

Mathematical Finance II

*(Financial Derivatives in
Mathematical Finance)*

MSDM 5058

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Options and Markets

The financial markets encounter are

- Stock,
- Bonds,
- Currency,
- Commodity,
- futures and Options

The principal market discussed is stocks (= equities)

What is a Financial Derivative?

- A *financial derivative* is an instrument whose value depends on, or is derived from, the value of another asset.
- *Examples:* futures, forwards, swaps, options, exotics...
- Various types of investment needs generate different kinds of Financial derivatives
- The principal derivatives discussed here are *options*

Why Derivatives Are Important

- Derivatives play a key role in transferring risks in the economy
- The underlying assets include stocks, bonds, currencies, interest rates, commodities, etc.
- Many financial transactions have embedded derivatives
- The real options approach to assessing capital investment decisions has become widely accepted

Ways Derivatives are used

- To hedge risk
- To reflect a view on the future
- To lock in an arbitrage profit
- To change the nature of a liability
- To change the nature of an investment without incurring the cost of selling one portfolio and buying another
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How Derivatives Are Traded

- On exchanges such as the Chicago Board Options Exchange (CBOE)
- In the over-the-counter (OTC) market where traders working for banks, fund managers and corporate treasurers contact each other directly

The OTC Market Prior to 2008

- Largely unregulated
- Banks acted as market makers quoting bids and offers
- Master agreements usually defined how transactions between two parties would be handled
- But some transactions were cleared through central counterparties (CCPs). A CCP stands between the two sides to a transaction in the same way that an exchange does

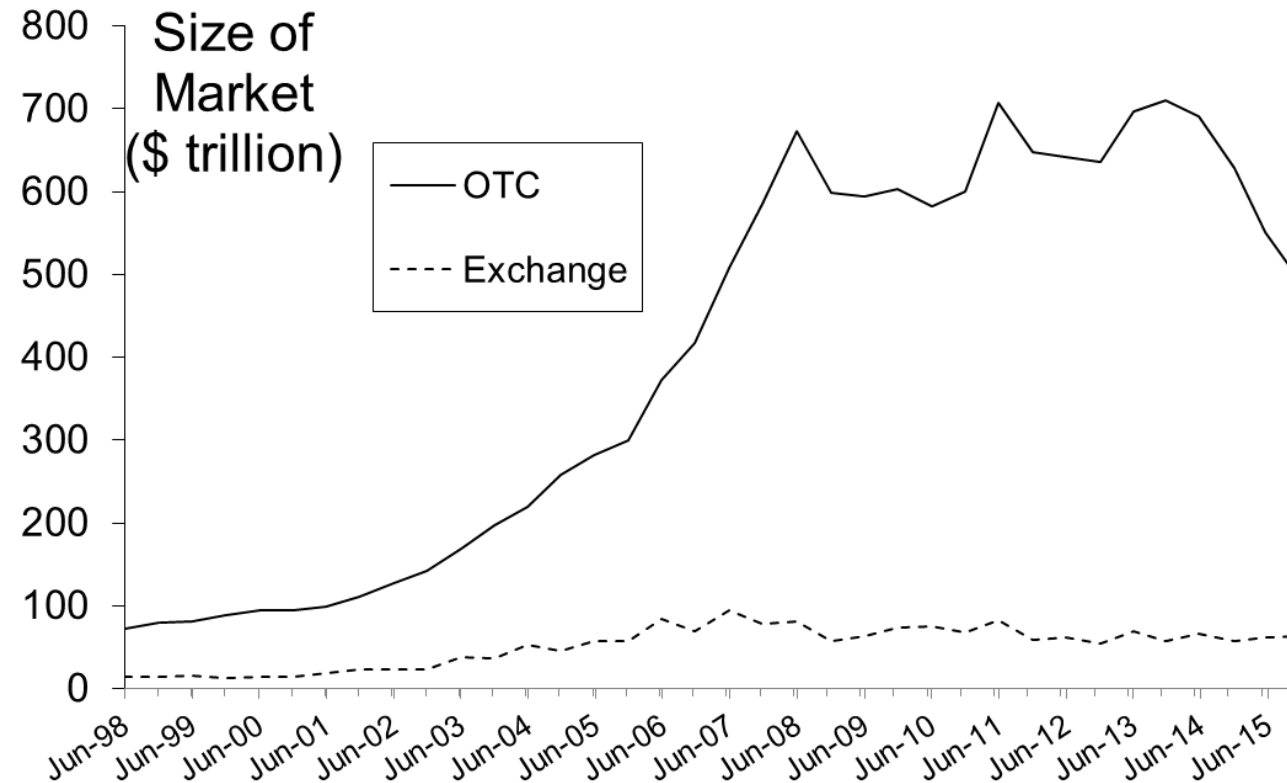
The Lehman Bankruptcy

- Lehman's filed for bankruptcy on September 15, 2008. This was the biggest bankruptcy in US history
- Lehman was an active participant in the OTC derivatives markets and got into financial difficulties because it took high risks and found it was unable to roll over its short term funding
- It had hundreds of thousands of transactions outstanding with about 8,000 counterparties
- Unwinding these transactions has been challenging for both the Lehman liquidators and their counterparties

Since 2008...

