Mathematical Finance III
(Binomial Model and Black-Scholes-Merton Model)

MSDM 5058

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Binomial Model

Recall: Option Pricing

- How to evaluate or price options?

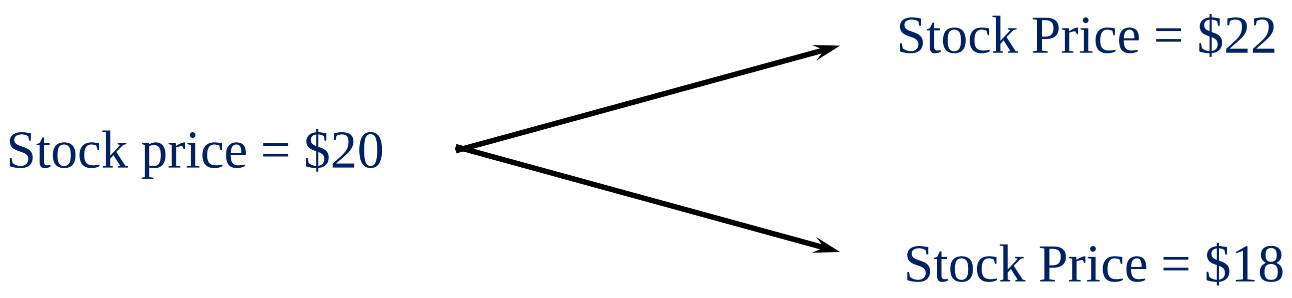
- Depends on the modeling of stock price

- A simple model of stock price is ***Binomial option pricing model (aka Binomial tree)***
  - The stock price is modeled as a ***random walk***
  - One-step model vs. multi-step model

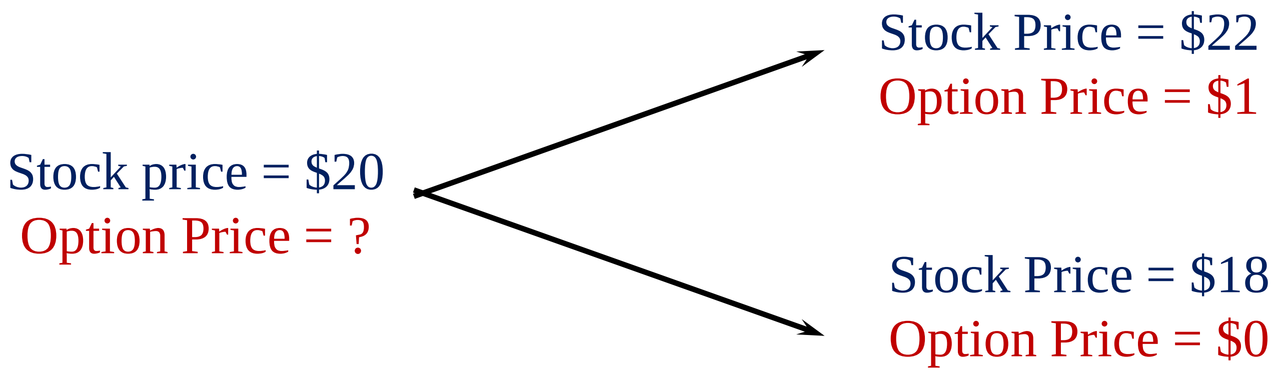
- Why is this simple model important?
  - Incorporate key insights such as no arbitrage pricing, risk-neutral pricing
  - Provide a powerful numerical option pricing method
  - Converge to the popular Black-Scholes-Merton (BSM) model

A Simple Binomial Model

- First consider a one-step binomial model
- A stock price is currently \$20
- In 3 months it will be either \$22 or \$18



- Consider a 3-month call option on the stock with the strike price of 21.



Setting Up a Riskless Portfolio

- Consider a portfolio: Long Δ shares and Short 1 call option

$$
\begin{tikzcd}[column sep=large]
 & 22\Delta - 1 \\
\bullet \arrow[ur] \arrow[dr] & \\
 & 18\Delta
\end{tikzcd}
$$

- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio and Option

Assume risk-free interest rate = 4%

- The riskless portfolio is:
  long 0.25 shares
  short 1 call option
- The value of the portfolio in 3 months is
  $22 \times 0.25 - 1 = 4.50$
- The ***value of the portfolio*** today is
  $4.5e^{-0.04\times 0.25} = 4.455$
- The value of the shares is
  5.000 ($= 0.25 \times 20$)
- The ***value of the option*** is therefore
  $5.000 - 4.455 = 0.545$

Setting up a Portfolio: Generalization and Definition

Consider a derivative that lasts for time T and is dependent on a stock . Define the following

S = present stock price ($20)

u = ratio of the stock price after 3 months to the present stock price (when price goes up, in the example, we assume 10% increase ==> 1.1)

d = ratio of the stock price after 3 months to the present stock price (when price goes down, in the example, we assume 10% decrease ==> 0.9)

Value of Option Now and at T

f = value of the option at present ($0.545)

f_u = value of the option at T (in our example, after 3 months) (when stock price goes up) ($1)

f_d = value of the option at T (in our example, after 3 months) (when stock price goes down) ($0)

Generalization

A derivative lasts for time T and is dependent on a stock

$$\begin{tikzcd}
 & \begin{matrix} S_0u \\ f_u \end{matrix} \\
\begin{matrix} S_0 \\ f \end{matrix} \arrow[ur] \arrow[dr] & \\
 & \begin{matrix} S_0d \\ f_d \end{matrix}
\end{tikzcd}$$

Value of a portfolio that is long Δ shares and short 1 derivative:

$$\begin{tikzcd}
 & S_0u\Delta - f_u \\
\cdot \arrow[ur] \arrow[dr] & \\
 & S_0d\Delta - f_d
\end{tikzcd}$$

Generalization (cont'd)

- The portfolio is riskless when $S_0u\Delta – f_u = S_0d\Delta – f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$

- Value of the portfolio at time T is $S_0u\Delta – f_u$
- Value of the portfolio today is $(S_0u\Delta – f_u)e^{–rT}$
- Another expression for the portfolio value today is $S_0\Delta – f$
- Hence

$$f = S_0\Delta – (S_0u\Delta – f_u)e^{–rT}$$

Generalization (cont'd)

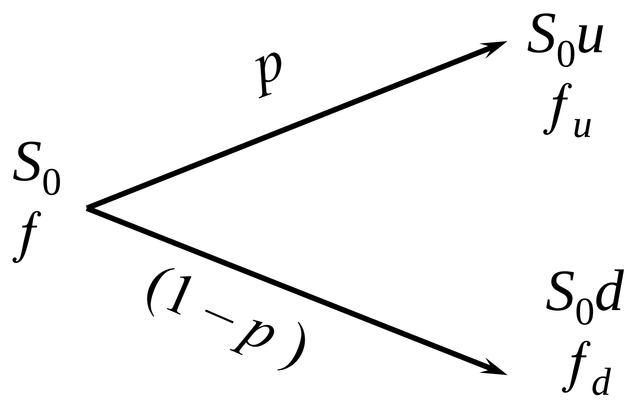
Substituting for Δ we obtain

$$f = [\ pf_u + (1 - p)f_d\ ]e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

- It is natural to interpret p and 1-p as probabilities of up and down movements
- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate



Derivation of f:

$$f = S_0\Delta - (S_0 u\Delta - f_u)e^{-rT} = S_0\Delta(1 - u\,e^{-rT}) + f_u\,e^{-rT}$$

$$= \frac{f_u - f_d}{u-d}\left(1 - ue^{-rT}\right) + f_u e^{-rT} = f_u\left[\frac{1 - ue^{-rT}}{u-d} + e^{-rT}\right] + f_d\left[\frac{ue^{-rT} - 1}{u-d}\right]$$

$$= f_u\left[\frac{1 - ue^{-rT} + ue^{-rT} - de^{-rT}}{u-d}\right] + f_d\left[\frac{ue^{-rT} - 1}{u-d}\right]$$

$$= f_u\left[\frac{1 - de^{-rT}}{u-d}\right] + f_d\left[\frac{ue^{-rT} - 1}{u-d}\right] = \left(f_u\left[\frac{e^{rT} - d}{u-d}\right] + f_d\left[\frac{u - e^{rT}}{u-d}\right]\right)e^{-rT}$$

$$= \left(f_u\left[\frac{e^{rT} - d}{u-d}\right] + f_d\left[1 - \frac{e^{rT} - d}{u-d}\right]\right)e^{-rT}$$

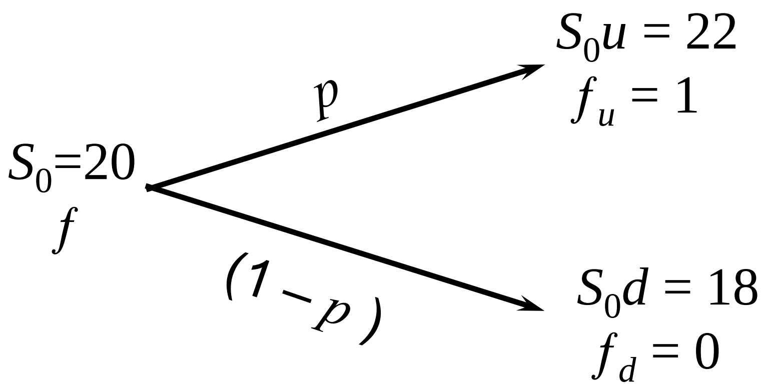
$$= [f_u\,p + f_d\,(1-p)]e^{-rT}$$

where $p = \frac{e^{rT} - d}{u-d}$

Risk-Neutral Valuation

- When the probability of an up and down movements are p and 1-p the expected stock price at time T is S_0e^{rT}
- This shows that the stock price earns the risk-free rate
- Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate
- This is known as using risk-neutral valuation

Original Example Revisited



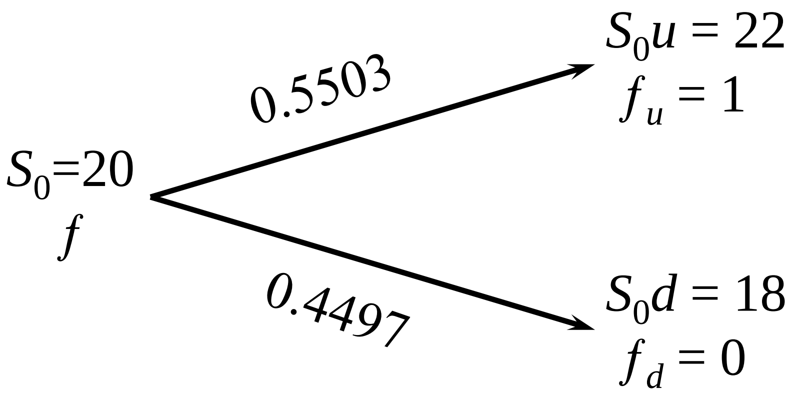
p is the probability that gives a return on the stock equal to the risk-free rate:

$20e^{0.04 \times 0.25} = 22p + 18(1 - p)$ so that $p = 0.5503$

Alternatively, we can directly use the formula for the risk-neutral probability p:

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.04 \times 0.25} - 0.9}{1.1 - 0.9} = 0.5503$$

Valuing the Option Using Risk-Neutral Valuation



Valuing the option by using

$$f = [\, pf_u + (1 - p)f_d \,]e^{-rT}$$

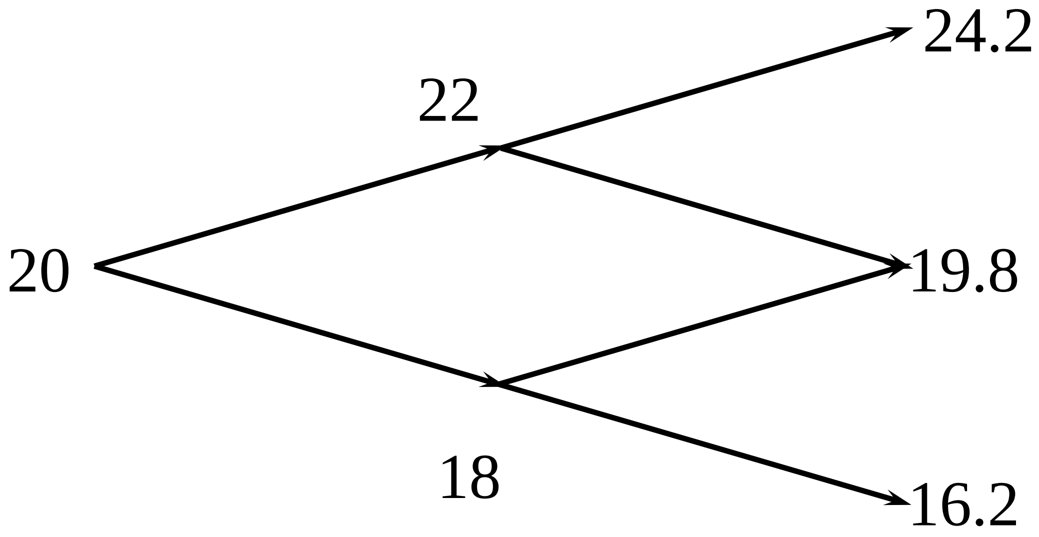
The value of the option is

$$e^{-0.04\times 0.25}(0.5503 \times 1 + 0.4497\times 0) = 0.545$$

Irrelevance of Stock's Expected Return

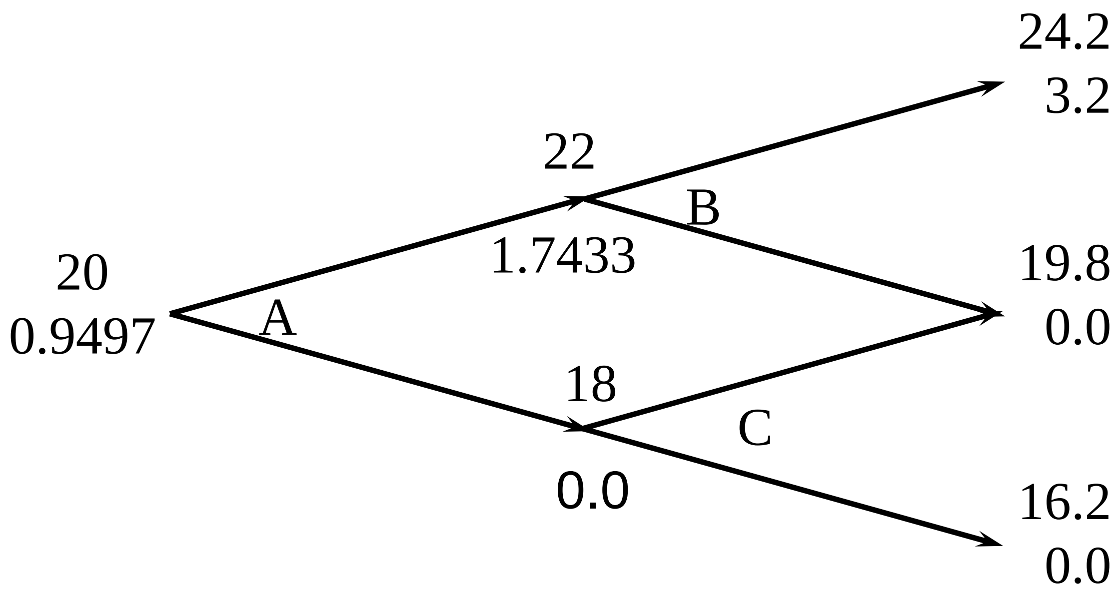
- When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant

A Two-Step Example



- K=21, r = 4%
- Each time step is 3 months

Valuing a Call Option



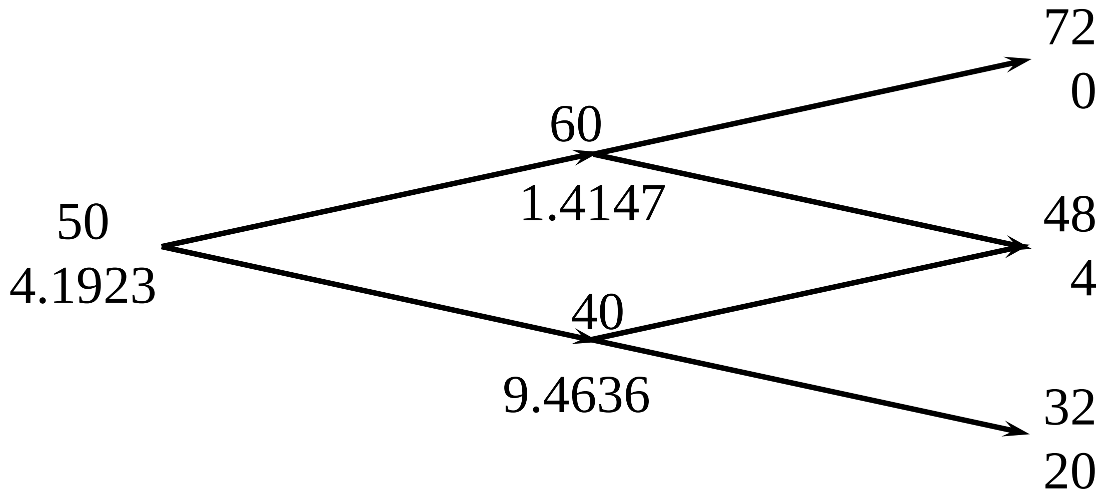
Value at node B

$= e^{-0.04\times0.25}(0.5503\times3.2 + 0.4497\times0) = 1.7433$

Value at node A

$= e^{-0.04\times0.25}(0.5503\times1.7433 + 0.4497\times0) = 0.9497$

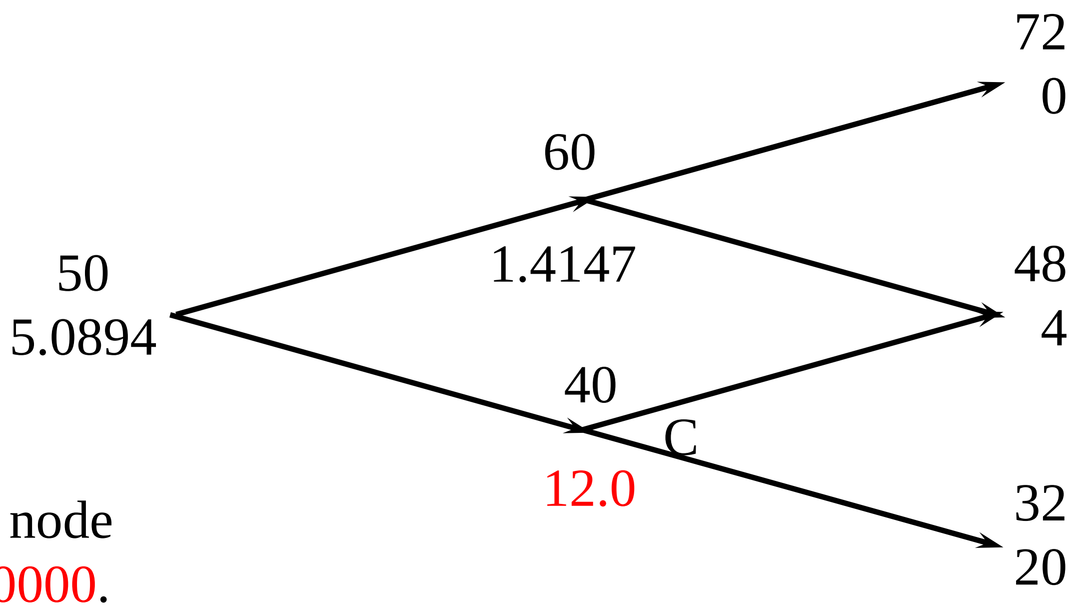
A Put Option Example



K = 52, time step =1yr

r = 5%, u =1.2, d = 0.8, p = 0.6282

What Happens When the Put Option is American



The American option increases the value at node C from 9.4636 to 12.0000.

This increases the value of the option from 4.1923 to 5.0894.

Delta

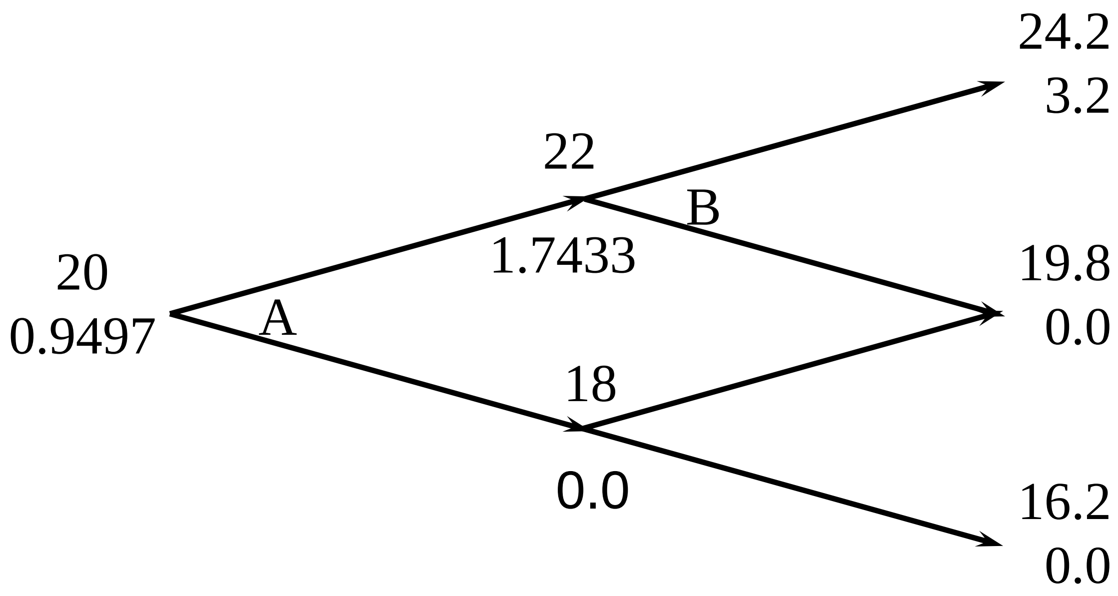
- Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

- Δ is the number of stocks we should hold to create a riskless portfolio
  - Construction of such riskless portfolio is called ***delta hedging***
  - Δ is positive for call and negative for put (why?)
- The value of Δ varies from node to node, implying that to maintain a riskless portfolio, one needs to adjust the holdings of stocks periodically

Delta (cont'd)

Let's revisit the example on p. 17



- The delta corresponding to stock price movements over the first time period is

$$(1.7433 - 0) / (22 - 18) = 0.4358$$

- The delta over the second time period is
  - ➢ $(3.2 - 0) / (24.2 - 19.8) = 0.7273$; *if an upward movement happens*
  - ➢ $(0 - 0) / (19.8 - 16.2) = 0$; *if a downward movement happens*

Choosing u and d

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$
$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the volatility and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein (1979)

The Binomial Tree Formula

When the length of the time step on a Binomial tree is Δt , we construct this Binomial tree by setting the parameters as follows.

