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Extending Kolkata Paise Restaurant Problem to Dynamic Matching in Mobility Markets

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Abstract

In mobility markets – especially vehicle for hire markets – drivers offer individual transportation by car to customers. Drivers individually decide where to go to pick up customers to increase their own utilization (probability of carrying a customer) and utility (profit). The utility drivers retrieve from customers comprises both costs of driving to another location and the revenue from carrying a customer and is thus not shared between different drivers. In this thesis, I present the Vehicle for Hire Problem (VFHP) as a generalization of the Kolkata Paise Restaurant Problem (KPRP) to evaluate different strategies for drivers in vehicle for hire markets. The KPRP is a multi-round game model presented by Chakrabarti et al. (2009) in which daily laborers constitute agents and restaurants constitute resources. All agents decide simultaneously, but independently where to eat. Every restaurant can cater only one agent and agents cannot divert to other resources if their first choice is overcrowded. The number of agents equals the number of resources. Also, there is a ranking of restaurants all agents agree upon, and no two resources yield the same utility. The VFHP relaxes assumptions on capacity and utility: Resources (customers) are grouped in districts, agents (drivers) can redirect to other resources in the same district. As the distance between agent and resource reduces the agent's utility and the location is not identical for all agents, the utility of a given resource is not identical for all agents. To study the impact of the different assumptions, I build four different model variants: Individual Preferences (IP) replaces the shared utility of the KPRP with uniformly distributed utilities per agent. The Mixed Preferences (MP) model variant uses the utility assumption of the VFHP, but the capacity of all districts remains 1. The Individual Preferences with Multiple Customers per District (IPMC) model variant groups customers in districts, and uses the uniform utilities introduced in the IP model variant. Mixed Preferences and Multiple Customers per District (MPMC) implements all assumptions of the VHFP. In this thesis, I study different strategies for the KPRP and all variants of the VFHP to build a foundation for an incentive scheme for dynamic matching in mobility markets. The strategies comprise history-dependent and utility-dependent strategies. In history-dependent strategies, agents incorporate their previous decisions and the utilization of resources in previous iterations in their decision. Agents adapting utility-dependent strategies choose the resource offering the highest utility with a given probability.

Keywords: vehicle for hire markets; distributed decision making; agent-based modelling; congestion game; limited rationality

1. Introduction

Mobility markets, or in particular vehicle for hire markets, comprise all modes of shared, but individual transportation with a driver, in particular with a short-term focus (e.g., taxis, Lyft, and Uber). In mobility markets, drivers individually decide where to look for customers. However, the average idle time of taxis is about 25–50% in most cities where data is available (Linne+Krause Marketing-Forschung, 2011; Cramer and Krueger, 2016; Linne+Krause Marketing-Forschung, 2016). Though excess capacity can partially explain these numbers, utilization could be increased, if drivers would be distributed across the city more efficiently. In contrast to underutilization, passengers have to wait for more than 20 minutes in approximately every third case in other cities (Rayle et al.,

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2014), suggesting that the drivers are not at the locations where they are needed.

To address these inefficiencies in vehicle for hire markets, coordinators could instruct drivers where to wait for customers. In current business models, however, this is not possible, since drivers are not employees of the coordinators. Hence, they try to maximize their individual profits by deciding independently where to look for customers without considering the social welfare or utilization of other agents. In practice, there are approaches like 'surge pricing' (price adapts dynamically to changes is demand and supply with the goal to influence demand and supply, e.g. increase supply by increased price) to respond to expected peaks in demand, though literature on the efficiency of different driver strategies is limited (Chen and Sheldon, 2015; Hall et al., 2015; Rogers, 2015). One, therefore, has to turn the attention to the coordination amongst drivers: Drivers maximize their individual utility, but their utility inversely depends on the number of agents selecting the same option. Thus, drivers benefit if there are less other drivers in the same district than available customers, thus, deciding against the crowd is beneficial. Alternatively, one could construct a game model derived from the *College Admission Problem* or *Stable Marriages Problem* (Gale and Shapley, 1962; Manlove and Sng, 2006; Abraham et al., 2007; Akbarpour et al., 2016). In these problems, agents try different matches until an optimal match is found. Yet, in vehicle for hire markets, I assume that redirecting to another resource, if the preferred resource is not available, is not an option, because of the costs and time constraints of redirecting (requires the agents to drive to another location consuming time and fuel).

To analyze the fundamental underlying problem, I propose a repeated non-cooperative game model to investigate different strategies in the coordination problem among drivers. It is a generalization of the Kolkata Paise Restaurant Problem (KPRP) (Chakrabarti et al., 2009) where agents repeatedly compete for a set of resources. As a foundation to be able to assess coordinators' incentives like 'surge pricing', one first needs to understand the fundamental impact of different driver strategies. I contribute to this research field by game model, relaxing assumptions of the KPRP. In contrast to existing research, I address both individual agent preferences and different resource capacities. Besides the game model, the contributions of this research are different mixed strategies for the model and an analysis of their impact on car utilization and driver utilities in different settings. These insights constitute building blocks for a characterization of favorable agent behavior to design incentive mechanisms to distribute drivers efficiently.

1.1. The Vehicle for Hire Problem and its Model Variants

In this thesis, I cover five different, but related model variants: The Kolkata Paise Restaurant Problem and four relaxations suited for mobility markets comprising the Vehicle for Hire Problem (VFHP).

> In Kolkata, there were very cheap and fixed-rate 'Paise Restaurants', popular among the daily laborers in the city. During lunch hours, the laborers used to walk down (to save the transport costs) to any of these restaurants and would miss the lunch if they arrived at a restaurant where their number is more than the capacity of the restaurant for such cheap lunch. Walking down to the next restaurant would mean failing to report back to the job in time! Paise means the smallest Indian coin and there were indeed some well known rankings of these restaurants as some of them would offer more tastier items compared to the others. (Chakrabarti et al., 2009, p. 2421)

The KPRP was first presented by Chakrabarti et al. (2009). In this model, N agents (that is daily laborers) aim at having lunch at one of the N restaurants. All agents gain the same utility from some restaurant, and all restaurants have mutually different utilities. Every agent aims at getting lunch at his preferred restaurant, but every restaurant can only cater a single agent. Thus, if more than 1 agent goes to some restaurant, some agents will not get lunch, as they cannot divert to another restaurant that same day. The KPRP is a repeated game with an infinite number of iterations.

In mobility markets, drivers $i \in I$ constitute agents and customers $j \in J$ (located in districts $k \in K$) constitute resources. Agents drive to resources. Agents carry resources (up to the capacity limit). For this thesis I relax two main assumptions: Agents no longer retrieve identical utility from a given resource, but one agent can prefer resource j and another agent can prefer resource $j' \neq j$ (with the highest utility determining preference). I present two different models: In the Individual Preferences model (IP), utilities are uniformly assigned to resources (customers). Thus, agent preferences are independent of each other. In the Mixed Preferences model (MP), utilities are calculated as a weighted average of an individual component (that is distance between agent and customer) and a shared component (that is the payoff). I further model increased capacity: Clustering customers $j \in J$ in districts $k \in K$ allows agents to divert to other customers inside the district they drove to. The average number of customers per district is φ, and the customers randomly "choose" the district they belong to, the number of customers per district is thus Gaussian distributed around φ. The Individual Preferences with

Multiple Trips per Customer model (IPMC) combines the IP model with the clustering concept: Agents gain random utilities from customers and customers belong to districts. In the Mixed Preferences with Multiple Trips per Customer model (MPMC), the utility is obtained as a weighted average of an individual component to model the distance and a shared component to model the payoff. The distance (and thus the individual component) is equal for all customers belonging to one district.

1.2. Outline of this Thesis

The remainder of this thesis is organized as follows: I first discuss related work in chapter 2, I then present the strategies (chapter 3). The successive chapters present the individual model variants and assess the performance of aforementioned strategies. Chapter 4 focuses on the KPRP, chapter 5 presents the IP model variant, chapter 6 gives insight in the MP model variant, chapter 7 concerns the IPMC model variant, and chapter 8 evaluates the MPMC model variant. To improve the reader's understanding, chapters 4-8 can be read independently from each other, as key concepts are presented in each of them. Chapter 9 discusses the results from chapters 4-8, and chapter 10 concludes this thesis.

2. Related Work

To my knowledge, no paper extends the KPRP for mobility markets. Relevant research is conducted in three fields: First, I give an overview of relevant game models in other application areas, in particular coordination games. Second, there is literature in optimization and operations research in the field of vehicle for hire markets. Third, I introduce basic literature of dynamic mechanism design.

2.1. Congestion Games

The presented model is a type of congestion game, a model for games in which agents should choose different alternatives to succeed first described by Rosenthal (1973). Mathematically, congestion games can be identified by their potential function and thus their pure-strategy Nash-equilibria; Congestion games are therefore also Potential games (Monderer and Shapley, 1996; Nash, 1951). Yet, such a Nash equilibrium is usually inefficient, as Correa et al. (2005) prove. Other congestion game models are the El Farol Bar Problem (Arthur, 1994), the KPRP (Chakrabarti et al., 2009), the Crowding Game (Milchtaich, 1996), and the minority game (Challet and Zhang, 1998).

The El Farol Bar Problem is a game model with N agents (scientists) and one resource (the bar in Santa Fe during Karaoke night). All agents aim at maximizing their profit. If more than $0.6 \cdot N$ agents go to the bar, it becomes overcrowded, and the agents would enjoy themselves more at home. If fewer agents go to the bar, they enjoy themselves more than if they stayed at home. Agents, therefore, coordinate themselves such that as many agents as possible (but less than $0.6 \cdot N$) go to the bar (Arthur, 1994).

The KPRP is the foundation game model for this thesis; the model is described in chapter 4 in more detail. Chakrabarti et al. (2009) and Ghosh et al. (2013) introduce strategies for increasing the utilization of the KPRP. Yang et al. (2016) study a generalization of the KPRP which is also aimed at dynamic markets: As a relaxation of the KPRP they study whether an agent should divert to another district or stay in the current one with different capacities for different districts. Agents are being replaced by others (which do not have the same prior knowledge) following a Poisson distribution. They prove the existence of a *Mean Field Equilibrium* (Lasry and Lions, 2007) for the *Threshold Strategy* (if a capacity threshold is exceeded at time t, agents stochastically divert to other districts) (Yang et al., 2016). This thesis on the opposite compares different strategies. Agarwal et al. (2016) generalize the KPRP to a *Majority Game*, in which they study convergence behavior given only few prior knowledge. In difference to the KPRP, capacity is not restricted, and in difference to the problem in mobility markets agents have no internal utility ranking, they aim at choosing with the herd.

The Crowding Game is a game model in which the utility of agents only depends on the number of agents also selecting the same option. If more agents select one option, the utility decreases (Milchtaich, 1996). The VFHP game model is similar to the Crowding Game as the number of agents decreases the utility (as the expected utility is divided among all agents selecting some resource), but this model also uses a basic utility which is not shared among agents.

The Minority Game is a game with N agents and two resources, and the utility for those agents choosing the resource with the lower occupancy is higher than the utility for those agents in the crowded resource (i.e. roads) (Challet and Zhang, 1998). In a recent study, "treatments" (which differ in the information given to participants) for the Minority Game were studied with experiments. The authors state that changing from one option to the other is not recommended regardless of prior knowledge (Chmura and Pitz, 2006). Because the Minority Game only allows two different payoffs from two different resources, I cannot directly transfer this insight to the Kolkata Paise Restaurant Problem in mobility markets.

2.2. Vehicle for Hire Market

There is only limited research work available on optimal distribution of drivers in vehicle for hire markets. Several studies focus on assigning drivers an optimal district where they await passengers (Lee et al., 2004; Seow et al., 2010); though, in most business models, drivers decide independently. Yang et al. (2005) study a model with varying demand and supply. Taxi drivers individually decide when to enter the market and when to leave it, resulting in a market equilibrium. This work does not study utility, but only utilization. Kim et al. (2011) propose an agent-based model incorporating real-world passenger travel pattern to predict the highest possible utility. Their model also incorporates districts ("areas") and varying utility functions over time, but tests for different criteria: Whilst I analytically derive utilization and utility for different strategies in a large environment, Kim et al. (2011) studies a setting with five nodes and retrieves utilization and passenger wait time for varying fleet sizes. Wong's primary criterion is reduced vacant mileage for taxis (Wong et al., 2015). He uses a two-step approach in which taxis can only divert to adjacent zones rather than all others. Trigo et al. (2006) uses Multi-Agent Markov Decision Processes to model drivers transporting passengers. This paper uses a cover story which is highly similar to ours, but rather than using stochastic strategies, Trigo et al. (2006) use a two-layered learning process. This thesis aims at improving the taxi allocation with respect to utilization fraction or utility assuming choice at discrete time steps. Li (2006) on the opposite studies strategies to minimize passenger waiting time or travel time, taxi idle time or non-live mileage with drivers deciding asynchronously. This thesis studies a large variety of strategies, Li (2006) restricts himself to three simple strategies. The paper concludes that returning to hotspots after serving a trip can increase all studied parameters. Similar results can also be seen in this thesis, as the utilization fraction increases after introducing multiple trips per district.

Li et al. (2011) present a model which predicts whether agents should wait for passengers stationary or continue driving to "hunt down" customers. They use data mining techniques with data on time, location, and strategy (hunt or wait). In the VFHP, all agents decide where to drive to (yet, the location might not change). Thus, the strategy of the VFHP dictates where to go rather than if to go to another location. The model by Li et al. (2011) cannot predict where taxi drivers should drive. Ge et al. (2010) build a recommender system to reduce the travel distance before carrying the next customer. This behavior is reflected by the VFHP game model, as the individual utility models distance. Yuan et al. (2011) extend the work by Ge et al. (2010) by also recommending optimal passenger behavior.

Alonso-Mora et al. (2017) postulate that it should be possible to replace 13,000 cabs in New York City by only 3,000 on-demand vehicles for ride-sharing, which would both reduce wait time and traffic congestion. Their calculations suggest that a better utilization fraction of cabs can be achieved, though ride-sharing is not considered in this thesis. Furthermore, using graph traversals for optimal distribution and routing of taxis is a solution a single driver cannot adopt, but only dispatchers.

Shi and Lian (2016) study the taxi transportation market from the opposite side as this thesis paper does: Passengers can decide whether or not they are queueing for a taxi (depending on the "queue length" (number of passengers) and the "buffer size" (number of cabs) at the taxi stand). The authors compare strategies of selfish and social passengers and options for the government to interfere.

Furthermore, there are several papers in the field of operations research which focus on the influence of regulation (taxi medallions, fixed rates) on the market (Cairns and Liston-Heyes, 1996; Arnott, 1996). In the VFHP game model, I assume that there are sufficient agents to carry every customer and sufficient customers such that every agent can carry a customer.

2.3. Dynamic Mechanism Design

There is early stage work on dynamic mechanism design in matching markets: If there is a dispatcher, he can make agents wait for a better suited trip. Kurino (2009) gives a dynamic version of the House Allocation Problem. Bloch and Houy (2012) periodically redistribute items between agents.

If agents are allowed to choose independently from a dispatcher, waiting time might influence their choice, reducing welfare. In this component – choosing the best individual option reduces social optimality – the problem described by Leshno (2012) is highly similar to the KPRP. Yet, unlike environments described in the paper (e.g., nursing homes, subsidized housing), there are no "overloaded waiting lists" (demand tremendously exceeds supply) in the taxi industry, as passengers usually have other means of transportation to choose from.

Social Welfare (benefit for the entire group) in transportation markets has been studied at the example of Rotterdam Port: Transportation tasks inside the port are assigned to trucks which are waiting for departure. The authors claim that a higher number of participants in general increases social welfare (as it is easier to adapt to peak load times), but agents might not continue participating if they assumed that the game put them at a disadvantage in comparison to other players. They, therefore, postulate an algorithm which ensures that agents are equally utilized (Ye and Zhang, 2016; Ye et al., 2017). In the KPRP on the opposite, I assume that the number of customers always equals the number of agents (agents will always participate), but agents are not assigned their trip.

Chen and Hu (2016) conduct research on market design in a market place with buyers and sellers such as Uber: In such markets, buyers wait for lower market prices while sellers wait for higher market prices. They conclude that fast changes in the market price (set by an intermediary) and price surges are not recommended, as participants might leave the market temporarily. This thesis on the opposite assumes myopic agents, who only plan ahead few time steps.

3. Strategies

In this thesis I consider seven strategies: No Learning (NL), Rank Dependent Choice (RD), Limited Learning (LL), One Period Repetition (OPR), Crowd Avoiding (CA), Stochastic Crowd Avoiding (SCA), and Stochastic Rank Dependent Choice (SRD). NL and RD are baseline strategies which represent basic behavior. RD, LL, OPR, and SRD incorporate the resource's utility in the agents' choices and are therefore utility-based. LL, OPR, CA, and SCA require knowledge about previous iterations and are therefore history-based.

The NL strategy dictates agents to randomly choose a resource in every iteration, regardless of history (hence the term "No Learning") or resource utility. Resources are either customers or districts (or restaurants in the KPRP). The strategy was first presented by Chakrabarti et al. (2009) in which restaurants comprise resources.

The second baseline strategy is the strategy RD. Agents always drive to the resource yielding them maximum utility. Agents thus receive maximum utility, if they carry a customer. If there are several resources yielding equal utility, agents decide randomly between all maximum utility resources. I introduce this strategy, as it mimics simple behavior if limited information is available: If agents do not know about the preferences or behavior of other agents, but assume that only a few agents share the same preference, the most simple approach is to always head for the preferred resource. It requires only very few computational power: Prior to the first iteration, agents calculate their preferred customer by comparing the utility of all resources. After driving there, they will remain in their position, requiring no recomputation at all. It also requires no information except the own utilities or preferences, making it suitable for large problem spaces.

Agents incorporating the LL strategy follow a two-step approach: (1) If an agent carried a customer at time t, he will drive to the highest utility resource at time $t+1$. (2) If an agent did not carry a customer at time t, he will randomly choose any other resource at time $t+1$. (If an agent was successful at the highest utility resource, he will return there in the next iteration). The LL strategy was presented by Chakrabarti et al. (2009) (named Limited Learning 1).

The OPR strategy requires agents to follow a three-step approach: (1) If an agent carried customer j at time t (but not at time $t-1$), he will return to this resource at time $t+1$ (*return*). (2) If an agent served the same resource j at time $t-1$ and t, he will compete for the highest utility customer at time $t+1$ (*improve*). (3) If an agent did not carry any customer at time t, he will randomly choose any resource which was vacant at time t in the next iteration (*random*). OPR was also introduced in Chakrabarti et al. (2009).

With the CA strategy agents only drive to resources which were vacant or had remaining capacity at time $t-1$. This strategy originates in a paper by Ghosh et al. (2013).

Agents using the SCA strategy stochastically decide whether to return to the same resource or to randomly turn to another resource. If a resource j does not exceed its capacity at time t, all agents driving to this resource j at time t will return there at time $t+1$. If the capacity is exceeded, all agents stochastically either return to j or drive to any other (randomly chosen) resource at time $t+1$ such that the expected number of agents in j equals its capacity (let the capacity be c_j and the number of agents at the resource be o_j: return with probability $\frac{c_j}{o_j}$, randomly choose another resource with probability $1-\frac{c_j}{o_j}$). The SCA strategy stems from Ghosh et al. (2013).