- OTC market has become regulated. Objectives:
 - Reduce systemic risk
 - Increase transparency
- In the U.S and some other countries, standardized OTC products must be traded on swap execution facilities (SEFs) which are electronic platforms similar to exchanges
- CCPs must be used to clear standardized transactions between financial institutions in most countries
- All trades must be reported to a central repository

Size of OTC and Exchange-Traded Markets



Source: Bank for International Settlements. Chart shows total principal amounts for OTC market and value of underlying assets for exchange market

Futures and Forward Contracts

- Agreement to buy or sell an asset for a certain price at a certain time
- Available on a wide range of assets
- Specifications need to be defined:
 - What can be delivered,
 - Where it can be delivered, &
 - When it can be delivered
- Settled daily
- Whereas a futures contract is traded on an exchange, a forward contract is traded OTC

Exchanges Trading Futures

- CME Group (formed when Chicago Mercantile Exchange and Chicago Board of Trade merged)
- InterContinental Exchange
- BM&F (Sao Paulo, Brazil)
- TIFFE (Tokyo)
- and many more

LONG and SHORT position

The party that has agreed to

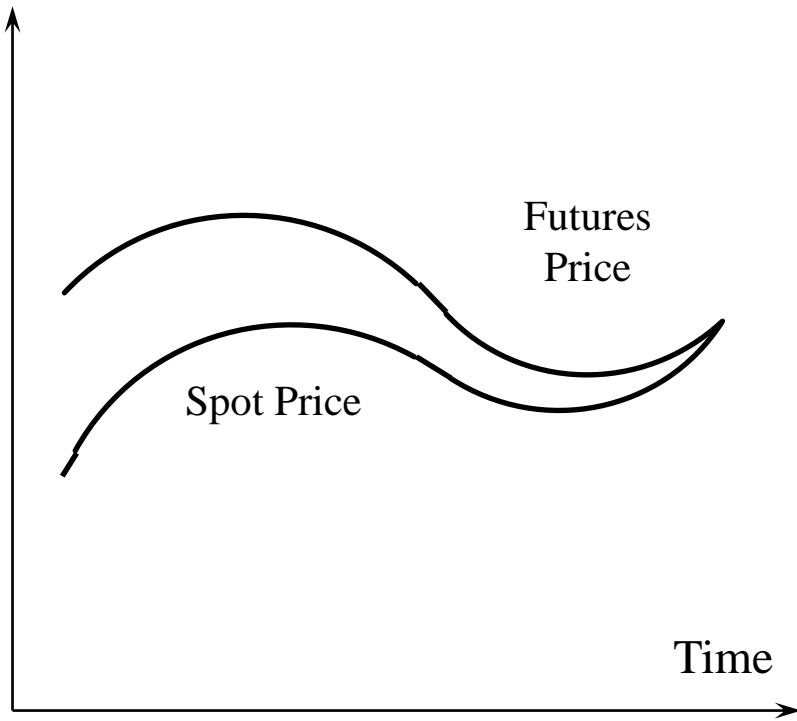
- BUY has what is termed a LONG position
- SELL has what is termed a SHORT position

Some Examples of Futures Contracts

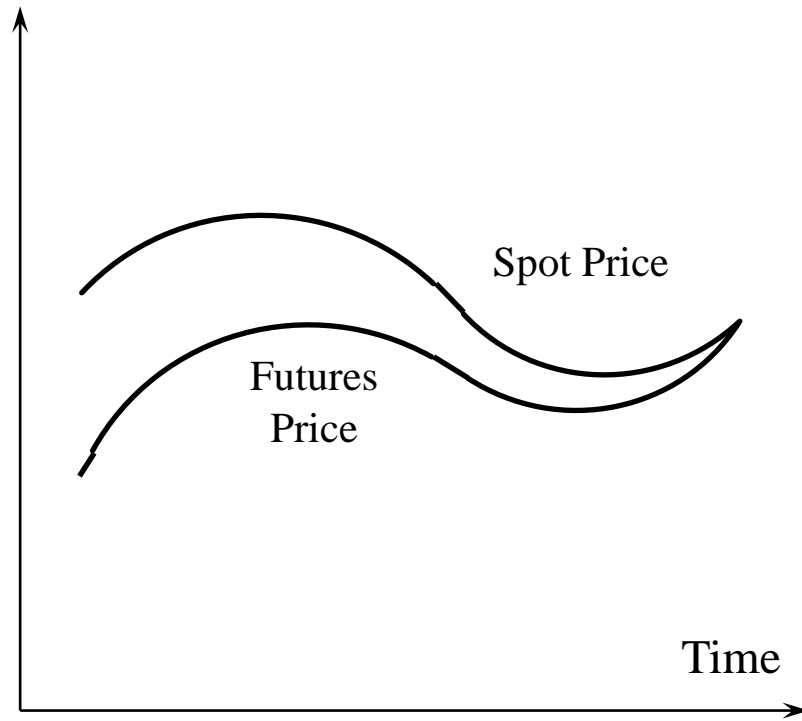
Agreement to:

- Buy 100 oz. of gold @ US\$1300/oz. in December
- Sell £62,500 @ 1.4500 US\$/£ in March
- Sell 1,000 bbl. of oil @ US\$50/bbl. in April

Convergence of Futures to Spot



(a)



