for every $G = (V, E)$, there exist $G^* = (V, E^*)$ with $E \subseteq E^*$ such that $r(G^*) = 1$.

<u>Proof:</u> From an arbitrary hypergraph G, let subset of hyperarcs whose perfect reciprocal opponents do not exist in a same hypergraph i.e., $E^{'} = \{e_k \mid (\langle T_k, H_k \rangle \notin E), (e_k \in E)\}$. Let set of additional hyperarcs that consist of perfectly opponents of E' as E^{opt}.

$$E^{opt} = \bigcup_{e_k \in E'} \langle T_k, H_k \rangle$$

Here, consider $E^* = E \cup E^{opt}$. For E^*, there always exist perfectly reciprocal opponent for each element (hyperarc). That is,

$$\text{Let } C_i = \{\langle T_i, H_i \rangle\} \quad \text{Then} \quad r(e_i|C_i) = \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right)$$
$$= \left(1 - \frac{0 + \cdots + 0}{|H_i| * \mathcal{L}_{max}}\right) = 1 \quad \because p_h = p_h^*$$

As $C_i = \{\langle T_i, H_i \rangle\}$, this candidate set always achieves maximum hyperarc reciprocity. In addition, this candidate set can always be found from E^* due to the max searching.

As every hyperarc reciprocity $r(e|C)$ is equal to 1, its hypergraph reciprocity $r(G)$ is also 1. Proof is done.

Failure of Axiom 8

B6. HyperRec with All the Arcs as Candidates: Even though there exist a perfect reciproal opponent of a specific hyperarc, transition probability cannot be identical to the optimal transition probability if there exist another inversely overlapping hyperarcs (see e_2 in Figure 4).

B7. HyperRec with Inversely Overlapping Arcs as Candidates: Similar to the previous case, if there exist multiple inversely overlapping hyperarcs, it inevitably uses all of them as candidate set. As a result, for such hyperarcs, $(1/|C_i|)^\alpha < 1$ is satisfied, which result in $r(e_i|C_i) < 1$. Consequently, overall hypergraph reciprocity becomes smaller than 1, which cannot be fully reciprocal.

2 Data Description

In this section, we provide detailed explanations regarding source of datasets and how we preprocess them.

metabolic dataset: There are two metabolic hypergraphs **iAF1260b** and **iJO1366**, which are provided from Yadati et al. [5]. We use them without preprocessing, since provided data format is exactly same as directed hypergraph structure. We only filter one hyperarc for each dataset, as these hyperarcs have abnormal head set and tail set size (second largest hyperarc's head set size is 8, but abnormal hyperarc head set size is greater than 20). Each node corresponds to the gene, and each hyperarc indicates the metabolic reaction among them. Reaction e_i indicates when genes in the head set H_i react together, they become the genes of tail set T_i.

email dataset: There are two email hypergraphs **email-enron** and **email-eu**. **Email-enron** dataset is from Chodrwo et al. [6]. We consider each email as a single hyperarc. Original hypergraph's hyperarc consist of a sender, receiver(s) and cc user(s). We compose head set H_i of a hyperarc (email) with receiver(s) and cc user(s) of corresponding email. Tail set T_i of that hyperarc consists of a single node, only a sender of that email. **Email-eu** dataset is from SNAP [7]. Original data is an email graph with the time stamp of when a corresponding email has been sent. It only has an information of a node v has sent an email to node u at the time stamp t. We consider di-arcs which have identical source node and time stamp are one single group of email, and we transform these di-arcs into one single hyperarc. Here, identical source node becomes a tail set, and each destination node of diarcs create a head set of corresponding hyperarc. Note that every hyperarc of this domain has size 1 tail set $|T_i| = 1 \; \forall i = \{1, \cdots, |E|\}$.

