MSDM 5058 Information Science
Assignment 1 (due 2^{nd} March, 2024)

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(1) **Bayes' Theorem**

A telegraphic communications system transmits the signals 1 and 0. Assume that due to noise, an average of 2/5 of the dots and 1/3 of the dashes are changed. Suppose that the ratio between the transmitted dots and the transmitted dashes is 5: 3. What is the probability that a received signal will be the same as the transmitted signal if (a) the received signal is a dot, (b) the received signal is a dash? (c) Assuming independence between signals, if the message dot-dot was received, what is the probability distribution of the four possible messages that could have been sent?

Solution:

Let *A* be the event that a dot is received, and *B* that a dash is received. One can make two hypotheses: H_1 that the transmitted signal was a dot; and H_2 that the transmitted signal was a dash. By assumption, one has $\mathrm{P}(H_1):\mathrm{P}(H_2) = 5:3$ and, $\mathrm{P}(H_1) + \mathrm{P}(H_2) = 1$. Therefore $\mathrm{P}(H_1)$= 5/8, $\mathrm{P}(H_2)$= 3/8. One knows that

$$\mathrm{P}(\mathrm{A}|H_1) = \frac{3}{5}\,; \quad P(A|H_2) = \frac{1}{3},$$
$$\mathrm{P}(\mathrm{B}|H_1) = \frac{2}{5}\,; \quad P(B|H_2) = \frac{2}{3}.$$

The probabilities of *A* and *B* are determined from the total probability formula:

$$\mathrm{P}(\mathrm{A}) = \frac{5}{8}\frac{3}{5} + \frac{3}{8}\frac{1}{3} = \frac{1}{2}\,; P(B) = \frac{5}{8}\frac{2}{5} + \frac{3}{8}\frac{2}{3} = \frac{1}{2}.$$

From Bayes' Theorem, the required probabilities are:

$$(\mathrm{a}) P(H_1|A) = \frac{P(H_1)P(A|H_1)}{P(A)} = \frac{\frac{5}{8}\frac{3}{5}}{\frac{1}{2}} = \frac{3}{4}\,;$$

$$\text{(b)}P(H_2|B) = \frac{P(H_2)P(B|H_2)}{P(B)} = \frac{\frac{3}{8}\frac{2}{3}}{\frac{1}{2}} = \frac{1}{2}.$$

(c) Given that 1-1 was received, the distribution of the four possibilities are:

Signal sent	Probability
$H_2 - H_2$	$\frac{1}{4}\frac{1}{4} = \frac{1}{16}$
$H_2 - H_1$	$\frac{1}{4}\frac{3}{4} = \frac{3}{16}$
$H_1 - H_2$	$\frac{1}{4}\frac{3}{4} = \frac{3}{16}$
$H_1 - H_1$	$\frac{3}{4}\frac{3}{4} = \frac{9}{16}$

(2) **Characteristic Function**

Find the (a) characteristic function of a random variable *X* with the probability density

$$f(x) = \frac{1}{2}e^{-|x|},$$

(b) from the characteristic function, find its mean and variance.

Solution:

$$M(x) = \int_{-\infty}^{+\infty} e^{iux} f(x)dx = \frac{1}{2}\int_{-\infty}^{+\infty} e^{iux-|x|}dx = \frac{1}{2}\int_{0}^{\infty} e^{(iu-1)x}dx + \frac{1}{2}\int_{-\infty}^{0} e^{(iu+1)x}dx$$

$$= \frac{1}{2}\left(\frac{1}{1-iu} - \frac{1}{1+iu}\right) = \frac{1}{1+u^2}$$

The n-th moment of X can be obtained by using

$$\langle x^n \rangle = \int_{-\infty}^{+\infty} x^n f(x)dx = (-i)^n \frac{d^n M(x)}{du^n}\bigg|_{u=0}$$

Therefore, the mean $\langle x \rangle = -i\frac{dM}{du}\Big|_{u=0} = \frac{2iu}{(1+u^2)^2}\Big|_{u=0} = 0$

Variance: $\langle x^2 \rangle - \langle x \rangle^2 = 2$

(3) **Autocorrelation**

Let *X*(*t*) be a normal stationary random function with zero expectation. Show that if

$$Z(t) = \frac{1}{2}\left[1 + \frac{X(t)X(t+\tau)}{|X(t)X(t+\tau)|}\right],$$

then

$$\bar{Z}(\mathrm{t}) = \frac{1}{\pi}\mathrm{acos}[-\rho(\tau)]$$

where $\rho(\tau)$ is the normalized correlation function of $X(t)$.

Solution:

The joint distribution of two normal random variables with correlation is given by

$$f(x, y) = \frac{1}{2\pi\sigma^2\sqrt{(1-\rho^2)}} e^{-\frac{(x^2+y^2)-2\rho xy}{2(1-\rho^2)\sigma^2}}$$

In our case here, when we look at the autocorrelation, the required expectation can be represented in the form,

$$\bar{Z}(\mathrm{t}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}\left[1 + \frac{xy}{|xy|}\right] f(x, y|t, t+\tau)dxdy$$

