MSDM5004
Numerical Methods and Modeling in Science
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Lecture 10

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2. Hyperbolic PDEs

Examples

Linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

Conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

Wave equation

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2\frac{\partial u^2}{\partial x^2} = 0$$

2.1. Characteristics and solutions

We focus on the linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad -\infty < x < \infty, \quad t > 0 \quad (1)$$

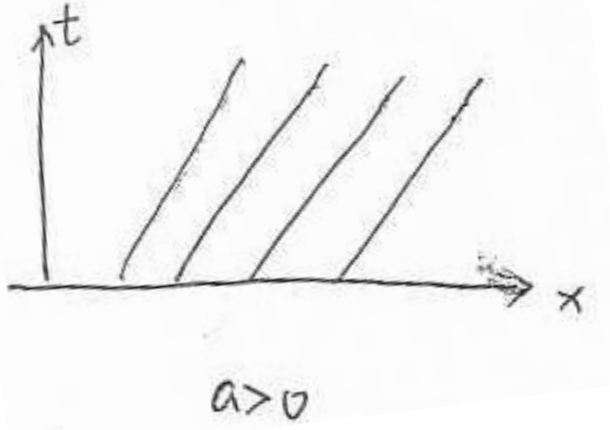
where *a* is a constant.

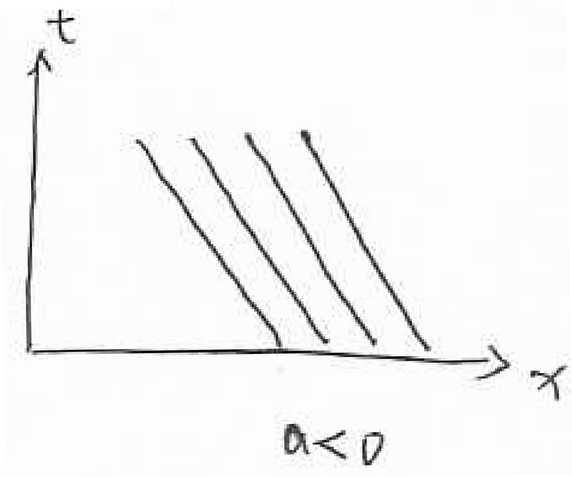
The characteristics of Eq. (1) are curves in the (x, t) plane satisfying

$$\frac{dx(t)}{dt} = a$$

If a is a constant, then the characteristics are straight parallel lines:

$$x - at = \text{constant}.$$





Along a characteristic, the solution of Eq. (1) is a constant.

In fact, if $\dfrac{dx(t)}{dt} = a$, then

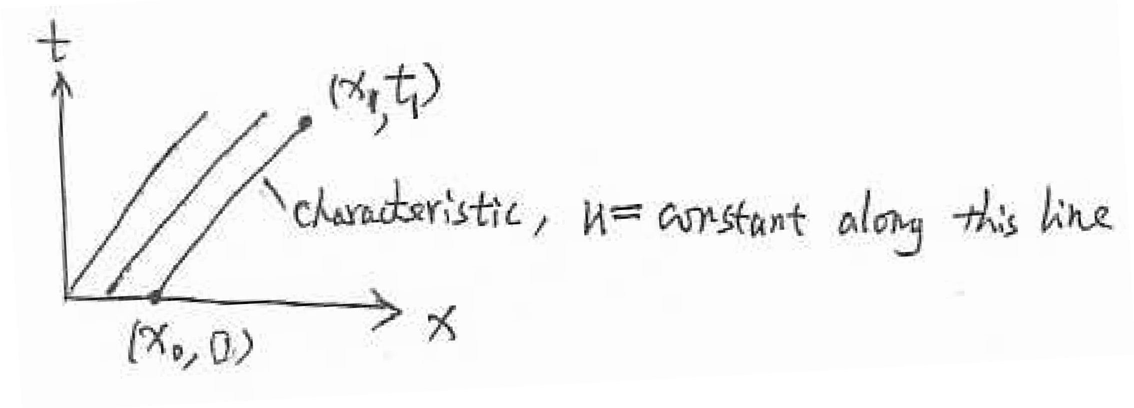
$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Thus $u(x(t),t)$ is constant along a characteristic $\dfrac{dx(t)}{dt} = a$.