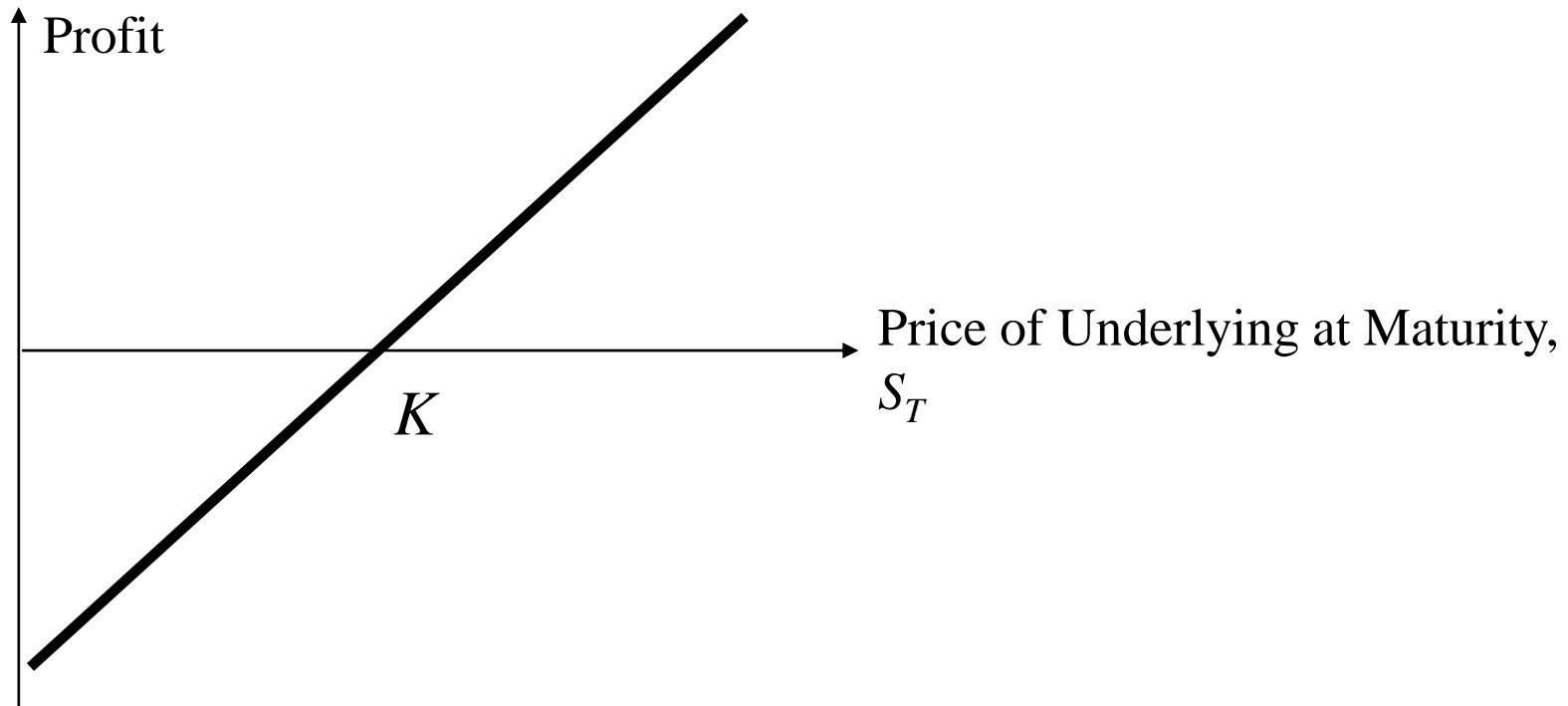
(b)

An Example:

- On May 3, 2016, the treasurer of a corporation enters into a long forward contract to buy £1 million in six months at an exchange rate of 1.4561
- This obligates the corporation to pay \$1,456,100 for £1 million on November 3, 2016
- What are the possible outcomes?

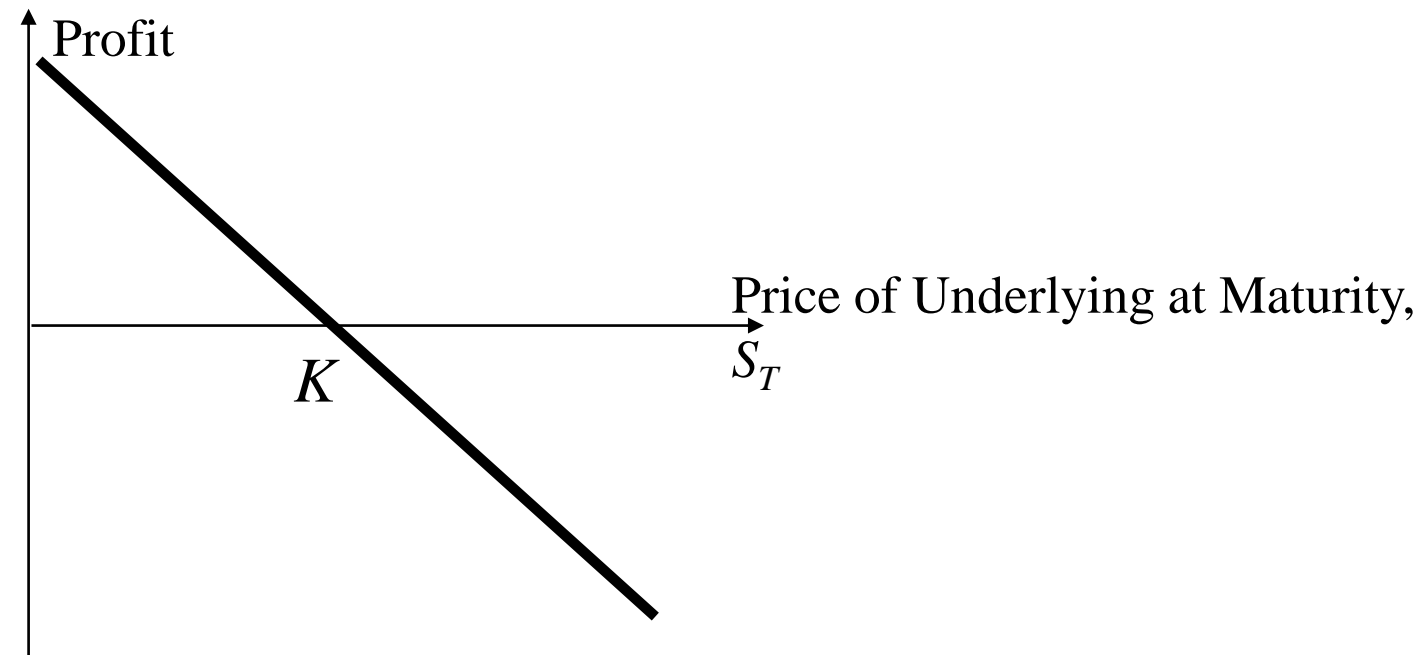
Profit from a Long Forward Position

(K = delivery price = forward price at time contract is entered into)



