Moving-average (MA) model

Model with finite time lags of memory!

Some daily stock returns have minor serial correlations. Can be modeled as MA or AR models.

MA(1) model

1. Form: $r_t = \mu + a_t - \theta a_{t-1}$

2. Stationarity: always stationary.

3. Mean (or expectation): $E(r_t) = \mu$.

4. Variance: $Var(r_t) = (1 + \theta^2)\sigma^2$.

5. Autocovariance:

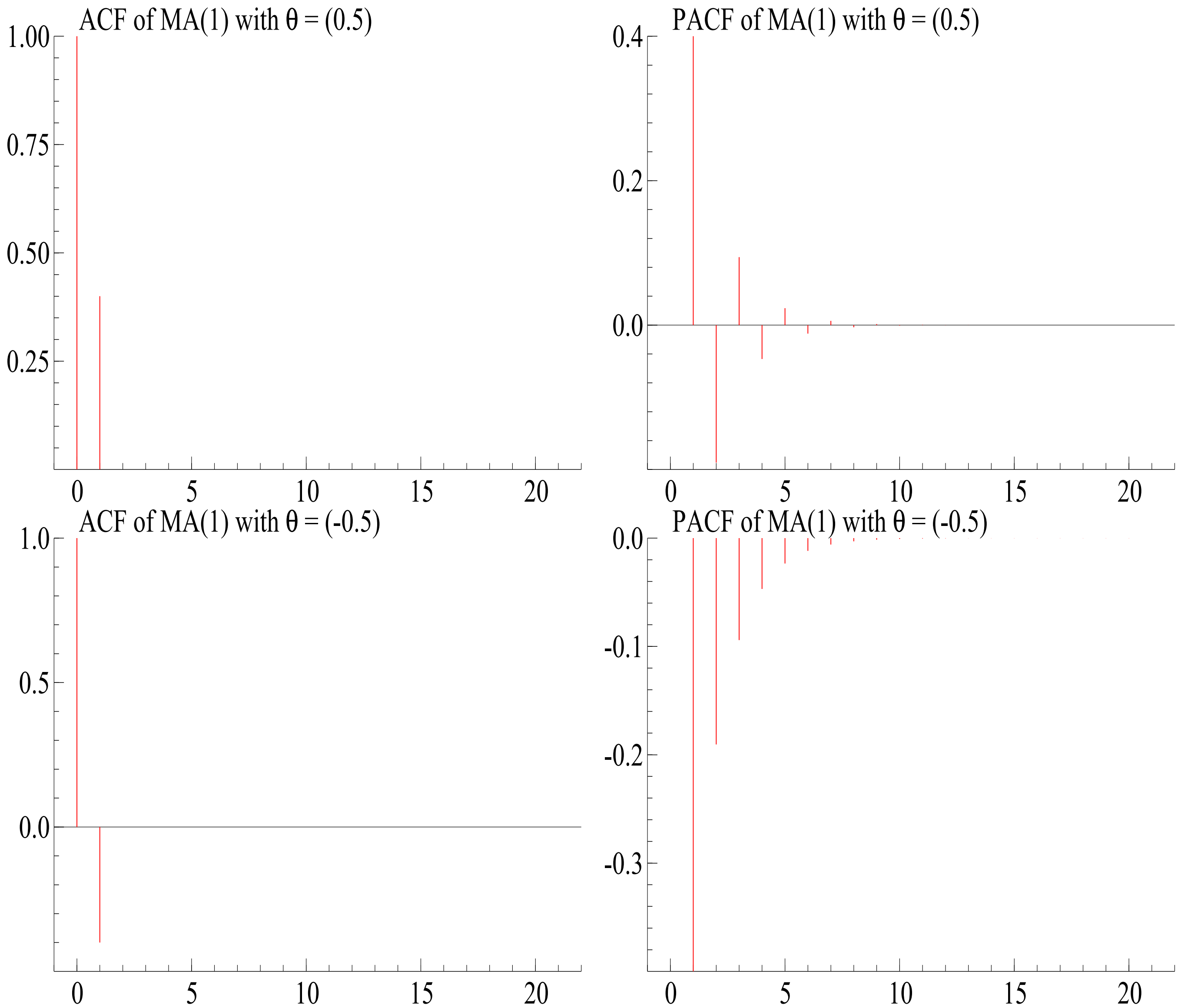
(a). Lag 1: $Cov(r_t, r_{t-1}) = -\theta\sigma^2$.

(b). Lag l: $Cov(r_t, r_{t-l}) = 0$ for $l > 1$.

Thus, r_t is not related to r_{t-2}, r_{t-3}, $\cdots$.

ACF: $\rho_1 = \frac{-\theta}{1+\theta^2}$, $\rho_l = 0$ for $l > 1$.

Finite memory! MA(1) models do not remember what happen two time periods ago.



ACF and PACF for MA(1) model

6. **Forecast (at origin $t = n$)**:

(a). 1-step ahead: $\hat{r}_n(1) = \mu - \theta a_n$. Why? Because at time n, a_n is known, but a_{n+1} is not.

(b). 1-step ahead forecast error: $e_n(1) = a_{n+1}$ with variance σ_a^2.

(c). Multi-step ahead: $\hat{r}_n(l) = \mu$ for $l > 1$.

Thus, for an MA(1) model, the multi-step ahead forecasts are just the mean of the series. Why? Because the model has memory of 1 time period.

(d). Multi-step ahead forecast error:

$$e_n(l) = a_{n+l} - \theta a_{n+l-1}.$$

(e). Variance of multi-step ahead forecast error:

$$(1 + \theta^2)\sigma_a^2 = \text{ variance of } r_t.$$

7. **Invertibility**:

Concept: r_t is a proper linear combination of a_t and the past observations $\{r_{t-1}, r_{t-2}, \cdots\}$.

Why is it important? It provides a simple way to obtain the shock a_t.

For an invertible model, the dependence of r_t on r_{t-l} converges to zero as l increases.

MA(1) model with condition $|\theta| < 1$:

$$a_t = (r_t - \mu) + \sum_{i=1}^{\infty} \theta^i (r_{t-i} - \mu)$$

or AR(∞) model

$$r_t = \frac{\mu}{1-\theta} + a_t - \sum_{i=1}^{\infty} \theta^i r_{t-i}.$$

Invertibility of MA models is the dual property of stationarity for AR models.

MA(2) model

1. Form: $r_t = \mu + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$, or

$$r_t = \mu + (1 - \theta_1 B - \theta_2 B^2) a_t.$$

2. Stationary with $E(r_t) = \mu$.

3. Invertibility:

all the roots of $\theta(z) = 1 - \theta_1 z - \theta_2 z^2 = 0$ lie outside the unit circle.

Decompose $1 - \theta_1 z - \theta_2 z^2 = (1 - \alpha_1 z)(1 - \alpha_2 z)$.