The solution of Eq.(1) with initial condition

$$u(x,0) = u_0(x), \quad -\infty < x < \infty \qquad (2)$$

can be obtained using this property.



$$u(x_1, t_1) = u(x_0, 0) = u_0(x_0)$$

Since $x_0 - a \cdot 0 = x_1 - at_1$, we have $\quad x_0 = x_1 - at_1.$

Therefore $u(x_1, t_1) = u_0(x_1 - at_1).$

That is, the solution of Eq. (1) with initial condition (2) is

$$u(x,t) = u_0(x - at).$$

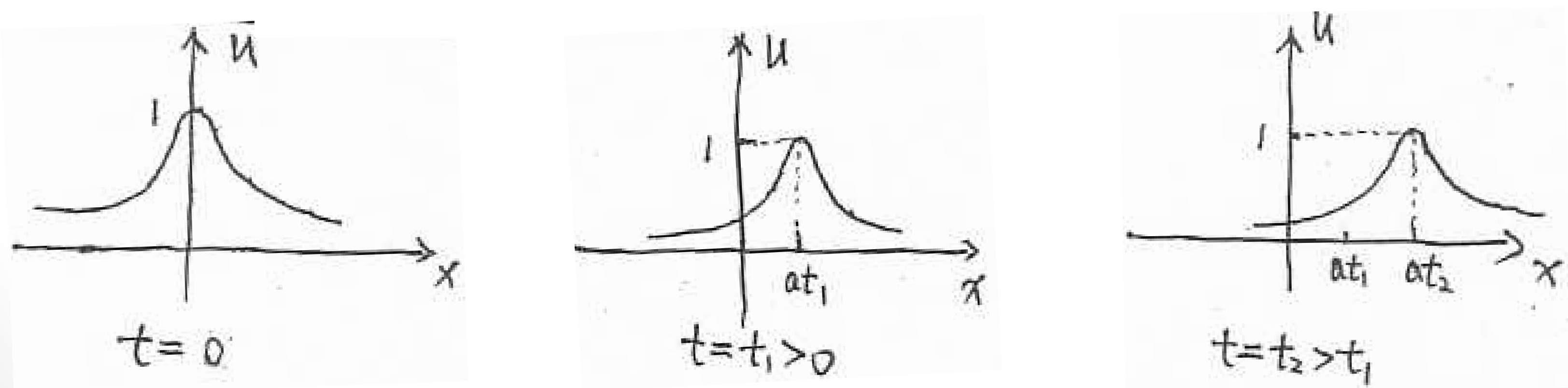
An example

$$u_0(x) = \frac{1}{x^2 + 1}$$

The solution of Eq (1) with this initial condition is

$$u(x,t) = u_0(x - at) = \frac{1}{(x - at)^2 + 1}$$

Solution u at different time (when $a > 0$):



The initial profile moves to the left (if *a>0*) with the speed *a*.

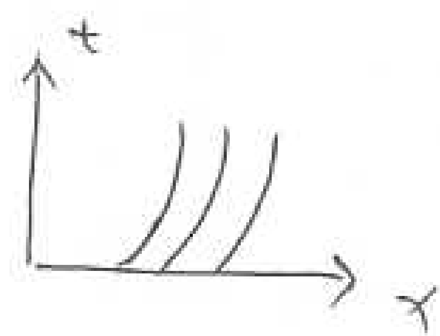
For a general linear PDE

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = 0$$

A characteristic $x(t)$ satisfies

$$\frac{dx(t)}{dt} = a(x,t).$$

It is not necessarily a straight line.



For the nonlinear conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$

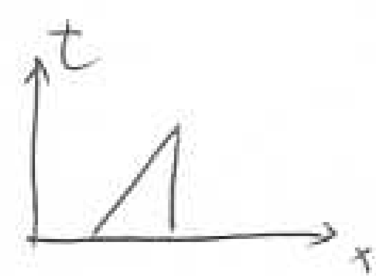
it can be written as

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0.$$

A characteristic $x(t)$ satisfies

$$\frac{dx(t)}{dt} = f'(u(x,t)).$$

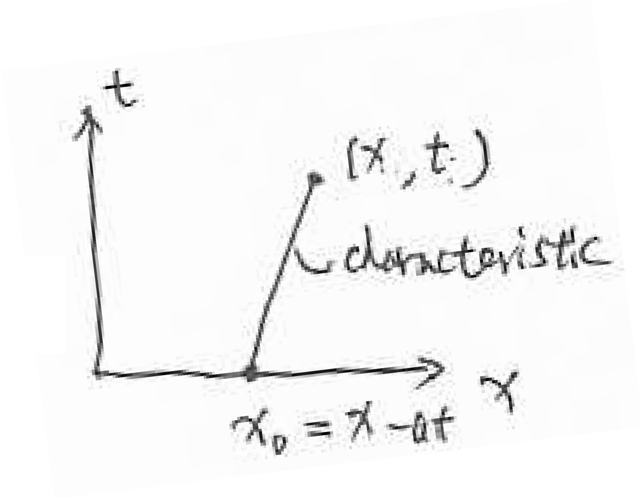
Characteristics may cross.