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{a-d}{u-d} \qquad a = e^{r\Delta t}$$

For European options under the Binomial model, for the first two time steps we have

$$f = [\, pf_u + (1-p)f_d \,]e^{-r\Delta t}$$

$$f_u = [\, pf_{uu} + (1-p)f_{ud} \,]e^{-r\Delta t}$$

$$f_d = [\, pf_{ud} + (1-p)f_{dd} \,]e^{-r\Delta t}$$

$$f = [\, p^2 f_{uu} + 2p\,(1-p)f_{ud} + (1-p)^2 f_{dd} \,]e^{-2r\Delta t}$$

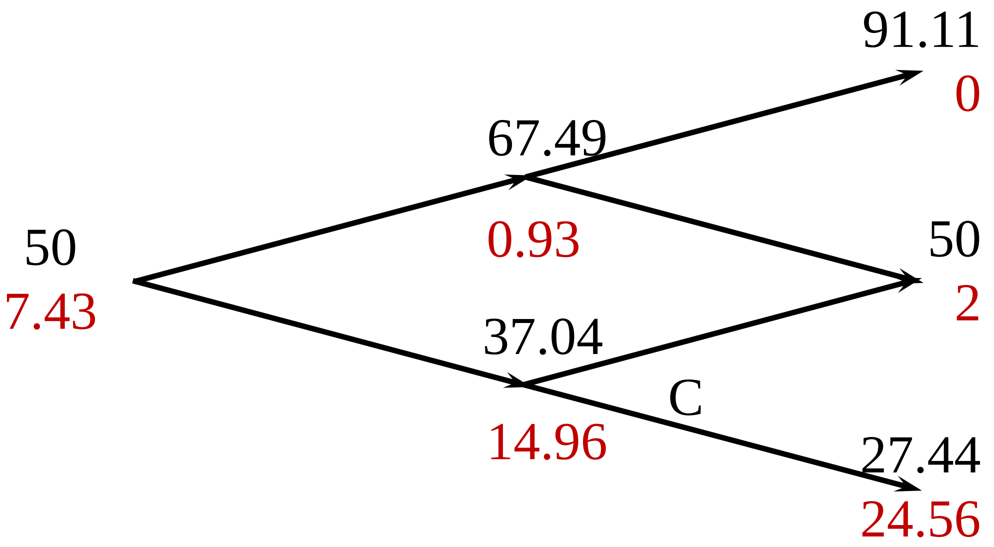
- ***Example***: Consider an American put option with parameters

$$S_0 = 50,\ K = 52,\ \Delta t = 1,\ T = 2,\ \underline{\mathrm{r}} = 0.05,\ \sigma = 0.3$$

- Then

$$u = e^{0.3\times 1} = 1.3499,\ d = 1/u = 0.7408,\ a = e^{0.05\times 1} = 1.0513$$

$$p = (1.0513 - 0.7488) / (1.3499 - 0.7408) = 0.5097.$$



Girsanov's Theorem

- Volatility is the same in the real world and the risk-neutral world
- We can therefore measure volatility in the real world and use it to build a tree for the an asset in the risk-neutral world

Assets Other than Non-Dividend Paying Stocks

For options on stock indices, currencies and futures the basic procedure for constructing the tree is the same except for the calculation of p

$$p = \frac{a - d}{u - d}$$

$a = e^{r\Delta t}$ for a non-dividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk-free rate

a = 1 for a futures contract

Wiener Processes and Itô's Lemma

Louis Bachelier
(1870-1946)

The first person to model the stochastic process (now called *Brownian motion*), which was part of his PhD thesis -- *The Theory of Speculation*, (published 1900) to evaluate stock options.

** Albert Einstein published his work on *Brownian motion* in 1905.**

Stochastic Processes

- Describes the way in which a variable such as a stock price, exchange rate or interest rate changes through time
- Incorporates uncertainties

Examples:

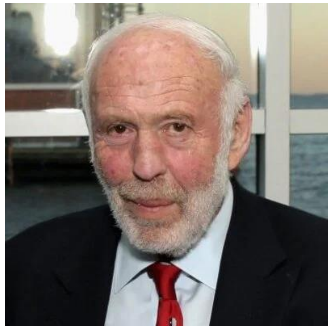
- Each day a stock price
    - increases by $1 with probability 30%
    - stays the same with probability 50%
    - reduces by $1 with probability 20%
- Each day a stock price change is drawn from a normal distribution with mean $0.2 and standard deviation $1

Markov Processes

- In a Markov process future movements in a variable depend only on where we are, not the history of how we got to where we are
- Is the process followed by the temperature at a certain place Markov?
- We assume that stock prices follow Markov processes

Markov Processes

How about
James Simons?
Net Worth $22 Billion



<u>*Weak-Form Market Efficiency:*</u>

- This asserts that it is impossible to produce consistently superior returns with a trading rule based on the past history of stock prices. In other words technical analysis does not work.
- A Markov process for stock prices is consistent with weak-form market efficiency

<u>*Example:*</u>

- A variable is currently 40
- It follows a Markov process
- Process is stationary (i.e. the parameters of the process do not change as we move through time)
- At the end of 1 year the variable will have, e.g. a normal probability distribution with mean 40 and standard deviation 10

Markov Processes

Questions:

- What is the probability distribution of the stock price at the end of 2 years?
- ½ years?
- ¼ years?
- Δt years?

Taking limits we have defined a continuous stochastic process

Variances & Standard Deviations:

- In Markov processes changes in successive periods of time are independent
- This means that variances ***are additive***
- Standard deviations ***are not additive***
- In our example it is correct to say that the variance is 100 per year.
- It is strictly speaking not correct to say that the standard deviation is 10 per year.

Wiener Processes (Brownian Motion)

- Define $\phi(m,v)$ as a normal distribution with mean m and variance v

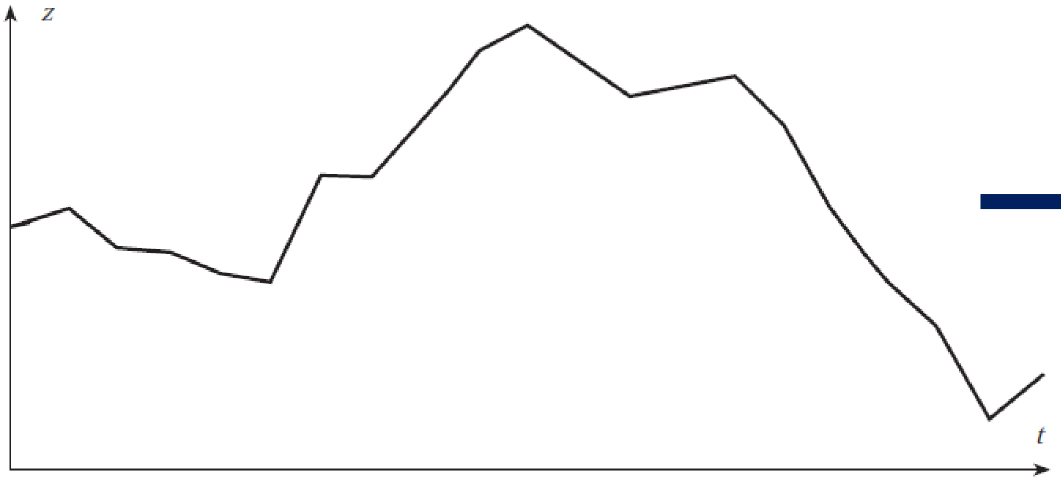
- A variable z follows a Wiener process if
  - The change in z in a small interval of time Δt is Δz
  - $\Delta z = \varepsilon\sqrt{\Delta t}$, where ε is $\phi(0,1)$
  - The values of Δz for any 2 different (non-overlapping) periods of time are independent

