MSDM 5003 Homework 1 Solution

1. (a) First, let us show that for two independent random variables X and Y with PDF $f_X(x)$ and $f_Y(y)$ respectively, the random variable $Z = X + Y$ has PDF $f_Z(z)$ given by

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\, dx. \tag{1}$$

The CDF for Z is given by

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy\, dx \\ &= \int_{-\infty}^{+\infty} f_X(x) \int_{-\infty}^{z-x} f_Y(y)\, dy\, dx. \end{aligned} \tag{2}$$

Let $F_Y(y)$ be the CDF of Y (and therefore anti-derivative of $f_Y(y)$), that is

$$\frac{dF_Y(y)}{dy} = f_Y(y). \tag{3}$$

Then,

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{+\infty} f_X(x) \Big[F_Y(y) \Big]_{-\infty}^{z-x} dx \\ &= \int_{-\infty}^{+\infty} f_X(x) [F_Y(z-x) - \underbrace{F_Y(-\infty)}_{0}]\, dx \\ &= \int_{-\infty}^{+\infty} f_X(x) F_Y(z-x)\, dx, \end{aligned} \tag{4}$$

$$\begin{aligned} f_Z(z) = \frac{dF_Z(z)}{dz} &= \frac{d}{dz} \int_{-\infty}^{+\infty} f_X(x) F_Y(z-x)\, dx \\ &= \int_{-\infty}^{+\infty} \frac{\partial}{\partial z} [f_X(x) F_Y(z-x)]\, dx \\ &= \int_{-\infty}^{+\infty} f_X(x) \frac{\partial F_Y(z-x)}{\partial z}\, dx \\ &= \int_{-\infty}^{+\infty} f_X(x) \frac{\partial F_Y(z-x)}{\partial (z-x)} \frac{\partial (z-x)}{\partial z}\, dx \\ &= \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\, dx. \end{aligned} \tag{5}$$

Now, suppose $X \sim \mathcal{N}\big(\mu_X, \sigma_X^2\big)$ and $Y \sim \mathcal{N}\big(\mu_Y, \sigma_Y^2\big)$ are independent and normally distributed with respective mean and variance. Let us compute the PDF of the random variable $Z = X + Y$.

$$f_Z(z) = \frac{1}{2\pi\sigma_X\sigma_Y} \int_{-\infty}^{+\infty} \exp\left\{ -\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{[(z-x)-\mu_Y]^2}{2\sigma_Y^2} \right\} dx \tag{6}$$

Let us take a closer look at the exponent.

$$
\begin{aligned}
& -\frac{(x-\mu_X)^2}{2\sigma_X^2}-\frac{[(z-x)-\mu_Y]^2}{2\sigma_Y^2} \\
=& -\frac{\sigma_Y^2\,(x-\mu_X)^2+\sigma_X^2\,[(z-x)-\mu_Y]^2}{2\sigma_X^2\sigma_Y^2} \\
=& -\frac{\sigma_Y^2\left(x^2-2\mu_X x+\mu_X^2\right)+\sigma_X^2\left(z^2-2zx+x^2-2\mu_Y z+2\mu_y x+\mu_Y^2\right)}{2\sigma_X^2\sigma_Y^2} \\
=& -\frac{\left(\sigma_X^2+\sigma_Y^2\right)x^2+\left[\sigma_X^2\,(-2z+2\mu_Y)-\sigma_Y^2\,(2\mu_X)\right]x+\left[\sigma_X^2\left(z^2-2\mu_Y z+\mu_Y^2\right)+\sigma_Y^2\mu_X^2\right]}{2\sigma_X^2\sigma_Y^2} \\
=& -\frac{\left(\sigma_X^2+\sigma_Y^2\right)x^2+2\left[\sigma_X^2\,(\mu_Y-z)-\sigma_Y^2\mu_X\right]x+\left[\sigma_X^2\,(z-\mu_Y)^2+\sigma_Y^2\mu_X^2\right]}{2\sigma_X^2\sigma_Y^2} \\
=& -\frac{\left[\sqrt{\sigma_X^2+\sigma_Y^2}\,x+\dfrac{\sigma_X^2\,(\mu_Y-z)-\sigma_Y^2\mu_X}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right]^2-\left[\dfrac{\sigma_X^2\,(\mu_Y-z)-\sigma_Y^2\mu_X}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right]^2+\left[\sigma_X^2\,(z-\mu_Y)^2+\sigma_Y^2\mu_X^2\right]}{2\sigma_X^2\sigma_Y^2}
\end{aligned}
\tag{7}
$$

