Mathematical Finance I

(Portfolio Management, Mean Variance Analysis and Sharpe Ratio)

MSDM 5058

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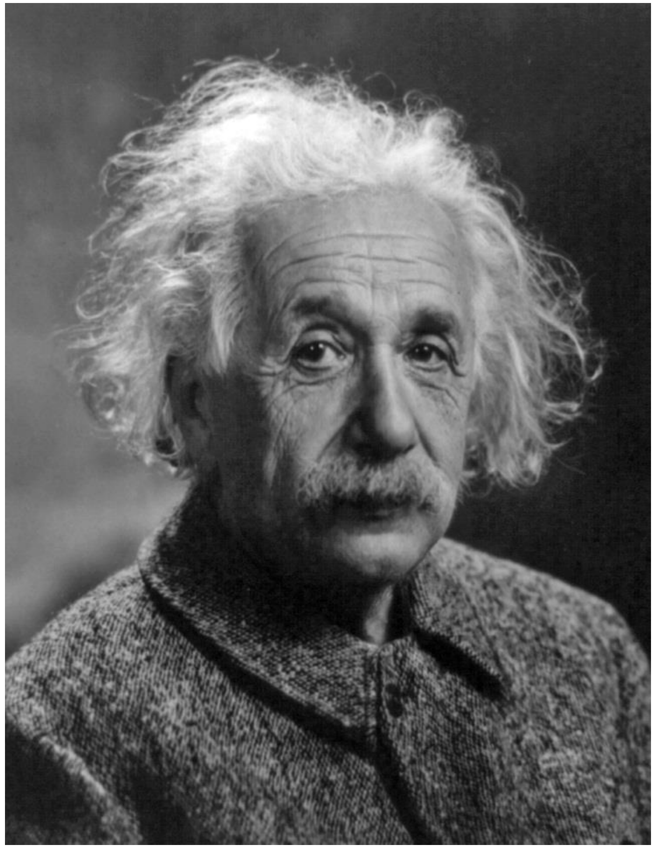
“I can calculate the motions of heavenly bodies, but not the madness of people.” --- A famous quote by Sir Isaac Newton after he lost 20,000 pounds (US$2.72 million in today's money) in the South Sea Bubble in the 1720s.



Einstein's Quotes:

The hardest thing to understand in the world is the income tax.

If I had to live my life over again, I would live it as a trader of goods.

The most powerful force in the universe is compound interest.

Ettore Majorana
1906-1938(?)

"There is an essential analogy between physics and the social sciences, between which an identity of value and method has turned out."

Daniel Bernoulli
(1700-1782)

His book “Specimen theoriae novae de mensura sortis (Exposition of a New Theory on the Measurement of Risk)”, published in 1738 in which the St. Petersburg paradox was studied, was the base of the economic theory of risk aversion, risk premium and utility.

Some Background Assumptions

- As an investor you want to maximize the returns for a given level of risk.
- Your portfolio includes all of your assets and liabilities.
- The relationship between the returns for assets in the portfolio is important.
- A good portfolio is not simply a collection of individually good investments.

Some Background Assumptions

- ***Risk Aversion***

  - Given a choice between two assets with equal rates of return, risk-averse investors will select the asset with the lower level of risk

  - Evidence
    - Many investors purchase insurance for: Life, Automobile, Health, and Disability Income.
    - Yield on bonds increases with risk classifications from AAA to AA to A, etc.

  - Not all Investors are risk averse
    - It may depend on the amount of money involved: Risking small amounts, but insuring large losses

Some Background Assumptions

- ***Definition of Risk***

  - *Uncertainty:* Risk means the uncertainty of future outcomes. For instance, the future value of an investment in Google's stock is uncertain; so the investment is risky. On the other hand, the purchase of a six-month CD (Certificate of Deposit) or bond has a certain future value; the investment is not risky.

  - *Probability:* Risk is measured by the probability of an adverse outcome. For instance, there is 40% chance you will receive a return less than 8%.

Markowitz Portfolio Theory

- ***Main Results***
  - Quantifies risk
  - Derives the expected rate of return for a portfolio of assets and an expected risk measure
  - Shows that the variance of the rate of return is a meaningful measure of portfolio risk
  - Derives the formula for computing the variance of a portfolio, showing how to effectively diversify a portfolio

Markowitz Portfolio Theory

- *Assumptions for Investors*
    1) Consider investments as probability distributions of expected returns over some holding period
    2) Maximize one-period expected utility
    3) Estimate the risk of the portfolio on the basis of the variability of expected returns
    4) Base decisions solely on expected return and risk
    5) Prefer higher returns for a given risk level. Similarly, for a given level of expected returns, investors prefer less risk to more risk

Using the five assumptions above, a single asset or portfolio of assets is considered to be efficient if no other asset or portfolio of assets offers higher expected return with the same (or lower) risk, or lower risk with the same (or higher) expected return.

Alternative Measure of Risk

- Variance or standard deviation of expected return
- Range of returns
- Returns below expectations
  - Semi-variance – a measure that only considers deviations below the mean
  - These measures of risk implicitly assume that investors want to minimize the damage from returns less than some target rate
- The Advantages of Using Standard Deviation of Returns
  - This measure is somewhat intuitive
  - It is a correct and widely recognized risk measure
  - It has been used in most of the theoretical asset pricing models

The Markowitz algorithm

- Markowitz algorithm provides a single period analysis tool in which the inputs provided by the user represent his/her probability beliefs about the upcoming period.

- In principle, the user should identify a number of distinct possible “outcomes” and assign a probability of occurrence for each outcome, and a return for each asset for each outcome.

- The expected return, standard deviation, and correlation matrix may then be computed using standard statistical formulae.

Expected Rates of Return

- For an Individual Asset
  - It is equal to the sum of the potential returns multiplied with the corresponding probability of the return

Example 1:

Probability	Possible Rate of Return (%)	Expected Security Return (%)
0.35	0.08	0.028
0.30	0.10	0.030
0.20	0.12	0.024
0.15	0.14	0.021
		$E[R]$ = 0.103=10.3%

Expected Rates of Return

If you want to construct a portfolio of n risky assets, what will be the expected rate of return on the portfolio if you know the expected rates of return on each individual assets?