2.2. Domain of dependence of the exact solution

We focus on the linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$



where *a* is a constant.

Solution:

$$u(x,t) = u_0(x_0) = u_0(x - at)$$

The solution depends on the initial value u_0 at $x - at$.

Information propagation speed =a, which is finite.

Domain of dependence

The set of all points in the space where the initial data at *t=0* may have some effect on the solution *u(x,t)*.

For

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

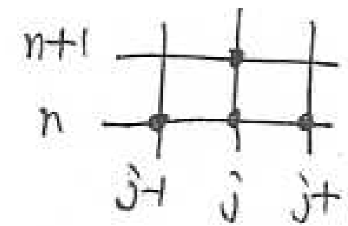
the domain of dependence of the solution *u(x,t)* is: *{x-at}*.

2.3. Some numerical schemes

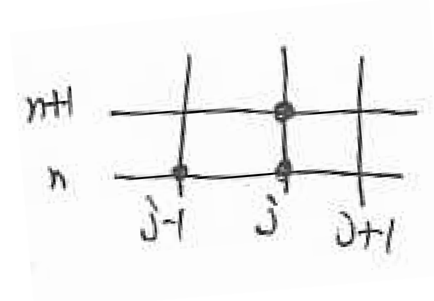
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Stencil

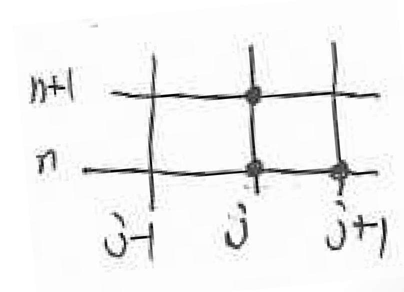
(1) $$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0$$



(2) $$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0$$



(3) $$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^n - U_j^n}{\Delta x} = 0$$



Let $\nu = \dfrac{\Delta t}{\Delta x}$.

Note: It is different from $\mu = \dfrac{\Delta t}{(\Delta x)^2}$ for the diffusion equation.

These schemes can be written as

$$(1)\quad U_j^{n+1} = U_j^n - \frac{a\nu}{2}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$(2)\quad U_j^{n+1} = U_j^n - a\nu\left(U_j^n - U_{j-1}^n\right)$$

$$(3)\quad U_j^{n+1} = U_j^n - a\nu\left(U_{j+1}^n - U_j^n\right)$$

2.4. Truncation error

(1) $$T(x_j, t_n) = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t} + a\frac{u(x_{j+1}, t_n) - u(x_{j-1}, t_n)}{2\Delta x}$$
$$= \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t + \frac{1}{6}a\frac{\partial^3 u}{\partial x^3}(\Delta x)^2 + O((\Delta t)^2) + O((\Delta x)^3).$$

First order in *t*, second order in *x*.

(2)
$$T(x_j,t_n) = \frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\Delta t} + a\frac{u(x_j,t_n) - u(x_{j-1},t_n)}{\Delta x}$$
$$= \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t - \frac{1}{2}a\frac{\partial^2 u}{\partial x^2}\Delta x + O((\Delta t)^2) + O((\Delta x)^2).$$

First order in both *t* and *x*.

(3)
$$T(x_j,t_n) = \frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\Delta t} + a\frac{u(x_{j+1},t_n) - u(x_j,t_n)}{\Delta x}$$
$$= \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t + \frac{1}{2}a\frac{\partial^2 u}{\partial x^2}\Delta x + O((\Delta t)^2) + O((\Delta x)^2).$$

First order in both *t* and *x*.

