MSDM 5003 Homework 3 Solution

1. (a) Before turning to numerical computation, let us first compute analytically the autocorrelation function
$$f(x) = \int_{-\infty}^{+\infty} s_o(y) s_o(y+x)\, dy \tag{1}$$
for the signal
$$s_o(x) = \begin{cases} \sin(2\pi x) & \text{when } -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$
Due to the factor $s_o(y)$ in the integrand, it is obvious that
$$f(x) = \int_{-3}^{3} s_o(y) s_o(y+x)\, dy. \tag{3}$$
Due to the factor $s_o(y+x)$ in the integrand, it is obvious that
$$f(x) = 0 \quad \forall \quad x \in (-\infty, -6] \cup [6, +\infty). \tag{4}$$
If $0 \le x \le 6$,
$$f(x) = \int_{-3}^{3-x} \sin(2\pi y)\sin(2\pi y + 2\pi x)\, dy + \int_{3-x}^{3} \sin(2\pi y)\, 0\, dy. \tag{5}$$
Using $\sin(a)\sin(b) = [\cos(a-b) - \cos(a+b)]/2$, we have
$$\begin{aligned} f(x) &= \frac{1}{2}\int_{-3}^{3-x} [\cos(2\pi x) - \cos(4\pi y + 2\pi x)]\, dy \\ &= \frac{1}{2}\left[\cos(2\pi x)\, y - \frac{1}{4\pi}\sin(4\pi y + 2\pi x)\right]_{-3}^{3-x} \\ &= -\left(\frac{x}{2} - 3\right)\cos(2\pi x) + \frac{1}{4\pi}\sin(2\pi x). \end{aligned} \tag{6}$$
Similarly, if $-6 \le x \le 0$,
$$\begin{aligned} f(x) &= \int_{-3-x}^{3} \sin(2\pi y)\sin(2\pi y + 2\pi x)\, dy \\ &= \frac{1}{2}\left[\cos(2\pi x)\, y - \frac{1}{4\pi}\sin(4\pi y + 2\pi x)\right]_{-3-x}^{3} \\ &= \left(\frac{x}{2} + 3\right)\cos(2\pi x) - \frac{1}{4\pi}\sin(2\pi x). \end{aligned} \tag{7}$$
In conclusion, we have
$$f(x) = \begin{cases} -\left(\frac{|x|}{2} - 3\right)\cos(2\pi |x|) + \frac{1}{4\pi}\sin(2\pi |x|) & \text{when } -6 \le x \le 6 \\ 0 & \text{otherwise} \end{cases} . \tag{8}$$
As expected, $f(x \le -6) = f(x \ge 6) = 0$ and $f(0) = 3 = \int_{-3}^{3} \sin^2(2\pi y)\, dy$.
For numerical computation, see either of the program files `autocorrelation_integrate.py` or `autocorrelation_vector.py`. The former shows the required computations clearly while the latter is optimised for speed. The programs are written in `Python` with the libraries `NumPy` and `Matplotlib` and were executed using `Spyder`.