– It is equal to the weighted average of the expected rates of return for the individual investments in the portfolio

– The formula

$$E(R_{port}) = \sum_{i=1}^{n} w_i E(R_i)$$

where w_i=percent of the portfolio in asset i; $E(R_i)$=expected rate of return for asset i

– See *Example 2*

Expected Rates of Return

Example 2:

Weight (w_j, % of portfolio)	Expected Security Return (R_j)	Expected Portfolio Return ($w_j \times R_j$)
0.20	0.10	0.020
0.30	0.11	0.033
0.30	0.12	0.036
0.20	0.13	0.026
		$E[R]$ = 0.115=11.5%

Individual Investment Risk Measure

- Variance
  - It is a measure of the variation of possible rates of return R_i, from the expected rate of return $[E(R_i)]$

$$Variance(\sigma^2) = \sum_{i=1}^{n}[R_i - E(R_i)]^2 P_i$$

where P_i is the probability of the possible rate of return R_i

- Standard Deviation (σ)
  – square root of the variance

Individual Investment Risk Measure

Example 3:

Possible Rate of Return (R_i)	Expected Return $E(R_i)$	$R_i - E(R_i)$	$[R_i - E(R_i)]^2$	P_i	$[R_i - E(R_i)]^2 P_i$
0.08	0.103	-0.023	0.005	0.35	0.000185
0.10	0.103	-0.003	0.000	0.30	0.000003
0.12	0.103	0.017	0.003	0.20	0.000058
0.14	0.103	0.037	0.0014	0.15	0.000205
					0.000451

Variance (σ^2) = 0.000451

Standard Deviation (σ) = 0.021237=2.1237%

Covariance of Returns and Correlation Coefficient

- For two assets, *i* and *j*, the covariance of rates of return is defined as:

$$Cov_{ij} = E[(R_i - E(R_i))(R_j - E(R_j))]$$

- Computing correlation coefficient from covariance

$$\rho_{ij} = \frac{Cov_{ij}}{\sigma_i \sigma_j}$$

ρ_{ij} = the correlation coefficient of returns

σ_i = the standard deviation of R_i

σ_j = the standard deviation of R_j

- The coefficient can vary in the range +1 to -1.
- A value of +1 would indicate perfect positive correlation. Returns for the two assets move together in a positively and *completely linear manner*.
- A value of –1 would indicate perfect negative correlation. Returns for two assets move together in a *completely linear manner, but in opposite directions*.

Correlation Coefficient

- Computing correlation coefficient from covariance

$$\rho_{ij} = \frac{Cov_{ij}}{\sigma_i \sigma_j}$$

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Standard Deviation of a Portfolio

The formula

$$\sigma_{port} = \sqrt{\sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov_{ij}}$$

where

σ_{port} = standard deviation of the portfolio

w_i = weights of the individual assets in the portfolio, determined by the proportion of value in the portfolio

σ_i^2 = variance of rates of return for asset i

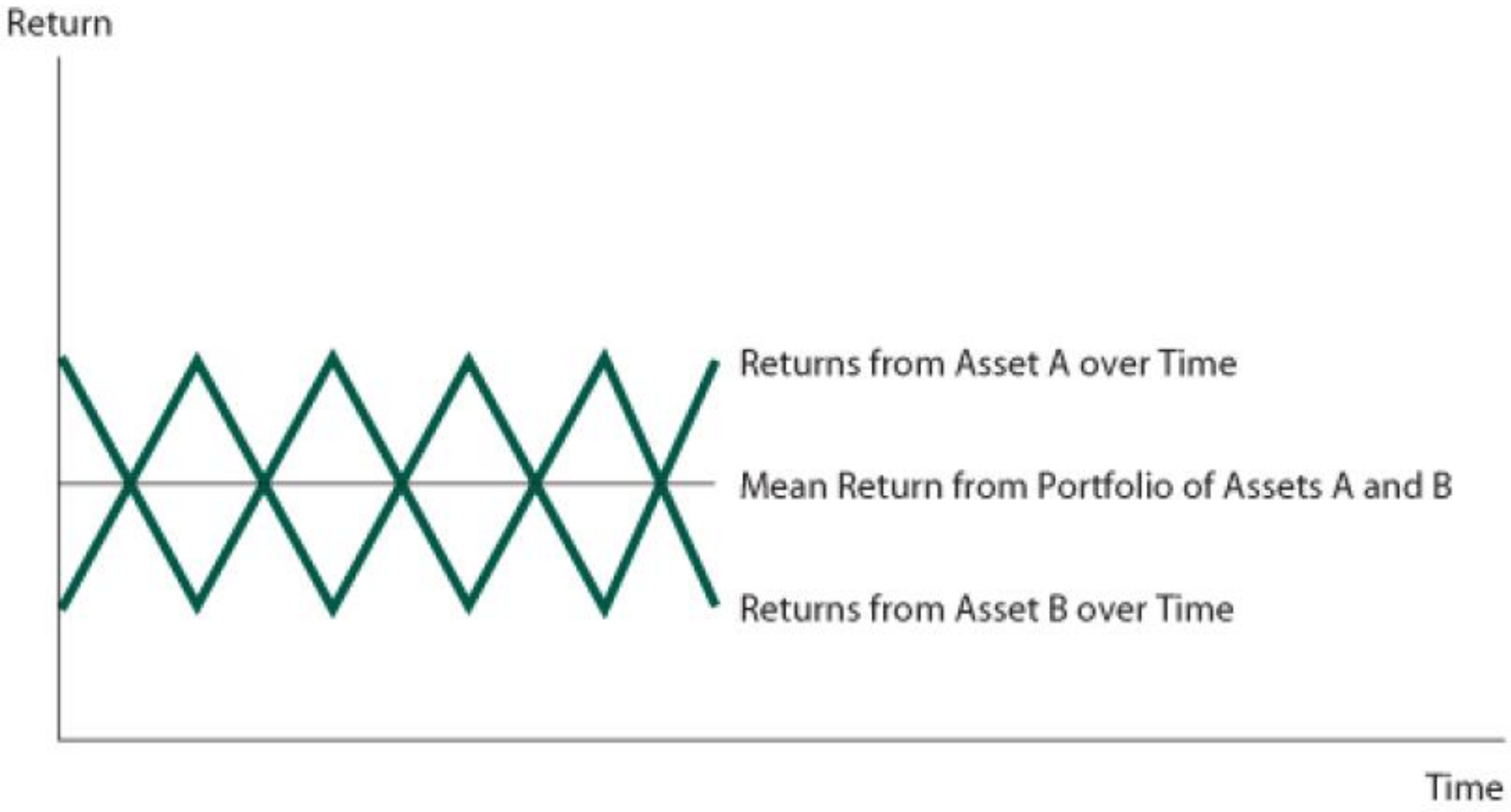
Cov_{ij} = covariance between rates of return for assets i and j, where $Cov_{ij} = \rho_{ij}\sigma_i\sigma_j$

Standard Deviation of a Portfolio

- Computations with a Two-Asset Portfolio
  - Any asset of a portfolio may be described by two characteristics:
    - The expected rate of return
    - The expected standard deviations of returns
  - The correlation, measured by covariance, affects the portfolio standard deviation
  - Low correlation reduces portfolio risk while not affecting the expected return
  - Negative correlation reduces portfolio risk
  - Combining two assets with +1.0 correlation will not reduce the portfolio standard deviation
  - Combining two assets with -1.0 correlation may reduce the portfolio standard deviation to zero

Standard Deviation of a Portfolio

Example 4: *Time Patterns of Returns for Two Assets with Perfect Negative Correlation*



