12. Gambling and rationality

Contents

(All variables are real and one-dimensional unless otherwise specified.)

Investing in a market is, after all, a sophisticated form of **gambling**. Humans' desire in earning effortlessly has led to the prosperity of **probability theory** in mathematics and **utility theory** in economics. While economics heavily relies on mathematics, mathematics does not always return comprehensible results to economics.

Trivia. "Gamble" is etymologically related to "game". They indeed share similar meanings. In this tutorial's context,

- a game involves two or more players, each of whom have a **free will** to use **strategies**, whereas
- a gamble is a unilateral process against a **system** that follows some **definite rules**. In finance, we call this system a "market".

1. St. Petersburg paradox

Let us first review the gamble in Section 4.1 of Tutorial 11. You can roll a fair dice for some times, and you win $\$X$ if you obtain X dots in the last roll. If you can roll the dice only once, you expect to earn

$$\$\frac{1}{6}(1+2+3+4+5+6) = \$3.5\,.$$

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This is the gamble's expected payoff. We may thus let it be the gamble's **value**, i.e. fair entrance fee. If the gamble requires an entrance fee—say, collected by a casino—a rational gambler will at most pay $\$3.5$ to play this gamble.

In this simple example, we equate a gamble's value with its expected payoff. This is alright when the expected outcome is **finite**. On the contrary, this raises nonsense if the expected payoff is

infinite. This is known as the **St. Petersburg paradox**.

1.1 St. Petersburg gamble

A Swiss mathematician Nicholas Bernoulli invented this gamble in 1713. However, the gamble got its name due to Nicholas's uncle, Daniel Bernoulli, another mathematician who lived in St. Petersburg and elaborated on the gamble in detail. (The Bernoulli family was a noble family with many prominent scholars.)

You can toss a fair coin as many times as possible until you get a tail, then you win $\$2^k$, where k is the total number of tosses. For example, if you get a tail right at the beginning, the gamble immediately stops, and you earn $\$2^1 = \$2$; if you get a head and then a tail, you earn $\$2^2 = \$4$. In general, you earn

$$\$\Pi\left(\underbrace{\mathrm{H}\ldots\mathrm{H}}_{k-1}\mathrm{T}\right) = \$2^k$$

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if your get a tail after $k-1$ consecutive heads.

What is the gamble's value? That is, how much should a rational person pay to try this gamble? Let us examine its expected payoff $\$E$.

$$\begin{aligned} E &= \Pi(\mathrm{T})\mathrm{P}(\mathrm{T}) + \Pi(\mathrm{HT})\mathrm{P}(\mathrm{HT}) + \Pi(\mathrm{HHT})\mathrm{P}(\mathrm{HHT}) + \cdots \\ &= 2^1 \times \frac{1}{2} + 2^2 \times \frac{1}{2^2} + 2^3 \times \frac{1}{2^3} + \cdots \\ &= 1 + 1 + 1 + \cdots \\ &= \infty \end{aligned}$$

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This calculation is sloppy, but rigorous mathematics does show that $E = \sum_{k=1}^{\infty} 2^k \left(\frac{1}{2}\right)^k$ diverges. Anyhow, the gamble has an infinite expected payoff, so a rational gambler should sacrifice an infinite amount of money for the gamble. Would you ever be this "rational"?

Expectation vs realization. Despite the infinite expected payoff, the number of tosses is certainly finite as the gamble terminates, so is the actual payoff that a gambler wins. This extreme case highlights the importance to remember that **an expected value may not be possibly realized**, just like you never get 3.5 dots from a fair dice. A French scientist Georges Buffon once carried out this experiment with his money: after 2084 gambles, he paid $10057 to the participants, i.e. around $5 per gamble.

1.2 The classical resolution

The St. Petersburg paradox is haunting for economists, mathematicians, and philosophers. They try hard to answer

- why the evaluation becomes absurd at an infinite expected payoff and
- what the gamble's value should be.

Daniel Bernoulli proposed his answer from an economic perspective in 1738, which is now regarded as the classical resolution. He argued that money has a **diminishing marginal utility**. The principle essentially says that a fixed amount of money means much more to the poor than to the rich. Therefore, he modelled the utility U of money $\$m$ with logarithm

$$U(\$m) = \log m$$

so that as the amount of money rises, its utility also rises but at a slower and slower pace.