The SRD strategies build upon the RD strategy, including some properties of the SCA strategy: Let the capacity of a resource j be c_j, and let the number of agents preferring resource j be p_j (agents who cannot retrieve higher utility from any other resource). Agents drive to their preferred resource if its capacity is not exceeded, that is $c_j \geq p_j$. Otherwise, they stochastically drive to j with probability $\frac{c_j}{p_j}$ and redirect to another resource with probability $1-\frac{c_j}{p_j}$. Thus, the expected number of agents preferring a resource j driving to that resource j is c_j, if at least c_j agent prefer j, and p_j otherwise. The resource agents divert to can be one of the following: (SRD1) Any customer which is noone's first choice; (SRD2) any other customer; (SRD3) his second choice customer; or (SRD4) the best customer which is noone's first choice. SRD3 and SRD4 are an extension of SRD2 and SRD1 respectively, increasing the average utility of successful agents, that is agents carrying a customer. If the first preferences of different agents are not independent, this likely also apply for the alternate preferences in SRD3 and SRD4, decreasing the utilization fraction. All SRD strategies require information about the first preferences of all other agents which can be acquired by a single iteration of RD upfront. Then all agents know how many other agents share the same top preference, making the second iteration identical (SRD2) or similar (SRD1, SRD3, SRD4) to SCA, as all agents redirect based upon the number of agents in the chosen district during the previous iteration. In addition to the number of agents preferring the same resource, the SRD1 strategy also requires information about the number of agents preferring all other resources which one could also retrieve in a single iteration of RD upfront.

Thus, the SRD1 strategy does not require too much information, if the number of iterations is sufficiently high to compensate for a potentially very low utility during the first iteration. The SRD2 strategy requires less information than the SRD1 strategy, as it only incorporates the number of agents preferring the resource they prefer themselves. It is thus beneficial if the information about other resources cannot be determined easily. The SRD3 strategy also requires only very few information (as much as SRD2), but the utility of successful agents is higher, as all successful agents receive a high utility (maximum or second highest utility). If the agent utilities of different agents are not stochastically independent, there can be a high number of resources noone drives to, neither as first nor as second preference. In many cases, the second preference of an agent is the first preference of another agent, thus not exploiting the full potential. In SRD4, the second preference is only chosen, if no agent prefers this resource. Thus, the set of first choice resources and the set of alternate choice resources do not intersect, making it impossible that alternate choice agents carry a customer who is preferred by another agent increasing the average utility. Yet, SRD4 requires more information about the preferences of other agents than SRD3. Thus, the existence of all strategies is justified by their different data requirements comparing to the expected performance. The performance of the different strategies with respect to the metrics utilization fraction and utility depends on the actual model variant.

4. Kolkata Paise Restaurant Problem

In their paper, Chakrabarti et al. (2009) discussed different strategies and provided simulations.

In the following, I will briefly reproduce their results analytically.

4.1. The Model

In the KPRP cover story, daily laborers $i \in I, |I| = N$ represent agents who select a restaurant $j \in J, |J| = N$ for lunch. Agents select (i.e. randomly) a restaurant to which they drive. Formally, I use $d(i,j)$ to represent that i goes to j.

$$d(i,j) = \begin{cases} 1 & \text{if agent } i \text{ goes to restaurant } j \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Definition 4.1)}$$

$$\forall j : o_j = \sum_{i \in I} d(i,j) \qquad \text{(Definition 4.2)}$$

Obviously, one agent can only go to one restaurant ($\forall i : \sum_{j \in J} d(i,j) = 1$). Every restaurant $j \in J$ can cater exactly one agent $i \in I$.

$$c(i,j) = \begin{cases} 1 & \text{if agent } i \text{ eats at restaurant } j \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Definition 4.3)}$$

If no agent went to j, j does not cater any agent, if more than one agent goes to restaurant j, only one will be served ($\forall j : c(i,j) = \min\left(\sum_{i \in I} d(i,j), 1\right)$). Agents can only eat at restaurants they went to ($\forall i, j : c(i,j) \leq d(i,j)$). The utility $u(i,j)$ agents receive from eating at a restaurant is a random permutation and is identical for all agents (resulting in a shared utility $u_s(j)$), that is $\forall j : u(i,j) = u_s(j)$ and $\forall j, j' : u_s(j) \neq u_s(j') \vee j = j'$). A daily laborer (agent) prefers a restaurant if no other restaurant yields higher utility for him. The number of agents preferring a restaurant j is denoted as p_j.

$$p(i,j) = \begin{cases} 1 & \text{if } \forall j' \in J \setminus \{j\} : u(i,j) \geq u(i,j') \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Definition 4.4)}$$

$$\forall j : p_j = \sum_{i \in I} p(i,j) \qquad \text{(Definition 4.5)}$$

The utilization fraction f is given as the number of agents getting lunch divided by the total number of agents. If an agent i gets lunch is given by $f(i)$ which is 0, if i ate at no restaurant, and 1 otherwise (as every agent can eat at maximum one restaurant).

$$f = \frac{1}{N} \cdot \sum_{i \in I} f(i) \qquad \text{(Definition 4.6)}$$

$$f(i) = \sum_{j \in J} c(i,j) \qquad \text{(Definition 4.7)}$$

The overall utility u is average utility per agent. The agent utility $u(i)$ is $u(i,j)$, if i eats at j and 0 otherwise.

$$u = \frac{1}{N} \cdot \sum_{i \in I} u(i) \qquad \text{(Definition 4.8)}$$

$$u(i) = \sum_{j \in J} u(i,j) \cdot c(i,j) \qquad \text{(Definition 4.9)}$$

In experiments and simulations, I further assume $N = 1000$ (1000 agents and 1000 customers), and that customers are indexed by their utility ($u_s(j) = \frac{j}{N}$). Thus, the utility is uniformly distributed such that $u_m = u_{max} = 1$ is the utility of agents eating at their preferred restaurant, and $u_{avg} = 0.5$ is the expected utility of agents eating at any other restaurant.

4.2. Theoretic Foundations

The capacity of all restaurants is 1. All agents prefer the same restaurant j_p. Thus, the probability that $j \in J$ is preferred by exactly p_j agents is 1 for j_p and $p_j = N$ and 0 otherwise.

4.3. No Learning

As a baseline comparison Chakrabarti et al. (2009) give an entirely random selection: In every iteration, every agent selects one of the restaurants at random.

In Chakrabarti et al. (2009) they give the formula equation 1 as probability $P\left(o_j\right)$ for o_j agents choosing the same restaurant, if on average λ agents go to the same restaurant. Equation 2 simplifies equation 1 by setting $\lambda = 1$. With $N \to \infty$, one can further simplify the formula using the Poisson Limit Theorem.

$$P\left(o_j\right) = \binom{\lambda N}{o_j} \frac{1}{N}^{o_j} \left(1 - \frac{1}{N}\right)^{\lambda N - o_j} = \frac{\lambda^{o_j}}{o_j!} e^{-\lambda} \tag{1}$$

$$= \binom{N}{o_j} \left(\frac{1}{N}\right)^{o_j} \left(1 - \frac{1}{N}\right)^{N - o_j} = \frac{1}{o_j!} e^{-1} \tag{2}$$

Therefore, $P(0)$ gives the probability of a restaurant being unoccupied any evening using this random stategy, making $1 - P(0) \approx 63.2\%$ the average utilization.

I, therefore, expect a Gaussian distribution around $f = f_{NL} = 63.2\%$ for the utilization fraction. As agents on average receive average utility (if they are successful), I conclude that the utility is $u = f \cdot u_{avg} = 0.316 \cdot u_{max}$.

4.4. Rank Dependent Choice

Agents $i \in I$ incorporating the RD strategy always turn to the restaurant j that yields them the highest utility $(d\left(i,j\right) = 1 \iff \forall j' : u\left(i,j\right) \geq u\left(i,j'\right))$.

In the KPRP, the restaurant with the highest utility and thus the first preference restaurant is identical for all agents $(\forall i, i' \in I : u\left(i,j\right) = u\left(i',j\right))$. Thus, all agents $i \in I$ go to the same restaurant j. This restaurant can only cater a single agent, resulting in a utilization fraction of $f = \frac{1}{N}$. For $N = 1000$, I, therefore, expect $f = f_{RD} = 0.1\%$. The (single) successful agent receives maximum utility, resulting in $u = 0.001 \cdot u_{max}$ on average.

4.5. Limited Learning

With this strategy, all agents choose a restaurant at random the first night. The utilization therefore is Gaussian distributed around 63.2%. During successive nights, all agents base their choice on whether they got dinner the previous day (Chakrabarti et al., 2009):

- If some agent got food at time t, he will choose the highest ranking restaurant at time $t+1$. (If an agent was successful at the highest utility restaurant, he will return there in the next iteration)
- If some agent did not get food at time t, he will randomly choose any other restaurant at time $t+1$.

The first case is irrelevant for the KPRP, as the utilization fraction for this part is $f_{RD} = \frac{1}{N}$ (with f_{RD} as the utilization fraction of the RD strategy or fraction of carried customers by an agent preferring them), with $N \to \infty$ the utilization fraction gets negligibly small (or $f_{RD} = 0.1\%$ for $N = 1000$). The second case is given by $\lambda = 1 - f$ in equation 1 (the ratio between agents and restaurants is $(1-f) : 1$). Chakrabarti et al. (2009) give the following recursion relation:

$$f_{t+1} = 1 - e^{-\lambda_t}; \lambda_t = 1 - f_t \tag{3}$$

In a more generalized fashion, I write:

$$f_t = \underbrace{f_{t-1} \cdot f_{RD}}_{\text{first try best}} + \underbrace{\left(1 - e^{-(1-f_{t-1})}\right)}_{\text{random or return}} \tag{4}$$

If one assumes that f converges as $f_{t+1} = f_t$, the utilization will be Gaussian distributed around an average value of $f = 43.3\%$ and $u = f \cdot u_{avg} = 0.212 \cdot u_{max}$.

4.6. One Period Repetition

All agents choose the restaurant randomly the first evening.

- If some agent got dinner at restaurant j at time t (but not at time $t-1$), he will return to this restaurant at time $t+1$ (*return*).
- If some agent got dinner at the same restaurant j at time $t-1$ and t, he will compete for the highest utility restaurant at time $t+1$ (*improve*).
- If some agent did not get dinner at any restaurant at time t, at time $t+1$ he will randomly choose any restaurant which was vacant at time t (*random*).

In their paper, Chakrabarti et al. (2009) both give the distribution and simulation results.

The probability distribution of utilizations is given by equation 6 with x_t being the fraction of agents returning to the same restaurant at time $t+1$, and thus the fraction of agents eating at a randomly chosen restaurant at time t. As all agents who do not eat at a restaurant at time $t-1$ choose a restaurant randomly and are successful with probability f_{NL}, Chakrabarti et al. (2009) assume that $x_t = (1 - x_{t-1}) \cdot f_{NL}$. x_t is also the fraction of agents improving at $t+2$ (in this case, the expected utilization is $f_{RD} = \frac{1}{x_t N}$, it can therefore be ignored if $N \to \infty$).

$$f_t = x_{t-1} + (1 - x_{t-1}) \cdot \left(1 - e^{-1}\right) \tag{5}$$

$$f_{t+1} = (1 - x_t) \cdot \left(1 - e^{-1}\right) + \left(1 - (1 - x_t) \cdot \left(1 - e^{-1}\right)\right) \cdot \left(1 - e^{-1}\right) \quad (6)$$

In their paper, Chakrabarti et al. (2009) conclude that the fixed point of this right half of both equations in 6 is at $x \approx 0.38$ or $f \approx 0.77$, a result I cannot replicate in simulations.

Their original formula is not replicable: It only considers those agents who are not eating at their preferred restaurant (the utilization fraction for these agents is added in the second term). From the remaining $(1 - f_{RD}) \cdot N$ agents, a fraction of x_{t-1} agents returns to the previously chosen restaurant, and a fraction x_{t-2} tries eating at the highest utility restaurant (yet unsuccessful, as all successful agents contribute utilization via the second term). Thus, a fraction of $(1 - x_{t-1} - x_{t-2})$ of all agents randomly chooses a restaurant. These agents are successful with probability $f_{NL} = 1 - e^{-1}$. Chakrabarti et al. (2009) do not deduct x_{t-2}, as these agents are unsuccessful. In the next iteration, those agents who successfully randomly choose a restaurant ($(1 - x_{t-1} - x_{t-2}) \cdot f_{NL}$), become x_t. Assuming that x_t converges to a stable state ($x_t = x_{t+1} = x_{t+2}$), I can drop subscript t, resulting in a fraction x. The corrected formula is given in equation 7.

$$f = \underbrace{(x + (1 - 2 \cdot x) \cdot f_{NL}) \cdot (1 - f_{RD})}_{\text{random, return, and improve}} + \underbrace{f_{RD}}_{\text{best}} \quad (7)$$

The fraction x is given by $x = (1 - 2x) \cdot \left(1 - e^{-1}\right) \approx 27.9\%$, and f_t decreases to $f = 55.8\%$. The utility is given as $u = 0.279 \cdot u_{max}$.

Yet, one should notice that this strategy is promising for vehicle for hire markets: The best (highest utility) resources are different for different agents, thus, this share is not "lost", but will be added.

4.7. Crowd Avoiding

Agents using the CA strategy only choose restaurants which did not serve customers the previous evening.

The probability $P(0)$ of a restaurant being vacant at time $t = 1$ after being empty at time $t = 0$ is given by equation 8. As the number of restaurants to choose from at time $t = 1$ is reduced from 1 to $1 - f$, the average number of agents per restaurant needs to be set to $\lambda = \frac{1}{1-f}$ to cater for this change (in equation 1).

$$P(0) = e^{-\lambda} = e^{-\frac{1}{1-f}} \quad (8)$$

Incorporating $f = 1 - P(0)$ and the fact that only $1 - f$ restaurants are available into equation 8, yields the following equation:

$$f = (1 - f)\left(1 - e^{-\frac{1}{1-f}}\right) \quad (9)$$

Equation 9 has two solutions at $f_1 \approx 0.457$ and $f_2 \approx 1.872$, the latter being discarded as the utilization fraction cannot exceed 1. The utilization fraction is therefore $f = 45.7\%$. As all agents who eat at any restaurant receive average utility, I conclude that the utility is $u = 0.229 \cdot u_{max}$.

4.8. Stochastic Crowd Avoiding

Ghosh et al. (2013) also introduced another strategy in which the probability of returning to some place inversely depends on the number of agents choosing this restaurant ($ret_j(t) = \frac{1}{o_j(t-1)}$ with ret_j the probability of returning to restaurant j and $o_j(t-1)$ the number of agents at restaurant j at time $t-1$). Alternatively, this agent will choose any other restaurant with equal probability $\frac{o_j(t-1)-1}{o_j(t-1)} \cdot \frac{1}{N-1}$.

In their paper, Ghosh et al. (2013) give an expected utilization fraction of $f \approx 80\%$. My simulations give an average utilization fraction of $\bar{f} = 0.735$. This is still better than random (the only better than average strategy), but it is not as good as expected.

Ghosh et al. (2013) define that a_i is the share of restaurants with i agents (in our model, i is o_j) and $a_i = 0 \; \forall i > 2$. Thus, $a_0 + a_1 + a_2 = 1$ (number of restaurants), and $a_1 + 2 \cdot a_2$ (number of agents). In every iteration, the share of vacant restaurants (a_0) is newly calculated, it comprises those restaurants which were empty the previous iteration (prev), minus those restaurants to which some agent drives to who went to an a_2 restaurant the previous iteration (new) and those a_2 restaurants in which both agents from the previous iteration divert and no agent goes to (both leave).

$$a_0 = \underbrace{a_0}_{\text{prev}} - \underbrace{a_0 \cdot a_2}_{\text{new}} + \underbrace{\frac{a_2}{4} - a_2 \frac{a_2}{4}}_{\text{both leave}} \quad (10)$$

I assume that the difference emerges from the fact that the authors ignored that more than two agents can head for in the same restaurant. They state that the influence of a_i for $i > 2$ is negligibly small), yet, using $a_0 = a_2 + 2 \cdot a_3 + 3 \cdot a_4 + \ldots$ the accumulated impact grows. In simulations with $N = 1000$ agents, I observed $o_j = 3$ in 3.39% of all restaurants and $o_j = 4$ in 0.42% of all restaurants, $o_j = 5$ to $o_j = 10$ occurred seldom, but still affected the final result.

The utility is $u = f \cdot u_{avg} = 0.368 \cdot u_{max}$.

4.9. Stochastic Rank Dependent Choice

Agents using the SRD strategy stochastically either eat at the highest-utility restaurant j_p or turn to another restaurant $j \in J$. As all N agents share the same first preference, the probability that some agent i goes to j_p is $\frac{1}{N}$.

In the SRD1 strategy, the other agents turn to all restaurants except j_p. On average, $N-1$ agents turn to $N-1$ restaurants, yielding an average utilization fraction of $1-e^{-1}$ (for those $N-1$ diverting agents). The total utilization fraction is therefore $f = \frac{1}{N} + \frac{N-1}{N} \cdot \left(1-e^{-1}\right) = 63.2\%$ and the utility is $u = 0.316 \cdot u_{max}$.

In the SRD2 strategy, redirecting agents turn to all restaurants $j \in J$ (including j_p). On average, $N-1$ agents turn to N restaurants, with $N \to \infty$ this yields and average utilization fraction of $1-e^{-1}$ for diverting agents and an overall utilization fraction of $f = 63.2\%$ and a utility of $u = 0.316 \cdot u_{max}$.

In the SRD3 strategy, diverting agents turn to their second choice (that is the restaurant yielding second highest utility). As all utilities are identical for all agents, this second preference is shared among all agents. Thus, all diverting agents go to the same restaurant j', resulting in a total utilization fraction of $f = \frac{2}{N} = 0.2\%$ for $N = 1000$ and a utility of $u = 0.002 \cdot u_{max}$.

The SRD4 strategy is identical to the SRD3 strategy for the KPRP, as the best vacant restaurant assuming all agents prefer the same restaurant is the restaurant that yields the second highest utility. I, therefore, conclude that the utilization fraction is $f = \frac{2}{N} = 0.2\%$ for $N = 1000$ and that the utilility is $u = 0.002 \cdot u_{max}$.

4.10. Results

Table 1 comprises analytical and simulation results of the previous sections (simulation for SCA, analytical otherwise).

For the KPRP, utilization fraction and utility are linearly dependent for most strategies ($u = f \cdot u_{avg}$). RD, SRD3 and SRD4 have $u = f \cdot u_{max}$, but the performance with respect to utilization fraction or utility of these strategies is insufficient. All strategies exceed the baseline comparison RD, but only SCA outperforms the baseline NL. SRD1 and SRD2 are as good as NL, but cannot outperform it. SRD1 and SRD2 as well as SRD3 and SRD4 perform pairwise equally well, as the alternate choice is identical for the KPRP.

5. Individual Preferences

In this chapter, I will apply the strategies introduced in chapter 3 to the IP model variant. Some of the aforementioned strategies do not draw upon the actual ranking; I can therefore safely assume that the utilization will be the same as in the KPRP with the given adjustments.

5.1. The Model

I formally define the IP game as follows:

The utility agents $i \in I, |I| = N$ receive from carrying some customer $j \in J, |J| = N$ is uniformly distributed, that is every agent associates every utility level between 0 and 1 with $\frac{1}{N}$ step size with some customer, but different agents may receive different payoff from the same customer. I assume strict utility levels (no two customers are associated with the same utility by some agent) and are therefore able to derive a preference ranking.