Profit from a Short Forward Position

(K = delivery price = forward price at time contract is entered into)



Options

- A *CALL* is an option to *BUY* a certain asset by a certain date for a certain price
- A *PUT* is an option to *SELL* a certain asset by a certain date for a certain price

Option Positions

LONG CALL

LONG PUT

SHORT CALL

SHORT PUT

Types of Options

- ***European options*** (simplest)
 - A European option can be exercised only at the end of its life
- ***American options*** (no restriction on time of processing the option)
 - An American option can be exercised at any time
- ***Asian option*** is more sophisticated in that the value can be some kind of average value of the asset in the history
- ***Barrier option*** is one that comes into existence or becomes worthless if the underlying asset reaches some prescribed value before expiry
- ***Lookback option*** is one with a price that depends on asset price (maximum or minimum)

Futures vs. Options

- A **FUTURES** contract gives the holder the *obligation* to buy or sell at a certain price
(you'll be punished if you do not follow the FUTURES contract)
- An **OPTION** contract gives the holder the *right* to buy or sell at a certain price
(no loss except the cost of writing the option contract if you don't process it)

Assets Underlying Exchange-traded Options

Stocks Foreign Currency, Stock Indices, Futures,

Specification of Exchange-traded options

Expiration date

Strike price

European or American or

Call or Put (option class)

Example: European Option

==> It is a contract between holder and writer

For Call option:

- The **holder** of the contract on the expiry date has the right to buy the underlying asset at a prescribed value (exercise price = strike price)
- The **writer** of the contract is the one who must sell the asset if the holder chooses to buy it

For Put option:

- It allows its **holder** to sell the asset on a certain future date for a prescribed amount (the strike price), although he does not need to sell it.
- The **writer** of the put option must always buy the asset at the strike price on the expiry date of the option if the holder chooses to sell it.

Cost of European Call option

- Since the option confers on its holder a right to buy but with no obligation to buy, he must pay for this flexibility at the time of opening the contract.
- Since the writer of the option must sell the asset if the holder chooses to buy it on the expiry date of the contract, he must be compensated for his loss of flexibility.

Market Value of European Option

Two questions are now raised:

- 1) How much would the holder be willing to pay for his right to buy?
- 2) How much should the writer charge so that his risk associated with his obligation be compensated?

Example: European Option

Question: How much is the following option on HK Bank worth, giving that the holder of this option may on the expiry date buy a share of HK Bank for \$100? The expiry date is set to be 12 months from now.

- *If the price of HK Bank share on the expiry date is \$130, then the holder can buy this share using only \$100, so he will make a profit of \$30*
- *If the price on the expiry date is \$90, then the holder will not buy this share for \$100. Therefore, the holder will not exercise his option*

Expected profit for holder of call option

- If the price of a stock A can only be \$90 or \$130 on expiry date of the option, with equal probability, then the expected profit is $0.5*0 + 0.5*30 = \$15$. Thus, the holder should be willing to pay for this option at \$15.
- If the share price on expiry date is \$130, he will make a net profit of $30 - 15 = \$15$, this means that he makes a *100% gain on his investment of paying for this option.*

Expected Loss for holder of call option

If the price of Stock A on expiry date of the option is less than the exercise price of \$100, the holder will lose all his initial premium of \$15

 ***A loss of 100%!!!***

Gearing

For an ordinary stock buyer, if today Stock A price is \$100, then if on the expiry date the price is \$90, he will lose \$10 or *10% of his investment*.

If the price on the expiry date is \$130, he will gain \$30, or *30% of his investment*.

—————→ *Options magnify effect of gain and loss!!!*

Dependence of Call Option Price

- Stock price at present
- Time before expiry date
- Randomness of stock price = volatility
- Bank interest rate

- Other factor...

What are options for?

- *Speculation*
- *Hedging (Risk Reduction)*

Types of Traders

- *Hedgers*

- Hedge the risk it is exposed to (avoid exposure to adverse movements in the asset price).

- *Speculators*

- Takes a view on the future direction of the market and gamble on this view and expect to earn money.

- *Arbitrageurs*

- Enter into transactions simultaneously in two or more markets to lock in a riskless profit.

