Discrete Convolution

Discrete Convolution

- Consider two infinite sequences of complex-valued functions g and h defined on the set of integers $\mathbb{Z}$
- For finite sequences one may extend the domain to $\mathbb{Z}$ by adding zeros (zero padding)
- The discrete convolution of g and h is given by

$$(g * h)[n] = \sum_{m=\cdots,-2,-1,0,1,2,\cdots} g[m]h[n-m] \qquad n \in \mathbb{Z}$$

if the limit of the infinite summation exists

- If $(g * h)[n]$ exists for all n, we say that the discrete convolution $g * h$ exists

Discrete Convolution

- $(g * h)[n]$, if exists, satisfies commutativity:

$$(g * h)[n] = \sum_{m=-\infty}^{\infty} g[m]h[n-m] = \sum_{n-m=-\infty}^{\infty} g[m]h[n-m]$$

$$= \sum_{n-m=-\infty}^{\infty} g[n-(n-m)]h[n-m]$$

$$= \sum_{m=-\infty}^{\infty} g[n-m]h[m]$$

$$= (h * g)[n]$$

Periodic Convolution

- If g and h are both periodic and have the same period N, their periodic convolution is defined by

$$(g * h)[n] = \sum_{m=0}^{N-1} g[m]h[n-m]$$

- It is readily observed that in this case $y[n] = (g * h)[n]$ is also periodic with period N

DFT of Periodic Convolution

$$Y[k] = \sum_{n=0}^{N-1} y[n] e^{2\pi i k n/N} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} g[m] h[n-m] e^{2\pi i k n/N}$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} h[n-m] e^{2\pi i k (n-m+m)/N} g[m]$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} h[n-m] e^{2\pi i k (n-m)/N} g[m] e^{2\pi i k m/N}$$

$$= \sum_{m=0}^{N-1} \left(\sum_{n-m=-m}^{n-m=N-1-m} h[n-m] e^{2\pi i k (n-m)/N} \right) g[m] e^{2\pi i k m/N}$$

$$= \sum_{m=0}^{N-1} \left(\sum_{p=-m}^{p=N-1-m} h[p] e^{2\pi i k p/N} \right) g[m] e^{2\pi i k m/N} = \sum_{m=0}^{N-1} \left(\sum_{p=0}^{N-1} h[p] e^{2\pi i k p/N} \right) g[m] e^{2\pi i k m/N}$$

$$= \left(\sum_{m=0}^{N-1} g[m] e^{2\pi i k m/N} \right) \left(\sum_{p=0}^{N-1} h[p] e^{2\pi i k p/N} \right) = G[k] H[k]$$

Sequences with Finite Support

- When h has finite support in the set $\{r, r+1, \cdots, s-1, s\}$ (representing a finite impulse response for instance), the infinite summation reduces to a finite summation:

$$(g * h)[n] = \sum_{m=r}^{s} g[n-m]h[m] = \sum_{m=n-s}^{n-r} g[m]h[n-m]$$

and hence $g * h$ exists

Sequences with Finite Support

- If in addition, g has finite support in the set $\{p, p+1, \cdots, q-1, q\}$, then $g * h$ has finite support in $\{p+r, p+r+1, \cdots, q+s-1, q+s\}$
- Note that the length of g is $M = q - p + 1$, the length of h is $K = s - r + 1$, and the length of $g * h$ is $N = (q+s) - (p+r) + 1 = M + K - 1$
- The linear convolution in this case is

$$(g * h)[n] = \sum_{m=r}^{s} g[n-m]h[m] = \sum_{m=p}^{q} g[m]h[n-m]$$

Linear Convolution and DFT

- One can compute the linear convolution directly by

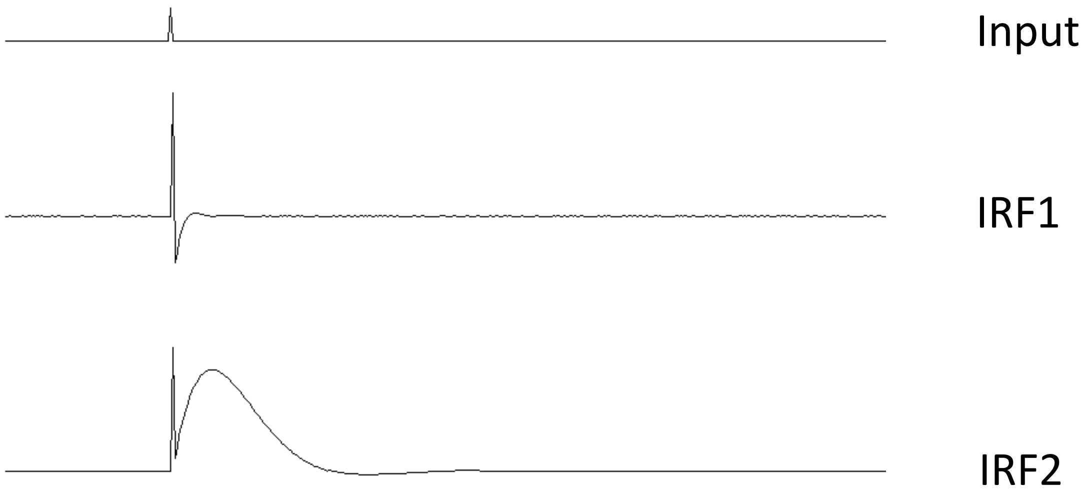
$$y[n] = (g * h)[n] = \sum_{m=r}^{s} g[n-m]h[m] = \sum_{m=p}^{q} g[m]h[n-m]$$

- But when K, M are large, it is typically done by computing the DFT by the efficient FFT algorithm

1. Choose $L \geq N = M + K - 1$
2. Pad both g and h with zeros to length L
3. Compute the DFTs $G[k]$ and $H[k]$, both of length L, for $g[n]$ and $h[n]$, respectively
4. Multiply them to get $Y[k] = G[k]H[k]$, which is also of length L
5. Compute the inverse DFT of $Y[k]$ to obtain $y[n]$ with length L