Properties of a Wiener Process:

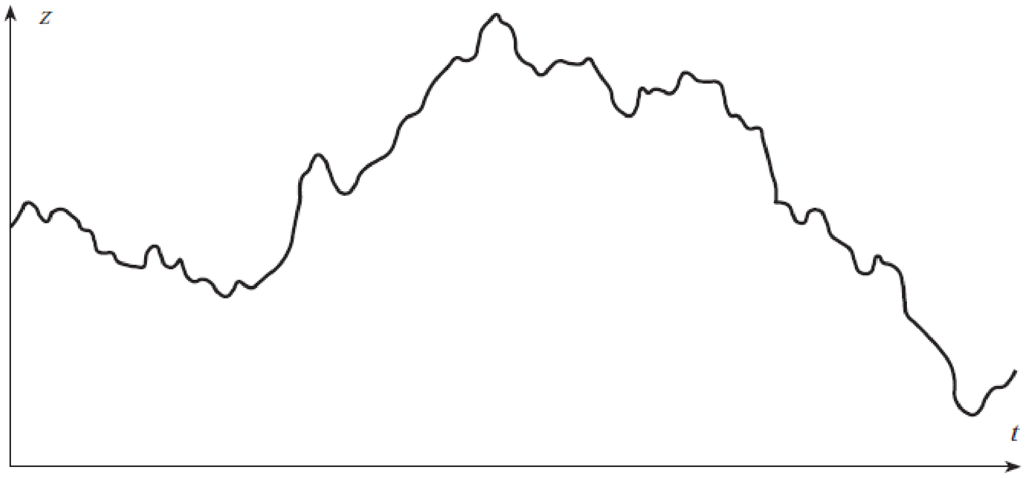
- Mean of $[z(T) - z(0)]$ is 0
- Variance of $[z(T) - z(0)]$ is T
- Standard deviation of $[z(T) - z(0)]$ is $\sqrt{T}$

Wiener Processes (cont'd)

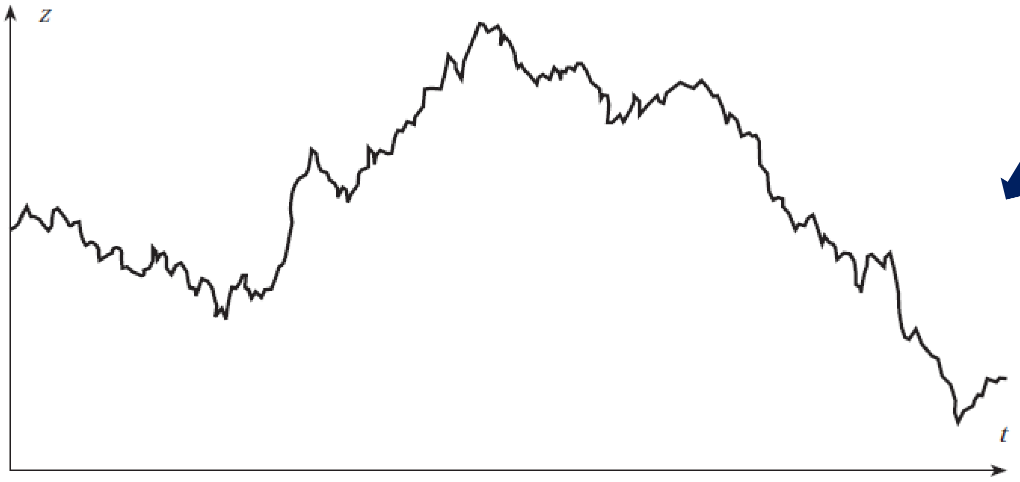
How a Wiener process is obtained when $\Delta t \rightarrow 0$



Relatively large value of Δt



Smaller value of Δt



The true process obtained as $\Delta t \rightarrow 0$

Generalized Wiener Processes

- A Wiener process has a drift rate (i.e. average change per unit time) of 0 and a variance rate of 1
- In a generalized Wiener process the drift rate and the variance rate can be set equal to any chosen constants

$$\Delta x = a\,\Delta t + b\,\varepsilon\sqrt{\Delta t}$$

- Mean change in x per unit time is a
- Variance of change in x per unit time is b^2

Generalized Wiener Processes (cont'd)

Taking Limits . . .:

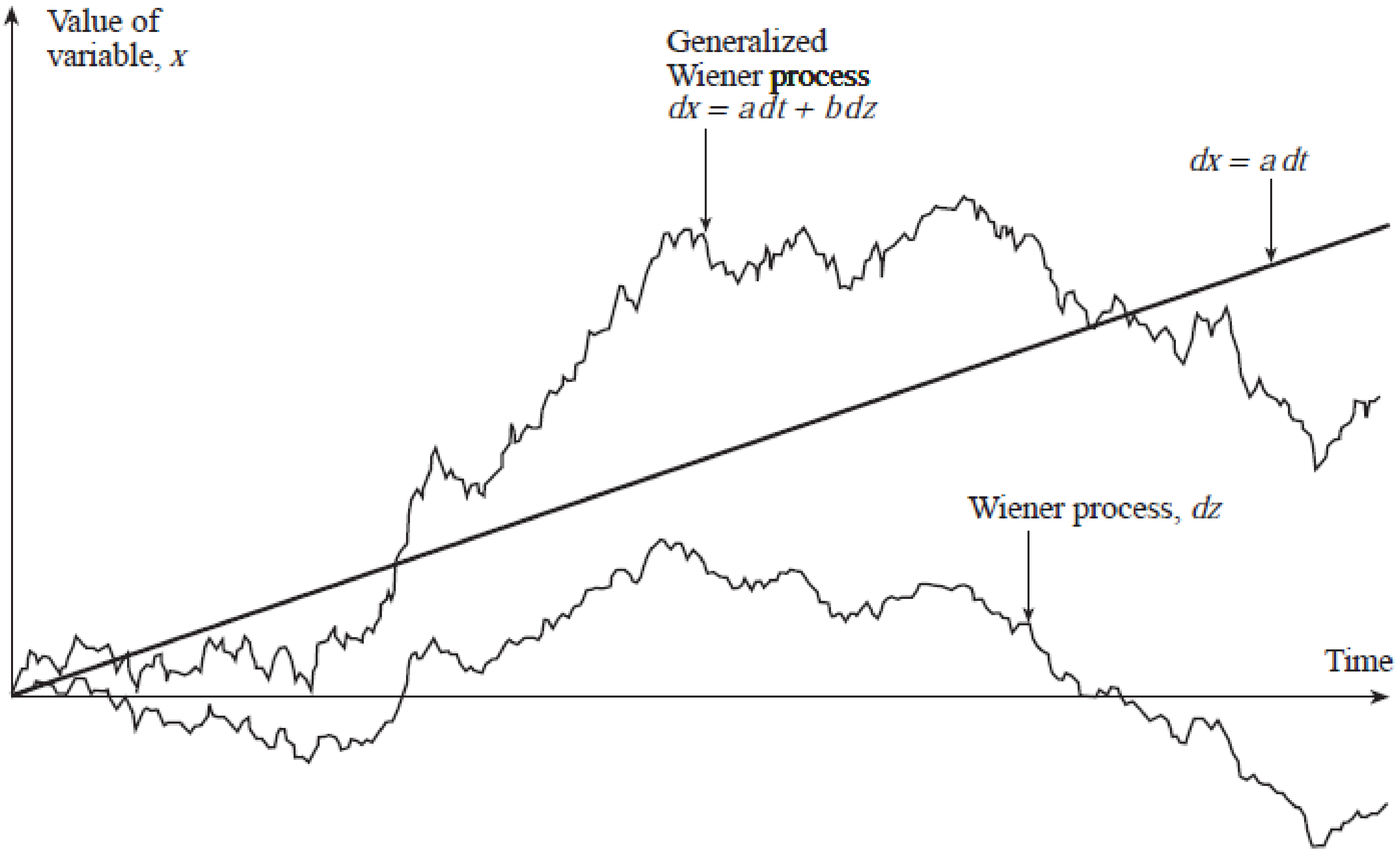
- What does an expression involving dz and dt mean?
- It should be interpreted as meaning that the corresponding expression involving Δz and Δt is true in the limit as Δt tends to zero
- In this respect, stochastic calculus is analogous to ordinary calculus

