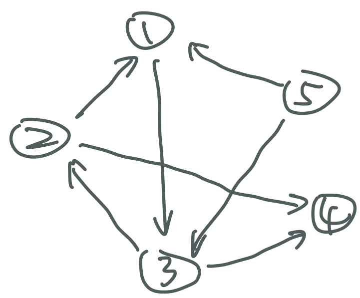
# MSDM 5056 Network Modeling
# Assignment 1 Solutions

(1) Adjacency matrix

(a) Directed. Because the adjacency matrix is asymmetric.

(b)




(c) In-degree sequence : $(2, 1, 2, 2, 0)$

Out-degree sequence : $(1, 2, 2, 0, 2)$

(d) $P_{in}(1) = \frac{1}{5}$ , $P_{in}(2) = \frac{3}{5}$

$P_{out}(1) = \frac{1}{5}$ $P_{out}(2) = \frac{3}{5}$

(2) Diameter

(a) 1

(b) 2

(c) $N-1$

(d) $2(L-1)$

(e) Since $L = \sqrt{N}$

So $D = 2(L-1) = 2(\sqrt{N}-1)$

$D \approx 2\sqrt{N}$, $(N >> 1)$

(f) $D = 3(L-1)$

Since $L = \sqrt[3]{N}$ and $N >> 1$. $D \approx 3 \cdot \sqrt[3]{N}$

(g) The number of vertices reachable in $d$ steps from the central vertex is $k(k-1)^{d-1}$ for $d \geq 1$

It can be proved using mathematical induction

Finding the diameter:

Since the total number of vertices is $n$.

So $n = \sum_{i=1}^{d} k(k-1)^{i-1} + 1$

$\Rightarrow \quad d = \log_{(k-1)} \frac{nk-2n-k+1}{k^2} + 1$

Because the 2 vertices with longest distance can reach each other in $2d$ steps

So $D = 2d = \left(\log_{(k-1)} \frac{nk-2n-k+1}{k^2} + 1\right) \times 2$

(3) Bipartite matrix

(a)

(i) $n_1 \times n_2$

(ii) $C_2^{n_1} + C_2^{n_2}$

(iii) In bipartite network, links in type1 equals to links in type 2, denoted to $L$

Moreover, $c_1 = \frac{L}{n_1}$, $c_2 = \frac{L}{n_2}$

so $c_2 = \frac{n_1}{n_2} c_1$

(b) Incident matrix:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

One-mode projections:


(4)

(a) $\langle k \rangle = \frac{1}{n} \sum_{i=1}^{n} k_i = \frac{1}{n}(n-1+n-1) = \frac{2(n-1)}{n}$

(b) Adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & & 0 & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & \ddots & 0 \\ 1 & 0 & \cdots & \cdots & & 0 \end{bmatrix}$$

Find the eigenvalues:

$$\det(A_{sn} - \lambda I) = \det(\lambda I - A_{sn}) = 0$$

$$\lambda I - A_{sn} = \begin{bmatrix} \lambda & -1 & -1 & \cdots & \cdots & -1 \\ -1 & \lambda & 0 & \cdots & \cdots & 0 \\ -1 & 0 & \lambda & & & \vdots \\ -1 & 0 & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ -1 & 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$

$$\det(\lambda I - A_{sn}) = -a_{21}C_{21} + a_{22}C_{22}$$

We can see from the matrix. $a_{21} = -1$, $a_{22} = \lambda$

$C_{22}$ is the determinant of $A_{s(n-1)}$ and $C_{21}$ is the det of

$$B = \begin{bmatrix} -1 & -1 & \cdots & \cdots & -1 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

So $\det B = -\lambda^{n-2}$

Therefore:

$$\det(\lambda I - A_{sn}) = (-1)(-\lambda^{n-2}) + \lambda \det(\lambda I - A_{s(n-1)})$$

$$= \lambda \det(\lambda I - A_{s(n-1)}) - \lambda^{n-2}$$

Letting $F_n = \det(\lambda I - A_{sn})$

So
$$F_n = \lambda F_{n-1} - \lambda^{n-2}$$
$$= \lambda(\lambda F_{n-2} - \lambda^{n-3}) - \lambda^{n-2}$$
$$= \lambda^2 F_{n-2} - \lambda^{n-2} - \lambda^{n-2}$$
$$= \lambda^2 F_{n-2} - 2\lambda^{n-2}$$

$$= \lambda^2(\lambda F_{n-3} - \lambda^{n-4}) - 2\lambda^{n-2}$$
$$= \lambda^3 F_{n-3} - \lambda^{n-2} - 2\lambda^{n-2}$$
$$= \lambda^3 F_{n-3} - 3\lambda^{n-2}$$
$$= \lambda^{n-3} F_3 - (n-3)\lambda^{n-2}$$

$$F_3 = \det(\lambda I - A_{S_3}) = \det\begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & 0 \\ -1 & 0 & \lambda \end{bmatrix}$$

$$= \lambda^3 - 2\lambda$$

So $F_n = \lambda^{n-3}(\lambda^3 - 2\lambda) - (n-3)\lambda^{n-2}$

$$= \lambda^n - 2\lambda^{n-2} - n\lambda^{n-2} + 3\lambda^{n-2}$$
$$= \lambda^n - (n-1)\lambda^{n-2}$$

