Seasonal Time Series

TS with periodic patterns, e.g., quarterly earnings

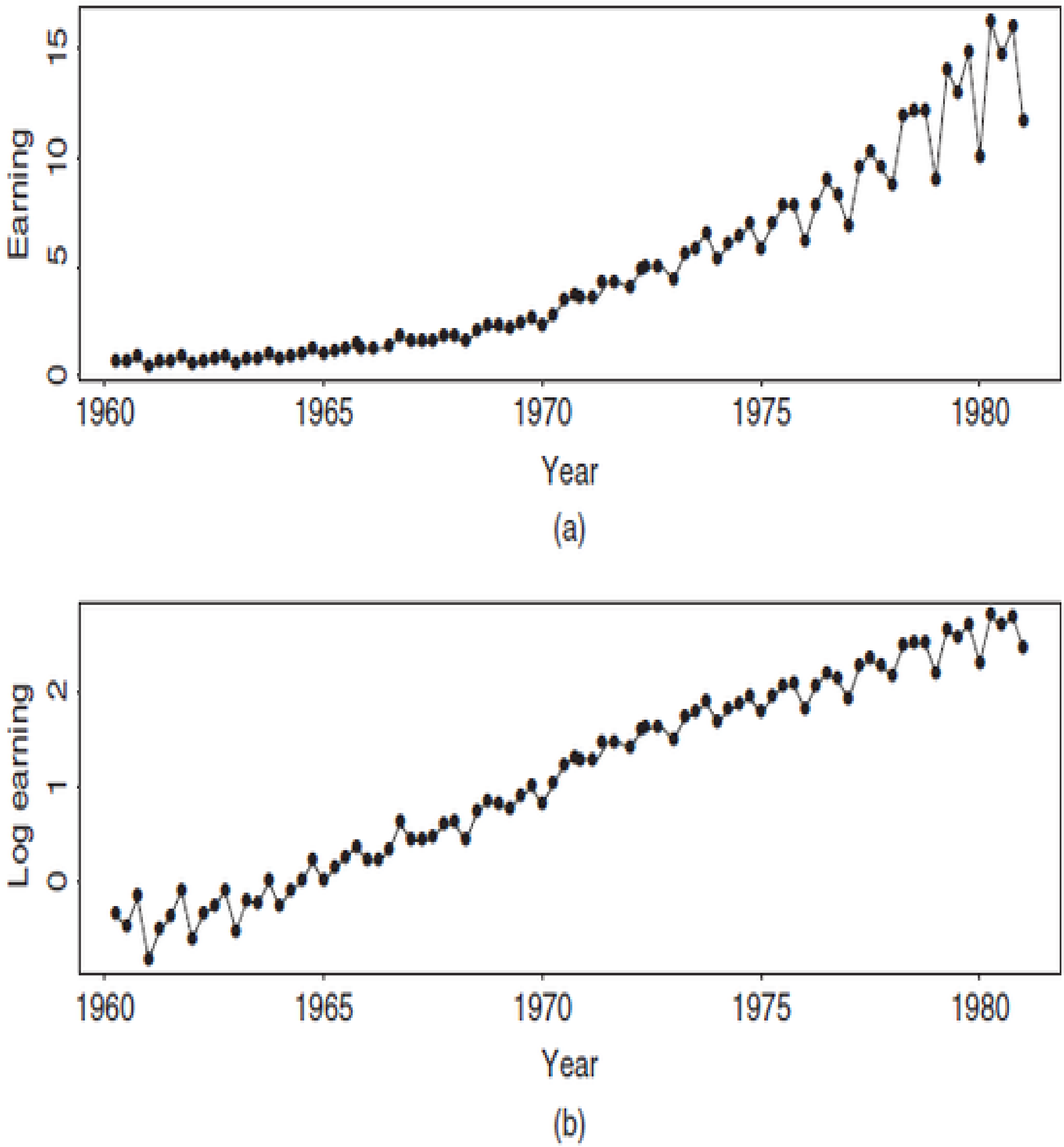

Figure 2.13 Time plots of quarterly earnings per share of Johnson & Johnson from 1960 to 1980: (a) observed earnings and (b) log earnings.

Multiplicative model

Ice cream model (for quarterly series)

Form:

$p_t - p_{t-1} - p_{t-4} + p_{t-5} = a_t - \theta_1 a_{t-1} - \theta_4 a_{t-4} + \theta_1 \theta_4 a_{t-5}$

or

$$(1-B)(1-B^4)p_t = (1-\theta_1 B)(1-\theta_4)a_t.$$

Define the differenced series w_t as

$w_t = p_t - p_{t-1} - p_{t-4} + p_{t-5} = (p_t - p_{t-1}) - (p_{t-4} - p_{t-5}).$

It is called regular and seasonal differenced series.

$$(1-B)(1-B^s)p_t = (1-\theta B)(1-\Theta B^s)a_t.$$

Define the differenced series w_t as

$w_t = p_t - p_{t-1} - p_{t-s} + p_{t-s-1} = (p_t - p_{t-1}) - (p_{t-s} - p_{t-s-1}).$

Thus,

$$w_t = (1-\theta B)(1-\Theta B^s)a_t.$$

$s---$seasonal period.