The Example Revisited:

- A stock price starts at 40 and has a probability distribution of $\phi(40,100)$ at the end of the year
- If we assume the stochastic process is Markov with no drift then the process is
- $dS = 10dz$
- If the stock price were expected to grow by \$8 on average during the year, so that the year-end distribution is $\phi(48,100)$, the process would be
- $dS = 8dt + 10dz$

Generalized Wiener Processes (cont'd)

Generalized Wiener process with a = 0.3 and b = 1.5



Generalized Wiener Processes (cont'd)

Why a Generalized Wiener Process is not appropriate for Stocks?

- For a stock price we can conjecture that its expected *percentage* change in a short period of time remains constant (not its expected actual change)

- We can also conjecture that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price

Itô Process

- In an Itô process the drift rate and the variance rate are functions of time

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

- The discrete time equivalent

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$$

is true in the limit as Δt tends to zero

Itô Process for Stock Prices

$$dS = \mu S dt + \sigma S dz$$

where μ is the expected return and σ is the volatility.

The discrete time equivalent is

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

The process is known as ***geometric Brownian motion***

Itô's Lemma

- If we know the stochastic process followed by *x*, Itô's lemma tells us the stochastic process followed by some function *G* (*x*, *t*). When $dx=a(x,t)\,dt+b(x,t)\,dz$ then

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdz$$

- Since a derivative is a function of the price of the underlying asset and time, Itô's lemma plays an important part in the analysis of derivatives

Itô's Lemma (cont'd)

Derivation:

Taylor's series expansion of $G(x, t)$ gives

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta x \Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \cdots$$

Ignoring terms of order higher than Δt, in ordinary calculus we have

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t$$

In stochastic calculus, this becomes

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2$$

Because Δx has a component of order $\sqrt{\Delta t}$

Itô's Lemma (cont'd)

Substituting for Δx, suppose $dx = a(x,t)dt + b(x,t)dz$
so that $\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$

Then ignoring terms of higher order than Δt

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\varepsilon^2\Delta t$$

For the $\varepsilon^2\Delta t$ Term, since $\varepsilon \approx \phi(0,1), E(\varepsilon) = 0$

$$E\left(\varepsilon^2\right) - [E(\varepsilon)]^2 = 1 \rightarrow E\left(\varepsilon^2\right) = 1$$

It follows that $E\left(\varepsilon^2\Delta t\right) = \Delta t$

The variance of Δt is Δt^2 and can be neglected. Hence

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\Delta t$$

Itô's Lemma (cont'd)

Taking Limits

$$\Delta G = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2 dt$$

Substituting

$$dx = a(x,t)dt + b(x,t)dz$$

We obtain

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$$

Ito's Lemma!

Itô's Lemma (cont'd)

Application of Itô's Lemma to a Stock Price Process:

The stock price process is

$$dS = \mu S dt + \sigma S dz$$

For a function *G* of *S* and *t*

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

Example:

1) The forward price of a stock for a contract maturing at time T

$$G = Se^{r(T-t)}\ ; \quad dG = (\mu - r) G dt + \sigma G dz$$

2) The log of a stock price

$$G = \ln S\ ; \quad dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz$$

The Black-Scholes-Merton Model

The Black-Scholes-Merton Model

The Stock Price Assumption:

- Consider a stock whose price is S
- In a short period of time of length Δt, the return on the stock is normally distributed:

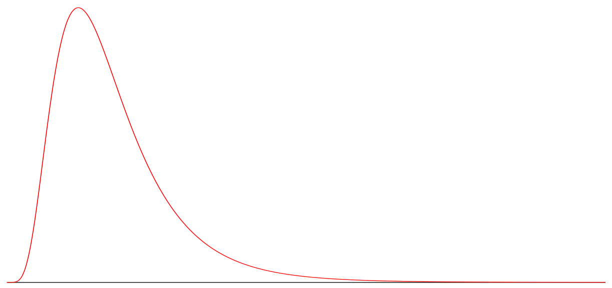
$$\frac{\Delta S}{S} = \phi\left(\mu \Delta t, \sigma^2 \Delta t\right)$$

where μ is expected return and σ is the volatility

The Black-Scholes-Merton Model (cont'd)

The Lognormal Property:

It follows from this assumption that



$$\ln S_T - \ln S_0 \approx \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

$$E[S_T] = S_0 e^{\mu T}$$

$$var[S_T] = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

or,

$$\ln S_T \approx \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

Since the logarithm of S_T is normal, S_T is lognormally distributed

The Black-Scholes-Merton Model (cont'd)

Continuously Compounded Return

If x is the realized continuously compounded return

$$S_T = S_0 e^{xT}$$

$$x = \frac{1}{T} \ln \frac{S_T}{S_0}$$

$$x \approx \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right)$$

The Black-Scholes-Merton Model (cont'd)

The Expected Return

- The expected value of the stock price is $S_0 e^{\mu T}$
- The expected return on the stock is $\mu - \sigma^2/2$ not μ

  This is because

$$\ln[E(S_T/S_0)] \text{ and } E[\ln(S_T/S_0)]$$

  are not the same

- μ is the expected return in a very short time, Δt, expressed with a compounding frequency of Δt

- $\mu - \sigma^2/2$ is the expected return in a long period of time expressed with continuous compounding (or, to a good approximation, with a compounding frequency of Δt)

The Black-Scholes-Merton Model (cont'd)

Mutual Fund Returns:

- Suppose that returns in successive years are 15%, 20%, 30%, −20% and 25% (ann. comp.)
- The arithmetic mean of the returns is 14%
- The return that would actually be earned over the five years (the geometric mean) is 12.4% (ann. comp.)
- The arithmetic mean of 14% is analogous to μ
- The geometric mean of 12.4% is analogous to $\mu - \sigma^2/2$

The Black-Scholes-Merton Model (cont'd)

The Volatility:

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year
- The standard deviation of the return in a short time period time Δt is approximately $\sigma\sqrt{\Delta t}$
- If a stock price is \$50 and its volatility is 25% per year what is the standard deviation of the price change in one day?

Estimating Volatility from Historical Data:

- Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years (e.g. for weekly data $\tau = 1/52$)
- Calculate the continuously compounded return in each interval as:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

- Calculate the standard deviation, *s*, of the u_i's
- The historical volatility estimate is

$$\hat{\sigma} = \frac{S}{\sqrt{\tau}} \qquad ; \qquad S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

The Black-Scholes-Merton Model (cont'd)

Nature of Volatility:

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed
- For this reason time is usually measured in "trading days" not calendar days when options are valued
- It is assumed that there are 252 trading days in one year for most assets

Example:

- Suppose it is April 1 and an option lasts to April 30 so that the number of days remaining is 30 calendar days or 22 trading days
- The time to maturity would be assumed to be 22/252 = 0.0873 years

The Black-Scholes-Merton Model (cont'd)

Underlying assumptions:

(1) Markets are efficient, implying that people are unable to consistently predict the direction of the market or an individual asset. Stock prices follow the continuous Ito process with μ and σ constant.