Every agent $i \in I$ drive to exactly one customer $j \in J$ ($\forall i : \sum_{j \in J} d(i,j) = 1$). I denote that i drives to j as $d(i,j) = 1$. The number of agents driving to some customer j is its occupancy o_j.

$$d(i,j) = \begin{cases} 1 & \text{if } i \text{ drives to } j, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(Definition 5.1)}$$

$$\forall j : o_j = \sum_{i \in I} d(i,j) \qquad \text{(Definition 5.2)}$$

Every agent drives to exactly one customer ($\forall i : \sum_{j \in J} d(i,j) = 1$). If more than one agent drives to some customer j, only one of the agents will be able to carry j; all others will run empty. I denote that agent i carries customer j as $c(i,j) = 1$.

$$c(i,j) = \begin{cases} 1 & \text{if } i \text{ carries } j, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(Definition 5.3)}$$

Obviously, an agent i can only carry a customer j, if he drives to j ($\forall i,j : c(i,j) \leq d(i,j)$), A customer j is carried by at most one agent, and if there is an agent i that drives to j, this customer will be carried ($\forall j : c(i,j) = \min\left(\sum_{i \in I} d(i,j), 1\right)$). Agents can either randomly or deterministically choose the customer they drive to. Every agent prefers one customer over all others, as it returns the highest utility for him (if no other agents were driving to the same customer). This customer j yields a higher utility than all other agents. The number of agents preferring some customer j is denoted as p_j.

$$p(i,j) = \begin{cases} 1 & \text{if } \forall j' \in J \setminus \{j\} : u(i,j) \geq u(i,j') \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Definition 5.4)}$$

$$\forall j : p_j = \sum_{i \in I} p(i,j) \qquad \text{(Definition 5.5)}$$

The utilization fraction is derived from the average number of agents carrying a customer.

Strategy	utilization f	utility u
NL	63.2%	0.316
RD	0.1%	0.001
LL	43.3%	0.212
OPR	55.8%	0.279
CA	45.7%	0.229
SCA	73.5%	0.368
SRD1	63.2%	0.316
SRD2	63.2%	0.316
SRD3	0.2%	0.002
SRD4	0.2%	0.002

Table 1: KPRP: Comparing Strategies

$$f = \frac{1}{N} \cdot \sum_{i \in I} f(i) \qquad \text{(Definition 5.6)}$$

$$f(i) = \sum_{j \in J} c(i, j) \qquad \text{(Definition 5.7)}$$

The utility is given as the average utility of all agents. The individual utility $u(i,j)$ an agent i receives from carrying a customer j is a random permutation for every customer ($\forall i : \forall j, j' : u(i,j) \neq u(i,j') \vee j = j'$).

$$u = \frac{1}{N} \cdot \sum_{i \in I} u(i) \qquad \text{(Definition 5.8)}$$

$$u(i) = \sum_{j \in J} u(i,j) \cdot c(i,j) \qquad \text{(Definition 5.9)}$$

In numerical experiments and simulations I use $|I| = |J| = N = 1000$ agents and customers, and a uniformly distributed utility (between $\frac{1}{N} \approx 0$ and $u_{max} = 1$). Agents i carrying their preferred customer ($\forall j : c(i,j) = p(i,j)$) receive an expected maximum utility $u_m = u_{max}$, agents carrying another (not preferred) customer ($\sum_{j \in J} c(i,j) = 1 \wedge \forall j : p(i,j) = 1 \Rightarrow c(i,j) = 0$) receive u_{avg}.

5.2. Theoretic Foundations

The capacity of all customers is 1. The agent preferences are randomly distributed, Thus, the probability that p_j agents prefer customer j is Poisson distributed around 1.

$$Pref\left(p_j\right) = \frac{1}{p_j!} \cdot e^{-1} \qquad (11)$$

5.3. No Learning

One of the best strategies for the *Kolkata Paise Restaurant Problem* with respect to the utilization fraction was to choose a restaurant randomly at every evening. I will therefore adopt this strategy for mobility markets.

With this strategy, every driver randomly selects the customer (independent of his individual preference ranking and the history). Thus, the utilization fraction is calculated as $f = 1 - e^{-1}$ and is therefore $f = f_{NL} = 63.2\%$.

As agents choose randomly, on average every driver can expect utility u_{avg} . As only 63.2% of all drivers can expect payoff (the others do not get a customer), only those can get payoff. The average utility is therefore given by equation 12. In the given experiment with $N = 1000$ agents, I, therefore, expect a Gaussian distributed utility around an average of $u = 0.316 \cdot u_{max}$.

$$u = u_{avg} \cdot f = u_{avg} \cdot \left(1 - e^{-1}\right) \qquad (12)$$

5.4. Rank Dependent Choice

The RD strategy is a second baseline comparison in addition to the NL strategy. Whilst the RD strategy was outperformed with respect to both metrics by all other strategies in the KPRP, the high number of distinct first preference resources makes it a reasonable choice in the IP model variant.

Assuming a random preference ranking, it would be beneficial to always try to get the maximum payoff, which – on average – should also yield an average utilization of $f = f_{RD} = 63.2\% = 1 - Pref(0) = 1 - e^{-1}$ with $Pref(0)$ being the probability that a customer is noone's first choice ($p_j = 0$). The expected average utility for successful agents – that is agents carrying a customer – increases from u_{avg} to u_{max}. In our example, this would be $u = 0.632 \cdot u_{max}$.

5.5. Limited Learning

Using the LL strategy, agents choose a customer randomly at time t and go to their highest utility customer at time $t+1$, if they got a tour at time t, otherwise they choose randomly again.

The utilization fraction can be given by the following formula:

$$f_t = \underbrace{f_{t-1} \cdot f_{RD}}_{\text{first try best}} + \underbrace{\left(1 - e^{-(1-f_{t-1})}\right)}_{\text{random or return}} \tag{13}$$

The left summand of the equation models all those agents which chose their top priority customer at time t after successfully choosing randomly (at time $t-1$ or earlier). The success rate for these agents is f_{RD} which is the utilization fraction of the RD strategy. The second summand of the equation comprises all those agents which choose randomly or which successfully chose their top priority at time $t-1$ and return there. Using this equation, the utilization fraction is $f = 70.2\%$.

One has to differentiate between those agents who return to their prioritized customer and those agents who randomly choose a customer, as both belong to the second summand of equation 13. Let's assume that all those agents who do not share their top priority with any other agent will be able to return there. The fraction of returning agents is, therefore, given as $r = Pref\,(1) = e^{-1}$ (probability that $p_j = 1$ agents prefer a customer j).

The utility is given by 14 which results in a utility of $u = 0.620 \cdot u_{max}$ for $N = 1000$ for the IP model.

$$\begin{aligned} u =& f \cdot f_{RD} \cdot u_m + \left(1 - e^{f-1}\right) \\ & \cdot \left(r \cdot u_m + (1-r) \cdot u_{avg}\right) \end{aligned} \tag{14}$$

5.6. One Period Repetition

Though the average utilization fraction was quite low for the One Period Repetition strategy in the KPRP, it can be a good solution for mobility markets: In the KPRP with identical rankings, the fraction of agents which headed for the best possible resource was usually lost (only one of them got dinner). This does not happen in mobility markets, as agents turn to different customers when going to their preferred resource.

Drawing upon the conclusions for the One Period Repetition in equation 7, I can assume that the new average utilization fraction is given by equation 16. Over time, all customers who are someone's first preference will be carried (second summand). f_{RD} is the utilization fraction of the RD strategy and, therefore, the fraction of customers carried by an agent preferring them. All other customers ($1 - f_{RD} = e^{-1}$) will be serviced during the random step and the improve step.

$$f = \left(x + (1-2x)\left(1 - e^{-1}\right)\right) \cdot (1 - f_{RD}) + f_{RD} \tag{15}$$

$$= \left(x + (1-2x)\left(1 - e^{-1}\right)\right) \cdot e^{-1} + \left(1 - e^{-1}\right) \tag{16}$$

Solving equation 16 yields an average utilization fraction $f = 83.7\%$.

The average utility is given by equation 5.6, in this formula, all those customers who are some agent's first preference will be serviced with maximum utility and all others will be serviced resulting in average utility for the respective agent. The result for this equation is $u = 0.728 \cdot u_{max}$.

$$\begin{aligned} u =& \left(x + (1-2x)\left(1 - e^{-1}\right)\right) \cdot e^{-1} \cdot u_{avg} + \\ & \left(1 - e^{-1}\right) \cdot u_m \end{aligned} \tag{17}$$

5.7. Crowd Avoiding

The strategy CA is identical to the one given in section 4.7 for the KPRP: All agents go to customers $j \in J$ who were vacant the previous iteration ($o_j = 0$ at time $t-1$).

As this is strategy is independent of the rank, the expected utilization fraction is $f = 45.7\%$ from equation 9, and the utility is $u = f \cdot u_{avg} = 0.229 \cdot u_{max}$ for $N = 1000$ agents.

5.8. Stochastic Crowd Avoiding

Like in the CA strategy (section 5.7), the strategy SCA for mobility markets works exactly like the one for the KPRP in section 4.8: The probability of returning to a customer the successive day is inversely dependent on the number of agents at this customer the previous day.

This strategy is also independent of the actual utility resulting in expected utilization fraction of $\bar{f} = 0.735$ and a utility of $u = f \cdot u_{avg} = 0.368 \cdot u_{max}$ for $N = 1000$ agents.

5.9. Stochastic Rank Dependent Choice

Assuming every agent knows the number of agents p_j with an identical highest-ranking customer, agents could head for this customer with a probability of $\frac{1}{p_j}$ and head for either

- any customer which is noone's first choice (SRD1)
- any other customer (SRD2)
- his second choice customer (SRD3)
- the best customer which is noone's first choice (SRD4)

with a probability of $1 - \frac{1}{p_j}$.

The expected utilization fraction f is the sum over the utilization given p_j agents preferring some customer j for all possible values of p_j. $F\left(p_j\right)$ is the expected fraction of customers being carried both in this customer and by switching to another customer (a more detailed description will follow in this section). $Pref\left(p_j\right)$ is the probability that some customer is preferred by p_j agents and is given by equation 11.

$$f = \sum_{p_j=1}^{N} Pref\left(p_j\right) \cdot F\left(p_j\right) \tag{18}$$

The fraction of agents servicing a customer given the number of agents preferring this customer p_j depends on the number of agents r_j switching ("redirecting") to another customer. Every r_j is associated with a probability $D\left(p_j, r_j\right)$ that r_j out of p_j agents divert to other customers. Every agent that switches to another customer yields utilization with probability s (success rate). In total r'_j/r''_j agents receive this payoff. If at least one agent remains at this prioritized customer, this agent (or one of these agents) i will receive utilization $f\left(i\right) = 1$. (In SRD2 and SRD3 it is possible that redirecting agents turn to a customer in which at least one agent remains. In this case, diverting agents can "bully out" other agents. This is included in the success rate s.)

$$F\left(p_j\right) = \sum_{r'_j=1}^{p_j} D\left(p_j, r'_j\right) \cdot s \cdot r'_j + \sum_{r''_j=0}^{p_j-1} D\left(p_j, r''_j\right) \tag{19}$$

The probability that r_j out of p_j agents redirect to another customer is given by $D\left(p_j, r_j\right)$. Agents service their top priority customer with $p = \frac{1}{p_j}$, otherwise they redirect. For larger r_j and p_j, one can apply the Poisson Limit Theorem.

$$D\left(p_j, r_j\right) = \binom{p_j}{r_j} \left(\frac{1}{p_j}\right)^{p_j - r_j} \left(1 - \frac{1}{p_j}\right)^{r_j} \tag{20}$$

$$= \frac{1}{\left(p_j - r_j\right)!} \cdot e^{-1} \tag{21}$$

The average utility is given by adapting equation 18. The utilization fraction for p_j agents preferring the same customer is replaced by the utility $U\left(p_j\right)$ which gives the corresponding utility.

$$u = \sum_{p_j=1}^{N} Pref\left(p_j\right) \cdot U\left(p_j\right) \tag{22}$$

$U\left(p_j\right)$ modifies $F\left(p_j\right)$ by introducing different expected utilities for successful agents: If an agent switches to another customer, he can only expect average utility u_{alt}, whilst staying with the top priority yields optimal utility u_m.

$$U\left(p_j\right) = \sum_{r'_j=0}^{p_j-1} D\left(p_j, r'_j\right) \cdot s \cdot r'_j \cdot u_{alt} + \sum_{r''_j=1}^{p_j} D\left(p_j, r''_j\right) \cdot u_m \tag{23}$$

The success rate s and the utilities u_m and u_{alt} depend on the behaviour of diverting agents. Table 2 lists these parameters, and they are discussed in the following sections.

5.9.1. Noone's First Choice (SRD1)

The success rate s is given by on average e^{-1} agents switching over to other (vacant) customers. On average, e^{-1} customers are vacant.

$$s = \left(1 - e^{-1}\right) \tag{24}$$

I, therefore, derive $f = 79.5\%$ and $u = 0.678 \cdot u_{max}$.

I further assume $u_m = u_{max} = 1$ and $u_{alt} = u_{avg} = 0.5$, as agents redirect to a randomly selected customer.

5.9.2. Any Other Customer (SRD2)

The success rate s for redirecting agents changes in comparison to the previous strategy: If an agent frequents a customer who is someone else's first preference, I cannot assume that the utilization is increased. On average, $e^{-1} \cdot N$ agents divert to other customers, and there are N customers these agents can divert to. The success rate is the probability that a diverting agent carries a customer j who is not preferred by any other agent ($p_j = 0$). On average, e^{-1} customers are not preferred by any agent. The probability that a customer j with $p_j = 0$ is not carried by another diverting agent is $e^{-\lambda}$ with λ the average number of diverting agents driving to a customer ($\lambda = \frac{1}{e^{-1}}$). Thus, the probability that at least one agent drives to some customer j is $1 - e^{-\frac{1}{e^{-1}}}$. The success rate is, therefore, given by equation 25.

$$s = e^{-1} \cdot \left(1 - e^{-\frac{1}{e^{-1}}}\right) \approx 0.347 \tag{25}$$

The expected maximum utility u_m is derived from the probability that $a = p_j - r_j$ agents remain with their shared first priority customer and another b agents get to this customer when selecting any other but their preferred customer. a agents remain if $r_j = p_j - a$ agents divert which is given by $D\left(p_j, p_j - a\right)$ from equation 21. The probability that b agents choose this customer randomly is given by equation 27 (swap to customer j). On average,

Strategy	s	u_m	u_{alt}	f	u
SRD1	0.623	1.00	0.50	79.5%	0.678
SRD2	0.347	1.00	0.50	69.0%	0.626
SRD3	0.347	1.00	1.00	69.0%	0.690
SRD4	0.623	1.00	1.00	79.5%	0.795

Table 2: IP: SRD Choice Strategy – Variables

e^{-1} of all agents divert to another customer, they choose from all N customers, thus, $\lambda = \frac{1}{e^{-1}} = e$.

$$u_m = \sum_{a=1}^{N}\sum_{b=1}^{N} D\left(p_j, p_j - a\right) \cdot P\left(b \text{ swap}\right) \cdot \frac{a \cdot u_{max} + b \cdot u_{alt}}{a+b} = 0.922 \tag{26}$$

$$P\left(b \text{ swap}\right) = \frac{\lambda^b}{b!} \cdot e^{-\lambda}, \lambda = e \tag{27}$$

The utilization fraction is, therefore, given as $f = 69.0\%$ and the utility is $u = 0.626 \cdot u_{max}$.

5.9.3. Second Choice Customer (SRD3)

In this strategy, every agent who knows that other agents share the same #1 priority decides to go to his #2 priority with probability $\frac{p_j - 1}{p_j}$ (with p_j from Definition 5.5).

Success rate $s = 0.347$ and expected utility for successful non-diverting agents $u_m = 1.0$ remain unchanged with respect to SRD2, but u_{alt} for successful diverting agents increases to u_m. In the numerical experiment, the top priority customer yields a utility of 1.0, the second best had a utility of 0.999. Thus, the payoff is always either 1 oder 0.999 (And, therefore, $f \cdot 0.999 < u < f \cdot 1$). With $N \to \infty$ I can assume $u_{alt} = u_{max}$.

The utilization fraction is $f = 69.0\%$ and the utility is $u = 0.690 \cdot u_{max}$.

5.9.4. Best Vacant Customer (SRD4)

Rather than choosing any vacant customer (like in the first case), or always the second best (regardless of other agents choosing this customer as #1) an agent chooses the best possible customer in which no other agent might be serving with maximum utility.

Mathematically, choosing this alternative customer is identical to randomly choosing any vacant customer (there are $e^{-1} \cdot N$ vacant customers, as the customers are assigned as a random permutation, one could also randomly draw these customers). Therefore, the success rate of diverting agents is $s = 1 - e^{-1}$ like in SRD1 (equation 24). The utilization fraction is, therefore, $f = 79.5\%$.

If an agent approaches his top priority customer and is the only one there, the utility will be given by u_{max}. If the agent diverts to another customer, the expected utility is slightly lower. The highest utility customer cannot be the best vacant customer. The second best customer is vacant with probability e^{-1}. The customer with the third highest utility is vacant with probability e^{-1}, but only is the best vacant customer, if the customer wiht the second highest utility is not vacant (with probability $1 - e^{-1}$). The l best customer is the best vacant customer if all $l - 2$ customers (all customers yielding a higher utility except the first preference customer) are not vacant and customer l is vacant. Customer l then yields a utility of $1 - \frac{l}{N}$.

$$u_{alt} = \sum_{l=2}^{N}\left(1 - \frac{l}{N}\right) \cdot e^{-1} \cdot \left(1 - e^{-1}\right)^{l-2} = u_{max} - \frac{1.7183}{N} \tag{28}$$

For $N = 1000$, the utility of the alternate choice is $u_{alt} = 99.8\%$. With increasing N, this deviation becomes negligible ($u_{alt} \approx u_{max}$). The utility is, therefore, $u = 0.795 \cdot u_{max}$.

5.10. Results

Table 3 lists utilization fraction and utility for all strategies in this setting.

The two baseline comparison strategies NL and RD perform equally well with respect to f, but RD outperms NL by orders of 2 concerning u, as all successful agents receive u_m (utility for agents carrying their preferred customer) rather than u_{avg} (average utility for agents carrying any customer). Except for CA, all strategies outperform NL and RD with respect to f (and NL with respect to u), but LL, SCA, and SRD2 fall behind RD with respect to utility, as agents receive a lower utility if they are successful (due to the fact that agents frequently choose a random customer). OPR performs best with respect to utilization but is outperformed by SRD4 regarding the utility. SRD1 and SRD4 as well as SRD2 and SRD3 show equal utilization, as the success rate is identical, but SRD3 and SRD4 outperform their counterparts on utility, as all agents receive (almost) u_{max}.

Strategy	utilization f	utility u
NL	63.2%	0.316
RD	63.2%	0.632
LL	70.2%	0.602
OPR	83.7%	0.728
CA	45.7%	0.229
SCA	73.5%	0.368
SRD1	79.5%	0.678
SRD2	69.0%	0.626
SRD3	69.0%	0.690
SRD4	79.5%	0.795

Table 3: IP: Comparing Strategies

6. Mixed Preferences

This section evaluates the performance regarding utilization and utility for the strategies defined in chapter 3 for the MP model: The distance to a customer is modeled as individual component in the utility of a customer, the payoff is modeled as the shared component.