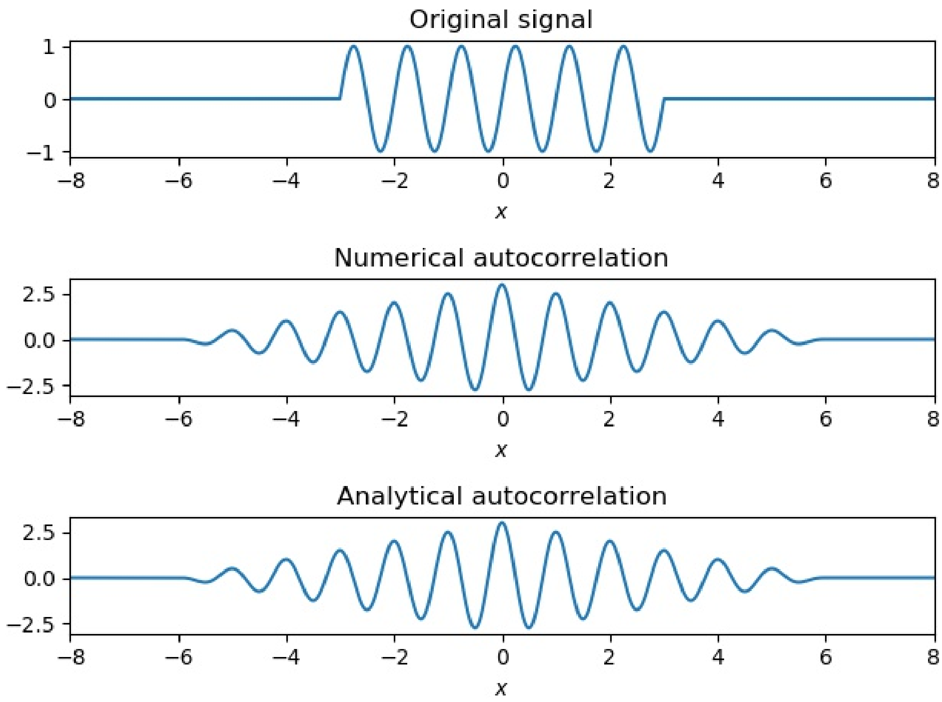

Figure 1: A signal and its autocorrelation. The autocorrelation obtained from analytical and numerical computations agree with each other.

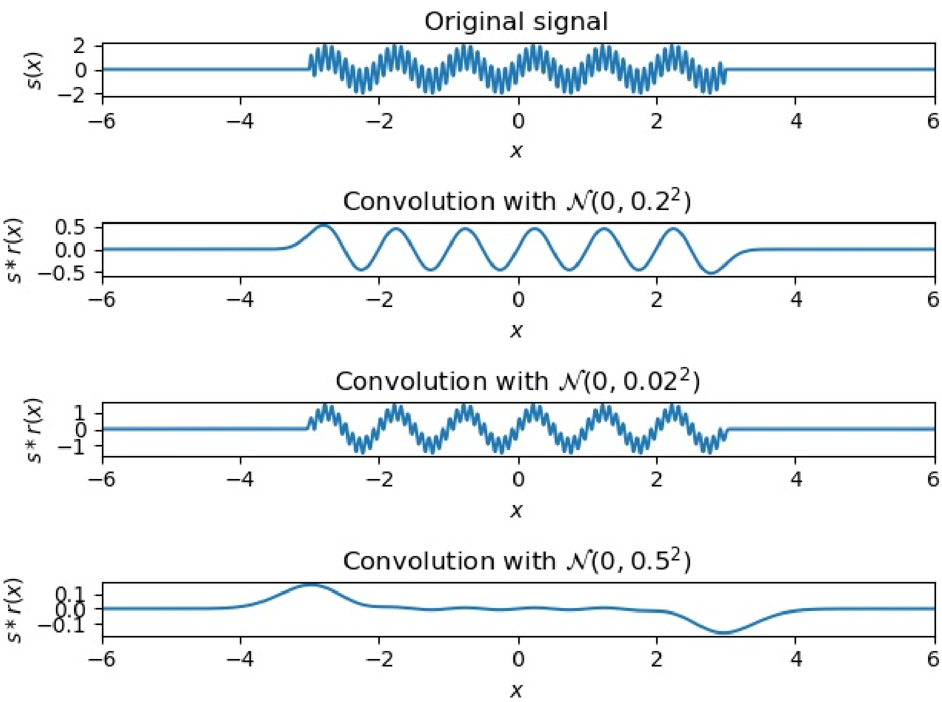

Figure 2: A signal and its convolution with $\mathcal{N}(0, \sigma^2)$ with $\sigma = 0.2, 0.02, 0.5$.

(b) See either of the program files `convolution_integrate.py` or `convolution_vector.py`. The former shows the required computations clearly while the latter is optimised for speed. The programs are written in `Python` with the libraries `NumPy` and `Matplotlib` and were executed using `Spyder`.