11. Binomial pricing

Contents

(All variables are real and one-dimensional unless otherwise specified.)

The theory of option pricing due to **no arbitrage** grew mature in 1970s.

- In 1973, Black and Scholes analyzed options with Brownian motion, and Merton extended their **Black-Scholes model** mathematically later. Scholes and Merton therefore won the Nobel Price in Economics in 1997, while Black had already passed away in 1995.
- In 1979, Cox, Ross, and Rubinstein formulated the **binomial model**. It turns out to be the discrete-time counterpart of the Black-Scholes model.
- In 1981, Harrison and Pliska derived the **fundamental theorem of asset pricing**. Its implications especially help simplify the binomial model's arguments.

1. The value of time

In a stable market, the safest form of investment must be saving money in a bank. The bank compounds the money M_0 with some annual **risk-free interest rate** r.

$$M_t = M_0\left(1+\frac{r}{n}\right)^{nt}$$

Here time t is measured in years, and n counts the times of compounding in a year. Strictly speaking, the formula holds only when nt is an integer. Now consider a very high frequency of compounding, i.e. $n \to \infty$. It defines the value of the money at any instant.

$$M(t) = M_0 e^{rt}$$

The principle of no arbitrage suggests that investments with the **same level of risk** should yield the **same level of return**. Consequently, if something **certainly** costs A after T years, it must cost $A_0 = Ae^{-rT}$ now.

2. The fundamental theorem

The **binomial** model requires that the stock's current price $S_0 = S$ either

- goes up to $S_1 = Su$ by a fraction u or
- goes down to $S_1 = Sd$ by another fraction d

at time T (in years). Both events' **probabilities** are **unknown**. They need not be equally probable. It turns out that we can price the option even though we do not know the probability $\mathrm{P}(S_1)$.

Consider a relevant option that will be exercised at T. If the price goes up, the option costs f_u; if the price goes down, the option costs f_d. We want to evaluate the option's current price f.

$$\begin{array}{ccccccc} & & Su & & & & f_u \\ & \nearrow & & & & \nearrow & \\ S & & & ; & f & & \\ & \searrow & & & & \searrow & \\ & & Sd & & & & f_d \end{array}$$

Consequently, a portfolio that longs D shares and shorts one option costs either $DSu - f_u$ or $DSd - f_d$ in the future. If $D = \dfrac{f_u - f_d}{Su - Sd}$, the portfolio has a **definite** future value

$$\Pi_T = DSu - f_u \equiv DSd - f_d \, .$$

Therefore, the portfolio currently costs $\Pi_0 = \Pi_T e^{-rT}$. Since we also know $\Pi_0 = DS - f$, the option's current price is argued to be

$$f = DS - (DSu - f_u)\, e^{-rT} \, .$$

If the option is not priced this way, there will be possibilities of arbitrage.

2.1 Risk-neutral probability

After some algebra, we can alternatively write

$$f = [p f_u + (1-p)\, f_d]\, e^{-rT}$$

as something similar to an expected value, where

$$p = \frac{e^{rT} - d}{u - d}$$

is called the **risk-neutral probability**. Risk-neutral probability is not probability—it may be negative or exceed one. On the other hand, it has an important property: at $r = 0$,

$$pSu + (1-p)\, Sd = S_0 \, .$$

The **fundamental theorem of asset pricing** states that because of this equality, we can manipulate p as the effective probability for the stock's price to go up, i.e.

$$p \overset{!}{=} \mathrm{P}(S_1 = Su) \equiv \mathrm{P}(f_1 = f_u) \, ,$$

and the resultant price of option f is **guaranteed to be free of arbitrage.**

2.2 Example: a one-step European call

A stock's current price is $S_0 = 1$, and it will either goes up to $S_1 = 2$ or down to $S_1 = 0.5$ tomorrow. A relevant European call option expires tomorrow with a strike price $K = 1$. How much does the option cost now? Consider $r = 0$.

This is the example in Section 3.2 of Tutorial 10.

Solution. First of all, the option either costs $C_1 = C_u = 1$ or $C_1 = C_d = 0$ tomorrow. The corresponding risk-neutral probability is

$$p = \frac{e^0 - 0.5/1}{2/1 - 0.5/1} = \frac{1}{3} \,.$$

This gives $C_0 = \left[pC_u + (1-p)\,C_d\right] e^0 = 1/3$, which equals the answer that we obtain through an argument of no arbitrage. But once we invoke the fundamental theorem, we do not need to somehow construct an equivalent portfolio with a price $\Pi_1 \equiv C_1$.

2.3 Example: a two-step European put

A stock's current price is $S = 50$. It goes either up or down by 20% every three months. A relevant European put expires six months later with a strike price $K = 52$. How much does the option cost now? Consider $r = 20\%$ per annum.