6.1. The Model

The MP game is defined as follows: Agents $i \in I, s.t. |I| = N$ drive to customers $j \in J, s.t. |J| = N$ ($d(i,j) = 1$), agents try to carry the customer they drive to ($c(i,j) = 1$), but one customer can only be carried by one agent ($\forall j : c(i,j) = \min \sum_{i \in I} d(i,j), 1$). Every agent drives to exactly one customer ($\forall i : \sum_{j \in J} d(i,j) = 1$), and o_j agents drive to customer j (occupancy of j). An agent i can only carry a customer j, if i drives to j ($\forall i,j : c(i,j) \leq d(i,j)$). The customer j that yields the highest utility for some agent i is preferred by i (denoted as $p(i,j) = 1$). The number of agents preferring some customer j is denoted as p_j.

$$d(i,j) = \begin{cases} 1 & \text{if } i \text{ drives to } j, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(Definition 6.1)}$$

$$\forall j : o_j = \sum_{i \in I} d(i,j) \qquad \text{(Definition 6.2)}$$

$$c(i,j) = \begin{cases} 1 & \text{if } i \text{ carries } j, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(Definition 6.3)}$$

$$p(i,j) = \begin{cases} 1 & \text{if } \forall j' \in J \setminus \{j\} : u(i,j) \geq u(i,j') \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Definition 6.4)}$$

$$\forall j : p_j = \sum_{i \in I} p(i,j) \qquad \text{(Definition 6.5)}$$

The utility an agent i receives from a customer j $u(i,j)$ is determined as the weighted average of two components: The individual utility $u_i(i,j)$ represents the inverse distance between agent and customer. The shared utility $u_s(j)$ is the utility which is identical to all agents $i \in I$. $u_i(i,j)$ is a uniform distribution in the range between 0 and 1 independently calculated for every agent, $u_s(j)$ is a uniform distribution in the range between 0 and 1.

$$u(i,j) = \alpha \cdot u_i(i,j) + (1-\alpha) \cdot u_s(j), 0 \leq \alpha \leq 1 \qquad \text{(Definition 6.6)}$$

The utilization fraction is calculated as the average number of agents carrying a customer (given by $f(i) = 1$) divided by the total number of agents N. The agent utilization $f(i)$ denotes if agent i carries any customer. The utility u is given by the average agent utility $u(i)$ which is 0 if agent i does not carry any customer and is $u(i,j)$ if i carries customer j.

$$f = \frac{1}{N} \cdot \sum_{i \in I} f(i) \qquad \text{(Definition 6.7)}$$

$$f(i) = \sum_{j \in J} c(i,j) \qquad \text{(Definition 6.8)}$$

$$u = \frac{1}{N} \cdot \sum_{i \in I} u(i) \qquad \text{(Definition 6.9)}$$

$$u(i) = \sum_{j \in J} u(i,j) \cdot c(i,j) \qquad \text{(Definition 6.10)}$$

For numerical experiments and simulations I assume that there are $N = 1000$ agents and customers. I further assume that $\alpha = 0.5$, resulting in the same influence for shared and individual utility. The individual utility is uniformly distributed between $\frac{1}{N}$ and $u_{max} = 1$. Every agent that is successful at the preferred customer receives on average u_m and every agent successful at a randomly chosen customer receives on average $u_{avg} = 0.5$. Without loss of generality, I further assume that customers are indexed by their shared utility ($u_s(j) = \frac{j}{N}$). Though deterministic rather than random, this does not influence numerical results (the index j is no more than a theoretical construct which one can fit to the utilities). It simplifies calculations, as one can easily iterate through all customers with a higher (or lower) shared utility.

6.2. Theoretic Foundations

The maximum utility an agent can achieve may be lower than $u_{max} = 1$ as the utility is built as the weighted sum of two uniformly distributed variables with maximum u_{max}.

6.2.1. Probability of a Customer with a given Shared Utility yielding Maximum Utility

It is possible that there is no longer a single customer yielding maximum utility, but there can be multiple customers with the same utility. A customer is part of the set of top customers for some agent if there is no customer who returns a higher utility for this agent.

For simplicity, I first consider random integers for the individual component rather than a random permutation for the shared component (no duplicates). With this simplification, the probability that the utility retrieved from one customer is higher than the utility retrieved from another customer is independent of the utility yielded by all other customers (otherwise, one had to ensure that no duplicates occurred).

I denote the probability $\Pi(j)$ that some customer j with shared utility component $u_s(j)$ is among the customers with highest utility for any agent $i \in I$. Assuming that $u(i,j) = \alpha \cdot u_i(i,j) + (1-\alpha) \cdot u_s(j)$ (Definition 6.6) and that $u_i(i,j)$ is random, I conclude that this probability only depends on the customer j.

$$\Pi(i,j) = \Pi(j) = P\left(\forall j' : u(i,j) \geq u(i,j')\right) \tag{29}$$

Without loss of generality, one can assume that $u_s(j) = \frac{j}{N}$. In the following I will use j as $u_s(j) \cdot N$. Numerically, I assume that every individual utility between $\frac{1}{N}$ and 1 is equally likely, I use $q \in 1 \ldots N$ to model all possible individual utilities ($q = u_i(i,j) \cdot N$). I separately calculate the probability that another customer yields higher utility for those customers with a higher ($\Pi_h(j,q)$) and a lower ($\Pi_l(j,q)$) shared utility component. The total number of customers considered in $\Pi_l(j,q)$ and $\Pi_h(j,q)$ is $N-1$, customers $j_l < j$ are considered in $\Pi_l(j,q)$, customers $j_h > j$ are considered in $\Pi_h(j,q)$.

$$\Pi(j) = \frac{1}{N} \sum_{q=1}^{N} \Pi_l(j,q)\, \Pi_h(j,q) \tag{30}$$

To derive the formulas for $\Pi_l(j,q)$ and $\Pi_h(j,q)$, I first consider a basic example: In an environment with $N = 5$ customers and agents, there is a customer $j = 3$ with shared utility $u_s(j) = \frac{3}{N}$ and an agent i assigning an individual utility $u_i(i,j) = \frac{3}{N}, q = 3$ to j. What is the probability that a customer with a lower shared utility $j_l \in \{1,2\}$ or a higher shared utility $j_h \in \{4,5\}$ is preferred over j by agent i? Agent i can assign any individual utility $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}$ to these customers $j' \in \{1,2,4,5\}$ (resulting in $q' \in \{1,2,3,4,5\}$). For every customer j' one determines the probability that this customer does not reach a higher utility than $u(i,j) = \alpha \cdot u_i(i,j) + (1-\alpha) \cdot u_s(j) = \frac{3}{N}$. In table 4 I display the (combined) utility of j' (multiplied by N for readability) and whether j or j' reaches a higher utility for agent i ($\rightarrow j$ and $\rightarrow j'$), depending on its individual utility q' that an agent i can derive from j' (leftmost column). The last row gives the probability that j' does not exceed j. As none of the other customers must reach a higher utility, I multiply the probabilities (that is $\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} = \frac{8}{125}$) to retrieve the probability that customer j reaches the highest utility for agent i, if agent i assigned him an individual utility of $\frac{q}{N} = \frac{3}{5}$. Obviously, one has to calculate the probability that j is the highest utility customer for all possible individual utilities, that is all values of $q \in \{1 \ldots N\}$.

$\Pi_l(j,q)$ is 1 if customer j has the lowest shared utility ($j = 1$) as there is no customer with a lower shared utility who could exceed the utility of customer j. Thus, j yields a higher utility than all customers with a lower shared utility. Otherwise, it is the product of the probabilities that the utility of j exceeds the utility of all customers $j' = j - j_l$ with a lower shared utility. The probability of exceeding any given other customer is given by $\frac{q+j_l}{N}$, but at most 1 ($\frac{N}{N}$). If a customer j' has a j_l lower shared utility than j, its individual utility must be at least $j_l + 1$ higher than the individual utility of j (q) to exceed j. I, therefore, calculate the probability that the individual utility of the other customer j' is not more than $q + j_l$.

$$\Pi_l(j,q) = \begin{cases} \prod\limits_{j_l=1}^{j-1} \frac{\min(N, q+j_l)}{N}, & \text{if } j > 1 \\ 1 & \text{otherwise} \end{cases} \tag{31}$$

$\Pi_h(j,q)$ is 1 if customer j has the highest shared component as no customer with a higher shared utility component exceeds the utility of j. Otherwise, it is the product of the probabilities that the utility of j exceeds every customer $j' = j + j_h$ with a higher shared utility. The probability of exceeding a given other customer is given by $\frac{q-j_h}{N}$, but is always non-negative. If a customer j' has a shared utility that is j_h higher than the one of j, its individual utility must be at most $j_h - 1$ lower than the individual utility of j (q). j', therefore, requires an indi-

Indiv. Utility $q' = u_i(i,j') \cdot N$	Lower $j'=1$	$j'=2$	Higher $j'=4$	$j'=5$
1	$1\rightarrow j$	$1.5\rightarrow j$	$2.5\rightarrow j$	$3\rightarrow j$
2	$1.5\rightarrow j$	$2\rightarrow j$	$3\rightarrow j$	$3.5\rightarrow j'$
3	$2\rightarrow j$	$2.5\rightarrow j$	$3.5\rightarrow j'$	$4\rightarrow j'$
4	$2.5\rightarrow j$	$3\rightarrow j$	$4\rightarrow j'$	$4.5\rightarrow j'$
5	$3\rightarrow j$	$3.5\rightarrow j'$	$4.5\rightarrow j'$	$5\rightarrow j'$
prob. $u(i,j) \geq u(i,j')$	$\frac{5}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$

Table 4: MP: Highest Utility Customer (Example)

vidual utility of $q - j_h + 1$ to exceed the utility of j. The combined utility is higher for j, if the individual utility of $j\prime$ is at most $q - j_h$.

$$\Pi_h(j,q) = \begin{cases} \prod_{j_h=1}^{N-j} \frac{\max(0,q-j_h)}{N}, & \text{if } j < N \\ 1 & \text{otherwise} \end{cases} \tag{32}$$

Incorporating equations 31 and 32 in equation 30 yields:

$$\Pi(j) = \begin{cases} \frac{1}{N} \cdot \sum_{q=1}^{N} \prod_{j_l=1}^{j-1} \frac{\min(N,q+j_l)}{N} \\ \prod_{j_h=1}^{N-j} \frac{\max(0,q-j_h)}{N}, & \text{if } 1 < j < N \\ \frac{1}{N} \cdot \sum_{q=1}^{N} \prod_{j_h=1}^{N-1} \frac{\max(0,q-j_h)}{N}, & \text{if } j = 1 \wedge N \neq 1 \\ \frac{1}{N} \cdot \sum_{q=1}^{N} \prod_{j_l=1}^{N-1} \frac{\min(N,q+j_l)}{N}, & \text{if } j = N \wedge N \neq 1 \\ 1 & \text{otherwise} \end{cases} \tag{33}$$

This equation 33 can be transformed to the random permutation case by decreasing the denominator as the number of options for the individual component of the other customer is reduced by the assignment to the first customer. This also decreases the numerator of the fraction in Π_l.

$$\Pi(j) = \begin{cases} \frac{1}{N} \cdot \sum_{q=1}^{N} \prod_{j_l=1}^{j-1} \frac{\min(N-1,q+j_l-1)}{N-1} \\ \prod_{j_h=1}^{N-j} \frac{\max(0,q-j_h)}{N-1}, & \text{if } 1 < j < N \\ \frac{1}{N} \cdot \sum_{q=1}^{N} \prod_{j_h=1}^{N-1} \frac{\max(0,q-j_h)}{N-1}, & \text{if } j = 1 \wedge \\ & N \neq 1 \\ \frac{1}{N} \cdot \sum_{q=1}^{N} \prod_{j_l=1}^{N-1} \frac{\min(N-1,q+j_l-1)}{N-1}, & \text{if } j = N \wedge \\ & N \neq 1 \\ 1 & \text{otherwise} \end{cases} \tag{34}$$

Given this approach, it might happen that two customers yield the same utility. The probability that the highest utility is shared among different customers decreases with $N \rightarrow \infty$. For $N = 1000$, approximately 3.9% of all agents prefer more than one customer (given by the sum of probabilities $\Pi(j)$ for all j).

With the above equation with $N = 1000$, I expect that 4.03% of all agents prefer the customer with the highest shared utility (that is $\max(j)$). For those 70 customers with the highest shared component the probability of an agent preferring them is greater than 0.1%, thus, on average, there is an agent for whom this customer yields the best possible utility.

6.2.2. Expected Number of Agents Sharing a Top Priority

The number of agents sharing the same top priority customer depends on the shared component of this customer. The customer with the highest possible shared utility will be chosen more often than the customer with the lowest shared utility.

$$Pref\left(p_j\right) = \binom{N}{p_j}\left(\Pi\left(j\right)\right)^{p_j}\left(1-\Pi\left(j\right)\right)^{N-p_j} \tag{35}$$

$$= \binom{N\Pi\left(j\right)}{p_j}\left(\frac{1}{N}\right)^{p_j}\left(1-\frac{1}{N}\right)^{N\Pi(j)-p_j} \tag{36}$$

$$= \frac{\left(\Pi\left(j\right)\right)^{p_j}}{p_j!}e^{-\Pi(j)} \tag{37}$$

6.2.3. Expected Number of Distinct Top Priorities

With the equation 37, it is now possible to calculate the probability that a customer is noone's first preference ($Pref\left(0\right)$ for $p_j = 0$) and the expected number of customers which are noone's preference (as the average probability).

I ignore duplicate first preferences and assume that a customer is selected with his associated probability of being first preference.

$$\text{no. of not pref. customers} = N - \sum_{j\in J} 1 - P_j\left(0\right) \tag{38}$$

The expected number of customers who are not preferred by any agent for $N = 1000$ is, therefore, 923 (or alternatively: I expect approximately 77 distinct first preferences).

6.2.4. Expected Utility of Top Priority Customers

The expected utility of a randomly selected customer is straight-forward: The average of two random numbers between $\frac{1}{N}$ and 1 is $u_{avg} = 0.5$ (for sufficiently large N). The expected utility for the first preference customer u_m is more elaborate: $u_m = u_{max} = 1$ can only be reached, if both the shared and the individual utility are maximum for an agent i and a customer j. Otherwise, the maximum agent utility is a weighted sum of $\frac{1}{N}\cdot\frac{j+q}{2}$ weighted by the probability that a customer yielding shared utility $\frac{j}{N}$ and individual utility $\frac{q}{N}$ (for agent i). For simplicity, I only consider the case $1 < j < N$; equation 39 needs to be adjusted accordingly to equation 34 to cater for $j = 1$ and $j = N$. For the defined numerical assumptions, the expected utility of top priority customers is $u_m = 0.92$.

$$u_m = \frac{1}{N}\sum_{q=1}^{N}\prod_{j_l=1}^{j-1}\frac{\min\left(N-1, q+j_l-1\right)}{N-1} \prod_{j_h=1}^{N-j}\frac{\max\left(0, q-j_h\right)}{N-1}\cdot\frac{j+q}{2} \tag{39}$$

6.3. No Learning

Agents incorporating the NL strategy randomly choose where to drive to. Thus, the number of agents per customer is Poisson distributed around 1. The number of agents carrying a customer equals the number of customers who are carried by some agent which is N minus the number of agents who are not carried by any agent ($\sum_{i\in I} c\left(i,j\right) = 0$). As the number of agents driving to some customer j is Poisson distributed, I conclude that the number of agents who do not carry any agent is $\left(1-e^{-1}\right)\cdot N$, resulting in a utilization fraction of $f = f_{NL} = 63.2\%$ and a utility of $u = f\cdot u_{avg} = 0.316\cdot u_{max}$.

6.4. Rank Dependent Choice

Obviously, only those customers who are some agent's first preference will be served with the RD strategy.

The utilization fraction is, therefore, given by equation 38 ($f = f_{RD} = 7.7\%$ for $N = 1000$). Those 7.7% of all agents will receive maximum utility, resulting in $u = f\cdot u_m = 0.075\cdot u_{max}$ (with $u_m = 0.92$ from equation 39).

6.5. Limited Learning

In the LL strategy, agents decide randomly on a customer until they are able to serve one. After that, agents try their preferred customer. If they are being "bullied" out, they return to selecting randomly. Those customers who are preferred by some agent ($j \in J | \exists i \in I : p\left(i,j\right) = 1$) will be carried in all iterations unless they did not carry any customer in the previous iteration $t-1$ ($f_{t-1}\cdot f_{RD}$). Agents who do not drive to their preferred customer randomly select any customer, resulting in $\left(1-e^{f_{t-1}-1}\right)$ as the number of agents in this phase is lower than the number of customers to choose from.

$$f_t = f_{t-1}\cdot f_{RD} + \left(1-e^{f_{t-1}-1}\right) \tag{40}$$

$$f = \lim_{t\to\infty} f_t \tag{41}$$

$$u = f\cdot f_{RD}\cdot u_m + \left(1-e^{f-1}\right)\cdot \left(r\cdot u_m + \left(1-r\right)\cdot u_{avg}\right) \tag{42}$$

Derived from equation 41 and **??** (with f_{RD} the number of customers who are preferred by some agent (or the utilization fraction of the RD strategy), $u_m = 0.97$ and $r = \sum_{j\in J}\Pi\left(j\right)\cdot e^{-\Pi(j)} = 0.01$), I deduce that the utilization fraction is $f = 45.5\%$ and that the average utility is $u = 0.246\cdot u_{max}$.

6.6. One Period Repetition

Using the OPR strategy, agents drive to their preferred customer after being successful with some randomly chosen customer for two iterations.

$f_{RD} \cdot N$ agents carry the customer they prefer (f_{RD} is the utilization fraction of the RD strategy). All other agents follow the three-step approach ((1) random, (2) return, and (3) improve). In every iteration a fraction $1-2x$ agents chooses randomly ($x = (1-2x) \cdot (1-e^{-1})$ successful), x agents return, and x agents drive to their preferred customer (which is already occupied by another agent, therefore not increasing the utilization). For $N = 1000$ and, therefore, $f_{RD} = 0.077$, the utilization fraction is $f = 56.6\%$.

$$f = f_{RD} + (1-f_{RD}) \cdot \left(x + (1-2x) \cdot \left(1-e^{-1}\right)\right) \quad (43)$$

The utility is calculated analogously, those f_{RD} agents carrying their preferred customer receive $u_m = 0.97$, the other agents carry a randomly selected customer and, therefore, receive u_{avg}. This results in $u = 0.320 \cdot u_{max}$.

6.7. Crowd Avoiding

Agents who follow the CA strategy randomly choose any customer who was not carried during the previous iteration. Thus, there are N agents driving to $(1-f_{t-1}) \cdot N$ customers.

$$\begin{aligned} f_t &= (1-f_{t-1}) \cdot (1 - \binom{N}{0} \cdot (1 - \frac{1}{(1-f_{t-1}) \cdot N})^N) \\ &= (1-f_{t-1}) \cdot \left(1 - e^{-(1-f_{t-1})}\right) \end{aligned} \quad (44)$$

I, therefore, conclude that $f = 45.7\%$. As all successful agents drive to a randomly chosen a customer, I assume that these agents receive u_{avg}. Thus, the utility is $u = 0.229 \cdot u_{max}$.

