Appendix 5

Return probability and spectral density

In this appendix, we give a simple derivation of the relation between the eigenvalue distribution $\rho(\lambda)$ (also called the spectral density) of a generic Laplacian operator L and the return probability $p_0(t)$ for the random walk described by the following master equation

$$\partial_t p(i,t|i_0,0) = -\sum_j L_{ij} p(j,t|i_0,0) \tag{A5.1}$$

with the initial condition $p(i,0|i_0,0) = \delta_{ii_0}$. Here the Laplacian can be either of the form (8.9) or of the form studied in Chapter 7 and Appendix 4, the important point being that $\sum_i L_{ij} = 0$ to ensure that the evolution equation (A5.1) preserves the normalization $\sum_i p(i,t|i_0,0) = 1$. The spectral density is defined as

$$\rho(\lambda) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle \tag{A5.2}$$

where the λ_i are the eigenvalues of the Laplacian and where the brackets denote the average over different realizations of the random network on which the random walk takes place.

The Laplace transform of $p(i,t|i_0,0)$ is defined as

$$\tilde{p}_{ii_0}(s) = \int_0^\infty \mathrm{d}t e^{-st} p(i,t|i_0,0). \tag{A5.3}$$

The Laplace transform of $\partial_t p(i,t|i_0,0)$ can be obtained by an integration by parts as

$$\int_0^\infty \mathrm{d}t e^{-st} \partial_t p(i,t|i_0,0) = -p(i,0|i_0,0) + s\tilde{p}_{ii_0}(s) = s\tilde{p}_{ii_0}(s) - \delta_{ii_0} \tag{A5.4}$$

where $\delta_{ii_0} = 1$ is $i = i_0$, and 0 otherwise. This allows us to rewrite Equation (A5.1) as

$$s\tilde{p}_{ii_0}(s) - \delta_{ii_0} = -\sum_j L_{ij} \tilde{p}_{ji_0}(s) \tag{A5.5}$$

or

$$\sum_j \left(s\delta_{ij} + L_{ij}\right) \tilde{p}_{ji_0}(s) = \delta_{ii_0}. \tag{A5.6}$$

This equality means that the matrix $\tilde{\mathbf{p}}(s)$ is therefore the inverse of the matrix $s\mathbf{I}+\mathbf{L}$, where $\mathbf{I}$ is the unit matrix ($I_{ij} = \delta_{ij}$).

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The probability of return is by definition the probability of returning to the initial point i_0, averaged over all initial nodes and over different realizations of the random network

$$p_0(t) = \left\langle \frac{1}{N} \sum_{i_0} p(i_0, t | i_0, 0) \right\rangle. \tag{A5.7}$$

Taking the Laplace transform of this equation leads to

$$\tilde{p}_0(s) = \left\langle \frac{1}{N} \sum_{i_0} \tilde{p}_{i_0 i_0}(s) \right\rangle = \left\langle \frac{1}{N} \mathrm{T}r \tilde{\mathbf{p}}(s) \right\rangle, \tag{A5.8}$$

where Tr denotes the trace operation. The trace of a matrix is moreover equal to the sum of its eigenvalues, and we can use the fact that $\tilde{\mathbf{p}}(s)$ is the inverse of $s\mathbf{I} + \mathbf{L}$, to obtain that its eigenvalues are given by $1/(s + \lambda_i)$, $i = 1, \ldots, N$. We therefore obtain

$$\tilde{p}_0(s) = \left\langle \frac{1}{N} \sum_i \frac{1}{s + \lambda_i} \right\rangle. \tag{A5.9}$$

We can now obtain $p_0(t)$ as the inverse Laplace transform of $\tilde{p}_0(s)$, which is given by the integral in the complex plane

$$p_0(t) = \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}s \mathrm{e}^{ts} \tilde{p}_0(s), \tag{A5.10}$$

where here $\mathrm{i}^2 = -1$ and c is a constant larger than any singularity of $\tilde{p}_0(s)$. This yields

$$\begin{aligned} p_0(t) &= \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}s \mathrm{e}^{ts} \left\langle \frac{1}{N} \sum_j \frac{1}{s + \lambda_j} \right\rangle \\ &= \left\langle \frac{1}{N} \sum_j \mathrm{e}^{-\lambda_j t} \right\rangle, \end{aligned} \tag{A5.11}$$

where we have used the residue theorem in order to obtain the last equality. Using the definition of $\rho(\lambda)$, this can then be rewritten as Equation (8.10)

$$p_0(t) = \int_0^{\infty} \mathrm{d}\lambda \mathrm{e}^{-\lambda t} \rho(\lambda).$$

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