Suppose the gamble requires an entrance fee $\$f$. If a person has wealth $\$w$ before the gamble, he has $\$\left(w + 2^k - f\right)$ after the gamble. Therefore, his wealth's utility on average changes by

$$\mathrm{E}(\Delta U) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left[\log\left(w + 2^k - f\right) - \log w\right] .$$

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Mathematics guarantees the sum's convergence. Bernoulli argued that a rational gambler should determine the gamble's value based on his wealth such that $\mathrm{E}(\Delta U) = 0$. For example, consider natural logarithm.

- A millionaire with $w = 10^6$ should pay at most $f \approx 20.9$ dollars.
- A "thousandaire" with $w = 10^3$ should pay at most $f \approx 11.0$ dollars.
- A "dollaraire" with $w = 10^0 = 1$ should pay at most $f \approx 2.82$ dollars. Because $f > w$, Bernoulli would suggest him borrowing the difference from others.

1.3 Other than Bernoulli

Some scholars have instead tried to attack the paradox from non-mathematical perspectives. There are two major approaches.

- **Dishonesty.** The casino of St. Petersburg could not in advance possess an infinite amount of money as the expected payoff promises, so a rational gambler should not consider this dishonest casino's gamble at all.
- **Time limit.** It takes infinite time to toss a coin for infinitely many times, so the gamble is unrealistic and ill-defined in the first place.

Buffon's resolution. Before drowning in the philosophical abyss, let us examine Buffon's (the rich scientist) interesting resolution. Buffon argued that for some sufficiently large n,

$\mathrm{P}\left(\underbrace{\mathrm{H}\ldots\mathrm{HT}}_{n}\right) = \frac{1}{2^{n+1}}$ $\mathrm{P}\left(\underbrace{\mathrm{H}\ldots\mathrm{HT}}_{n}\right) = \frac{1}{2^{n+1}}$ becomes so small that a normal person naturally ignores, yielding

$$E = \sum_{k=1}^{n} 2^k \left(\frac{1}{2}\right)^k = n\,.$$

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How do we determine this n, though? Buffon suggested a somewhat crazy idea: a 56-year-old man normally does not believe that he is going to die within the next 24 hours, while Buffon's data showed the frequency was in fact $1/10189$, so any probability smaller than the value should be regarded zero. Hence, $n = 13$.

1.4 Russian gambles

Bernoulli's resolution is not perfect. One can easily refute his use of utility function by changing the unit of prize from dollars to "units of utility": instead of winning $\$2^k$, a gambler now wins 2^k units of utility (whatever this actually means). This setting forces us to stick to the original divergent expected payoff:

$$E = \sum_{k=1}^{\infty} 2^k \left(\frac{1}{2}\right)^k$$

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units of utility. The situation gets even more confusing when the St. Petersburg gamble is compared with its variants, which I collectively call "Russian gambles".

- **Petrograd.** In the Petrograd gamble, you win $2^k + 1$ units of utility. Its expected payoff is

$$E_{\text{Petr}} = \sum_{k=1}^{\infty} \left(2^k + 1\right) \left(\frac{1}{2}\right)^k .$$

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- **Leningrad.** In the ith-order Leningrad gamble, you first win 2^k units of utility; if $k \geq i$, you can play an extra St. Petersburg gamble. Its expected payoff is

$$E_{\text{Lenin},i} = E + E \times \mathrm{P}(k \geq i) = \left(1 + \frac{1}{2^{i-1}}\right) E .$$

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- **Moscow.** The Moscow gamble follows the same rules as the St. Petersburg gamble, but it uses an biased coin with probability of head $\mathrm{P(H)} = 0.6$ instead. Its expected payoff is

$$E_{\text{Mos}} = \sum_{k=1}^{\infty} 2^k \times 0.6^{k-1} \times 0.4 = \frac{2}{3} \sum_{k=1}^{\infty} 1.2^k .$$

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These Russian gambles are all intuitively more valuable than the St. Petersburg gamble. However, their expected payoffs all diverge, and it is mathematically ill to compare divergences. How can we refine the formulation so that the mathematics matches the intuition? Which of these gambles is the most valuable then?