6.8. Stochastic Crowd Avoiding

Using the SCA strategy, agents either return to the same customer or drive to any other customer depending on the number of agents driving to the customer they drove to in the previous iteration. If at time $t-1$ agent i drove to customer j ($d(i,j) = 1$) and the occupancy of customer j is $o_j = 1$, agent i returns to customer j at time t. If agent i drove to customer j at time $t-1$ and the occupancy $o_j > 1$, i returns there with probability $\frac{1}{o_j}$ and randomly chooses any other customer at time t with probability $\frac{o_j - 1}{o_j}$.

In simulations with $N = 1000$, $u_{max} = 1$ and $u_{avg} = 0.5$ I observe a utilization fraction of $f = 73.5\%$ and a utility of $u = 0.368 \cdot u_{max}$.

6.9. Stochastic Rank Dependent Choice

With this strategy, the probability of driving to the top customer depends on the number of agents which share the same top priority.

Analytically, one can assume that the function of the utilization fraction has to incorporate the no longer random number of agents preferring some customer. $Pref(p_j)$ is the probability that a customer j is preferred by exactly p_j agents (derived from equation 37). $F(p_j)$ is the expected utilization, if p_j agents prefer customer j. As $Pref(p_j)$ is used to weight $F(p_j)$, one has to divide by $\sum_{j \in J} \sum_{p_j=1}^{N} Pref(p_j) = \sum_{j \in J} \Pi(j) \approx N$.

$$f = \frac{1}{\sum_{j \in J} \Pi(j)} \sum_{j \in J} \left(\sum_{p_j=1}^{N} Pref(p_j) \cdot F(p_j) \right) \quad (45)$$

$F(p_j)$ includes the probability that r_j agents divert to other customers (with probability $D(p_j, r_j) = ((p_j - r_j)!)^{-1} \cdot e^{-1}$).

$$F(p_j) = \sum_{r'_j=1}^{p_j} D\left(p_j, r'_j\right) \cdot s \cdot r'_j + \sum_{r''_j=0}^{p_j-1} D\left(p_j, r''_j\right) \quad (46)$$

The average utility is calculated by adapting equations 45 and 46 such that it incorporates different utility levels regarding on the agent's type of choice (remain with their top priority resulting in u_m or diverting to alternative resources resulting in u_{alt}).

$$u = \frac{1}{\sum_{j \in J} \Pi(j)} \sum_{j \in J} \left(\sum_{p_j=1}^{N} Pref(p_j) \cdot U(p_j) \right) \quad (47)$$

$$\begin{aligned} U(p_j) = &\sum_{r'_j=0}^{p_j-1} D\left(p_j, r'_j\right) \cdot s \cdot r'_j \cdot u_{alt} + \\ &\sum_{r''_j=1}^{p_j} D\left(p_j, r''_j\right) \cdot u_m \end{aligned} \quad (48)$$

s, u_m, and u_{alt} depend on the actual strategy. Table 5 compares the variables for SRD1 and SRD2.

6.9.1. Noone's First Choice Customer

In this strategy, agents choose those customers who are not preferred by any agent ($j \in J$, s.t. $\sum_{i \in I} p(i,j) = 0$).

As the number of diverting agents on average equals the number of customers who are not preferred by any agent, I can assume that a fraction of $s = 0.632$ of all diverting agents successfully carries another customer

Strategy	s	u_m	u_{alt}	f	u
SRD1	0.632	0.92	0.46	89.8%	0.521
SRD2	0.661	0.92	0.50	88.0%	0.512

Table 5: MP: SRD Strategy – Variables

(success rate). The utilization fraction is, therefore, $f = 63.8\%$. The utility of diverting agents (alternate utility) is u_{alt}. One cannot assume $u_{alt} = 0.5$, as only those customers with a lower shared component and therefore a lower utility are being selected as noone's preference. For $N = 1000$, I assume $u_{alt} = 0.46$, as on average 77 of the highest utility customers cannot be selected. The expected maximum utility is $u_m = 0.92$, according to equation 39, the utility is thus $u = 0.326 \cdot u_{max}$.

6.9.2. Any Other Customer

The SRD2 strategy dictates diverting agents to choose any other customer, regardless of the preferences of other agents or own preferences. The success rate $s = 0.611$ therefore derived from equation 1 with $\lambda = 1 - f_{RD}$ as $(1 - f_{RD})\, N$ agents divert to N customers. f_{RD} is the utilization fraction of the RD strategy and can be interpreted as the fraction of customers who can carry their preferred customer in the SRD strategy.

$$s = (1 - f_{RD}) \cdot \left(1 - e^{-\frac{1}{1-f_{RD}}}\right) \tag{49}$$

Thus, the expected utilization fraction is $f = 61.9\%$. All agents carrying their preferred customer ($i \in I, \text{s.t.} \forall j \in J : c\,(i,j) = p\,(i,j)$) can expect $u_m = 0.92$ (as in equation 39). Diverting agents can expect $u_{alt} = 0.462$. The expected average utility is $u = 0.330 \cdot u_{max}$.

6.9.3. Second Choice Customer

In the SRD3 strategy, diverting agents drive to the customer yielding them the second highest utility. For this strategy, the utilization rises only slightly in comparison to the RD strategy, as those 92.3% of all agents who randomly choose not to service the top ranked customer will go to the second ranked customer, which in most cases is someone else's top priority or overlaps with another agent's second priority.

The number of distinct second preferences is around 93 for $N = 1000$. Yet, many of these customers are some other agent's first preference. The expected number of customers which are either first or second preference is, therefore, ≈ 94 (in simulations).

Simulations suggest a utilization fraction of $\bar{f} = 9.4\%$ and an average utility of $u = 0.091 \cdot u_{max}$.

6.9.4. Best Vacant Customer

A similar explanation holds for the strategy SRD4 (Best Vacant Customer): Even if agents only turn to customers who are noone's first preference, they will most likely be competing there, as those customers will also be much alike.

The total number of distinct customers in the best vacant customer choice is approx. 74 with $N = 1000$. With ≈ 77 distinct first preference customers, there are around 151 customers the agents choose from.

The actual utilization is lower, as agents do not distribute themselves uniformly. In simulations, the utilization fraction was $\bar{f} = 12.1\%$ and the utility was $u = 0.115 \cdot u_{max}$.

6.10. Results

The utilization fraction and utility for all considered strategies can be found in table 6.

All strategies which do not incorporate the utility (NL, CA, SCA) are obviously not affected by mixed utilities. LL, OPR, RD, and SRD on the opposite worsen (moderately to dramatically) in comparison to the *Individual Preferences* setting. Only one of the rank dependent strategies outperforms both baseline comparisons: SRD1 (and with respect to utility OPR as well). As the redirection option for SRD3 and SRD4 is correlated to the first choice, and due to the low number of distinct first preferences, those strategies fall behind SRD1 and SRD2. With the decreased performance of rank dependent strategies (most "first preference selections" do not increase utility and utilization), SCA becomes the best strategy concerning both utilization fraction and utility.

7. Individual Preferences with Multiple Customers per District

In this model variant I assume that there are several customers in one district, thus, an agent always has several customers from which he can carry one even if the preferred one is not available. I assume that every district on average has the same number of customers, but as customers randomly spawn in some district, there can also be less or more customers in a district. Agents select a customer and drive to the district in which the selected customer is located in.

Strategy	utilization f	utility u
NL	63.2%	0.316
RD	7.7%	0.075
LL	45.5%	0.246
OPR	56.6%	0.320
CA	45.7%	0.229
SCA	73.5%	0.368
SRD1	63.8%	0.326
SRD2	61.9%	0.313
SRD3	9.4%	0.091
SRD4	12.1%	0.115

Table 6: MP: Comparing Strategies

7.1. The Model

In the IPMC model variant, customers are located to districts. Agents $i \in I, |I| = N$ drive to their preferred customer and are able to divert to other customers in the same district at no cost. I denote that some customer $j \in J, |J| = N$ is located in a district $k \in K, |K| = D = \frac{N}{\varphi}$ as $b(j,k) = 1$ (j "belongs to" k). Every customer j belongs to exactly one district k ($\forall j : \sum_{k \in K} b(j,k) = 1$), and c_k customers are located in district k (capacity of k).

$$b(j,k) = \begin{cases} 1 & \text{if } j \text{ is in } k \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Definition 7.1)}$$

$$\forall k : c_k = \sum_{j \in J} b(j,k) \qquad \text{(Definition 7.2)}$$

Agents drive to customers. I denote this relation as $d(i,j) = 1$. Every agent drives to exactly one customer ($\forall i : \sum_{j \in J} d(i,j) = 1$). As agents are able to divert to other customers in the same district, I extend $d(i,j) = 1$ as the notion that agent i drives to customer j to $d(i,k) = 1$ to denote that agent i drives to the district k that j is located in ($d(i,j) = 1 \wedge b(j,k) = 1 \Rightarrow d(i,k) = 1$). As the customer j that agent i originally drove to exactly one district k, I conduct that every agent drives to exactly one district ($\forall i : \sum_{k \in K} d(i,k) = 1$). The number of agents driving to some district k yields the occupancy o_k.

$$d(i,j) = \begin{cases} 1 & \text{if } i \text{ drives to } j \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Definition 7.3)}$$

$$\forall j : o_j = \sum_{i \in I} d(i,j) \qquad \text{(Definition 7.4)}$$

$$d(i,k) = \begin{cases} 1 & \text{if } i \text{ drives to } k \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Definition 7.5)}$$

$$\forall k : o_k = \sum_{i \in I} d(i,j) \qquad \text{(Definition 7.6)}$$

Agents can carry any customer that awaits a ride in the district k that agent i drove to. I denote that agent i carries customer j as $c(i,j) = 1$. One customer can only be carried by one agent ($\forall j : \sum_{i \in I} c(i,j) \leq 1$) and one agent i can carry at most one customer ($\forall i : \sum_{j \in J} c(i,j) \leq 1$). Agents can only carry customers located in the district they drove to ($c(i,j) \leq \sum_{k \in K} d(i,k) \cdot b(j,k)$). If agents are able to carry any customer, they prefer carrying him over not carrying anyone. Thus, the total number of customers carried from one district k is the minimum of the number of customers in k (capacity c_k) and the number of agents driving to k (occupancy o_k) ($\forall k : \sum_{\substack{i \in I \\ j \in J}} c(i,j) \cdot b(j,k) = \min(c_k, o_k)$).

$$c(i,j) = \begin{cases} 1 & \text{if } i \text{ carries } j \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Definition 7.7)}$$

Agents can either drive to their preferred customer or district or randomly choose a resource. I use $p(i,j) = 1$ to denote that i prefers j (j yields more utility for i than any other customer). This is the case if no other customer j' results in a higher utility. The number of agents preferring j is given as p_j. Analogously, I define p_k as the number of agents preferring any customer that are located in district k.

$$p(i,j) = \begin{cases} 1 & \text{if } \forall j' : u(i,j) \geq u(i,j') \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Definition 7.8)}$$

$$\forall j : p_j = \sum_{i \in I} p(i,j) \qquad \text{(Definition 7.9)}$$

$$\forall k : p_k = \sum_{\substack{i \in I \\ j \in J}} p(i,j) \cdot b(j,k) \qquad \text{(Definition 7.10)}$$

$u(i,j)$ is a random permutation individually assigned for every agent ($\forall i \in I : \forall j, j' \in J : u(i,j) = u(i,j') \Rightarrow j =$

j'). As agents who select their preferred resource choose whichever customer results in the highest utility for them and drive to the corresponding district, I define that the utility of a district k is determined by the highest utility of any customer in k.

$$\forall k : u(i,k) = \max_{j \in J} \left(u(i,j) \cdot b(j,k) \right) \tag{Definition 7.11}$$

One calculates the utilization fraction as the share of successful agents, that is agents who carry some customer. The utility is the average of all agent utilities $u(i)$. $u(i)$ is the utility agent i receives. If i does not carry any customer, the agent utility is $u(i) = 0$, otherwise it is the utility $u(i,j)$ of the customer j that agent i carries.

$$f = \frac{1}{N} \cdot \sum_{i \in I} f(i) \tag{Definition 7.12}$$

$$f(i) = \sum_{j \in J} c(i,j) \tag{Definition 7.13}$$

$$u = \frac{1}{N} \cdot \sum_{i \in I} u(i) \tag{Definition 7.14}$$

$$u(i) = \sum_{j \in J} u(i,j) \cdot c(i,j) \tag{Definition 7.15}$$

In simulations and numerical experiments, I assume that there are $N = 1000$ agents and customers in $D = 200$ districts (on average $\varphi = 5$ customers per district), that the utility is uniformly distributed between $\frac{1}{N}$ and $u_{max} = 1$. Every agent that is successful in the preferred district receives on average u_m and every agent successful at a randomly chosen district receives on average $u_{avg} = 0.5$.

7.2. Theoretic Foundations

7.2.1. Capacity: Number of Customers per District

In theory, there can be $0 \ldots N$ customers in one district, though both extremes are highly unlikely. Assuming that there are φ customers on average per district ($N = \varphi D$), the probability $C(c_k)$ for capacity c_k is given by equation 50. In this case φ is the average number of customers per district (in numerical experiments and simulations: $\varphi = 5$).

$$\begin{aligned} C(c_k) &= \binom{\varphi D}{c_k} \cdot \left(\frac{1}{D}\right)^{c_k} \cdot \left(1 - \frac{1}{D}\right)^{\varphi D - c_k} \\ &= \frac{\varphi^{c_k}}{c_k!} \cdot e^{-\varphi} \end{aligned} \tag{50}$$

7.2.2. Occupancy: Number of Agents per District (based upon Capacity)

As agents choose a customer and then drive to the corresponding district, the probability that o_k agents drive to district k depends on its capacity c_k. With N agents and N customers, the number of agents in district k with c_k customers is Gaussian distributed around c_k.

$$O(o_k, c_k) = \frac{{c_k}^{o_k}}{o_k!} e^{-c_k} \tag{51}$$

7.2.3. Same First Preference

The probability that a district with capacity c_k is preferred by p_k agents is calculated as a Gaussian distribution around c_k, as agents randomly "choose" their preferred customer.

$$Pref(c_k, p_k) = \frac{c_k^{p_k}}{p_k!} \cdot e^{-c_k} \tag{52}$$

7.2.4. Expected Utility of Top Priority Customers

The expected maximum utility depends on the capacity c_k: If an agent i enters a district with c_k customers and he carries any customer in this district, there is a $\frac{1}{c_k}$ chance that the customer j that i carries is his preferred customer yielding a utility of u_{max} and a $1 - \frac{1}{c_k}$ chance that i carries any other customer, yielding a utility of on average u_{avg}.

$$u_m(c_k) = \frac{1}{c_k} \cdot u_{max} + \frac{c_k - 1}{c_k} \cdot u_{avg} \tag{53}$$

The expected maximum utility u_m in random processes is calculated by weighting $u_m(c_k)$ by the probability of c_k and the expected number of successful agents $o_k \leq c_k$. I, therefore, conclude $u_m = 0.59$ if c_k of district k is unknown.

7.3. No Learning

In this strategy, every agent randomly decides which district he will go to by randomly selecting a customer j and driving to the district k that j is located in. The agent is then randomly assigned a customer from the selected district. If there are less or equal agents than customers ($o_k \leq c_k$), every agent will be assigned a customers. Otherwise, there is a $\frac{c_k}{o_k}$ probability for every agent to actually be assigned a customer.

Agents select customers and drive to the corresponding districts rather than districts directly, as this increases the utilization fraction and utility, as every district is – on average – chosen by as many drivers as it can cater (instead of φ drivers on average per district). In the appendix I calculate the utilization fraction and utility for district-based choice (**??**). To derive the utilization fraction, I calculate the expected number of not carried customers for every possible capacity c_k ($\sum_{o_k=0}^{c_k-1} O(o_k, c_k)(c_k - o_k)$) and derive the number of carried customers from it. The probability of capacity c_k is derived from equation 50 and the probability of occupancy o_k is derived from equation 51.

$$f = \frac{1}{\varphi} \sum_{c_k=1}^{N} C(c_k) \cdot \left(c_k - \sum_{o_k=0}^{c_k-1} O(o_k, c_k)(c_k - o_k) \right) \tag{54}$$

$$= \frac{1}{\varphi} \sum_{c_k=1}^{N} \frac{\varphi^{c_k}}{c_k!} e^{-\varphi} \cdot \left(c_k - \sum_{o_k=0}^{c_k-1} \frac{c_k^{o_k}}{o_k!} e^{-c_k} (c_k - o_k) \right) \tag{55}$$

The utilization fraction is, therefore, $f = f_{NL} = 83.0\%$. With average utility for all successful agents, the expected utility is $u = f \cdot u_{avg} = 41.5\%$ for $N = 1000$.

7.4. Rank Dependent Choice

I now consider the strategy in which every agent drives to the district which provides him with the best possible utility that is the district containing the customer yielding the highest utility. There are different possible approaches to choosing the best district: Choose the district with the highest average utility from all customers in this district or choose the district which contains ones (individual) #1 priority customer. The first corresponds to selecting a district in *No Learning*, the second to selecting a customer. I only consider the latter as it results in a higher utilization and utility. Yet, one can find some insight on the first in the appendix.

The utilization fraction is the same as for the NL strategy (given by equation 55), as the preferred customer is randomly selected (resulting in $f = f_{RD} = 83.0\%$). The utility increases slightly in comparison to *No Learning*, as the probability of serving the top priority customer is increased.

$$u = \frac{1}{\varphi} \cdot \sum_{c_k=1}^{N} C(c_k) \left(c_k - \sum_{o_k=0}^{c_k} O(o_k, c_k)(c_k - o_k) \right) \cdot u_m(c_k) \tag{56}$$

For $N = 1000$ and $\varphi = 5$ this results in an average utility of $u = 0.495 \cdot u_{max}$.

7.5. Limited Learning

In the LL strategy, every agent first chooses a customer at random and – after carrying a customer – continues with the highest ranked district. With multiple customers in a district, one has to choose which district one deems #1 priority (district containing highest utility customer).

The utilization fraction f depends on the fraction of agents servicing their top district for the first time and the fraction of agents who either randomly choose a district or return to the best possible district. From equation 55 I derive $f_{RD} = 83.0\%$ which is the fraction of customers carried by an agent preferring them, f_t is calculated iteratively. On average $(1 - f_{t-1}) \cdot N$ customers are not carried by first agents choosing their preferred customer the first time (and thus belong to the first summand of the equation). Thus, on average $\lambda = (1 - f_{t-1})$ customers per district are not carried by agents belonging to the left summand of the equation.

$$f_t = \underbrace{f_{t-1} \cdot f_{RD}}_{\text{first try best}} + \underbrace{\frac{1}{\varphi} \cdot \sum_{c_k=1}^{N} C(c_k) \left(c_k - \sum_{o_k=0}^{c_k-1} O(o_k, \lambda) \cdot (c_k - o_k) \right)}_{\text{random or return}} \tag{57}$$

f converges towards $f = 85.2\%$ for $f_{RD} = 0.830$.