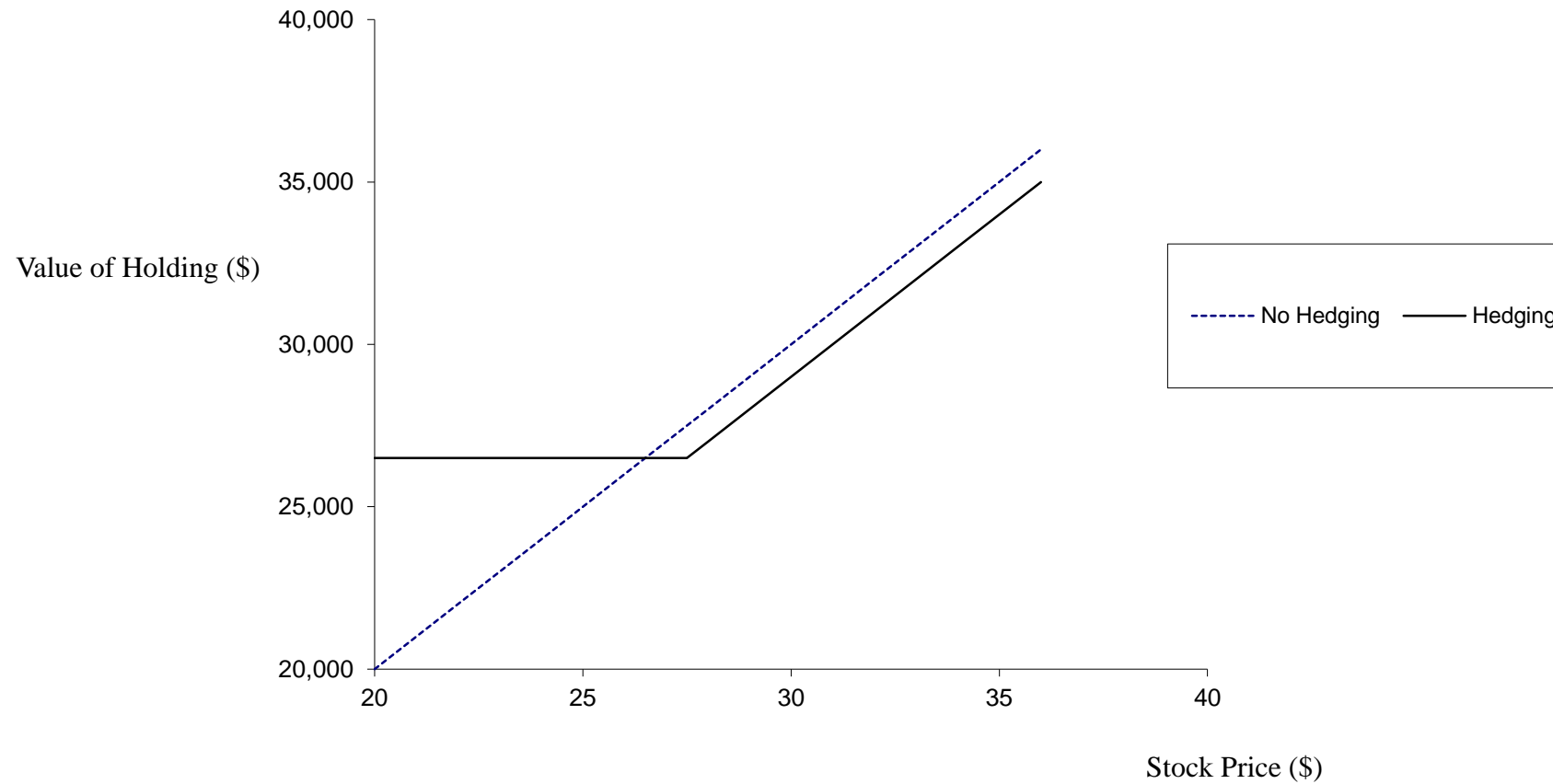
HEDGING

Hedging is a strategy that tries to limit risks in financial assets. For example, by taking advantage of the correlation between stock price and put option to reduce risk.

Hedging Examples

- A US company will pay £10 million for imports from Britain in 3 months and decides to hedge using a long position in a forward contract
- An investor owns 1,000 Microsoft shares currently worth \$28 per share. A two-month put with a strike price of \$27.50 costs \$1. The investor decides to hedge by buying 10 contracts

Value of Microsoft Shares with and without Hedging



Speculation: For holder of option

- If I think that Stock A price will rise, and I want to earn more by gearing, then I will buy a call option at a low strike price
 - If I think that the stock price will fall in the future, I can make more by buying a put option at a high strike price.
- ➡ *Holder of options is a cheap way of exposing a portfolio to a larger amount of risk.*

Speculation Example

- An investor with \$2,000 to invest feels that a stock price will increase over the next 2 months. The current stock price is \$20 and the price of a 2-month call option with a strike of 22.50 is \$1
- What are the alternative strategies?

Arbitrage

The simultaneous purchase and sale of an asset in order to profit from a difference in the price.

It is a trade that profits by exploiting price differences of identical or similar financial instruments, on different markets or in different forms.

Arbitrage exists as a result of market inefficiencies; it provides a mechanism to ensure prices do not deviate substantially from fair value for long periods of time.

Arbitrage Example

- A stock price is quoted as £100 in London and \$150 in New York
- The current exchange rate is 1.5300
- What is the arbitrage opportunity?

Gold: An Arbitrage Opportunity?

Suppose that:

- The spot price of gold is US\$1,200
- The 1-year forward price of gold is US\$1,300
- The 1-year US\$ interest rate is 5% per annum

Is there an arbitrage opportunity?

Is there an arbitrage opportunity if the 1-year forward price of gold is US\$1,200

The Future Price of Gold

If the spot price of gold is S and the future price for a contract deliverable in T years is F , then

$$F = S (1+r)^T$$

where r is the 1-year (domestic currency) risk-free rate of interest.

In our examples, $S = 1200$, $T = 1$, and $r = 0.05$ so that

$$F = 1200(1+0.05) = 1,260$$

Oil: An Arbitrage Opportunity?

Suppose that:

- The spot price of oil is US\$50
- The quoted 1-year futures price of oil is US\$60
- The 1-year US\$ interest rate is 5% per annum
- The storage costs of oil are 2% per annum

Is there an arbitrage opportunity?

Is there an arbitrage opportunity if the quoted 1-year futures price of oil is US\$40

Key idea of Arbitrage Pricing Theory

- ***No Free Lunch Theorem:***

In equilibrium, a portfolio of stocks that entails a zero net investment and that has no risk should have a zero expected rate of return

- What does this mean?

There is No chance to make an INSTANTANEOUS risk free profit

Risk reduction: For Holder of Option

Hold a put option to guard against unexpected fall in stock values

Suppose the stock price falls, then the value of a put option will go up and the holder of put can sell it at the higher strike price compared to the declining stock value.

