Nonlinear Time Series Models

Does nonlinearity exist in financial TS?

Yes, especially in volatility & high-freq data

We focus on simple nonlinear models.

What is a linear time series?

$$x_t = \mu + \sum_{i=0}^{\infty} \psi_i a_{t-i},$$

where μ is a constant, ψ_i are real numbers with $\psi_0 = 1$, and $\{a_t\}$ is an iid $(0, \sigma_a^2)$.

General concept: let F_{t-1} be info. available at time $t-1$.

Conditional mean:

$$\mu_t = E(x_t|F_{t-1}) \equiv g(F_{t-1}),$$

Conditional variance:

$$\sigma_t^2 = Var(x_t|F_{t-1}) \equiv h(F_{t-1}),$$

where $g(.)$ and $h(.)$ are well-defined functions with $h(.) > 0$. For a linear series, $g(.)$ is a linear function of F_{t-1} and $h(.) = \sigma_a^2$. a. Statistics literature: focuses on $g(.)$, See the book by Tong (Oxford University Press, 1990)

Econometrics literature: focuses on $h(.)$

1. Threshold AR (1) [TAR(1)] model:

Tong (1978), Chan (1993), Chan and Tsay (1998).

Example: 2-regime AR(1) model

$$x_t = \begin{cases} -1.5x_{t-1} + a_t & \text{if } \; x_{t-1} < 0, \\ 0.5x_{t-1} + a_t & \text{if } \; x_{t-1} \geq 0, \end{cases}$$

where a_t's are iid N(0, 1).

Here the delay is 1 time period, x_{t-1} is the **threshold** variable, and the threshold is 0. The threshold divides the x_{t-1}-space into two regimes with Regime 1 denoting $x_{t-1} < 0$.

What is so special about this model?

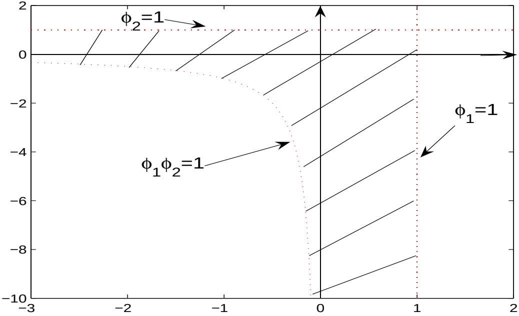
Special features of the model: (a) asymmetry in rising and declining patterns, (b) the mean of x_t is not zero even though there is no

Model:

$$r_t = \begin{cases} \phi_1 r_{t-1} + a_t, & \text{if} \quad r_{t-1} > r, \\ \phi_2 r_{t-1} + a_t, & \text{if} \quad r_{t-1} \leq r, \end{cases}$$

where a_t is i.i.d. r.v. with mean 0 and each having a strictly positive density $f(\cdot)$ on R. Then, $\{r_t\}$ is stationary and ergodic if and only if

$$\phi_1 < 1, \phi_2 < 1 \text{ and } \phi_1 \phi_2 < 1.$$



2
0
−2
−4
−6
−8
−10
−3
−2
−1
0
1
2
ϕ2=1
ϕ1=1
ϕ1ϕ2=1

Higher-order Threshold AR (p) [TAR(p)] model:

$$r_t = \begin{cases} \mu_1 + \phi_{11} r_{t-1} + \cdots + \phi_{1p} r_{t-p} + a_t, & \text{if} \quad r_{t-d} > r, \\ \mu_2 + \phi_{21} r_{t-1} + \cdots + \phi_{2p} r_{t-p} + a_t, & \text{if} \quad r_{t-d} \leq r, \end{cases}$$

where a_t is i.i.d. r.v. with mean 0 and each having a strictly positive density $f(\cdot)$ on R. Then, $\{r_t\}$ is stationary and ergodic if