2. Kelly strategy

Now let us leave the philosophical matters aside and turn to a more practical discussion of gambling. An American physicist John Kelly, a colleague of Shannon, derived the optimal strategy of gambling with communication theory. This is the so-called **Kelly strategy**. (It is said that Shannon later tried the Kelly strategy in Las Vegas and won a lot.)

We will focus on its simplest application, i.e. on **binary** gambles in which a gambler bet on one of two outcomes.

2.1 Binary gambles

For example, you have wealth V_0 before the gamble, then you are given a biased coin with a probability of head $p \geq 0.5$. You can bet $V_0 f$ for some fraction f on which face you are going to get.

- If you are correct, you first get back $V_0 f$ and then win $V_0 fb$, where b is called the **net odds** in gambling.
- If you are wrong, you simply lose $V_0 f$.

As the coin favours head, you should always bet on a head so that your probability of winning becomes p. Still, how much should you bet? The Kelly strategy says that the optimal fraction is

$$f^* = p - \frac{1-p}{b}\,.$$

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For example, if the coin is in fact fair, $p = 0.5$ reduces the formula to $f^* = \frac{b-1}{2b}$. Then, $b = 2$ gives $f^* = 25\%$, whereas $b = 5$ gives $f^* = 40\%$. If $b = 1$, which gambling jargon calls "even odds", the gamble is worthless for a rational gambler as $f^* = 0$.

In a long run, a gambler who adjusts his amount of betting according to the Kelly strategy will have the greatest amount of money among all who bet on the same outcomes.

2.2 Derivation

Utility theory. Your wealth becomes $V_0\,(1 + fb)$ upon winning or $V_0\,(1 - f)$ upon losing. Therefore, following Bernoulli's philosophy on economics, the expected **log-utility** of your wealth after the gamble is

$$E = p \log\left[V_0\,(1 + fb)\right] + (1-p)\log\left[V_0\,(1-f)\right].$$

$$E = p \log\left[V_0(1 + fb)\right] + (1-p)\log\left[V_0(1-f)\right].$$

The optimal fraction f^* maximizes E and thus solves

$$\frac{\mathrm{d}E}{\mathrm{d}f} \sim \frac{pb}{1+fb} - \frac{1-p}{1-f} = 0\,.$$

$$\frac{\mathrm{d}E}{\mathrm{d}f} \sim \frac{pb}{1+fb} - \frac{1-p}{1-f} = 0\,.$$

(The proportionality depends on the base of the logarithm.) A rearrangement gives $f^* = p - \frac{1-p}{b}$.

Log-return. Although the utility-based proof is easy, Kelly **avoided** utility function: why should one attach a logarithm to money? He in fact derived his formula with another approach, but it turns out to be consistent with utility theory.

Suppose you have played the gamble for N times, in which you win W times and lose $L = N - W$ times. Your wealth becomes

$$V_N = (1+fb)^W (1-f)^L V_0$$

dollars. Kelly considered its **logarithmic rate of return** or simply **log-return** G.

$$G = \lim_{N\to\infty} \frac{1}{N} \log \frac{V_N}{V_0} = \lim_{N\to\infty} \left[\frac{W}{N} \log(1+fb) + \frac{L}{N} \log(1-f) \right]$$

(Kelly in fact used base-two logarithm, so the unit of G is a bit.) As $N \to \infty$, the frequencies of winning and losing converge to their probability, meaning that $W/N = p$ and $L/N = 1 - p$. Therefore, the log-return reduces to

$$G = p \log(1+fb) + (1-p) \log(1-f),$$

which is, again, maximized upon the optimal fraction $f^* = p - \frac{1-p}{b}$.

2.3 Gambles as channels

More profoundly (in my opinion), Kelly identified the general relationship between gambling and Shannon's **communication theory**. He brilliantly modelled gambles with **channels**.

