10. Mathematical finance

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(All variables are real and one-dimensional unless otherwise specified.)

Trivia. Google Books gives somewhat more results for "mathematical finance" than "financial mathematics", but the latter term was more prevalent before 2000.

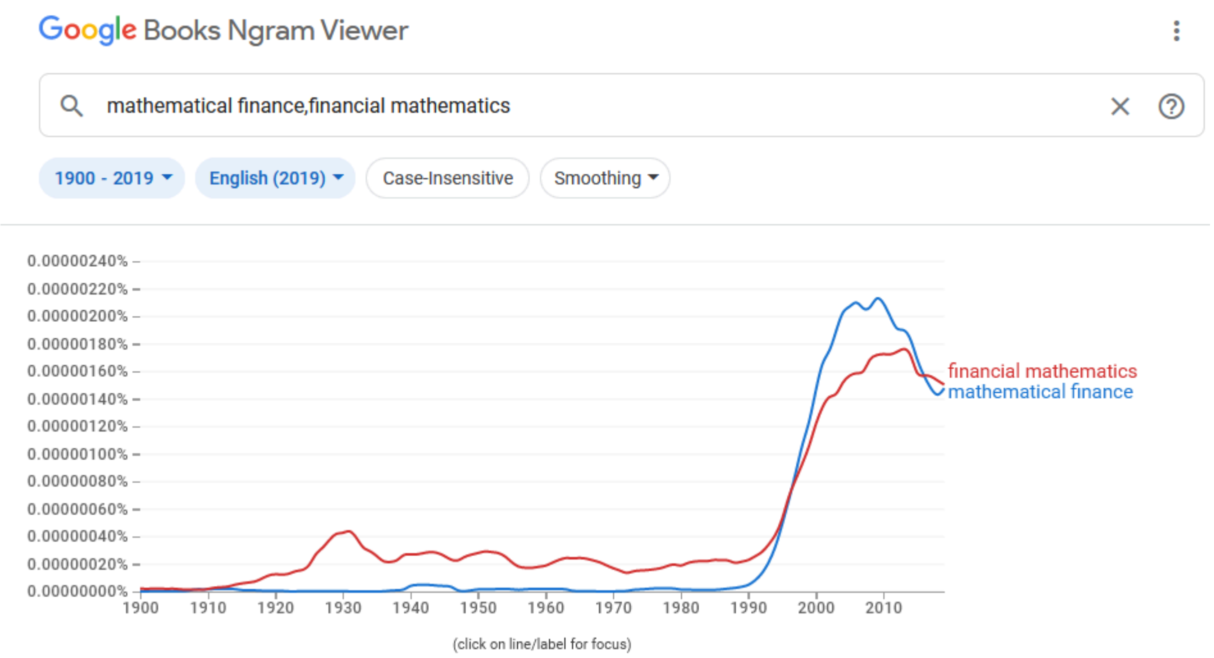

Fig. 1 Percentage of books with the two terms per year of publication. Retrieved from [**Google Books Ngram Viewer** (https://books.google.com/ngrams/graph?content=mathematical+finance%2Cfinancial+mathematics&year_start=1900&year_end=2019&corpus=en-2019&smoothing=3)](https://books.google.com/ngrams/graph?content=mathematical+finance%2Cfinancial+mathematics&year_start=1900&year_end=2019&corpus=en-2019&smoothing=3).

1. Portfolio theory

Harry Markowitz pioneered the portfolio theory in 1952, for which he won the Nobel Prize of Economics in 1990. (He originally studied philosophy and physics but later switched to economics.) This section discusses Markowitz's version of portfolio theory and ignores its descendants.

1.1 Return

Let S be the discrete-time series of a stock's daily price, so S_t is its price on the tth day. We are often interested in its **return**, which is defined as

$$R_t^{(1)}(n) = \frac{S_t - S_{t-n}}{S_{t-n}} = \frac{S_t}{S_{t-n}} - 1 \qquad \text{or}$$
$$R_t^{(2)}(n) = \ln\left(\frac{S_t}{S_{t-n}}\right).$$

The second definition may be specifically called **log-return**. They are numerically close because Taylor expansion gives $\ln x \approx x - 1$. Behind the numerical similarity, the ordinary return implies

$$S_t = \left[1 + R_t^{(1)}(n)\right] S_{t-n}$$

and thus models a stock that grows linearly like **simple interest**, whereas the log-return implies

$$S_t = S_{t-n} e^{R_t^{(2)}(n)} \approx S_{t-n} \left[1 + \frac{R_t^{(2)}(n)}{n}\right]^n$$

and thus models a stock that grows exponentially like **compound interest**.

