5056 Network Assignment 2 Solution

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(1) Matrix formalism

a) For an undirected network of N nodes, the degree can be written in terms of the adjacency matrix as

$$k_i = \sum_{j=1}^{N} A_{ij}$$

Hence;

$$\vec{k} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & \cdots & & A_{2n} \\ \vdots & & & \\ A_{m1} & \cdots & & A_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \vec{A}\vec{1}$$

b) For an undirected network, L is the links of network, we have

$$2L = \sum_{i=1}^{N} k_i$$

$$= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{bmatrix}$$

$$= \vec{1}^T \vec{k} = \vec{1}^T \vec{A} \vec{1}$$

So, $L = \frac{1}{2} \vec{1}^T \vec{A} \vec{1}$

c) If node i and j have common neighbors, the path of i and j is 2, so the number of common neighbors is the number of paths of leaght 2, So

$$N = \sum_{k=1}^{N} A_{ik} A_{kj} = A^2$$

(2)

a) in-degree sequences : $\{0, 2, 1, 1\}$

out-degree sequences : $\{2, 0, 1, 1\}$

b) $p_{in}(0) = \frac{1}{4}$. $p_{in}(1) = \frac{1}{2}$ $p_{in}(2) = \frac{1}{4}$, $p_{in}(3) = 0$

$p_{out}(0) = \frac{1}{4}$ $p_{out}(1) = \frac{1}{2}$ $p_{out}(2) = \frac{1}{4}$ $p_{out}(3) = 0$

c)

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

we can get $A^2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $A^n = \vec{0}$ for $n > 2$

So

$$\vec{x} = \beta(\vec{I} - \alpha\vec{A})^{-1}\vec{1} = \beta \sum_{n=0}^{\infty} \alpha^n \vec{A}^n \vec{1}$$

$$= \beta \cdot (\alpha^0 \vec{A}^0 \vec{1} + \alpha^1 \vec{A} \vec{1} + \alpha^2 \vec{A}^2 \vec{1})$$

$$= \beta(\vec{1} + \alpha \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix})$$

$$= \begin{bmatrix} \beta(1+2\alpha+2\alpha^2) \\ \beta \\ \beta(1+\alpha) \\ \beta(1+\alpha) \end{bmatrix}$$

(3) Closeness centrality

According to the definition, the closeness centrality is

$C_i = N\left[\sum_{j=1}^{N} d_{ij}\right]^{-1}$, where d_{ij} is the length of shortest path between i and j.

Denoting the sum of length of shortest paths from vertice 1 to the other vartices in n_1 as D_{n_1} and the sum of length of shortest paths from vertice 2 to the other vertices in n_2 as D_{n_2}, We can get sum of length of shortest paths from vertice 2 to n_1 is $D_{n_1}+n_1$ and the sum of length of shortest paths from vertice 1 to n_2 is $D_{n_2}+n_2$

So, $C_1 = n(D_{n_1}+D_{n_2}+n_2)^{-1}$ (1)

$C_2 = n(D_{n_1}+n_1+D_{n_2})^{-1}$ (2)

From (1), we can get $D_{n_1}+D_{n_2} = \frac{n}{C_1} - n_2$.

