MSDM 5056 Network Modeling

Assignment 1 (due 27th September, 2023)

Submit your assignment solution on canvas. You may discuss with others or seek help from your TA, but should not directly copy from others. Otherwise, it will be considered as plagiarism.

(1) **Adjacency matrix.**

Consider the following adjacency matrix of a network

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

a) Is the network directed or undirected? (Explain why).

b) Draw the network.

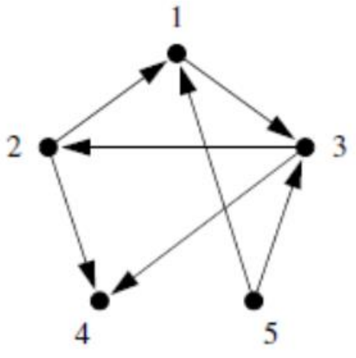
c) List the in-degree sequence and the out-degree sequence of the network

d) Determine the in-degree distribution and the out-degree distribution.

Solution:

a) The network is directed because the adjacency matrix is asymmetric.

b)



c) The in-degree sequence is given by {2; 1; 2; 2; 0}.

The out-degree sequence is given by {1; 2; 2; 0; 2}.

d) The in degree distribution is given by P(0) = 1/5; P(1) = 1/5; P(2) =3/5, and $P(k)$ = 0 for k > 2.

The out-degree distribution is given by P(0) = 1/5; P(1) = 1/5; P(2) =3/5, and $P(k)$ = 0 for k > 2.

(2) **Diameter.**

One can calculate the diameter of certain types of network exactly. Assume that each of the following networks has network size *N*.

a) What is the diameter of a fully connected network?

b) What is the diameter of a star network?

c) What is the diameter of a linear chain of *N* nodes? (Figure 1 below)

d) What is the diameter *D* of a square portion of square lattice, with *L* nodes along each side (figure 2 below)?

e) Consider the expression found in (d), show that the leading term of D in terms of the total number of nodes *N* in the network, in the limit *N* >> 1

$$D \approx 2\sqrt{N}$$

f) What is the diameter of the corresponding hypercubic lattice in *d* dimensions (*d* =3 corresponds to a cubic lattice) with *L* nodes along each side, in the limit *L* >> 1 (hence *N* >> 1)?

(g) A Cayley tree is a symmetric regular tree in which each vertex is connected to the same number *k* of others, until we get out to the leaves, as shown in Figure 3. (*k* = 3 in this case.) Show that the number of vertices reachable in *d* steps from the central vertex is $k(k-1)^{d-1}$ for $d \geq 1$. Hence find an expression for the diameter of the network in terms of k and the number of vertices *n*.

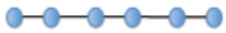


Figure 1

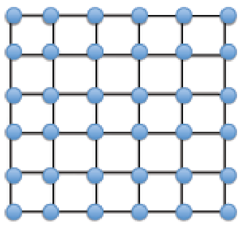


Figure 2

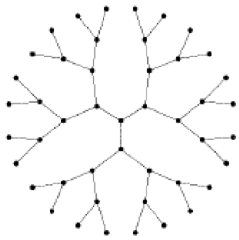


Figure 3

Solution:

a) Since a fully connected network is a network in which every node is liked to every other node, any two nodes are therefore at distance *d* = 1. The diameter is *D* = 1.

b) A star network is a network in which a central node is linked to *N*-1 leaves nodes of degree 1. Therefore the diameter of the star network is D = 2.

c) In a chain of *N* nodes the maximum distance is the distance between the two nodes at the edges of the chain. Therefore *D* = *N*-1.

d) In a square portion of a square lattice, the most distant pair of points are the ones formed by the nodes at the two opposite corners of the square lattice. These nodes are at distance *d* = 2(*L*-1), therefore the diameter of the network is *D* = 2(*L*-1).

e) Since the number of nodes $N = L^2$, and we have *D* = 2(*L*-1). In the limit *N*>>1, we have $D = 2(\sqrt{N} - 1) \cong 2\sqrt{N}$.

f) In a similar fashion, since $\mathrm{L} = \sqrt[d]{N}$, we have $D = \mathrm{d}(L - 1) \approx \mathrm{d}\sqrt[d]{N}$.

g) In the first step, there are *k* vertices connected to the central vertex. In the next step, each of the *k* vertices can connect to (*k* – 1) vertices, and in the third step, there will be (k – 1) that connect to each of the *k* (*k* – 1), which gives $k(k-1)^2$. In the *d* step, there will be k vertices in the outer most layer. The total number of vertices from the central vertex to the outer most layer is a sum of

$$n = 1 + k\sum_{i=1}^{d}(k-1)^{i-1} = 1 + \frac{k[(k-1)^d - 1]}{(k-1) - 1}$$

which gives,

$$d = \frac{\ln[n(k-2) + 2] - \ln k}{\ln(k-1)}$$

When k = 2, it will become a chain like Figure 1, and one can use e.g., L'Hopital's rule to verify that n = 1 + 2*d*; or *d* = (*n* – 1)/2. The diameter is equal to 2*d*.