Standard Deviation of a Portfolio

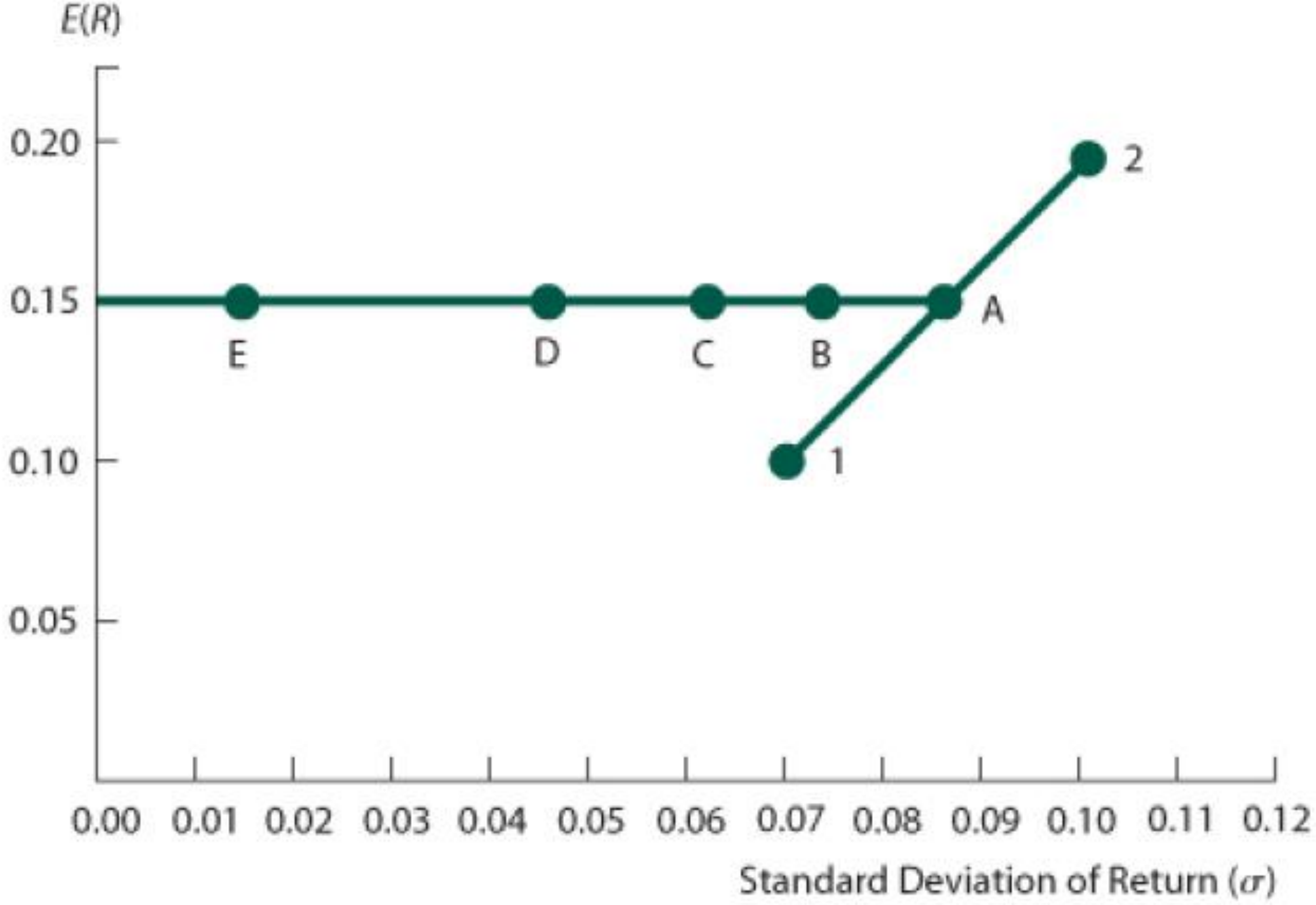
Two Stocks with Different Returns and Risk

Asset	$E(R_i)$	w_i	σ_i^2	σ_i
1	.10	.50	.0049	.07
2	.20	.50	.0100	.10

Case	Correlation Coefficient	Covariance
A	+1.00	.0070
B	+0.50	.0035
C	0.00	.0000
D	-0.50	-.0035
E	-1.00	-.0070

Standard Deviation of a Portfolio

Example 5: *Risk-Return for Portfolios with Different Returns, Standard Deviations, and Correlations*



Standard Deviation of a Portfolio

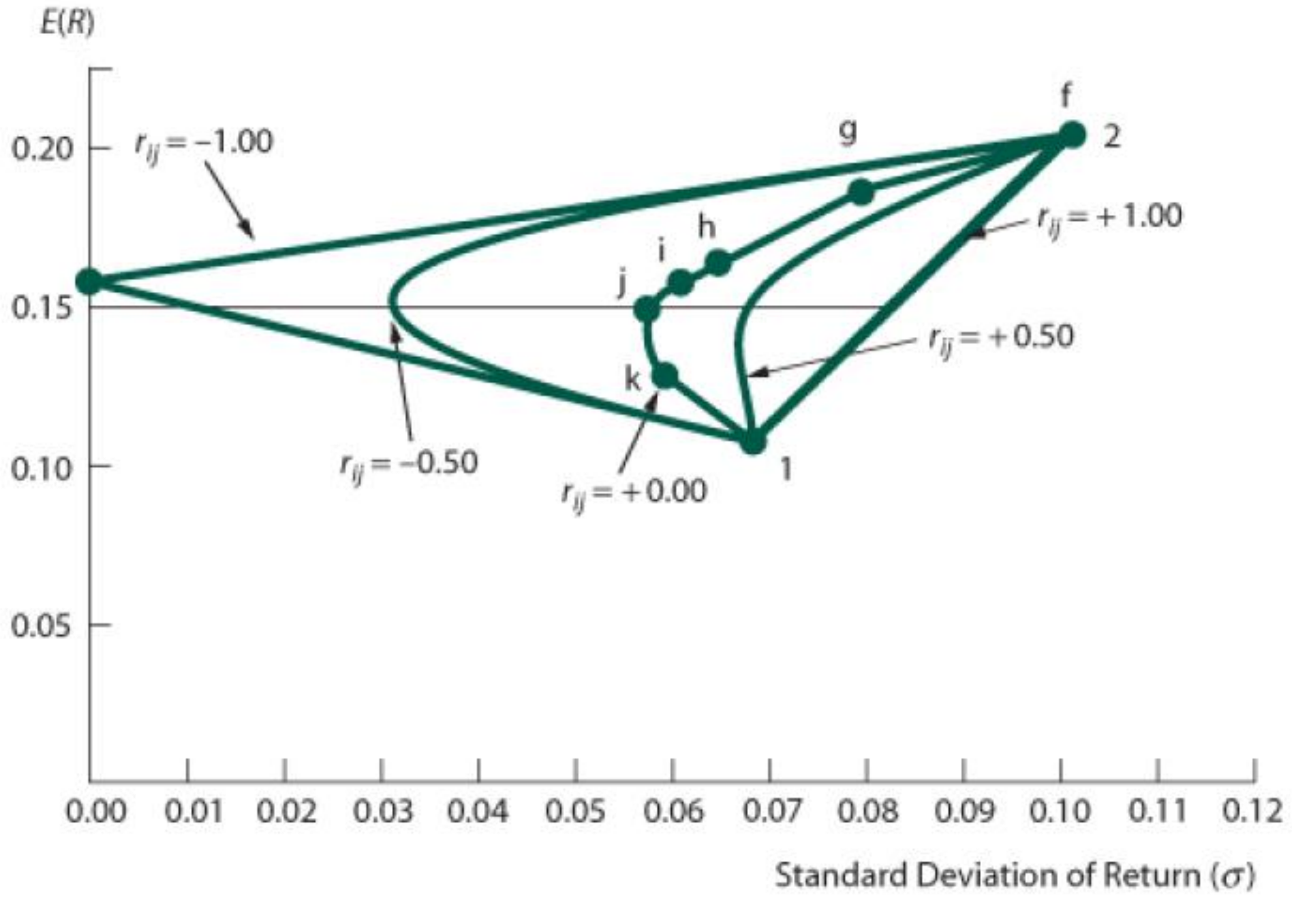
- Constant Correlation with Changing Weights
  - Assume the correlation is 0 in the earlier example and let the weight vary as shown below.
  - Portfolio return and risk are (Example 7):