1.2 Efficient frontier

Suppose two stocks S_1 and S_2 give daily (i.e. $n = 1$) returns R_1 and R_2. (The subscripts are now indices instead of time.) Let μ_i and σ_i^2 be the mean and the variance of R_i, then let σ_{12} be the covariance between R_1 and R_2. While μ_i measures a stock's expected return, σ_i^2 measures the stock's **risk**. A large σ_i^2 implies strong fluctuation in R_i, so it becomes hard to predict its value and thus risky to invest in S_i.

We may combine S_1 and S_2 as a **portfolio**

$$S_p = A_1 S_1 + A_2 S_2$$

for $A_1 \in [0, 1]$ and $A_2 \equiv 1 - A_1$. Hence the portfolio's expected return

$$\mu_p = A_1 \mu_1 + A_2 \mu_2$$

is bounded in $[\min(\mu_1, \mu_2), \max(\mu_1, \mu_2)]$. As μ_p is a linear function in A_1, the portfolio's risk

$$\sigma_p^2 = A_1^2 \sigma_1^2 + A_2^2 \sigma_2^2 + 2 A_1 A_2 \sigma_{12}$$

is a quadratic function in μ_p. After some expansion, you will get the formula of the portfolio's **efficient frontier**: $\sigma_p^2 = a u_p^2 + b u_p + c$ for

$$\begin{cases} a &= \dfrac{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}{(\mu_1 - \mu_2)^2} \\ b &= \dfrac{-2\left[\mu_2\sigma_1^2 + \mu_1\sigma_2^2 - (\mu_1 + \mu_2)\,\sigma_{12}\right]}{(\mu_1 - \mu_2)^2} \\ c &= \dfrac{\mu_2^2\sigma_1^2 + \mu_1^2\sigma_2^2 - 2\mu_1\mu_2\sigma_{12}}{(\mu_1 - \mu_2)^2} \end{cases}.$$

The portfolio consequently possesses the **minimum risk** $\sigma_p^{*2} = c - \dfrac{b^2}{4a}$ at $u_p^* = -\dfrac{b}{2a}$. Therefore, $A_1 = \dfrac{u_p^* - u_2}{u_1 - u_2}$ yields the **minimum-risk portfolio** as long as it satisfies the boundary condition $A_1 \in [0, 1]$. If it violates the condition, the minimum-risk portfolio simply consists of 100% of the lower-risk stock.

2. Introduction to options

An **option** is a contract whose value is derived (hence a **derivative**) from an **underlying** asset. The buyer of an option has the **freedom** to exercise its terms on its **exercise date**; if the buyer chooses to exercise the terms, the seller must follow. Because the buyer has gained advantage from the seller, the buyer has to pay the seller a **premium** to compensate for the **deprived freedom**.

If a contract obliges both its buyer and seller to exercise its terms, the contract is called a **forward**.

2.1 Types of options

Two basic dimensions of an option are **long** vs **short** and **call** vs **put**. They cross-classify four basic types of options, viz. long calls, long puts, short calls and short puts.

- **Long vs short.** While discussing trading, finance prefers the verbs "**long**" and "**short**" to the verbs "buy" and "short". To long an option means to buy it, whereas to short an option means to sell it. The one who longs is a **holder**, and the one who shorts is a **writer**.
- **Call vs put.** A **call** option grants the holder a right to buy the writer's asset, and the writer must sell if the holder buys. On the other hand, a **put** option grants the holder a right to sell asset to the writer, and the writer must buy if the holder sells. Practically, the holder and the writer trade the equivalent amount of money instead of the underlying asset.

2.2 Styles of options

An option only specifies an **expiry date** and a **strike price** K, whereas its **style** defines its possible exercise dates and its payoff at exercise. The two fundamental styles are **European** and **American** options, which are collectively called the **vanilla options**.

