MSDM 5058 Information Science
Assignment 2 (due 23th March, 2024)

Submit your assignment solution on canvas. You may discuss with others or seek help from your TA, but should not directly copy from others. Otherwise, it will be considered as plagiarism.

(1) **Average Entropy**

Let $H(p) = -p \log_2 p - (1-p) \log_2 (1-p)$ be the entropy function of a binary source. (a) Use $\log_2 3 = 1.585$, to evaluate $H(1/3)$. (b) Calculate the average entropy $H(p)$ when the probability p is chosen uniformly in the range $0 \le p \le 1$. (*Hint: For* (*b*)*, use* $\log_2 x = \frac{\ln x}{\ln 2}$*, where ln is the natural logarithm.*)

(2) **Mutual Information for correlated normal distributions**

X and Y are two correlated random normal variables with the following joint probability distributions

$$\binom{X}{Y} \sim N_2\left(0, \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}\right)$$

Evaluate $I(X; Y)$ and comment on the cases when $\rho = 1$, 0 and -1.

(3) **Channel Capacity**

Consider a binary asymmetric communication channel, whose input source is the alphabet $X = \{0, 1\}$ with probabilities $\{0.5, 0.5\}$; whose output alphabet is $Y = \{0, 1\}$; and whose channel matrix is

$$\begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix}$$

Where α is the probability of transmission error when sending $X = 0$ and β is the probability of transmission error when sending $X = 1$.

(a) What is the entropy of the source, $H(X)$?

(b) What is the probability distribution of the outputs, $p(Y)$, and the entropy of this output distribution, $H(Y)$?

(c) What is the joint probability distribution for the source and the output, $p(X, Y)$, and what is the joint entropy, $H(X, Y)$?

(d) What is the mutual information of this channel, $I(X; Y)$, as a function of α and β?

(e) How many combinations of (α, β) are there for which the mutual information of this channel is maximal? What are those values, and what then is the capacity of

such a channel in bits?

(f) What condition do (α, β) satisfy when the capacity of this channel is minimal? What is the channel capacity in that case?

(4) **Shannon, Fano and Huffman codes**

A source X has an alphabet of seven characters {a, b, c, d, e, f, g}, with the corresponding probability {0.01, 0.24, 0.05, 0.20, 0.47, 0.01, 0.02}.

(a) What is the entropy of X?

(b) What is the expected length of the set of codewords by using the Shannon, Fano and Huffman code, respectively?

(c) Give an example of a set of codewords for X using the Shannon code, Fano code and Huffman code respectively and explain how you get it.

(d) Which code is the optimal code for X and how much greater is its expected length than the entropy of X?