MSDM5004
Numerical Methods and Modeling in Science
Spring 2024

Lecture 14

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3. Elliptic PDEs

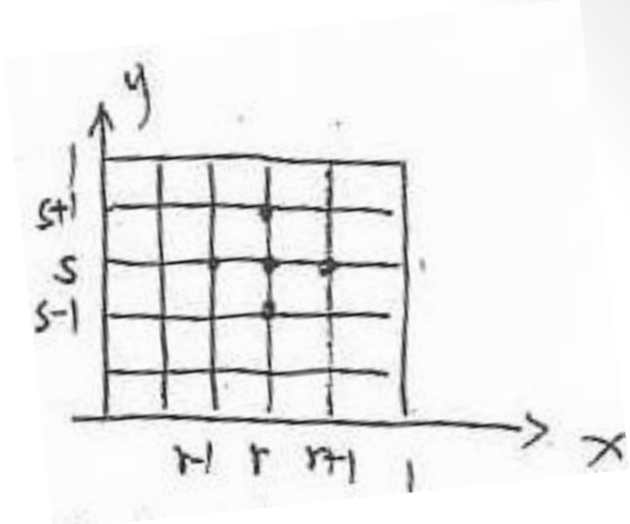
A model problem

$$
\begin{aligned}
u_{xx} + u_{yy} + f(x,y) = 0, &\quad (x,y) \in \Omega, \\
u = 0, &\quad (x,y) \in \partial\Omega,
\end{aligned}
$$

$$\Omega := (0,1) \times (0,1)$$

Notations: $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}, \quad u_{yy} = \dfrac{\partial^2 u}{\partial y^2}$

Uniform mesh $\Delta x = \Delta y = 1/J$



Five-point scheme

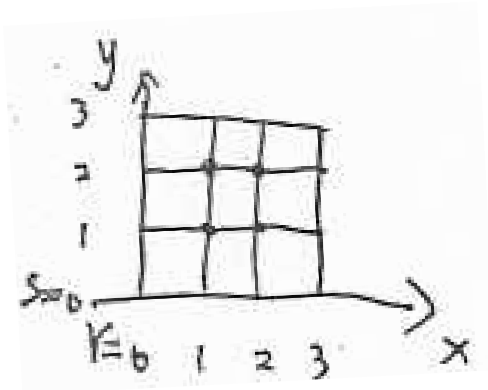
$$\frac{U_{r+1,s} + U_{r-1,s} + U_{r,s+1} + U_{r,s-1} - 4U_{r,s}}{(\Delta x)^2} + f_{r,s} = 0.$$

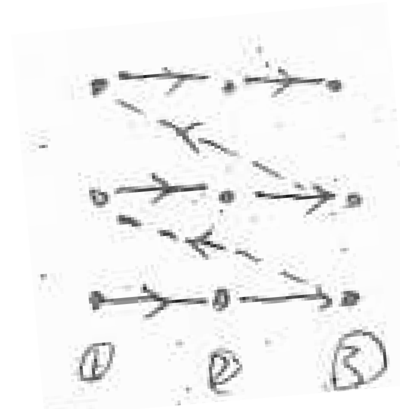
$r = 1, 2, \ldots, J-1 \quad s = 1, 2, \ldots, J-1$

Obtained by central difference approximations of u_{xx} and u_{yy}

$$\frac{U_{r+1,s} - 2U_{r,s} + U_{r-1,s}}{(\Delta x)^2} + \frac{U_{r,s+1} - 2U_{r,s} + U_{r,s-1}}{(\Delta y)^2} + f_{r,s} = 0$$

e.g. J=3





Natural order:

$$U_{1,1}, U_{2,1}, U_{3,1}, \cdots, U_{J-1,1}, U_{1,2}, U_{2,2}, \cdots, U_{J-1,2}, \cdots, U_{1,J-1}, U_{2,J-1}, \cdots, U_{J-1,J-1}$$

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{1,2} \\ U_{2,2} \end{bmatrix} = \begin{bmatrix} -f_{1,1}(\Delta x)^2 - g_{1,0} - g_{0,1} \\ -f_{2,1}(\Delta x)^2 - g_{2,0} - g_{3,1} \\ -f_{1,2}(\Delta x)^2 - g_{0,2} - g_{1,3} \\ -f_{2,2}(\Delta x)^2 - g_{3,2} - g_{2,3} \end{bmatrix}$$