$$\max\{\sum_{i=1}^{p} |\phi_{1p}|, \sum_{i=1}^{p} |\phi_{2p}|\} < 1.$$

2 Threshold DAR [TDAR(p)] model (Li and Ling 2010)

$$r_t = \begin{cases} \phi_{10} + \sum_{j=1}^{p} \phi_{1j}\, r_{t-j} + \varepsilon_t \sqrt{\alpha_{10} + \sum_{j=1}^{p} \alpha_{1j}\, r_{t-j}^2}, & \text{if } r_{t-d} \le r, \\ \\ \phi_{20} + \sum_{j=1}^{p} \phi_{2j}\, r_{t-j} + \varepsilon_t \sqrt{\alpha_{20} + \sum_{j=1}^{p} \alpha_{2j}\, r_{t-j}^2}, & \text{if } r_{t-d} > r, \end{cases}$$

where ε_t is iid with mean 0 and variance 1.

One of sufficient condition of the stationarity and ergodicity is

$$\left\{ \sum_{i=1}^{p} \max(|\phi_{1i}|, |\phi_{2i}|) \right\}^2 + \sum_{i=1}^{p} \max(\alpha_{1i}, \alpha_{2i}) < 1,$$

Quasi-maximum likelihood estimation

$$L_n(\boldsymbol{\theta}) = \sum_{t=1}^{n} \varphi_t(\boldsymbol{\theta}),$$

where $\boldsymbol{\theta} = (\boldsymbol{\lambda}', r)' = (\boldsymbol{\phi}_1', \boldsymbol{\alpha}_1', \boldsymbol{\phi}_2', \boldsymbol{\alpha}_2', r)'$ is the parameter with $\boldsymbol{\phi}_i = (\phi_{i0}, \phi_{i1}, ..., \phi_{ip})'$ and $\boldsymbol{\alpha}_i = (\alpha_{i0}, \alpha_{i1}, ..., \alpha_{ip})'$ for $i = 1, 2$,

$$\begin{aligned}\varphi_t(\boldsymbol{\theta}) = & -\frac{1}{2}\left\{\log(\boldsymbol{\alpha}_1'\mathbf{Y}_{t-1}) + \frac{(r_t - \boldsymbol{\phi}_1'\mathbf{y}_{t-1})^2}{\boldsymbol{\alpha}_1'\mathbf{Y}_{t-1}}\right\}\mathbb{1}(r_{t-d} \leq r) \\ & -\frac{1}{2}\left\{\log(\boldsymbol{\alpha}_2'\mathbf{Y}_{t-1}) + \frac{(r_t - \boldsymbol{\phi}_2'\mathbf{y}_{t-1})^2}{\boldsymbol{\alpha}_2'\mathbf{Y}_{t-1}}\right\}\mathbb{1}(r_{t-d} > r)\end{aligned}$$

with $\mathbf{y}_{t-1} = (1, r_{t-1}, ..., r_{t-p})'$ and $\mathbf{Y}_{t-1} = (1, r_{t-1}^2, ..., r_{t-p}^2)'$.

Let Θ be the parameter space of $\boldsymbol{\theta}$.

$$\widehat{\boldsymbol{\theta}}_n = \arg\max_{\boldsymbol{\theta} \in \Theta} L_n(\boldsymbol{\theta})$$

is the QMLE of $\boldsymbol{\theta}_0 = (\boldsymbol{\lambda}_0', r_0)'$.

Under some conditions, we can show that $\widehat{\theta}_n \to \theta_0$, a.s. as $n \to \infty$.

Assume that there exist nonrandom vectors $\mathbf{w}^* = (1, w_1, ..., w_p)'$ with $w_d = r_0$ and $\mathbf{W}^* = (1, W_1, ..., W_p)'$ with $W_d = r_0^2$ such that

$$\left\{(\phi_{10} - \phi_{20})'\mathbf{w}^*\right\}^2 + \left\{(\alpha_{10} - \alpha_{20})'\mathbf{W}^*\right\}^2 > 0.$$

Under some Assumptions, we can show that

$$n(\widehat{r}_n - r_0) = O_p(1);$$

$$\sqrt{n}(\widehat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda}_0) \Longrightarrow \text{normal},$$

$$n(\widehat{r}_n - r_0) \Longrightarrow M_-.$$

What is M_-?