$k = \bar{k}\frac{2\pi}{L}$, $\bar{k}$ is an integer

2.5. Stability

Fourier analysis $\quad U_j^n = [\lambda(k)]^n e^{ikx_j}$

(1) $\quad \lambda(k) = 1 + ia\nu \sin k\Delta x$

$$|\lambda(k)|^2 = 1 + a^2\nu^2 \sin^2 k\Delta x$$

When $k\Delta x = \frac{\pi}{2}$, $\quad |\lambda(k)|^2 = 1 + a^2\nu^2 > 1.$

Thus $|\lambda(k)|^n$ is unbounded for large n.

This scheme (central difference scheme) is always unstable.

(2) $\lambda(k) = 1 - a\nu\left(1 - e^{-ik\Delta x}\right)$

$$|\lambda(k)|^2 = 1 - 4a\nu(1 - a\nu)\sin^2\frac{k\Delta x}{2}$$

This scheme is stable if and only if $|\lambda(k)| \leq 1$, which is equivalent to

$$a\nu(1 - a\nu) \geq 0.$$

$a > 0$, stability condition $a\nu \leq 1$.

$a < 0$, always unstable.

(3) $$\lambda(k) = 1 - a\nu\left(e^{ik\Delta x} - 1\right)$$

$$|\lambda(k)|^2 = 1 + 4a\nu(1 + a\nu)\sin^2\frac{k\Delta x}{2}$$

$a > 0$, always unstable.

$a < 0$, stability condition $|a|\nu \leq 1$.

Therefore,

- When $a>0$, use scheme (2);
- When $a<0$, use scheme (3).

This scheme is called upwind scheme.

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

The upwind scheme is

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\Delta t} + a\dfrac{U_j^n - U_{j-1}^n}{\Delta x} = 0, & \text{if } a > 0 \\ \dfrac{U_j^{n+1} - U_j^n}{\Delta t} + a\dfrac{U_{j+1}^n - U_j^n}{\Delta x} = 0, & \text{if } a < 0 \end{cases}$$

or

$$\begin{cases} U_j^{n+1} = U_j^n - a\nu\left(U_j^n - U_{j-1}^n\right), & \text{if } a > 0 \\ U_j^{n+1} = U_j^n - a\nu\left(U_{j+1}^n - U_j^n\right), & \text{if } a < 0 \end{cases}$$

First order in both *t* and *x*.

Stability condition $|a|\nu \leq 1.$

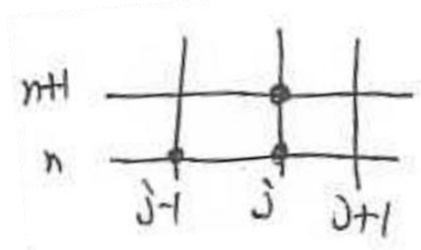
2.6. The CFL condition

Stability condition

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Consider the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0 \qquad (2)$$



$$U_j^{n+1} = U_j^n - a\nu\left(U_j^n - U_{j-1}^n\right) \qquad \nu = \frac{\Delta t}{\Delta x}$$

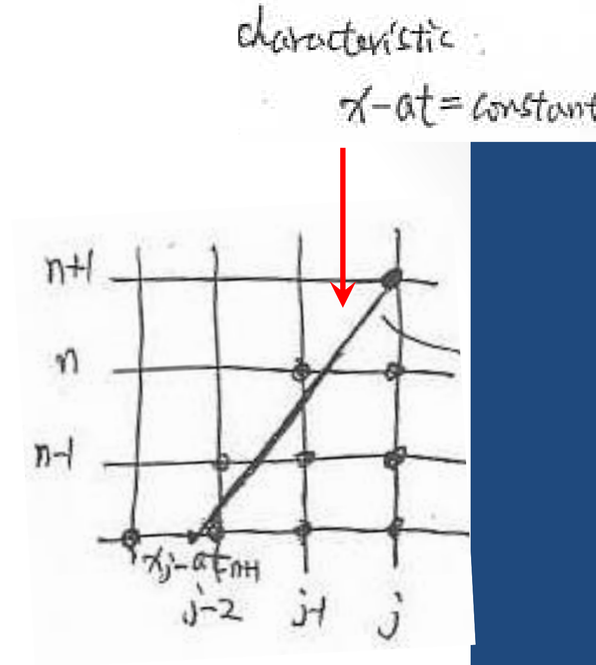
$$U_j^{n+1} = (1 - a\nu)U_j^n + a\nu U_{j-1}^n$$