Truncation error

$$T_{r,s} = \frac{u(x_{r+1}, y_s) + u(x_{r-1}, y_s) + u(x_r, y_{s+1}) + u(x_r, y_{s-1}) - 4u(x_r, y_s)}{(\Delta x)^2} + f(x_r, y_s)$$

$$= \frac{1}{12}(\Delta x)^2 \left(u_{xxxx} + u_{yyyy}\right)_{r,s} + O((\Delta x)^4)$$

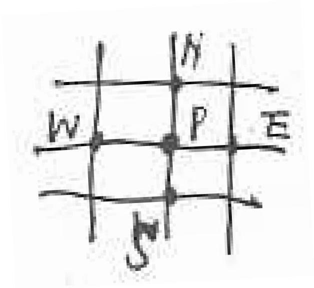
$$|T_{r,s}| \le T := \frac{1}{12}(\Delta x)^2 \left(M_{xxxx} + M_{yyyy}\right)$$

$$M_{xxxx} \equiv \max_{\Omega} |u_{xxxx}|, \quad M_{yyyy} \equiv \max_{\Omega} |u_{yyyy}|$$

Iterative methods for solving the linear system

Write the five-point scheme as

$$\frac{u_E - 2u_P + u_W}{(\Delta x)^2} + \frac{u_N - 2u_P + u_S}{(\Delta x)^2} + f_P = 0$$



$$4u_P - (u_E + u_W + u_N + u_S) = (\Delta x)^2 f_P$$

Moving the values on the boundary to the right

$$\tilde{c}_P U_P - [\tilde{c}_E U_E + \tilde{c}_W U_W + \tilde{c}_N U_N + \tilde{c}_S U_S] = b_P$$

If E is on the boundary, $U_E = g_E$

$$4U_P - (0 + U_W + U_N + U_S) = (\Delta x)^2 f_P + g_E$$

$$\Rightarrow \tilde{C}_P = 4, \quad \tilde{C}_E = 0, \quad \tilde{C}_W = \tilde{C}_N = \tilde{C}_S = 1, \quad b_P = (\Delta x)^2 f_P + g_E$$

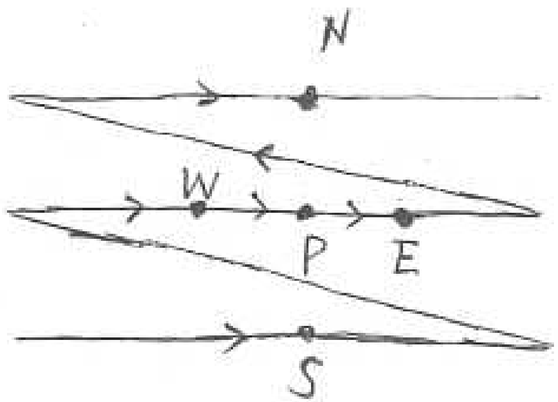
Jacobi iteration

$$U_P^{(n+1)} = (1/\tilde{c}_P)[b_P + \tilde{c}_E U_E^{(n)} + \tilde{c}_W U_W^{(n)} + \tilde{c}_N U_N^{(n)} + \tilde{c}_S U_S^{(n)}]$$

Gauss-Seidel iteration

$$U_P^{(n+1)} = (1/\tilde{c}_P)[b_P + \tilde{c}_E U_E^{(n)} + \tilde{c}_W U_W^{(n+1)} + \tilde{c}_N U_N^{(n)} + \tilde{c}_S U_S^{(n+1)}]$$

Natural order:



SOR

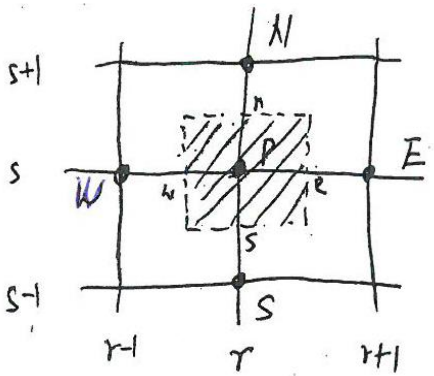
$$U_P^{(n+1)} = (1-\omega)U_P^{(n)} + (\omega/\tilde{c}_P)[b_P + \tilde{c}_E U_E^{(n)} + \tilde{c}_W U_W^{(n+1)} + \tilde{c}_N U_N^{(n)} + \tilde{c}_S U_S^{(n+1)}]$$