Case	W_1	W_2	$E(R_i)$	$E(\sigma_{port})$
f	0.00	1.00	0.20	0.1000
g	0.20	0.80	0.18	0.0812
h	0.40	0.60	0.16	0.0662
i	0.50	0.50	0.15	0.0610
j	0.60	0.40	0.14	0.0580
k	0.80	0.20	0.12	0.0595
l	1.00	0.00	0.10	0.0700

Standard Deviation of a Portfolio

Example 6: *Portfolio Risk-Return Plots for Different Weights when*

ρ_{ij} = 1.0; 0.5; 0.0; -0.5; -1.0



Standard Deviation of a Portfolio

- An *n*-Asset Portfolio
  - The results presented earlier for the two-asset portfolio can be extended to a portfolio of *n* assets
  - As more assets are added to the portfolio, more risk will be reduced with everything else being the same
  - The general computing procedure is still the same, but the amount of computation will increase rapidly
  - For the three-asset portfolio, the computation will double in comparison with the two-asset portfolio

Estimation Issues

- Results of portfolio allocation depend on accurate statistical inputs

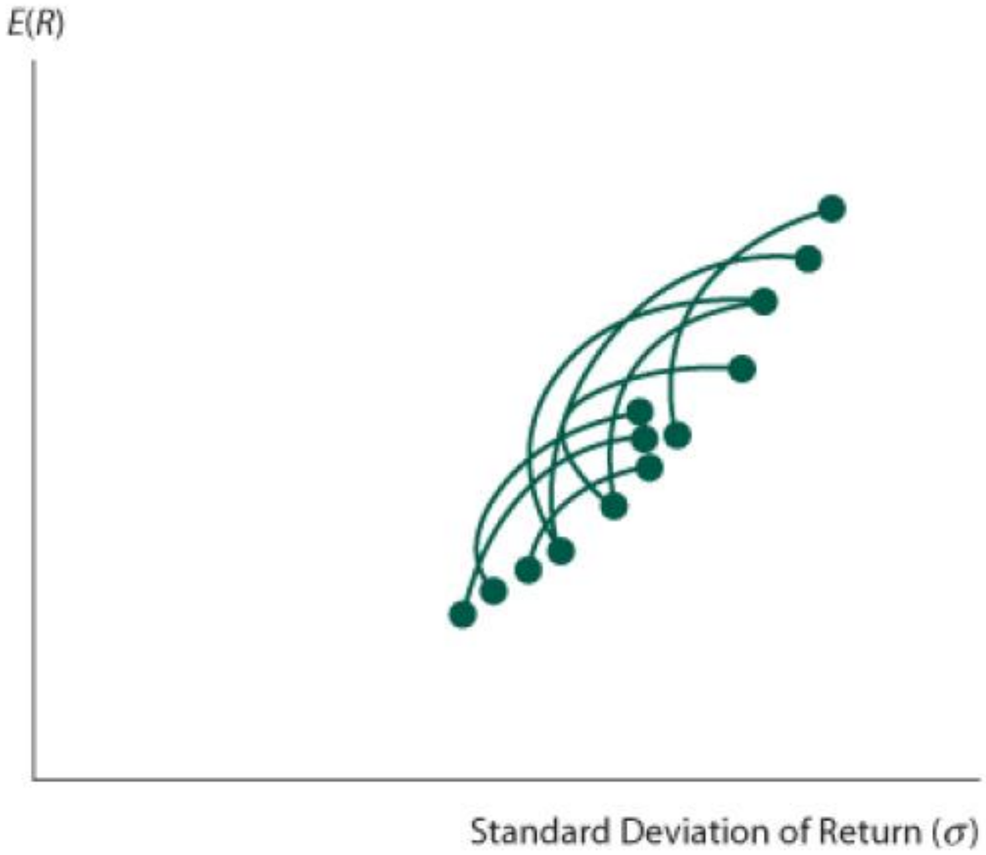
- Estimates of
  - Expected returns
  - Standard deviation
  - Correlation coefficient
    - Among entire set of assets
    - With 100 assets, 4,950 correlation estimates

- Estimation of risk refers to potential errors

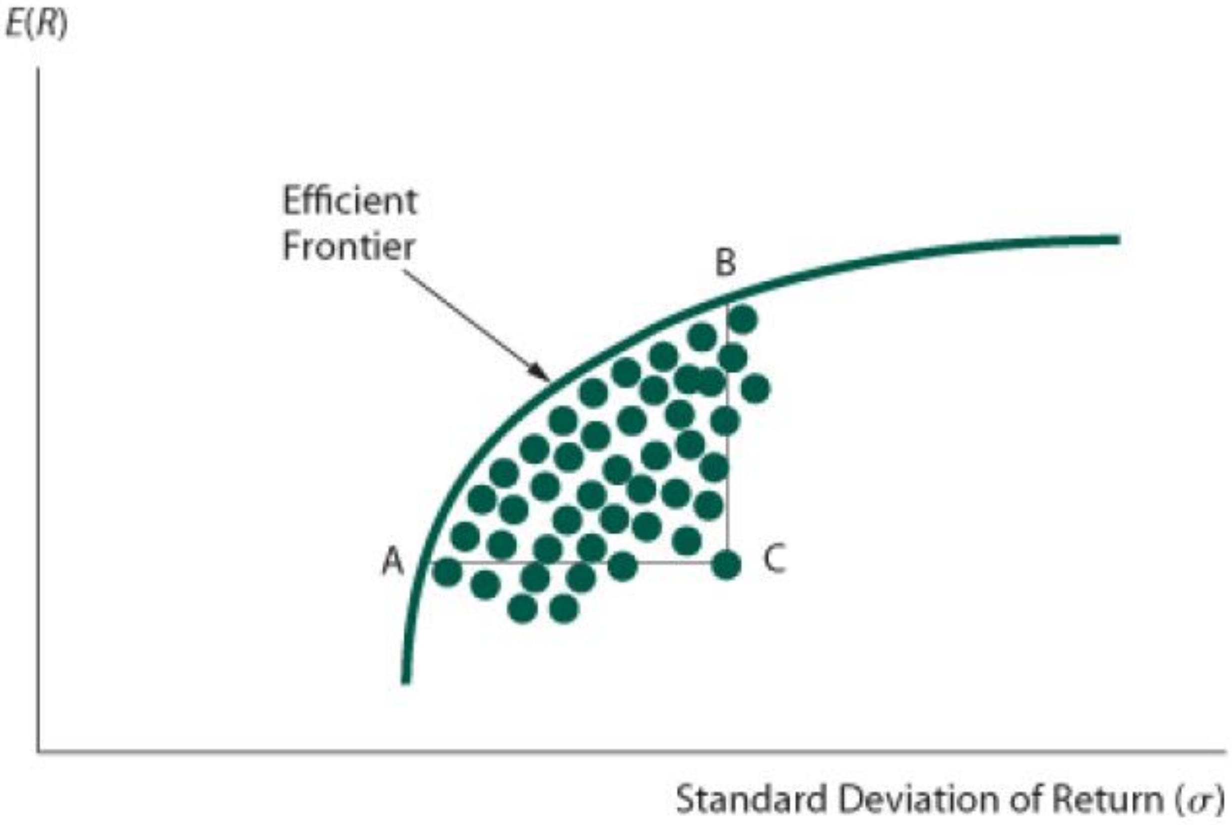
The Efficient Frontier

The efficient frontier is the set of optimal portfolios that offer the highest expected return for a given level of risk or the lowest risk for a given level of expected return.