Define a two-sided compound Poisson process (CPP) $\wp(z)$ as

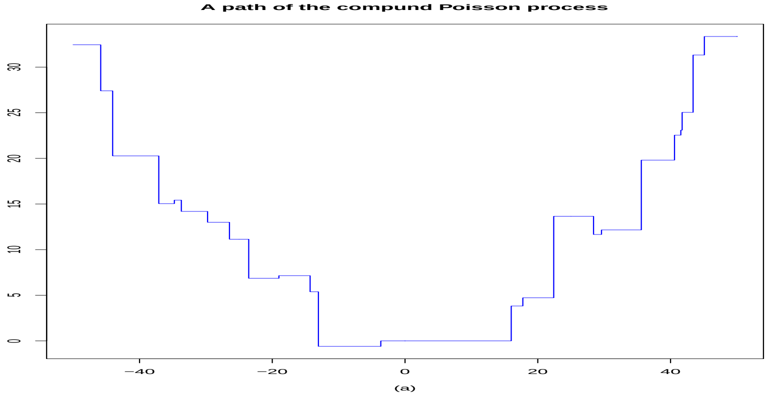
$$\wp(z) = \mathbb{1}(z < 0) \sum_{i=1}^{N_1(-z)} Y_i + \mathbb{1}(z \geq 0) \sum_{j=1}^{N_2(z)} Z_j, \ \ z \in \mathbb{R},$$

where $\{N_1(z), z \geq 0\}$ and $\{N_2(z), z \geq 0\}$ are two independent Poisson processes with $N_1(0) = N_2(0) = 0$ a.s. and with identical jump rate $\pi(r_0)$, where $\pi(\cdot)$ is the density of y_1.

$\{Y_i\}$ and $\{Z_i\}$ are iid with the same dist. as $\zeta_{1t}|_{y_{t-d}=r_0}$ and $\zeta_{2t}|_{y_{t-d}=r_0}$, resp., where

$$\begin{aligned} \zeta_{1t} &= \log \frac{\alpha_{20}^{\mathrm{T}} X_{t-1}}{\alpha_{10}^{\mathrm{T}} X_{t-1}} + \frac{\left\{(\phi_{10} - \phi_{20})^{\mathrm{T}} Y_{t-1} + \varepsilon_t \sqrt{\alpha_{10}^{\mathrm{T}} X_{t-1}}\right\}^2}{\alpha_{20}^{\mathrm{T}} X_{t-1}} - \varepsilon_t^2, \\ \zeta_{2t} &= \log \frac{\alpha_{10}^{\mathrm{T}} X_{t-1}}{\alpha_{20}^{\mathrm{T}} X_{t-1}} + \frac{\left\{(\phi_{10} - \phi_{20})^{\mathrm{T}} Y_{t-1} - \varepsilon_t \sqrt{\alpha_{20}^{\mathrm{T}} X_{t-1}}\right\}^2}{\alpha_{10}^{\mathrm{T}} X_{t-1}} - \varepsilon_t^2. \end{aligned}$$

$\wp(z) \to \infty$ a.s. as $z \to \pm\infty$. There exists a unique random interval $[M_-, M_+)$ on which the process $\wp(z)$ attains its global minimum.



Simulation studies

To assess the performance of the QMLE in finite samples, we use sample sizes $n = 100$, 200, 400 and 800, each with replications 1000 for the following TDAR(1) model

$$y_t = \begin{cases} 1 - 0.6 y_{t-1} + \varepsilon_t \sqrt{(1 + 0.5 y_{t-1}^2)}, & \text{if } y_{t-1} \leq 0, \\ \\ -1 - 0.2 y_{t-1} + \varepsilon_t \sqrt{(0.5 + 0.3 y_{t-1}^2)}, & \text{if } y_{t-1} > 0. \end{cases}$$

The distribution of ε_t is $\mathcal{N}(0,1)$, t_5 and Exp.