A more general elliptic equation

$$\nabla\cdot(a\nabla u)+f=0 \quad \text{in } \Omega$$

$$a(x,y)\geq a_0>0$$

constant



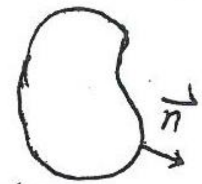
▨ control volume, denoted by V

integrating the equation on V

$$\int_V \nabla\cdot(a\nabla u)\,dx\,dy+\int_V f\,dx\,dy=0$$

Using Gauss theorem: $\int_V \nabla\cdot\vec{w}\,dx\,dy = \int_{\partial V} \vec{w}\cdot\vec{n}\,dl$ $\quad$ $\vec{n}$: outer normal vector

$$\int_V \nabla\cdot(a\nabla u)\,dx\,dy = \int_{\partial V} a\nabla u\cdot\vec{n}\,dl = \int_{\partial V} a\frac{\partial u}{\partial n}\,dl$$



($\nabla u\cdot\vec{n} = \frac{\partial u}{\partial n}$ normal derivatives)

The equation becomes

$$\int_{\partial V} a\frac{\partial u}{\partial n}\,dl + \int_V f\,dx\,dy = 0$$

$$\int_{\partial V} a\frac{\partial u}{\partial n}\, dl = -\int_{x_{r-\frac{1}{2}}}^{x_{r+\frac{1}{2}}} a(x, y_{s-\frac{1}{2}})\frac{\partial u}{\partial y}(x, y_{s-\frac{1}{2}})\,dx$$

$$x_{r-\frac{1}{2}} \le x \le x_{r+\frac{1}{2}} \quad y = y_{s-\frac{1}{2}} \qquad \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial y}$$

$$+\int_{y_{s-\frac{1}{2}}}^{y_{s+\frac{1}{2}}} a(x_{r+\frac{1}{2}}, y)\frac{\partial u}{\partial x}(x_{r+\frac{1}{2}}, y)\,dy$$

$$x = x_{r+\frac{1}{2}} \quad y_{s-\frac{1}{2}} \le y \le y_{s+\frac{1}{2}} \qquad \frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}$$

$$+\int_{x_{r-\frac{1}{2}}}^{x_{r+\frac{1}{2}}} a(x, y_{s+\frac{1}{2}})\frac{\partial u}{\partial y}(x, y_{s+\frac{1}{2}})\,dy$$

$$x_{r-\frac{1}{2}} \le x \le x_{r+\frac{1}{2}} \quad y = y_{s+\frac{1}{2}} \qquad \frac{\partial u}{\partial n} = \frac{\partial u}{\partial y}$$

$$-\int_{y_{s-\frac{1}{2}}}^{y_{s+\frac{1}{2}}} a(x_{r-\frac{1}{2}}, y)\frac{\partial u}{\partial x}(x_{r-\frac{1}{2}}, y)\,dy$$

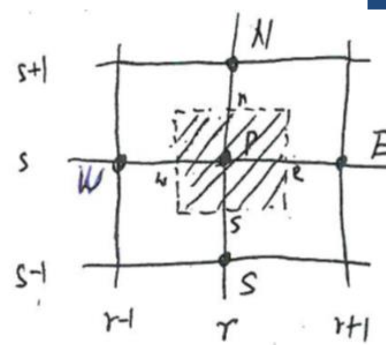
$$x = x_{r-\frac{1}{2}} \quad y_{s-\frac{1}{2}} \le y \le y_{s+\frac{1}{2}} \qquad \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}$$

$$\approx -a_{r,s-\frac{1}{2}}\frac{U_{r,s}-U_{r,s-1}}{\Delta y}\Delta x$$

$$+a_{r+\frac{1}{2},s}\frac{U_{r+1,s}-U_{r,s}}{\Delta x}\Delta y$$

$$+a_{r,s+\frac{1}{2}}\frac{U_{r,s+1}-U_{r,s}}{\Delta y}\Delta x$$

$$-a_{r-\frac{1}{2},s}\frac{U_{r,s}-U_{r-1,s}}{\Delta x}\Delta y$$



$$\int_V f\,dx\,dy \approx f_{r,s}\,\Delta x\,\Delta y$$

As a result, the numerical scheme is

$$\frac{a_{r+\frac{1}{2},s}(U_{r+1,s}-U_{r,s}) - a_{r-\frac{1}{2},s}(U_{r,s}-U_{r-1,s})}{(\Delta x)^2}$$

$$+\ \frac{a_{r,s+\frac{1}{2}}(U_{r,s+1}-U_{r,s}) - a_{r,s-\frac{1}{2}}(U_{r,s}-U_{r,s-1})}{(\Delta y)^2} + f_{r,s} = 0$$