citation dataset: There are two citation hypergraphs **citation-data mining** and **citation-software**. Creating directed hypergraph from normal citation network has been suggested from [8], and we follow the proposed steps. Let authors of publications as a nodes. Assume a paper A which is written by $\{v_1, v_2, v_3\}$ has cited another paper B, which is written by $\{v_4, v_5\}$. Here, we consider each author group of publications as a set (this set can be either head set or tail set). In this case, citation $B \leftarrow A$ becomes one hyperarc e_1. Set $\{v_4, v_5\}$ becomes a head set H_1 of this hyperarc e_1, and $\{v_1, v_2, v_3\}$ becomes a tail set of T_2 this hyperarc e_1. Now, consider a paper B has cited paper C which is written by $\{v_6, v_7\}$ (let this hyperarc e_2). Now, $\{v_4, v_5\}$ becomes a tail set T_2 of a new hyperarc e_2. We use a DBLP citation dataset [9] for citation network. Among them, we choose two sub fields in a computer science, data mining

Table 1: Hypergraph reciprocity $r(G)$ of 11 datasets with $\alpha \approx 0$, 0.5 and 1.

		metabolic		email		citation		qna		bitcoin		
		iAF1260b	iJO1366	enron	eu	data mining	software	math	server	2014	2015	2016
$r(G)$	$\alpha \approx 0$	21.455	22.533	59.001	79.416	12.078	15.316	9.608	13.219	10.829	6.923	3.045
	$\alpha = 0.5$	17.756	18.497	49.321	65.405	10.840	13.984	9.283	13.196	10.654	6.845	2.988
	$\alpha = 1.0$	16.654	17.385	44.299	58.069	10.585	13.704	9.236	13.193	10.606	6.828	2.977

Table 2: Hypergraph reciprocity $r(G)$ robustness regarding different α. Although their absolute value may be different as shown in the Table 1, their relation and rank among them are still strongly maintained as all of the pearson correlation and spearman rank correlation is greater than 0.99.

		Pearson Correlation	**Spearman Rank Correlation**
$r(G)$	$\alpha :\approx 0 \leftrightarrow 0.5$	0.999	1.0
	$\alpha :\approx 0 \leftrightarrow 1.0$	0.999	0.991
	$\alpha : 0.5 \leftrightarrow 1.0$	0.999	0.991

and software engineering. In order to select papers of these two sub fields, we choose the veneus of data mining and software engineering at [10]. Then, we adopt papers which are published in the selected venues. At last, we filter papers which contain more than 10 authors to minimize the effect of such outliers of gigantic size.

question answering dataset: There are two question answering hypergraphs **qna-math** and **qna-server**. We create directed hypergraphs from the log data of question answering sites *stack exchange* provided at [11]. Among various domains, we choose *math-overflow*, a site which covers mathematical questions, and *server-fault*, a site which treats server related issues. Original dataset contains the posts of the site, which consist of a single questioner and answerer(s). We treat each user as a node, and each post as a hyperarc. For each post (hyperarc), a questioner of that post becomes a head set, and answerer(s) consist a tail set. We only use the posts which contain at least one answerer. Note that every hyperarc of this domain has a head set size of 1 $|H_i| = 1 \quad \forall i = \{1, \cdots, |E|\}$.

bitcoin transaction dataset: There are three bitcoin transaction hypergraphs **bitcoin-2014**, **bitcoin-2015**, and **bitcoin-2016**. Original dataset is from [12], which has provided first 1,500,000 transactions at 11/2014, 06/2015, and 01/2016. Here, each account (user) becomes a node, and each transaction becomes a hyperarc. As multiple users can involve in a single transaction, entities who send coins compose a tail set, and entities who receive coins compose a head set of corresponding transaction. As change-making is also treated as a transaction, we filter transactions where the head set and the tail set are the same.

3 Experiments

In this section, we introduce entire experimental results which are omitted in the main paper due to space limit.

3.1 Robustness of α

In the main paper, we demonstrate the robustness of α, i.e., which is a size penalty scalar of the candidate set, with one data from each domain. We discover that even each hypergraph's absolute reciprocity may vary with different value of α (see Table 1), their rank or relational position across different domains do not change. This can be verified by the high pearson correlation and rank correlation among different setting of αs (see Table 2).

To investigate hyperarc level reciprocity robustness regarding α, we measure the pearson correlation and spearman rank correlation between hyperarc reciprocity vectors within same dataset but different value of α. Although absolute value of each hyperarc reciprocity value may differ, their general tendency remain similar since measured statistics are greater than 0.9 except for the email dataset (see Table 3).

In sum, we observe that both hypergraph reciprocity and hyperarc reciprocity is robust to the different values of α (general tendency is maintained), although their absolute value may vary with respect to it.

