Reciprocity in Directed Hypergraphs: Measures, Findings, and Generators (Online Appendix)

1 Proof of the Theorems

In this section, we formalize our proposed measure and provide proof of theorems in the main paper.

1.1 Preliminaries of the Proofs

In this subsection, we first give the general form of our proposed measure. Then, we introduce several important characteristics of *Jensen-Shannon Divergence(JSD)* [1], which play key roles in the proofs. After that, we examine how these concepts are applied to our measure.

The proposed reciprocity measure **HyperRec** for hyperarc $r(e_i|C_i)$ and hypergraph $r(G)$ is defined as

$$r(e_i|C_i) := \max_{C_i \subseteq E} \left(\frac{1}{|C_i|}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}}\right), \tag{1}$$

$$r(G) := \frac{1}{|E|} \sum_{i=1}^{|E|} r(e_i|C_i), \tag{2}$$

where $\mathcal{L}(p_h, p_h^*)$ denotes the Jensen-Shannon Divergence [1] between a transition probability p_h distribution and an optimal transition probability distribution p_h^*. Here, the transition probability of a target arc e_i with a candidate set that consists of a single candidate arc $\{e_i'\}$ is defined as

$$p_h(v) = \begin{cases} \frac{1}{|H_i'|} & \text{if } v \in H_i' \\ 0 & \text{otherwise} \end{cases}$$

This is a case where $v_h \in \{H_i \cap T_i'\}$. For each $v_h \in H_i \setminus T_i'$, $p_h(v) = 1$ if $v = v_{sunk}$ and 0 elsewhere. For the target arc e_j with an arbitrary number of candidate arcs $C_j = \{e_{j1}, \cdots e_{jk}\}$, the transition probability is defined formally as

$$p_h(v) = \begin{cases} p_{h,1}(v) & \text{if } v_h \in \bigcup_{e_s \in C_j} T_s, \\ p_{h,2}(v) & \text{otherwise} \end{cases}$$

where

$$p_{h,1}(v) = \frac{\sum_{e_s \in C_j} \left(\frac{\mathbf{1}[v_h \in T_s, v \in H_s]}{|H_s|}\right)}{\sum_{e_s \in C_j} (\mathbf{1}[v_h \in T_s])}, \quad p_{h,2}(v) = \begin{cases} 1 & \text{if } v = v_{sunk} \\ 0 & \text{otherwise,} \end{cases}, \quad \mathbf{1}[\text{TRUE}] = 1, \quad \text{and } \mathbf{1}[\text{FALSE}] = 0.$$

Throughout the proof, we use these formal expressions for both single-candidate-arc and multiple-candidate-arc cases.

We now discuss the theoretical aspect of $JSD(P||Q)$. The general formula of JSD for the discrete space is

$$\mathcal{L}(P, Q) = \sum_{i=1}^{|V|} \ell(p_i, q_i), \tag{3}$$

where

$$\ell(p_i, q_i) = \frac{p_i}{2} \log \frac{2p_i}{p_i + q_i} + \frac{q_i}{2} \log \frac{2q_i}{p_i + q_i}. \tag{4}$$

A. 1. For any two discrete probability distributions P and Q, $0 \leq JSD(P||Q) \leq \log 2$ holds, as shown in [1].

A. 2. For two discrete probability distributions P and Q where their non-zero probability domains do not overlap (i.e., $p_i q_i = 0, \forall i = \{1, \cdots, |V|\}$), the overall JSD between two probability distributions is maximized as $\log 2$.

Proof: Let $\mathcal{X}_p$ be the domain where P has non-zero probabilities, and let $\mathcal{X}_q$ be the domain where Q has non-zero probabilities. As $\mathcal{X}_p$ and $\mathcal{X}_q$ do not overlap, Eq. (3) is summarized as

$$\sum_{i \in \mathcal{X}_p} \frac{p_i}{2} \log 2 + \sum_{i \in \mathcal{X}_q} \frac{q_i}{2} \log 2 = \frac{\log 2}{2} \left(\sum_{i \in \mathcal{X}_p} p_i + \sum_{i \in \mathcal{X}_q} q_i \right) = \frac{\log 2}{2} \times (1+1) = \log 2.$$

A. 3. Consider two discrete probability distributions P and Q. If there exists a value where both P and Q have non-zero probabilities, then $JSD(P||Q) < \log 2$ holds.

Proof: Let k be a value where $p_k q_k \neq 0$ holds. Then Eq. (3) can be summarized as

$$\mathcal{L}(p,q) = \sum_{i \in \mathcal{X}_p \setminus k} \frac{p_i}{2} \log 2 + \sum_{i \in \mathcal{X}_q \setminus k} \frac{q_i}{2} \log 2 + \left(\frac{p_k}{2} \log \frac{2p_k}{p_k + q_k} + \frac{q_k}{2} \log \frac{2q_k}{p_k + q_k} \right). \tag{5}$$

In order to show that Eq. (3) is smaller than $\log 2$, it is required to show that

$$\begin{aligned}
&\left(\frac{p_k}{2} \log 2 + \frac{q_k}{2} \log 2 \right) - \left(\frac{p_k}{2} \log \frac{2p_k}{p_k + q_k} + \frac{q_k}{2} \log \frac{2q_k}{p_k + q_k} \right) > 0 \\
&\equiv \frac{p_k}{2} \log \left(1 + \frac{q_k}{p_k} \right) + \frac{q_k}{2} \log \left(1 + \frac{p_k}{q_k} \right) > 0 \quad (\because p_k, q_k > 0).
\end{aligned}$$

As the log function has a positive real value if its input is greater than 1, the last inequality holds. Thus, we can conclude that $JSD(P||Q) < \log 2$.

We use above three statements (**A. 1**, **A. 2**, and **A. 3**) to derive additional three statements as follows.

A. 4. If the target arc's head set and candidate arc's tail set do not overlap, the target arc's reciprocity becomes zero. Formally,

$$\text{If } |H_i \cap \bigcup_{e_k \in C_i} T_k| = 0 \quad \text{then} \quad r(e_i|C_i) = 0.$$

Proof: For this case, as mentioned in the main paper, the transition probability is heading toward sunken node v_{sunk}. On the other hand, optimal transition probability p^* is heading toward $v \in T_i$ where $v_{sunk} \notin T_i$. Thus, the non-zero probability domain of transition probability and optimal transition probability is not overlapped, where the probabilistic distance between them is maximized as $\mathcal{L}_{max}$ by **A. 2**. This happens for all H_i, in sum,

$$\begin{aligned}
r(e_i|C_i) &= \left(\frac{1}{|C_i|} \right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}_{max}}{|H_i| * \mathcal{L}_{max}} \right) \\
&= \left(\frac{1}{|C_i|} \right)^{\alpha} \times \left(1 - \frac{|H_i| * \mathcal{L}_{max}}{|H_i| * \mathcal{L}_{max}} \right) = \left(\frac{1}{|C_i|} \right)^{\alpha} \times 0 = 0
\end{aligned}$$

A. 5. If the target arc's tail set and candidate arc's head set do not overlap, the target arc's reciprocity becomes zero. Formally,

$$\text{If } |T_i \cap \bigcup_{e_k \in C_i} H_k| = 0 \quad \text{then} \quad r(e_i|C_i) = 0.$$

Proof: Similar to the **A. 4**, non-zero probability domain of transition probability and optimal probability is not overlapped since $|T_i \cap \bigcup_{e_k \in C_i} H_k| = 0$. Again, probabilistic distance is maximized as $\mathcal{L}_{max}$.
This happens for all H_i, in sum,

$$\begin{aligned}
r(e_i|C_i) &= \left(\frac{1}{|C_i|} \right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}_{max}}{|H_i| * \mathcal{L}_{max}} \right) \\
&= \left(\frac{1}{|C_i|} \right)^{\alpha} \times \left(1 - \frac{|H_i| * \mathcal{L}_{max}}{|H_i| * \mathcal{L}_{max}} \right) = \left(\frac{1}{|C_i|} \right)^{\alpha} \times 0 = 0
\end{aligned}$$

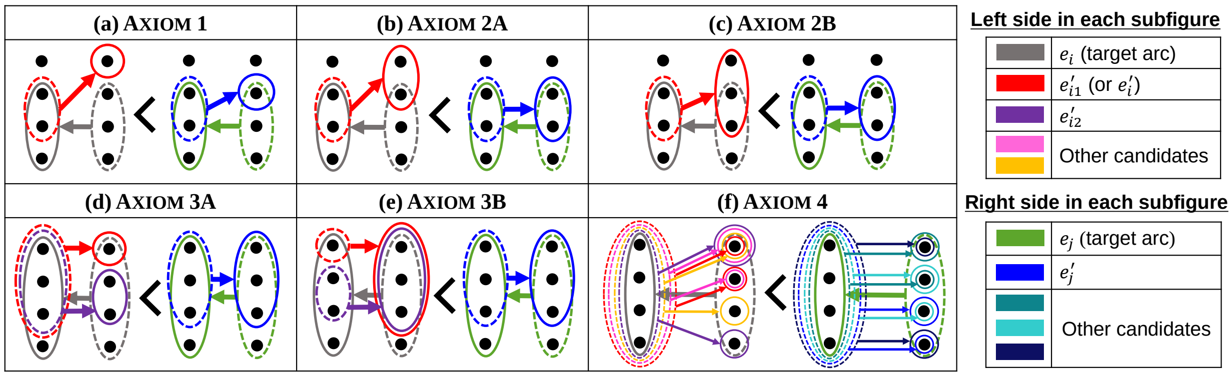


Figure 1: Examples for **AXIOMS** 1-4. In each subfigure, the reciprocity of the arc e_i on the left side should be smaller than that of the arc e_j on the right side.

A. 6. If the target arc's head set and candidate arc's tail set overlap and the target arc's tail set and candidate arc's head set overlap, the target arc's reciprocity is greater than zero. Formally,

$$\text{If} \sum_{e_k \in C_i} |H_i \cap T_k| \times |T_j \cap H_k| \geq 1 \quad \text{then} \quad r(e_i|C_i) > 0.$$

Proof: According to the statement, there exist at least one candidate arc e_k whose tail set overlaps to target arc's head set $|T_k \cap H_i| \geq 1$, and head set also overlaps to target arc's tail set $|H_k \cap T_i| \geq 1$. Thus, for $v_h \in H_i \cap T_k$, p_h and p_h^* have a common non-zero probability domain, which implies $\boldsymbol{\mathcal{L}(p_h, p_h^*) < \log 2}$ by **A. 3**. In sum, we can derive the following inequality (**see the inequality sign between two middle terms**).

$$\begin{aligned} r(e_i|C_i) &= \left(\frac{1}{|C_i|}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right) \\ &> \left(\frac{1}{|C_i|}\right)^{\alpha} \times \left(1 - \frac{|H_i| * \mathcal{L}_{max}}{|H_i| * \mathcal{L}_{max}}\right) = \left(\frac{1}{|C_i|}\right)^{\alpha} \times 0 = 0 \end{aligned}$$

1.2 Proof of the Theorem 1.

For the **AXIOM** 1, we simply show that former's reciprocity gets zero $r(e_i|C_i) = 0$, while latter's reciprocity get's a positive value $r(e_j|C_j) > 0$. For the other **AXIOMS** , we first show the relationship between **AXIOMS** and the probabilistic distance (Step A). After, we show that a less reciprocal case has a higher probabilistic distance between observed transition probability and optimal transition probability for every head-set node of a target arc (Step B). An overview of **AXIOMS** is illustrated in Figure 1.