Solution. Let $u = 1.2$ and $d = 0.8$. The stock's price will be $Su^2 = 72$, $Sud = 48$, or $Sd^2 = 32$ after six months. A European put's payoff is $\max\left(K - S, 0\right)$, so it may cost $P_{uu} = 0$, $P_{ud} = 4$, or $P_{dd} = 20$ at expiry.

$$S: \begin{pmatrix} 50 & \rightarrow & 60 & \rightarrow & 72 \\ \downarrow & & \downarrow & & \\ 40 & \rightarrow & 48 & & \\ \downarrow & & & & \\ 32 & & & & \end{pmatrix} \quad ; \quad P: \begin{pmatrix} P & \rightarrow & P_u & \rightarrow & 0 \\ \downarrow & & \downarrow & & \\ P_d & \rightarrow & 4 & & \\ \downarrow & & & & \\ 20 & & & & \end{pmatrix}$$

The corresponding risk-neutral probability is

$$q = \frac{e^{rT} - d}{u - d} = \frac{e^{0.2\times 3/12} - 0.8}{1.2 - 0.8} \approx 0.628 \,,$$

where $T = 3/12$ years are the length of **one** time-step. Therefore,

$$\begin{cases} P_u &= [qP_{uu} + (1-q)\,P_{ud}]\,e^{-rT} &\approx& 1.41 \\ P_d &= [qP_{ud} + (1-q)\,P_{dd}]\,e^{-rT} &\approx& 9.46 \end{cases}$$

and $P = [qP_u + (1-q)\,P_d]\,e^{-rT} \approx 4.19$. We can alternatively write down

$$P = \left[q^2 P_{uu} + 2q\,(1-q)\,P_{ud} + (1-q)^2 P_{dd}\right] e^{-2rT}\,,$$

where we may interpret q^2 as $\mathrm{P}\left(Su^2\right)$ and $(1-q)^2$ as $\mathrm{P}\left(Sd^2\right)$. Similarly, we may interpret $2q\,(1-q)$ as $\mathrm{P}(Sud)$, and the leading 2 corresponds to the two possible cases to reach Sud, i.e. up-then-down ($S \to Su \to Sud$) and down-then-up ($S \to Sd \to Sud$).

3. Path dependence

Asian options and lookback options feature **path dependence**. Therefore, they must be priced according to the full **history** of their underlying stock's price.

3.1 Example: two-step Asian put

Consider Section 2.3's stock again.

$$S: \begin{pmatrix} 50 & \to & 60 & \to & 72 \\ \downarrow & & \downarrow & & \\ 40 & \to & 48 & & \\ \downarrow & & & & \\ 32 & & & & \end{pmatrix}$$

A relevant Asian put option with a strike price $K = 52$ expires six months later, and it can be only exercised at expiry. The option's payoff is $\max\left(K - A, 0\right)$, where A is the stock's average price throughout. How much does the option cost now? Consider $r = 20\%$ per annum.

Solution. The risk-neutral probability remains unchanged:

$$q = \frac{e^{rT} - d}{u - d} \approx 0.628\,.$$

The changed parts are the option's payoffs at exercise. There are four possible histories for S, and each history leads to a different payoff.

$$\begin{array}{rcrcr} S \to Su \to Su^2 &:& A_{uu} \approx 60.7 &\Rightarrow& P_{uu} = 0 \\ S \to Su \to Sud &:& A_{ud} \approx 52.7 &\Rightarrow& P_{ud} = 0 \\ S \to Sd \to Sdu &:& A_{du} = 46 &\Rightarrow& P_{du} = 6 \\ S \to Sd \to Sd^2 &:& A_{dd} \approx 40.7 &\Rightarrow& P_{dd} \approx 11.3 \end{array}$$

Although $Sud = Sdu$, $P_{ud} \neq P_{du}$. Finally, apply the canonical formula:

$$P = \left[q^2 P_{uu} + q(1-q) P_{ud} + (1-q) q P_{du} + (1-q)^2 P_{dd} \right] e^{-2rT} \approx 2.69 \,.$$

4. Optimal stopping

American options can be exercised **before expiry**, so their holders must determine when it is the most profitable to exercise them. This problem falls into the realm of **optimal stopping**, a branch of mathematical theory that studies the "best" time to stop a random process. The meaning of "best" can vary among people. Here we assume **risk-neutrality**: we aim at a profit as high as possible without taking excessive risk.

4.1 A simple dice game

Let us first illustrate the concept of optimal stopping with a simple dice game. You can roll a fair dice for up to **three** times. You win $\$X$ once you stop throwing, where X is the number of dots that you **last** get. How should you play this game if you are risk-neutral?

If you can only play this game **once**, you expect to earn

$$\$\frac{1+2+3+4+5+6}{6} = \$3.5 \,.$$

This is trivial. Now, what if you can play this game **twice**? Suppose your first get 1. You may either

- stop now to earn $\$1$ or
- play again and hope to earn $\$3.5$.

You should choose the second option to improve your profit if you are rational. On the other hand, suppose you first get 6. You may similarly either

- stop now to earn $\$6$ or
- play again and hope to earn $\$3.5$.

Playing again probably makes you lose the already good profit, so you should stop playing. Therefore, you play again if you get $X \in \{1, 2, 3\}$ but stop playing if you get $X \in \{4, 5, 6\}$.

