MSDM 5056 Network Modeling
Assignment 3 (due 25th October, 2023)

Submit your assignment solution on canvas. You may discuss with others or seek help from your TA, but should not directly copy from others. Otherwise, it will be considered as plagiarism.

(1) **Properties of random networks**

Consider a random network in the ensemble *G(n; p)* with $n = 9.6 \times 10^6$ nodes, and its linking probability is $p = 10^{-4}$.

(a) Calculate the average degree $\langle k \rangle$= c, of this network.

(b) Calculate the standard deviation σ_P using the approximated degree distribution discussed in the lecture notes. (*Refer to p.20 of lecture 6*)

(c) Assume that you observe a node with degree 1,000. How many standard deviation is this observation from the mean? Is this an expected observation or is this an unexpected observation? (*Refer to p.21 of lecture 6*)

(2) **More properties of random networks**

Consider again a random network *G(n; p),* with mean degree *c*.

(a) Show that in the limit of large *n,* the expected number of triangles in the network is $c^3/6$. This implies that the number of triangles is constant in the limit of large *n*.

(b) Show that the expected number of connected triples in the network is $nc^2/2$.

(c) Calculate the clustering coefficient C and confirm that it agrees for large *n* with the expression on p.19 in Lecture 6.

(3) **Generating function of random graphs**

A random graph ensemble *G(n; p)* with $p = \frac{c}{n-1}$ when *n* → ∞ can be approximated by a Poisson distribution given by

$$P(k) = \frac{1}{k!} c^k e^{-c} \tag{1}$$

(a) Calculate the generating function

$$G_0(x) = \sum_{k=0}^{\infty} P(k) x^k \tag{2}$$

for the Poisson degree distribution in eq. (1).

(b) Using the properties of the generating functions, evaluate $\langle k \rangle$ and $\langle k(k-1) \rangle$ of the degree distribution $P(k)$ given by eq. (1), where $\langle ... \rangle$ denotes the expectation value of the quantity inside the brackets.

In many real world calculations, if we go along the edge of one node and reach another node, we are not so interested in the total degree of the vertex at the end of an edge but rather in the number of edges attached to that vertex other than the one we arrived along. The number of edges attached to a vertex other than the edge we arrived along is called the ***excess degree*** of the vertex and is just one less than the total degree. The excess degree probability distribution $Q(k)$ is given by

$$Q(k) = \frac{(k+1)\ P(k+1)}{\langle k \rangle} \tag{3}$$

where $P(k+1)$ is the probability of a node having degree $k + 1$, and $\langle k \rangle$ is the average degree of the graph.

(c) Substitute the Poisson distribution in eq. (1) into (3). What do you get for $Q(k)$?

(d) Calculate the generating function of this excess degree distribution

$$G_1(x) = \sum_{k=0}^{\infty} Q(k) x^k \tag{4}$$

for the Poisson degree distribution in eq. (1).

(e) Compare the results of (a), (c) and (d), what can you conclude?

(4) **The average number of friends of your friends**

Given a random uncorrelated network with degree distribution $P(k)$.

a) Show that

$$\frac{\langle k^2 \rangle}{\langle k \rangle} \geq \langle k \rangle$$

where the equal sign holds only for regular networks (all the nodes with the same degree). (*Hint: You can begin by showing* $\langle k^2 \rangle \geq \langle k \rangle^2$.)

b) Show that this result justify the social network paradox summarized in the following sentence: Your friends have more friends than you do. Assume for simplicity that the social network is not regular, and that the network is uncorrelated. (*Refer to p.42 of lecture 6 for the calculation of the average degree of the neighbor of a node in an uncorrelated random network.*)