Example

- The impulse response function (IRF) of a dynamic system is its output when the input is an impulse
- For discrete-time systems, an impulse can be modeled as the Kronecker delta (c.f. a Dirac delta function for continuous-time systems)
- Hence if the input is g and the IRF is h, for linear time-invariant (LTI) systems, the output is the linear convolution $g * h$
- For finite impulse response (FIR), h has finite support

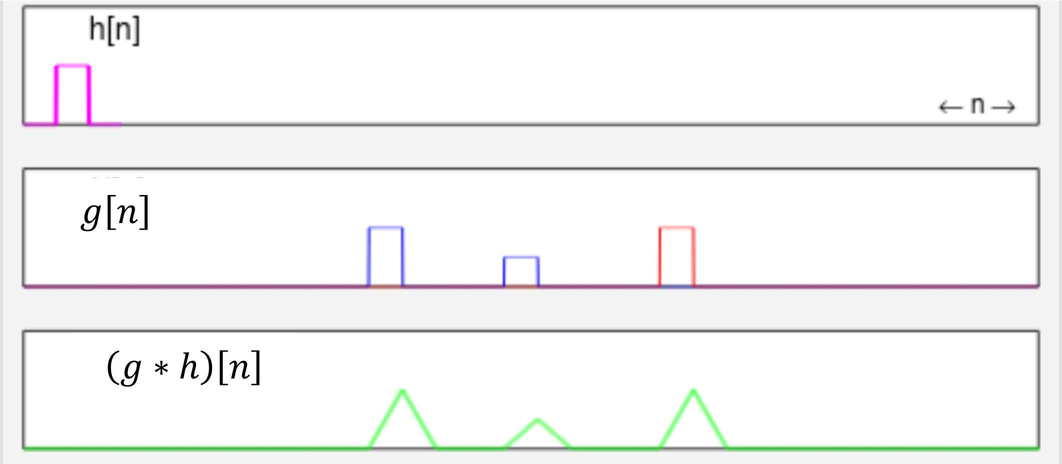


Example

- Let h be an FIR with length K: $\{h[0], h[1], \cdots, h[K-1]\}$
- This represents the output when the input is $\delta[n] = \delta_{n0} = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$
- Then for finite support input $\{g[p], g[p+1], \cdots, g[p+M-1]\}$ with length M, the output has finite support with length $N = M + K - 1$:

$$y[n] = (g * h)[n] = \sum_{m=0}^{K-1} g[n-m]h[m]$$

$$\rightarrow \{y[p], y[p+1], \cdots, y[p+M+K-2]\}$$



Example

- Alternatively, use DFT

1. Choose $L = M + K - 1$
2. Pad g with zeros to
   $g[p \leq n \leq p + L - 1]$: $\{g[p], g[p+1], \cdots, g[p+M-1], 0, 0, \cdots, 0\}$
3. Pad h with zeros to
   $$h[0 \leq n \leq L - 1]: \{h[0], h[1], \cdots, h[K-1], 0, 0, \cdots, 0\}$$
4. Compute the DFTs $G[k]$ and $H[k]$, both of length L, for $g[n]$ and $h[n]$, respectively
5. Multiply them to get $Y[k] = G[k]H[k]$, which is also of length L
6. Compute the inverse DFT of $Y[k]$ to obtain $y[n]$ with length L to get $\{y[p], y[p+1], \cdots, y[p+L-1]\}$

Periodic Summation

- If h is periodic function with period N, and if $(g * h)[n]$ exists, then

$$(g * h)[n] = \sum_{m=-\infty}^{\infty} g[m]h[n-m]$$
$$= \sum_{m=0}^{N-1} \left(\sum_{k=-\infty}^{\infty} g[m+kN] \right) h[n-m]$$

- It can be observed that $(g * h)[n]$ is also periodic with period N
- The periodic summation of g:

$$g_N[m] = \sum_{k=-\infty}^{\infty} g[m+kN]$$

if considered as a function of m, is also periodic with period N

Periodic Summation

- Note that by commutativity, we also have

$$(g * h)[n] = \sum_{m=-\infty}^{\infty} g[n-m]h[m]$$

$$= \sum_{m=0}^{N-1} \left(\sum_{k=-\infty}^{\infty} g[n-m-kN] \right) h[m]$$

Circular Convolution (Cyclic Convolution)

- The idea is to extend the definition to functions g and h that both may in general be aperiodic
- Def: Given a parameter N, the circular convolution of g and h is defined to be

$$(g_N * h)[n] = \sum_{m=-\infty}^{\infty} g_N[n-m]h[m]$$

$$= \sum_{m=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} g[n-m-kN] \right) h[m]$$

Circular Convolution (Cyclic Convolution)

- Note that

$$(g_N * h)[n] = \sum_{m=-\infty}^{\infty} g_N[n-m]h[m]$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=0}^{N-1} g_N[n-m-kN]h[m+kN] = \sum_{k=-\infty}^{\infty} \sum_{m=0}^{N-1} g_N[n-m]h[m+kN]$$

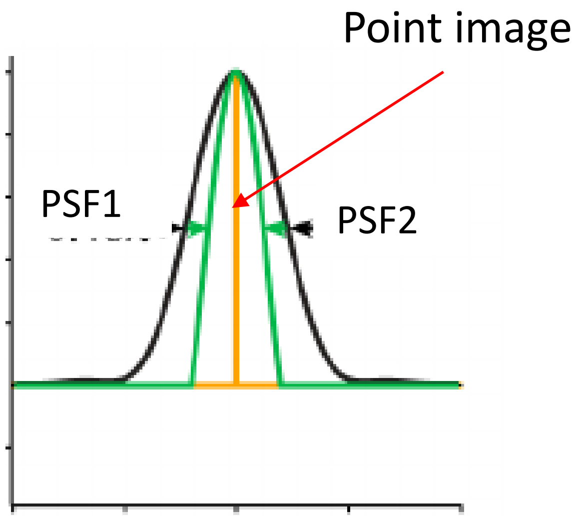
$$= \sum_{m=0}^{N-1} \sum_{k=-\infty}^{\infty} g_N[n-m]\, h[m+kN] = \sum_{m=0}^{N-1} \left(\sum_{k=-\infty}^{\infty} h[m+kN] \right) g_N[n-m]$$

$$= \sum_{m=0}^{N-1} g_N[n-m]h_N[m] = \sum_{m=n-N+1}^{n} g_N[m]h_N[n-m]$$

$$= \sum_{m=0}^{N-1} g_N[m]h_N[n-m] = (g * h_N)[n]$$

Example

- Image deconvolution: 1-D black-and-white "photo" for example
- Photo is blurred due to the error that each point on the photo gets smeared out to some "smearing distribution" → Point spread function (PSF)