Example 7:
Numerous Portfolio Combinations of Available Assets



Example 8:
Efficient Frontier for Alternative Portfolios



Mathematical Analysis of Mean-Variance Optimization

Consider a portfolio consisting stocks S_1 and S_2 with the proportion invested in S_1 is denoted by W_1 and proportion invested in S_2 by W_2.

Let's denote $u_i = E(R_i) =$ expected return on stock S_i after one year.

$$\sigma_i^2 = \sigma^2(R_i) = \text{risk on Stock } S_i$$

$$\sigma_{ij} = Cov(R_i, R_j)$$

Then, $u_p =$ expected return of our portfolio is

$$u_P = E(W_1 R_1 + W_2 R_2) = W_1 u_1 + W_2 u_2$$

Since $W_1 + W_2 = 1$, we can solve for W_i.

Mathematical Analysis of Mean-Variance Optimization

We assume our two stocks are different with different returns: $u_1 \neq u_2$, then

$$u_P = W_1 u_1 + (1 - W_1) u_2 \;\rightarrow\; u_P = W_1 (u_1 - u_2) + u_2$$

$$W_1 = \frac{u_P - u_2}{u_1 - u_2} \;\text{ and }\; W_2 = 1 - W_1 = \frac{u_1 - u_P}{u_1 - u_2}$$

Now, what is the risk of this portfolio?

Risk of Portfolio =

$$\sigma_P^2 = \sigma^2{}_{(W_1 R_1 + W_2 R_2)} = W_1^2 \sigma_1^2 + W_2^2 \sigma_2^2 + 2 W_1 W_2 \sigma_{12}$$

- This expression is ***quadratic*** in the portfolio mean u_p and we can solve this quadratic equation

Mathematical Analysis of Mean-Variance Optimization

Let's work out the coefficient (a, b, c). (*Derive this yourself*)

$$\sigma_P^2 = au_P^2 + bu_P + c$$

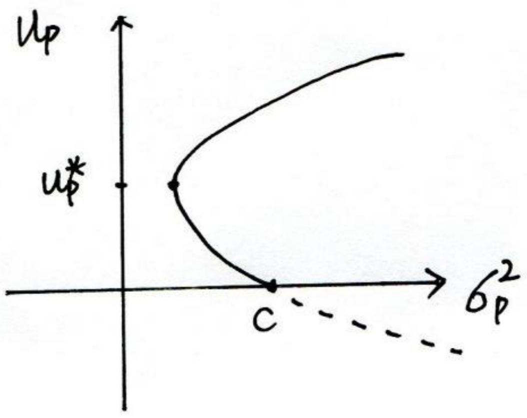
$$a = \frac{\left(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}\right)}{(u_1 - u_2)^2}$$

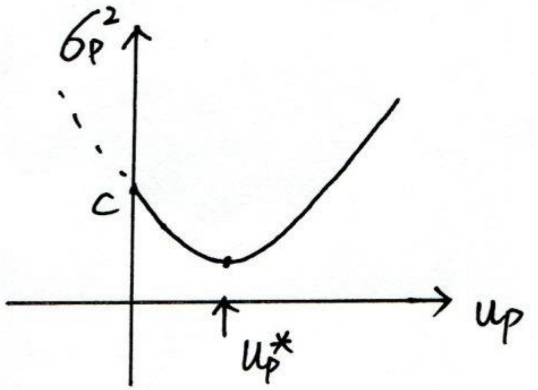
$$b = \frac{-2\left(u_2\sigma_1^2 + u_1\sigma_2^2 - u_2\sigma_{12} - u_1\sigma_{12}\right)}{(u_1 - u_2)^2}$$

$$c = \frac{u_2^2\sigma_1^2 + u_1^2\sigma_2^2 - 2u_1u_2\sigma_{12}}{(u_1 - u_2)^2}$$

Mathematical Analysis of Mean-Variance Optimization

If we plot u_P vs. σ_P^2, we have a rotated parabola (this is the usual way to show the ***Efficient Frontier***)





What is u_P^* ?

$$\frac{d\sigma_P^2}{du_P} = 0 \text{ at } u_P^*$$

$$\frac{d\sigma_P^2}{du_P} = \frac{d}{du_P}\left(au_P^2 + bu_P + c\right) = 2au_P + b = 0 \text{ at } u_P = u_P^*$$

$$\text{==>}\ u_P^* = -\frac{b}{2a}$$

$$\sigma_P^2(u_P = u_P^*) = c - \frac{b^2}{4a}$$

Mathematical Analysis of Mean-Variance Optimization

What is the meaning of this special point $\left(u_P^*, \sigma_P^{*2}\right)$?

At this point, the portfolio has the minimum risk!

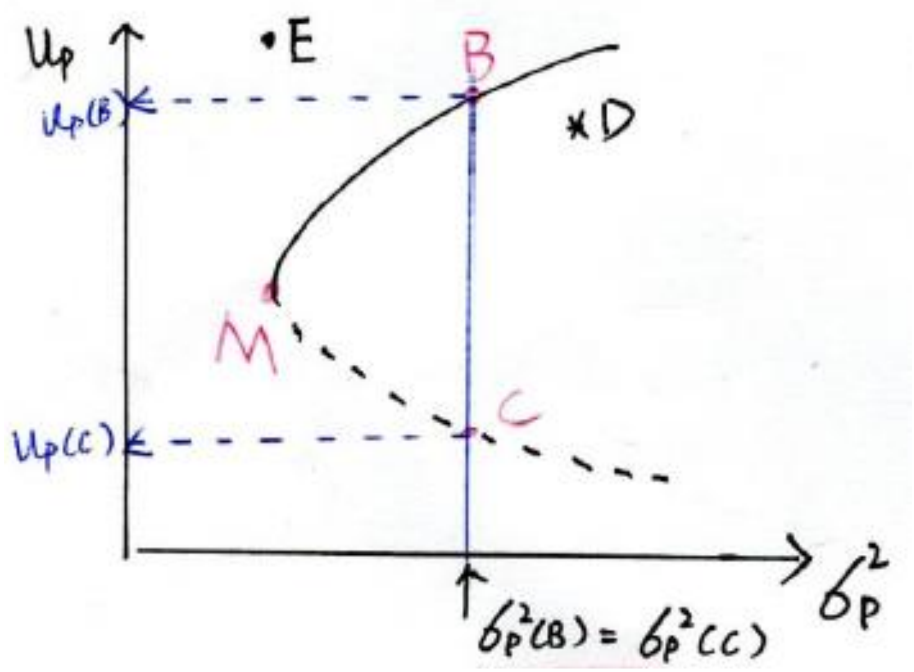
***Note that at* $\left(u_P^*, \sigma_P^{*2}\right)$*, we also have* $\frac{d^2\sigma_P^2}{du_P^2} > 0$ *, since a > 0.*

From this analysis, the minimum risk portfolio has a risk less than any other combination of Stock 1 and Stock 2.