Then $|\alpha_1| < 1$ and $|\alpha_2| < 1$.

$$r_t = \mu + (1 - \alpha_1 B)(1 - \alpha_2 B) a_t.$$

Let $u_t = (1 - \alpha_2 B) a_t$. Then

$$r_t = \mu + u_t - \alpha_1 u_{t-1} \; and \; u_t = a_t - \alpha_2 a_{t-1}.$$

Given $\{r_t\}$, we can invert $\{u_t\}$ and then invert $\{a_t\}$.

4. Variance: $Var(r_t) = (1 + \theta_1^2 + \theta_2^2)\sigma_a^2$.

ACF: $\rho_1 \neq 0$ and $\rho_2 \neq 0$, but $\rho_l = 0$ for $l > 2$.

5. Forecasts: go the the mean after 2 periods.

MA(q) model

Form: $r_t = \mu + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}$, or

$$r_t = \mu + (1 - \theta_1 B - \theta_2 B^2 \cdots - \theta_q B^q) a_t.$$

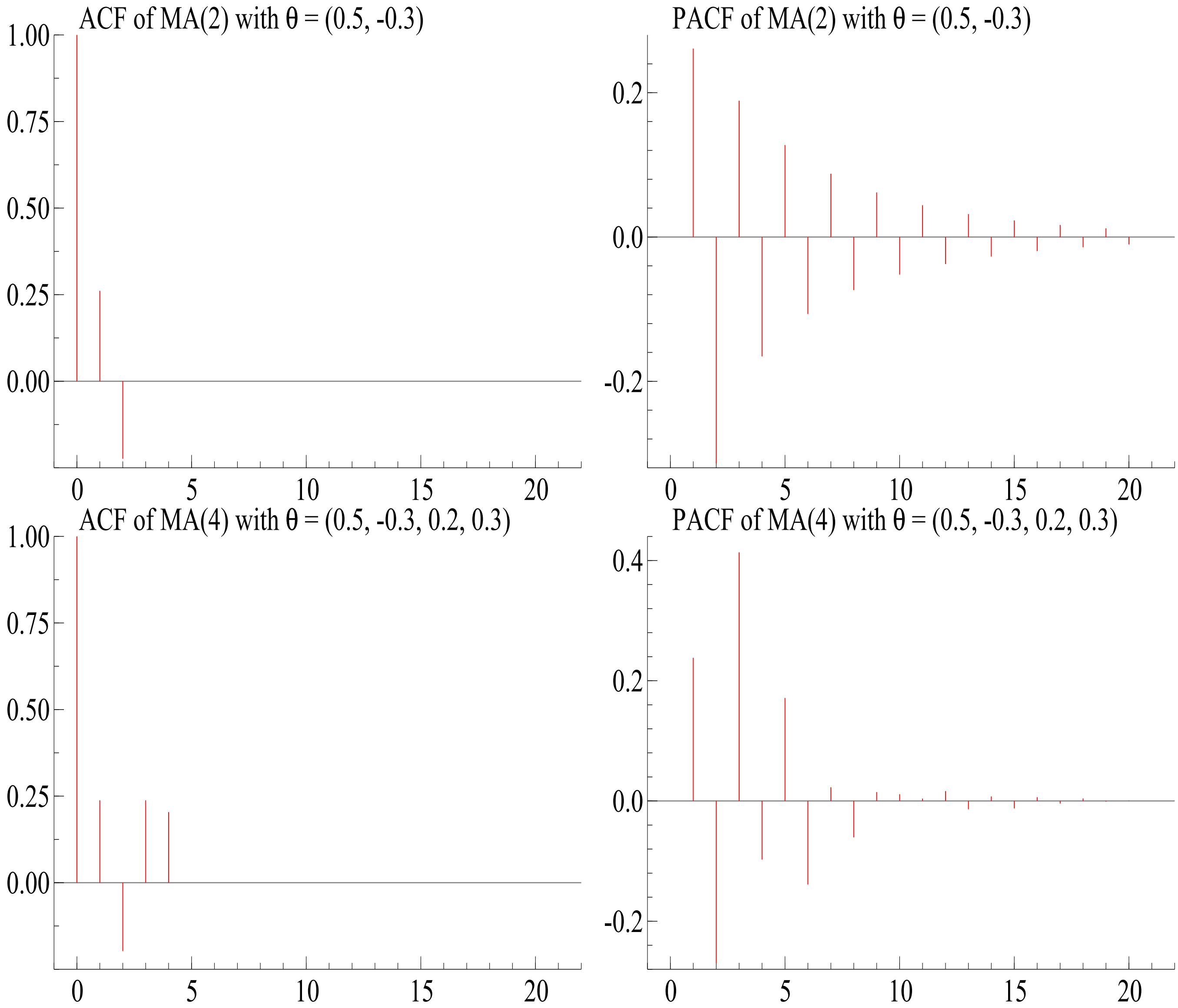
Invertibility:

all the roots of $\theta(z) = 1 - \theta_1 z - \theta_2 z^2 - \cdots - \theta_q z^q = 0$ lie outside the unit circle.

Building an MA model

1. Specification order q: Use sample ACF

Sample ACFs are all small after lag q for an $MA(q)$ series. (See test of ACF.)



ACF and PACF for MA(p) model

Constant term? Check the sample mean.

2. Estimation:

Assume that $\{r_1, \cdots, r_n\}$ is from the MA(1) model:

$$r_t = -\theta_0 a_{t-1} + a_t \ \ with \ \ |\theta_0| < 1.$$

Conditional LSE– minimizer of

$$S_n(\theta) = \sum_{t=1}^{n} [r_t - (-\theta a_{t-1})]^2,$$

where θ is the unknown parameter of θ_0. Note that

$$a_1 = r_1 + \theta_0 a_0$$

$$a_2 = r_2 + \theta_0 a_1$$

$$\cdots$$

$$a_t = r_t + \theta_0 a_{t-1}.$$

Let $a_0 = 0$ and replace θ_0 by θ. Denote

$$a_1(\theta) = r_1 + \theta \times 0$$

$$a_2(\theta) = r_2 + \theta a_1(\theta)$$

$$\cdots$$

$$a_t(\theta) = r_t + \theta a_{t-1}(\theta).$$

Thus, conditional LSE– minimizer of

$$S_n(\theta) = \sum_{t=1}^{n} [a_t(\theta)]^2.$$

Note that

$$a_t = r_t + \sum_{i=1}^{t-1} \theta_0^i r_{t-1} + \theta_0^t a_0,$$

$$a_t(\theta) = r_t + \sum_{i=1}^{t-1} \theta^i r_{t-i} + \theta^t a_0.$$

The initial value a_0 does not affect the estimator, asymptotically.