Example

- Let g represent the true image and h be the point spread function of a point image with unit brightness at 0
- Then the brightness of the blurred image is given by the convolution

$$y[n] = (g * h)[n] = \sum_{m=-\infty}^{\infty} h[n-m]g[m]$$

Example

- Now in order to be able to apply periodic convolution theorem, we approximate the linear convolution by circular convolution

$$y[n] = \sum_{m=0}^{M-1} h_M[n-m]g[m]$$

where

$$h_M[m] = \sum_{k=-\infty}^{\infty} h[m+kM]$$

is the periodic summation of the PSF with period M

- The approximation works because usually the PSF h is sharply peaked at 0, with width much smaller than the size of the image
- It is expected that the approximation will only lead to noticeable visual artifact near the boundary of the image
- The sharper the PSF, the smaller the visual artifacts

Example

- Since

$$y[n] = \sum_{m=0}^{M-1} h_M[n-m]g[m] = \sum_{m=0}^{M-1} h[n-m]g_M[m]$$

one can equivalently interpret it as repeating the true image with period M


Latent Clear Image


Blurred Image


Restored Image with boundary artifacts

Example

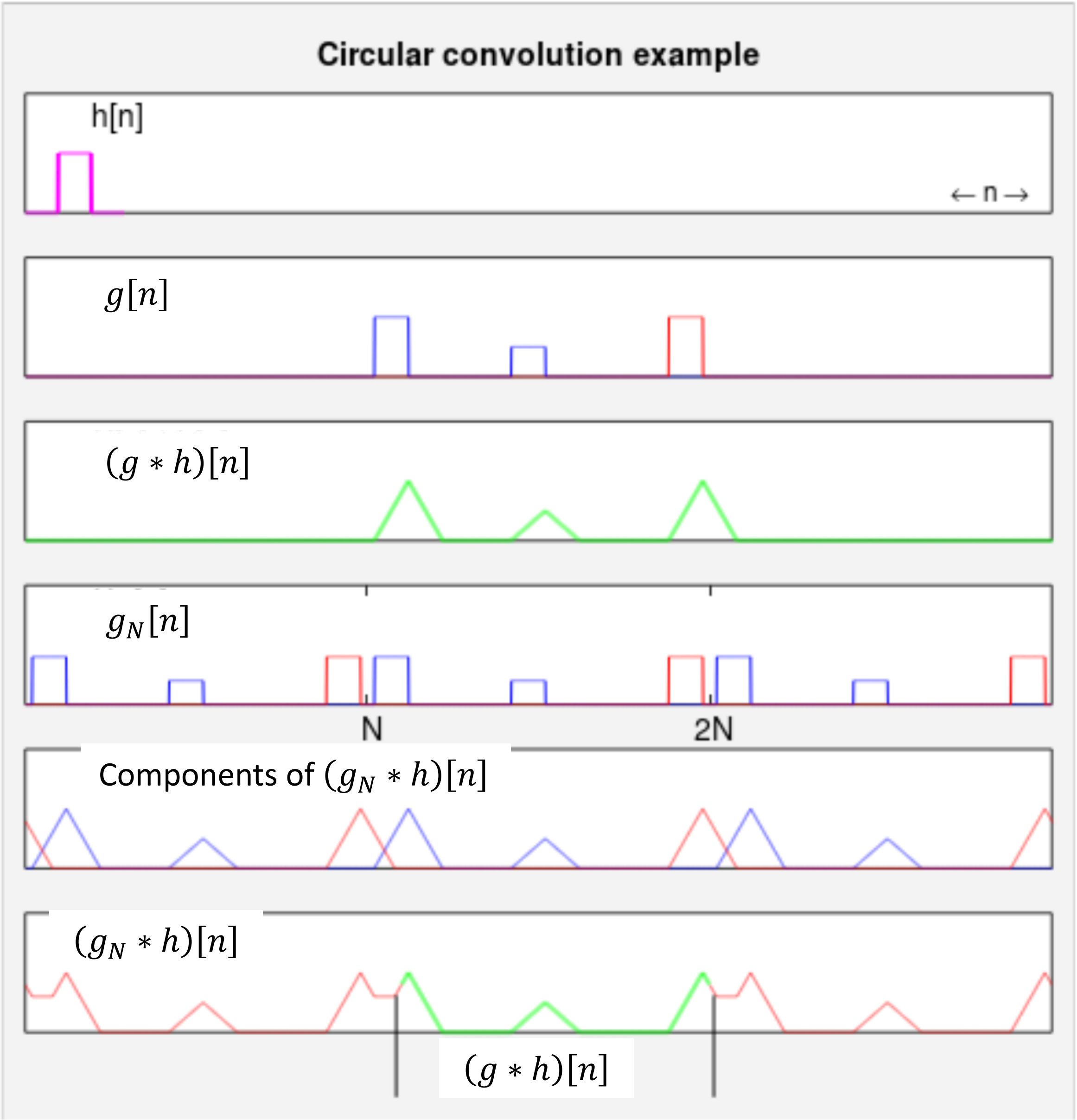
- Therefore, by the theorem of periodic convolution, we have

$$Y[k] = G[k]H[k]$$

where G, H, and Y are the FT of g, h_M, and y, respectively

- The deconvolution can be done as follows:

1. Set y and h_M both with length M
2. Compute Y and H, both with length M, by DFT
3. Compute G, with length M, by $G[k] = \frac{Y[k]}{H[k]}$
4. Compute g, with length M, by inverse DFT



Circular convolution example
h[n]
← n →
$g[n]$
$(g * h)[n]$
$g_N[n]$
N
2N
Components of $(g_N * h)[n]$
$(g_N * h)[n]$
$(g * h)[n]$

Partial Differential Equation (PDE)
(2^{nd} order)

Mathematical Classification:

- Hyperbolic: Involves 2^{nd} derivatives of opposite signs. E.g. wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

- Parabolic: 1^{st} order derivative in 1 variable, 2^{nd} order in others. E.g. diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right)$$

- Elliptic: 2^{nd} derivatives of variables with the same sign. E.g. Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x, y)$$

Numerical Classification

- Initial value problem:
  - Know the solution at an initial “time”.
  - Want to know how the solution “evolves” subject to some B.C. (not on “time”)
  - e.g. Diffusion and wave equation

- Boundary value problem:
  - Know the boundary conditions
  - Want to know the solution at interior points
  - Types of B.C.:

    (1) Periodic: Assume periodic solution

    (2) Dirichlet: Function values specified on a large closed surface

    (3) Neumann: Specified the values of the normal gradients of the function on the boundary

    (4) Mixed

Boundary Value Problem

Principal method: Finite differencing

Model problem:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \rho(x,y)$$

by

- $x_j = x_0 + j\Delta \qquad j = 0,1,\ldots,J$
- $y_l = y_0 + l\Delta \qquad l = 0,1,\ldots,L$
- $\Delta =$ grid spacing
- $\phi_{j,l} = \phi(x_j, y_l)$
- $\rho_{j,l} = \rho(x_j, y_l)$

Finite differencing: Use central difference to approximate 2nd derivatives

$$\frac{\phi_{j+1,l} - 2\phi_{j,l} + \phi_{j-1,l}}{\Delta^2} + \frac{\phi_{j,l+1} - 2\phi_{j,l} + \phi_{j,l-1}}{\Delta^2} = \rho_{j,l}$$

$$\Rightarrow \qquad \phi_{j+1,l} + \phi_{j-1,l} + \phi_{j,l+1} + \phi_{j,l-1} - 4\phi_{j,l} = \rho_{j,l}\Delta^2$$

Turned into a set of Linear Algebraic Equations in ϕ_{ij}.

The equation holds at the points

$$j = 1,2,\ldots,J-1$$

$$l = 1,2,\ldots,L-1$$

B.C. to determine the equations for $j = \{0, J\}$ and $l = \{0, L\}$.

How to incorporate B.C.? Move ϕ at the boundary to the RHS.

- Dirichlet B.C. (ϕ known at boundary):

$$\phi_{2,l} + \phi_{0,l} + \phi_{1,l+1} + \phi_{1,l-1} - 4\phi_{1,l} = \rho_{1,l}\Delta^2$$

$$\Rightarrow \qquad \phi_{2,l} + \phi_{1,l+1} + \phi_{1,l-1} - 4\phi_{1,l} = \rho_{1,l}\Delta^2 - \phi_{0,l}$$

$$\phi_{J,l} + \phi_{J-2,l} + \phi_{J-1,l+1} + \phi_{J-1,l-1} - 4\phi_{J-1,l} = \rho_{J-1,l}\Delta^2$$

$$\Rightarrow \qquad \phi_{J-2,l} + \phi_{J-1,l+1} + \phi_{J-1,l-1} - 4\phi_{J-1,l} = \rho_{J-1,l}\Delta^2 - \phi_{J,l}$$

$$\phi_{j+1,1} + \phi_{j-1,1} + \phi_{j,2} + \phi_{j,0} - 4\phi_{j,1} = \rho_{j,1}\Delta^2$$

$$\Rightarrow \qquad \phi_{j+1,1} + \phi_{j-1,1} + \phi_{j,2} - 4\phi_{j,1} = \rho_{j,1}\Delta^2 - \phi_{j,0}$$

$$\phi_{j+1,L-1} + \phi_{j-1,L-1} + \phi_{j,L} + \phi_{j,L-2} - 4\phi_{j,L-1} = \rho_{j,L-1}\Delta^2$$

$$\Rightarrow \qquad \phi_{j+1,L-1} + \phi_{j-1,L-1} + \phi_{j,L-2} - 4\phi_{j,L-1} = \rho_{j,L-1}\Delta^2 - \phi_{j,L}$$

How to incorporate B.C.? Move ϕ at the boundary to the RHS.

- Neumann B.C. ($\nabla\phi = g$ known at boundary):

Approximated by:

$$\phi_{1,l} - \phi_{0,l} = g_{0,l}\Delta$$

$$\phi_{2,l} + \phi_{0,l} + \phi_{1,l+1} + \phi_{1,l-1} - 4\phi_{1,l} = \rho_{1,l}\Delta^2$$

$$\phi_{2,l} + \phi_{1,l+1} + \phi_{1,l-1} - 3\phi_{1,l} = \rho_{1,l}\Delta^2 + g_{0,l}\Delta$$

$$\phi_{J,l} - \phi_{J-1,l} = g_{J,l}\Delta$$

$$\phi_{J,l} + \phi_{J-2,l} + \phi_{J-1,l+1} + \phi_{J-1,l-1} - 4\phi_{J-1,l} = \rho_{J-1,l}\Delta^2$$

$$\phi_{J-2,l} + \phi_{J-1,l+1} + \phi_{J-1,l-1} - 3\phi_{J-1,l} = \rho_{J-1,l}\Delta^2 - g_{J,l}\Delta$$

$$\phi_{j,1} - \phi_{j,0} = g_{j,0}\Delta$$

$$\phi_{j+1,1} + \phi_{j-1,1} + \phi_{j,2} + \phi_{j,0} - 4\phi_{j,1} = \rho_{j,1}\Delta^2$$

$$\phi_{j+1,1} + \phi_{j-1,1} + \phi_{j,2} - 3\phi_{j,1} = \rho_{j,1}\Delta^2 + g_{j,0}\Delta$$

$$\phi_{j,L} - \phi_{j,L-1} = g_{j,L}\Delta$$

$$\phi_{j+1,L-1} + \phi_{j-1,L-1} + \phi_{j,L} + \phi_{j,L-2} - 4\phi_{j,L-1} = \rho_{j,L-1}\Delta^2$$

$$\phi_{j+1,L-1} + \phi_{j-1,L-1} + \phi_{j,L-2} - 3\phi_{j,L-1} = \rho_{j,L-1}\Delta^2 - g_{j,L}\Delta$$