Now consider another gamble with a "digital coin", which shows either 0 or 1. The net odds are b, and you bet a fraction f of your wealth per gamble. You win if you bet on the correct outcome, otherwise you lose. Imagine that a prophet knows the coin's outcome X. He sends you his prophecy through a binary channel, then you receive Y at the other end.

$$\begin{array}{cccc} X=0 & \rightarrow & \rightarrow & Y=0 \\ & \searrow & \nearrow & \\ & \nearrow & \searrow & \\ X=1 & \rightarrow & \rightarrow & Y=1 \end{array}$$

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If the channel were so reliable that $Y \equiv X$, you would bet all of your wealth on Y per gamble because you are going to win for sure. The channel is generally noisy, though. The transmission may fail and yield $Y = 1 - X$. Before betting, you need to decipher the true source signal X based on the received signal Y. Intuitively, the higher $\mathrm{P}(X = x \mid Y)$ is, the more you should bet on x.

Suppose you have received $Y = y$ from the prophet, then you bet a fraction $f_{0|y}$ of your wealth on $X = 0$ and its complement $f_{1|y} = 1 - f_{0|y}$ on $X = 1$ so that you surely win with one of the two bets. Once the coin is tossed and reveals $X = x$, your wealth becomes $V_1 = f_{x|y}\,(1+b)\,V_0$. Let $a = 1 + b$ be the **odds** (while b is the net odds), meaning that you win α units, including the bet, per unit of bet. As a result, you possess

$$\begin{aligned} V_N &= V_0\left(f_{0\mid 0}a\right)^{N_{0,0}}\left(f_{1\mid 0}a\right)^{N_{1,0}}\left(f_{0\mid 1}a\right)^{N_{0,1}}\left(f_{1\mid 1}a\right)^{N_{1,1}} \\ &= V_0\prod_{x=0}^{1}\prod_{y=0}^{1}\left(f_{x\mid y}a\right)^{N_{x,y}} \end{aligned}$$

after $N = N_{0,0} + N_{0,1} + N_{1,0} + N_{1,1}$ gambles as you have observed $Y = y$ and then correctly bet on $X = x$ for $N_{x,y}$ times. This implies your log-return is

$$\begin{aligned} G &= \lim_{N\to\infty}\frac{1}{N}\sum_{x,y} N_{x,y}\log\left(f_{x\mid y}a\right) \\ &= \sum_{x,y} p_{x,y}\log\left(f_{x\mid y}a\right) \\ &= \sum_{x,y} p_{x,y}\log f_{x\mid y} + \sum_{x,y} p_{x,y}\log a \end{aligned}$$

for $p_{x,y} \equiv \mathrm{P}(X = x \cap Y = y)$. In an even more general setting, a depends on the outcome, so $a \rightarrow a_x$. (For instance, betting on different teams' victory in a football match gives you different amounts of money.)

$$\therefore G = \sum_{x,y} p_{x,y} \log f_{x|y} + \sum_{x,y} p_{x,y} \log a_x$$
$$= \sum_{x,y} p_{x,y} \log f_{x|y} + \sum_{x} p_x \log a_x$$

$$\therefore G = \sum_{x,y} p_{x,y} \log f_{x \mid y} + \sum_{x,y} p_{x,y} \log a_x$$
$$= \sum_{x,y} p_{x,y} \log f_{x \mid y} + \sum_{x} p_x \log a_x$$

You need to maximize G by varying $f_{x|y}$, whereas a_x is out of your control. Subject to the constraint $f_{0|y} + f_{1|y} = 1$, the first term attains its maximum at

$$f^*_{x|y} = \mathrm{P}(X = x \mid Y = y)\,.$$

$$f^*_{x \mid y} = \mathrm{P}(X = x \mid Y = y)\,.$$

An ordinary coin. In Section 2.1's gamble, you should simultaneously bet a fractional wealth

$$f^*_{\mathrm{H}} = \mathrm{P}(Y=0) f^*_{0|0} + \mathrm{P}(Y=1) f^*_{0|1} = \mathrm{P}(X=0) \equiv \mathrm{P}(\mathrm{H}) = p$$