1.2.1 Proof of How HyperRec Satisfies Axiom 1

[AXIOM1]: *Existence of Inverse Overlap*

Assume $C_i = \{e'_i\}$ and $C_j = \{e'_j\}$. Then

$$\min(|H_i \cap T'_i|, |T_i \cap H'_i|) = 0 \text{ and } \min(|H_j \cap T'_j|, |T_j \cap H'_j|) \geq 1$$

Then, the following inequality holds.

$$r(e_i|C_i) < r(e_j|C_j)$$

PROOF: We first show $r(e_i|C_i) = 0$. Term $\min(|H_i \cap T'_i|, |T_i \cap H'_i|) = 0$ implies at least one of $H_i \cap T'_i$ or $T_i \cap H'_i$ is an empty set. By the **A. 4** and **A. 5**, we can derive that reciprocity for this case is equal to zero $r(e_i|C_i) = 0$.

Now we show that $r(e_j|C_j) > 0$. As $\min(|H_j \cap T'_j|, |T_j \cap H'_j|) \geq 1$, we can guarantee that both $H_j \cap T'_j$ and $T_j \cap H'_j$ are non-empty set. By using the **A. 6**, we prove the inequality $r(e_j|C_j) > 0$.

In conclusion as $r(e_i|C_i) = 0$ and $r(e_j|C_j) > 0$, we can derive $r(e_i|C_i) < r(e_j|C_j)$.

Note that from AXIOM 2-4, we assume two target arcs e_i and e_j have equal size $|H_i| = |H_j|$ and $|T_i| = |T_j|$.

1.2.2 Proof of How HYPERREC Satisfies AXIOM 2

[AXIOM2A]: *Degree of Inverse Overlap: More Overlap*

Let $C_i = \{e'_i\}$ and $C_j = \{e'_j\}$ where

$$|H'_i| = |H'_j| \text{ and } |T'_i| = |T'_j|$$
$$\text{(I)} \quad 0 < |H'_i \cap T_i| < |H'_j \cap T_j| \text{ and } 0 < |T'_i \cap H_i| \leq |T'_j \cap H_j| \quad \text{or}$$
$$\text{(II)} \quad 0 < |H'_i \cap T_i| \leq |H'_j \cap T_j| \text{ and } 0 < |T'_i \cap H_i| < |T'_j \cap H_j|$$

Then, the following inequality holds:

$$r(e_i|C_i) < r(e_j|C_j).$$

PROOF: As $|C_i| = |C_j| = 1$, cardinality penalty terms on both side can be discarded.

$$r(e_i|C_j) = \left(\frac{1}{|C_i|}\right)^{\alpha} \left(1 - \frac{\sum_{v_i \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}}\right) = 1 - \frac{\sum_{v_i \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}} \quad \text{Same for } r(e_j|C_j) \text{ with subscript } j.$$

<u>Step A:</u> Here, we are trying to show $r(e_i|C_i) < r(e_j|C_j)$, which is equivalent to

$$\frac{\sum_{v_i \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}} > \frac{\sum_{v_j \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| \cdot \mathcal{L}_{max}}$$

As $|H_i| = |H_j|$, denominator of both terms are identical. By removing them, inequality can be summarized as

$$\equiv \sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*) > \sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*) \tag{6}$$

Each head set can be divided into two parts ; $H_k \setminus T'_k$ and $H_k \cap T'_k$ $\forall k = i, j$.

For $H_k \setminus T'_k$, as described in **A. 4**, probabilistic distance gets maximized as $\mathcal{L}_{max} = \log 2$.

For $H_k \cap T'_k$, by using $T_k \cap H'_k \neq \emptyset$, we can derive, $\mathcal{L}(p_h, p_h^*) < \log 2$ satisfies for $\forall v_h \in H_k \cap T'_k$ where $\forall k = i, j$ by the **A. 6**. One more notable fact is that as there is a single candidate arc in both cases, $\mathcal{L}(p_h, p_h^*)$ is the same for every $v_h \in H_k \cap T'_k$. Here, let $p_{h,i}$ be the transition probability regarding a target arc e_i and its candidate set C_i. We summarize the main inequality (6) as follows.

$$\begin{aligned}
&\equiv \sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*) > \sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*) \\
&\equiv |H_i \setminus T'_i| * \log 2 + |H_i \cap T'_i| * \mathcal{L}(p_{h,i}, p_{h,i}^*) > |H_j \setminus T'_j| * \log 2 + |H_j \cap T'_j| * \mathcal{L}(p_{h,j}, p_{h,j}^*)
\end{aligned}$$

Further inequality can be developed differently according to the condition (whether it is (I) or (II)).

For the case of (I), the intersection of the target arc's head set and the candidate arc's tail set is larger in e_j than e_i. As a result, the inequality is re-written as follows.

$$\begin{aligned}
&\equiv |H_i \setminus T'_i| * \log 2 + |H_i \cap T'_i| * \mathcal{L}(p_{h,i}, p_{h,i}^*) > |H_j \setminus T'_j| * \log 2 + |H_j \cap T'_j| * \mathcal{L}(p_{h,i}, p_{h,i}^*) \\
&\geq |H_j \setminus T'_j| * \log 2 + |H_j \cap T'_j| * \mathcal{L}(p_{h,j}, p_{h,j}^*)
\end{aligned}$$

For the case of (II), the intersection of the target arc's head set and the candidate arc's tail set can be identical, while the target arc's tail set and the candidate arc's head set intersection is greater in e_j than e_i. Then, the inequality is re-written as follows.

$$\begin{aligned}
&\equiv |H_i \setminus T'_i| * \log 2 + |H_i \cap T'_i| * \mathcal{L}(p_{h,i}, p_{h,i}^*) \geq |H_j \setminus T'_j| * \log 2 + |H_j \cap T'_j| * \mathcal{L}(p_{h,i}, p_{h,i}^*) \\
&> |H_j \setminus T'_j| * \log 2 + |H_j \cap T'_j| * \mathcal{L}(p_{h,j}, p_{h,j}^*)
\end{aligned}$$

Here, one can notice that if $\mathcal{L}(p_{h,i}, p_{h,i}^*) > \mathcal{L}(p_{h,j}, p_{h,j}^*)$ holds, then both cases (**I and II**) can be satisfied. Formally,

$$\begin{aligned}
&\equiv |H_j \setminus T'_j| * \log 2 + |H_j \cap T'_j| * \mathcal{L}(p_{h,i}, p_{h,i}^*) > |H_j \setminus T'_j| * \log 2 + |H_j \cap T'_j| * \mathcal{L}(p_{h,j}, p_{h,j}^*) \\
&\equiv |H_j \cap T'_j| * \mathcal{L}(p_{h,i}, p_{h,i}^*) > |H_j \cap T'_j| * \mathcal{L}(p_{h,j}, p_{h,j}^*) \\
&\equiv \mathcal{L}(p_{h,i}, p_{h,i}^*) > \mathcal{L}(p_{h,j}, p_{h,j}^*)
\end{aligned}$$

Step B: We only need to show that the probabilistic distance between transition probability and optimal transition probability is greater in e_i than e_j. Denote the intersection as $F_1 = |T_i \cap H_i'| \ < \ F_2 = |T_j \cap H_j'|$. We can decompose domain $v \in V$ into four parts as

$$v \in T_k \cap H_k' \quad v \in T_k \setminus H_k' \quad v \in H_k' \setminus T_k \quad v \in V \setminus \{H_k' \cup T_k\} \quad \forall k = i, j$$

For the last term, both transition probability and optimal of it have a value of zero $p_h(v) = p_h^*(v) = 0$, which result in no penalty. We only need to consider first three terms for comparison. Here, the probabilistic distance can be explicitly written as

$$\mathcal{L}(p_{h,i}, p_{h,i}^*) = |F_1| * \ell\left(\frac{1}{T}, \frac{1}{A}\right) + (A - |F_1|) * \frac{1}{2A}\log 2 + (T - |F_1|) * \frac{1}{2T}\log 2$$
$$\mathcal{L}(p_{h,j}, p_{h,j}^*) = |F_2| * \ell\left(\frac{1}{T}, \frac{1}{A}\right) + (A - |F_2|) * \frac{1}{2A}\log 2 + (T - |F_2|) * \frac{1}{2T}\log 2$$

where ℓ denotes JSD's functional form explained in equation (4). Let $A = |S_i'| = |S_j'|$ and $T = |T_i| = |T_j|$. By showing $\mathcal{L}(p_{h,i}, p_{h,i}^*) - \mathcal{L}(p_{h,j}, p_{h,j}^*) > 0$, proof can be done. Formally,

$$\begin{aligned}
&\equiv \mathcal{L}(p_{h,i}, p_{h,i}^*) - \mathcal{L}(p_{h,j}, p_{h,j}^*) > 0 \\
&= (|F_1| - |F_2|) * \ell\left(\frac{1}{T}, \frac{1}{A}\right) + (|F_2| - |F_1|) * \left(\frac{1}{2A} + \frac{1}{2T}\right)\log 2 > 0 \\
&\equiv (|F_2| - |F_1|) * \left(\frac{1}{2A}\log 2 + \frac{1}{2T}\log 2 - \ell\left(\frac{1}{T}, \frac{1}{A}\right)\right) > 0 \\
&\equiv \left(\frac{1}{2A}\log 2 - \frac{1}{2A}\log\frac{2T}{A+T} + \frac{1}{2T}\log 2 - \frac{1}{2T}\log\frac{2A}{A+T}\right) > 0 \quad \because (|F_2| - |F_1|) > 0 \\
&\equiv \frac{1}{2A}\log\frac{A+T}{T} + \frac{1}{2T}\log\frac{A+T}{A} > 0
\end{aligned}$$

As $x > 1 \Rightarrow \log x > 0$, inequality holds. The proof is done.

[AXIOM2B]: *Degree of Inverse Overlap: Small difference*

Let $C_i = \{e_i'\}$ and $C_j = \{e_j'\}$ where

$$|H_i'| > |H_j'| \text{ and } |T_i'| = |T_j'| \quad 0 < |H_i' \cap T_i| = |H_j' \cap T_j| \text{ and } 0 < |T_i' \cap H_i| = |T_j' \cap H_j|$$

Then, the following inequality holds.

$$r(e_i|C_i) < r(e_j|C_j)$$

PROOF: As $|C_i| = |C_j| = 1$, cardinality penalty term is being erased for both terms.

