GARCH Models–Conditional Heteroscedastic Models

What is stock volatility?

$$r_t = \mu_t + \sigma_t \epsilon_t.$$

where $\sigma_t^2 = Var(r_t|F_{t-1}) = Var(a_t|F_{t-1})$.

Answer: conditional standard deviation σ_t of stock returns

Why is volatility important?

Has many important applications

Option (derivative) pricing, e.g., Black-Scholes formula

Risk management, e.g. value at risk (VaR)

Asset allocation, e.g., minimum-variance portfolio; see pages 184-185 of Campbell, Lo and MacKinlay (1997).

Interval forecasts

A key characteristic: Not directly observable!!

How to calculate volatility?

1. Use high-frequency data: French, Schwert & Stambaugh (1987); see Section 3.15.

Realized volatility of daily returns in recent literature.

Use daily high, low, and closing prices.

2. Implied volatility of options data.

3. Econometric modeling

We focus on the econometric modeling first. Use of high frequency data will be discussed later.

Basic idea of econometric modeling Shocks of asset returns are NOT serially correlated, but dependent.

See ACF of squared and absolute returns of some stocks.

Testing for the ARCH effect

When $\mu_t = \mu$, let $\xi_t = (r_t - \mu)^2$.

If there is not ARCH effect, then the ACF of ξ_t are all zero.

McLeod and Li (1983): use the Ljung-Box to test the null H_0: the ACF ρ_k of ξ_t are all zero, i.e,

$$H_0 : \rho_1 = \cdots = \rho_m = 0.$$

Let $\widehat{\rho}_k$ be the sample ACF of $\{\xi_t\}$. We use the Ljung-Box:

$$Q(m) = n(n-1) \sum_{k=1}^{m} \frac{\widehat{\rho}_k^2}{n-k} \sim \chi^2(m).$$

Engle (1982): assume that

$$\xi_t = \alpha_1 \xi_{t-1} + \cdots + \alpha_m \xi_{t-m} + e_t = \beta' X_t + e_t,$$

where $\beta = (\alpha_1, \cdots, \alpha_m)'$ and $X_t = (\xi_{t-1}, \cdots, \xi_{t-m},$ and use LM test for the null:

$$H_0 : \alpha_1 = \cdots = \alpha_m = 0.$$

The LM test statistic:

$$LM = \left(\sum_{t=1}^{n} X_t'\right) \left[\sum_{t=1}^{n} X_t X_t'\right]^{-1} \left(\sum_{t=1}^{n} X_t\right) \sim \chi^2(m).$$

Basic structure

$$r_t = \mu_t + \sigma_t \epsilon_t,$$
$$\mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i}.$$

where $\sigma_t^2 = Var(r_t|F_{t-1}) = Var(a_t|F_{t-1})$.

Two general categories:

Fixed function and

Stochastic function of the available information.

Univariate volatility models:

1. Autoregressive conditional heteroscedastic (ARCH) model of Engle (1982),

2. Generalized ARCH (GARCH) model of Bollerslev (1986),

3. GARCH-M models

4. IGARCH models

5. Exponential GARCH (EGARCH) model of Nelson (1991),

6. Threshold GARCH model of Zakoian (1994) or GJR model of Glosten, Jagannathan, and Runkle (1993).

7. Conditional heteroscedastic ARMA (CHARMA) model of Tsay (1987),

8. Random coefficient autoregressive (RCA) model of Nicholls and Quinn (1982),

9. Stochastic volatility (SV) models of Melino and Turnbull (1990), Harvey, Ruiz and Shephard (1994), and Jacquier, Polson and Rossi (1994).

ARCH model

$$a_t = \sigma_t \epsilon_t, \ \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2,$$

where $\{\epsilon_t\}$ is a sequence of iid r.v. with mean 0 and variance 1, $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i > 0$.

GARCH Model

$$\begin{aligned} a_t &= \sigma_t \epsilon_t, \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{i=1}^{s} \beta_i \sigma_{t-i}^2, \end{aligned}$$

where $\{\epsilon_t\}$ is defined as before, $\alpha_0 > 0$, $\alpha_i \geq 0$ and $\beta_j \geq 0$.

Distribution of ϵ_t: Standard normal, standardized Student-t, generalized error dist (GED), or skewed Student$-t$.

Re-parameterization: Let $\eta_t = a_t^2 - \sigma_t^2$. $\{\eta_t\}$ uncorrelated series. The GARCH model becomes

$$a_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 - \sum_{i=1}^{s} \beta_i \eta_{t-i} + \eta_t.$$

This is an ARMA form for the squared series a_t^2.