$$f^*_{\mathrm{H}} = \mathrm{P}(Y=0) f^*_{0 \mid 0} + \mathrm{P}(Y=1) f^*_{0 \mid 1} = \mathrm{P}(X=0) \equiv \mathrm{P}(\mathrm{H}) = p$$

on a head and the remaining part $f^*_{\mathrm{T}} = 1 - p \equiv P(\mathrm{T})$ on a tail. However, you have to reinterpret this strategy since you are going to bet only f^* on a head. If the outcome is a head and you have bet on a head, you possess

$$V_{1,\mathrm{H}} = f^*_{\mathrm{H}} a_{\mathrm{H}} V_0 \equiv \left(1 + f^* b\right) V_0\,.$$

This gives $f^*_{\mathrm{H}} a_{\mathrm{H}} \equiv 1 + f^* b$ and thus recovers

$$f^* = \frac{f^*_{\mathrm{H}} a_{\mathrm{H}} - 1}{b} = \frac{p\,(b+1) - 1}{b} = p - \frac{1-p}{b}\,.$$

$$f^* = \frac{f^*_{\mathrm{H}} a_{\mathrm{H}} - 1}{b} = \frac{p(b+1) - 1}{b} = p - \frac{1-p}{b}\,.$$

Mutual information. What does the expression $G = \sum_{x,y} p_{x,y} \log f_{x \mid y} + \sum_x p_x \log a_x$ remind you of? Kelly argued that it resembles **mutual information**. While the first term is maximized at $f^*_{x|y} = \mathrm{P}(X = x \mid Y = y)$, the second term is maximized at $a_x = 1/p_x$ (assuming that we can control a_x).

$$\therefore \max G = -\mathrm{H}(X \mid Y) + \mathrm{H}(X) \equiv \mathrm{I}(X; Y)$$

In this regard, the Kelly strategy best enhances your mutual information with the all-knowing prophet, who is telling you the true outcome.

3. Newcomb's paradox

Economics and finance always assume that everyone is rational when they make decisions. Despite this assumption, we may still be unable to conclude what people **should do** or **will do** in certain situations. After all, we cannot even judge what is a rational action (recall Ellsberg's paradox in Tutorial 11). **Newcomb's paradox** further highlights the fragile nature of rationality.

Newcomb's problem. Alice plays a gamble hosted by Bob. Alice sees two boxes X and Y. Box X is transparent with \$1000 inside, while Box Y is opaque. At the same time, Bob tells Alice that Box Y contains either nothing or \$1 million. Alice is allowed to pick either box Y only or both boxes, then she can win the money inside. Hence, Alice's payoff Π in the four possible scenarios can be summarized as

Π	$\mathrm{Y} = 0$	$\mathrm{Y} = 1000$
Y	0	1000
$\mathrm{X} + \mathrm{Y}$	1	1001

(in units of \$1000).

However, it turns out that the money in box Y depends on Bob's advanced brain-scanning machine, which accurately predicts Alice's will. It does not need to be perfectly accurate—as a matter of fact, it has predicted correctly in 90% of its previous tests. If it predicts that Alice will pick box Y only, box Y contains \$1 million, otherwise box Y contains nothing. As a result, Alice expects the following payoffs.

$$\begin{cases} \mathrm{E}(\mathrm{Y}) & = & 0.1 \times 0 + 0.9 \times 1000 & = & 900 \\ \mathrm{E}(\mathrm{X}+\mathrm{Y}) & = & 0.9 \times 1 + 0.1 \times 1001 & = & 101 \end{cases}$$

The paradox. On one hand, if Alice's rationality is based on the fact that $\Pi(\mathrm{X}+\mathrm{Y} \mid \mathrm{Y} = m) > \Pi(\mathrm{Y} \mid \mathrm{Y} = m)$ regardless of m, she will conclude that it is better to pick both boxes. On the other hand, if Alice's rationality is based on the fact that $\mathrm{E}(\mathrm{Y}) > \mathrm{E}(\mathrm{X}+\mathrm{Y})$, she will conclude that it is better to pick box Y only. Which argument should Alice follow and why? Psychological experiments find that around half of the people follow the first idea and around half follow the second idea, and both groups think the other group is stupid.

What is your choice?