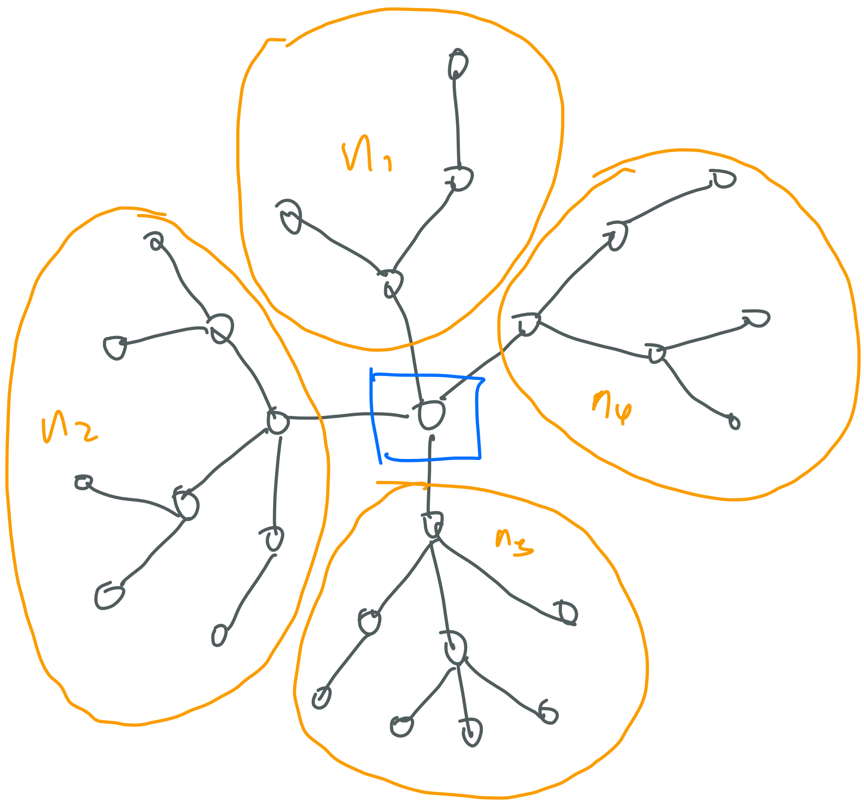
From (2). we can get $D_{n_1}+D_{n_2} = \frac{n}{C_2} - n_1$

So, $\frac{n}{C_1} - n_2 = \frac{n}{C_2} - n_1$

As a result, $\frac{1}{C_1} + \frac{n_1}{n} = \frac{1}{C_2} + \frac{n_2}{n}$

(4) Betweeness centrality

(a)



According to the definition of betweenness centrality.

$$C_B(i) = \sum_{j<k} \frac{g_{jk}(i)}{g_{jk}}$$, where $g_{jk}(i)$ is the paths between j and k that pass through i and g_{jk} is the paths between j and k.

Note that in a tree network, any two vertice have only one path. According to the given information, the path between a vertice in one region and another vertice in any other regions must pass through the particular vertice

Because I don't need to normalize with respect to the total number of path

So. $x = g_{jk}(i) = \frac{1}{2}\sum_{\substack{p\neq q\\p=1\\q=1}}^{k} n_p \cdot n_q$

$$= \frac{1}{2}\left(\sum_{\substack{p=1\\q=1}}^{k} n_p \cdot n_q - \sum_{m=1}^{k} n_m^2\right)$$

$$= \frac{1}{2}\left(\sum_{p=1}^{k} n_p \cdot \sum_{q=1}^{k} n_q - \sum_{m=1}^{k} n_m^2\right)$$

$$= \frac{1}{2}\left[(n-1)^2 - \sum_{m=1}^{k} n_m^2\right]$$

(b) A line graph is a special tree. We denote the two seperated region as n_1 and n_2 after removing ith vertice. So $n_1+n_2=n-1$. and $n_1 \cdot n_2 = (i-1)(n-i)$ for $i = 1, 2 \cdots n$

So $x(i) = \frac{1}{2}\left[(n-1)^2 - \sum_{m=1}^{k} n_m^2\right]$

$$= \frac{1}{2}\left[(n-1)^2 - (n_1^2+n_2^2)\right]$$

$$= \frac{1}{2}\left[(n-1)^2 - \left[(n_1+n_2)^2 - 2n_1 n_2\right]\right]$$

$$= \frac{1}{2}\left[(n-1)^2 - (n-1)^2 + 2(i-1)(n-i)\right]$$

$$= (i-1)(n-i) \quad \text{for all } i = 1, 2, \cdots, n$$

(5)

(a) The edge B-C should be weak tie. Because if B-C is strong tie, it will violate the Strong Triadic Closure assumption since there is no edges E-C and B-F.

(b) Node C and E violate the Strong Triadic Closure property because there has to be an edge connecting C and D is edge C-E is strong tie.