To calculate the utility u, I adapt equation 57 to incorporate whether agents expect maximum utility $u_m(c_k)$ or average utility u_{avg}. All those agents who carry a customer from their highest utility district receive on average $u_m(c_k)$. As the right half of the equation comprises both those agents who randomly choose any resource and those, who return to their highest utility customer, I have to differentiate between those groups by introducing r as the fraction of agents returning to their highest utility resource. r is calculated as the fraction of customers in not overutilized districts ($r = \sum_{c_k=1}^{N} \sum_{p_k=0}^{c_k} Pref(p_k, c_k) = 0.621$). Thus, I derive $u = 0.500 \cdot u_{max}$.

$$u = f \cdot f_{RD} \cdot u_m(c_k) + \\ + \frac{1}{\varphi} \cdot \sum_{c_k=1}^{N} C(c_k) \left(c_k - \sum_{o_k=0}^{c_k-1} O(o_k, \lambda) \cdot (c_k - o_k) \right) \cdot \\ (r \cdot u_m(c_k) + (1 - r) \cdot u_{avg}) \tag{58}$$

7.6. One Period Repetition

Agents applying the OPR strategy choose the district containing their top priority customer after returning once to a successful random district choice.

Drawing upon the results from section 5.6 I calculate the utilization fraction and the utility as follows. f_{RD}

agents carry a customer from their preferred district, all other agents follow a three step approach: (1) random choice (with a success probability of f_{NL}), (2) return to the same district (certainly successful, that is $f(i) = 1$, as randomly choosing agents only drive to previously not carried customers), and (3) try best district (with a success rate of 0, as the agent would otherwise belong to those f_{RD} agents who are constantly successful). In every iteration, a share x of all agents is in step (2) and (3), and a share of $1-2x$ is in step (1) (successful with probability f_{NL}, resulting in $x = (1-2x) \cdot f_{NL} \approx 0.312$). f_{RD} is the utilization of the RD strategy and f_{NL} is the utilization of the NL strategy.

$$f = (x + (1-2x) \cdot f_{NL}) \cdot (1 - f_{RD}) + f_{RD} \tag{59}$$

$$u = (x + (1-2x) \cdot f_{NL}) \cdot (1 - f_{RD}) \cdot u_{avg} + u_m \cdot f_{RD} \tag{60}$$

Thus, I expect a utilization fraction of $f = 93.6\%$. The average utility is $u = 0.547 \cdot u_{max}$.

7.7. Crowd Avoiding

Using the strategy CA, agents only choose from customers which have not been carried the previous time step and drive to the district the selected customer is located in. This yields a weighted selection of the districts with too few agents. The number of customers which can be chosen at some time t is the number of customers not chosen at time $t-1$. Those remaining customers are located in different districts. On average, a fraction of $\lambda = \frac{1}{1-f}$ of all customers remain vacant. I assume that these remaining customers are Gaussian distributed across districts, resulting in $\lambda \cdot c_k$ customers remaining per district.

$$f = (1-f) \cdot \left(\sum_{c_k=1}^{N} C(c_k) \cdot \left(c_k - \sum_{o_k=0}^{c_k-1} O(o_k, \lambda \cdot c_k) \cdot (c_k - o_k) \right) \right) \tag{61}$$

With the above assumptions, one can derive $f = 49.7\%$. As all agents randomly decide upon a resource, I conduct $u = 0.249 \cdot u_{max}$.

7.8. Stochastic Crowd Avoiding

With this strategy, agents deterministically return to the same district, if the capacity of district was not exceeded in the previous iteration. Otherwise, agents stochastically return to the same district or drive to any other district.

There are two different choice mechanisms: Returning if the customer is not taken by others or returning if the district has remaining capacity. In the appendix, I introduce a customer-based decision but will continue with a district-based decision in this chapter.

If the number of agents in a district does not exceed the number of customers, this agent will return there. Otherwise, the agent will move towards another customer with $p = 1 - \frac{c_k}{o_k}$ and return to the same district with $p = \frac{c_k}{o_k}$. The customer is then chosen at random from all available customers. In simulations, the utilization fraction is $\bar{f} = 93.8\%$. The utility is average for all agents serving a customer that time step and, therefore, $u = 0.469 \cdot u_{max}$ for $N = 1000$.

7.9. Stochastic Rank Dependent Choice

This strategy vastly builds upon the strategy *Rank Dependent Choice*. Yet, all those drivers who prefer an overcrowded district will not carry a customer with a given probability. With *Stochastic Rank Dependent Choice*, these drivers are now diverted to another district with some probability $p = \frac{c_k - p_k}{p_k}$. The district to divert to is either a district which has remaining capacity, any other district, the #2 district, or the highest utility district which has remaining capacity. The overall utilization fraction f is calculated as a generalization of equation 18.

$$f = \sum_{c_k=0}^{N} C(c_k) \cdot \sum_{p_k=0}^{N} Pref(p_k, c_k) \cdot F(c_k, p_k) \tag{62}$$

$$u = \sum_{c_k=0}^{N} C(c_k) \cdot \sum_{p_k=0}^{N} Pref(p_k, c_k) \cdot U(c_k, p_k) \tag{63}$$

$Pref(p_k, c_k)$ is the probability that p_k agents prefer a district with capacity c_k (equation 52). $C(c_k)$ is the probability that the capacity of some district k is c_k (given by equation 50). The utilization fraction function $F(c_k, p_k)$ calculates the expected utilization, if p_k agents prefer a district k with capacity c_k (including r_k agents redirecting to other districts with probability $D(c_k, p_k, r_k)$).

$$F(c_k, p_k) = \begin{cases} p_k & \text{if } p_k \leq c_k \\ \sum_{r_k=0}^{c_k} D(c_k, p_k, r_k) \cdot \\ (s \cdot r_k + c_k) \\ + \sum_{r_k=c_k+1}^{p_k} D(c_k, p_k, r_k) \cdot \\ (s \cdot r_k + (p_k - r_k)) & \text{otherwise} \end{cases} \tag{64}$$

$$D\left(c_k, p_k, r_k\right) = \binom{p_k}{r_k} \cdot \left(\frac{p_k - c_k}{p_k}\right)^{r_k} \cdot \left(1 - \frac{p_k - c_k}{p_k}\right)^{p_k - r_k} = \frac{\left(p_k - c_k\right)^{r_k}}{r_k!} \cdot e^{c_k - p_k} \tag{65}$$

The success rate s depends on the strategy and its associated behavior in case of swapping.

The utility function $U\left(c_k, p_k\right)$ is given by adapting equation 64 accordingly to equation 22:

$$U\left(c_k, p_k\right) = \begin{cases} p_k \cdot u_m & \text{if } p_k \leq c_k \\ \sum_{r_k=0}^{p_k} D\left(c_k, p_k, r_k\right) \cdot \\ \left(s \cdot r_k \cdot u_{alt} + c_k \cdot u_m\right) \\ + \sum_{r_k=0}^{p_k} D\left(c_k, p_k, r_k\right) \cdot \\ \left(s \cdot r_k \cdot u_{alt} + \left(p_k - r_k\right) \cdot u_m\right) & \text{otherwise} \end{cases} \tag{66}$$

Table 7 lists the variables s, u_{max}, and u_{alt} for the different SRD strategies.

In strategies SRD1 and SRD4, I assume that $s = 0.595$ as given by equation 67. On average $0.17N = (1 - 0.83)\,N$ agents divert to other districts. Thus, $0.17N$ customers are not being serviced by an agent to whom they are first preference. I furthermore assume that these customers are Gaussian distributed across all districts.

$$s = \sum_{c_k=1}^{N} \frac{\varphi^{c_k}}{c_k!} \cdot e^{-\varphi} \sum_{o_k=0}^{c_k-1} \frac{c_k^{o_k}}{o_k!} \cdot e^{-c_k}, \varphi = 5 \cdot 0.17 \tag{67}$$

In strategies SRD2 and SRD3, the success rate is $s = 0.442$. In this case, I calculate the expected number of previously not serviced customers ($c'_k = c_k - o_k + r_k$) and the probability that these customers are serviced by r'_k agents who divert to district k.

$$s = \sum_{c_k=1}^{N} \sum_{c'_k}^{c_k} P\left(c'_k\right) \left(c'_k - \sum_{r'_k=0}^{c'_k-1} \frac{\left(\left(1 - f_{RD}\right) \cdot c_k\right)^{r'_k}}{r'_k!} \cdot e^{-\left(\left(1 - f_{RD}\right) \cdot c_k\right)}\right) \tag{68}$$

The utility u_m is derived from section 7.2.4. In strategies SRD3 and SRD4 I also use this value u_m for u_{alt} (the alternative choice utility), for SRD1 and SRD2 I set $u_{alt} = u_{avg}$.

7.10. Results

Table 8 lists utilization and utility for all disussed strategies for the IPMC model variant.

In the IPMC setting, OPR outperforms all other strategies regarding the utility and is outperformed by SCA concerning f by only a slight margin. All strategies except CA exceed the utilization of the baseline comparisons NL and RD, with respect to utility, SCA also falls behind RD (and RD outperforms NL). I assume that a higher average number of customers per district φ further increases the numbers for utilization and utility, this comparison is, therefore, purely relative. In comparison to the previously presented IP and MP model variants, the utility values for different strategies in the IPMC models are close to each other, as average utility and expected utility of a top priority customer are rather close.

8. Mixed Preferences with Multiple Customers per District

8.1. The Model

In the MPMC model, customers are located in districts (“belong to”) and the utility consists of a customer-specific (“shared”) component and an “individual” component that is based on customer and agent. The shared utility models the payoff an agent receives from carrying a customer. All agents would receive the same payoff if they carried this customer. The individual component models the costs to get to the pickup location which is identical for all customers in one district but varies between different agents.

In the MPMC model, customers $j \in J, |J| = N$ are “clustered” in districts $k \in K, |K| = D = \frac{N}{\varphi}$. One average φ customers await a driver in one district. As customers are located in a randomly drawn district, the number of customers in a district is Gaussian-distributed around φ. Customers $j \in J$ belong to the district $k \in K$ in which they await a driver. Let's denote this as $b\left(j,k\right) = 1$. Every agent is located in exactly one district ($\forall j : \sum_{k \in K} b\left(j,k\right) = 1$) and the number of customers that are located in a district k is its capacity c_k.

$$b\left(j,k\right) = \begin{cases} 1 & \text{if } j \text{ is in } k \\ 0 & \text{otherwise} \end{cases} \tag{Definition 8.1}$$

$$\forall k : c_k = \sum_{j \in J} b\left(j,k\right) \tag{Definition 8.2}$$

Agents $i \in I, |I| = N$ select customers $j \in J$ ($d\left(i,j\right) = 1$) and drive to the district k that j is located in. Every agent drives to exactly one customer ($\forall i : \sum_{j \in J} d\left(i,j\right) = 1$), and the number of agents driving to customer j is denoted as occupancy o_j. In the MPMC model, agents can divert

Strategy	s	u_m	u_{alt}	f	u
SRD1	0.595	0.59	0.50	89.8%	0.521
SRD2	0.442	0.59	0.50	88.0%	0.512
SRD3	0.442	0.59	0.59	88.0%	0.519
SRD4	0.595	0.59	0.59	89.8%	0.530

Table 7: IPMC: SRD Strategy – Variables

Strategy	utilization f	utility u
NL	83.0%	0.415
RD	83.0%	0.495
LL	85.2%	0.500
OPR	93.6%	0.547
CA	49.7%	0.249
SCA	93.8%	0.469
SRD1	89.8%	0.521
SRD2	87.2%	0.508
SRD3	87.2%	0.515
SRD4	89.8%	0.530

Table 8: IPMC: Comparing Strategies

to other customers that belong to the same district at no cost; I, therefore, extend $d(i,j)$ to $d(i,k)$ to denote that agent i drives to district k. Every agent i drives to exactly one district k ($\forall i : \sum_{k\in K} d(i,k) = 1$). If an agent drives to a customer j, he also drives to the district k that j belongs to ($d(i,j) = 1 \wedge b(j,k) = 1 \Rightarrow d(i,k)$). The occupancy o_k of district k is the number of agents i driving to k.

$$d(i,j) = \begin{cases} 1 & \text{if } i \text{ drives to } j \\ 0 & \text{otherwise} \end{cases} \quad \text{(Definition 8.3)}$$

$$\forall j : o_j = \sum_{i\in I} d(i,j) \quad \text{(Definition 8.4)}$$

$$d(i,k) = \begin{cases} 1 & \text{if } i \text{ drives to } k \\ 0 & \text{otherwise} \end{cases} \quad \text{(Definition 8.5)}$$

$$\forall k : o_k = \sum_{i\in I} d(i,j) \quad \text{(Definition 8.6)}$$

As agents independently decide upon the customer or district they drive to, distributions in which too many agents drive to some customers and too few customers drive to some other agents can and do frequently occur. I further introduce the notion $c(i,j) = 1$ to denote that agent i carries customer j. An agent i can carry a customer j, if i drives to the district k that j belongs to ($c(i,j) \leq \sum_{k\in K} d(i,k) \cdot b(j,k)$). One agent i can carry at most one customer j ($\forall j : \sum_{i\in I} c(i,j) \leq 1$) and one customer j can be carried by at most one agent i ($\forall i : \sum_{j\in J} c(i,j) \leq 1$). In every district, agents carry as many customers as possible, no agent refuses to carry a customer remaining at this district. Thus, the number of customers carried per district is either capacity c_k or occupancy o_k ($\forall k : \sum_{\substack{i\in I \\ j\in J}} c(i,j) \cdot b(j,k) = \min(c_k, o_k)$).

$$c(i,j) = \begin{cases} 1 & \text{if } i \text{ carries } j \\ 0 & \text{otherwise} \end{cases} \quad \text{(Definition 8.7)}$$

Agents can either drive to their preferred customer or a randomly drawn customer (given by the strategy). For every agent i there exists a customer j whom he prefers over all other customers, as this customer yields the highest utility for him. A customer j is preferred by p_j agents. Agents prefer the district their preferred customer belongs to. A district k is preferred by p_k agents.

$$p(i,j) = \begin{cases} 1 & \text{if } \forall j' : u(i,j) \geq u(i,j') \\ 0 & \text{otherwise} \end{cases}$$

(Definition 8.8)

$$\forall j : p_j = \sum_{i\in I} p(i,j) \quad \text{(Definition 8.9)}$$

$$\forall k : p_k = \sum_{\substack{i\in I \\ j\in J}} p(i,j) \cdot b(j,k) \quad \text{(Definition 8.10)}$$

The utility an agent i can gain from carrying customer j depends on both an individual and a shared utility component ($u_i(i,j), u_s(j) = u_s(i,j) \, \forall i$). Both utilities are uniformly distributed between 0 and 1.

$$u(i,j) = \alpha \cdot u_i(i,j) + (1-\alpha) \cdot u_s(j), 0 \le \alpha \le 1 \quad \text{(Definition 8.11)}$$

$$\forall j, j' \in J : \forall k \in K : b(j,k) = b(j',k) \Rightarrow u_s(j) = u_s(j') \quad \text{(Definition 8.12)}$$

In the MPMC game model, the individual utility is identical for all customers which are located in a given district as the driving distance between agent and customer is identical for all customers in the same location (district).

$$\forall k \in K : u_i(i,j) = u_i(i,k) \vee b(j,k) = 0 \quad \text{(Definition 8.13)}$$

I define that the utility of a district k is given by the utility of the customer yielding the highest utility (see Proposition 8.2.2). The highest utility customer is defined as $b_1(j,k) = 1$. Obviously, the "best" customer j (customer with highest utility) must be located in district k, and there must not be any other customer j' that also belongs to k that yields a higher shared utility.

$$\forall k : u(i,k) = \max_{j \in J} (u(i,j) \cdot b(j,k)) \quad \text{(Definition 8.14)}$$

$$b_1(j,k) = \begin{cases} 1, & \text{if } b(j,k) = 1 \wedge \\ & (u_s(j) \ge u_s(j') \vee b(j',k) = 0 \, \forall j') \\ 0, & \text{otherwise} \end{cases} \quad \text{(Definition 8.15)}$$

The utilization fraction is calculated as the average of all agent utilizations. The agent utilization $f(i)$ defines whether an agent i carries any customer. The utility is calculated as the average of all agent utilities $u(i)$. $u(i)$ is 0, if i does not carry any customer and the utility of the customer j that i carries ($u(i,j)$) otherwise.

$$f = \frac{1}{N} \cdot \sum_{i \in I} f(i) \quad \text{(Definition 8.16)}$$

$$f(i) = \sum_{j \in J} c(i,j) \quad \text{(Definition 8.17)}$$

$$u = \frac{1}{N} \cdot \sum_{i \in I} u(i) \quad \text{(Definition 8.18)}$$

$$u(i) = \sum_{j \in J} u(i,j) \cdot c(i,j) \quad \text{(Definition 8.19)}$$

For numerical experiments and simulations I assume that there are $N = 1000$ agents and customers in $D = 200$ districts (on average $\varphi = 5$ customers per district), that $\alpha = 0.5$, that the individual utility is uniformly distributed between $\frac{1}{N}$ and $u_{max} = 1$ (with step size $\frac{1}{D}$, as the individual utility is calculated on a district basis) and every agent that is successful at the preferred customer receives on average u_m and every agent successful at a randomly chosen customer receives on average $u_{avg} = 0.5$, and that customers are indexed by their utility ($u_s(j) = \frac{j}{N}$).

8.2. Theoretic Foundations

8.2.1. Capacity: Number of Customers per District

The capacity c_k that is the number of customers belonging to district k is given as a Gaussian distribution around the average number of customers per district φ, as customers randomly choose the district they belong to. Thus, the probability for capacity c_k is calculated as follows:

$$\begin{aligned} C(c_k) &= \binom{\varphi D}{c_k} \cdot \left(\frac{1}{D}\right)^{c_k} \cdot \left(1 - \frac{1}{D}\right)^{\varphi D - c_k} \\ &= \frac{\varphi^{c_k}}{c_k!} \cdot e^{-\varphi} \end{aligned} \quad (69)$$

8.2.2. Highest Utility Customer and District

Proposition: In the MPMC partial game model, agents only prefer the customer j with the highest shared utility in district k. If another customer j' who belongs to the same district k has a higher shared utility, j is not preferred by any agent.

Proof. Assume that $j, j' \in J$ are customers, $k \in K$ is the district both customers belong to such that $b(j,k) = 1$ and $b(j',k) = 1$. Assume that j a higher utility than j' ($u(i,j) < u(i,j')$). An agent i chooses the district which yields the highest utility, assume that this district is k ($p(i,k) = 1$). Thus, $\forall k' \in K \setminus \{k\} : u(i,k) \ge u(i,k')$. From definition Definition 8.14 I know that the utility of a district is given by the highest utility of any of the customers belonging to it. I assume that this customer is j.

$$\begin{aligned} u(i,j) &> u(i,j') \\ &\mid \text{ with Definition 8.11} \end{aligned} \quad (70)$$

$$\begin{aligned} \alpha \cdot u_i(i,j) + (1-\alpha) \cdot u_s(i,j) &> \\ \alpha \cdot u_i(i,j') + (1-\alpha) \cdot u_s(i,j') & \\ &\mid \text{ with Definition 8.13} \end{aligned} \quad (71)$$

$$\alpha \cdot u_i(i,k) + (1-\alpha) \cdot u_s(i,j') > \\ \alpha \cdot u_i(i,k) + (1-\alpha) \cdot u_s(i,j') \mid -\alpha \cdot u_i(i,k) \tag{72}$$

$$u_s(i,j) > u_s(i,j') \tag{73}$$

The probability that a customer j yields the highest utility in his district k (is the "best" customer) is denoted as $B_1(j,c_k)$ and is calculated as the probability that all customers j_h with a higher shared utility $u_s(j_h) > u_s(j)$ choose other districts ($\forall j_h : b(j_h,k) = 0$), j belongs to k ($b(j,k) = 1$) and exactly $c_k - 1$ customers j_l with lower shared utility choose this district k. Without loss of generality, I assume that there are $N - j$ customers with higher shared utility and $j - 1$ customers with lower shared utility (one assigns the identifiers j to customers based on their shared utility component). N is the number of customers and the number of agents ($|I| = |J| = N$), and D is the number of districts ($|K| = D = \frac{N}{\varphi}$).

$$B_1(j,c_k) = \underbrace{\binom{j}{c_k} \frac{1}{D}^{c_k} \frac{D-1}{D}^{j-c_k}}_{j_l \leq j} \underbrace{\frac{D-1}{D}^{N-j}}_{j_h > j} \\ = \binom{j}{c_k - 1} \frac{1}{D}^{c_k} \frac{D-1}{D}^{N-j} = \frac{\left(\frac{j}{D}\right)^{c_k}}{(c_k)!} \cdot e^{-\frac{j}{D}} \tag{74}$$

If the capacity of (another) district is unknown, one can use a generalization of equation 74. $B_1(j)$ ensures that all customers j_h with a higher shared utility component choose other districts and all those j_l with lower shared utility component are being ignored.

$$B_1(j) = \binom{N-j}{0} \frac{1}{D}^0 \left(1 - \frac{1}{D}\right)^{N-j} = \frac{D-1}{D}^{N-j} \\ = \frac{\frac{N}{\varphi} - 1}{\frac{N}{\varphi}}^{N-j} \tag{75}$$

8.2.3. Same First Preference

In the MPMC model, the probability that district k yields maximum utility is no longer equal for all $k \in K$, as the utility depends on a shared component all agents agree upon.