A European option can be exercised **only** on its expiry date, whereas an American option can be exercised at **any** time before its expiry date. Some analysts further distinguish an **Bermudan** option from an American one: instead of any time, it can be exercised only at a **finite** set of time-points before its expiry date.

2.3 Payoffs of vanilla options

Let S be the price of a **vanilla call**'s underlying asset. If $S < K$, the holder should not exercise the option, otherwise he buys the asset more expensively than buying from the market. In other words, the option is worthless for the holder if $S < K$. However if $S > K$, the holder should exercise the option and will earn $S - K$. These combine to mean that a call option costs

$$C(S, K) = \max\left(S - K, 0\right)$$

at exercise. For a **vanilla put**, the holder should exercise the option only if $S < K$, otherwise he sells the asset cheaper than selling to the market. Hence its payoff is similarly argued.

$$P(S, K) = \max\left(K - S, 0\right)$$

Let $V \in \{C, P\}$ be a vanilla option's payoff. On one hand, the holder H pays the writer W a premium p; on the other hand, the writer loses V to the holder as long as the holder exercises the option rationally. This trade can be written symbolically as $H \underset{V}{\overset{p}{\rightleftharpoons}} W$.

Put-call parity. Observe the identity for a vanilla option.

$$S + P(S, K) - C(S, K) \equiv K$$

This implies that a portfolio with one long asset, one long put and one short call always costs K on its exercise date. Therefore, the portfolio is **risk-free**.

2.4 Exotic options

Options that are not European or American may be referred to as **exotic options** although they may still be classified as "European-like" or "American-like" according to their possible exercise dates. They derive their payoffs very differently. Here we consider calls only.

- **Binary.** The first kind is **cash-or-nothing**, with which the holder earns a fixed amount of money if $S > K$. The second kind is **asset-or-nothing**, with which the holder earns the asset's value if $S > K$.

- **Asian.** The holder earns $\max\left[A(t,T)-K,0\right]$, where $A(t,T)$ is the asset's **average** price between time t and the exercise date T. It may be called a **Russian** option if t is the option's beginning time.
- **Lookback**. The holder earns $\max\left[M(t,T)-K,0\right]$, where $M(t,T)$ is the asset's **maximum** or **minimum** price between time t and the exercise date T.

An Asian option and a lookback option are **path-dependent** because they cannot be priced without the **history** of S. In addition to redefining payoffs, some exotic options impose rules to alter how they are exercised.

- **Barrier.** The holder can exercise the option only if the asset's price has touched (or never touched) a **barrier** price before expiry.
- **Israeli,** aka a **game option**. The writer can exercise the option for the holder in advance. Because the holder has lost his freedom, the writer has to pay him an extra **compensation**.

3. Pricing in a market

Consider a stock with a current price $S_0=1$. It either rises to $S_1=2$ or drops to $S_1=1/2$ with equal probabilities tomorrow. A relevant European call with $K=1$ expires tomorrow, so its payoff is

$$C_1=\begin{cases}1 & (S_1=2)\\ 0 & (S_1=1/2)\end{cases}$$

at exercise. However, how much does the option cost currently—what is C_0? We first simplify the scenario by neglecting the value of time, meaning that the money saved in a bank remains constant.

While an option's payoff is obvious at exercise, it gets nontrivial to evaluate its payoff **before exercise**. This section first examines the **philosophical principles** behind **pricing**, leaving the actual mathematics for the next tutorial.

3.1 The traditional approach

The traditional answer is $C_0=\mathrm{E}(C_1)=1/2$: the present price amounts to the **expected future price** so that a holder of the call is expected to **neither gain nor lose** through his investment. Although it looks fair, this principle of pricing means that investing in an **risky** option is as worthless as saving money in a bank. The investor is not awarded for his **risk-loving** attitude at all. This consequently violates the principle of **no arbitrage**.

3.2 Arbitrage

Arbitrage means "earning more without taking more risk". This happens when an asset is marked two different prices at the same time. For example, market A marks an exchange rate "1 apple : 1 banana", and market B marks an exchange rate "1 banana : 2 apples". Then, an investor can earn 1 apple for free by

- selling 1 apple for 1 banana in market A and
- selling 1 banana for 2 apples in market B.