Similarly, for MA(q) model:

$$a_t(\theta) = r_t - \mu + \theta_1 a_{t-1}(\theta) + \theta_2 a_{t-2}(\theta) + \cdots + \theta_q a_{t-q}(\theta).$$

Conditional LSE– minimizer of

$$S_n(\theta) = \sum_{t=1}^{n} [a_t(\theta)]^2,$$

where $\theta = (\mu, \theta_1, \cdots, \theta_q)'$. Conditional on $r_t = 0$ and $a_t = 0$ for $t \leq 0$. Denote the minimizer by $\widehat{\theta}$. Then $a_t(\widehat{\theta})$ is called residual.

$$\widehat{\sigma}_a^2 = \frac{1}{n} \sum_{t=1}^{n} a_t(\widehat{\phi})^2.$$

Theory:

$$\sqrt{n}(\widehat{\theta} - \theta_0) \sim N(0, \widehat{\Omega}).$$

$$\sqrt{n}(\widehat{\theta}_i - \theta_{0i}) \sim N(0, \widehat{\sigma}_{ii}).$$

where

$$\widehat{\Omega} = \left(\sum_{t=1}^{n} \frac{\partial a_{t-1}(\widehat{\theta})}{\partial \theta} \frac{\partial a_{t-1}(\widehat{\theta})}{\partial \theta'} \right)^{-1} \widehat{\sigma}_a^2$$

and $\widehat{\sigma}_{ii}$ is the (i,i)-element of $\widehat{\Omega}$.

In R, s.e. is $\sqrt{\widehat{\sigma}_{ii}/n}$.

$$H_0 : \theta_{0i} = 0 \ v.s. \ H_a : \theta_{0i} \neq 0$$

If $|\widehat{\theta}_i| > 1.96 s.e. \approx 2 s.e.$, then reject H_0. θ_{0i} is considered to non-zero at significance level $\alpha = 0.05$.

3. Model checking: Ljung-Box text to examine residuals (to be white noise)

$a_t(\widehat{\theta}) = r_t - \widehat{\mu} + \widehat{\theta}_1 a_{t-1}(\widehat{\theta}) + \widehat{\theta}_2 a_{t-2}(\widehat{\theta}) + \cdots + \widehat{\theta}_q a_{t-q}(\widehat{\theta}),$ where $\widehat{\theta}$ is the MLE.

4. Forecast: use the residuals as $\{a_t\}$ (which can be obtained from the data and fitted parameters) to perform forecasts.

ARMA(1,1) model

1. Form: $(1 - \phi_1 B)r_t = \phi_0 + (1 - \theta B)a_t$, or

$$r_t = \phi_1 r_{t-1} + \phi_0 + a_t - \theta_1 a_{t-1}.$$

A combination of an AR(1) on the LHS and an MA(1) on the RHS.

2. Stationarity: same as AR(1)

3. Invertibility: same as MA(1)

4. Mean: as AR(1), i.e. $E(r_t) = \frac{\phi_0}{1-\phi_1}$

5. Variance: given in the text

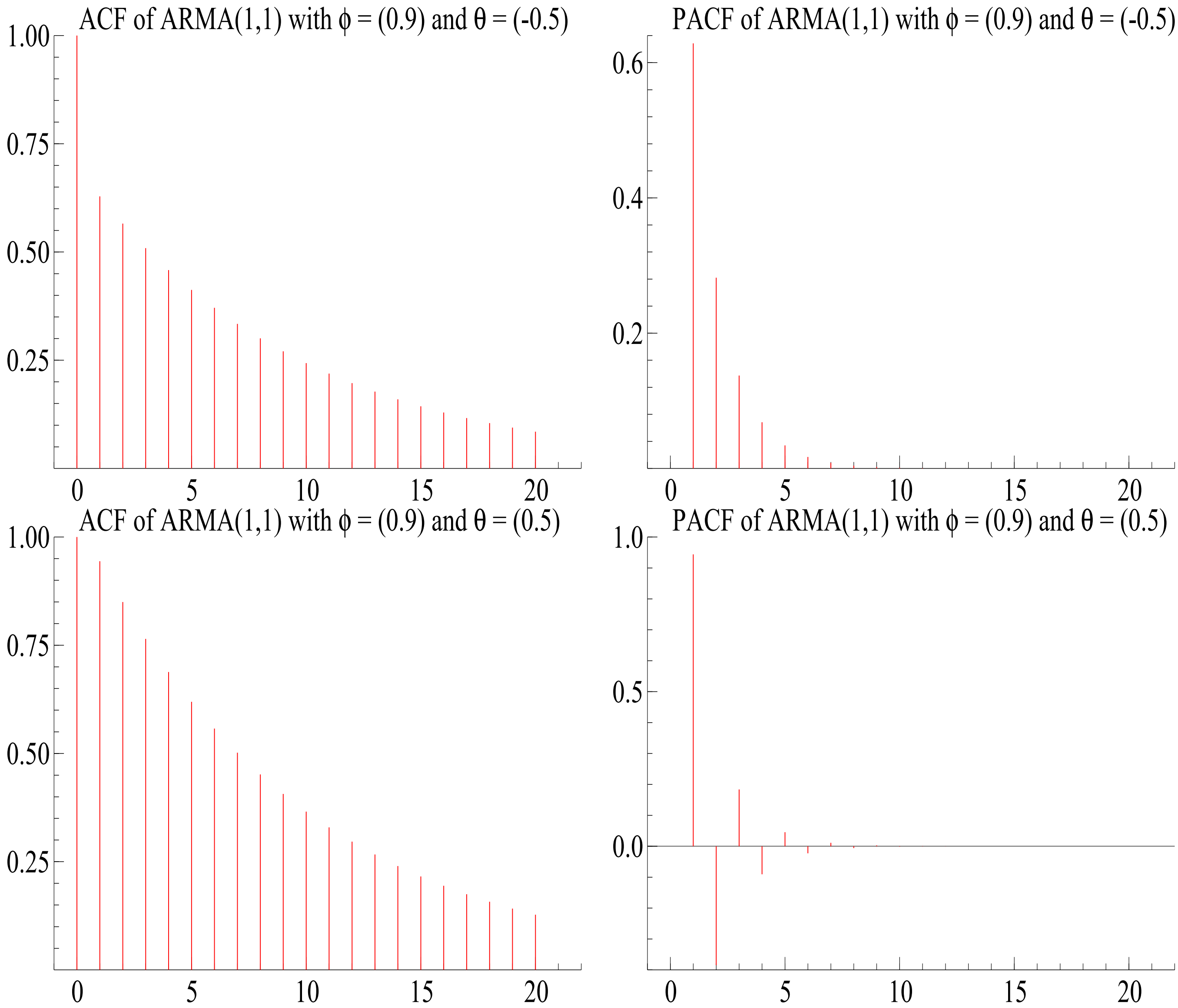
6. ACF: Satisfies $\rho_k = \phi_1 \rho_{k-1}$ for $k > 1$, but

$$\rho_1 = \phi_1 - [\theta_1 \sigma_a^2 / Var(r_t)] \neq \phi_1.$$

This is the difference between AR(1) and ARMA(1,1) models.