Risk reduction against stock price falls

What happens to a portfolio that contains both real stock shares and put options?

If the portfolio contains only stock shares, the value of the portfolio will fall when stock price falls.

On the other hand, if I am also holder of put options, the value of the options will rise as stock price falls, therefore, the value of the portfolio containing both stock shares and put options will not change greatly even if the stock price falls.

➡ *Hence a combination of shares with holders of put option will reduce the risk of stock price fall.*

Risk free profit

If one can find the correct ratio of stock share and put options, such that a small unpredictable movement in the value of the stock price will not result in any unpredictable movement in the value of the portfolio, then this ratio is instantaneously ***Risk Free***

How to Insure Stock Against Falling Prices?

Suppose that I already have a sizable amount of Stock A shares as a long term investment, and I want to insure against a temporary fall in the share price, I also do *not* want to sell it and buy it back later, what can I do?

—→ *Become a holder of a put option with a high strike price*

Holder and Writer

Holder of a call option always wants the asset price to rise for then he can buy the asset at a low cost (relative low strike price) for something on the expiry date that will be worth more.

Holder of a put option always wants the asset price to fall so that he can sell it at a high price (relative high strike price) for something on the expiry date that will be worth less.

Writer of a call option always wants the asset price to fall for then the holder of call will not buy (for his strike price is high compared to the price on the expiry date) and the writer gets the premium from the holder of the call option.

Writer of a put option always wants the asset price to rise so that the holder of put will not sell (for his strike price is low compared to the price on the expiry date) and the writer gets the premium from the holder of the put option.

Why would anyone write a Call Option?

- Writer of a Call Option expects
the Stock Price to Fall
- Same view as the holder of a put option
- Take a short position of the underlying asset

Why would anyone write a Put Option?

- Writer of a Put Option expects
the Stock Price to Rise
- Same view as the holder of a call option
- Take a long position of the underlying asset

Geared-Increase as Stock Price Rises

If stock price rises, then the value of a call option will go up for then the holder of call can buy it at the lower strike price compared to the increasing stock value.

Portfolio of Stock and Call

If the portfolio contains only stock shares, the value of the portfolio will rise when stock price rises.

On the other hand, if I am also a holder of call options, the value of the options will rise more than the real stock because of gearing effect.

Thus, the value of the portfolio containing both stock shares and call options will increase more rapidly if the stock price rises.

Hence, a combination of shares with holders of call option will increase the yield of a stock price rise.

Bullish View on the Market

First View: Price goes up : Bullish Market

I can become a *holder* of call options (with a low strike price), or a writer of a put option (with a high strike price).

In either case, when the stock price does actually go up, I will make a profit.

Bearish View on The Market

Second View: Price goes down: Bearish Market

I can become a *writer* of call options (with a high strike price), or a holder of a put option (with a low strike price).

In either case, when the stock price does actually go down, I will make a profit.

Two Kinds of Risk:

- ***Specific risk*** is associated with a single asset (e.g., Sino-Land) or a single sector of the market (e.g., Chemical sector).
Specific risk can be reflected in the large volatility of its stock price, as unstable management would affect the company, but not so much the market.
- ***Non-specific Risk*** is associated with factors affecting the whole market.
Example: A change in the interest rate will affect all sectors of the economy.

Call Option

If $S > K$ at expiry (current value is greater than exercise price), it makes sense to exercise the call.

If $S < K$ at expiry, we will not exercise the call, for it will incur loss $= K - S$

In general, the profit from the exercise of a call option is either $S - K$ or 0

$$C(S, T) = \max(S - K, 0)$$

Put Option

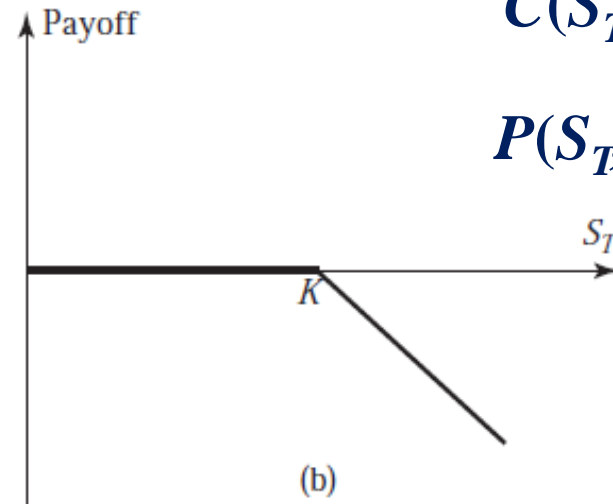
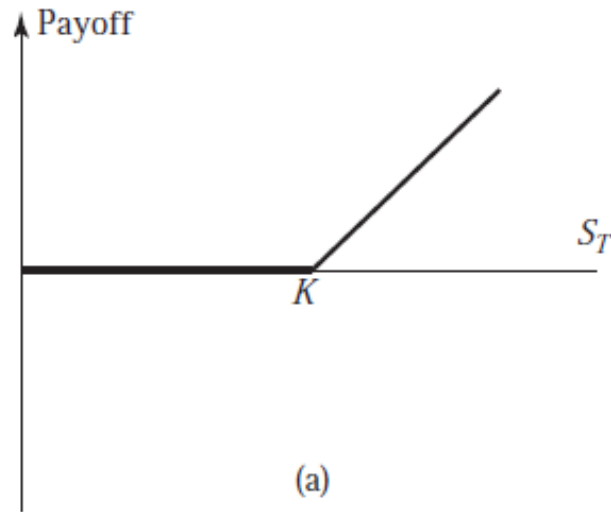
If $S > K$ at expiry, we will not exercise the put for it will incur loss = $S - K$, (Profit = 0)