Define the 1D array $\phi_{j,l} \to$ 1D vector ϕ_i

$$\phi_{j,l} \quad \Rightarrow \qquad \phi_i$$

$$\begin{cases} j = 1,2,\ldots J-1 \\ l = 1,2,\ldots,L-1 \end{cases} \quad \Rightarrow \quad i = j(L-1) + l$$

Total number of elements: $(J-1)(L-1)$

The FDEs can be written in matrix form $M\phi = S$.

What is the form of M?

Take a simple case, a rectangular grid in 2D with Dirichlet B.C. M is tridiagonal with fringes.

$$\begin{pmatrix}
-4 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \\
1 & -4 & 1 & \cdots & \cdots & 1 & & \\
\vdots & 1 & -4 & 1 & \cdots & \cdots & 1 & \\
\vdots & \vdots & 1 & -4 & 1 & & & \ddots \\
1 & \vdots & \vdots & 1 & -4 & \ddots & & \\
\vdots & 1 & \vdots & & 1 & \ddots & & \\
\vdots & & 1 & & & \ddots & & \\
 & & & \ddots & & & & \\
 & & & & & & &
\end{pmatrix}$$

Methods for solving $M\phi = S$:

1. Spectral method
2. Relaxation
   - Write M in 2 parts

$$M = N - L$$

     where N can be inverted easily.
   - Then $N\phi = L\phi + S$
   - Choose initial guess $\phi^{(0)}$, then improve it iteratively by

$$N\phi^{(r)} = L\phi^{(r-1)} + S$$

3. Direct method
   - Solve $M\phi = S$ directly.
   - Large matrix, e.g., 100×100 grids $\Rightarrow$ 10^4unknowns, 10000×10000 matrix
   - Must use sparsity of M.

Spectral Method for Boundary Value Problems

Fourier Transform (FT) method:

Recall: FT pairs are defined by

$$\tilde{h}(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} dt \quad \text{and} \quad h(t) = \int_{-\infty}^{\infty} \tilde{h}(f) e^{-2\pi i f t} df$$

For example, consider the ODE

$$\frac{d^2 x(t)}{dt^2} + k^2 x(t) = f(t)$$

By F.T. $x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{x}(\omega) e^{-i\omega t} d\omega$ and $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t} d\omega$

Transforming the ODE into

$$-\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \omega^2 \tilde{x}(\omega) e^{-i\omega t} d\omega + k^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \omega^2 \tilde{x}(\omega) e^{-i\omega t} d\omega = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t} d\omega$$

$$-\omega^2 \tilde{x}(\omega) + k^2 \tilde{x}(\omega) = \tilde{f}(\omega)$$

$$\tilde{x}(\omega) = \frac{\tilde{f}(\omega)}{k^2 - \omega^2}$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{f}(\omega)}{k^2 - \omega^2} e^{-i\omega t} d\omega$$

Discrete Fourier Transform

- Discrete points $\alpha = 0,1,2,\ldots,N-1$
- Similar to basis in linear space:

$$u_k(\alpha) = \frac{1}{\sqrt{N}}\exp(\,2\pi ik\alpha/N)$$

- Orthonormal: $\sum_\alpha u_k(\alpha)u_{k'}^*(\alpha) = \delta_{kk'}$
- Complete: $\sum_k u_k(\alpha)u_k^*(\alpha') = \delta_{\alpha\alpha'}$

Then $\forall f(\alpha)$

$$\begin{aligned} f(\alpha) &= \sum_{\alpha'} \delta_{\alpha\alpha'} f(\alpha') \\ &= \sum_{\alpha'}\sum_k u_k(\alpha)u_k^*(\alpha')f(\alpha') \\ &= \sum_k \left[\sum_{\alpha'} u_k^*(\alpha')f(\alpha')\right] u_k(\alpha) \end{aligned}$$

- $f(\alpha) = \sum_{k=0}^{N-1} \tilde{f}(k)e^{2\pi ik\alpha/N}$
- $\tilde{f}(k) = \frac{1}{N}\sum_{\alpha=0}^{N-1} f(\alpha)e^{-\frac{2\pi ik\alpha}{N}}$

with $\alpha = 0,1,2,\ldots,N-1$ and $k = 0,1,2,\ldots,N-1$

In 2D:

$$\phi_{jl} = \frac{1}{JL}\sum_{m=0}^{J-1}\sum_{n=0}^{L-1}\tilde{\phi}_{mn}e^{-2\pi i jm/J}e^{-2\pi i ln/L}$$

$$\tilde{\phi}_{mn} = \sum_{j=0}^{J-1}\sum_{l=0}^{L-1}\phi_{jl}e^{2\pi i jm/J}e^{2\pi i ln/L}$$

$$\rho_{jl} = \frac{1}{JL}\sum_{m=0}^{J-1}\sum_{n=0}^{L-1}\tilde{\rho}_{mn}e^{-2\pi i jm/J}e^{-2\pi i ln/L}$$

$$\tilde{\rho}_{mn} = \sum_{j=0}^{J-1}\sum_{l=0}^{L-1}\rho_{jl}e^{2\pi i jm/J}e^{2\pi i ln/L}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \rho(x,y)$$

$$\phi_{j+1,l} + \phi_{j-1,l} + \phi_{j,l+1} + \phi_{j,l-1} - 4\phi_{j,l} = \rho_{jl}\Delta^2$$

with periodic boundary conditions:

$$\phi_{j+J,l} = \phi_{j,l} \qquad \phi_{j,l+L} = \phi_{j,l}$$

Note that the periodic boundary condition determines the solution only up to an arbitrary constant. Hence one is free to set arbitrary value for $\tilde{\phi}_{00}$. Different $\tilde{\phi}_{00}$ leads to ϕ_{jl} which differ by a constant only.