PACF: does not cut off at finite lags.

7. Forecast: MA(1) affects the 1-step ahead forecast. Others are similar to those of AR(1) models.



ACF and PACF for ARMA(1,1) model

ARMA(p,q) model:

Assume that $r_1, r_2, \cdots, r_n$ follow a stationary and invertible **ARMA**(p, q) model:

$$\begin{aligned} r_t &= \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \cdots + \phi_p r_{t-p} \\ &\quad + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}, \end{aligned}$$

where all the roots of

$$\phi_p(z) = 1 - \sum_{i=1}^{p} \phi_i z^i \; and \; \theta_q(z) = 1 - \sum_{i=1}^{q} \theta_i z^i$$

lie outside the unit circle, and they have no common roots.

Representations:

$$\phi_p(z)\theta_q^{-1}(z) = 1 - \sum_{i=1}^{\infty} \pi_i z^i,$$

$\pi_i = O(\rho^i)$ with $\rho \in (0, 1)$. AR representation:

$$r_t = \theta_q^{-1}(1)\phi_0 + a_t + \pi_1 r_{t-1} + \pi_2 r_{t-2} + \cdots.$$

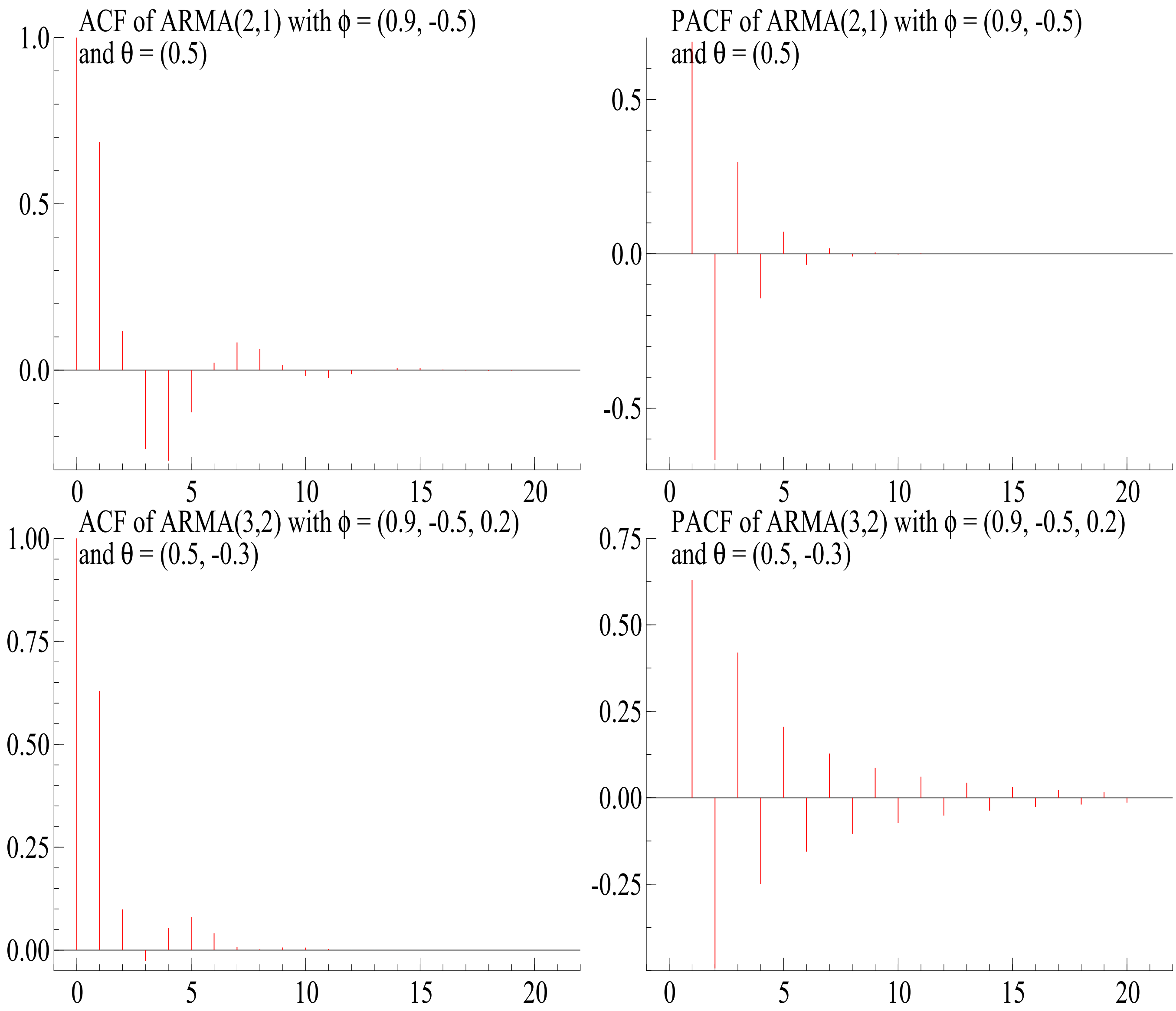
It tells how r_t depends on its past values.

$$\phi_p^{-1}(z)\theta_q(z) = 1 + \sum_{i=1}^{\infty} \psi_i z^i,$$

$\psi_i = O(\rho^i)$ with $\rho \in (0, 1)$. MA representation:

$$r_t = \phi_0 \phi_p^{-1}(1) + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots.$$

It tells how r_t depends on the past shocks.



ACF and PACF for ARMA(p) model

Specification: use AIC to determine p and q.

Estimation: conditional or exact likelihood method

Model checking: as before

Forecast: as before.

The MA representation is particularly useful in computing variances of forecast errors.

Procedure for Buliding an ARMA Model

We have log-return: $r_1, r_2, \cdots, r_n$ which follow a stationary and invertible **ARMA** model:

$$\begin{aligned} r_t &= \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \cdots + \phi_p r_{t-p} \\ &\quad + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}\,, \end{aligned}$$

where $a_t \sim i.i.d. N(0, \sigma_a^2)$. We need to find

p, q, $\phi_1, \cdots, \phi_p$, $\theta_1, \cdots, \theta_q$, μ and σ_a^2, which are called unknown parameters.

How to find?

Step 1. Determine (p, q) by ACF and PACF, or AIC and BIC. If it is not easy to find p and q, you can try some different (p, q).

Step 2. Estimate $\phi_1, \cdots, \phi_p$, $\theta_1, \cdots, \theta_q$, μ and σ_a^2. Methods are LSE or MLE.

Step 3. Checking whether or not (p, q) is correct.

If it is not correct, try some different (p, q) and then go to **Step 2**.

Even if it is correct, we still need to try some different (p, q) in practice.

Step 4. In general, the correct (p, q) is not unique. We can select the best one by **AIC** and **BIC** criteria.

The above procedure is called building a model, fit a model, model fitting, or modeling.

