MSDM 5056 Network Modeling

Assignment 2 (due 11th October, 2023)

Submit your assignment solution on canvas. You may discuss with others or seek help from your TA, but should not directly copy from others. Otherwise, it will be considered as plagiarism.

(1) **Matrix formalism**

Let **A** be the adjacency matrix of an undirected network of size N and **1** be the column vector whose elements are all l, i.e.

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Show that (*refer to Lecture 2, p.30*)

a) the vector $\boldsymbol{k}$ whose elements are the degrees k_i of the vertices, i.e.,

$$\mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{pmatrix}$$

is given by $\boldsymbol{k} = \mathbf{A1}$.

b) the number L of links in the network is given by

$$L = \frac{1}{2}\mathbf{1}^T\mathbf{A1}$$

c) the matrix **N** whose element N_{ij} is equal to the number of common neighbors of nodes i and j can be written as

$$\mathbf{N} = \mathbf{A}^2$$

(2) **Structure and centrality measures for a directed network**

Consider the adjacency matrix A of a network with 4 nodes given by

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(a) Write the in-degree sequence $\{k_1^{in}, k_2^{in}, k_3^{in}, k_4^{in}\}$ and the out-degree sequence $\{k_1^{out}, k_2^{out}, k_3^{out}, k_4^{out}\}$.

(b) Write the in-degree distribution of the network $P^{in}(k)$ for k = 0, 1, 2 3 and the out-degree distribution of the network $P^{out}(k)$ for k = 0, 1, 2, 3.

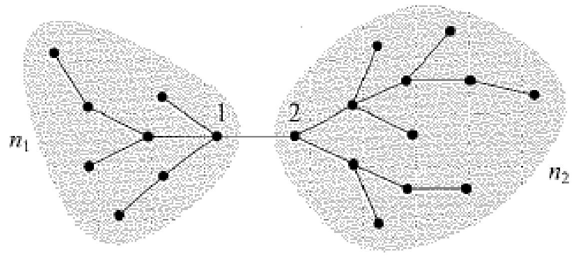
(c) The Katz centrality vector x has elements x_i indicating the Katz centrality of node i = 1, 2, …, N. Calculate x using the following definition

$$x = \beta(\mathbf{I} - \alpha\mathbf{A})^{-1}\mathbf{1} = \beta \sum_{n=0}^{\infty} \alpha^n \mathbf{A}^n \mathbf{1} \tag{1}$$

where $\alpha > 0$ and $\beta > 0$ and $\mathbf{1}$ is the column vector with elements $1_i = 1 \ \forall i = 1, 2, \dots, N$ and $\mathbf{I}$ denotes the $N \times N$ identity matrix.

(3) **Closeness centrality**

Consider an undirected tree of n vertices. A particular edge in the tree joins vertices 1 and 2 and divides the tree into two disjoint regions of n_1 and n_2 vertices as sketched



Show that the closeness centralities c_1 and c_2 of the two vertices are related by

$$\frac{1}{c_1} + \frac{n_1}{n} = \frac{1}{c_2} + \frac{n_2}{n}$$

(*Hint*: Begin with the definition of closeness centrality on p.34 of Lecture 3)

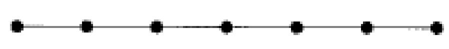
(4) **Betweenness centrality**

Consider an undirected (connected) tree of n vertices. Suppose that a particular vertex in the tree has degree k, so that its removal would divide the tree into k disjoint regions, and suppose that the sizes of those regions are $n_1, n_2, \dots, n_k$.

a) Show that the betweenness centrality x of the vertex, as defined on p.29 of Lecture 3 is (i.e., we define the paths pass through the particular node that connect any two other nodes, and no need to normalize with respect to the total number of paths.)

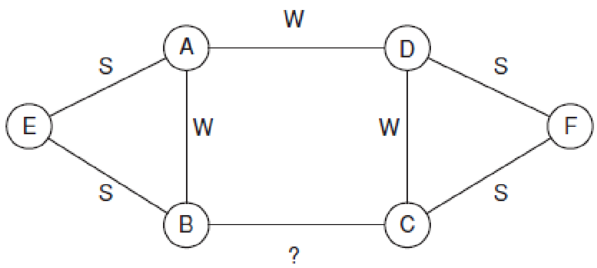
$$x = \frac{1}{2}\left[(n-1)^2 - \sum_{m=1}^{k} n_m^2\right]$$

b) Hence, or otherwise, calculate the betweenness of the i^{th} vertex from the end of a "line graph" of n vertices, i.e., n vertices in a row like this:



(5) **Strong and Weak Ties**

a) Consider the graph below, in which each edge – except the edge connecting nodes B and C – is labeled as a strong tie (S) or a weak tie (W). According to the theory of strong and weak ties, using the Strong Triadic Closure assumption, how would you expect the edge connecting B and C to be labeled? Give a brief (one- to two-sentence) explanation for your answer.



b) In the social network depicted below, in which each edge is labeled as either a strong or weak tie, which two nodes violate the Strong Triadic Closure property? Provide an explanation for your answer.