Letting $F_n = 0$

$$\lambda^n - (n-1)\lambda^{n-2} = \lambda^{n-2}(\lambda^2 - (n-1)) = 0$$

So $\lambda^2 - (n-1) = 0$ or $\lambda^{n-2} = 0$

$\lambda = \pm\sqrt{n-1}$ or $\lambda = 0$

So, the eigenvalue $\lambda_1 = \sqrt{n-1}$, $\lambda_2 = -\sqrt{n-1}$. $\lambda_3 = 0$

(C)

$$L = D - A = \begin{bmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_n \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & & 0 \\ \vdots & \vdots & & 0 & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 & -1 & -1 & \cdots & -1 \\ -1 & k_2 & 0 & \cdots & 0 \\ -1 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & \\ -1 & 0 & \cdots & & k_n \end{bmatrix} = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -1 & 0 & \cdots & & 1 \end{bmatrix}$$

Find the eigenvalue :

$$\lambda I - A_{Sn} = \begin{bmatrix} \lambda-n+1 & 1 & 1 & \cdots & 1 \\ 1 & \lambda-1 & 0 & \cdots & 0 \\ 1 & 0 & \lambda-1 & & \vdots \\ \vdots & & & \ddots & \\ 1 & 0 & \cdots & & \lambda-1 \end{bmatrix}$$

$$\det(\lambda I - A_{Sn}) = -a_{21}C_{21} + a_{22}C_{22} = 0$$

From the matrix, we know $a_{21} = 1$, $a_{22} = \lambda - 1$

$C_{22}$ is det of $\lambda I - A_{S(n-1)}$ and $C_{21}$ is det of matrix

$$B = \begin{bmatrix} 1 & 1 & \cdots & & 1 \\ 0 & \lambda-1 & 0 & & 0 \\ 0 & 0 & \lambda-1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & & \lambda-1 \end{bmatrix}$$

So $\det B = (\lambda-1)^{n-2}$

Therefore $\det(\lambda I - A_{Sn}) = -(\lambda-1)^{n-2} + (\lambda-1)\det(\lambda I - A_{S(n-1)})$

Letting $F_n = \det(\lambda I - A_{Sn})$

So $F_n = (\lambda-1)F_{n-1} - (\lambda-1)^{n-2}$

$$= (\lambda-1)\left[(\lambda-1)F_{n-2} - (\lambda-1)^{n-3}\right] - (\lambda-1)^{n-2}$$

$$= (\lambda-1)^2 F_{n-2} - (\lambda-1)^{n-2} - (\lambda-1)^{n-2}$$

$$= (\lambda-1)^2 F_{n-2} - 2(\lambda-1)^{n-2}$$

$$= (\lambda-1)^{n-3} F_3 - (n-3)(\lambda-1)^{n-2}$$

$$F_3 = \det(\lambda I - A_{S_3}) = \det \begin{bmatrix} \lambda-n+1 & 1 & 1 \\ 1 & \lambda-1 & 0 \\ 1 & 0 & \lambda-1 \end{bmatrix}$$

$$= (\lambda-n+1)(\lambda-1)^2 - (\lambda-1) - (\lambda-1)$$

$$= (\lambda-1)[(\lambda^2-1) - n(\lambda-1) - 2]$$

So $F_n = (\lambda-1)^{n-3}[(\lambda-1)[(\lambda^2-1) - n(\lambda-1) - 2] - (n-3)(\lambda-1)^{n-2}$

$$= (\lambda-1)^{n-2}[(\lambda^2-1) - n(\lambda-1) - 2] - (n-3)(\lambda-1)^{n-2}$$

$$= (\lambda-1)^{n-2}[(\lambda^2-1) - n(\lambda-1) - n + 1]$$

$$= (\lambda-1)^{n-2} \cdot (\lambda^2 - n\lambda) = 0$$

So $\lambda - 1 = 0$ or $\lambda^2 - n\lambda = 0$

$\lambda = 1$ or $\lambda = 0$ or $\lambda = n$

So the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = n$

Many eigenvalue equals to 1 means that the graph is probably sparsely connected.