Mathematical Analysis of Mean-Variance Optimization

(W_1, W_2) are defined on this parabola: $\sigma_P^2 = au_P^2 + bu_P + c$

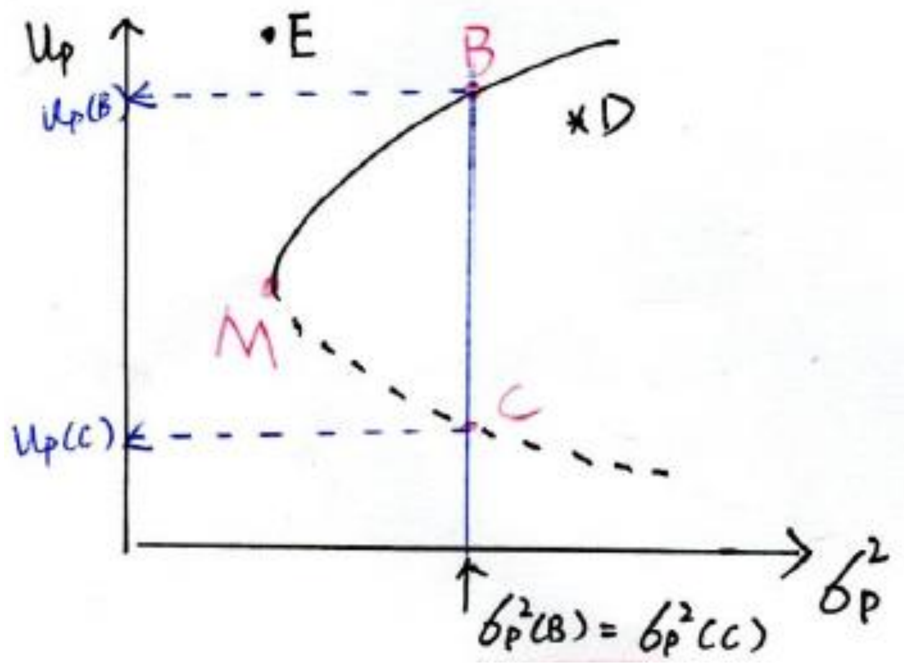
Consider two different portfolios B and C
An investor using B will get a higher return $u_P\,(B)$ than using C, which gives a return $u_P\,(C)$.
However, the risk bored by the investor using portfolio B is the same as the risk bored by the investor using portfolio C.
==> ***Thus, B is preferred by the investor.***
This illustrates a simple example of ***"efficient frontier"***.



Mathematical Analysis of Mean-Variance Optimization

If we compare a portfolio *B* with *D*, again *B* is preferred since it yields higher return and lower risk.

Note that the portfolio mixture (W_1, W_2) does not permit one to choose a point E.
From this picture, all combination of S_1 and S_2 ($0 < W_1 < 1$, $W_2 = 1 - W_1$) is bounded by the upper branch of the parabola (BM). This is called the efficient frontier.



For an investor with a specific risk bearing level, "σ_P^2", he can choose a point on the efficient frontier.

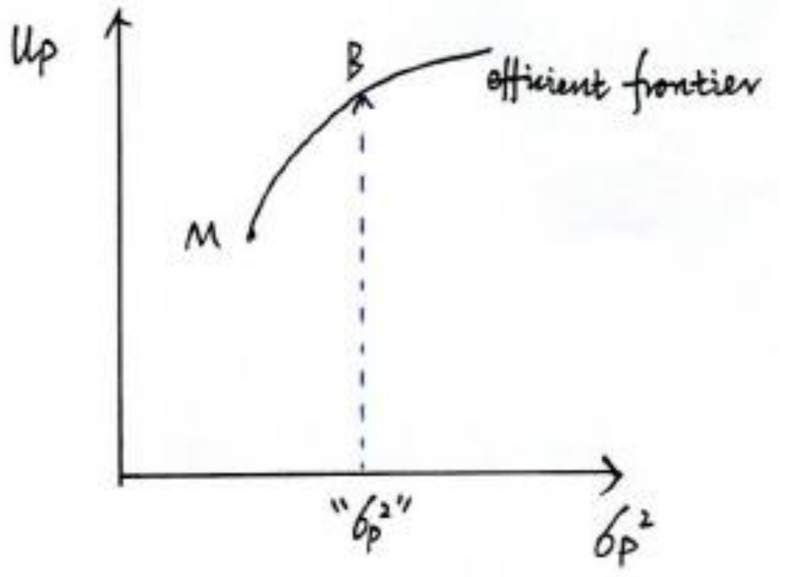
Mathematical Analysis of Mean-Variance Optimization

We say that B is better than M?

In general, one cannot say *B* is better than *M* because *M* offers less risk at the price of lower return, while *B* offers higher return at the price of higher risk.

The two portfolios *M* and *B* cannot win over the other.

Neither *M* nor *B* dominates any points on the efficient frontier.



Thus, one can also say that the efficient frontier consists of the set of portfolios that are "non-dominated", but they all are better off than others not on the frontier.

Underlying hypothesis of Mean Variance Optimization

When you use historical data to provide the MVO inputs, you assume that

- The returns in the different periods are independent.
- The returns in the different periods are drawn from the same statistical distribution.
- The N periods of available data provide a sample of this distribution.

Three Problems associated with the underlying hypothesis of MVO

The hypotheses listed above may simply not be true due to an often occurring phenomenon called mean reversion

1. Mean Reversion
2. Error in the estimated mean
3. The true long term return

1. Mean Reversion

The most serious inaccuracies arise from a phenomenon called ***mean reversion***, in which a period, or periods, of superior (inferior) performance of a particular asset tend to be followed by a period, or periods, of inferior (superior) performance.

Suppose, for example, you have used 5 years of historical data as MVO inputs for the upcoming year. The outputs of the algorithm will favor those assets with high expected return, which are those which have performed well over the past 5 years.

Yet if mean reversion is in effect, these assets may well turn out to be those that perform most poorly in the upcoming year.

2. Error in the Estimated Mean

Even if you believe that the returns in the different periods are independent and identically distributed (iid), you are of necessity using the available data to estimate the properties of this statistical distribution.

If you take the expected return for a given asset to be the simple average R of the N historical values, and the standard deviation to be the root mean square deviation from this average value, then statistics tells us that the one standard deviation error in the value R as an estimate of the mean is the standard deviation divided by the square root of N.

If N is not sufficiently large, then this error can distort the results of the MVO analysis considerably.

3. The True Long Term Return

If one applies the MVO method in period after period, then the inputs which you use in each period will be more or less the same.

Consequently, the outputs in each period will also be much the same, and so, by repeatedly applying your single period strategy, you will effectively be pursuing a multi-period strategy in which you rebalance your portfolio to a specified allocation at the beginning of each period.

It is then reasonable to hope that the expected return given by the Markowitz algorithm for your chosen portfolio will be the return that would actually have been obtained by this rebalancing strategy in the past, and thus also, by hypothesis, in the future.