The average number of agents choosing a district k with individual utility $u_i(k) = u_i(j)$ is denoted as $\varphi \cdot \Pi'(k)$. $\Pi'(k)$ is calculated as the product of probabilities that no other customer j_l, j_h yields a higher utility $u(i,j_l)$, $u(i,j_h)$ for any agent i and is best in his district for all customers $j \in J$.

$$\Pi'(j) = \prod_{j_l=1}^{j-1} P\left(u(i,j) \geq u(i,j_l) \vee \sum_{k=1}^{D} b_1(j_l,k) = 0\right) \cdot \\ \prod_{j_h=j+1}^{N} P\left(u(i,j) \geq u(i,j_h) \vee \sum_{k=1}^{D} b_1(j_h,k) = 0\right) \tag{76}$$

Numerically, I adapt equation 76 as follows: I iterate through all customers with lower $j' = j - j_l$ and higher $j' = j + j_h$ shared utility component assuming $\forall j \in J : u_s(j) = \varphi \frac{k}{D} \vee b(j,k) = 0$, and weighting individual and shared utility component equally ($\alpha = 0.5$). A customer $j' = j - j_l$ (shared utility is $j_l \cdot \frac{1}{N}$ lower if an agent carries j' than if he carried j) does not exceed the utility of j if its individual utility is less than $(j_l - 1) \cdot \frac{1}{\varphi}$ higher. Assuming that individual utilities are represented by q ($u_i(i,k) = \varphi \cdot q \cdot \frac{1}{N}$), one can derive that the individual utility of the district k' that j' is located in must not be higher than $\varphi q + j_l$. Analogously, the individual utility of a customer j exceeds the utility of $j' = j + j_h$ (customer with higher shared utility) if the individual utility is correspondingly lower that is lower by $\varphi \cdot q - j_h$. If j' does not yield the highest shared utility in its district, I do not consider it.

$$\Pi'_l(j,\varphi,q) = \\ \prod_{j_l=1}^{j-1} \underbrace{\left(1 - \underbrace{(B_1(j-j_l))}_{j' \text{ best}} \cdot \underbrace{\left(1 - \underbrace{\frac{\min(N, \varphi \cdot q + j_l)}{N}}_{u(i,j) \geq u(i,j')}\right)}_{u(i,j) < u(i,j')}\right)}_{j' \text{ not best or } u(i,j) \geq u(i,j')}$$

$$\Pi'_h(j,\varphi,q) = \prod_{j_h=1}^{N-j} (1 - B_1(j+j_h) \\ \cdot \left(1 - \frac{\max(0, \varphi \cdot q - j_h)}{N}\right))$$

$$\Pi''(j) = \sum_{q=1}^{D} \Pi'_l(j,\varphi,q) \cdot \Pi'_h(j,\varphi,q)$$

$$\Pi'(j) = \frac{D}{\sum_{j''=1}^{N} \Pi''(j'')} \cdot \Pi''(j) \tag{77}$$

I, therefore, expect $\varphi\Pi'(j) = \varphi\Pi'(k)$ ($b_1(j,k) = 1$) agents preferring the district k in which j is the highest utility customer. Yet, the actual number of agents preferring k is Gaussian distributed around $\varphi\Pi'(k)$. $Pref(p_k, \varphi\Pi'(j))$ is the probability that district k with the highest utility customer j is preferred by exactly p_k agents.

$$Pref\left(p_k, \varphi\Pi'(j)\right) = \frac{\left(\varphi\Pi'(j)\right)^{p_k}}{p_k} \cdot e^{-\varphi\Pi'(j)} \tag{78}$$

8.2.4. Occupancy

The occupancy of district k depends on the type of choice: If agents decide randomly, the average number of agents in district k is its capacity c_k, otherwise, it is the expected number of agents preferring it ($\varphi\Pi'(k)$). In the following, λ_k is the expected number of agents driving to district k.

$$O\left(o_k, \lambda_k\right) = \frac{\lambda_k^{o_k}}{o_k} \cdot e^{-\lambda_k} \tag{79}$$

8.2.5. Expected Utility of Top Priority Customers

In the MPMC setting, all agents agree upon the same "best" customer inside a district (Proposition 8.2.2).

To calculate the expected agent utility, I assume that every customer in a district with capacity c_k and highest utility customer j yields on average $u_e(j,k)$ to the agent carrying him. $\bar{u}_i(j)$ is the average individual utility of the district k that j is located in.

$$u_e(j,k) = \frac{1}{c_k}\left(u_s(j) + \bar{u}_i(j)\right) + \frac{c_k - 1}{c_k} \cdot \left(\frac{u_s(j)}{2} + \bar{u}_i(j)\right) \tag{80}$$

$$\bar{u}_i(j) = \frac{1}{\Pi'(j)} \cdot \sum_{k=1}^{D} \underbrace{\varphi \cdot k}_{\text{utility}} \cdot \underbrace{\Pi'_l(j,\varphi,k) \cdot \Pi'_h(j,\varphi,k)}_{\text{if successful}} \tag{81}$$

The average utility of an agent who carries a customer from his preferred district $u_m = 0.785$ is a weighted average of all possible $u_e(j,k)$ (weighted by the probability $B_1(j,c_k)$).

8.3. No Learning

Using the NL strategy, all agents drive to a randomly selected customer. Thus, individual utility levels are irrelevant. The utilization fraction depends on (1) the capacity c_k of district k (associated with probability $C(c_k)$) and (2) the occupancy o_k of district k (associated with probability $O(o_k, c_k)$). $C(c_k) = \frac{\varphi_k^c}{c_k!} \cdot e^{-\varphi}$ is the probability that c_k customers are randomly assigned the same district given an average of φ customers per district. $O(o_k, c_k) = \frac{c_k^{o_k}}{o_k!} \cdot e^{-c_k}$ is the probability of o_k agents randomly driving to district k containing c_k customers.

$$f = \frac{1}{\varphi} \cdot \sum_{c_k=1}^{N} C(c_k) \cdot \left(c_k - \underbrace{\sum_{o_k=0}^{c_k-1} O(o_k, c_k) \cdot (c_k - o_k)}_{\text{expected remaining capacity}} \right) \tag{82}$$

Thus, the utilization fraction is $f = f_{NL} = 83.0\%$. As all agents decide randomly where to drive to, all successful agents will receive average utility u_{avg}. The utility is, therefore, $u = 0.415 \cdot u_{max}$.

8.4. Rank Dependent Choice

In the MPMC model, for every customer j, I calculate the average number of agents driving there if this customer yields the highest utility of all customers in its district k with c_k customers.

The probability that a district is being selected utility-dependent only depends on the customer with the highest shared utility component $u_s(j)$ in this district and the individual utility $u_i(i,k)$ of the district but is ignorant about the number of customers in this district and all other customers' shared utility component.

Bearing that in mind I define the utilization fraction f as follows. The probability of being the customer with the highest utility is $B_1(j, c_k)$ and the average number of agents driving to a district k containing customer j is $\varphi\Pi'(j)$. All agents drive to their preferred customer ($\forall i,j : d(i,j) = p(i,j)$).

$$f = \frac{1}{N} \cdot \sum_{j=1}^{N} \sum_{c_k=1}^{j} B_1(j, c_k) \cdot \left(c_k - \sum_{p_k=0}^{c_k-1} Pref\left(p_k, \varphi \cdot \Pi'(j)\right) \cdot (c_k - p_k) \right) \tag{83}$$

The utilization fraction of agents using the RD strategy is, thus, $f = f_{RD} = 30.6\%$. The expected average utility $u = 0.240 \cdot u_{max}$ is given by adapting equation 83 with the expected utility $u_e(j,k)$ for all successful agents.

$$u = \frac{1}{N} \sum_{j \in J} \sum_{c_k=1}^{j} B_1(j, c_k) \cdot \left(c_k - \sum_{p_k=0}^{c_k-1} Pref\left(p_k, \varphi\Pi'(j)\right) \cdot (c_k - p_k) \cdot u_e(j,k) \right) \tag{84}$$

8.5. Limited Learning

Using the strategy LL, agents first drive to the district a randomly selected customer is located in. Agents who carried a customer at time t drive to their preferred customer at time $t+1$. The utilization fraction for the MPMC model is calculated as follows: The left summand comprises those agents who were successfully carrying a randomly chosen customer in the previous iteration (f_{t-1}) and now drive to their preferred resource. These agents are successful with probability f_{RD}. f_{RD} is the utilization fraction of the RD strategy and thus the number of customers who are preferred by any agent. The right summand comprises all other agents driving to the remaining districts. The average number of customers per district is adapted to $\lambda = \varphi\left(1 - f_{t-1}\right)$ as the expected number of remaining customers is reduced.

$$f_t = f_{t-1} \cdot f_{RD} + \frac{1}{\varphi} \cdot \sum_{c_k=1}^{N} C\left(c_k\right) \left(c_k - \sum_{o_k=0}^{c_k-1} O\left(o_k, \lambda\right) \cdot \left(c_k - o_k\right)\right) \tag{85}$$

For the expected utility one has to differentiate between randomly choosing agents and those who return to their preferred district, as both groups are comprised in the right summand of equation 85. Of these agents, a fraction of $\bar{r} = 0.186$ return to their preferred district, $1 - r$ choose randomly.

$$u = f \cdot f_{RD} \cdot u_m + \frac{1}{\varphi} \cdot \sum_{c_k=1}^{N} C\left(c_k\right) \left(c_k - \sum_{o_k=0}^{c_k-1} O\left(o_k, \lambda\right) \cdot \left(c_k - o_k\right)\right) \cdot \left(\bar{r} \cdot u_m + (1-\bar{r}) \cdot u_{avg}\right) \tag{86}$$

From equations 85 and 86 I derive $f = 57.0\%$ and $u = 0.357 \cdot u_{max}$.

8.6. One Period Repetition

Agents adopting the OPR strategy randomly choose a resource at time t, and return there at time $t+1$ if they were successful at time t. At time $t+2$, agents drive to their preferred customer (after being successful at time t and $t+1$).

The utilization fraction and utility are calculated as follows. $x = (1-2x) \cdot f_{NL}$ is the fraction of agents who return to the same district and who improve by driving to their preferred resource after returning to a random resource. $1 - 2x$ agents randomly select any customer, $x = (1-2x) \cdot f_{NL}$ of these agents are successful. f_{NL} is the utilization fraction of the NL strategy and, therefore, randomly behaving agents. Further, all districts are utilized up to $\min\left(c_k, p_k\right)$, which comprises f_{RD}. These $f_{RD} \cdot N$ agents constantly remain with their preferred district (f_{RD} is the utilization fraction of the RD strategy).

$$f = (x + (1-2x) \cdot f_{NL}) \cdot (1 - f_{RD}) + f_{RD} \tag{87}$$

$$u = (x + (1-2x) \cdot f_{NL}) \cdot (1 - f_{RD}) \cdot u_{avg} + u_m \cdot f_{RD} \tag{88}$$

With $f_{NL} = 0.830$, and $f_{RD} = 0.308$ this results in $f = 73.9\%$ and $u = 0.457 \cdot u_{max}$.

8.7. Crowd Avoiding

The CA strategy ignores the utility or "rank" of customers; agents only drive to customers who were not carried in the previous iteration. On average, agents choose from of $\lambda = \frac{1}{1-f}$ of all customers, resulting in $\lambda \cdot c_k$ customers remaining per district.

$$f = (1 - f) \cdot \left(\sum_{c_k=1}^{N} C\left(c_k\right) \cdot \left(c_k - \sum_{o_k=0}^{c_k-1} O\left(o_k, \lambda \cdot c_k\right) \cdot \left(c_k - o_k\right)\right)\right) \tag{89}$$

I, therefore, conclude that the utilization fraction is $f = 49.7\%$, and that the utility is $u = 0.249 \cdot u_{max}$.

8.8. Stochastic Crowd Avoiding

Agents applying the SCA strategy either return to the same resource in the next iteration or divert to other resources. An agent i remains at district k, if k's capacity is not fully used ($o_k \leq c_k$), or with probability $\frac{c_k}{o_k}$. If an agent i does not return to the same district, he randomly selects any resource $k \in K$.

Simulations suggest a utilization fraction of $\bar{f} = 93.8\%$ and a utility of $u = 0.469 \cdot u_{max}$.

8.9. Stochastic Rank Dependent Choice

The strategy SRD dictates that agents stochastically either drive to their preferred district or any other district, depending on the number of agents with the same preference (p_k for $p\left(i,k\right) = 1$). Diverting agents drive to (1) any underutilized district, (2) any other district, (3) the district yielding second highest utility, or (4) the underutilized district that yields the highest utility.

The overall utilization fraction f for every strategy is calculated as a generalization of equation 18.

The utilization fraction sums up the expected number of agents carrying a customer ($F\left(c_k, p_k\right)$) for the number of agents preferring district k (p_k with probability

$Pref\left(p_k, \varphi\Pi'(j)\right)$), the capacity of this district (c_k with probability $C(k)$), and the customer yielding highest utility j. The utility function u analogously sums up all individual utilities $U(c_k, p_k)$ analogously.

$$f = \sum_{j \in J} \sum_{c_k=0}^{N} C(c_k) \cdot \sum_{p_k=0}^{N} Pref\left(p_k, \varphi\Pi'(j)\right) \cdot F(c_k, p_k) \tag{90}$$

$$u = \sum_{j \in J} \sum_{c_k=0}^{N} C(c_k) \cdot \sum_{p_k=0}^{N} Pref\left(p_k, \varphi\Pi'(j)\right) \cdot U(c_k, p_k) \tag{91}$$

The utilization function $F(c_k, p_k)$ is p_k, if the capacity is not exceeded by those agents preferring district k. In this case, no agent diverts and thus all agents can carry a preferred customer. Otherwise, one sums up the utilization retrieved from r_k agents redirecting for all $r_k \leq p_k$ weighted by the probability $D(p_k, c_k, r_k)$ that r_k agents divert in a district k containing c_k customers that is preferred by p_k agents and is calculated as a Poisson distribution around $p_k - c_k$ ($D(c_k, p_k, r_k) = \frac{(p_k - c_k)^{r_k}}{r_k!} \cdot e^{c_k - p_k}$). $\min(c_k, p_k - r_k)$ agents remaining at district k carry a customer in this district. If less agents divert than required, not all of them will be able to carry a customer, but all c_k customers will be carried. If more agents divert than required, all $p_k - r_k$ agents carry a customer, but not all customers are carried. Those r_k agents who redirect to another district can increase the utilization, if they are able to carry the customer they divert to. The probability of carrying a customer as a diverting agent is given by success rate s. SRD2 and SRD3 allow diverting agents to drive to fully capacitated districts. Yet, for calculating the utilization fraction I assume without loss of generality that not diverting agents favorably carry customers. Diverting agents receive a certain utilization depending on the success rate s which varies depending on the strategy and its associated behavior in case of swapping. The success rate factors in that diverting agents can only be successful if no other agent is "bullied out" his preferred district.

$$F(c_k, p_k) = \begin{cases} p_k & \text{if } p_k \leq c_k \\ \sum\limits_{r_k=0}^{c_k} D(c_k, p_k, r_k) \cdot \\ (s \cdot r_k + c_k) \\ + \sum\limits_{r_k=c_k+1}^{p_k} D(c_k, p_k, r_k) \cdot \\ (s \cdot r_k + (p_k - r_k)) & \text{otherwise} \end{cases} \tag{92}$$

The utility function $U(c_k, p_k)$ is given by adapting equation 92 to cater for varying utility levels. Agents carrying a customer from their preferred district receive on average a utility of u_m (from section 8.2.5), diverting agents receive on average u_{alt} if they are successful. u_{alt} depends on the strategy.

$$U(c_k, p_k) = \begin{cases} p_k \cdot u_m & \text{if } p_k \leq c_k \\ \sum\limits_{r_k=0}^{p_k} D(c_k, p_k, r_k) \cdot \\ (s \cdot r_k \cdot u_{alt} + c_k \cdot u_m) \\ + \sum\limits_{r_k=0}^{p_k} D(c_k, p_k, r_k) \cdot \\ (s \cdot r_k \cdot u_{alt} + (p_k - r_k) \cdot u_m) & \text{otherwise} \end{cases} \tag{93}$$

Table 9 compares the variables s, u_m, and u_{alt} for strategies SRD1 and SRD2. Strategies SRD3 and SRD4 perform worse than random, as first preference and alternative choice are not independent of each other (thus, diverting agents r_k are not uniformly distributed, making it impossible to analytically derive a success rate s). In simulations, the utilization fraction of SRD3 is $\bar{f} = 36.7\%$, and its utility is $u = 0.283 \cdot u_{max}$. The utilization of strategy SRD4 is $\bar{f} = 47.4\%$, and its utility is $u = 0.366 \cdot u_{max}$.

In strategy SRD1, I assume that the success rate is $s = 0.866$ as given by equation 94. On average $0.697N = (1 - 0.303)N$ agents divert to other districts. Thus, $0.697N$ customers are not being carried by an agent to whom they are first preference. I furthermore assume that these customers are Gaussian distributed across all districts, resulting in on average $\lambda = \varphi \cdot 0.697$ customers per district. The success rate s is calculated as the utilization fraction of the NL strategy with a reduced number of customers per district.

$$s = \sum_{c_k=1}^{N} \frac{\lambda^{c_k}}{c_k!} \cdot e^{-\lambda} \sum_{o_k=0}^{c_k-1} \frac{c_k^{o_k}}{o_k!} \cdot e^{-c_k} \tag{94}$$

In strategy SRD2, the success rate is $s = 0.850$. In this case, I calculate the expected number of previously not carried customers ($c'_k = c_k - o_k + r_k$ with probability $P\left(c'_k\right)$) and the probability that these customers are carried by r'_k agents who divert to district k.

$$s = \sum_{c_k=1}^{N} \sum_{c'_k}^{c_k} P\left(c'_k\right) \left(c'_k - \sum_{r'_k=0}^{c'_k-1} \frac{\left((1 - f_{RD}) \cdot c_k\right)^{r'_k}}{r'_k!} \cdot e^{-\left((1-f_{RD}) \cdot c_k\right)} \right) \tag{95}$$

The utility u_m is the utility of strategy RD for those who are successful. I set $u_{alt} = u_{avg}$, as the alternate choice is independent from the actual utility.