Table 3: Hyperarc reciprocity $r(e|C)$ robustness regarding different α values. Their relational tendency is strongly maintained, as pearson and spearman rank correlation are greater than 0.9 for the most of the cases.

		metabolic		email		citation		qna		bitcoin		
		iAF1260b	iJO1366	enron	eu	data mining	software	math	server	2014	2015	2016
Pearson	$\alpha :\approx 0 \leftrightarrow 0.5$	0.961	0.957	0.928	0.836	0.984	0.986	0.994	0.999	0.997	0.998	0.997
	$\alpha :\approx 0 \leftrightarrow 1.0$	0.916	0.913	0.828	0.678	0.973	0.977	0.992	0.999	0.995	0.997	0.996
	$\alpha : 0.5 \leftrightarrow 1.0$	0.985	0.986	0.975	0.967	0.998	0.998	0.999	0.999	0.999	0.999	0.999
Spearman Rank	$\alpha :\approx 0 \leftrightarrow 0.5$	0.973	0.969	0.947	0.817	0.998	0.998	0.999	0.999	0.999	0.999	0.999
	$\alpha :\approx 0 \leftrightarrow 1.0$	0.925	0.918	0.869	0.721	0.996	0.996	0.999	0.999	0.999	0.999	0.999
	$\alpha : 0.5 \leftrightarrow 1.0$	0.975	0.973	0.978	0.983	0.999	0.999	0.999	0.999	0.999	0.999	0.999

3.2 Observation 1

In the main paper, we report that there exist a significant differece in hypergraph reciprocity between real-world hypergraph and null (randomized) hypergraph (see main paper for detail information regarding null hypergraph). We verify this characteristic by doing Z-test as follows.

- We create 30 randomized hypergraphs, $r(G_{null,i})\ \forall i = \{1, \cdots 30\}$ for each dataset.
- With created 30 randomized hypergraphs, we find average of null hypergraph reciprocity values $\overline{r(G_{null})}$ and standard deviation of them $sd(r(G_{null}))$.
- By using the central limit theorem, we approximate the equation 17 to standard normal distribution, where $r(G)$ is a real-world hypergraph's reciprocity, and $\xrightarrow{d}$ indicates a converge in distribution.

$$Z_{(real-null)} = \frac{\overline{r(G_{null,obs})} - r(G)}{\frac{sd(r(G_{null}))}{\sqrt{30}}} \xrightarrow{d} N(0,1) \tag{17}$$

- At last, we can perform hypothesis test where its null hypothesis μ_0 and alternative hypothesis μ_a are ;

μ_0 : Null hypergraph's average hypergraph reciprocity is identical to the real-world hypergraph's average reciprocity ;
μ_a : Null hypergraph's average hypergraph reciprocity is smaller than real-world hypergraph's average reciprocity ;

$$\mu_0 : \overline{r(G_{null})} = r(G)$$
$$\mu_a : \overline{r(G_{null})} < r(G)$$

- From the computed $Z_{(real-null)}$, we can derive the p-value as $P(z < Z_{(real-null)})$ where $P(z)$ is a cumulative probability distribution of standard normal distribution.

We set significance level of the testing as 0.01, which is more rigid than common setting 0.05. For all the cases, we adopt the alternative hypothesis, which implies $\overline{r(G_{null})}$ is statistically-significantly smaller than $\overline{r(G)}$. Detailed numerical results of the test are illustrated in the Table 4. In sum, we discover that real world hypergraph's reciprocity is significantly large than null (randomized) hypergraph.

Table 4: P-value testing result of 11 datasets. Null hypotheses are all rejected, which implies real-world hypergraphs are significantly more reciprocal than null hypergraphs. P-value smaller than 0.00001 are all marked as 0.0000*

	metabolic		email		citation		qna		bitcoin		
	iAF1260b	iJO1366	enron	eu	data mining	software	math	server	2014	2015	2016
Z-stat	-1502.52	-1789.79	-241.13	-3835.98	-17548.20	-9605.19	-8884.98	-88965.12	-691316.77	-555709.95	-325308.06
P-value	0.0000*	0.0000*	0.0000*	0.0000*	0.0000*	0.0000*	0.0000*	0.0000*	0.0000*	0.0000*	0.0000*
Null hypothesis	Reject	Reject	Reject	Reject	Reject	Reject	Reject	Reject	Reject	Reject	Reject