Forecasting Intervals

Assume that $r_1, r_2, \cdots, r_n$ follow a stationary and invertible **ARMA** model:

$$\begin{aligned} r_t &= \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \cdots + \phi_p r_{t-p} \\ &\quad + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}. \end{aligned}$$

Then

$$\widehat{r}_n(l) = E(r_{n+l} | r_n, r_{n-1}, \cdots).$$

(Recall AR(1) model

$$r_{n+l} = \phi_0 + \phi_1 r_{t-1} + a_t.$$

Starting at the origin n, the forecasting value of r_{n+l} is $\widehat{r}_n(l) = \phi_1 \widehat{r}_n(l-1))$.

Let r_t have MA representation:

$$r_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots .$$

Forecasting error:

For a l-step ahead forecast, the forecast error is

$$e_n(l) = a_{n+l} + \psi_1 a_{n+l-1} + \cdots + \psi_{l-1} a_{n+1}.$$

Forecasting variance:

$$Var[e_n(l)] = (1 + \psi_1^2 + \cdots + \psi_{l-1}^2)\sigma_a^2.$$

Forecast interval (limit) (FI):

$$\left[\widehat{r}_n(l) - N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2},\ \widehat{r}_n(l) + N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2}\right]$$

where $N_{\frac{\alpha}{2}}$ is the $\alpha/2$-quantile of the standard normal distribution,

i.e., $P(N > N_{\frac{\alpha}{2}}) = \alpha/2$.

When $\alpha = 0.05$, $N_{\frac{\alpha}{2}} = 1.96$.

See simulation example.

Non-stationary Time Series Models

Random walk

Form $p_t = p_{t-1} + a_t$

Unit root? It is an AR(1) model with coefficient $\phi_1 = 1$.

Nonstationary: Why? Because the variance of r_t diverges to infinity as t increases.

Strong memory: sample ACF approaches 1 for any finite lag.

Random walk with drift

Form: $p_t = \mu + p_{t-1} + a_t,\ \mu \neq 0$.

Has a unit root

Nonstationary

Strong memory

Has a time trend with slope μ. Why?

Differencing

1st difference: $r_t = p_t - p_{t-1}$

If p_t is the log price, then the 1st difference is simply the log return. Typically, 1st difference means the change or increment of the original series.

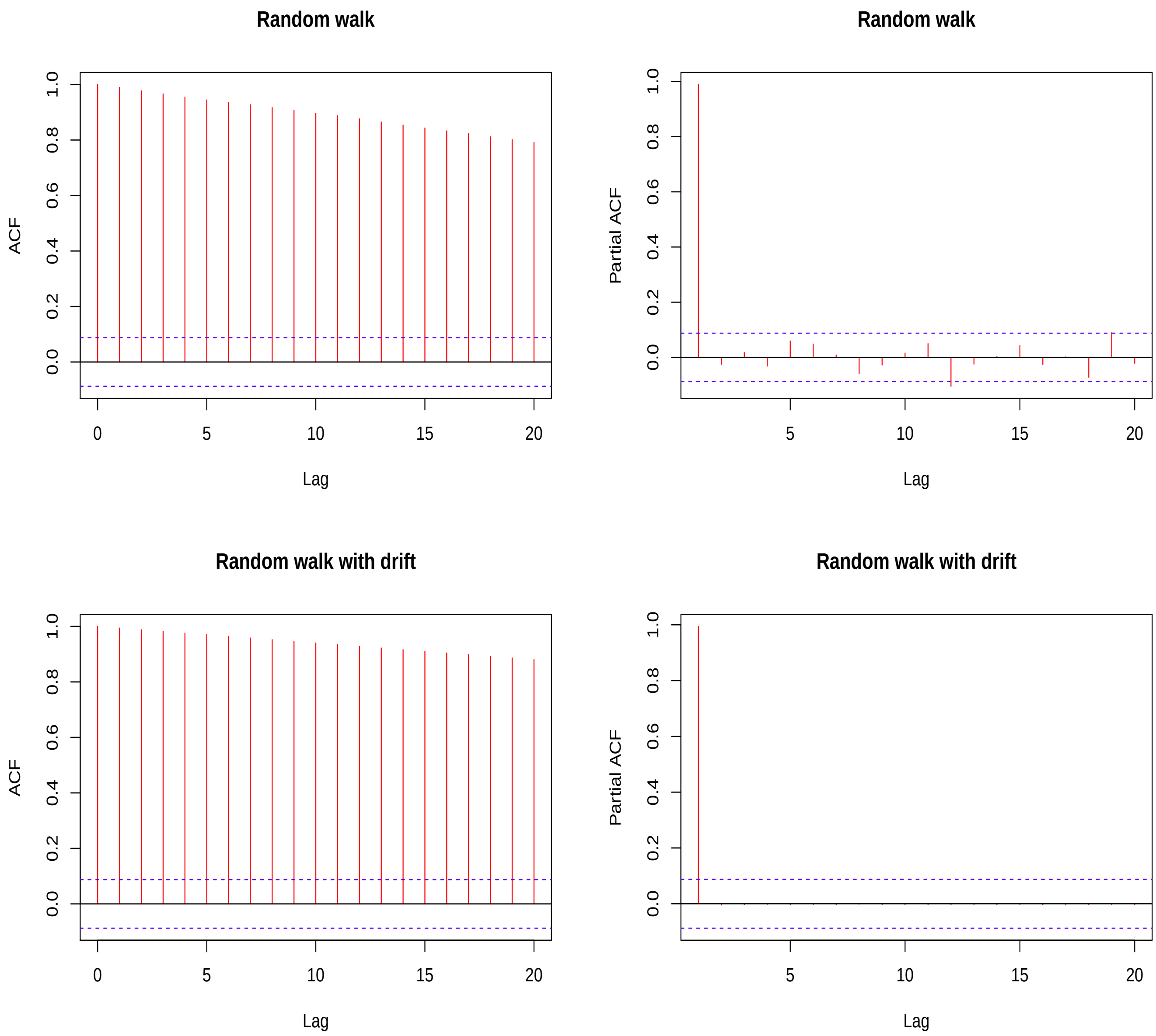
Example Simulated 100 values from

$$(1 - B)p_t = a_t \,,$$

and

$$(1 - B)p_t = 4 + a_t \,,$$

Show the sample ACF and PACF.