Table 1: Simulation studies for model (2.1) with $\boldsymbol{\theta}_0 = (1, -0.6, 1, 0.5, -1, -0.2, 0.5, 0.3, 0)'$ when ε_t is standard normal.

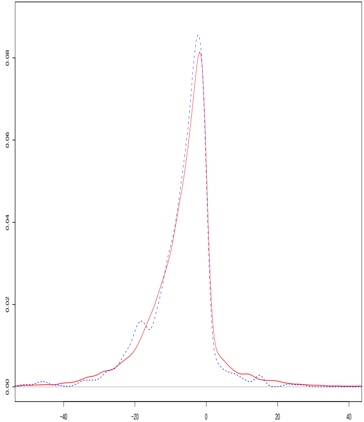
n		ϕ_{10}	ϕ_{11}	α_{10}	α_{11}	ϕ_{20}	ϕ_{21}	α_{20}	α_{21}	r
	EM	1.0477	-0.5741	0.8650	0.4786	-1.0173	-0.1935	0.4180	0.2923	-0.0528
100	ESD	0.3542	0.2547	0.4112	0.2148	0.2555	0.1632	0.2288	0.1082	0.1242
	ASD	0.3203	0.2363	0.3965	0.2116	0.2361	0.1550	0.2182	0.1029	0.1012
	EM	1.0253	-0.5851	0.9398	0.4865	-1.0050	-0.1983	0.4596	0.2939	-0.0250
200	ESD	0.2337	0.1664	0.2931	0.1547	0.1692	0.1086	0.1579	0.0749	0.0548
	ASD	0.2239	0.1670	0.2768	0.1501	0.1639	0.1088	0.1511	0.0725	0.0506
	EM	1.0227	-0.5909	0.9734	0.4988	-1.0135	-0.1970	0.4861	0.2971	-0.0127
400	ESD	0.1605	0.1182	0.1977	0.1069	0.1132	0.0771	0.1088	0.0506	0.0256
	ASD	0.1575	0.1171	0.1951	0.1051	0.1152	0.0764	0.1064	0.0510	0.0253
	EM	1.0042	-0.6006	0.9973	0.4926	-1.0026	-0.1996	0.4946	0.2971	-0.0061
800	ESD	0.1080	0.0811	0.1391	0.0750	0.0830	0.0540	0.0778	0.0377	0.0140
	ASD	0.1110	0.0825	0.1376	0.0741	0.0813	0.0539	0.0751	0.0360	0.0127

Table 2: Empirical quantiles of M_-.

α	0.5%	1%	2.5%	5%	95%	97.5%	99%	99.5%
$N(0,1)$	-45.02	-38.20	-30.38	-24.25	5.77	12.50	21.54	28.81
ST(5)	-52.47	-46.91	-37.16	-29.66	8.75	19.23	33.56	46.25
Dexp	-65.14	-56.61	-44.80	-34.44	11.86	22.93	37.78	51.14

Table 3: Coverage probabilities.

ε_t	α	100	200	400	800
	0.01	0.979	0.986	0.989	0.984
$N(0, 1)$	0.05	0.932	0.940	0.944	0.946
	0.10	0.880	0.893	0.900	0.887
	0.01	0.970	0.980	0.984	0.987
ST(5)	0.05	0.906	0.925	0.934	0.949
	0.10	0.859	0.871	0.884	0.886
	0.01	0.970	0.969	0.987	0.991
Dexp	0.05	0.919	0.922	0.942	0.945
	0.10	0.845	0.878	0.886	0.892



(a)

An Empirical Example

Weekly closing prices of Hang Seng Index: 01/2000 –12/ 2007.

418 observations.

$P_t --$ the weekly closing price at time t.