Let us first simplify the last two terms in the numerator.

$$
\begin{aligned}
& -\left[\frac{\sigma_X^2\,(\mu_Y-z)-\sigma_Y^2\mu_X}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right]^2+\left[\sigma_X^2\,(z-\mu_Y)^2+\sigma_Y^2\mu_X^2\right] \\
=& \frac{-\sigma_X^4\,(\mu_Y-z)^2+2\sigma_X^2\sigma_Y^2\mu_X\,(\mu_Y-z)-\sigma_Y^4\mu_X^2+\left(\sigma_X^4+\sigma_X^2\sigma_Y^2\right)(z-\mu_Y)^2+\left(\sigma_X^2\sigma_Y^2+\sigma_Y^4\right)\mu_X^2}{\sigma_X^2+\sigma_Y^2} \\
=& \frac{2\sigma_X^2\sigma_Y^2\mu_X\,(\mu_Y-z)+\sigma_X^2\sigma_Y^2\,(z-\mu_Y)^2+\sigma_X^2\sigma_Y^2\mu_X^2}{\sigma_X^2+\sigma_Y^2} \\
=& \frac{\sigma_X^2\sigma_Y^2\left[2\mu_X\,(\mu_Y-z)+\left(z^2-2\mu_Y z+\mu_Y^2\right)+\mu_X^2\right]}{\sigma_X^2+\sigma_Y^2} \\
=& \frac{\sigma_X^2\sigma_Y^2\left[z^2+(-2\mu_X-2\mu_Y)\,z+\left(2\mu_X\mu_Y+\mu_Y^2+\mu_X^2\right)\right]}{\sigma_X^2+\sigma_Y^2} \\
=& \frac{\sigma_X^2\sigma_Y^2\left[z^2-2\,(\mu_X+\mu_Y)\,z+(\mu_X+\mu_Y)^2\right]}{\sigma_X^2+\sigma_Y^2} \\
=& \frac{\sigma_X^2\sigma_Y^2\,[z-(\mu_X+\mu_Y)]^2}{\sigma_X^2+\sigma_Y^2}
\end{aligned}
\tag{8}
$$

Now, let us put together our results.

$$
\begin{aligned}
f_Z(z) &= \frac{1}{2\pi\sigma_X\sigma_Y}\int_{-\infty}^{+\infty}\exp\left\{-\frac{\left[\sqrt{\sigma_X^2+\sigma_Y^2}\,x+\dfrac{\sigma_X^2(\mu_Y-z)-\sigma_Y^2\mu_X}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right]^2}{2\sigma_X^2\sigma_Y^2}-\frac{[z-(\mu_X+\mu_Y)]^2}{2\left(\sigma_X^2+\sigma_Y^2\right)}\right\}dx \\
&= \frac{1}{2\pi\sigma_X\sigma_Y}\exp\left\{-\frac{[z-(\mu_X+\mu_Y)]^2}{2\left(\sigma_X^2+\sigma_Y^2\right)}\right\}\int_{-\infty}^{+\infty}\exp\left\{-\left[\frac{\sqrt{\sigma_X^2+\sigma_Y^2}}{\sqrt{2}\sigma_X\sigma_Y}x+\frac{\sigma_X^2(\mu_Y-z)-\sigma_Y^2\mu_X}{\sqrt{2}\sigma_X\sigma_Y\sqrt{\sigma_X^2+\sigma_Y^2}}\right]^2\right\}dx
\end{aligned} \tag{9}
$$