Step A: Overall inequality can be re-written as

$$r(e_i|C_i) < r(e_j|C_j) \equiv 1 - \frac{\sum_{v_i \in H_i}\mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}} < 1 - \frac{\sum_{v_j \in H_j}\mathcal{L}(p_h, p_h^*)}{|H_j| \cdot \mathcal{L}_{max}}$$

Similar to the previous proof, $|H_i| = |H_j|$, and $\mathcal{L}(p_h, p_h^*)$ among different $v_h \in H_k \cap T_k'$ are all identical. In addition, $|H_i \cap T_i'| = |H_j \cap T_j'|$, $|T_i'| = |T_j'|$, number of target arc's head set nodes v_h that satisfy $\mathcal{L}(p_h, p_h^*) < \log 2$ are identical in both terms. Thus,

$$\begin{aligned}
&\equiv r(e_i|C_i) < r(e_j|C_j) \equiv \frac{\sum_{v_j \in H_j}\mathcal{L}(p_h, p_h^*)}{|H_j| \cdot \mathcal{L}_{max}} < \frac{\sum_{v_i \in H_i}\mathcal{L}(p_h, p_h^*)}{|H_i| \cdot \mathcal{L}_{max}} \\
&\equiv |H_j \cap T_j'| * \mathcal{L}(p_{h,j}, p_{h,j}^*) < |H_i \cap T_i'| * \mathcal{L}(p_{h,i}, p_{h,i}^*) \\
&\equiv \mathcal{L}(p_{h,j}, p_{h,j}^*) < \mathcal{L}(p_{h,i}, p_{h,i}^*)
\end{aligned}$$

Step B: We only need to show that the probabilistic distance between transition probability and optimal transition probability is greater in e_i than e_j. Let $A = |H'_i| \; > \; B = |H'_j|$. We can decompose domain $v \in V$ into four parts as

$$v \in T_k \cap H'_k \quad v \in T_k \setminus H'_k \quad v \in H'_k \setminus T_k \quad v \in V \setminus \{H'_k \cup T_k\} \quad \forall k = i, j$$

Here, JSD in the second term and fourth term are identical for both cases. That is, we only need to compare probabilistic distances which are related to the first and third parts of the above four domains.

$$\begin{aligned}
&\equiv \mathcal{L}(p_{h,j}, p^*_{h,j}) < \mathcal{L}(p_{h,i}, p^*_{h,i}) \\
&\equiv F * \ell\left(\frac{1}{B}, \frac{1}{T}\right) + \frac{B-F}{2B}\log 2 < F * \ell\left(\frac{1}{A}, \frac{1}{T}\right) + \frac{A-F}{2A}\log 2
\end{aligned}$$

where $F = |H'_i \cap T_i| = |H'_j \cap T_j|$ and $T = |T_i| = |T_j|$. Note that $\boldsymbol{A > B}$. Inequality can be summarized as

$$\begin{aligned}
&\equiv \mathcal{L}(p_{h,i}, p^*_{h,i}) - \mathcal{L}(p_{h,j}, p^*_{h,j}) \\
&\equiv \frac{F}{2}\left(\frac{1}{B} - \frac{1}{A}\right)\log 2 > F * \left(\ell\left(\frac{1}{B}, \frac{1}{T}\right) - \ell\left(\frac{1}{A}, \frac{1}{T}\right)\right)
\end{aligned}$$

To simplify the equation, we unfold $\ell(p, q)$ as follows.

$$\begin{aligned}
&\equiv \frac{F}{2}\left(\frac{1}{B} - \frac{1}{A}\right)\log 2 > F * \left(\ell\left(\frac{1}{B}, \frac{1}{T}\right) - \ell\left(\frac{1}{A}, \frac{1}{T}\right)\right) \\
&\equiv \frac{1}{2}\left(\frac{1}{B} - \frac{1}{A}\right)\log 2 > \frac{1}{2T}\log\frac{2B}{B+T} + \frac{1}{2B}\log\frac{2T}{B+T} - \frac{1}{2T}\log\frac{2A}{A+T} - \frac{1}{2A}\log\frac{2T}{A+T} \\
&\equiv \left(\frac{1}{B} - \frac{1}{A}\right)\log 2 > \frac{1}{T}\log\frac{2B}{B+T} + \frac{1}{B}\log\frac{2T}{B+T} - \frac{1}{T}\log\frac{2A}{A+T} - \frac{1}{A}\log\frac{2T}{A+T} \\
&\equiv 0 > \frac{1}{T}\log\frac{B}{B+T} + \frac{1}{B}\log\frac{T}{B+T} - \frac{1}{T}\log\frac{A}{A+T} - \frac{1}{A}\log\frac{T}{A+T} \qquad \because \text{pull 2 inside the log term out} \\
&\equiv 0 > \frac{1}{T}\log\frac{B(A+T)}{A(B+T)} - \frac{1}{B}\log\frac{B+T}{T} + \frac{1}{A}\log\frac{A+T}{T} \qquad \text{multiply } T \text{ in both terms} \\
&\equiv 0 > \log\frac{AB+BT}{AB+AT} + \frac{T}{A}\log(1+\frac{A}{T}) - \frac{T}{B}\log(1+\frac{B}{T})
\end{aligned}$$

We show the last inequality by splitting it to two terms ; $\log\frac{AB+BT}{AB+AT} < 0$ and $\frac{T}{A}\log(1+\frac{A}{T}) - \frac{T}{B}\log(1+\frac{B}{T}) < 0$. The first term is trivial since

$$\frac{AB+BT}{AB+AT} < 1 \quad \because B < A$$

Regarding the second term, note the form of the function.

$$\frac{T}{A}\log(1+\frac{A}{T}) - \frac{T}{B}\log(1+\frac{B}{T}) < 0 \tag{7}$$

Here, terms regarding A and B have forms of $f(x) = \frac{1}{x}\log(1+x)$. By using the fact that $A > B$, if a function $f(x)$ is a decreasing function at $x > 0$, inequality (7) is satisfied. For this, we show $f'(x) < 0 \quad x > 0$,

$$\begin{aligned}
f'(x) &= -\frac{1}{x^2}\log(x+1) + \frac{1}{x(x+1)} < 0 \quad \text{multiply } x^2(x+1) \text{ in both terms} \\
f'(x) &= -(x+1)\log(x+1) + x < 0 \quad \text{where } f'(0) = 0 \\
f''(x) &= -1 - \log(x+1) + 1 < 0 \quad \forall x > 0
\end{aligned}$$

As $f''(x) < 0$, and $f'(x) = 0$ we can derive $f'(x) < 0$ for $x > 0$. This implies a function $f(x)$ is a decreasing function. Thus we have shown inequality (7) satisfies, The proof is done.

1.2.3 Proof of How HYPERREC Satisfies AXIOM 3

[AXIOM3A]: *Number of Candidates where Tail Sets are Fixed.*

Let $e'_k \subseteq_{(R)} e_k$ indicates $H'_k \subseteq T_k$ and $T'_k \subseteq H_k$.
Assume $C_i = \{e'_{i1}, e'_{i2}\}$ and $C_j = \{e'_j\}$ where

$$e'_{i1} \subseteq_{(R)} e_i \quad e'_{i2} \subseteq_{(R)} e_i \quad e'_j \subseteq_{(R)} e_j \quad T'_{i1} = T'_{i2} \quad |T'_{i1}| = |T_j|$$
$$H'_{i1} \cap H'_{i2} = \emptyset \quad \text{and} \quad |(H'_{i1} \cup H'_{i2}) \cap T_i| = |H'_j \cap T_j|$$

Then, the following inequality holds.

$$r(e_i|C_i) < r(e_j|C_j)$$

PROOF: Unlike previous cases where there exists only a single hyperarc in a candidate set, there are two arcs in a former case. Thus, cardinality penalty terms are not discarded. Two reciprocity terms $r(e_i|C_i)$ and $r(e_j|C_j)$ becomes

$$r(e_i|C_i) = \left(\frac{1}{2}\right)^{\alpha} \times \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right)$$
$$r(e_j|C_j) = \left(1 - \frac{\sum_{v_h \in H_j} \mathcal{L}((p_h, p_h^*))}{|H_j| \times \mathcal{L}_{max}}\right)$$

<u>Step A:</u> From the inequality $r(e_i|C_i) < r(e_j|C_j)$, we can derive that

$$\begin{aligned}
&\equiv r(e_i|C_i) < r(e_j|C_j) \\
&= \left(\frac{1}{2}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right) < \left(1 - \frac{\sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| \times \mathcal{L}_{max}}\right) \\
&\equiv \left(\frac{1}{2}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right) < \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right) \leq \left(1 - \frac{\sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| \times \mathcal{L}_{max}}\right)
\end{aligned}$$

The last less or equal relation can be induced because $\alpha > 0$. In sum, proof can be summarized as

$$\frac{\sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| \times \mathcal{L}_{max}} \leq \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}} \quad \text{as } |H_i| = |H_j| \text{, this can be re-written as}$$

$$\sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*) \leq \sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*) \tag{8}$$

For the target arc e_i, as $T'_{i1} = T'_{i2}$, each $p_h \quad \forall v_h \in H_i$ has a same distribution. For the target arc e_j, there is only one single candidate arc, still each $p_h \quad \forall v_h \in H_i$ has a same distribution. That is, inequality (8) can be summarized as

$$\begin{aligned}
&\equiv \sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*) \leq \sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*) \\
&\equiv |H_j \cap T'_j| * \mathcal{L}(p_{h,j}, p_{h,j}^*) \leq |H_i \cap T'_{i1}| * \mathcal{L}(p_{h,i}, p_{h,i}^*)
\end{aligned}$$

Since $e'_{i1} \subseteq_{(R)} e_i, \quad e'_{i2} \subseteq_{(R)} e_i, \quad e'_j \subseteq_{(R)} e_j$ and $|T'_j| = |T'_{i1}|$, last inequality can be written by

$$\begin{aligned}
&\equiv |H_j \cap T'_j| * \mathcal{L}(p_{h,j}, p_{h,j}^*) \leq |H_i \cap T'_{i1}| * \mathcal{L}(p_{h,i}, p_{h,i}^*) \\
&\equiv |T'_j| * \mathcal{L}(p_{h,j}, p_{h,j}^*) \leq |T'_{i1}| * \mathcal{L}(p_{h,i}, p_{h,i}^*) \\
&\equiv \mathcal{L}(p_{h,j}, p_{h,j}^*) \leq \mathcal{L}(p_{h,i}, p_{h,i}^*)
\end{aligned}$$

The proof can be done by showing the last inequality, $\mathcal{L}(p_{h,j}, p_{h,j}^*) \leq \mathcal{L}(p_{h,i}, p_{h,i}^*)$.