$y_t = 100(\log P_t - \log P_{t-1})$.

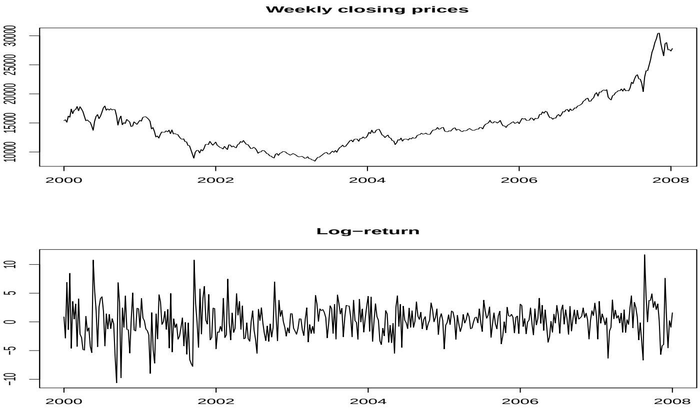


Figure 1: Time plots of the weekly closing prices and the log-returns for Hang Seng Index.

Tsay's (1986)—- the nonlinearity

McLeod-Li test— the ARCH effect.

$$y_t = \begin{cases} \underset{(0.317)}{-0.238} - \underset{(0.149)}{0.154} y_{t-1} + \underset{(0.088)}{0.264} y_{t-2} + \varepsilon_t \sigma_t, & \text{if } y_{t-1} \leq \underset{(0.423)}{0}, \\ \\ \underset{(0.250)}{-0.104} + \underset{(0.092)}{0.096} y_{t-1} - \underset{(0.061)}{0.068} y_{t-2} + \varepsilon_t \sigma_t, & \text{if } y_{t-1} > 0, \end{cases}$$

with

$$\sigma_t^2 = \begin{cases} \underset{(1.102)}{4.402} + \underset{(0.165)}{0.513} y_{t-1}^2 + \underset{(0.124)}{0.178} y_{t-2}^2 + \underset{(0.085)}{0.105} y_{t-3}^2, & \text{if } y_{t-1} \leq 0, \\ \\ \underset{(0.658)}{4.000} + \underset{(0.059)}{0.075} y_{t-2}^2 + \underset{(0.078)}{0.134} y_{t-3}^2, & \text{if } y_{t-1} > 0. \end{cases}$$

Ljung-Box test statistic $Q(k)$

McLeod-Li test statistic $Q^2(k)$

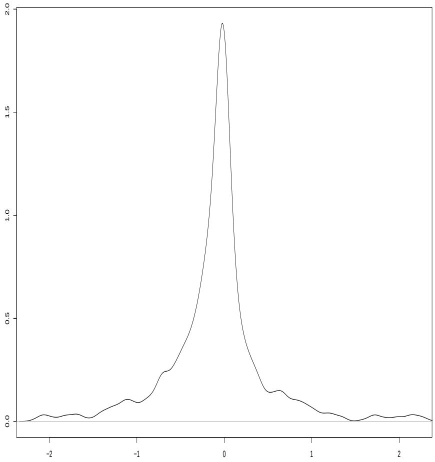
$k = 6, 12.$

The p-values of $Q(6)$, $Q(12)$, $Q^2(6)$ and $Q^2(12)$ are 0.72, 0.45, 0.84 and 0.53, respectively.

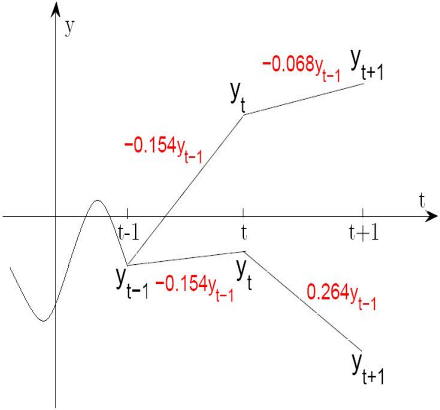
The fit is adequate at the significance level 5%.