3.3 Observation 2

In the main paper, we discover that in the real-world hypergraphs, hyperarcs of non-zero reciprocity tend to have higher head set out degree distribution and tail set in degree distribution than zero reciprocity hyperarcs. Here, head set out degree and tail set in degree are defined as

$$\text{Head set out-degree: } d_{H,out}(e_i) = \frac{1}{|H_i|} \sum_{v \in H_i} d_{out}(v) \tag{18}$$

$$\text{Tail set in-degree: } d_{T,in}(e_i) = \frac{1}{|T_i|} \sum_{v \in T_i} d_{in}(v) \tag{19}$$

From most of the dataset (except for the tail set in degree distribution of question answering server dataset), we can observe the tendency that real world hypergraph's non-zero reciprocity hyperarcs are having higher head set out degree distribution and tail set in degree distribution than zero reciprocity hyperarcs. Such phenomenon is not clear in the null hypergraphs (see Figure 5).

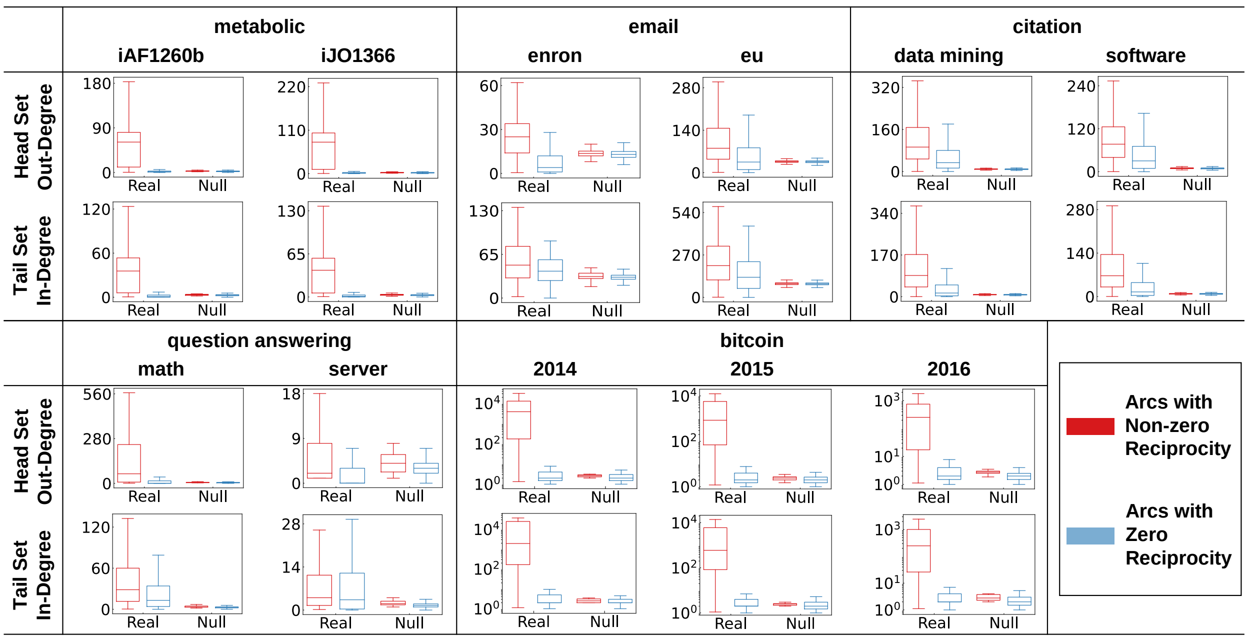


Figure 5: Result of observation 2 in the entire dataset: Except for the question answering server tail set in degree, arcs with non zero reciprocity have higher head set out degree and tail set in degree than arcs with zero reciprocity. Such phenomenon is not observed from null randomized hypergraphs.