U_j^{n+1} depends on U_j^n and U_{j-1}^n

U_j^n depends on U_j^{n-1}, U_{j-1}^{n-1}

U_{j-1}^n depends on $U_{j-1}^{n-1}, U_{j-2}^{n-1}$

$\Rightarrow U_j^{n+1}$ depends on $U_j^{n-1}, U_{j-1}^{n-1}, U_{j-2}^{n-1}$



$a > 0$

Repeat this consideration, we can find that

$$U_j^{n+1} \text{ depends on } U_j^0, U_{j-1}^0, U_{j-2}^0, \cdots, U_{j-n-1}^0.$$

Numerical domain of dependence of U_j^{n+1} is

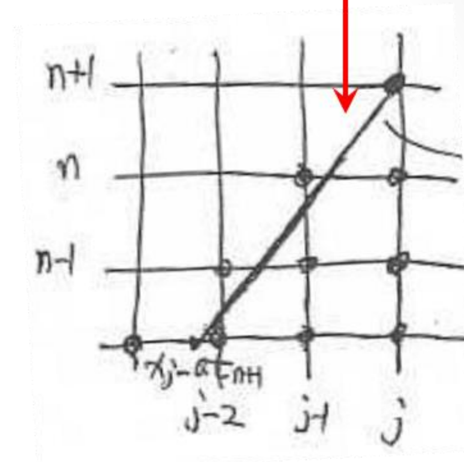
$$\{x_{j-n-1}, x_{j-n}, \cdots, x_{j-1}, x_j\}$$

which for small Δx is approximately the interval $x_{j-n-1} \leq x \leq x_j$.

characteristic:
$x - at = \text{constant}$

The exact solution $u(x_j, t_{n+1})$ depends on the value at

$$(x_j - at_{n+1}, 0).$$



The domain of dependence of $u(x_j, t_{n+1})$ is

$$\{x_j - at_{n+1}\}.$$

A **necessary** condition for stability: $a > 0$

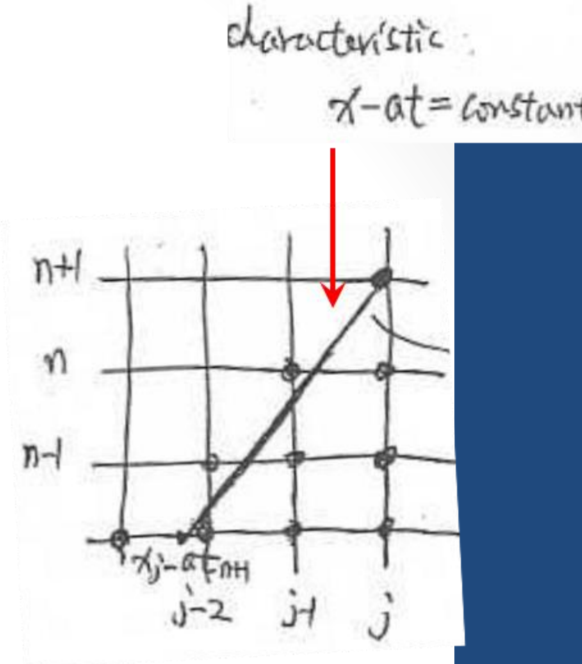
Domain of dependence is inside the numerical domain of dependence.

This is the CFL condition for stability.

CFL: Courant, Freidrichs, Lewy

The CFL condition of this scheme is $x_{j-n-1} \le x_j - at_{n+1} \le x_j$.

CFL condition $\qquad x_{j-n-1} \leq x_j - at_{n+1} \leq x_j.$

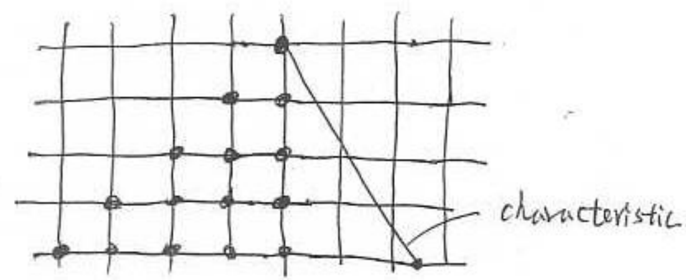


If $a > 0$, the right inequality is always true; the left inequality requires that $a\Delta t \leq \Delta x$, or $a\nu \leq 1$.

Therefore, when $a > 0$, the CFL condition is $a\nu \leq 1$.