Step B: In order to show $\mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i})$, we should clarify the transition probability of e_i, since it is different from the previous arcs with single candidate arc. That is

$$p_{h,i}(v) = \begin{cases} \frac{1}{2|H'_{i1}|} & \text{if } v \in H'_{i1} \\ \frac{1}{2|H'_{i2}|} & \text{if } v \in H'_{i2} \\ 0 & \text{otherwise} \end{cases}$$

Since $e'_{i1} \subseteq_{(R)} e_i, \quad e'_{i2} \subseteq_{(R)} e_i, \quad e'_j \subseteq_{(R)} e_j$, probabilistic domain can be separated as

$$\begin{array}{l} v \in H'_{i1} \quad v \in H'_{i2} \quad v \in T_i \setminus \{H'_{i1} \cup H'_{i2}\} \quad v \in V \setminus T_i \quad \text{for } e_i \\ v \in H'_j \quad v \in T_j \setminus H'_j \quad v \in V \setminus T_j \quad \text{for } e_j \end{array}$$

As $|T_i| = |T_j|$, the last term is identical for both cases. By using this information, we can formally express $\mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i})$ by the below inequality. Since $|H_{i1} \cup H_{i2}| = |H_j|$ where $|H_{i1} \cap H_{i2}| = 0$, we derive $|H_{i1}| + |H_{i2}| = |H_j|$. By using it, we simplify the size of each set as follows ; $|H_j| = A, |H_{i1}| = B, |H_{i2}| = A - B$, and $|T_i| = |T_j| = T$.

$$\begin{aligned}
&\equiv \mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i}) \\
&\equiv A * \ell\left(\frac{1}{T}, \frac{1}{A}\right) + \frac{T-A}{2T}\log 2 \leq B * \ell\left(\frac{1}{T}, \frac{1}{2B}\right) + (A-B) * \ell\left(\frac{1}{T}, \frac{1}{2(A-B)}\right) + \frac{T - B - (A-B)}{2T}\log 2 \\
&\equiv A * \ell\left(\frac{1}{T}, \frac{1}{A}\right) \leq B * \ell\left(\frac{1}{T}, \frac{1}{2B}\right) + (A-B) * \ell\left(\frac{1}{T}, \frac{1}{2(A-B)}\right) \\
&\equiv \frac{A}{2T}\log\frac{2A}{A+T} + \frac{A}{2A}\log\frac{2T}{A+T} \quad \text{unfolding } \ell(p,q) \\
&\leq \frac{B}{2T}\log\frac{4B}{2B+T} + \frac{B}{4B}\log\frac{2T}{2B+T} + \frac{(A-B)}{2T}\log\frac{4(A-B)}{2(A-B)+T} + \frac{(A-B)}{4(A-B)}\log\frac{2T}{2(A-B)+T}
\end{aligned}$$

If we pull out $\log 2$ terms from both sides

$$\left(\frac{A}{2T} + \frac{A}{2A}\right)\log 2 \qquad \left(\frac{B}{2T} + \frac{B}{4B} + \frac{A-B}{2T} + \frac{B-A}{4(B-A)}\right)\log 2$$

we notice that LHS and RHS have identical terms. By erasing them, original inequality can be re-written as

$$\begin{aligned}
&\equiv \frac{A}{2T}\log\frac{A}{A+T} + \frac{A}{2A}\log\frac{T}{A+T} \\
&\leq \frac{B}{2T}\log\frac{2B}{2B+T} + \frac{B}{4B}\log\frac{T}{2B+T} + \frac{(A-B)}{2T}\log\frac{2(A-B)}{2(A-B)+T} + \frac{(A-B)}{4(A-B)}\log\frac{T}{2(A-B)+T}
\end{aligned}$$

Instead of directly comparing every term, we partially compare inequality as

$$\frac{A}{2T}\log\frac{A}{A+T} \leq \frac{B}{2T}\log\frac{2B}{2B+T} + \frac{(A-B)}{2T}\log\frac{2(A-B)}{2(A-B)+T} \tag{9}$$

$$\frac{1}{2}\log\frac{T}{A+T} \leq \frac{1}{4}\log\frac{T}{2B+T} + \frac{1}{4}\log\frac{T}{2(A-B)+T} \tag{10}$$

We first show the inequality (9). By multiplying $2T$ on both sides, we get

$$\begin{aligned}
&\frac{A}{2T}\log\frac{A}{A+T} \leq \frac{B}{2T}\log\frac{2B}{2B+T} + \frac{(A-B)}{2T}\log\frac{2(A-B)}{2(A-B)+T} \\
&\equiv A\log\frac{A}{A+T} \leq B\log\frac{2B}{2B+T} + (A-B)\log\frac{2(A-B)}{2(A-B)+T}
\end{aligned}$$

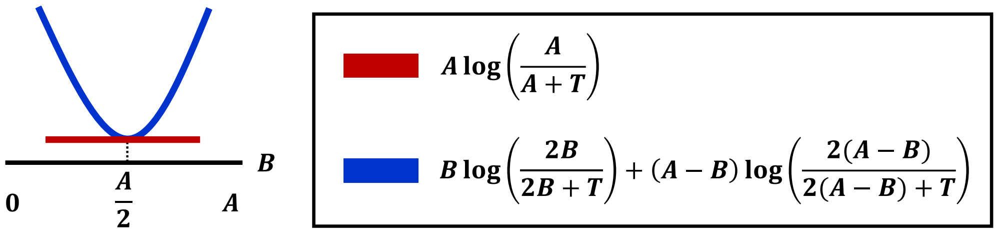


Figure 2: Result of the inequality (9)

Here, we prove this inequality by using the functional form of $f(B) = B\log\frac{2B}{2B+T} + (A-B)\log\frac{2(A-B)}{2(A-B)+T}$ where B lies in $0 < B < A$.

$$
\begin{aligned}
f(B) &= B[\log(2B) - \log(2B+T)] + (A-B)[\log(2(A-B)) - \log(2(A-B)+T)] \\
\frac{\partial f(B)}{\partial B} &= \log(2B) - \log(2B+T) + B\left(\frac{2}{2B} - \frac{2}{2B+T}\right) \\
&\quad - \log(2(A-B)) + \log(2(A-B)+T) - (A-B)\left(\frac{2}{2(A-B)} - \frac{2}{2(A-B)+T}\right)
\end{aligned}
$$

By cancelling out several identical terms, we can derive

$$
\begin{aligned}
f'(B) &= \log(2B) - \log(2B+T) - \frac{2B}{2B+T} - \log(2(A-B)) + \log(2(A-B)+T) + \frac{2(A-B)}{2(A-B)+T} \\
&\equiv \log\frac{2B}{2B+T} - \frac{2B}{2B+T} - \log\frac{2(A-B)}{2(A-B)+T} + \frac{2(A-B)}{2(A-B)+T}
\end{aligned}
$$

Note that the functional form of the last equation is $f'(B) = \log x - x - (\log y - y)$. Denote $x = \frac{2B}{2B+T}$ and $y = \frac{2(A-B)}{2(A-B)+T}$ who lie in $0 < x, y < 1$. Here, we derive the functional from of $f(B)$ by using following facts.

- If we plug in $B = \frac{A}{2}$, we have $f'(B) = 0$.
- $\log x - x$ is an increasing function for $0 < x < 1$.
- If $0 < B < \frac{A}{2}$, then $(\log y) - y > (\log x) - x$, which implies $f'(B) < 0$.
- If $\frac{A}{2} < B < A$, then $(\log x) - x > (\log y) - y$, which implies $f'(B) > 0$.

In sum, we can derive that

$$
f'(B) = \begin{cases} < 0 & \text{if } 0 < B < \frac{A}{2} \\ = 0 & \text{if } B = \frac{A}{2} \\ > 0 & \text{if } \frac{A}{2} < B < A \end{cases}
$$

Thus, we infer that $f(B)$ is a convex function at $0 < B < A$. Value of $f(B = \frac{A}{2}) = \frac{A}{2}\log\frac{A}{A+T} + \frac{A}{2}\log\frac{A}{A+T} = A\log\frac{A}{A+T}$. Rewind that our original goal is to show the below inequality.

$$
\equiv A\log\frac{A}{A+T} \leq B\log\frac{2B}{2B+T} + (A-B)\log\frac{2(A-B)}{2(A-B)+T}
$$

That is, the RHS term of inequality is a convex function that has its minimum value at $B = A/2$, and its minimum value is equal to the LHS term. Thus we can guarantee the inequality (9) is satisfied. The overview of this result is illustrated in Figure 2.

Now we show the inequality (10).

$$\begin{aligned}
&\equiv \frac{1}{2}\log\frac{T}{A+T} \leq \frac{1}{4}\log\frac{T}{2B+T} + \frac{1}{4}\log\frac{T}{2(A-B)+T} \\
&\equiv \frac{1}{4}\log\frac{T}{A+T} + \frac{1}{4}\log\frac{T}{A+T} \leq \frac{1}{4}\log\frac{T}{2B+T} + \frac{1}{4}\log\frac{T}{2(A-B)+T} \\
&\equiv \frac{1}{4}\log\frac{T^2}{(A+T)(A+T)} \leq \frac{1}{4}\log\frac{T^2}{(2B+T)(2(A-B)+T)}
\end{aligned}$$

This can be shown by comparing denominator of both terms, we can rewrite last inequality by

$$\begin{aligned}
&\equiv (2B+T)(2(A-B)+T) \leq (A+T)(A+T) \\
&= 4B(A-B) + 2AT + T^2 \leq A^2 + 2AT + T^2 \\
&\equiv A^2 - 4AB + 4B^2 = (A-2B)^2 \geq 0
\end{aligned}$$

where equality holds at $B = A/2$. As both inequality (9) and (10) hold, where equality occur at $B = A/2$ in both terms, $\mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i})$ is satisfied, and the proof is done.

[AXIOM3.B]: *Number of Candidates where Head Sets are Fixed.*

Let $e'_k \subseteq_{(R)} e_k$ indicates $H'_k \subseteq T_k$ and $T'_k \subseteq H_k$.
Assume $C_i = \{e'_{i1}, e'_{i2}\}$ and $C_j = \{e'_j\}$ where

$$\begin{gathered}
e'_{i1} \subseteq_{(R)} e_i \quad e'_{i2} \subseteq_{(R)} e_i \quad e'_j \subseteq_{(R)} e_j \quad H'_{i1} = H'_{i2} \quad |H'_{i1}| = |H_j| \\
T'_{i1} \cap T'_{i2} = \emptyset \quad |(T'_{i1} \cup T'_{i2}) \cap H_i| = |T'_j \cap H_j|
\end{gathered}$$

Then, the following inequality holds.

$$r(e_i|C_i) < r(e_j|C_j)$$

PROOF: Setting is all the same as the proof of **AXIOM3.A** until the inequality (8).
Step A: Following the proof of **AXIOM3.A**, below two inequalities are equivalent.

$$r(e_i|C_i) < r(e_j|C_j) \equiv \sum_{v_h \in H_i} \mathcal{L}(p_h, p^*_h) \leq \sum_{v_h \in H_j} \mathcal{L}(p_h, p^*_h)$$

Here, note that as $H'_{i1} = H'_{i2}$, two candidate arcs of e_i have the same transition probability. Furthermore, as $T'_{i1} \cap T'_{i2} = \emptyset$, transition probability for every $v_h \in H_i \cap \{T'_{i1} \cup T'_{i2}\}$ are identical. Thus, let transition probability of e_i in this case as $p_i(v)$, which is

$$p_i(v) = p_{h,i}(v) \quad \forall v_h \in H_i \cap \{T'_{i1} \cup T'_{i2}\}$$

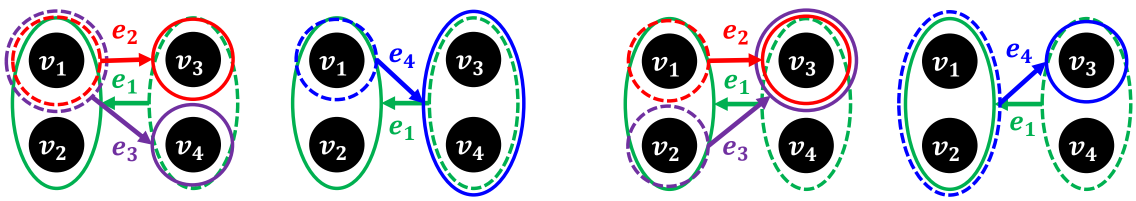
As $|H_i \cap \{T'_{i1} \cup T'_{i2}\}| = |H_j \cap T'_j|$, above final inequality can be summarized as

$$\mathcal{L}(p_{h,i}, p^*_{h,i}) \leq \mathcal{L}(p_{h,j}, p^*_{h,j})$$

Step B: Note that $e'_{i1} \subseteq_{(R)} e_i \quad e'_{i2} \subseteq_{(R)} e_i \quad e'_j \subseteq_{(R)} e_j$ and $|H'_{i1}| = |H'_j|$, probabilistic distance at e_i and e_j are identical.

$$\mathcal{L}(p_{h,i}, p^*_{h,i}) = \mathcal{L}(p_{h,j}, p^*_{h,j})$$

Although their probabilistic distance are identical, penalty term $\left(\frac{1}{|C_i|}\right)^{\alpha} = \left(\frac{1}{2}\right)^{\alpha} < 1 \quad \forall \alpha > 0$ enables the inequality $r(e_i|C_i) < r(e_j|C_j)$. The proof is done.