ACF and PACF for ARIMA(1,1,0) model

Unit-root test

Let p_t be the log price of an asset. To test that p_t is not predictable (i.e. has a unit root), we consider the AR(1) model:

$$p_t \;=\; \phi_1 p_{t-1} + e_t,$$

The hypothesis of interest is

$$H_0 : \phi_1 = 1 \;\; vs \;\; H_a : \phi_1 < 1.$$

Dickey-Fuller test is the usual t-ratio of the OLS estimate of ϕ_1 being 1:

$$\rho_n = n(\widehat{\phi}_1 - 1) = \frac{n \sum_{t=1}^{n} p_{t-1} e_t}{\sum_{t=1}^{n} p_{t-1}^2},$$

$$\tau_n = n(\widehat{\phi}_1 - 1)\sqrt{\sum_{t=1}^{n} p_{t-1}^2 / n^2}.$$

In SAS:

ZM: zero mean or no intercept case:

$$p_t = \phi_1 p_{t-d} + e_t.$$

When $d = 1$, test statistics:

$$\rho_n \to_d \frac{\int_0^1 B(\tau)dB(\tau)}{\int_0^1 B^2(\tau)d\tau},$$

$$\tau_n \to_d \frac{\int_0^1 B(\tau)dB(\tau)}{\sqrt{\int_0^1 B^2(\tau)d\tau}},$$

where $B(\tau)$ is the standard Brownian motion on $C[0,1]$

SM: single mean or intercept case.

$$p_t = \phi_0 + \phi_1 p_{t-d} + e_t.$$

TR: intercept and deterministic time trend case.

$$p_t = \phi_0 + \gamma t + \phi_1 p_{t-d} + e_t.$$

In R:

Let $\Delta p_t = p_t - p_{t-1}$ and rewrite AR(1) model as

$$\Delta p_t = \delta p_{t-1} + e_t,$$

where $\delta = \phi_1 - 1$. The hypothesis of interest is

$$H_0 : \delta = 0 \ \ vs \ \ H_a : \delta < 0.$$

Test statistic:

$$\tau_n \to_d \frac{\int_0^1 B(\tau) dB(\tau)}{\sqrt{\int_0^1 B^2(\tau) d\tau}}.$$

The general form of Augmented Dickey-Fuller test-s:

$$\Delta p_t = \phi_0 + \gamma t + \delta p_{t-1} + \sum_{i=1}^{p} \phi_i^* \Delta p_{t-i} + e_t,$$

where $\phi_0 = \gamma = 0$. Using LSE, a test statistic as τ_n is constructed for H_0 v.s. H_a.

Autoregressive Integrated Moving-average (ARIMA) Model

The General ARIMA Model

Let p_t be a General Stochastic Trend Model:

$$(1-B)^d p_t = r_t, \qquad d \geq 1.$$

If r_t is a weakly stationary and invertible ARMA model,

$$\phi_p(B) r_t = \theta_q(B) a_t,$$

then:

$$\phi_p(B)(1-B)^d p_t = \theta_q(B) a_t\,,$$

p_t is called ARIMA(p, d, q) model.(?)

If r_t is the following ARMA model,

$$\phi_p(B) r_t = \theta_0 + \theta_q(B) a_t\,,$$

then,

$$\phi_p(B)(1-B)^d p_t = \theta_0 + \theta_q(B) a_t\,,$$

p_t is called ARIMA(p, d, q) model.

ARIMA(0, 1, 0) model:

$(1-B)p_t = a_t$

or $p_t = p_{t-1} + a_t$ (random walk)

$p_t = \theta_0 + p_{t-1} + a_t$.

The ARIMA(0, 1, 1) or IMA(1, 1) Model:

$$(1-B)p_t = (1-\theta B)a_t \,.$$

or

$$p_t = p_{t-1} - \theta a_{t-1} + a_t \,,$$

where $|\theta| < 1$.

Expansion:

$$a_t = p_t - (1-\theta)\sum_{j=1}^{\infty} \theta^{j-1} p_{t-j}.$$

or

$$p_t = (1-\theta)\sum_{j=1}^{\infty} \theta^{j-1} p_{t-j} + a_t.$$

Example: Simulated 100 values from three models:

ARIMA(1, 1, 0) model:

$(1 - 0.8B)(1 - B)p_t = a_t$,

ARIMA(0, 1, 1) model:

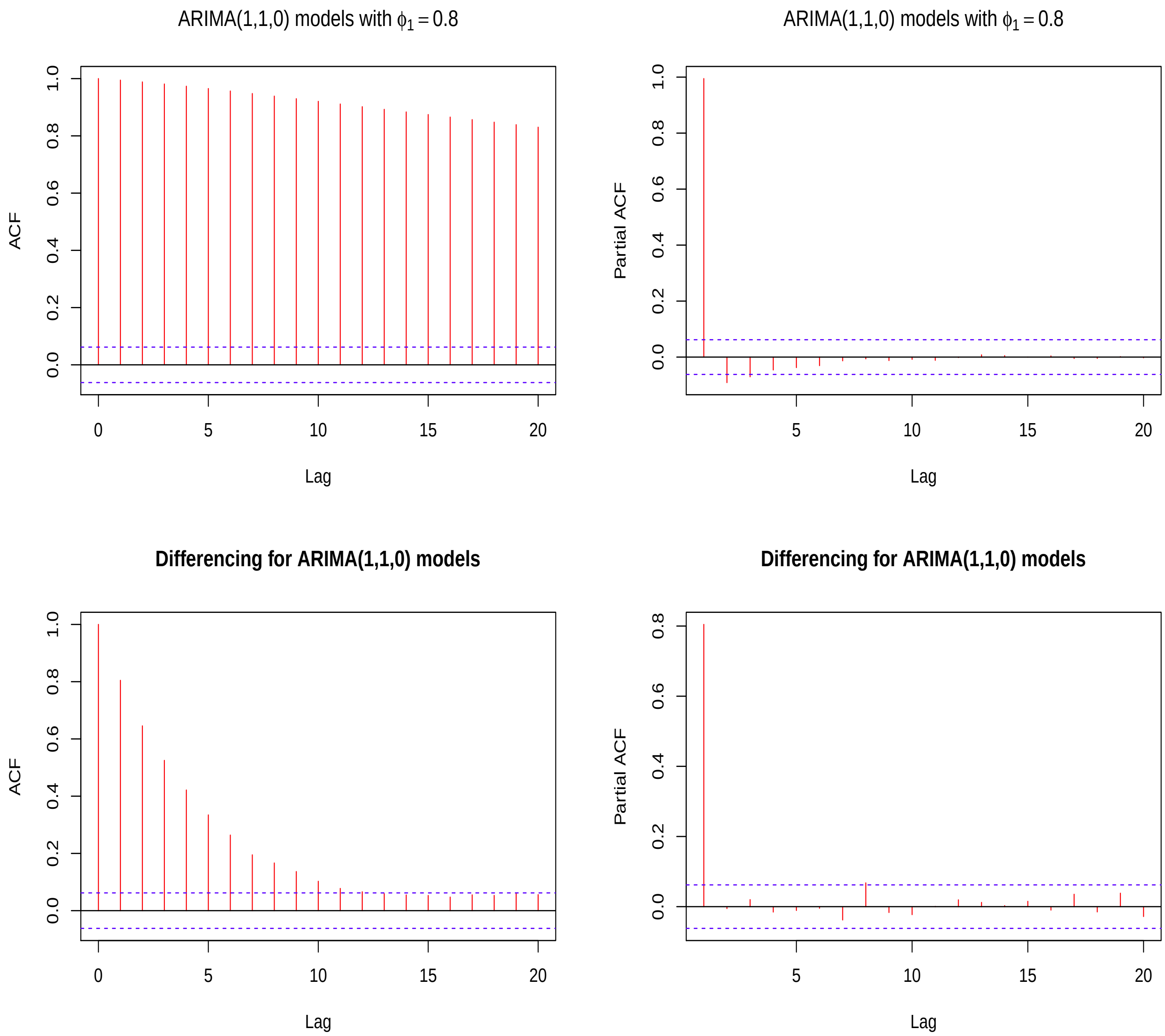
$(1 - B)p_t = (1 - 0.75B)a_t$,

ARIMA(1, 1, 1) model:

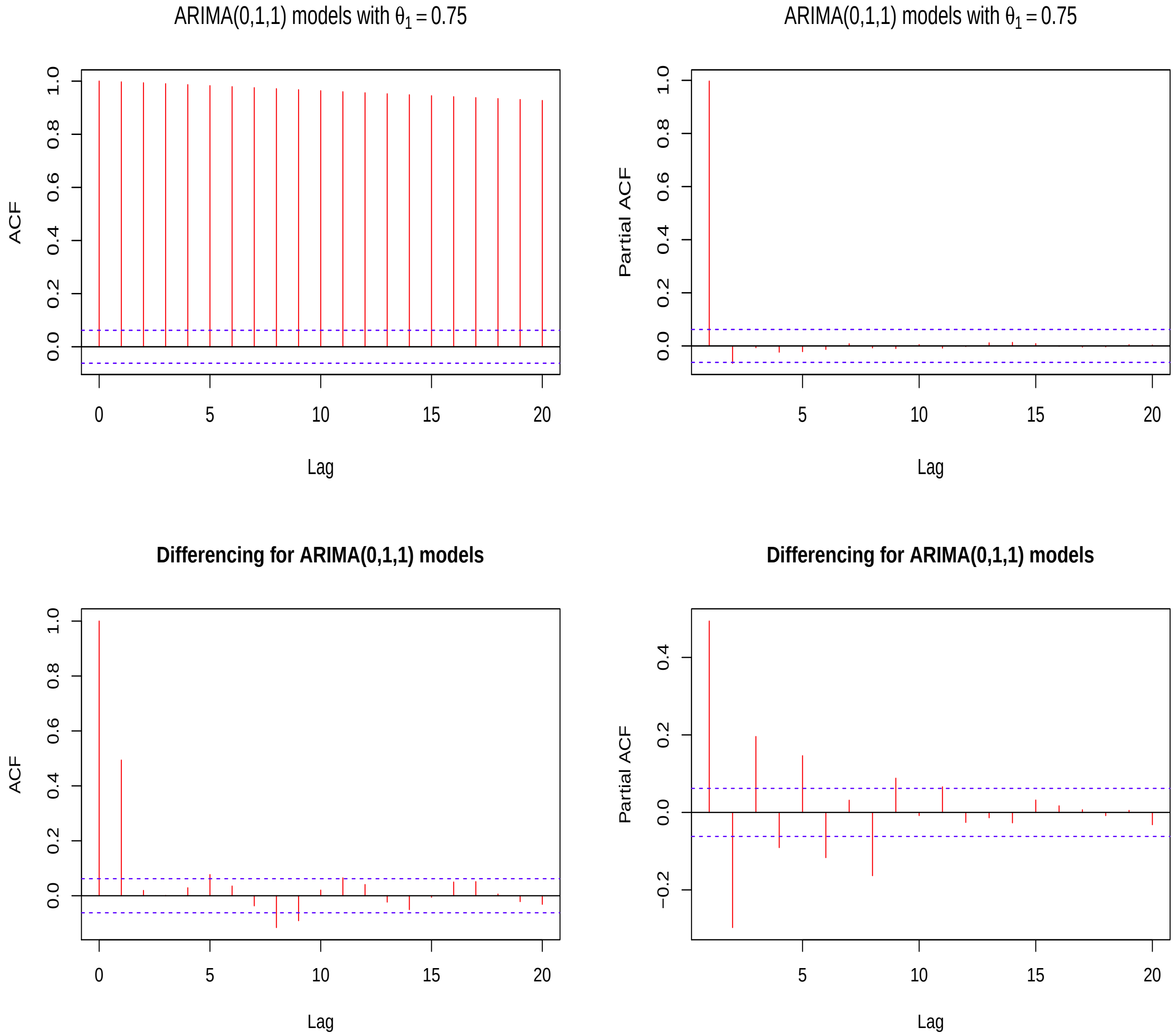
$(1 - 0.9B)(1 - B)p_t = (1 - 0.5B)a_t$,

a. Show the sample ACF and PACF.

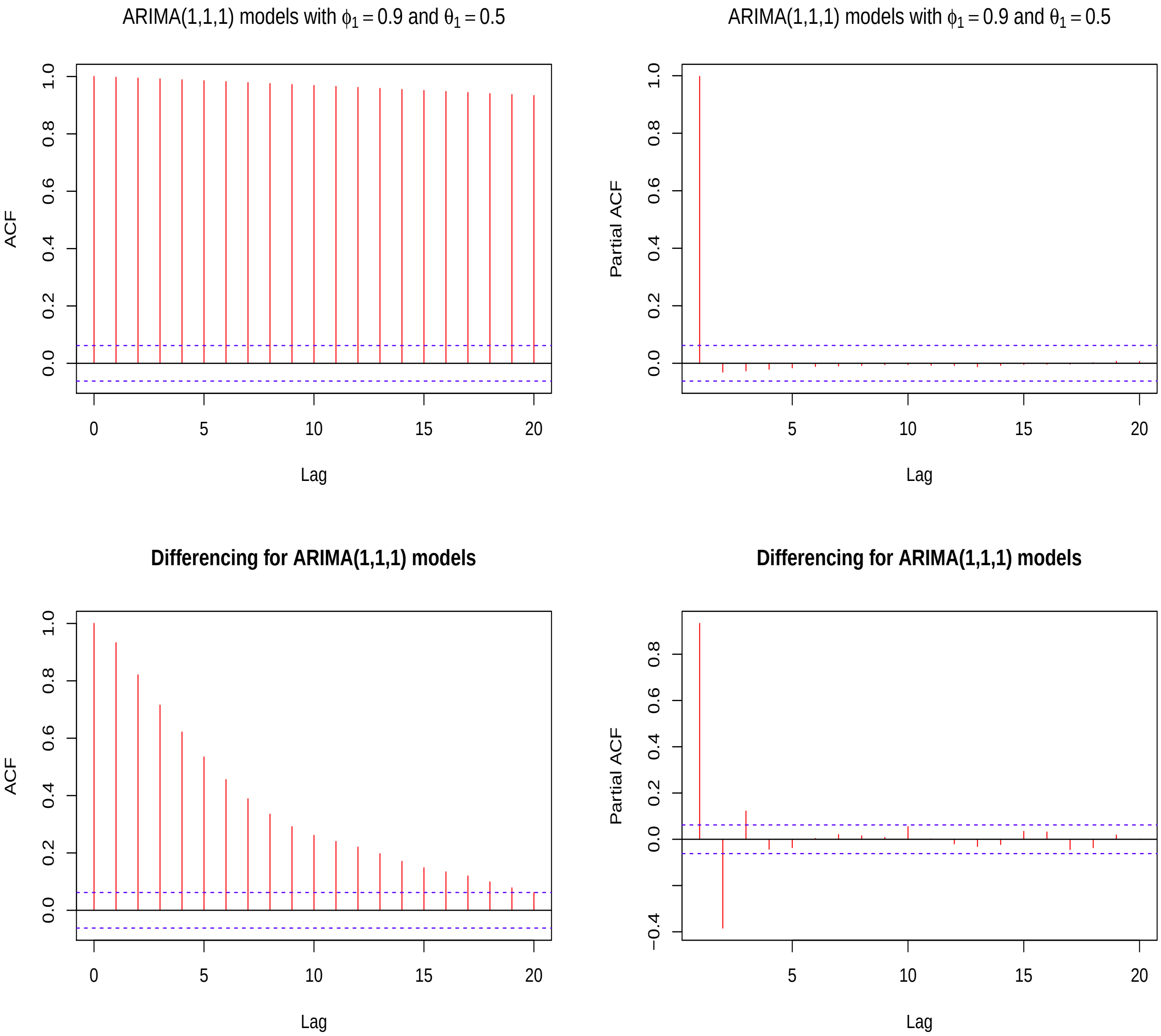
b. Let $r_t = (1 - B)p_t$. Show the sample ACF and PACF of r_t.