Unfortunately this is generally not true in practice!!!

Sharpe Ratio – Performance Evaluation

- Can anyone consistently earn an “excess” return, thereby “beating” the market?
- *Performance Evaluation* is a term for assessing how well a money manager achieves a balance between high returns and acceptable risks.
- The *raw return* on a portfolio, R_{port}, is simply the total percentage return on a portfolio.
- The raw return is a *naive* performance evaluation measure because:
  - *The raw return has no adjustment for risk.*
  - *The raw return is not compared to any benchmark, or standard.*
- Therefore, the usefulness of the *raw return* on a portfolio is limited.

Sharpe Ratio – Performance Evaluation

The Sharpe Ratio

- The *Sharpe ratio* is a reward-to-risk ratio that focuses on *total risk*.

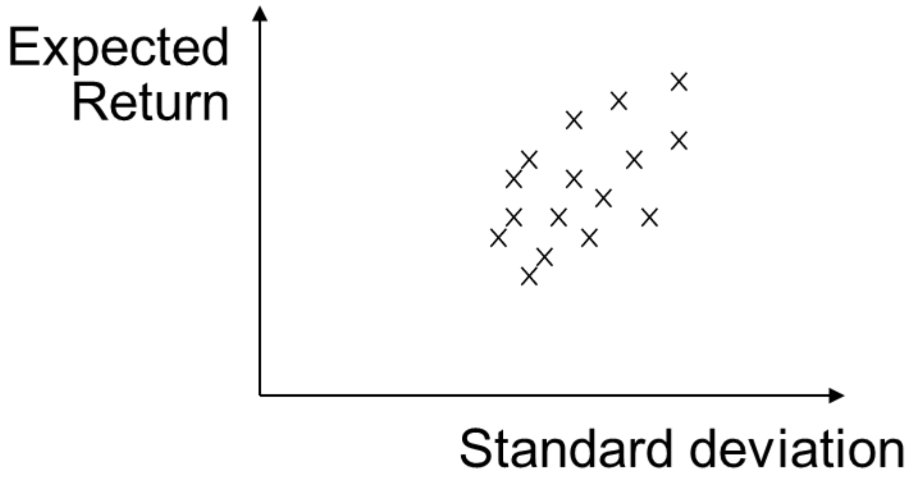
- It is computed as a portfolio's risk premium divided by the standard deviation for the portfolio's return.

$$Sharpe\ Ratio = \frac{R_p - R_f}{\sigma_p}$$

Sharpe Ratio – Performance Evaluation

The Sharpe Ratio

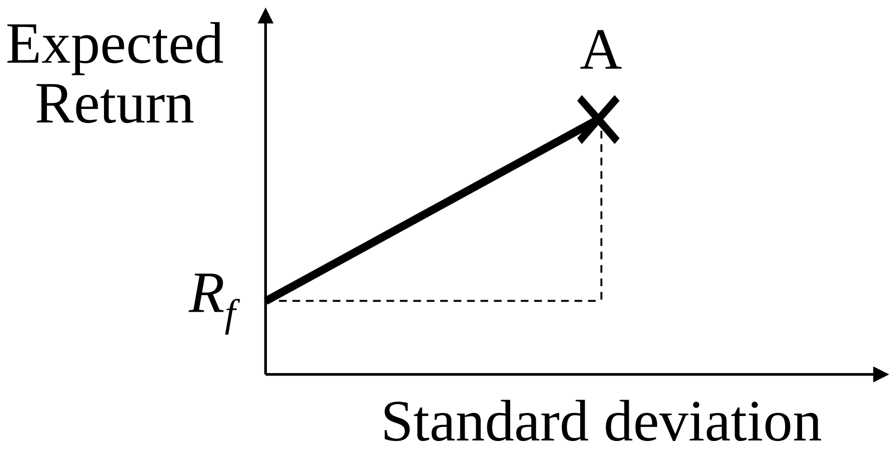
- Appropriate for the evaluation of an entire portfolio.
- Penalizes a portfolio for being undiversified, because in general, total risk ~ systematic risk only for relatively well-diversified portfolios.
- Allocating funds to achieve the highest possible Sharpe ratio is said to be *Sharpe-optimal*.
- To find the Sharpe-optimal portfolio, first look at the plot of the possible risk-return possibilities, i.e., the investment opportunity set.



Sharpe Ratio – Performance Evaluation

The Sharpe Ratio

- The slope of a straight line drawn from the risk-free rate to where the portfolio plots gives the Sharpe ratio for that portfolio.