(a) Axiom3.A (b) Axiom3.B

Figure 3: Counterexamples for baselines. Left hypergraph of each subfigure is G_1 and right hypergraph of each subfigure is G_2.

Violations of the Axiom 3 of Baseline Measures

B1. Pearcy et al. [2]: They use clique expansion which bi-cliques a hypergraph to the weighted digraph (see related work section of the main paper). We provide a simple example of the **AXIOMS** where hypergraphs become indistinguishable after clique expansion (see Figure 3). Note that as [2] does not propose the arc level reciprocity, we compare cases of $r(G_1)$ and $r(G_2)$ where $G_1 = (V_1 = \{v_1, v_2, v_3, v_4\}, E_1 = \{e_1, e_2, e_3\})$ and $G_2 = (V_2 = \{v_1, v_2, v_3, v_4\}, E_2 = \{e_1, e_4\})$. We show the counterexample by proving $r(G_1) = r(G_2)$.

Here for Figure 3 (a), resulting clique expansion $\bar{A}_1$ and $\bar{A}_2$ becomes

$$\bar{A}_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad \bar{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

One can notice that resulting clique expanded adjacency matrices are identical. Here, perfect reciprocal hypergraphs $\bar{A}_1'$ and $\bar{A}_2'$ are also the same.

$$\bar{A}_1' = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \quad \bar{A}_2' = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

Thus, resulting reciprocity $r(G_1) = \frac{tr(\bar{A}_1^2)}{tr(\bar{A}_1'^2)}$ and $r(G_2) = \frac{tr(\bar{A}_2^2)}{tr(\bar{A}_2'^2)}$ are surely identical. That is, **AXIOM3.A** which should be $r(G_1) < r(G_2)$ has been violated.

For Figure 3 (b), resulting clique expansion $\bar{A}_1$ and $\bar{A}_2$ become

$$\bar{A}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad \bar{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Similar to the previous case, resulting clique expanded adjacency matrices are identical. Furthermore, perfect reciprocal hypergraphs $\bar{A}_1'$ and $\bar{A}_2'$ are also the same.

$$\bar{A}_1' = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad \bar{A}_2' = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Here again, resulting reciprocity $r(G_1) = \frac{tr(\bar{A}_1^2)}{tr(\bar{A}_1'^2)}$ and $r(G_2) = \frac{tr(\bar{A}_2^2)}{tr(\bar{A}_2'^2)}$ are identical. Thus, **AXIOM3.B** which should be $r(G_1) < r(G_2)$ has been violated.

In sum, B1. Pearcy et al. [2] satisfies neither **AXIOM3.A** nor **AXIOM3.B**.

B2. Ratio of Covered Pairs: Unlike the previous case, we compare hyperarc level reciprocity. Here, we show that this metric results in $r(e_1|C_1 = \{e_2, e_3\}) = r(e_1|C_1 = \{e_4\})$, which is again a violation of **AXIOM3**. For Figure 3 (a), each pair interaction can be defined as

$$R(e_1) = \{\langle v_1, v_3\rangle, \langle v_1, v_4\rangle, \langle v_2, v_3\rangle, \langle v_2, v_4\rangle\}.$$
$$R(e_2) = \{\langle v_3, v_1\rangle\} \quad R(e_3) = \{\langle v_4, v_1\rangle\} \quad R(e_4) = \{\langle v_3, v_1\rangle, \langle v_4, v_1\rangle\}$$
$$R^{-1}(e_2) = \{\langle v_1, v_3\rangle\} \quad R^{-1}(e_3) = \{\langle v_1, v_4\rangle\} \quad R^{-1}(e_4) = \{\langle v_1, v_3\rangle, \langle v_1, v_4\rangle\}$$

Thus, each reciprocity is defined as

$$r(e_1|C_1 = \{e_2, e_3\}) = \frac{2}{4} = 0.5$$
$$r(e_1|C_1 = \{e_4\}) = \frac{2}{4} = 0.5$$

In conclusion, we have shown $r(G_1) = r(G_2)$, which is a violation of the **AXIOM3.A**. Similarly, for Figure 3 (b), interaction pairs are defined as

$$R(e_1) = \{\langle v_1, v_3\rangle, \langle v_1, v_4\rangle, \langle v_2, v_3\rangle, \langle v_2, v_4\rangle\}.$$
$$R(e_2) = \{\langle v_3, v_1\rangle\} \quad R(e_3) = \{\langle v_3, v_2\rangle\} \quad R(e_4) = \{\langle v_3, v_1\rangle, \langle v_3, v_2\rangle\}$$
$$R^{-1}(e_2) = \{\langle v_1, v_3\rangle\} \quad R^{-1}(e_3) = \{\langle v_2, v_3\rangle\} \quad R^{-1}(e_4) = \{\langle v_1, v_3\rangle, \langle v_2, v_3\rangle\}$$

Similarly, each reciprocity is defined as

$$r(e_1|C_1 = \{e_2, e_3\}) = \frac{2}{4} = 0.5$$
$$r(e_1|C_1 = \{e_4\}) = \frac{2}{4} = 0.5$$

Here again, identical reciprocity values on both examples, which is a violation of the **AXIOM3.B.**. We conclude that **B2. Ratio of Covered Pairs** also violates **AXIOM3.B.**.

B5. HYPERREC w/o Size Penalty: While proving general **HYPERREC**, original inequality $r(e_i|C_i) < r(e_j|C_j)$ has been relaxed to $\mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i})$ with the help of the size penalty term $(1/|C_i|)^\alpha$. On the other hand, if we remove this size penalty term, the equality in $\mathcal{L}(p_{h,j}, p^*_{h,j}) \leq \mathcal{L}(p_{h,i}, p^*_{h,i})$ cannot guarantees the original reciprocity inequality. However, we have shown several cases where $\mathcal{L}(p_{h,j}, p^*_{h,j}) = \mathcal{L}(p_{h,i}, p^*_{h,i})$ occur, which result in the violation of **AXIOM3**. In sum, **HYPERREC** without the size penalty term $(1/|C_i|)^\alpha$ cannot satisfies the **AXIOM3**.

1.2.4 Proof of How HYPERREC Satisfies AXIOM 4

[AXIOM4]: *Bias in the Candidate Arcs*

For two target arcs e_i and e_j,

$$|C_i| = |C_j| = |T_i| = |T_j| \quad |T'_i| = |T'_j|$$
$$T'_i = H_i \ , \ H'_i \subset T_i \ , \ |H'_i| = 2, \forall e'_i \in C_i$$
$$T'_j = H_j \ , \ H'_j \subset T_j \ , \ |H'_j| = 2, \forall e'_j \in C_j$$

$$\exists u, v \in T_i \quad \text{where} \quad |\{u \in e'_i \mid e'_i \in C_i\}| \neq |\{v \in e'_i \mid e'_i \in C_i\}| \tag{11}$$
$$\forall u, v \in T_j \quad \text{where} \quad |\{u \in e'_j \mid e'_j \in C_j\}| = |\{v \in e'_j \mid e'_j \in C_j\}| \tag{12}$$

Then, the following inequality holds.

$$r(e_i|C_i) < r(e_j|C_j)$$

PROOF: This **AXIOM** implies if there exists a bias in the candidate arcs, (roughly speaking, candidate arcs are concentrated on specific nodes of the target arc only) it is less reciprocal than the case where candidate arcs are uniformly pointing the target arc's tail set nodes.

Step A: General form can be written as

$$
\begin{aligned}
&\equiv r(e_i|C_i) < r(e_j|C_j) \\
&= \left(\frac{1}{|C_i|}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right) < \left(\frac{1}{|C_j|}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| * \mathcal{L}_{max}}\right)
\end{aligned}
$$

As $|C_i| = |C_j|$, $|H_i| = |H_i|$, $T_i' = H_i$, and $T_j' = H_j$, above inequality can be summarized as

$$
\begin{aligned}
&\equiv r(e_i|C_i) < r(e_j|C_j) \\
&\equiv \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right) < \left(1 - \frac{\sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*)}{|H_j| * \mathcal{L}_{max}}\right) \\
&\equiv \sum_{v_h \in H_j} \mathcal{L}(p_h, p_h^*) < \sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*) \\
&\equiv \mathcal{L}(p_{h,j}, p_{h,j}^*) < \mathcal{L}(p_{h,i}, p_{h,i}^*)
\end{aligned}
$$

Step B: Here, we prove the **AXIOM** by showing that given e_j is an optimally reciprocal case from the **AXIOM**'s setting, and other cases are inevitably less reciprocal than it. For the target arc e_k $\forall k = i, j$, corresponding candidate arcs are having head set of size 2, and given candidate set has cardinality of $|C_k| = |T_k|$. In this setting, Eq (12) (about e_j) implies that every node in target arc's tail set $v \in T_i$ is included in candidate arcs' head set twice. Thus,

- There are $|T_j|$ number of candidate arcs, whose heat set sizes are all 2.
- Every tail set node of target arc is being involved in the candidate arc's head set twice. $|T_j| * 2$
- Every candidate arc's head set is a subset of the target arc's tail set.

From above three statements, the transition probability can be induced as

$$
p_{h,j} = \begin{cases} \frac{1}{2|T_j|} + \frac{1}{2|T_j|} = \frac{1}{|T_j|} & \text{if } v \in T_j \\ 0 & \text{otherwise} \end{cases}
$$

Note that this is identical to the optimal transition probability.

Now, consider the case of e_i. Here, as there exists at least one inequality between the number of inclusion of each target arc's tail set node to the candidate arcs' head set, the transition probability cannot be uniform as in the case of e_j.

Let's assume $v_{j1}' \in T_i$ belongs to the candidate arcs' head set $K \neq 2$ times. Then, the transition probability at v_{j1}' is defined as $p(v_{j1}') = \frac{1}{2|T_i|} * K \neq \frac{1}{|T_i|}$. This result indicates the transition probability of e_i is not optimal at all. Thus, we can guarantee the following inequality. The proof is done.

$$
\mathcal{L}(p_{h,j}, p_{h,j}^*) < \mathcal{L}(p_{h,i}, p_{h,i}^*) \quad \forall v_{h,i} \in T_i' \;\; \forall v_{h,j} \in T_j'
$$

Violations of the Axiom 4 of Baseline Measures

B2. Ratio of Covered Pairs: If every pair of target arc has been covered, then reciprocity for such a case is always equal to 1. That is, how many times each node has been pointed is indistinguishable from this metric. Here, reciprocity for both cases become $r(e_i|C_i) = r(e_j|C_j)$, which violates the **AXIOM4**.