3.4 Observation 3

In the main paper, we discover that in the real-world hypergraphs, nodes with balanced in degree and out degree tend to be involved in high reciprocity hyperarcs. To quantify the degree balance of nodes, we measure following statistic for each node $v \in V$,

$$Bal(v) = log(d_{in}(v)) - log(d_{out}(v))$$

If $Bal(v) \approx 0$, a node v's in degree and out degree is similar. On the other hand, $Bal(v) \to -\infty$ implies a node v is having higher out degree than in degree. Inversely, for the case of $Bal(v) \to +\infty$, a node v has higher in degree than out degree.

As arc reciprocity is defined on hyperarc level, we map each hyperarc reciprocity to node by

$$r(v) = \frac{1}{|E_v|} \sum_{e_k \in E_v} r(e_k)$$

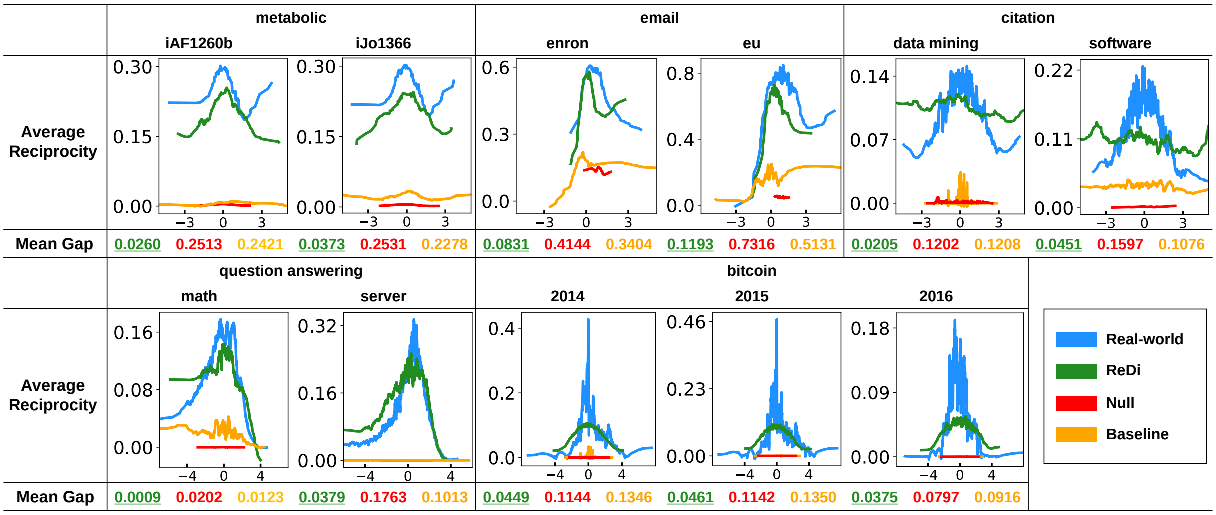


Figure 6: Result of observation 3 in the entire dataset of real-world, **REDI**, null, and baseline: For the real world hypergraph, we can observe bell-shaped plot which has highest point around zero x value, which implies nodes with balanced degree tend to be involved in high reciprocity hyperarcs. Hypergraphs generated by **REDI** also show such similar pattern for most of the dataset except for the citation networks. However, null hypergraphs and baseline hypergraphs do not show such tendency.

where $E_v = \{e_k : v \in (H_k \cup T_k)\}$ is the set of arcs where v is included in its head set or its tail set.

We draw line plot with $Bal(v)$ and $r(v)$, where each coordinate of plot becomes $(x, y) = (Bal(v), r(v)) \;\; \forall i = \{1, \cdots, |V|\}$. For the x values with multiple y values, i.e., for $u, v \in V$, $Bal(v) = Bal(u), \quad r(v) \neq r(u)$, we use average values of their reciprocity values. That is, for a specific x coordinate x_j, corresponding y coordinate is

$$y_j = \frac{1}{|V'_j|} \sum_{v \in V'_j} r(v) \quad V'_j = \{v_k \in V \mid Bal(v_k) = x_j\}$$

As shown in the Figure 6, we can observe that real-world hypergraphs tend to show bell-shaped plot which has its maximum point around zero x value (see the blue lines of the figure). This indicates if a node's degree is relatively balanced i.e., $Bal(v) \approx 0$, this nodes tends to be included in high reciprocity arcs. Such pattern is ambiguous for null hypergraphs.