Remark: If we write the equation as the follows and perform discretization:

$$a\Delta u + \nabla a \cdot \nabla u + f = 0$$

There will be some additional restriction on $\Delta x, \Delta y$

Boundary conditions

Dirichlet boundary condition: $u(x,y)=g(x,y) \quad (x,y)\in\partial\Omega$

Neumann boundary condition: $\frac{\partial u}{\partial n}=g \quad (x,y)\in\partial\Omega$

Robin boundary condition: $\alpha_1\frac{\partial u}{\partial n}+\alpha_0 u=g \quad (x,y)\in\partial\Omega$
(mixed boundary condition)

$\alpha_0, \alpha_1 \geqslant 0, \ \alpha_0+\alpha_1>0$

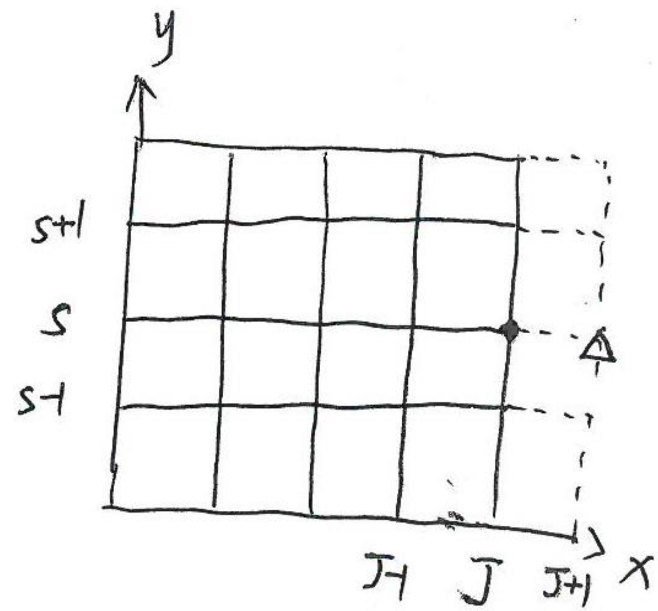
$$u_{xx}+u_{yy}+f(x,y)=0 \qquad (x,y)\in\Omega$$

$$\alpha_1 \frac{\partial u}{\partial n}+\alpha_0 u = g \qquad (x,y)\in\partial\Omega$$

where $\Omega=(0,1)\times(0,1)$.

consider the boundary condition at (x_J, y_s).

add one more point (x_{J+1}, y_s) with the solution $u_{J+1,s}$ on it.



the boundary condition can be approximated by

$$\alpha_1 \frac{U_{J+1,s} - U_{J-1,s}}{2\Delta x} + \alpha_0 U_{J,s} = g_{J,s}. \qquad \text{(second order)}$$

assume the equation holds on (x_J, y_s)

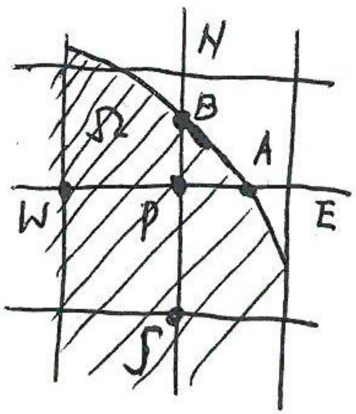
$$\frac{U_{J+1,s} - 2U_{J,s} + U_{J-1,s}}{(\Delta x)^2} + \frac{U_{J,s+1} - 2U_{J,s} + U_{J,s-1}}{(\Delta y)^2} + f_{J,s} = 0.$$

We can use these two equations to eliminate $U_{J+1,s}$.

General boundaries

$$u_{xx} + u_{yy} + f = 0 \qquad (x,y) \in \Omega$$
$$u = g \qquad (x,y) \in \partial\Omega$$

where Ω has curved boundary $\partial\Omega$.



Consider the equation at point P

$$PA = \theta \Delta x \qquad 0 < \theta < 1$$

$$u_A = \left[u + \theta\Delta x\, u_x + \tfrac{1}{2}(\theta\Delta x)^2 u_{xx} + \cdots\right]_P$$
$$u_W = \left[u - \Delta x\, u_x + \tfrac{1}{2}(\Delta x)^2 u_{xx} + \cdots\right]_P$$

Eliminating u_x, we have

$$[u_{xx}]_P \approx \frac{u_A + \theta u_W - (1+\theta) u_P}{\frac{1}{2}\theta(1+\theta)(\Delta x)^2} \qquad \text{(first order)}$$

u_{xx} at point p