(3) **Bipartite matrix.**

a) Consider a bipartite network with its two types of vertices, and supposed that there are n_1 vertices of type 1 and n_2 vertices of type 2.

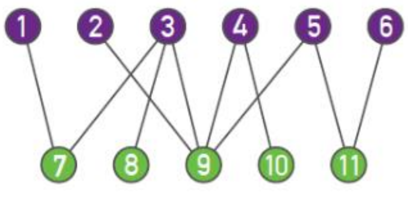
i) What is the maximum number of links L_{max} the network can have?

ii) How many links cannot occur compared to a non-bipartite network of size $n = n_1 + n_2$?

iii) Show that the mean degrees c_1 and c_2 of the two types are related by

$$c_2 = \frac{n_1}{n_2} c_1 .$$

b) Consider the following undirected bipartite network.



Find the incidence matrix of this bipartite network. Then draw the one-mode projections of the bipartite network.

Solution:

a) i) The maximum number of links is $L_{max} = n_1 \times n_2$.

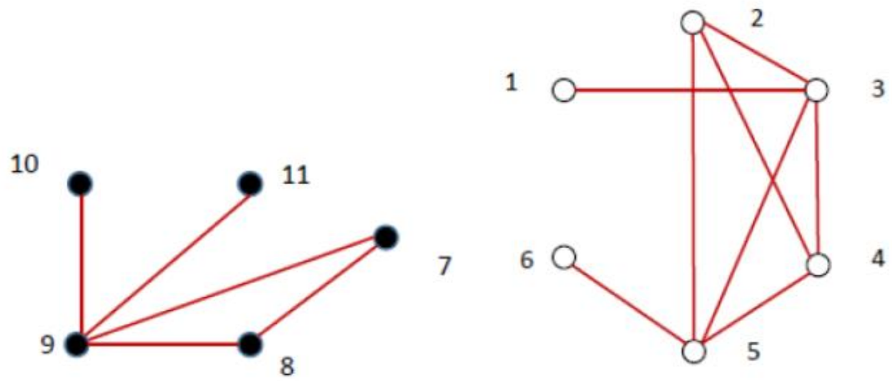
ii) The maximum number of links for a non-bipartite network is $L_{max} = \frac{1}{2}(n_1 + n_2)(n_1 + n_2 - 1)$. Therefore, the difference is $\frac{1}{2}[n_1(n_1 - 1) + n_2(n_2 - 1)]$.

iii) Since the total number of degree of the two types are equal, i.e., $\sum_i k_i = n_1 c_1 = n_2 c_2$, we therefore have $c_2 = \frac{n_1}{n_2} c_1$.

b) The incidence matrix B is

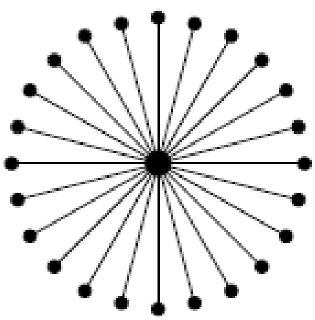
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c) The one mode projections are shown below.



(4) **Spectrum of star graphs**

A star graph consists of a single central node with $n - 1$ other nodes connected to it



(a) What is the average degree of the above star graph?

(b) Find the eigenvalues of the adjacency matrix of the star graph.

(c) Now find the eigenvalues of the Laplacian of the above network. What do its eigenvalues tell you about the graph? (*Refer to p.56 of Lecture 2*)

Solution:

(a) The average degree of a star graph is $2\times\frac{n-1}{n} = 2-\frac{2}{n}$.

(b) The adjacency matrix takes the form

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The eigenvalues are obtained by finding the roots of the determinant

$$\begin{vmatrix} -\lambda & 1 & 1 & \cdots & 1 \\ 1 & -\lambda & 0 & \cdots & 0 \\ 1 & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -\lambda \end{vmatrix}$$

which gives,

$$\lambda^n - (\mathrm{n}-1)\lambda^{n-2} = 0 \rightarrow \lambda^{n-2}\,[\lambda^{n-2} - (n-1)] = 0.$$

Therefore, the eigenvalues are $\sqrt{n-1}$, 0, $-\sqrt{n-1}$, with a multiplicity of $n-2$ for the eigenvalue equals 0.

(c) The Laplacian takes the form

$$\begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Its eigenvalues are obtained by finding the roots of the determinant

$$\begin{vmatrix} n-1-\lambda & -1 & -1 & \cdots & -1 \\ -1 & 1-\lambda & 0 & \cdots & 0 \\ -1 & 0 & 1-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1-\lambda \end{vmatrix}$$

This gives,

$$(n-1-\lambda)(1-\lambda)^{n-1} - (n-1)(1-\lambda)^{n-2} = 0 \rightarrow (1-\lambda)^{n-2}\lambda(\lambda-n) = 0.$$

The eigenvalues are n, 1, 0, with a multiplicity of $n-2$ for the eigenvalue equals 1. The second eigenvalue is equal to 1, which is larger than zero. This means the graph is a connected graph, consisting of only one single component. Notice that all the eigenvalues are larger than or equal to 0. There is only one zero eigenvalue, again meaning that there is only one component in the graph.