3.5 REDI's Reproducibility of Observation 2

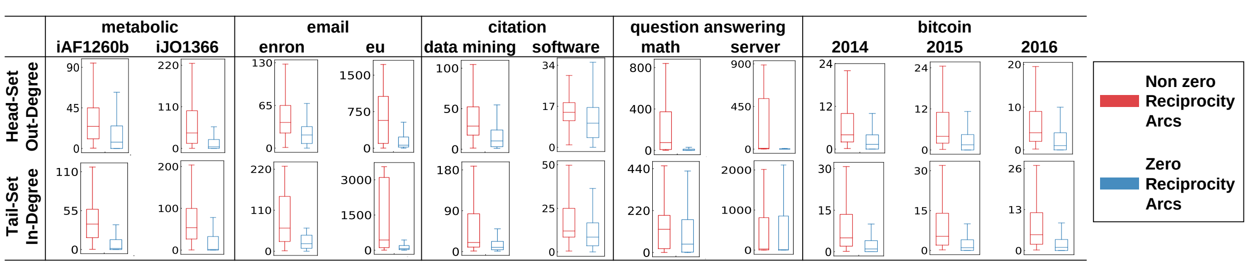


Figure 7: Result of observation 2 in the generated hypergraphs by **REDI**. Similar to the real-world hypergraphs, non-zero reciprocity hyperarcs tend to have higher head set out degree and tail set in degree than zero reciprocity hyperarcs in most of the dataset.

Part of the generation result of proposed generator **REDI** regarding observation 2 is reported in the main paper. We examine how much reciprocal pattern regarding observation 2 has been maintained in **REDI** from entire dataset. As shown in the Figure 7, reciprocal patterns of observation 2 has been maintained in most of the dataset, except for the citation software dataset where its tendency of head set out degree is little unclear.

One interesting fact is that tail set in degree distribution of real-world question answering server dataset is similar between non-zero reciprocity hyperarcs and zero reciprocity hyperarcs (see Figure 5). This phenomenon also appear in the generated question answering server dataset (see Figure 7).

3.6 REDI's Reproducibility of Observation 3

Part of the generation result of proposed generator **REDI** regarding observation 3 is reported in the main paper. Similar to the previous subsection, we investigate how much reciprocal pattern regarding observation 3 has been maintained in **REDI**'s hypergraph at the entire dataset. General bell-shaped tendency is also observed in the **REDI**'s hypergraph from the most of the dataset (except for the citation network), while null hypergraphs and baseline generator fail in capturing this characteristic.

In order to quantitatively compare three methods' (**REDI**, null, and baseline) reproducibility regarding observation 3, we measure the *mean gap* between the plots of real world hypergraph and generated hypergraphs. *mean gap* is defined as follows. Denote a specific hypergraph G's plot's x coordinate as X_G and y coordinate as Y_G. Then,

$$X_{G_{real}} = [x_1, x_2, \cdots, x_K], \quad Y_{G_{real}} = [y_1, y_2, \cdots, y_K]$$
$$X_{G_{syn}} = [x'_1, x'_2, \cdots, x'_M], \quad Y_{G_{syn}} = [y'_1, y'_2, \cdots, y'_M]$$
$$X' = X_{G_{real}} \cap X_{G_{syn}} \quad Y'_{real} = \{y_k \in Y_{G_{real}} \mid x_k \in X'\} \quad Y'_{syn} = \{y'_k \in Y_{G_{syn}} \mid x'_k \in X'\}$$
$$Y'_{REAL} = \text{Vector of } Y'_{real}\text{, which is sorted according to each element's corresponding x-value}$$
$$Y'_{SYN} = \text{Vector of } Y'_{syn}\text{, which is sorted according to each element's corresponding x-value}$$
$$\textit{mean gap} = \frac{1}{|Y'_{REAL}|} \sum_{i=1}^{|Y'_{REAL}|} (Y'_{REAL,i} - Y'_{SYN,i})^2$$

This *mean gap* measures the average vertical distance between real-world hypergraph's degree balance plot and generated hypergraph's degree balance plot. Among three methods, **REDI** shows the minimum distance from the real world hypergraph, which indicates that **REDI** captures the general tendency most well among three generation methods (see numeric values of Figure 6).

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