The next step is of course to carry out the integral. Let

$$
u=\frac{\sqrt{\sigma_X^2+\sigma_Y^2}}{\sqrt{2}\sigma_X\sigma_Y}x+\frac{\sigma_X^2(\mu_Y-z)-\sigma_Y^2\mu_X}{\sqrt{2}\sigma_X\sigma_Y\sqrt{\sigma_X^2+\sigma_Y^2}}, \tag{10}
$$

then $u=-\infty$ when $x=-\infty$; $u=+\infty$ when $x=+\infty$; $dx=\dfrac{\sqrt{2}\sigma_X\sigma_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\,du$. The integral simplifies to

$$
\int_{-\infty}^{+\infty}\exp\left(-u^2\right)\frac{\sqrt{2}\sigma_X\sigma_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\,du=\frac{\sqrt{2}\sigma_X\sigma_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\int_{-\infty}^{+\infty}\exp\left(-u^2\right)du=\frac{\sqrt{2\pi}\sigma_X\sigma_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}. \tag{11}
$$

Finally, we obtain

$$
f_Z(z)=\frac{1}{\sqrt{2\pi\left(\sigma_X^2+\sigma_Y^2\right)}}\exp\left\{-\frac{[z-(\mu_X+\mu_Y)]^2}{2\left(\sigma_X^2+\sigma_Y^2\right)}\right\}. \tag{12}
$$

In other words, if $X\sim\mathcal{N}\left(\mu_X,\sigma_X^2\right)$ and $Y\sim\mathcal{N}\left(\mu_Y,\sigma_Y^2\right)$, $Z=X+Y\sim\mathcal{N}\left(\mu_X+\mu_Y,\sigma_X^2+\sigma_Y^2\right)$. Applying what we have just proved to the variables $X\sim\mathcal{N}(0,1)$, $Y\sim\mathcal{N}(1,2)$ and $Z=X+Y$, we immediately conclude that $Z\sim\mathcal{N}(0+1,1+2)=\mathcal{N}(1,3)$.

(b) See the program file `add_two_gaussian_variables.py`. The program is written in `Python` with the libraries `NumPy` and `Matplotlib` and was executed using `Spyder`. 50000 samples were respectively generated for X and Y. According to the maximum likelihood estimation, to fit a normal distribution to the generated samples for Z, the mean and variance should be chosen to be the mean and variance of the samples, which were 0.9978785055860286 and 3.0002427891910344 respectively. In other words, the fitted normal distribution for Z was $\mathcal{N}(0.99788, 3.00024)$, which agrees with the theoretical distribution of $\mathcal{N}(1,3)$. As can be seen from Fig. 2, the variable Z does behave as normally distributed and the two PDFs for the fitted and theoretical distributions almost overlap each other exactly.