The fundamental **assumption** of finance is **the absence of long-term arbitrage**, which informally means that we cannot buy a **free lunch** in a market. In the example of apples and bananas, if many investors utilize the stated strategy of arbitrage, the **demand** of bananas should rise in the market A, whereas their **supply** should rise in market B. According to laws of economics, the price of bananas must rise in market A but drop in market B, so the two markets must eventually reach the same exchange rate (say "1 banana : 1.5 apples").

(That said, real exchange rates of currencies never match exactly in different markets, but their differences are so small that arbitrage is infeasible after considering transaction costs.)

The principle of no arbitrage. Now return to the option's price C_0. If an investor borrows $1/3$ units from a bank today and buys $2/3$ units of the stock, his portfolio costs

$$\Pi_1 = \frac{2}{3}S_1 - \frac{1}{3} = \begin{cases} 1 & (S_1 = 2) \\ 0 & (S_1 = 1/2) \end{cases} \equiv C_1$$

tomorrow. The principle of no arbitrage hence requires

$$C_0 \equiv \Pi_0 = \frac{2}{3}S_0 - \frac{1}{3} = \frac{1}{3}$$

and implies that if the option only costs $C_0 = 1/2$ today, an investor can long the portfolio and short one call today to earn

$$(\Pi_1 - C_1) - (\Pi_0 - C_0) = 0 - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{6}$$

tomorrow without any risk.

The concept of pricing due to no arbitrage emerges in 1970s, but its mathematics remains a bit mysterious until 1980s: how to construct the portfolio $\Pi = 2S/3 - 1/3$ in advance so that we can define $C_0 = \Pi_0$? What is the equally worthy portfolio for a general option? This involves the **fundamental theorem of asset pricing**, the key topic of the next tutorial.

3.3 Market efficiency

Nonetheless, the principle of no arbitrage is useless unless we can assess C_1, which requires us to be able to somehow **predict** S_1. But can we really predict this future price? After all, while the market is constantly marking prices for stocks, why is each stock marked this price but not that price? This is another important philosophical question in finance. Long story short, we **cannot** predict anything if the market is **efficient**.

An efficient market reacts against investors' action so quickly that **no one** can consistently make a **meaningful analysis**. Why? If a wise investor accurately predicted $S_1 > S_0$ and decided to buy the stock, other equally wise investors would adopt the same strategy. As a result, the stock's demand goes up, so does S_0 until $S_0 \approx S_1$ soon, when it is no longer profitable to buy the stock.

Now rephrase everything in terms of information H. Since a consistently good analysis is impossible, the stock's future price $S(t > 0)$ contains more information than the past data $D(t \leq 0)$ that any wise investor would choose to analyze, while the data is **at most** as informative as $S(t = 0)$. In other words, however wise he is, an investor can at most infer $S(t = 0)$ from $D(t \leq 0)$.

$$H[D(t \leq 0)] \leq H[S(t = 0)] < H[S(t > 0)]$$

The exact scope of $D(t \leq 0)$ defines three levels of efficiency.

- **Weak.** $D(t \leq 0)$ represents the stock's **history** $S(t \leq 0)$.
  - If a market is weakly efficient, an analysis on $S(t \leq 0)$ does not predict the future $S(t > 0)$ better than an analysis solely based on $S(t = 0)$, which has contained all the information that $S(t \leq 0)$ can give.
    - This implies that stock's price follows the AR(1) model as $S(t = 1)$ depends on $S(t = 0)$ only.
    - An investor can only predicts better if he knows more than the stock's history, e.g. knowing other stocks' history as well.
- **Semi-strong**. $D(t \leq 0)$ represents **all public information** in the world.
  - If a market is semi-strongly efficient, an investor never predicts better than other investors unless he knows some "secrets" about the stock.
- **Strong**. $D(t \leq 0)$ represents **all information** in the world, **public and private**.
  - If a market is strongly efficient, no one in the world can predict better than others at all—unless he is very **lucky**.

After all, are real markets efficient? The controversial **efficient-market hypothesis** says that they are (without specifying the level), but no one can prove or disprove this claim.

4. Risk and uncertainty

We have been exploiting the concept "risk" some time without clarifying its meaning. This section is going to reveal some subtle ambiguities in its meaning.