Focus on a GARCH(1,1) model:

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \tag{1}$$

1. $E(a_t) = 0$,

2. Model (1) has the following expansion (or solution):

$$\sigma_t^2 = \alpha_0 \left[1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \left(\alpha_1 \epsilon_{t-i}^2 + \beta_1 \right) \right]$$

if and only if

$$E \ln(\alpha_1 \epsilon_t^2 + \beta_1) < 0, \tag{2}$$

and the solution is unique an d stationary and ergodic.

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Math Proof. First, if (2) holds, then there exists a $\gamma > 0$ such that

$$\rho \equiv E|\alpha_1\epsilon_t^2 + \beta_1|^\gamma < 1.$$

We now write (1) as

$$\begin{aligned}
\sigma_t^2 &= \alpha_0 + \left(\alpha_1\epsilon_{t-1}^2 + \beta_1\right)\sigma_{t-1}^2 \\
&= \alpha_0 + \alpha_0\left(\alpha_1\epsilon_{t-1}^2 + \beta_1\right) \\
&\quad + \left(\alpha_1\epsilon_{t-1}^2 + \beta_1\right)\left(\alpha_1\epsilon_{t-2}^2 + \beta_1\right)\sigma_{t-2}^2 \\
&= \cdots \\
&= \alpha_0\left[1 + \sum_{j=1}^{m}\prod_{i=1}^{j}\left(\alpha_1\epsilon_{t-i}^2 + \beta_1\right)\right] \\
&\quad + \prod_{i=1}^{m+1}\left(\alpha_1\epsilon_{t-i}^2 + \beta_1\right)\sigma_{t-m-1}^2.
\end{aligned}$$

Let

$$S_{mt} = \alpha_0\left[1 + \sum_{j=1}^{m}\prod_{i=1}^{j}\left(\alpha_1\epsilon_{t-i}^2 + \beta_1\right)\right].$$

For any $m, n > 0$, we have

$$\begin{aligned} E\left|S_{m+n,t} - S_{mt}\right|^\gamma &= \alpha_0^\gamma E\left|\sum_{j=m+1}^{m+n}\prod_{i=1}^{j}\left(\alpha_1\epsilon_{t-i}^2 + \beta_1\right)\right|^\gamma \\ &\leq \alpha_0^\gamma \sum_{j=m+1}^{m+n}\prod_{i=1}^{j} E\left|\alpha_1\epsilon_t^2 + \beta_1\right|^\gamma \\ &= O(\rho^n). \end{aligned}$$

Thus, by Cauchy criterion, we can show that

$$S_{mt} \to S_{\infty t} \equiv \sigma_t^2, \text{ a.s. and in } L^\gamma, \text{ as } m \to \infty.$$

We can verify that σ_t^2 is the solution of model (1) and is stationary and ergodic with $E\sigma_t^{2\gamma} < \infty$.

Suppose that we have another solution σ_t^{*2} to model (1). Then,

$$\begin{aligned} \Delta\sigma_t^2 &\equiv \sigma_t^2 - \sigma_t^{*2} = (\alpha_1\epsilon_{t-1}^2 + \beta_1)\Delta\sigma_{t-1}^2 = \cdots \\ &= \prod_{i=1}^{m}(\alpha_1\epsilon_{t-i}^2 + \beta_1)\Delta\sigma_{t-m}^2 \end{aligned}$$

and

$$E\left|\Delta\sigma_t^2\right|^\gamma \leq \prod_{i=1}^{m} E\left|\alpha_1\epsilon_t^2 + \beta_1\right|^\gamma E\left|\Delta\sigma_{t-m}^2\right|^\gamma \leq C\rho^m \to 0,$$

as $m \to \infty$.

Thus, $\Delta\sigma_t = 0$ a.s., i.e., $\sigma_t = \sigma_t^*$ a.s..

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3. $Var(a_t) = \alpha_0/(1-\alpha_1-\beta_1)$ if $0 < \alpha_1 + \beta_1 < 1$.

Weak stationarity condition: $0 \leq \alpha_1,\ \beta_1 \leq 1$, $(\alpha_1 + \beta_1) < 1$.

$[\sum_{i=1}^{\max(m,s)}(\alpha_i + \beta_i) < 1$ for GARCH (m,s)].