$$\phi_{j+1,l} + \phi_{j-1,l} + \phi_{j,l+1} + \phi_{j,l-1} - 4\phi_{j,l} = \rho_{jl}\Delta^2$$

$$\frac{1}{JL}\sum_{m=0}^{J-1}\sum_{n=0}^{L-1}\tilde{\phi}_{mn}e^{-\frac{2\pi i(j+1)m}{J}}e^{-\frac{2\pi i l n}{L}} + \frac{1}{JL}\sum_{m=0}^{J-1}\sum_{n=0}^{L-1}\tilde{\phi}_{mn}e^{-\frac{2\pi i(j-1)m}{J}}e^{-\frac{2\pi i l n}{L}}$$

$$+\frac{1}{JL}\sum_{m=0}^{J-1}\sum_{n=0}^{L-1}\tilde{\phi}_{mn}e^{-\frac{2\pi i j m}{J}}e^{-\frac{2\pi i(l+1)n}{L}} + \frac{1}{JL}\sum_{m=0}^{J-1}\sum_{n=0}^{L-1}\tilde{\phi}_{mn}e^{-\frac{2\pi i j m}{J}}e^{-\frac{2\pi i(l-1)n}{L}} - 4\frac{1}{JL}\sum_{m=0}^{J-1}\sum_{n=0}^{L-1}\tilde{\phi}_{mn}e^{-\frac{2\pi i j m}{J}}e^{-\frac{2\pi i l n}{L}}$$

$$= \Delta^2\frac{1}{JL}\sum_{m=0}^{J-1}\sum_{n=0}^{L-1}\tilde{\rho}_{mn}e^{-\frac{2\pi i j m}{J}}e^{-\frac{2\pi i l n}{L}}$$

$$\Rightarrow \sum_{m=0}^{J-1}\sum_{n=0}^{L-1}\tilde{\phi}_{mn}e^{-\frac{2\pi i j m}{J}}e^{-\frac{2\pi i l n}{L}}(e^{-\frac{2\pi i m}{J}} + e^{\frac{2\pi i m}{J}} + e^{-\frac{2\pi i n}{L}} + e^{\frac{2\pi i n}{L}} - 4) = \Delta^2\sum_{m=0}^{J-1}\sum_{n=0}^{L-1}\tilde{\rho}_{mn}e^{-\frac{2\pi i j m}{J}}e^{-\frac{2\pi i l n}{L}}$$

$$\Rightarrow \left(e^{-\frac{2\pi i m}{J}} + e^{\frac{2\pi i m}{J}} + e^{-\frac{2\pi i n}{L}} + e^{\frac{2\pi i n}{L}} - 4\right)\tilde{\phi}_{mn} = \Delta^2\tilde{\rho}_{mn}$$

$$\Rightarrow 2\left[\cos\left(\frac{2\pi m}{J}\right) + \cos\left(\frac{2\pi n}{L}\right) - 2\right]\tilde{\phi}_{mn} = \Delta^2\tilde{\rho}_{mn}$$

For $m = n = 0$, $\tilde{\phi}_{00}$ is arbitrary.

- However, if $\tilde{\rho}_{00} \neq 0$, then there is no solution. In Poisson problem, if the net source is not zero, there is no solution satisfying the periodic boundary condition.
- Assuming $\tilde{\rho}_{00} = 0$, then the solution is

$$\tilde{\phi}_{mn} = \frac{\tilde{\rho}_{mn}\Delta^2}{2\left(\cos\frac{2\pi m}{J} + \cos\frac{2\pi n}{L} - 2\right)}$$

for $(m, n) \neq (0,0)$, and $\tilde{\phi}_{00}$ is arbitrary.

<u>Procedure</u>:

Step 1) Compute DFT of ρ

$$\tilde{\rho}_{mn} = \sum_{j=0}^{J-1}\sum_{l=0}^{L-1} \rho_{jl} e^{\frac{2\pi i m j}{J}} e^{\frac{2\pi i n l}{L}}$$

Step 2) Compute $\tilde{\phi}_{mn}$ from $\tilde{\rho}_{mn}$ for $(m, n) \neq (0,0)$ and set arbitrary value for $\tilde{\phi}_{00}$

Step 3) Compute ϕ_{ij} using inverse DFT

$$\phi_{jl} = \frac{1}{JL}\sum_{m=0}^{J-1}\sum_{n=0}^{L-1} \tilde{\phi}_{mn} e^{-\frac{2\pi i j m}{J}} e^{-\frac{2\pi i l n}{L}}$$

How to deal with other boundary conditions?

Dirichlet boundary condition

Suppose $\phi = 0$ at the boundaries ($j = 0, J, l = 0, L$), use sine transform

$$\phi_{jl} = \frac{2}{J}\frac{2}{L}\sum_{m=1}^{J-1}\sum_{n=1}^{L-1}\tilde{\phi}_{mn}\sin\frac{\pi jm}{J}\sin\frac{\pi ln}{L}$$

$$\tilde{\phi}_{mn} = \sum_{j=1}^{J-1}\sum_{l=1}^{L-1}\phi_{jl}\sin\frac{\pi jm}{J}\sin\frac{\pi ln}{L}$$

$$\rho_{jl} = \frac{2}{J}\frac{2}{L}\sum_{m=1}^{J-1}\sum_{n=1}^{L-1}\tilde{\rho}_{mn}\sin\frac{\pi jm}{J}\sin\frac{\pi ln}{L}$$

$$\tilde{\rho}_{mn} = \sum_{j=1}^{J-1}\sum_{l=1}^{L-1}\rho_{jl}\sin\frac{\pi jm}{J}\sin\frac{\pi ln}{L}$$

Proof of DST:

I shall prove 1D. Generalization to 2D is left as exercise.