Using the simulation method with 1000 replications,

AD is 0.423 and 95% C.I. of r_0 is $[-1.338, 1.190]$



$$y_t = \begin{cases} -0.238 - 0.154 y_{t-1} + 0.264 y_{t-2} + \varepsilon_t \sigma_t, & \text{if } y_{t-1} \leq 0, \\ -0.104 + 0.096 y_{t-1} - 0.068 y_{t-2} + \varepsilon_t \sigma_t, & \text{if } y_{t-1} > 0 \end{cases}$$



4. Other models:

a. AR-TGARCH(1,1) model

Example: Daily log returns of IBM stock from July 3, 1962 to December 31, 2003 for 10,446 observations. See Figure 4.3 of the text (p. 162).

AR(2)-GARCH(1,1) model:

$$r_t = 0.062 - 0.024 r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t,$$

$$\sigma_t^2 = 0.037 + 0.077 a_{t-1}^2 + 0.913 \sigma_{t-1}^2.$$

Std residuals: $Q(10) = 5.19(0.88)$ and $Q(20) = 24.38(0.23)$

Sq. std. res.: $Q(10) = 11.67(0.31)$ and $Q(20) = 18.25(0.57)$.

AR(2)-TGARCH(1,1) model

$$
\begin{aligned}
r_t &= 0.033 - .023 r_{t-2} + a_t, \;\; a_t = \sigma_t \epsilon_t, \\
\sigma_t^2 &= 0.075 + 0.041 a_{t-1}^2 + 0.903 \sigma_{t-1}^2 \\
&\quad + (0.03 a_{t-1}^2 + 0.062 \sigma_{t-1}^2) I(a_{t-1} \leq 0).
\end{aligned}
$$

Rewrite the TGARCH(1,1) as

$$
\sigma_t^2 = \begin{cases} 0.075 + 0.071 a_{t-1}^2 + 0.965 \sigma_{t-1}^2 & \textit{if} \;\; a_{t-1} \leq 0 \\ 0.075 + 0.041 a_{t-1}^2 + 0.907 \sigma_{t-1}^2 & \textit{if} \;\; a_{t-1} > 0, \end{cases}
$$

The asymmetry in volatility is clearly seen. When $a_{t-1} < 0$, $\alpha + \beta = 1.036 > 1$. However, $\alpha + \beta = 0.949 < 1$ when a_{t-1} is positive.

b. Markov switching model: Two-state MS model:

$$x_t = \begin{cases} c_1 + \sum_{i=1}^{p} \phi_{1,i} x_{t-i} + a_{1t} & \text{if } s_t = 1, \\ c_2 + \sum_{i=1}^{p} \phi_{2,i} x_{t-i} + a_{2t} & \text{if } s_t = 2, \end{cases}$$

where s_t assumes values in $\{1, 2\}$ and is a first-order Markov chain with trans. prob.

$$P(s_t = 2 | s_{t-1} = 1) = w_1, \ P(s_t = 1 | s_{t-1} = 2) = w_2,$$

where $0 \leq w_1 \leq 1$ is the probability of switching out State 1 from time t - 1 to time t. A large w1 means that it is easy to switch out State 1, i.e. cannot stay in State 1 for long. The inverse, $1/w_1$, is the expected duration (number of time periods) to stay in State 1. Similar idea applies to w_2.

Example: Growth rate of US quarterly real GNP 47-91. See Figure 4.4 of the textbook (p.166).

	c_i	ϕ_1	ϕ_2	ϕ_3	ϕ_4	σ_i	w_i
				State 1			
est	.909	.265	.029	$-.126$	$-.110$	.816	.118
s.e	.202	.113	.126	.103	.109	.125	.053
				State 2			
est	$-.420$	.216	.628	$-.073$	$-.097$	1.017	.286
s.e	.324	.347	.377	.364	.404	.293	.064

c. Neural networks, see textbook.