If $S < K$ at expiry, we will exercise the put to make a profit = $K - S$

In general, the profit from the exercise of a put option is $K - S$ or 0

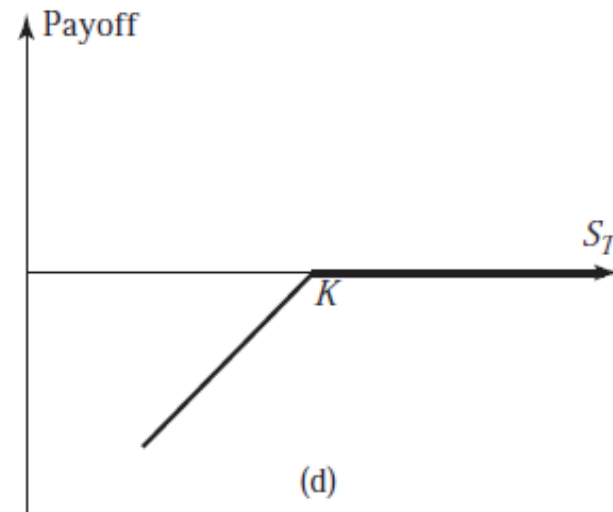
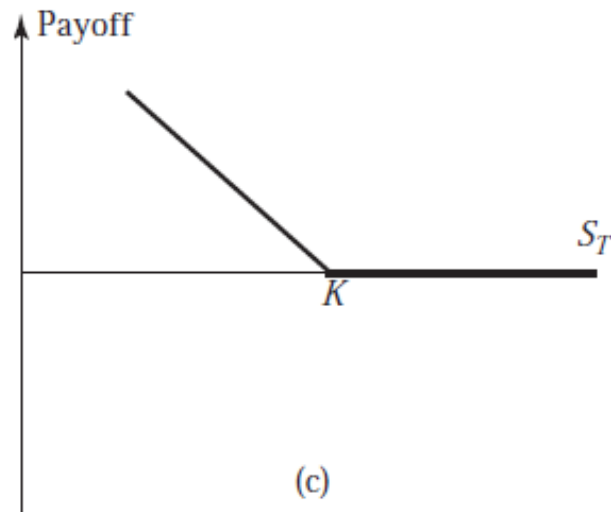
$$**$P(S, T) = \max (K - S, 0)$**$$

Payoff Diagram



$$C(S_T, T) = \max(S_T - K, 0)$$

$$P(S_T, T) = \max(K - S_T, 0)$$



Payoffs from positions in European options: (a) long call; (b) short call; (c) long put; (d) short put. Strike price K ; price of asset at maturity S_T

Option Pricing, Interest Rate and Present Value

Option involves the future price of a stock, compared to the present reasonable price for the option.

In considering paying for an option, we must consider the choice of putting the money in the bank to earn interest.

This leads to the question of *present value*.

How much would I pay NOW to receive a guaranteed amount K at the future time T , if the interest rate for the period is r ?

We assume a constant interest rate r for the entire period, but a changing interest rate can also be considered.

Value of the payoff

If interest rate is a constant r , money in the bank $M(t)$ will grow exponentially:

$$\frac{dM}{M} = r dt \rightarrow M = C e^{rt}$$

Since at time T , $M = K$

$$M = (K e^{-rT}) e^{rt} = K e^{-r(T-t)}$$

If interest rate is a function of time $r = r(t)$, then

$$M = K e^{-\int_t^T r(t') dt'}$$

Risk Free Investment

Risk free investment gives a guaranteed return

- government bond
- term deposit in a bank
-

***The greatest risk free return that one can make in a portfolio of assets is the same as the return from a term deposit in a bank.*

*==> It is **risk taking** that can generate higher return.*

Value of an Option

Value of an option is denoted as V

- $V=C(S, t)$: value of a call option
- $V=P(S, t)$: value of a put option
- S = current value of the underlying asset,
- t = time