B3. Penalized Ratio of Covered Pairs: Although size penalty has been added to **B2**, cardinality of candidate sets are identical for both cases. Thus, this also has the same issue, $r(e_i|C_i) = r(e_j|C_j)$, which violates the **AXIOM4**.

1.2.5 Proof of How HYPERREC Satisfies AXIOM 5

[AXIOM5]: *Upper and Lower Bounds of Hyperarc Reciprocity*

For every hyperarc $e_i \in E$ with $C_i \subseteq E$, following range should holds.

$$0 \leq r(e_i|C_i) \leq 1$$

PROOF: We begin from the $\mathcal{L}(p, q)$.

$$\begin{aligned}
&\equiv 0 \leq \frac{\mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}} \leq 1 \quad \because 0 \leq \mathcal{L}(p,q) \leq \log 2 \\
&\equiv 0 \leq \sum_{v_h \in H_i} \frac{\mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}} \leq |H_i| \\
&\equiv 0 \leq \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}} \leq 1 \\
&\equiv 0 \leq 1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}} \leq 1 \\
&\equiv 0 \leq \left(\frac{1}{|C_i|}\right)^{\alpha} \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right) \leq 1 \quad \because \alpha > 0
\end{aligned}$$

The proof is done.

Violations of the Axiom 5 of Baseline Measures

B4. HYPERREC w/o Normalization: If we have similar steps without $|H_i|$ in the denominator, we get

$$\begin{aligned}
&\equiv 0 \leq \frac{\mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}} \leq 1 \quad \because 0 \leq \mathcal{L}(p,q) \leq \log 2 \\
&\equiv 0 \leq \sum_{v_h \in H_i} \frac{\mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}} \leq |H_i| \\
&\equiv 0 \leq |H_i| - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}} \leq |H_i| \\
&\equiv 0 \leq \left(\frac{1}{|C_i|}\right)^{\alpha} \left(|H_i| - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{\mathcal{L}_{max}}\right) \leq |H_i| \quad \because \alpha > 0
\end{aligned}$$

We can see the reciprocity lies between $0 \leq r(e_i|C_i) \leq |H_i|$, which is a violation of the **AXIOM5**. This contains the problem that it is hard to compare different hyperarcs' reciprocity who have different head set sizes, since the hyperarc reciprocity's range depends on the size of its head set.

1.2.6 Proof of How HYPERREC Satisfies AXIOM 6

[AXIOM6]: *Inclusion of Graph Reciprocity*

Digraph reciprocity is a special case of directed hypergraph reciprocity. That is, for $G = (V, E)$ if every hyperarc's head set and tail set size are equal to 1, $|H_i| = |T_i| = 1 \ \forall e_i \in E$ (i.e., digraph), following equality should hold.

$$r(G) = \frac{|E^{\leftrightarrow}|}{|E|} \quad \text{where } E^{\leftrightarrow} = \{e_i \mid \exists e_k = \langle H_k = T_i, T_k = H_i \rangle, (e_i, e_k \in E)\} \tag{13}$$

Note that this reciprocity measure is identical to the normal digraph reciprocity [3, 4].

PROOF: Hypergraph level reciprocity of **HYPERREC** is defined as

$$r(G) = \frac{1}{|E|}\sum_{i=1}^{|E|} r(e_i \mid C_i)$$

As every hyperarc's head set size is equal to 1, each $r(e_i|C_i)$ can be written as

$$r(e_i|C_i) = \max_{C_i \subseteq E} \left(\frac{1}{|C_i|}\right)^{\alpha} (\mathcal{L}(p_h, p_h^*)) \quad \text{where } \{v_h\} = H_i$$

Here, the optimal transition probability is defined as

$$p_h^*(v) = \begin{cases} 1 & \text{if } \{v\} = T_i \\ 0 & \text{otherwise} \end{cases}$$

One can notice that only one arc is required for the candidate set since there is no partially covered arc (i.e., $T_i' \subset H_i,\ H_i' \subset T_i$ does not exist), and we only need to decide whether an arc inversely-overlap or not, because max term will filter arcs which are not necessary.

For $C_i = \{e_k\}$ where $T_k \neq H_i$, by the **A. 4**, reciprocity for such hyperarc gets zero.

For $C_i = \{e_k\}$ where $T_k = H_i$, transition probability for the target arc e_i is defined as

$$p_h^*(v) = \begin{cases} 1 & \text{if } \{v\} = H_k \\ 0 & \text{otherwise} \end{cases}$$

Here, the transition probability and the optimal transition probability become identical if and only if $H_k = T_i$. Otherwise, as their non-zero probability domain does not overlap, reciprocity gets zero.

In sum, $r(e_i|C_i)$ is formally re-written as

$$r(e_i|C_i) = \begin{cases} 1 & \text{if } H_k = T_i \text{ and } T_k = H_i \\ 0 & \text{otherwise} \end{cases} \quad \text{where } C_i = \{e_k\}$$

Thus, $r(e_i|C_i)$ works as an indicator function which gives a value of 1 if there exists its inverse-pair, $\exists e_k \in E$ where $e_k = \langle H_k = T_i, T_k = H_i \rangle$ and zero elsewhere. Formally,

$$\begin{aligned} r(G) &= \frac{1}{|E|}\sum_{i=1}^{|E|} r(e_i|C_i) \\ &= \frac{1}{|E|}\sum_{i=1}^{|E|} \mathbb{1}((\exists e_k = \langle H_k = T_i, T_k = H_i \rangle), (e_k \in E)) \quad \text{where } \mathbb{1}(\text{TRUE}) = 1 \quad \mathbb{1}(\text{FALSE}) = 0 \\ &= \frac{|E^{\leftrightarrow}|}{|E|} \end{aligned}$$

which is identical to the digraph reciprocity measure. The proof is done.

Violations of the Axiom 6 of Baseline Measures

B1. Pearcy et al. [2]: We provide a simple counterexample. See the case illustrated in Figure 4

Digraph reciprocity for Figure 4 can be computed as

$$E = \{e_1, e_2, e_3\}, \quad E^{\leftrightarrow} = \{e_1, e_2\} \quad \rightarrow \quad r(G) = \frac{2}{3}$$

In this case, clique expanded adjacency matrix of actual data and its optimal case are

$$\bar{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \bar{A}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

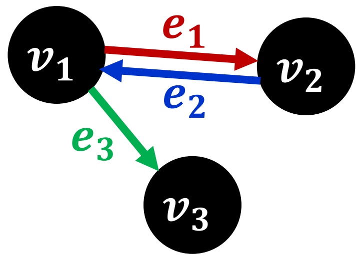


Figure 4: Counterexample why certain baselines fail in **AXIOM 6**.

The reciprocity becomes $\frac{2}{4} = 0.5$ as $tr(\bar{A}^2) = 2$ and $tr(\bar{A}'^2) = 4$. Since $\frac{2}{4} \neq \frac{2}{3}$, **B1. Pearcy et al. [2]** violates the **AXIOM 6**.

B6. HYPERREC with All Arcs as Candidates: As we put all the hyperarcs in C_i, this measure cannot achieves hyperarc reciprocity of 1 even in the case where there exists its perfectly reciprocal opponent.

In Figure 4, although e_2 has its perfectly reciprocal arc i.e., e_1, it takes e_3 also in consideration, which results in the transition probability of $p_1 = [v_1, v_2, v_3, v_{sunk}] = [0, 0.5, 0.5, 0]$. Here, final reciprocity becomes $r(e_2|C_2 = \{e_1, e_3\}) = 0.7842 \neq 1$, which implies this metric cannot works as a proper indicator function. Thus, **AXIOM 6** cannot be achieved.

1.2.7 Proof of How HYPERREC Satisfies AXIOM 7

[AXIOM7]: *Upper and Lower Bounds of Hypergraph Reciprocity*

The reciprocity of any hypergraph should be within a fixed range. Specifically, for any hypergraph G, $0 \leq r(G) \leq 1$ should hold.

PROOF: Hypergraph level reciprocity of **HYPERREC** is defined as

$$r(G) = \frac{1}{|E|} \sum_{i=1}^{|E|} r(e_i|C_i)$$

Here, by the **AXIOM5**, we verify that an arbitrary hyperarc reciprocity $r(e_i|C_i)$ lies in $[0, 1]$. From this fact, we can derive that

$$\begin{aligned}
&0 \leq r(e_i|C_i) \leq 1 \quad \forall i = \{1, \cdots, |E|\} \\
&0 \leq \sum_{i=1}^{|E|} r(e_i|C_i) \leq |E| \\
&0 \leq \frac{1}{|E|} \sum_{i=1}^{|E|} r(e_i|C_i) \leq 1
\end{aligned}$$

The proof is done.

1.2.8 Proof of How HYPERREC Satisfies AXIOM 8

[AXIOM8]: *Surjection of Reciprocity*

The maximum reciprocity, which is 1 by the **AXIOM7**, should be reachable from any hypergraph by adding specific hyperarcs. That is, for every $G = (V, E)$, there exists $G^* = (V, E^*)$ with $E \subseteq E^*$ such that $r(G^*) = 1$.

PROOF: From an arbitrary hypergraph G, let a subset of hyperarcs whose perfect reciprocal opponents do not exist in the same hypergraph i.e., $E^{'} = \{e_k \mid (\langle T_k, H_k\rangle \notin E), (e_k \in E)\}$. Let a set of additional hyperarcs that consist of perfectly reciprocal opponents of E' as E^{opt}.

$$E^{opt} = \bigcup_{e_k \in E'} \langle T_k, H_k \rangle$$

Here, let $E^* = E \cup E^{opt}$. For E^*, there always exists perfectly reciprocal opponent for each element (hyperarc).

$$\text{Let } C_i = \{\langle T_i, H_i \rangle\} \quad \text{Then} \quad r(e_i|C_i) = \left(1 - \frac{\sum_{v_h \in H_i} \mathcal{L}(p_h, p_h^*)}{|H_i| * \mathcal{L}_{max}}\right)$$
$$= \left(1 - \frac{0 + \cdots + 0}{|H_i| * \mathcal{L}_{max}}\right) = 1 \quad \because p_h = p_h^*$$

As $C_i = \{\langle T_i, H_i \rangle\}$, the target arc e_i can always achieves maximum hyperarc reciprocity. In addition, the corresponding candidate set can always be found from E^* due to the max searching. As every hyperarc reciprocity $r(e|C)$ is equal to 1, its hypergraph reciprocity $r(G)$ is also 1. The proof is done.

Violations of the Axiom 8 of Baseline Measures

B6. HYPERREC with All the Arcs as Candidates: Even though there exists a perfect reciprocal opponent of a specific hyperarc, transition probability cannot be identical to the optimal transition probability if there exist another inversely overlapping hyperarcs (see target arc e_2's case in Figure 4).

B7. HYPERREC with Inversely Overlapping Arcs as Candidates: Similar to the previous case, if there exist multiple inversely overlapping hyperarcs, it inevitably uses all of them as a candidate set. As a result, for such hyperarcs, cardinality penalty gets smaller than 1 i.e., $(1/|C_i|)^{\alpha} < 1$, resulting in $r(e_i|C_i) < 1$. Consequently, overall hypergraph reciprocity becomes smaller than 1, which cannot be fully reciprocal.