- Here we have ϕ_j with $j = 0, 1, \cdots, J$, and $\phi_0 = \phi_J = 0$
- Odd extension: ϕ_j with $j = -1, -2, \ldots, -J+1$ and $\phi_{-j} = -\phi_j$
- Period = $2J$

$$
\begin{aligned}
\tilde{\phi}_k' &= \sum_{j=-J+1}^{J} \phi_j \exp\left(i \frac{2\pi kj}{2J}\right) \\
&= \sum_{j=-J+1}^{-1} \phi_j \exp\left(i \frac{2\pi kj}{2J}\right) + \sum_{j=1}^{J-1} \phi_j \exp\left(i \frac{2\pi kj}{2J}\right) \\
&= \sum_{j=1}^{J-1} \phi_j \left[\exp\left(i \frac{2\pi kj}{2J}\right) - \exp\left(-i \frac{2\pi kj}{2J}\right)\right] \\
&= 2i \sum_{j=1}^{J-1} \phi_j \sin\left(\frac{\pi kj}{J}\right)
\end{aligned}
$$

Define $\tilde{\phi}_k = -\frac{i}{2} \tilde{\phi}_k{}'$

$$
\tilde{\phi}_k = \sum_{j=1}^{J-1} \phi_j \sin\left(\frac{\pi kj}{J}\right)
$$

It is readily observed that $\tilde{\phi}_k$ has period $2J$.

- $\tilde{\phi}_{-k} = -\tilde{\phi}_k$
- $\tilde{\phi}_0 = \tilde{\phi}_J = 0$

The inverse transform is

$$\begin{aligned}
\phi_j &= \frac{1}{2J} \sum_{k=-J+1}^{J} \tilde{\phi}'_k \exp\left(-i\frac{2\pi kj}{2J}\right) \\
&= \frac{1}{2J} 2i \sum_{k=-J+1}^{J} \tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right) \\
&= \frac{i}{J}\left[\sum_{k=-J+1}^{-1} \tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right) + \sum_{k=1}^{J-1} \tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right)\right] \\
&= \frac{i}{J}\left[-\sum_{k=1}^{J-1} \tilde{\phi}_k \exp\left(i\frac{2\pi kj}{2J}\right) + \sum_{k=1}^{J-1} \tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right)\right] \\
&= \frac{i}{J}(-2i) \sum_{k=1}^{J-1} \tilde{\phi}_k \sin\left(\frac{\pi kj}{J}\right) \\
&= \frac{2}{J} \sum_{k=1}^{J-1} \tilde{\phi}_k \sin\left(\frac{\pi kj}{J}\right)
\end{aligned}$$

Procedure:

Step 1) Compute DST of ρ

$$\tilde{\rho}_{mn} = \sum_{j=1}^{J-1} \sum_{l=1}^{L-1} \rho_{jl} \sin\frac{\pi jm}{J} \sin\frac{\pi ln}{L}$$

Step 2) Compute $\tilde{\phi}_{mn}$ from $\tilde{\rho}_{mn}$ for $(m, n) \neq (0,0)$

$$\tilde{\phi}_{mn} = \frac{\tilde{\rho}_{mn}\Delta^2}{2\left(\cos\frac{\pi m}{J} + \cos\frac{\pi n}{L} - 2\right)}$$

Step 3) Compute ϕ_{ij} using inverse DST

$$\phi_{jl} = \frac{2}{J}\frac{2}{L} \sum_{m=1}^{J-1} \sum_{n=1}^{L-1} \tilde{\phi}_{mn} \sin\frac{\pi jm}{J} \sin\frac{\pi ln}{L}$$

Note:

(1) In Step 2), the angles in cosine are halved because the periods are doubled.

(2) Because $\tilde{\rho}_{00} = 0$, and so solution always exists.

For inhomogeneous B.C.:

- Suppose $\phi = 0$ on all boundaries except $\phi = f(y)$ on $x = J\Delta$
- Write the solution as

$$\phi = \phi^I + \phi^B$$

where $\phi^I = 0$ on all boundaries and $\phi^B = 0$ everywhere except on the boundaries. i.e.

$$\phi^B_{jl} = \begin{cases} f_l & \text{if } j = J \\ 0 & \text{otherwise} \end{cases}$$

$$\nabla^2 \phi = \rho$$
$$\nabla^2 \phi^I = -\nabla^2 \phi^B + \rho$$

Finite differencing

$$\phi^I_{j+1,l} + \phi^I_{j-1,l} + \phi^I_{j,l+1} + \phi^I_{j,l-1} - 4\phi^I_{jl}$$
$$= -\left(\phi^B_{j+1,l} + \phi^B_{j-1,l} + \phi^B_{j,l+1} + \phi^B_{j,l-1} - 4\phi^B_{jl}\right) + \rho_{jl}\Delta^2 = \begin{cases} -f_l + \rho_{J-1,l}\Delta^2 & \text{if } \ j = J-1 \\ \rho_{jl}\Delta^2 & \text{otherwise} \end{cases}$$

$\Rightarrow$ Reduced to the original problem, with the source modified by $\rho_{J-1,l} \to \rho_{J-1,l} - f_l/\Delta^2$

Neumann boundary condition

Suppose $\nabla\phi = 0$ on boundaries, use cosine transform

$$\phi_{jl} = \frac{2}{J}\frac{2}{L}\sum_{m=0}^{J}{}''\sum_{n=0}^{L}{}''\tilde{\phi}_{mn}\cos\frac{\pi jm}{J}\cos\frac{\pi ln}{L}$$

$$\tilde{\phi}_{mn} = \sum_{j=0}^{J}{}''\sum_{l=0}^{L}{}''\phi_{jl}\cos\frac{\pi jm}{J}\cos\frac{\pi ln}{L}$$

$$\rho_{jl} = \frac{2}{J}\frac{2}{L}\sum_{m=0}^{J}{}''\sum_{n=0}^{L}{}''\tilde{\rho}_{mn}\cos\frac{\pi jm}{J}\cos\frac{\pi ln}{L}$$

$$\tilde{\rho}_{mn} = \sum_{j=0}^{J}{}''\sum_{l=0}^{L}{}''\rho_{jl}\cos\frac{\pi jm}{J}\cos\frac{\pi ln}{L}$$

$\sum''$: $\begin{matrix} j, m = 0 \text{ or } J \\ l, n = 0 \text{ or } L \end{matrix}$ $\Rightarrow$ multiplied by 1/2. c.f. $\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \neq 0 \\ 1 & m = n = 0 \end{cases}$

Again, the boundary condition $\nabla\phi = 0$ only determines the solution up to an arbitrary constant. One is free to choose arbitrary values for $\tilde{\phi}_{00}$.