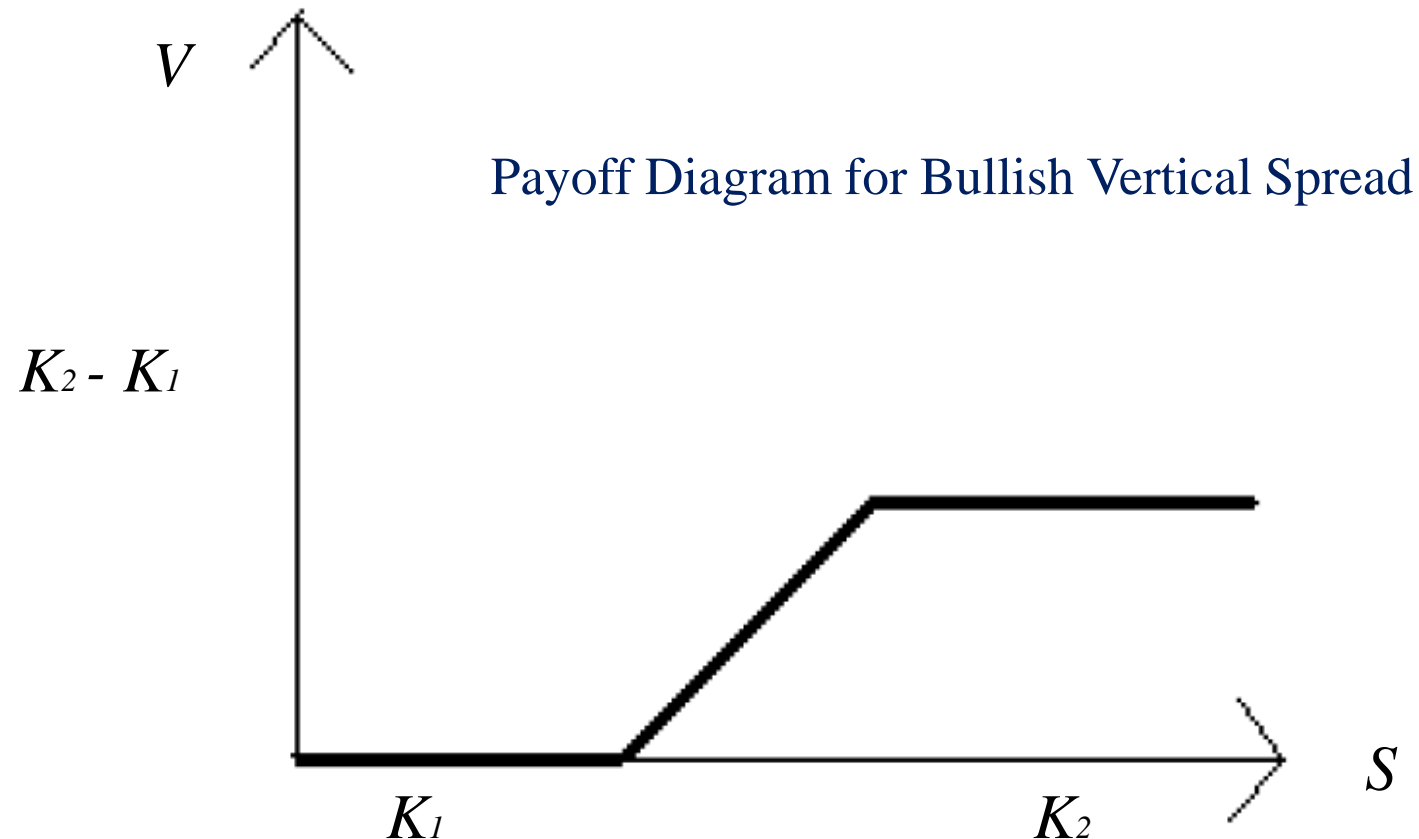
Excluding S and t , V also depends on

- σ = the volatility of the asset
- K = the exercise price
- T = the expiry time
- r = the interest rate

Example: Bullish Vertical Spread

Buy one call and write one call with the same expiry date but a larger exercise price (this is one of the simplest combination of options)

$$V = \max(S - K_1, 0) - \max(S - K_2, 0), \text{ where } K_2 > K_1$$



Put Call Parity: Risk Elimination

Long one asset & Long one put & Short one call

$$\Pi = S + P - C$$

Recall that

$$C(S, t) = \max(S - K, 0)$$

$$P(S, t) = \max(K - S, 0)$$

$$\text{If } S \leq K, \text{ then } \Pi = S + (K - S) - 0 = K$$

$$\text{If } S \geq K, \text{ then } \Pi = S + 0 - (S - K) = K$$

****How much will I pay for a portfolio that gives a *guaranteed* K at time $t = T$?**

What does the Arbitrage Pricing Theory Say?

The Arbitrage Pricing Model says return from this put-call parity portfolio must be the same as the return from a bank deposit.

$$\Pi = S + P - C = Ke^{-r(T-t)}$$

Why?

*If this were not true, then arbitrageurs will make an **instantaneous riskless profit** by buying and selling options and stock, at the same time borrowing or lending money in the correct proportions.*

How?

Borrow money from the Bank at rate r . The present value is

$$M = Ke^{-r(T-t)}$$

*Use this M to make the **Put-Call Parity***

If the value of the Put-Call Parity portfolio is greater than K at expiry time T , then there is a lock in a profit today with zero payoff in the future.

Pricing of Put Call Parity

Put-call parity always yields value K , and at interest rate r , the present value of put-call parity is

$$**$S \geq K, \text{ then } \Pi = S + 0 - (S - K) = K$**$$

Thus, if there is no arbitrage, (no riskless profit) we must have

$$**$\Pi = S + P - C = Ke^{-r(T-t)}$**$$

Put-Call Parity (No Dividends) : An Example

- Consider the following 2 portfolios:
 - Portfolio A: European call on a stock
+ zero-coupon bond that pays K at time T
 - Portfolio B: European put on the stock + the stock
- At time T , the value of $A = \max(S_T, K)$ = the value of B

		If $S_T > K$	If $S_T < K$
Portfolio A	Call option	$S_T - K$	0
	Zero-coupon bond	K	K
	Total value at T	S_T	K
Portfolio B	Put option	0	$K - S_T$
	Stock	S_T	S_T
	Total value at T	S_T	K

- To avoid arbitrage, the values of A and B at time 0 should be also the same, i.e., $C + Ke^{-rT} = P + S_0$ (**The Put-Call Parity**)

Arbitrage Opportunities : An Example

- Suppose that $C = 5$ $S_0 = 50$
 $T = 0.25$ $r = 6\%$
 $K = 48$ $D = 0$
- What are the arbitrage possibilities if $P = 4$?
(Hint: The value of A < the value of B, so buy A short B)

	At time 0	At time T if $S_T < K$	At time T if $S_T > K$
Long Call	-5	0	-48 + stock
Short Put	4	-48 + stock	0
Short stock	50	- stock	- stock
Investment	49	$49 e^{0.06 \times 0.25} = 49.74$	
Net profit		1.74	

Arbitrage Opportunities : An Example (cont'd)

- What if $P = 2$?
(Hint: The value of $A >$ the value of B , so short A buy B)

	At time 0	At time T if $S_T < K$	At time T if $S_T > K$
Short Call	5	0	48 – stock
Long Put	–2	48 – stock	0
Buy stock	–50	stock	stock
Investment	–47	$-47 e^{0.06 \times 0.25} = -47.71$	
Net profit		0.29	