(2) The short selling of securities with full use of proceeds is permitted.

(3) There are no transaction costs or taxes. All securities are perfectly divisible.

(4) There are no dividends during the life of the derivative. European exercise terms are used; where the option can only be exercised on the expiration date

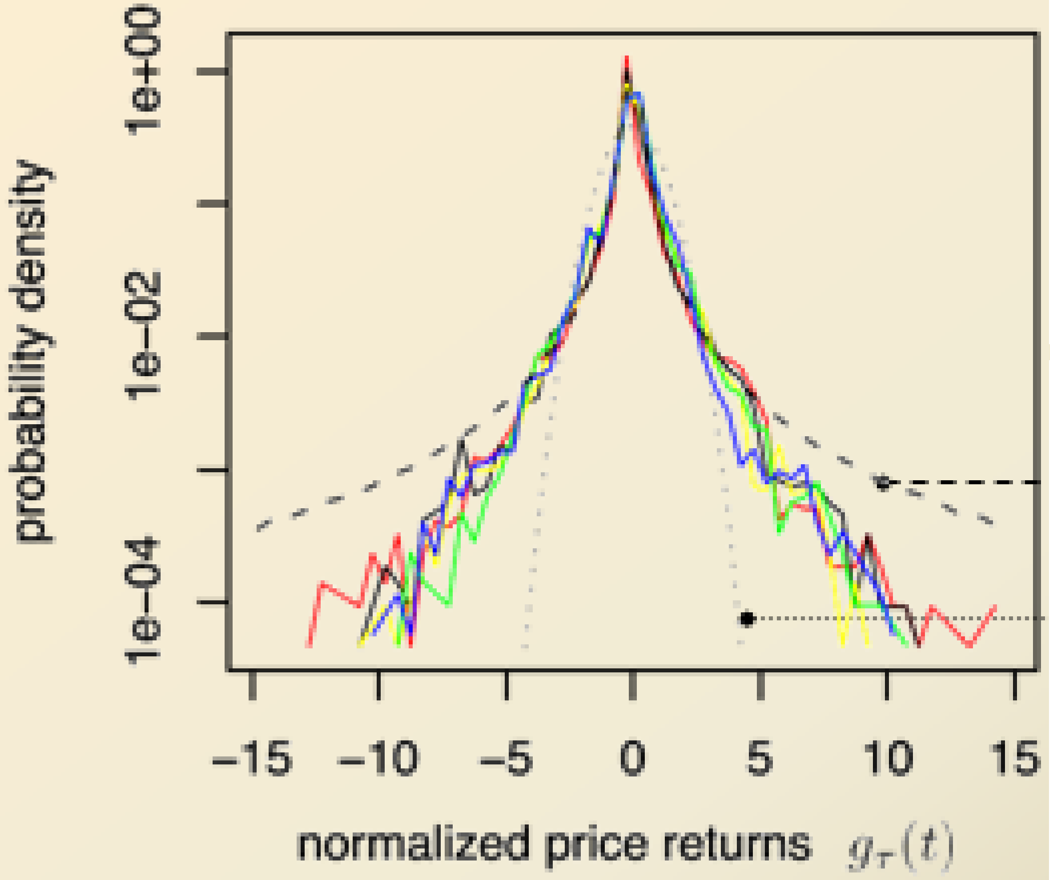
(5) There are no riskless arbitrage opportunities.

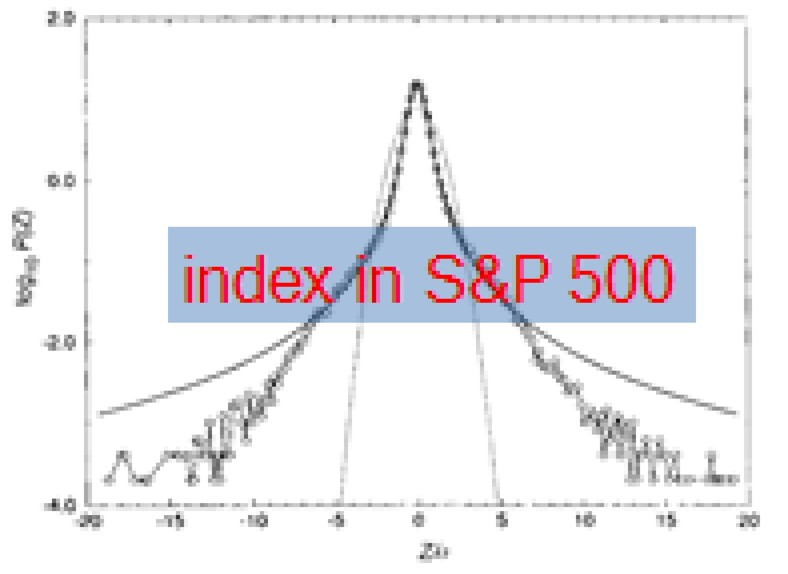
(6) Security trading is continuous.

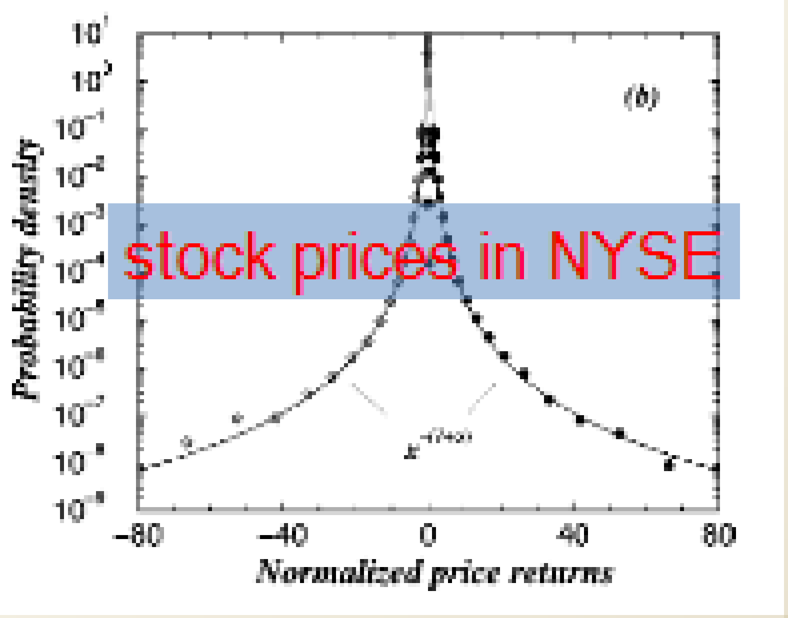
(7) The risk-free rate of interest, r, is known and remains constant for all maturities.

Recall: Heavy Tail (Stylized Fact)

Price Fluctuation







The Black-Scholes-Merton Model (cont'd)

The Concepts Underlying Black-Scholes-Merton:

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes-Merton differential equation

The Black-Scholes-Merton Model (cont'd)

Derivation of the Black-Scholes-Merton Differential Equation:

$$\Delta S = \mu S \Delta t + \sigma S \Delta z\,; \Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z$$

We set up a portfolio consisting of

$$-1\!:\text{derivative} \;\; ; \;\; +\frac{\partial f}{\partial S}\!:\text{shares}$$

This gets rid of the dependence on Δz.