Since $\frac{1}{2}\left[1 + \frac{xy}{|xy|}\right]$ is identically equal to zero if the signs of x and y are different, and equal to one otherwise, we have

$$\bar{Z}(\mathrm{t}) = \int_{-\infty}^{0}\int_{-\infty}^{0} f(x, y|t, t+\tau)\, dxdy + \int_{0}^{\infty}\int_{0}^{\infty} f(x, y|t, t+\tau)\, dxdy$$

$$= 2\int_{0}^{\infty}\int_{0}^{\infty} f(x, y|t, t+\tau)\, dxdy$$

Change of variables to r and θ, and let $x = r\cos\theta$, $y = r\sin\theta$,

$$\frac{2}{2\pi\sigma^2\sqrt{1-\rho^2}}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty} e^{-\frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\sigma^2}} rdrd\theta$$

$$= \frac{2}{2\pi\sigma^2\sqrt{1-\rho^2}}\int_{0}^{\frac{\pi}{2}}\left[\frac{(1-\rho^2)\sigma^2}{1-\rho\sin 2\theta}\right]d\theta\int_{0}^{\infty} e^{-z^2}d(z^2)$$

$$= \frac{\sqrt{1-\rho^2}}{\pi}\int_{0}^{\frac{\pi}{2}}\frac{d\theta}{1-\rho\sin 2\theta} = \frac{1}{\pi}\mathrm{atan}\left[\frac{\tan\theta-\rho}{\sqrt{1-\rho^2}}\right]\Bigg|_{0}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \mathrm{atan}\left[\frac{-\rho}{\sqrt{1-\rho^2}}\right]\right]$$

where we have again use change of variables $z^2 = \frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\sigma^2}$ when integrating over r. It is now easy to convert the above result to show that $\bar{Z}(\mathrm{t}) = \frac{1}{\pi}\mathrm{acos}[-\rho(\tau)]$.

(4) **Radar System**

Consider a radar system that uses radio waves to detect aircrafts. The system receives a signal and, based on the received signal, it needs to decide whether an aircraft is present or not. Let X be the received signal. Suppose that we know

$X = W$, if no aircraft is present.

$X = 1 + W$, if an aircraft is present.

where $W \sim N(0, \sigma^2 = \frac{1}{9})$. Thus, we can write $X = \theta + W$, where $\theta = 0$ if there is no aircraft, and $\theta = 1$ if there is an aircraft. Suppose that we define H_0 and H_1 as follows:

H_0 (Null hypothesis): No aircraft is present.

H_1 (Alternative hypothesis): An aircraft is present.

(a) Write the null hypothesis, H_0, and the alternative hypothesis, H_1, in terms of possible values of θ.
(b) Design a level 0.05 test ($\alpha = 0.05$) to decide between H_0 and H_1.
(c) Find the probability of type II error, β, for the above test in b). Note that this is the probability of missing a present aircraft.
(d) If we observe $X = 0.6$, is there enough evidence to reject H_0 at significance level $\alpha = 0.01$?
(e) If we would like the probability of missing a present aircraft to be less than 5%, what is the smallest significance level that we can achieve?
(f) What is the likelihood ratio for ***false alarm***?

Solution:

(a) The null hypothesis corresponds to $\theta = 0$ and the alternative hypothesis corresponds to $\theta = 1$. Thus, we can write

H_0 (Null hypothesis): No aircraft is present: $\theta = 0$.

H_1 (Alternative hypothesis): An aircraft is present: $\theta = 1$.

Note that here both hypotheses are simple.

(b) To decide between H_0 and H_1, we look at the observed data. Here, the situation is relatively simple. The observed data is just the random variable X. Under H_0, $X \sim N(0,1/9)$, and under H_1, $X \sim N(1,1/9)$. Thus, we can suggest the following test: We choose a threshold c. If the observed value of X is less than c, we choose H_0 (i.e., $\theta = 0$). If the observed value of X is larger than c, we choose H_1 (i.e., $\theta = 1$). To choose c, we use the required α:

$$\alpha = P(\text{type I error}) = P(\text{Reject } H_0|H_0) = P(X > c|H_0) = P(W > c)$$
$$= 1 - \Phi(3c)$$

For $\alpha = 0.05$, $\Phi(3c) = 0.95$, or $c = 0.548$

(c) For type II error, we have

$$\beta = P(\text{type II error}) = P(\text{accept } H_0|H_1) = P(X < c|H_1) = P(1 + W < c)$$
$$= P(W < c - 1) = \Phi(3(c - 1))$$

Since $c = 0.548$, we therefore have $\beta = 0.088$.

(d) In part (b), we have $\Phi(3c) = 1 - \alpha$. For $\alpha = 0.01$, $\Phi(3c) = 0.99$, or $c = 0.775$, which is larger than 0.6. Thus, we cannot reject H_0 at significance level $\alpha = 0.01$.

(e) We here want $\beta = \Phi\big(3(c - 1)\big) \leq 0.05$, which gives $c \leq 0.452$. Substituting into the equation for α, we have

$$\alpha = P(\text{type I error}) = 1 - \Phi(3c) = 1 - \Phi(3 \times 0.452) = 0.0875$$

Therefore, the smallest significance level that we can achieve is α = 0.0875.

(f) The probability function of H_0 is $f_Z(z|H_0) = \frac{3}{\sqrt{2\pi}} e^{-\frac{9z^2}{2}}$; and the probability function of H_1 is $f_Z(z|H_1) = \frac{3}{\sqrt{2\pi}} e^{-\frac{9(z-1)^2}{2}}$. Thus, the likelihood ratio for false alarm is

$$\Lambda(Z) = \frac{f_Z(z|H_0)}{f_Z(z|H_1)} = \frac{\frac{3}{\sqrt{2\pi}} e^{-\frac{9z^2}{2}}}{\frac{3}{\sqrt{2\pi}} e^{-\frac{9(z-1)^2}{2}}} = e^{\frac{9(1-2z)}{2}}$$

Therefore, we will accept H_0 (i.e., a false alarm) if

$$e^{\frac{9(1-2z)}{2}} \geq c$$

Or, equivalently, $z \leq \frac{1}{2}\left(1 - \frac{2}{9}\ln c\right)$, where c is a preset threshold.