ACF of w_t has a nice symmetric structure (see the text).

Let $w_t = (1-B)(1-B^s)p_t$. Then the autocovariance of w_t can be found to be

$$
\begin{aligned}
\gamma_0 &= (1+\theta^2)(1+\Theta^2)\sigma_a^2, \\
\gamma_1 &= -\theta(1+\Theta^2)\sigma_a^2, \\
\gamma_{s-1} &= \theta\Theta\sigma_a^2, \\
\gamma_s &= -\Theta(1+\theta^2)\sigma_a^2, \\
\gamma_{s+1} &= \theta\Theta\sigma_a^2, \\
\gamma_j &= 0, \quad \text{otherwise.}
\end{aligned}
$$

The ACF becomes:

$$
\begin{aligned}
\rho_1 &= \frac{-\theta}{1+\theta^2}, \\
\rho_{s-1} &= \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)} = \rho_{s+1}, \\
\rho_s &= \frac{-\Theta}{1+\Theta^2}, \\
\rho_j &= 0, \quad \text{otherwise.}
\end{aligned}
$$

Usually, we can take $w_t = (1-B)(1-B^s)p_t$.

w_t is called (pure) seasonal ARMA$(P,Q)_s$ model if

$$\Phi_P(B^s)w_t = \phi_0 + \Theta_Q(B^s)a_t,$$

where

$$\Phi_P(B^s) = 1 - \sum_{i=1}^{P} \Phi_i B^{is},$$

$$\Theta_Q(B^s) = 1 - \sum_{i=1}^{Q} \Theta_i B^{is}.$$

Box-Jenkins multiplicative seasonal ARIMA$(p, d, q) \times (P, D, Q)_s$ model:

$$\Phi_P(B^s)\phi_p(B)(1-B)^d(1-B^s)^D p_t$$
$$= \phi_0 + \theta_q(B)\Theta_Q(B^s)a_t,$$

where

$$\phi_p(B) = 1 - \sum_{i=1}^{p} \phi_i B^i,$$

$$\theta_q(B) = 1 - \sum_{i=1}^{q} \theta_i B^i.$$

This model is widely applicable to many many seasonal time series.

Forecasts: exhibit the same pattern as the observed series.

Empirical Example. International Airline Passengers data in Box and Jenkins (1976).

X_t= the number of Passengers in the $t-$th month.

Step 1. Make a transformation: $p_t = \log(X_t)$.

Step 2. Remove nonstationary or seasonal components:

$$\begin{aligned} w_t &= (1-B)p_t, \text{ or} \\ w_t &= (1-B^{12})p_t, \text{ or} \\ w_t &= (1-B)(1-B^{12})p_t. \end{aligned}$$

Step 3. Note that $w_t = (1-B)(1-B^{12})p_t$ is stationary. So we use the seasonal ARMA model to fit the data:

$$\Phi_P(B^{12})\phi_p(B)w_t = \theta_q(B)\Theta_Q(B^{12})a_t,$$

or

$$\Phi_P(B^{12})\phi_p(B)w_t = \theta_q(B)\Theta_Q(B^{12})a_t.$$

Now, the problem is how to find p, q, P and Q !!!

Step 4. Look at the ACF and PACF of W_t or try some different p, q, P and Q.

For example, we try the model:

$$(1 - \phi B)(1 - \Phi B^{12})w_t = a_t.$$

Step 5. Estimate the parameters in the model.

How to estimate? CLSE, ULSE or MLE methods. The results are:

$$\phi = -0.38, \qquad \Phi = -0.5.$$

Step 6. Diagnostic checking.

Calculate the residuals:

$$\widehat{a}_t = (1 + 0.38B)(1 + 0.52B^{12})w_t.$$

As for ARMA model, if $\{\widehat{a}_t\}$ are close to white noises, the model is correct.

Step 7. If the model is wrong, we should try other models. Even if it is correct, we still need to try some possible models.

For example, we try another model:

$$w_t = (1 - \theta B)(1 - \Theta B^{12})a_t.$$

Through **Step 5**, we obtain:

$$\theta = 0.4, \qquad \Theta = 0.61.$$

Through **Step 6**, we know this model is correct, too.

Step 8. Model selection: **AIC**, **BIC**, or **SBC**.

The final model is:

$$w_t = (1 - 0.40B)(1 - 0.61B^{12})a_t,$$

or

$$\begin{aligned}(1 - B)(1 - B^{12}) \log(X_t) \\ = (1 - 0.40B)(1 - 0.61B^{12})a_t.\end{aligned}$$

Step 9. Forecasting.