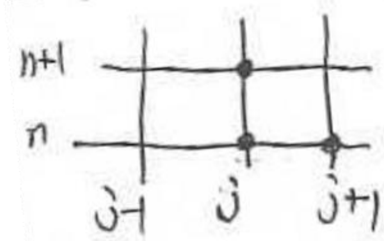
CFL condition $\quad x_{j-n-1} \le x_j - at_{n+1} \le x_j.$

If $a < 0$, the right inequality is always false, which means that this scheme is always unstable.



For the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^n - U_j^n}{\Delta x} = 0 \qquad (3)$$

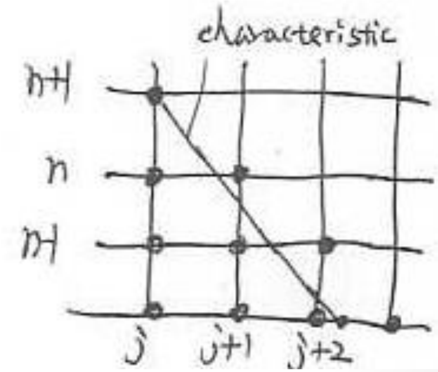


$$U_j^{n+1} = U_j^n - a\nu\left(U_{j+1}^n - U_j^n\right)$$

The domain of dependence of $u(x_j, t_{n+1})$ $\{x_j - at_{n+1}\}.$

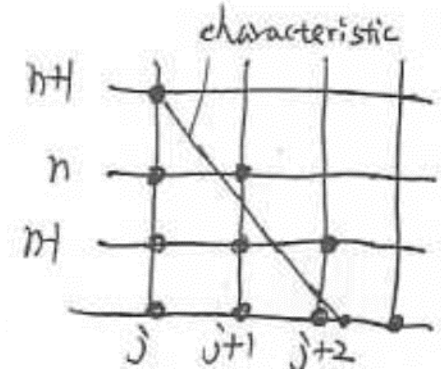
Numerical domain of dependence $x_j \leq x \leq x_{j+n+1}$

CFL condition $x_j \leq x_j - at_{n+1} \leq x_{j+n+1}$



If $a > 0$, the left inequality is always false, which means that this scheme is always unstable.

If $a < 0$, the CFL condition requires that $|a|\nu \leq 1$.



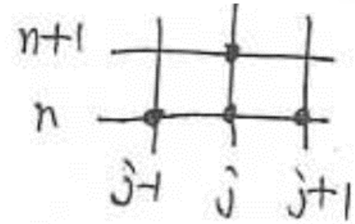
Therefore, when $a > 0$, we can use scheme (2), and when $a < 0$, we can use scheme (3). In both cases, the CFL condition is $|a|\nu \leq 1$.

This is the upwind scheme.

The CFL condition of the upwind scheme is the same as the stability condition obtained using Fourier analysis.

Now consider the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0 \qquad (1)$$

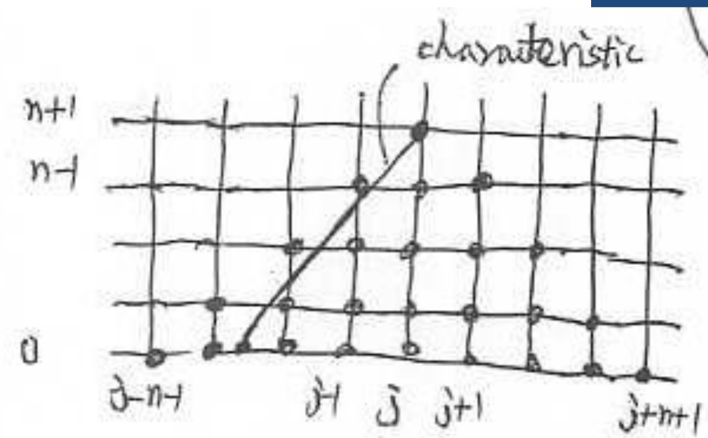


The domain of dependence of $u(x_j, t_{n+1})$

$$\{x_j - at_{n+1}\}.$$

Numerical domain of dependence

$$x_{j-n-1} \le x \le x_{j+n+1}$$



CFL condition $\quad x_{j-n-1} \le x_j - at_{n+1} \le x_{j+n+1}$

or $\qquad |a|\nu \le 1.$

The CFL condition is only a necessary condition. Using Fourier analysis, we know that this central difference scheme is always unstable.