4.1 Risk as uncertainty

In portfolio theory, we measure the "risk" of a stock S with its **sample variance**

$$\hat{\sigma}_S^2 \sim E_t\left(x_S^2\right) - E_t^2\left(x_S\right),$$

where x_S is the stock's return and E_t represents a **time average**. (The proportionality accounts for Bessel's correction.) For example, if a stock $S = \{1, 2, 3, 4, 5\}$ gives $x_S = \{1, 1/2, 1/3, 1/4\}$ and $\hat{\sigma}_S^2 \approx 0.113$, we conclude that "the stock has 0.113 units of risk".

On the other hand, consider a government bond $B = \{1, 2, 3, 4, 5\}$, which grows **definitely** by one unit every day. Since it is by definition **risk-free**, the bond has 0 units of risk despite $\hat{\sigma}_B^2 \approx 0.113 = \hat{\sigma}_S^2$. What happens behind the discrepancy? Despite their identical variance, a bond is risk-free because we know its price **certainly**, whereas the stock is risky because we do not. In this regard, an asset's risk quantifies the **uncertainty** we feel about its price.

Now let us formulate the matter more precisely. An asset's risk at time t should be fundamentally defined with its **population variance**

$$\sigma^2(t) = E_\Omega\left[X^2(t)\right] - E_\Omega^2\left[X(t)\right],$$

where E_Ω denotes an **ensemble average**, which averages over all possible realizations of its argument. For example, $E_\Omega[X(t)]$ checks the value of X at time t in every possible world and averages them. (Hence X is still a random variable.) For the bond, $X_B(t)$ takes one possible value only, so $\sigma_B^2(t) \equiv 0$. In contrast, it is impractical to measure $\sigma_S^2(t)$ because of the **uncountable** possibilities of $X_S(t)$. However, we can approximate $\sigma_S^2(t)$ with $\hat{\sigma}_S^2$ if

- t is so large that the number of possibilities of $X_S(t)$ is also large and
- $\hat{\sigma}_S^2$ is computed with a large sample size.

Symbolically,

$$\begin{cases} |X_S(t)| & \gg & 1 \\ |x_S| & \gg & 1 \end{cases} \quad \Rightarrow \quad \sigma_S^2(t) = \hat{\sigma}_S^2(x_S).$$

In mathematical physics, the equality between time average and ensemble average is related to the **ergodicity theorem**.

4.2 Risk versus uncertainty

However, risk and uncertainty are not always synonyms in economics. The economist Frank Knight first distinguishes the two ideas. He describes "risk" as **probabilistic** knowledge but "uncertainty" as the **lack** of knowledge. With his dichotomy, an investor who minimizes risk may act very differently from one who minimizes uncertainty. The **Ellsberg's paradox** (popularized by the economist Daniel Ellsberg) highlights their distinction.

Ellsberg's paradox. There are $R = 30$ red balls, G green balls, and B blue balls in a bag, where $G + B = 60$, so there are $N = R + G + B = 90$ balls in total. Ellsberg lets you play two guessing games, and you win $\$100$ in each game if you guess correctly. In the first game, you guess whether you are going to draw

1. a red ball or
2. a blue ball.

In the second game, you guess whether you will draw

1. a red ball or a green ball,
2. a blue ball or a green ball.

Economics always assumes that a rational person will take decision to **minimize risk**, i.e. a rational person is **risk-averse** in professional terms. In Ellsberg's games, a rational player should either choose the first guess for both games or choose the second guess for both games because

$$\frac{R}{N} < \frac{B}{N} \Leftrightarrow \frac{R+G}{N} < \frac{B+G}{N}.$$

This is an argument with probabilistic knowledge. On the contrary, Ellsberg's experiment finds that most people, perhaps including you, choose the first guess for the first game but the second guess for the second game.

Can we therefore say that most people are irrational? No, they do think about how to maximize their profits before guessing. But instead of being risk-averse, it turns out that a rational person tends to be **uncertainty-averse** (or **ambiguity-averse**), i.e. he prefers minimizing uncertainty per choice: $R = 30$ is more certain than $B = ?$ in the first game, whereas $B + G = 60$ is more certain than $R + G = ?$ in the second game.