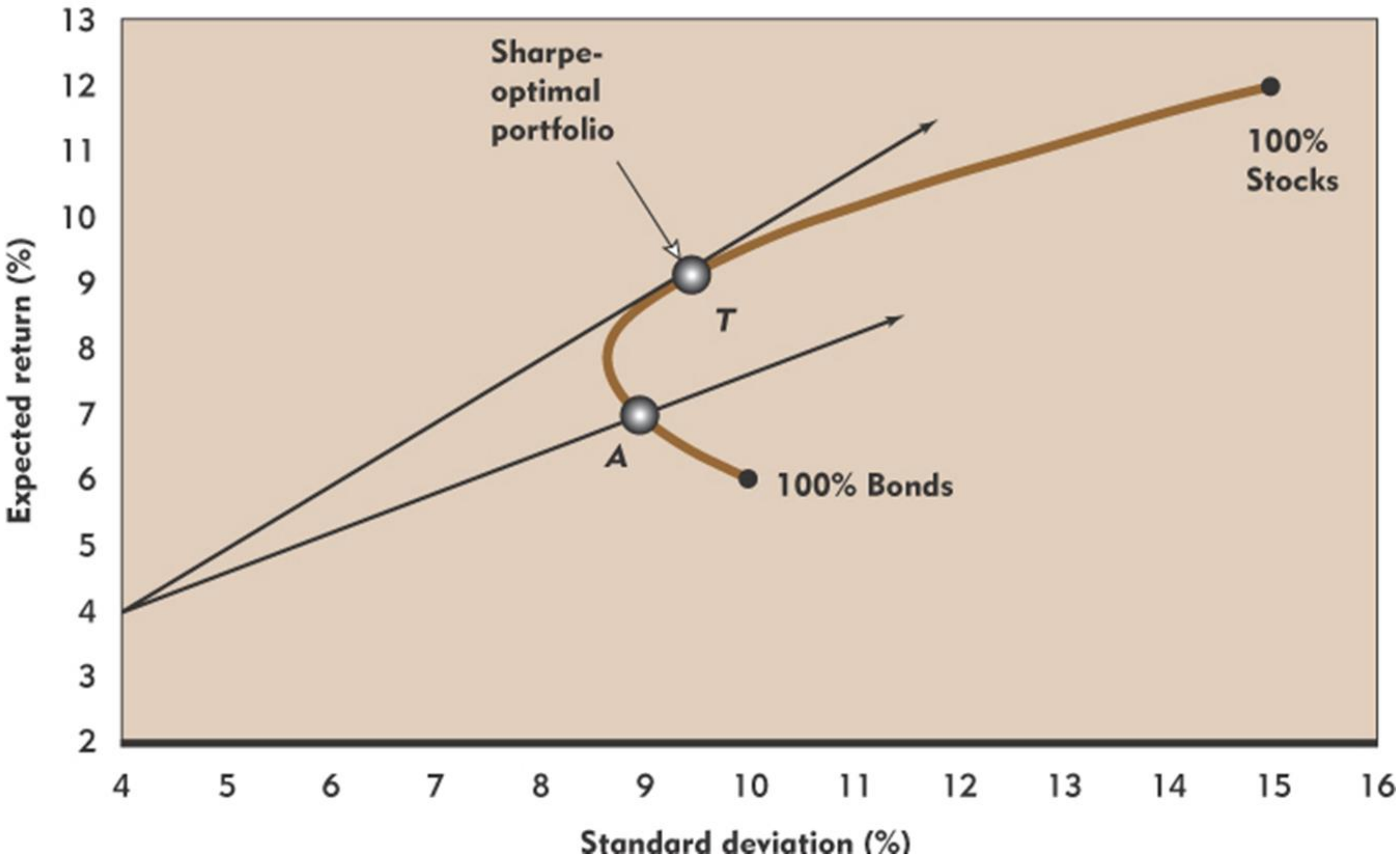


$$Slope = \frac{E(R_A) - R_f}{\sigma_A}$$

- The portfolio with the ***steepest*** slope is the Sharpe-optimal portfolio.

Sharpe Ratio – Performance Evaluation

The Sharpe Ratio



Sharpe Ratio – Performance Evaluation

The Sharpe Ratio

For a 2-asset portfolio

Portfolio Return:

$$E(R_{port}) = w_1 E(R_1) + (1 - w_1)E(R_2) \;\; ; \quad w_1 + w_2 = 1$$

Portfolio Variance:

$$\sigma_{port}^2 = w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + 2w_1(1 - w_1)\, Cov_{12}$$

Sharpe Ratio:

$$\frac{E(R_{port}) - R_f}{\sigma_{port}} = \frac{w_1 E(R_1) + (1 - w_1)E(R_2) - R_f}{\sqrt{w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + 2w_1(1 - w_1)\, Cov_{12}}}$$

Portfolio Optimization: An Example of Convex Optimization

Recall: General optimization problem in standard form

Minimize$_x$ $\quad f_0(x)$

subject to $\quad f_i(x) \leq 0 \quad i = 1, \ldots, m$

$\quad h_j(x) = 0 \quad j = 1, \ldots, p$

where $x = (x_1, \ldots, x_n)$ is the set of optimization variables.

Goal: *Find an optimal solution x^* that minimizes $f_0(x)$ while satisfying all the constraints.*

Portfolio Optimization: An Example of Convex Optimization

Recall: Convex optimization problem in standard form

Minimize$_x$ $\quad f_0\,(x)$

subject to $\quad f_i\,(x) \leq 0 \qquad i = 1, \ldots, m$

$\quad Ax = b$

where $f_0, f_1, \ldots, f_m$ are convex and equality constraints are affine.

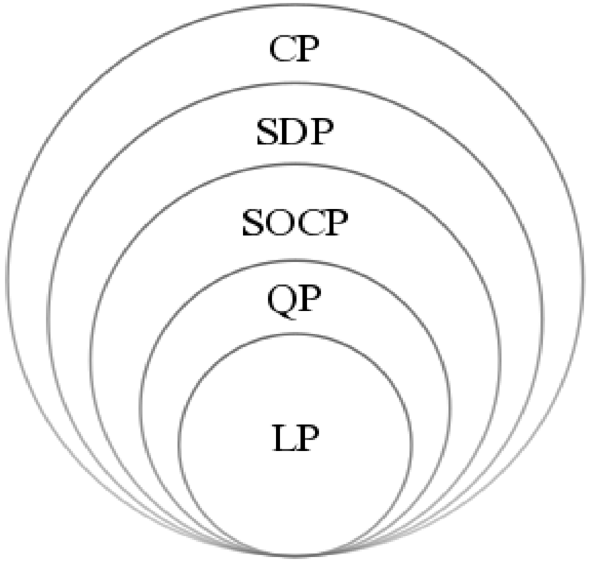
➡ Any *locally* optimal point of a convex problem is *globally* optimal.

Portfolio Optimization: An Example of Convex Optimization

Recall: Convex optimization problem in standard form

The following problem classes are all convex optimization problems, or can be reduced to convex optimization problems via simple transformations:

- Least squares
- Linear programming
- Convex quadratic minimization with linear constraints
- Quadratic minimization with convex quadratic constraints
- Conic optimization
- Geometric programming
- Second order cone programming
- Semidefinite programming
- Entropy maximization with appropriate constraints



Hierarchy of convex optimization problems. (LP: linear program, QP: quadratic program, SOCP second-order cone program, SDP: semidefinite program, CP: cone program.)

Portfolio Optimization: An Example of Convex Optimization

Recall: Quadratic Programming problem in standard form

$$\begin{aligned} &\text{Minimize}_x && (1/2)\, x^T P x + q^T x + r \\ &\text{subject to} && Gx \leq h \\ & && Ax = b \end{aligned}$$

This is a convex optimization problem iff P is positive semidefinite and the constraint functions are affine.

Portfolio Optimization: An Example of Convex Optimization

Recall:

$$\sigma_{port} = \sqrt{\sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov_{ij}}$$

One can rewrite $\sigma_{port}{}^2$ in the following form, $\boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{A}\, \boldsymbol{w}$, where $\boldsymbol{A}$ is the covariance matrix.

Portfolio Optimization: An Example of Convex Optimization

In Modern Portfolio Theory, one is interested in allocating an amount of money into a set of N stocks with different weights. Let us assume that the portfolio $w \in R^N$ is the amount of normalized money invested in each stock and the sum of this N dimensional vector is 1. This portfolio optimization process can be formulated into a convex optimization problem as follows:

$$\text{Minimize}_w \quad \boldsymbol{w^T A\, w}$$

$$\text{subject to} \quad \boldsymbol{\mu^T w} \geq \beta$$
$$\boldsymbol{w} \geq 0$$
$$\boldsymbol{w^T.1 = 1}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{A}$ are the given vector of expected stock returns and covariance matrix of the stock returns respectively. The objective $\boldsymbol{w^T A\, w}$ is the portfolio variance, the first constraint is the portfolio expected return, , and β is the parameter that controls the lower bound of expected return.

Portfolio Optimization: An Example of Convex Optimization

The goal here is to minimize the risk it takes under the certain level of money one earns. The solution to the above problem, $w^{\star}$, is called the "Markowitz portfolio" or the "Mean-variance portfolio."

In this example, we will use Python and the CVXPY package to solve this problem given that $0 \leq \beta \leq 0.05$, with the following values of $\boldsymbol{A}$ and $\boldsymbol{\mu}$:

$$\boldsymbol{A} = \begin{bmatrix} 1.0 & 0.015 & -0.02 \\ 0.015 & 1.0 & -0.1 \\ -0.02 & -0.1 & 1.0 \end{bmatrix}$$

$$\boldsymbol{\mu} = [0.01, 0.06, 0.005]$$

We will plot the portfolio expected return ($\boldsymbol{\mu}^T\boldsymbol{w}^*$) vs. portfolio volatility ($\sqrt{\boldsymbol{w}^T\boldsymbol{A}\,\boldsymbol{w}}$).