The value of the portfolio Π is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

The Black-Scholes-Merton Model (cont'd)

Derivation of the Black-Scholes-Merton Differential Equation (cont'd):

The return on the portfolio must be the risk-free rate. Hence

$$\Delta\Pi = r\Pi\Delta t$$

$$-\Delta f + \frac{\partial f}{\partial S}\Delta S = r\left(-f + \frac{\partial f}{\partial S}S\right)\Delta t$$

We substitute for Δf and ΔS in this equation to get the Black-Scholes equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

The Black-Scholes-Merton Model (cont'd)

The Black-Scholes-Merton Equation:

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In a forward contract the boundary condition is $f = S - K$ when $t = T$
- The solution to the equation is

$$f = S - K e^{-r(T-t)}$$

The Black-Scholes-Merton Model (cont'd)

The Black-Scholes-Merton Formulas for Options:

$$C(S,t) = S_0 N[d_1] - Ke^{-rT} N[d_2] = e^{-rT} N[d_2] \left[\frac{S_0 e^{rT} N[d_1]}{N[d_2]} - K \right]$$

$$P = Ke^{-rT} N[-d_2] - S_0 N[-d_1]$$

where N[.] is the standard normal cumulative distribution function and

$$d_1 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \;; d_2 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

e^{-rT}: Present value factor
K: Strike price paid if option is exercised
$N[d_2]$: Probability of exercise

$\frac{S_0 e^{rT} N[d_1]}{N[d_2]}$: Expected stock price in a risk-neutral world if option is exercised

The Black-Scholes-Merton Model (cont'd)

Properties of Black-Scholes Formula:

- As S_0 becomes very large C tends to $S_0 – Ke^{-rT}$ and P tends to zero
- As S_0 becomes very small C tends to zero and P tends to $Ke^{-rT} – S_0$
- What happens as S becomes very large?
- What happens as T becomes very large?

The Black-Scholes-Merton Model (cont'd)

Risk-Neutral Valuation:

- The variable μ does not appear in the Black-Scholes-Merton differential equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation

Applying Risk-Neutral Valuation:

- Assume that the expected return from the stock price is the risk-free rate
- Calculate the expected payoff from the option
- Discount at the risk-free rate

The Black-Scholes-Merton Model (cont'd)

Valuing a Forward Contract with Risk-Neutral Valuation

- Payoff is $S_T - K$
- Expected payoff in a risk-neutral world is $S_0 e^{rT} - K$
- Present value of expected payoff is

$$e^{-rT}[S_0 e^{rT} - K] = S_0 - Ke^{-rT}$$

The Black-Scholes-Merton Model (cont'd)

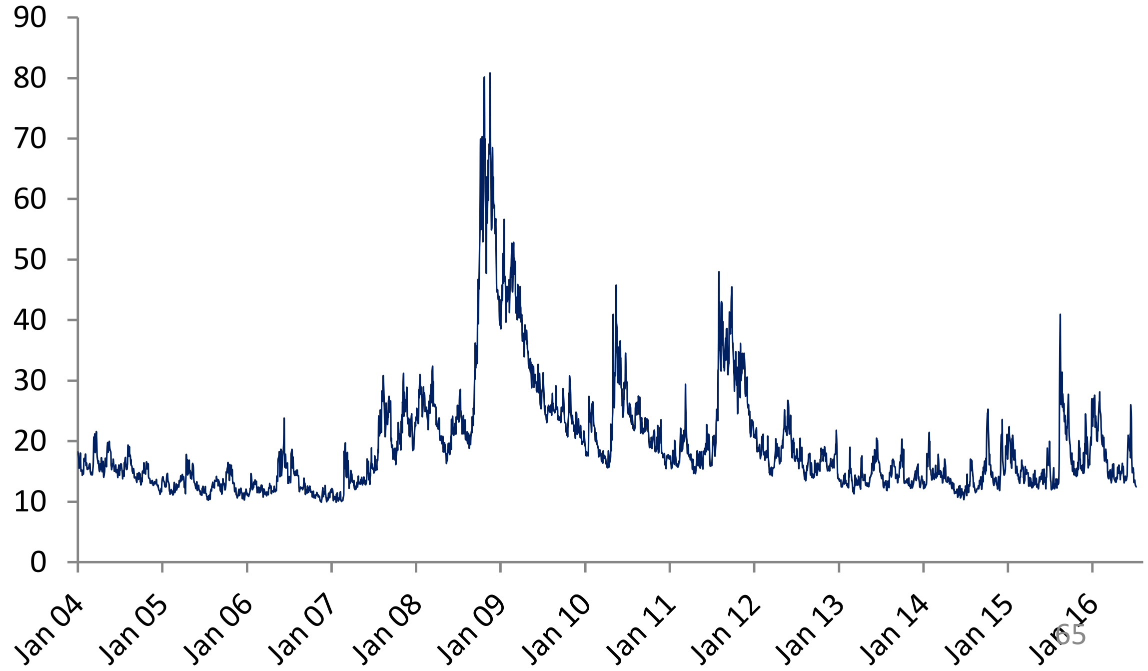
Implied Volatility

- The implied volatility of an option is the volatility for which the Black-Scholes-Merton price equals the market price
- There is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

The Black-Scholes-Merton Model (cont'd)

The VIX S&P500 Volatility Index

The CBOE publishes indices of implied volatility. The most popular index, the SPX VIX, is an index of the implied volatility of 30-day options on the S&P 500 calculated from a wide range of calls and puts. It is sometimes referred to as the ***"fear factor."*** An index value of 15 indicates that the implied volatility of 30-day options on the S&P 500 is estimated as 15%.



Black-Scholes Equation (Physics Viewpoint)

From Black-Scholes Equation to Diffusion Equation

$$-\frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

Let $u = \ln\left(\frac{S}{c}\right)$; $\frac{\partial u}{\partial S} = \frac{1}{S}$, where c is a constant.

Define $\tilde{V}(u,t) = V(S,t)$

$$\frac{\partial \tilde{V}}{\partial S} = \frac{\partial \tilde{V}}{\partial u}\frac{\partial u}{\partial S} = \frac{1}{S}\frac{\partial \tilde{V}}{\partial u}\ ;\ \frac{\partial^2 \tilde{V}}{\partial S^2} = \frac{1}{S^2}\left(\frac{\partial^2 \tilde{V}}{\partial u^2} - \frac{\partial \tilde{V}}{\partial u}\right)$$

$$\frac{\partial \tilde{V}}{\partial t} = r\tilde{V} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial \tilde{V}}{\partial u} - \frac{1}{2}\sigma^2\frac{\partial^2 \tilde{V}}{\partial u^2}$$

Black-Scholes Equation (Physics Viewpoint)

$$\tilde{V} = e^{-r(T-t)} y(u,t)$$

$$\frac{\partial y}{\partial t} = -\left(r - \frac{\sigma^2}{2}\right)\frac{\partial y}{\partial u} - \frac{1}{2}\sigma^2 \frac{\partial^2 y}{\partial u^2}$$

$$u' = u\frac{\left(r - \sigma^2/2\right)}{\sigma^2/2} \;;\; t' = \frac{\left(r - \sigma^2/2\right)}{\sigma^2/2}(T-t)$$

$$\frac{\partial \hat{y}}{\partial t'} = \frac{\partial \hat{y}}{\partial u'} + \frac{\partial^2 \hat{y}}{\partial u'^2} \;;\ \text{with}\ \hat{y}(u',t') = y(u,t);$$

Let $z = u' + t'$; $\tilde{y}(z,t') = \tilde{y}(u'+t',t') = \hat{y}(u,t)$, gives

Diffusion Equation with unit diffusion coefficient $\rightarrow$ $$\frac{\partial \tilde{y}}{\partial t'} = \frac{\partial^2 \tilde{y}}{\partial z^2}$$ $\rightarrow$ *t'* → *it'* becomes a Schrodinger Equation