Strategy	s	u_m	u_{alt}	f	u
SRD1	0.866	0.79	0.50	78.5%	0.438
SRD2	0.850	0.72	0.50	77.3%	0.432

Table 9: MPMC: SRD Strategy – Variables

8.10. Results

Table 10 shows utilization and utility for the previously examined strategies in the MPMC model.

Of the two baseline comparisons, NL outperforms RD both with respect to utilization and utility, as the number of districts containing a preferred customer is lower than a random selection of districts. None of the rank dependent strategies (LL, OPR, SRD1-SRD4) reach the utilization of the NL strategy, but OPR, SRD1 and SRD2 outperform NL with respect to utility. SCA performs best both with respect to utilization and utility.

9. Critical Discussion

In the previous sections I observe that utilization fraction and utility of some strategy vastly depend on the model variant: In general, one can state that using districts (IPMC, MPMC) improves both optimization criteria. Obviously, if there was only a single district ($D = 1, \varphi = N$) in which all customers are located, one can expect a utilization fraction of $f = 1$ regardless of the implemented strategy, as all agents can divert to other customers in the same district until every customer is carried. If there are no districts, the utilization fraction is determined by the KPRP, or the IP and MP model variant, depending on the other assumptions. I thus advise "clustering" the resources (customers) based on proximity, for example by using taxi stands. They allow agents to serve another customer in the same district if another agent already carries the selected customer. I notice that all strategies always perform at least as good in IP and IPMC as in their mixed preferences counterpart. Obviously, NL, CA and SCA are not affected, as agents never deterministically drive to their preferred resource, but utilization fraction and average agent utility for the other strategies decrease when introducing mixed preferences as the number of distinct highest utility resources decreases. The number of distinct highest utility resources depends on the probability that a resource is preferred by any given customer which is not identical for all resources in the MP and MPMC model variant but depends on the shared utility component. Due to this, exceeding f_{NL} with rank dependent strategies becomes difficult for $\alpha = 0.5$. With increasing α the number of distinct highest utility resources decreases, resulting in a decreasing utility of all rank-dependent strategies, as shown in appendix **??**. Thus, I conclude that high individual utility components are preferred by agents, as the probability of being able to carry the preferred customer increases. In mobility markets – that is vehicle for hire markets – I derive that one would prefer a high influence of the cost of driving to the pickup location which can either be achieved by revenue in a small range or by high distances to the pickup location. Alternatively, a coordination instance could impose personalized incentives, causing agents to distribute themselves in balance with customers.

I also observe that stochastic rank dependent strategies (SRD) outperform their strict counterpart (RD). This is because a fraction of agents chooses its top preference, whilst the other agents can receive utility from another resource. I observe that SRD1 (and SRD4 in IP and IPMC) perform best with respect to utilization fraction f (most customers are carried). SRD1 and SRD4 outperform SRD2 and SRD3 in the IP and IPMC model variants of the VFHP, and SRD1 outperforms SRD2 in the MP and MPMC model variants, as the success rate of redirecting agents is higher. In IP and IPMC, SRD4 outperforms SRD1 with respect to utility, as agents always choose a district yielding high utility. SRD4 performs poorly for mixed utility models (MP, MPMC), as most agents share the same highest utility district with remaining utility. Yet, SRD2 and SRD3 require less information about the preferences of other agents and are therefore preferred in environments without full information.

The CA strategy outperforms the NL strategy in none of the models and is more complex as it requires information about the occupancy rate of all resources, making it unsuitable for implementation. The LL strategy is outperformed by the OPR strategy in all models, making it less attractive for implementation. Yet, the two-step approach is easier to establish in a larger group of agents. From comparing the strategies LL and OPR I conclude that waiting for m periods before choosing the highest utility resource further improves both optimization criteria (strategy m-Period Repetition, mPR). I observe that OPR and SCA perform best regarding the utilization fraction and utility. Yet, agents will not be able to carry their top priority customer with SCA in most cases (probability $\frac{1}{N}$). My findings recommend that taxi drivers consider both history and associated utility when choosing a customer or resource.

Yet, my model draws a rather theoretical picture of the reality: I assume that utilities $u_s(j)$ and $u_i(i,j)$ are uniformly distributed and random ($\frac{1}{N} \ldots 1$ with step size

Strategy	utilization f	utility u
NL	83.0%	0.415
RD	30.6%	0.240
LL	57.0%	0.357
OPR	73.9%	0.457
CA	49.7%	0.249
SCA	93.8%	0.469
SRD1	78.5%	0.438
SRD2	77.3%	0.432
SRD3	36.7%	0.283
SRD4	47.4%	0.366

Table 10: MPMC: Comparing Strategies

$\frac{1}{N}$), allowing for an analytical approach. In most cities, one would rather assume a majority of customers returning a low or medium utility and only very few trips with very high utility. Also, assuming Gaussian-distributed numbers of customer per district is a major abstraction, in reality, a small number of hot spots such as airports or railway stations draw more attention than a large number of residential neighborhoods. Yet, the VFHP game model I discussed in chapters 5-8 can easily be adapted by exchanging $C(c_k)$ by more suitable functions for the given distribution. In the MP and MPMC model variants, I model the distance between agent i and customer j as $u_i(i,j)$. In real world examples, $u_i(i,j)$ depends on the history, as agents move through the city. Also, two adjacent resources will result in similar utilities for all agents which is not reflected in the presented model. Though, my model allows for extensions addressing these limitations.

In reality, the individual utility of agents – that is distance between agent and resource – changes in every iteration, as agents drive to customers. Thus, the utility agents can derive from customers has to be recalculated in every iteration. Yet, varying utilities do not influence the general idea VFHP game model; one only had to retrieve information about the preferences of all other agents in every iteration. Another abstraction concerns the timing between agents: One cannot assume that all agents select a resource at the same time. One could impose a discrete time model assuming that every agent drives to one customer per discrete time step, but as driving to a customer takes differently long depending on the distance. In the VFHP game model, it is sufficient to assume that the number of customers and agents is identical in all iterations, but several of the history-dependent strategies (LL, OPR, SCA) will perform differently for agents who did not participate in the previous iteration, as these agents will have to select a random resource rather than using a more promising selection. For example, agents implementing the OPR strategy receive a certain utilization of $f(i) = 1$ from customer j in the "return" phase, as no other agent drives to this customer j if this resource was occupied in the previous iteration. Yet, if agent i returns to a customer after pausing for several iterations, it is possible that another agent chose this resource as well, reducing utilization fraction and utility. Also, drivers who did not carry a customer will be able to drive to another customer directly after, whilst agents carrying a customer first have to finish this trip and are thus not available during the next iteration. One can extend the VHFP game model with a "continue carrying" phase for agents, in which they are utilized ($f(i) = 1$) and the utility the carried customer yields is divided up over the all iterations this trip takes. Customers disappear after being carried, and new customers appear frequently. As the shared utility of customers is the expected revenue, the VFHP game model can easily incorporate appearing and disappearing customers. Also, the expected utility yielded by customers can be difficult to determine, as individual behavior cannot be predicted precisely. It is possible to predict general tendencies (e.g., customers at airports often travel downtown and thus quite far), but for other locations, one cannot predict precise travel distances or patterns of customers (e.g. in city centers, most customers travel short distance, but few customers need longer transport, yet, it is difficult to predict when exactly customers require these longer trips). The IP and IPMC model variant do not use shared utilities in terms of customer revenue and are therefore more suited if the utility is unknown. In more rural areas, the expected number of customers in a district can be below 1, but the VFHP assumes discrete numbers of customers per district. Whilst rounding is reasonable for larger numbers of customers per district, rounding will frequently result in no expected customers in rural areas. There, vehicles for hire are usually called by phone. Thus, a dispatcher sends a driver to pick up this customer. The VHFP on the opposite mimics taxi hailing or calling a nearby taxi via app, if no dispatcher is available.

Despite the above limitations, the VHFP presents a suitable game model for agent behavior in vehicle for hire markets and lays ground work for improving utilization and utility in mobility markets.

10. Conclusion and Future Work

In this thesis I analysed two different models for mobility markets, the Kolkata Paise Restaurant Problem (KPRP) and four model variants of the Vehicle for Hire Problem (VFHP). To adapt the KPRP for mobility markets, I gradually drop or alter the assumptions of the KPRP: Agents no longer agree upon the resources' utilities (IP and MP model variants), and resources are "clustered" in districts, allowing agents to deviate from their first choice (IPMC and MPMC model variants). Further, I compared those five models by testing for utilization fraction and utility for agents using one of seven different strategies. Three of these strategies stem from Chakrabarti et al. (2009), two further strategies were introduced by Ghosh et al. (2013). I developed the strategies RD and SRD to specifically address the requirements of dynamic mobility markets. In dynamic matching markets, the behavior of other agents in previous iterations cannot determine the utility agents associate with resources in the future with absolute certainty as agents and customers enter and leave the market at will, calling for history-independent rank-dependent strategies.

Future research will be conducted on (1) behavior of agents, if two or more strategies are implemented in one market and the influence on utilization fraction and utility, (2) performance of the discussed strategies in practice, (3) incentive mechanisms and their effect in practice, and (4) the influence of the rise of autonomous cars and successive merge of the vehicle for hire and the car-sharing market.

If agents apply different strategies, the overall utilization fraction and utility might increase or decrease. Also, the utility could be unevenly distributed. For example, if $N-1$ agents play NL in the KPRP and one agent plays RD, this agent can expect a higher utility than the other agents ($0.632 \cdot u_{max}$ vs. $0.316 \cdot u_{max}$). Unilateral deviation can therefore be beneficial for agents. In the CA strategy, unilaterally deviating agents can implement a strategy in which they only choose from previously occupied resources, if only one agent deviates, he is guaranteed a utilization of $f(i) = 1$. The OPR strategy retrieves its high utility from agents not randomly choosing resources which were served by other agents the previous iteration, including those agents who constantly carry their preferred customer. Single agents implementing a NL strategy reduce the number of agents returning to their preferred resource, decreasing the performance of the OPR strategy.

This thesis focuses on the performance of several strategies in theoretical settings. As discussed in chapter 9, utilization and utility can vary as the assumptions of the VFHP deviate from reality. With real world data on the location of customers during a given time frame and the routes of drivers, one can evaluate whether the strategies improve current driver behavior. With insight from this data analysis, one can improve the strategies presented in this thesis and continue with incentive mechanisms to enforce beneficial behavior.

One can use the knowledge about the theoretic (and real world) performance of different strategies to incentivize behavior that is beneficial for the entire group. As discussed in chapter 9, agents incorporating the strategy OPR achieve a high utilization fraction and high utility in the IP and IPMC model. The strategy dictates a three-step approach: A random choice of a resource, returning to this resource once, and driving to the preferred resource. Yet, agents might be reluctant to wait for one iteration prior to driving to the preferred resource (e.g. due to missing trust in other agents, bounded rationality). For these agents, a coordination instance can offer incentives to return to the same resource.

Developments in the field of autonomous cars will most likely result in the end of vehicle for hire markets in its current setup, as drivers are no longer required, but cars independently carry passengers. Another industry that develops towards autonomous vehicles for passenger transportation is the car-sharing market in which passengers can rent cars for a short period (i.e. for one-way trips in major cities). The vehicle for hire market and car-sharing market steer towards offering the same service, if drivers become obsolete. Obviously, strategies and algorithms to redirect agents will become increasingly important; future research should therefore focus on improving the basic strategies presented in this thesis.

References

Abraham, D. J., Irving, R. W., Kavitha, T., and Mehlhorn, K. Popular matchings. *SIAM Journal on Computing*, 37(4):1030–1045, 2007.

Agarwal, S., Ghosh, D., and Chakrabarti, A. S. Self-organization in a distributed coordination game through heuristic rules. *arXiv preprint arXiv:1608.00213*, 2016.

Akbarpour, M., Li, S., and Oveis Gharan, S. Thickness and information in dynamic matching markets. *Available at SSRN 2394319*, 2016.

Alonso-Mora, J., Samaranayake, S., Wallar, A., Frazzoli, E., and Rus, D. On-demand high-capacity ride-sharing via dynamic trip-vehicle assignment. *Proceedings of the National Academy of Sciences*, page 201611675, 2017.

Arnott, R. Taxi travel should be subsidized. *Journal of Urban Economics*, 40(3):316–333, 1996.

Arthur, W. B. Inductive reasoning and bounded rationality. *The American economic review*, 84(2):406–411, 1994.

Bloch, F. and Houy, N. Optimal assignment of durable objects to successive agents. *Economic Theory*, 51(1):13–33, 2012.

Cairns, R. D. and Liston-Heyes, C. Competition and regulation in the taxi industry. *Journal of Public Economics*, 59(1):1–15, 1996.

Chakrabarti, A. S., Chakrabarti, B. K., Chatterjee, A., and Mitra, M. The kolkata paise restaurant problem and resource utilization. *Physica A: Statistical Mechanics and its Applications*, 388(12):2420–2426, 2009.

Challet, D. and Zhang, Y.-C. On the minority game: Analytical and numerical studies. *Physica A: Statistical Mechanics and its applications*, 256(3):514–532, 1998.

Chen, M. K. and Sheldon, M. Dynamic pricing in a labor market: Surge pricing and flexible work on the uber platform. Technical report, Mimeo, UCLA, 2015.

Chen, Y. and Hu, M. Pricing and matching with forward-looking buyers and sellers. 2016.

Chmura, T. and Pitz, T. Successful strategies in repeated minority games. *Physica A: Statistical Mechanics and its Applications*, 363(2): 477–480, 2006.

Correa, J. R., Schulz, A. S., and Stier-Moses, N. E. On the inefficiency of equilibria in congestion games. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 167–181. Springer, 2005.

Cramer, J. and Krueger, A. Disruptive change in the taxi business: The case of uber. *American Economic Review*, 106(5):177–182, 2016.

Gale, D. and Shapley, L. S. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

Ge, Y., Xiong, H., Tuzhilin, A., Xiao, K., Gruteser, M., and Pazzani, M. An energy-efficient mobile recommender system. In *Proceedings of the 16th ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 899–908. ACM, 2010.

Ghosh, A., Biswas, S., Chatterjee, A., Chakrabarti, A. S., Naskar, T., Mitra, M., and Chakrabarti, B. K. Kolkata paise restaurant problem: An introduction. In *Econophysics of Systemic Risk and Network Dynamics*, pages 173–200. Springer, 2013.

Hall, J., Kendrick, C., and Nosko, C. The effects of uber's surge pricing: A case study. *The University of Chicago Booth School of Business*, 2015.

Kim, H., Yang, I., and Choi, K. An agent-based simulation model for analyzing the impact of asymmetric passenger demand on taxi service. *KSCE Journal of Civil Engineering*, 15(1):187–195, 2011.

Kurino, M. House allocation with overlapping agents: A dynamic mechanism design approach. Technical report, Jena economic research papers, 2009.

Lasry, J.-M. and Lions, P.-L. Mean field games. *Japanese journal of mathematics*, 2(1):229–260, 2007.

Lee, D.-H., Wang, H., Cheu, R., and Teo, S. Taxi dispatch system based on current demands and real-time traffic conditions. *Transportation Research Record: Journal of the Transportation Research Board*, (1882): 193–200, 2004.

Leshno, J. D. Dynamic matching in overloaded systems. *URL hp://www. people. fas. harvard. edu/jleshno/papers. html*, 2:7, 2012.

Li, B., Zhang, D., Sun, L., Chen, C., Li, S., Qi, G., and Yang, Q. Hunting or waiting? discovering passenger-finding strategies from a large-scale real-world taxi dataset. In *Pervasive Computing and Communications Workshops (PERCOM Workshops), 2011 IEEE International Conference on*, pages 63–68. IEEE, 2011.

Li, S. S. H. *Multi-attribute taxi logistics optimization*. PhD thesis, Massachusetts Institute of Technology, 2006.

Linne+Krause Marketing-Forschung. Gutachten über die wirtschaftliche lage des hamburger taxi-gewerbes. http://hh-taxi.de/wp-content/Taxi_Gutachten_L&K_2011.pdf, 2011. retrieved at 12.02.2017.

Linne+Krause Marketing-Forschung. Untersuchung zur wirtschaftlichkeit des berliner taxigewerbes. http://www.stadtentwicklung.berlin.de/verkehr/politik/taxi/download/untersuchung_wirtschaftlichkeit_taxi_berlin.pdf, 2016. retrieved at 12.02.2017.

Manlove, D. F. and Sng, C. T. Popular matchings in the capacitated house allocation problem. In *European Symposium on Algorithms*, pages 492–503. Springer, 2006.

Milchtaich, I. Congestion games with player-specific payoff functions. *Games and economic behavior*, 13(1):111–124, 1996.

Monderer, D. and Shapley, L. S. Potential games. *Games and economic behavior*, 14(1):124–143, 1996.

Nash, J. Non-cooperative games. *Annals of mathematics*, pages 286–295, 1951.

Rayle, L., Shaheen, S., Chan, N., Dai, D., and Cervero, R. App-based, on-demand ride services: Comparing taxi and ridesourcing trips and user characteristics in san francisco. *University of California, Berkeley, United States Rogers, B.(2015) The social costs of Uber. James E. Beasley School of Law, Temple University, Philadelphia, United States*, 2014.

Rogers, B. The social costs of uber. *U. Chi. L. Rev. Dialogue*, 82:85, 2015.

Rosenthal, R. W. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.

Seow, K. T., Dang, N. H., and Lee, D.-H. A collaborative multiagent taxi-dispatch system. *IEEE Transactions on Automation Science and Engineering*, 7(3):607–616, 2010.

Shi, Y. and Lian, Z. Optimization and strategic behavior in a passenger–taxi service system. *European Journal of Operational Research*, 249(3): 1024–1032, 2016.

Trigo, P., Jonsson, A., and Coelho, H. Coordination with collective and individual decisions. In *Advances in Artificial Intelligence-IBERAMIA-SBIA 2006*, pages 37–47. Springer, 2006.

Wong, R., Szeto, W., and Wong, S. A two-stage approach to modeling vacant taxi movements. *Transportation Research Procedia*, 7:254–275, 2015.

Yang, H., Ye, M., Tang, W. H.-C., and Wong, S. C. A multiperiod dynamic model of taxi services with endogenous service intensity. *Operations research*, 53(3):501–515, 2005.

Yang, P., Frazier, P. I., et al. Mean field equilibria for competitive exploration in resource sharing settings. In *Proceedings of the 25th International Conference on World Wide Web*, pages 177–187. International World Wide Web Conferences Steering Committee, 2016.

Ye, Q. C. and Zhang, Y. Participation behavior and social welfare in repeated task allocations. In *Agents (ICA), IEEE International Conference on*, pages 94–97. IEEE, 2016.

Ye, Q. C., Zhang, Y., and Dekker, R. Fair task allocation in transportation. *Omega*, 68:1–16, 2017.

Yuan, J., Zheng, Y., Zhang, L., Xie, X., and Sun, G. Where to find my next passenger. In *Proceedings of the 13th international conference on Ubiquitous computing*, pages 109–118. ACM, 2011.