Exercises

1. Find the ACF and PACF and plot the ACF ρ_k for $k = 0, 1, 2, 3, 4$, and 5 for each of the following models:

(a) $r_t - 0.5r_{t-1} = a_t$,

(b) $r_t + 0.98r_{t-1} = a_t$,

(c) $r_t - 1.3r_{t-1} + 0.4r_{t-2} = a_t$.

2. For each of the following models,

$$\begin{array}{ll} (i) & (1 - 0.9B)(r_t - 10) = a_t, \\ (ii) & r_t = 10 - 0.9a_{t-1} + a_t, \\ (ii) & (1 - 0.5B)(r_t - 10) = a_t - 0.9a_{t-1}, \end{array}$$

where $\sigma_a^2 = 2$. Given $r_1 = 1.2$ and $r_2 = 0.1$, find the $l-$step ahead forecast values and forecast variances for $l = 1, 2, 3, 4$.

3. Find the ACF and PACF for $k = 0, 1, 2, 3$ and 4 for each of the following models:

(a) $r_t = (1 - 0.8B)a_t$,

(b) $r_t = (1 - 1.2B + 0.5B^2)a_t$,

4. Verify whether or not the following models are stationary and/ or invertible:

$$\begin{array}{ll} (a) & (1 - B)r_t = (1 - 1.5B)a_t, \\ (b) & (1 - 0.8B)r_t = (1 - 0.5B)a_t, \\ (c) & (1 - 1.1B + 0.8B^2)r_t = (1 - 1.7B + 0.72B^2)a_t, \\ (d) & (1 - 0.6B)r_t = (1 - 1.2B + 0.2B^2)a_t. \end{array}$$

5. Consider the two models:

$$\begin{array}{ll} (a) & (1 - 0.43B)(1 - B)r_t = a_t, \\ (b) & (1 - B)r_t = (1 - 0.43B)a_t, \end{array}$$

where a_t is i.i.d. $N(0, 1)$. Given the observations $r_{49} = 33.4$ and $r_{50} = 33.9$, compute their forecasts $r_{50}(l)$, for $l = 1, 2, 3, 4$, and the corresponding 90% forecast intervals.

6. Find the ACF for the following seasonal models:

$$\begin{array}{ll} (a) & r_t = (1 - \theta_1 B)(1 - \Theta_1 B)a_t, \\ (b) & (1 - \Phi_1 B^s)r_t = (1 - \theta_1 B)a_t, \\ (c) & (1 - \Phi_1 B^s)(1 - \phi_1 B)r_t = a_t, \end{array}$$

where $a_t \sim iidN(0, \sigma^2)$.

7. Consider the ARCH model:

$$a_t = \eta_t \sigma_t, \ \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2.$$

Show that the unconditional variance of a_t is $Var(a_t) = \alpha_0/(1-\alpha_1)$, where $\alpha_0 > 0$, $0 \le \alpha_1 < 1$ and η_t is i.i.d $N(0,1)$.

8. Give the stationarity condition and its representation in terms of $\{\eta_t\}$ for the GARCH model:

$$a_t = \eta_t \sigma_t, \ \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

where $\alpha_0 > 0$, $\alpha_1, \beta_1 \ge 0$, and η_t is i.i.d $N(0,1)$. Furthermore, give Ea_t^4 and the prediction of the conditional variances σ_{t+s}^2.

9. Give the stationarity and invertibility conditions, MA and AR representation and ACFs of the seasonal ARMA models:

$$\begin{array}{ll} (a). & y_t = \phi y_{t-s} + a_t, \\ (b). & y_t = \theta a_{t-s} + a_t, \end{array}$$

where $\{a_t$ is white noise and variance σ_a^2.

10. Consider the following EGARCH(1,1) model

$$a_t = \sigma_t \epsilon_t, \quad (1-\beta B)\ln(\sigma_t^2) = \alpha_0 + \alpha g(\epsilon_{t-1}),$$

where $\epsilon_t \sim N(0,1)$ and $E(|\epsilon_t|) = \sqrt{2/\pi}$ and

$$g(\epsilon_t) = \theta \epsilon_t + [|\epsilon_t| - E(|\epsilon_t|)].$$

Show the representation of $\ln(\sigma_t^2)$ in terms of ϵ_t and give its mean and variance.