Proof of DCT:

I shall prove 1D. Generalization to 2D is left as exercise.

- Here we have ϕ_j with j=0,1,…J
- Even extension: ϕ_j with $j = -1, -2, \dots, -J+1$ and $\phi_{-j} = \phi_j$
- Period = 2J

$$
\begin{aligned}
\tilde{\phi}'_k &= \sum_{j=-J+1}^{J} \phi_j \exp\left(i\frac{2\pi kj}{2J}\right) \\
&= \sum_{j=-J+1}^{-1} \phi_j \exp\left(i\frac{2\pi kj}{2J}\right) + \phi_0 + \sum_{j=1}^{J-1} \phi_j \exp\left(i\frac{2\pi kj}{2J}\right) + (-1)^k \phi_J \\
&= \phi_0 + (-1)^k \phi_J + \sum_{j=1}^{J-1} \phi_j \left[\exp\left(i\frac{2\pi kj}{2J}\right) + \exp\left(-i\frac{2\pi kj}{2J}\right)\right] \\
&= 2\left[\frac{1}{2}\phi_0 + \frac{(-1)^k}{2}\phi_J + \sum_{j=1}^{J-1} \phi_j \cos\left(\frac{\pi kj}{J}\right)\right] \\
&= 2\sum_{j=0}^{J} {}''\phi_J \cos\left(\frac{\pi kj}{J}\right)
\end{aligned}
$$

Define $\tilde{\phi}_k = \frac{1}{2}\tilde{\phi}_k{}'$

$$\tilde{\phi}_k = \sum_{j=1}^{J-1}{}''\phi_j \cos\left(\frac{\pi kj}{J}\right)$$

It is readily observed that $\tilde{\phi}_k$ has period $2J$ and $\tilde{\phi}_{-k} = \tilde{\phi}_k$.

The inverse transform is

$$\phi_j = \frac{1}{2J}\sum_{k=-J+1}^{J} \tilde{\phi}_k' \exp\left(-i\frac{2\pi kj}{2J}\right)$$

$$= \frac{1}{2J}2\sum_{k=-J+1}^{J} \tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right)$$

$$= \frac{1}{J}\left[\sum_{k=-J+1}^{-1} \tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right) + \tilde{\phi}_0 + \sum_{k=1}^{J-1} \tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right) + (-1)^k\tilde{\phi}_J\right]$$

$$= \frac{1}{J}\left[\tilde{\phi}_0 + (-1)^k\tilde{\phi}_J + \sum_{k=1}^{J-1} \tilde{\phi}_k \exp\left(i\frac{2\pi kj}{2J}\right) + \sum_{k=1}^{J-1} \tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right)\right]$$

$$= \frac{2}{J}\left[\frac{1}{2}\tilde{\phi}_0 + \frac{(-1)^k}{2}\tilde{\phi}_J + \sum_{k=1}^{J-1} \tilde{\phi}_k \cos\left(\frac{\pi kj}{J}\right)\right]$$

$$= \frac{2}{J}\sum_{k=1}^{J-1}{}''\tilde{\phi}_k \cos\left(\frac{\pi kj}{J}\right)$$

Procedure:

Step 1) Compute DCT of ρ

$$\tilde{\rho}_{mn} = \sum_{j=0}^{J}{}'' \sum_{l=0}^{L}{}'' \rho_{jl} \cos\frac{\pi jm}{J} \cos\frac{\pi ln}{L}$$

Step 2) Compute $\tilde{\phi}_{mn}$ from $\tilde{\rho}_{mn}$ for $(m,n) \neq (0,0)$

$$\tilde{\phi}_{mn} = \frac{\tilde{\rho}_{mn}\Delta^2}{2\left(\cos\frac{\pi m}{J} + \cos\frac{\pi n}{L} - 2\right)}$$

And set $\tilde{\phi}_{00} =$ Arbitrary number

Step 3) Compute ϕ_{ij} using inverse DCT

$$\phi_{jl} = \frac{2}{J}\frac{2}{L} \sum_{m=0}^{J}{}'' \sum_{n=0}^{L}{}'' \tilde{\phi}_{mn} \cos\frac{\pi jm}{J} \cos\frac{\pi ln}{L}$$

Note:

(1) In Step 2), the angles in cosine are halved because the periods are doubled.

(2) Must have $\tilde{\rho}_{00} = 0$ for solution to exist.

For inhomogeneous B.C.:

- Suppose $\nabla\phi = g(y)$ at $x = 0$
- B.C. $\frac{\phi_{1,l}-\phi_{-1,l}}{2\Delta} = g_l$
- Write the solution as

$$\phi = \phi^I + \phi^B$$

where $\nabla\phi^I = 0$ on all boundaries and $\phi^B = 0$ everywhere except just outside the boundaries

$$\nabla\phi = g(y) = \nabla\phi^I + \nabla\phi^B = \nabla\phi^B$$
$$\Rightarrow \phi^B_{-1,l} = -2g_l\Delta$$

Finite differencing:

$$\phi^I_{j+1,l} + \phi^I_{j-1,l} + \phi^I_{j,l+1} + \phi^I_{j,l-1} - 4\phi^I_{jl} = \begin{cases} 2g_l\Delta + \rho_{0,l}\Delta^2 & j = 0 \\ \rho_{j,l}\Delta^2 & \text{otherwise} \end{cases}$$