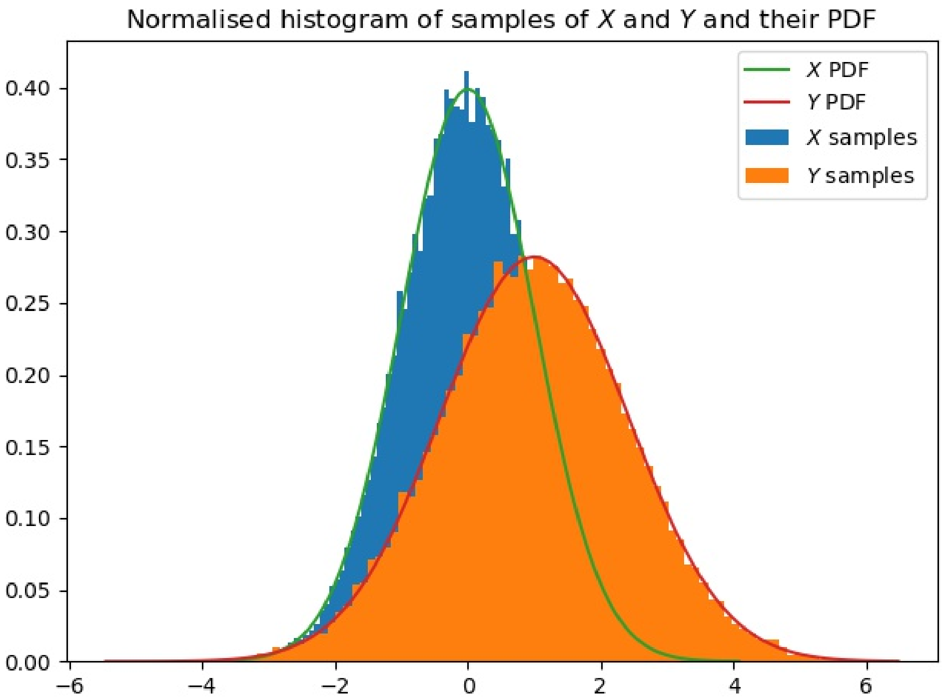

Figure 1: Normalised histogram of the generated samples for the two independent normally distributed random variables $X \sim \mathcal{N}(0, 1)$ and $Y \sim \mathcal{N}(1, 2)$ and their PDF.

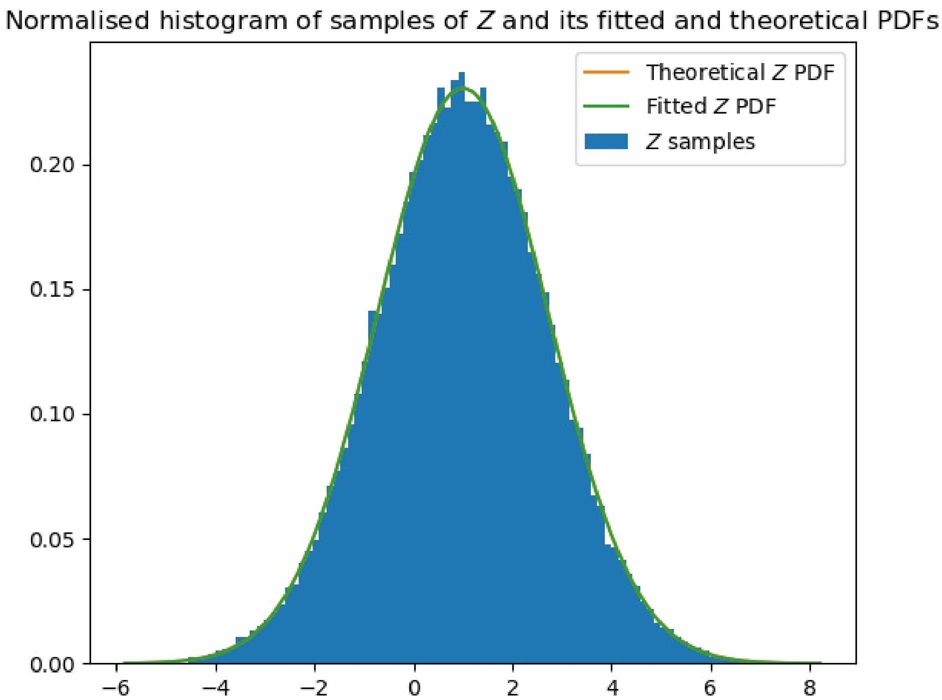

Figure 2: Normalised histogram of the samples for the random variable $Z = X + Y$ constructed using the generated samples for X and Y shown in Fig. 1. The numerically fitted and theoretical PDFs for Z are also plotted.