11. Consider the following bivariate VAR model:

$$\begin{aligned} y_{1t} &= 0.3 y_{1,t-1} + 0.8 y_{2,t-1} + a_{1t}, \\ y_{2t} &= 0.9 y_{1,t-1} + 0.4 y_{2,t-1} + a_{2t}, \end{aligned}$$

with $E(a_{1t}a_{1\tau}) = 1$ if $t = \tau$ and 0 otherwise, $E(a_{2t}a_{2\tau}) = 2$ if $t = \tau$ and 0 otherwise, and $E(a_{1t}a_{2\tau}) = 0$ for all t and τ.

(a). Is this system stationary?

(b). Calculate the two-step ahead forecast variance for variable $y_{1,t+2}$, that is

$$E[y_{1,t+2} - E(y_{1,t+2}|Y_t, Y_{t-1}, \cdots)]^2,$$

where $Y_t = (y_{1t}, y_{2t})'$.

12. Write down the bivariate system into an VAR model and show that it is not stationary:

$$\begin{aligned} y_{1t} &= \gamma y_{2t} + \varepsilon_{1t} \\ y_{2t} &= y_{2,t-1} + \varepsilon_{2t} \end{aligned}$$

where $\gamma \ne 0$, ε_{1t} and ε_{2t} being uncorrelated white noise processes.

13. Show that the following VAR model

$$\mathbf{y_t} = \sum_{\mathbf{i=1}}^{\mathbf{p}} \mathbf{\Phi_i y_{t-i}} + \varepsilon_\mathbf{t}$$

can be written as following VCE model:

$$\Phi(B)\mathbf{y_t} = \mathbf{\Phi^*(B)(1-B)y_t + \Phi(1)By_t},$$

where $\Phi^*(B) = \mathbf{I_m} - \sum_{\mathbf{i=1}}^{\mathbf{p-1}} \mathbf{\Phi_i^* B^i}$ with $\Phi_i^* = -\sum_{j=i+1}^{p} \Phi_j$.

14. Consider the two dimensional vector AR(2) model:

$$\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} + \begin{bmatrix} -0.2 & 0.1 \\ 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{bmatrix} + \begin{bmatrix} 0.8 & 0.7 \\ -0.4 & 0.6 \end{bmatrix} \begin{bmatrix} Z_{1,t-2} \\ Z_{2,t-2} \end{bmatrix},$$

where $\{(a_{1t}, a_{2t})'\}$ is a sequence of i.i.d. standard normal random vectors. Show that it is a partially non-stationary AR model

15. Determine the stationarity and invertibility of the following two-dimensional vector models and find their correlation matrix function, ρ_k, for $k = \pm 1, \pm 2, \pm 3$:

(a). $(I - \Phi_1 B)Z_t = a_t$, where $\Phi_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.6 \end{bmatrix}$ and $\Sigma = I$,

(b). $(I - \Phi_1 B)Z_t = a_t$, where $\Phi_1 = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.8 \end{bmatrix}$ and $\Sigma = I$,

(c). $Z_t = (I - \Theta_1 B)a_t$, where $\Theta_1 = \begin{bmatrix} 0.6 & 1.2 \\ 0.4 & 0.8 \end{bmatrix}$ and $\Sigma = I$.

16. Consider the process

$$\begin{aligned} Z_{1t} &= Z_{1,t-1} + a_{1t} + \theta a_{1,t-1}, \\ Z_{2t} &= \phi Z_{1t} + a_{2t}, \end{aligned}$$

where $|\phi| < 1$, $|\theta| < 1$ and $a_t = [a_{1t}, a_{2t}]' \sim N(0, \Sigma)$.

(a) Write the process in a vector form,

(b) Is the process $[Z_{1t}, Z_{2t}]'$ stationary and invertible?

(c). Write down the model for the vector of the first differences $(I-B)Z_t$, where $Z_t = [Z_{1t}, Z_{2t}]'$. Is the resulting model stationary and invertible?

17. Show that the process $y_t = z_t - z_{t-1}$ is weakly stationary, where $z_t = 0.9z_{t-1} + a_t$ and $\{a_t\}$ is a white noise series.

18. You need to review all the simple theory we taught in our lectures