Appendix 4

Laplacian matrix of a graph

The discrete Laplacian on a network and acting on a function ϕ is given by

$$\Delta\phi(v) = \sum_{w \in \mathcal{V}(v)} (\phi(w) - \phi(v)) \,. \tag{A4.1}$$

From this definition, one introduces the Laplacian matrix of a graph given by $\mathbf{L} = -\mathbf{\Delta}$, which can be rewritten as

$$\mathbf{L} = \mathbf{D} - \mathbf{X} \tag{A4.2}$$

where $\mathbf{D}$ is the diagonal degree matrix with elements $D_{ij} = \delta_{ij} k_i$ and $\mathbf{X}$ is the adjacency matrix. The Laplacian matrix has thus diagonal elements equal to the degree $L_{ii} = k_i$ and is the opposite of the adjacency matrix for off diagonal elements $L_{i \neq j} = -x_{ij}$. It is therefore symmetric if the graph is undirected. This matrix is a central concept in spectral graph analysis (Mohar, 1997). For undirected graphs, some important properties appear, such as the following:

- If the graph is an infinite square lattice grid, this definition of the Laplacian can be shown to correspond to the continuous Laplacian.
- The Laplacian matrix L being symmetric, has real positive eigenvalues $0 \leq \lambda_1 \leq \ldots \leq \lambda_N$.
- The multiplicity of 0 as an eigenvalue of L is equal to the number of connected components of the graph.
- The second smallest eigenvalue is called the algebraic connectivity. It is non-zero only if the graph is formed of a single connected component. The magnitude of this value reflects how well connected the overall graph is, and has implications for properties such as synchronizability and clustering.

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