2 Data Description

In this section, we provide the sources of the considered datasets and describe how we preprocess them.

Metabolic datasets: We use two metabolic hypergraphs, **iAF1260b** and **iJO1366**, which are provided by Yadati et al. [5]. They are provided in the form of directed hypergraphs, and they do not require any pre-processing. We remove one hyperarc from each dataset since their head set or tail set is abnormally large. Specifically, the size of their head sets is greater than 20, while the second largest one is 8. Each node corresponds to a gene, and each hyperarc indicates a metabolic reaction among them. Specifically, a hyperarc e_i indicates that a reaction among the genes in the tail set T_i results in the genes in the head set H_i.

Email datasets: We use two email hypergraphs, **email-enron** and **email-eu**. The **Email-enron** dataset is provided by Chodrwo et al. [6]. We consider each email as a single hyperarc. Specifically, the head set is composed of the receiver(s) and cc-ed user(s), and the tail set is composed of the sender. The **Email-eu** dataset is from SNAP [7]. The original dataset is a dynamic graph where each temporal edge from a node u to a node v at time t indicates that u sent an email to v at time t. The edges with the same source node and timestamp are replaced by a hyperarc, where the tail set consists only of the source node and the head set is the set of destination nodes of the edges. Note that every hyperarc in these datasets has a unit tail set, i.e., $|T_i| = 1, \forall i = \{1, \cdots, |E|\}$.

Citation datasets: We use two citation hypergraphs, **citation-data mining** and **citation-software**, which we create from pairwise citation networks, as suggested by Yadati et al. [8]. Nodes are the authors of publications. Assume that a paper A, which is co-authored by $\{v_1, v_2, v_3\}$, cited another paper B, which is co-authored by $\{v_4, v_5\}$. Then, this citation leads to a hyperarc where the head set is $\{v_4, v_5\}$ and the tail set is $\{v_1, v_2, v_3\}$. As pairwise citation networks, we use subsets of a DBLP citation dataset [9]. The subsets consist of papers published in the venues of data mining and software engineering, respectively.[1] In addition, we filter out all papers co-authored by more than 10 authors to minimize the impact of such outliers.

Question answering datasets: We use two question answering hypergraphs, **qna-math** and **qna-server**. We create directed hypergraphs from the log data of a question answering site, *stack exchange*, provided at [11]. Among various

[1] We use the venues listed at [10].

Table 1: Hypergraph reciprocity $r(G)$ of 11 datasets with $\alpha \approx 0$, 0.5 and 1.

		metabolic		email		citation		qna		bitcoin		
		iAF1260b	iJO1366	enron	eu	data mining	software	math	server	2014	2015	2016
$r(G)$	$\alpha \approx 0$	21.455	22.533	59.001	79.416	12.078	15.316	9.608	13.219	10.829	6.923	3.045
	$\alpha = 0.5$	17.756	18.497	49.321	65.405	10.840	13.984	9.283	13.196	10.654	6.845	2.988
	$\alpha = 1.0$	16.654	17.385	44.299	58.069	10.585	13.704	9.236	13.193	10.606	6.828	2.977

Table 2: Hypergraph reciprocity $r(G)$ and the choice of α. Although their absolute value may differ as shown in Table 1, their ranks and linear relation among different datasets are maintained, since all of the measured Pearson correlation and Spearman rank correlation are greater than 0.99.

		Pearson Correlation	**Spearman Rank Correlation**
$r(G)$	$\alpha :\approx 0 \leftrightarrow 0.5$	0.999	1.0
	$\alpha :\approx 0 \leftrightarrow 1.0$	0.999	0.991
	$\alpha : 0.5 \leftrightarrow 1.0$	0.999	0.991

domains, we choose *math-overflow*, which covers mathematical questions, and *server-fault*, which treats server related issues. The original log data contains the posts of the site, and one questioner and one or more answerers are involved with each post. We ignore all posts without any answerer. We treat each user as a node, and we treat each post as a hyperarc. For each hyperarc, the questioner of the corresponding post composes the head set, and the answerer(s) compose the tail set. Note that every hyperarc in these datasets has a unit head set, i.e., 1 $|H_i| = 1, \forall i = \{1, \cdots, |E|\}$.

Bitcoin transaction dataset: We use three bitcoin transaction hypergraphs, **bitcoin-2014**, **bitcoin-2015**, and **bitcoin-2016**. The original datasets are provided by Wu et al. [12], and they contain first 1,500,000 transactions in 11/2014, 06/2015, and 01/2016, respectively. We model each account as a node, and we model each transaction as a hyperarc. As multiple accounts can be involved in a single transaction, the accounts from which the coins are sent compose the tail set, and the accounts to which the coins sent compose the head set. We remove all transactions where the head set and the tail set are exactly the same.

3 Experiments

In this section, we provide full experimental results, which are omitted in the main paper due to the space limit.

3.1 Reciprocity and the Choice of α

In the main paper, we demonstrate that although absolute reciprocity values vary depending on α i.e., size penalty scalar of candidate sets (see Table 1), their ranks in real-world hypergraphs remain almost the same. In addition to the five data reported in the main paper, we do the same process with omitted six datasets. This phenomenon is maintained for the entire data, supported by the fact that both the Pearson correlation and Spearman rank correlation value among different αs are near 1 (see Table 2).

To investigate a similar characteristic on hyperarc level reciprocity, we measure the Pearson and Spearman rank correlation between hyperarc reciprocity $r(e|C)$ vectors within the same dataset but with different values of α. Although the absolute value of each hyperarc reciprocity may differ, their general tendency remains similar since measured correlation statistics are greater than 0.9, except for the email dataset (see Table 3).

In sum, we observe that both hypergraph reciprocity $r(G)$ and hyperarc reciprocity $r(e|C)$ are robust to the choice of α (the general tendency is maintained), although their absolute value may vary concerning it.

3.2 Observation 1

In the main paper, we report that the hypergraph reciprocity is several orders of magnitude greater in real-world hypergraphs than in corresponding null hypergraphs (refer to the main paper for information related to the null hypergraph). To statistically verify this statement, we perform Z-test as follows.

Table 3: Hyperarc reciprocity $r(e|C)$ and the choice of α. Although their absolute value may differ as shown in the Table 1, their ranks and linear relation within dataset are maintained, since all of the measured Pearson correlation and Spearman rank correlation are greater than 0.99.

		metabolic		email		citation		qna		bitcoin		
		iAF1260b	iJO1366	enron	eu	data mining	software	math	server	2014	2015	2016
Pearson	$\alpha :\approx 0 \leftrightarrow 0.5$	0.961	0.957	0.928	0.836	0.984	0.986	0.994	0.999	0.997	0.998	0.997
	$\alpha :\approx 0 \leftrightarrow 1.0$	0.916	0.913	0.828	0.678	0.973	0.977	0.992	0.999	0.995	0.997	0.996
	$\alpha : 0.5 \leftrightarrow 1.0$	0.985	0.986	0.975	0.967	0.998	0.998	0.999	0.999	0.999	0.999	0.999
Spearman Rank	$\alpha :\approx 0 \leftrightarrow 0.5$	0.973	0.969	0.947	0.817	0.998	0.998	0.999	0.999	0.999	0.999	0.999
	$\alpha :\approx 0 \leftrightarrow 1.0$	0.925	0.918	0.869	0.721	0.996	0.996	0.999	0.999	0.999	0.999	0.999
	$\alpha : 0.5 \leftrightarrow 1.0$	0.975	0.973	0.978	0.983	0.999	0.999	0.999	0.999	0.999	0.999	0.999

- We create 30 randomized hypergraphs, $r(G_{null,i})\ \forall i = \{1, \cdots 30\}$ for each dataset.
- With created 30 randomized hypergraphs, we measure an average of null hypergraph reciprocity values $\overline{r(G_{null})}$ and a standard deviation of them $sd(r(G_{null}))$.
- By using the central limit theorem, we approximate an Eq. (14) to the standard normal distribution, where $r(G)$ is a real-world hypergraph's reciprocity, and $\xrightarrow{d}$ indicates a convergence in distribution.

$$Z_{(real-null)} = \frac{\overline{r(G_{null,obs})} - r(G)}{\frac{sd(r(G_{null}))}{\sqrt{30}}} \xrightarrow{d} N(0,1) \tag{14}$$

- At last, we perform hypothesis test where its null hypothesis μ_0 and alternative hypothesis μ_a are ;

μ_0 : Null hypergraphs' average hypergraph reciprocity is identical to the real-world hypergraph's average reciprocity ;
μ_a : Null hypergraphs' average hypergraph reciprocity is smaller than real-world hypergraph's average reciprocity ;

$$\mu_0 : \overline{r(G_{null})} = r(G)$$
$$\mu_a : \overline{r(G_{null})} < r(G)$$

- From the computed $Z_{(real-null)}$, we can derive the p-value as $P(z < Z_{(real-null)})$ where $P(z)$ is a cumulative probability distribution of standard normal distribution.

We set the significance level of the testing as 0.01, which is a more rigid standard than the common setting of 0.05. For all the cases, we adopt the alternative hypothesis, which implies $\overline{r(G_{null})}$ is statistically-significantly smaller than $\overline{r(G)}$. Detailed numerical results of the test are illustrated in Table 4. In sum, we discover that the hypergraph reciprocity is statistically-significantly greater in real-world hypergraphs than in corresponding null hypergraphs.

Table 4: P-value testing result from 11 datasets. Null hypotheses are all rejected, which implies real-world hypergraphs are significantly more reciprocal than null hypergraphs. A p-value smaller than 0.00001 is marked as 0.0000^{*}

	metabolic		email		citation		qna		bitcoin		
	iAF1260b	iJO1366	enron	eu	data mining	software	math	server	2014	2015	2016
Z-stat	-1502.52	-1789.79	-241.13	-3835.98	-17548.20	-9605.19	-8884.98	-88965.12	-691316.77	-555709.95	-325308.06
P-value	0.0000^{*}	0.0000^{*}	0.0000^{*}	0.0000^{*}	0.0000^{*}	0.0000^{*}	0.0000^{*}	0.0000^{*}	0.0000^{*}	0.0000^{*}	0.0000^{*}
Null hypothesis	Reject	Reject	Reject	Reject	Reject	Reject	Reject	Reject	Reject	Reject	Reject

3.3 Observation 2

In the main paper, we discover that in real-world hypergraphs, hyperarcs of non-zero reciprocity tend to have a higher head set out-degree and tail set in-degree than zero reciprocity hyperarcs. Here, head set out-degree and tail set in-degree are defined as

$$\text{Head set out-degree: } d_{H,out}(e_i) = \frac{1}{|H_i|} \sum_{v \in H_i} d_{out}(v) \tag{15}$$

$$\text{Tail set in-degree: } d_{T,in}(e_i) = \frac{1}{|T_i|} \sum_{v \in T_i} d_{in}(v) \tag{16}$$

From most of the dataset (except for the tail set in-degree distribution of question answering server dataset), we can observe the tendency that real-world hypergraphs' non-zero reciprocity hyperarcs are having a higher degree than zero reciprocity hyperarcs. Such a phenomenon is not clear in null hypergraphs (see Figure 5).