The strategy is valid for a game with at most two trials. What if you are given another trial, i.e. at most three trials? We need to check to expected profit in a two-trial game, which is no longer $\$3.5$ but

$$\$\frac{3.5+3.5+3.5+4+5+6}{6} = \$4.25 \,.$$

The three 3.5's are the expected profits when you first get $X \in \{1, 2, 3\}$ and thus choose to play again. In other words, you expect to earn $\$4.25$ if you can play twice. Now we apply the same analysis: if you first get $X \in \{1, 2, 3, 4\}$, you should play again because you expect to earn more in the remaining trials. After the first trial, the three-trial game reduces to a two-trial game, so the overall strategy is

- to stop if you get 5 or 6 in the first trial and then
- to stop if you get 4, 5, or 6 in the second trial.

In general, if you are risk-neutral, **stop playing once the current profit outperforms the expected profit.**

4.2 Example: a two-step American put

An American option can be priced with the same philosophy. Again, take Section 2.3's stock as an example.

$$S:\begin{pmatrix} 50 & \rightarrow & 60 & \rightarrow & 72 \\ \downarrow & & \downarrow & & \\ 40 & \rightarrow & 48 & & \\ \downarrow & & & & \\ 32 & & & & \end{pmatrix}$$

A relevant American put has a strike price $K = 52$. It expires six months later and can be exercised every three months. How much does the option cost now? Consider $r = 20\%$ per annum.

(Some prefer classifying this kind of option as a Bermudan option because it can be exercised only at certain moments instead of any time within the six months.)

Solution. The American option's payoff at expiry is equivalent to its European counterpart's: $P_{uu} = 0$, $P_{ud} = 4$, or $P_{dd} = 20$. The difference lies upon P_u and P_d.

$$P:\begin{pmatrix} P & \rightarrow & P_u & \rightarrow & 0 \\ \downarrow & & \downarrow & & \\ P_d & \rightarrow & 4 & & \\ \downarrow & & & & \\ 20 & & & & \end{pmatrix}$$

First consider P_u. The option costs P_u when the stock's price goes up to $Su = 60$. At this moment, its holder may either

- exercise the option immediately or
- wait three more months and exercise it at expiry.

The option costs $\max\left(K - Su, 0\right) = 0$ if the holder takes the first choice, whereas the option costs $\left[qP_{uu} + (1-q)\,P_{ud}\right]e^{-rT} \approx 1.41$ if he takes the second choice. (Remember that the option's style does not change the risk-neutral probability and $q \approx 0.628$.) Therefore, a risk-neutral investor should wait until expiry, yielding

$$\begin{aligned} P_u &= \max\left\{\max\left(K - Su, 0\right), \left[qP_{uu} + (1-q)\,P_{ud}\right]e^{-rT}\right\} \\ &\approx \max\left(0, 1.41\right) = 1.41\,. \end{aligned}$$

Now we can analyze P_d with the same logic, when the stock's price is $Sd = 40$. If the investor exercises the option then, the option costs $\max\left(K - Sd, 0\right) = 12$. This is higher than the original risk-neutral expected payoff $\left[qP_{ud} + (1-q)\,P_{dd}\right]e^{-rT} \approx 9.46$, so the investor should exercise the option immediately, yielding

$$\begin{aligned} P_d &= \max\left\{\max\left(K - Sd, 0\right), \left[qP_{ud} + (1-q)\,P_{dd}\right]e^{-rT}\right\} \\ &\approx \max\left(12, 9.46\right) = 12\,. \end{aligned}$$

Finally, we can deduce P. (Note that It is possible to exercise the option right at the beginning.)

$$\begin{aligned} P &= \max\left\{\max\left(K - S, 0\right), \left[qP_u + (1-q)\,P_d\right]e^{-rT}\right\} \\ &\approx \max\left(2, 5.09\right) = 5.09 \end{aligned}$$

This price turns out to be higher than its European counterpart's current price $P^{\text{Euro}} \approx 4.19$. In general, an American option is more worthy because it grants the holder more **freedom**.

4.3 Example: a two-step Israeli put

Israeli options are a sophisticated variety of American options. While it is identical to an American option from the holder's perspective, an Israeli option allows the writer to exercise it for the holder at certain moments, but then the writer must pay a **compensation** to the holder for his deprived freedom.

For example, Section 4.2's American put can be transformed to an Israeli put by merely adding a condition: the writer can exercise it once the holder declares not to exercise it at the third month; if so, the writer is obliged to additionally pay the holder a compensation $c = 1$. With all other factors unchanged, how much does this new option cost now?

$$S : \begin{pmatrix} 50 & \rightarrow & 60 & \rightarrow & 72 \\ \downarrow & & \downarrow & & \\ 40 & \rightarrow & 48 & & \\ \downarrow & & & & \\ 32 & & & & \end{pmatrix} \quad ; \quad P : \begin{pmatrix} P & \rightarrow & P_u & \rightarrow & 0 \\ \downarrow & & \downarrow & & \\ P_d & \rightarrow & 4 & & \\ \downarrow & & & & \\ 20 & & & & \end{pmatrix}$$

Challenge. $P \approx 4.84$. It is lower than the original American put's current price $P^{\text{Amer}} \approx 5.09$ because a rational writer can wisely stop the option to minimize his loss.