4. Under normality, if $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$, then

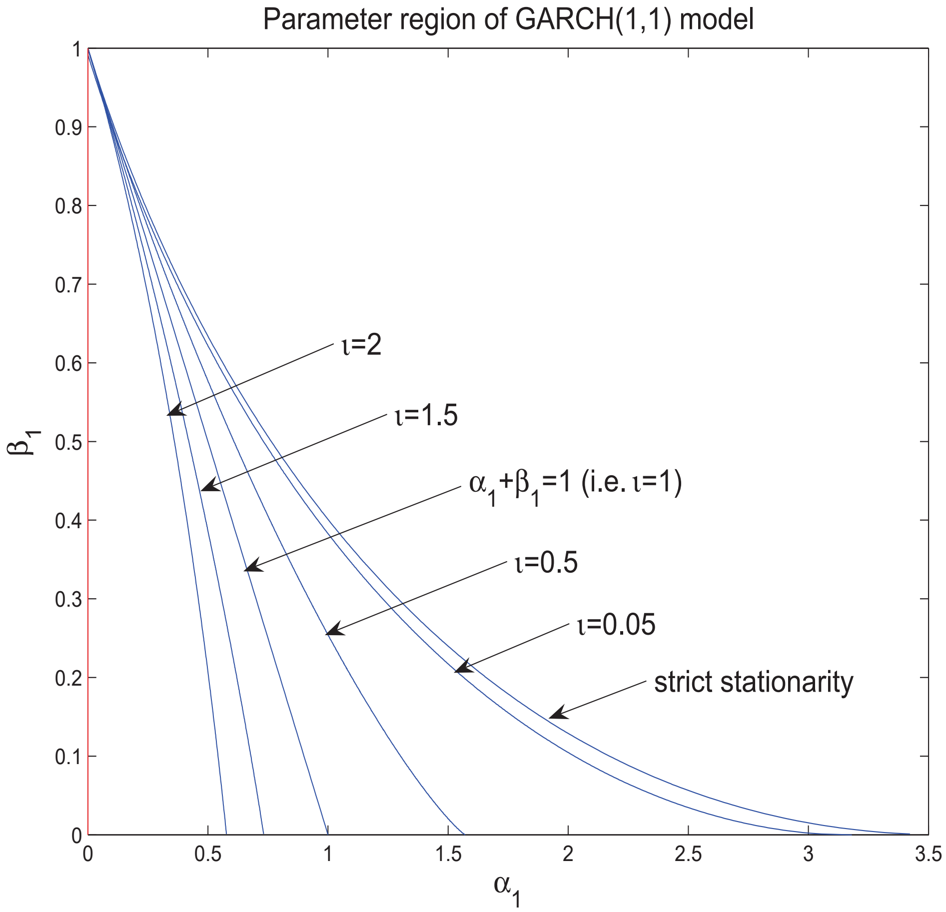
$$\frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1-(\alpha_1+\beta_1)^2]}{1-(\alpha_1+\beta_1)^2 - 2\alpha_1^2} > 3.$$

This will give heavy tail feature and Volatility clusters.

When $\beta_1 = 0$,

$$m_4 = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)},$$

provided $0 < \alpha_1 < 1/3$.



Regions

$$D_\iota = \{(\alpha_1, \beta_1):\ E|a_t|^{2\iota} < \infty\}$$

$$D = \{(\alpha_1, \beta_1):\ E \ln|\alpha_1 \epsilon_t^2 + \beta_1| < 0\}$$

5. Volatility forecast:

For 1-step ahead forecast,

$$\widehat{\sigma}_n^2(1) = \alpha_0 + \alpha_1 a_n^2 + \beta_1 \sigma_n^2.$$

For multi-step ahead forecasts, use $a_t^2 = \sigma_t^2 \epsilon_t^2$ and rewrite the model as

$$\sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2 + \alpha_1 \sigma_t^2(\epsilon_t^2 - 1).$$

2-step ahead volatility forecast

$$\widehat{\sigma}_n^2(2) = \alpha_0 + (\alpha_1 + \beta_1)\widehat{\sigma}_n^2(1).$$

In general, we have

$$\widehat{\sigma}_n^2(l) = \alpha_0 + (\alpha_1 + \beta_1)\widehat{\sigma}_n^2(l-1), \ l > 1.$$

This result is exactly the same as that of the AR model: $r_t = \alpha_0 + (\alpha_1 + \beta_1) r_{t-1} + a_t$.

IGARCH model:

An IGARCH model is as follows:

$$a_t = \sigma_t \epsilon_t, \ \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1) a_{t-1}^2.$$

In this case,

$$\hat{\sigma}_n^2(l) = \hat{\sigma}_n^2(1) + (l - 1)\alpha_0,$$

where n is the forecast origin. The effect of $\hat{\sigma}_n^2(1)$ on future is persistent, and the volatility forecasts form a straight line with slope α_0.