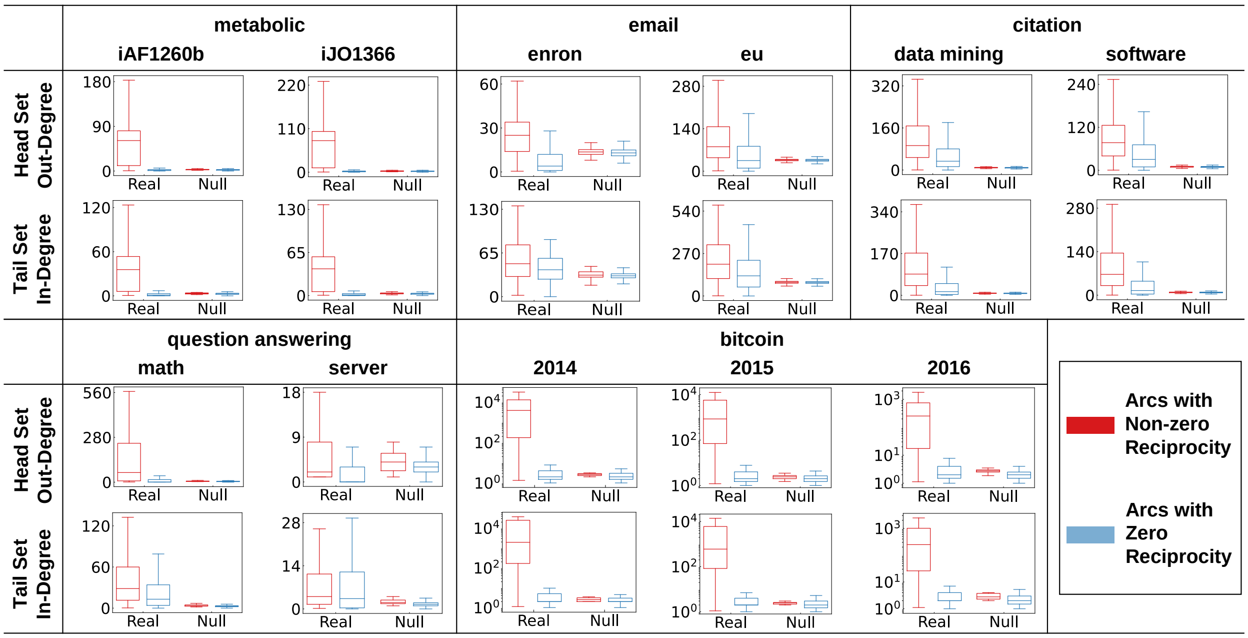


Figure 5: Result of observation 2 in the entire dataset: Except for the question answering server dataset, arcs with non-zero reciprocity have the higher head set out-degree and tail set in-degree than arcs with zero reciprocity. Such phenomenon is not observed from null randomized hypergraphs.

3.4 Observation 3

In the main paper, we discover that in real-world hypergraphs, nodes with balanced in-degree and out-degree tend to be involved in high reciprocity hyperarcs. To quantify the degree balance of nodes, we measure the following statistic for each node $v \in V$,

$$Bal(v) = log(d_{in}(v)) - log(d_{out}(v))$$

If $Bal(v) \approx 0$, a node v's in degree and out degree is similar. On the other hand, $Bal(v) \to -\infty$ implies a node v is having higher out-degree than in-degree. Inversely, for the case of $Bal(v) \to +\infty$, a node v has a higher in-degree than out-degree. As the arc reciprocity is defined on the hyperarc level, we map each hyperarc reciprocity to nodes by

$$r(v) = \frac{1}{|E_v|} \sum_{e_k \in E_v} r(e_k)$$

where $E_v = \{e_k : v \in (H_k \cup T_k)\}$ is the set of hyperarcs where v is included in its head set or tail set.

We draw the line plot with $Bal(v)$ and $r(v)$, where each coordinate of the plot becomes $(x, y) = (Bal(v), r(v))\ \forall i = \{1, \cdots, |V|\}$. For x values with multiple y values, i.e., for $u, v \in V$, $Bal(v) = Bal(u), \quad r(v) \neq r(u)$, we use an average of their reciprocity values. That is, for a specific x coordinate x_j, the corresponding y coordinate is

$$y_j = \frac{1}{|V_j'|} \sum_{v \in V_j'} r(v) \quad V_j' = \{v_k \in V \mid Bal(v_k) = x_j\}$$

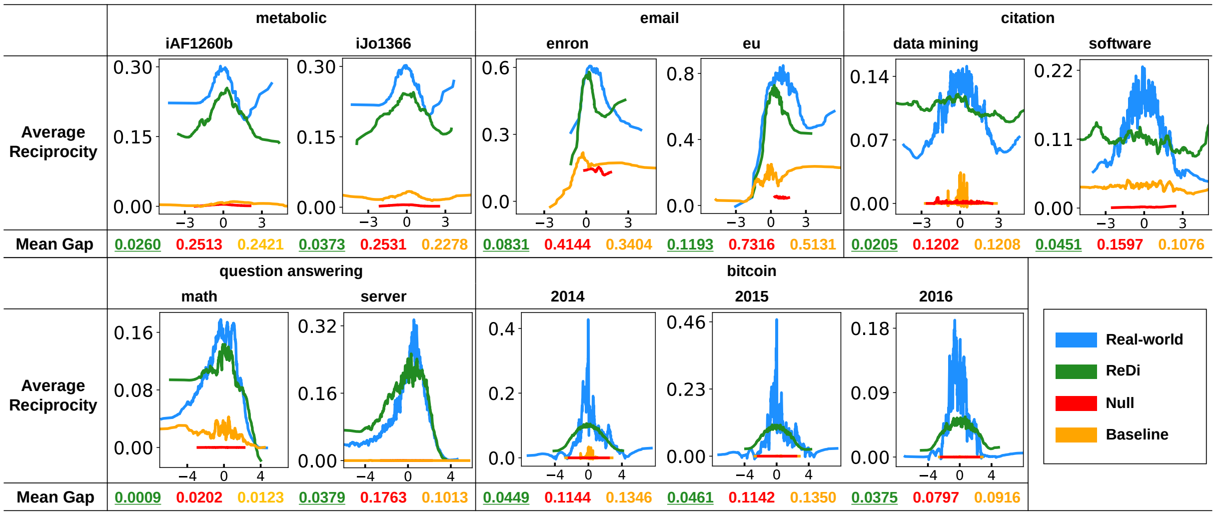


Figure 6: Result of observation 3 in the entire dataset of real-world, **REDI**, null, and baseline: For real-world hypergraphs, we can observe bell-shaped plots which have the highest point around zero x value, which implies nodes with a balanced degree tend to be involved in high reciprocity hyperarcs. Hypergraphs generated by **REDI** show similar patterns for most of the data except for the citation dataset. However, null hypergraphs and baseline hypergraphs do not show such a tendency.

At last, we apply the Savitzky–Golay filter [13] for smoothing curves. As shown in Figure 6, we can observe that real-world hypergraphs tend to show bell-shaped plots which have their maximum point around zero x value (see blue lines). This indicates the node level reciprocity gets larger as nodes' in- and out-degrees become balanced i.e., $Bal(v) \approx 0$. On the other hand, such a pattern is ambiguous in null hypergraphs.

3.5 REDI's Reproducibility of Observation 2

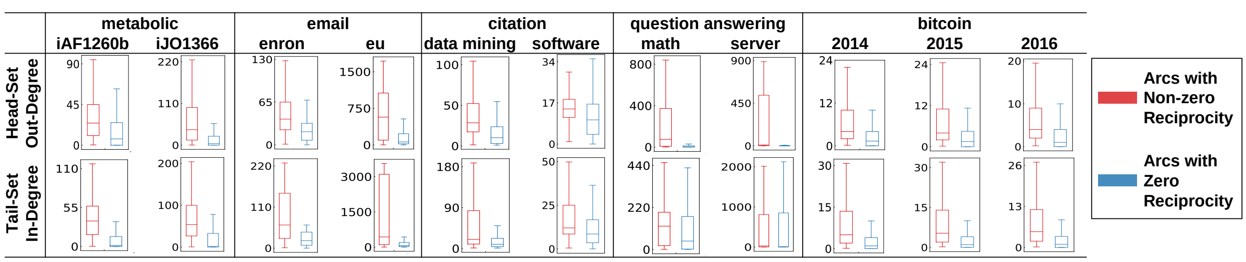


Figure 7: Result of observation 2 in generated hypergraphs by **REDI**. Similar to real-world hypergraphs, non-zero reciprocity hyperarcs tend to have a higher head set out-degree and tail set in-degree than zero reciprocity hyperarcs in most of the datasets.

Part of the generation result of the proposed generator **REDI** regarding observation 2 is reported in the main paper. We examine how much reciprocal pattern regarding observation 2 has been maintained in **REDI** from the entire dataset. As shown in Figure 7, reciprocal patterns of observation 2 have been maintained in most of the data, except for the citation software dataset where its tendency of a head set out-degree is a little unclear.

One interesting fact is that tail set in-degree distribution of the real-world question answering server dataset is similar between non-zero reciprocity hyperarcs and zero reciprocity hyperarcs (see Figure 5). This phenomenon also appears in the generated question answering server dataset (see Figure 7).

3.6 **REDI**'s Reproducibility of Observation 3

Part of the generation result of the proposed generator **REDI** regarding observation 3 is reported in the main paper. Similar to the previous subsection, we investigate how much reciprocal pattern regarding observation 3 has been maintained in **REDI**'s hypergraph at the entire dataset. General bell-shaped tendency is also observed in the **REDI**'s hypergraph from most of the dataset (except for the citation network), while null hypergraphs and baseline generator fail in capturing this characteristic.

To quantitatively compare three methods' (**REDI**, null, and baseline) reproducibility regarding observation 3, we measure the *mean gap* between plots of real-world hypergraphs and generated hypergraphs. *mean gap* is defined as follows. Denote a specific hypergraph G's plot's x coordinate as X_G and y coordinate as Y_G. Then,

$$X_{G_{real}} = [x_1, x_2, \cdots, x_K], \quad Y_{G_{real}} = [y_1, y_2, \cdots, y_K]$$
$$X_{G_{syn}} = [x'_1, x'_2, \cdots, x'_M], \quad Y_{G_{syn}} = [y'_1, y'_2, \cdots, y'_M]$$
$$X' = X_{G_{real}} \cap X_{G_{syn}} \quad Y'_{real} = \{y_k \in Y_{G_{real}} \mid x_k \in X'\} \quad Y'_{syn} = \{y'_k \in Y_{G_{syn}} \mid x'_k \in X'\}$$
$$Y'_{REAL} = \text{Vector of } Y'_{real}\text{, which is sorted according to each element's corresponding x-value}$$
$$Y'_{SYN} = \text{Vector of } Y'_{syn}\text{, which is sorted according to each element's corresponding x-value}$$
$$\textit{mean gap} = \frac{1}{|Y'_{REAL}|} \sum_{i=1}^{|Y'_{REAL}|} (Y'_{REAL,i} - Y'_{SYN,i})^2$$

This *mean gap* measures the average vertical distance between real-world hypergraphs' degree balance plots and generated hypergraphs' degree balance plots. Among three methods, **REDI** shows the minimum distance from the real-world hypergraph, which implies that **REDI** captures the general tendency most well among three generation methods (see numeric values in Figure 6).

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