ACF and PACF for ARIMA(1,1,0) model



ACF and PACF for ARIMA(0,1,1) model



ACF and PACF for ARIMA(1,1,1) model

Minimum Mean Square Error Forecasts for ARIMA models

A. Model:

Let p_t be ARIMA(p, d, q) model with $d \neq 0$,

$$\phi_p(B)(1-B)^d p_t = \theta_q(B) a_t.$$

where all the roots of $\phi_p(z) = 0$ and $\theta_q(z) = 0$ lie outside the unit circle.

Given the observations: $p_n, p_{n-1}, \cdots,$

how to forecast $p_{n+1}, \cdots, p_{n+l}, \cdots$?

B. Minimum Mean Square Error Forecasts:

$$\widehat{p}_n(l) = E(p_{n+l} | p_n, p_{n-1}, \cdots).$$

C. Computation of forecast:

Denote

$$\begin{aligned}
\Psi(B) &= \phi_p(B)(1-B)^d \\
&= 1 - \Psi_1 B - \Psi_2 B^2 - \cdots - \Psi_{p+d} B^{p+d}. \\
p_t &= \Psi_1 p_{t-1} + \Psi_2 p_{t-2} + \cdots + \Psi_{p+d} p_{t-p-d} \\
&\quad + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}.
\end{aligned}$$

Formulas:

$$\begin{aligned}\widehat{p}_n(l) &= \Psi_1\widehat{p}_n(l-1)+\cdots+\Psi_{p+d}\widehat{p}_n(l-p-d)\\ &\quad +\widehat{a}_n(l)-\theta_1\widehat{a}_n(l-1)-\cdots-\theta_q\widehat{a}_n(l-q).\end{aligned}$$

where

$$\widehat{p}_n(j) = \begin{cases} E(p_{n+j}|p_n, p_{n-1}, \cdots) & \text{if} \quad j=1,2,\cdots,l. \\ p_{n+j} & \text{if} \quad j=0,-1,-2,\cdots. \end{cases}$$

$$\widehat{a}_n(j) = \begin{cases} 0 & \text{if } j=1,2,\cdots,l. \\ a_{n+j} & \text{if } j=0,-1,-2,\cdots. \end{cases}$$

D. Forecast error:

$$e_n(l) = p_{n+l} - \widehat{p}_n(l) = \sum_{j=0}^{l-1} \psi_j a_{n+l-j}\,,$$

where ψ_i can be calculated, recursively:

$$\psi_j = \sum_{i=0}^{j-1} \pi_{j-i}\psi_i\,, \qquad j=1,2,\cdots,l-1.$$

π_j is the coefficients of the expansion:

$$\pi(B) = \frac{\phi_p(B)(1-B)^d}{\theta_q(B)} = 1 - \sum_{j=1}^{\infty} \pi_j B^j.$$

$$p_{t+l} = \sum_{j=1}^{\infty} \pi_j p_{t+l-j} + a_{t+l}.$$

E. Forecast variance:

$$\text{Var}[e_n(l)] = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2.$$

F. Forecast interval (limit) (FI):

$$\left[\widehat{p}_n(l) - N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2},\ \widehat{p}_n(l) + N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2}\right]$$

where $N_{\frac{\alpha}{2}}$ is the $\alpha/2$-quantile of the standard normal distribution,

In SAS, the output gives the forecasting intervals for AR(p), MA(q), and ARMA(p,q) models.

Note that $r_t = p_t - p_{t-1}$, where $p_t = \ln P_t$. SAS outputs give the forecasting value of p_{n+l}, i.e., $\widehat{p}_n(l)$, and the forecasting intervals of p_{n+l}, i.e., $[L_\alpha(l), U_\alpha(l)]$.

Then the forecast value of P_{n+l} is

$$\begin{aligned}
\widehat{P}_n(l) &= E(P_{n+l}|F_n) = E(e^{p_{n+l}}|F_n) \\
&= e^{\widehat{p}_n(l)}E(e^{p_{n+l}-\widehat{p}_n(l)}|F_n) \\
&= e^{\widehat{p}_n(l)}E(e^{e_n(l)}|F_n) \\
&= e^{\widehat{p}_n(l)}E(e^{e_n(l)}) \\
&= \exp\{\widehat{p}_n(l) + \frac{1}{2}var(e_n(l))\}.
\end{aligned}$$

Math: normal and lognormal dists

If $X \sim N(\mu, \sigma^2)$,

$Y = \exp(X)$ is lognormal r.v. with mean and variance

$$E(Y) = \exp(\mu + \frac{\sigma^2}{2}),$$

$$V(Y) = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1].$$

The forecasting interval of P_{n+l} is

$$\left[e^{L_\alpha(l)},\, e^{U_\alpha(l)}\right].$$

Example 1. Consider ARIMA(0, 1, 1) model:

$$(1 - B)p_t = (1 - \theta B)a_t.$$

Example 2. Consider ARIMA(1, 1, 1) model:

$$(1 - \phi B)(1 - B)p_t = (1 - \theta B)a_t.$$