1. $H(p) = -p \cdot \log_2 p - (1-p)\log_2(1-p)$

(a). $H(\frac{1}{3}) = \frac{1}{3}\log_2 3 - \frac{2}{3}(1-\log_2 3)$

$= \frac{1}{3} \times 1.585 - \frac{2}{3} + \frac{2}{3} \times 1.585$

$= 1.585 - \frac{2}{3} \approx 0.91833$

(b). $H(p) = \int_0^1 -x\log_2 x - (1-x)\log_2(1-x)\, dx$

$= \int_0^1 -x \cdot \frac{\ln x}{\ln 2} - (1-x)\frac{\ln(1-x)}{\ln 2}\, dx$

$= -\frac{2}{\ln 2}\int_0^1 x\ln x\, dx$

$= -\frac{2}{\ln 2} \cdot \left[\frac{x^2}{2}\ln x \Big|_0^1 - \int_0^1 \frac{x}{2} dx \right]$

$= -\frac{1}{2\ln 2}$.

2. $I(X;Y) = \iint p(x,y) \cdot \log \frac{p(x,y)}{p(x) \cdot p(y)}\, dxdy$

$= \iint p(x,y) \cdot \ln\left[\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \cdot \exp\left\{ -\frac{1}{2\sigma^2(1-\rho^2)}\left[x^2+y^2-2\rho xy\right] \right\} \Big/ \frac{1}{2\pi\sigma^2} \cdot \exp\left(-\frac{1}{2\sigma^2}(x^2+y^2)\right) \right] dxdy$

$= \iint p(x,y) \cdot \ln\left[\frac{1}{\sqrt{1-\rho^2}} \cdot \exp\left\{ -\frac{1}{2\sigma^2(1-\rho^2)} \left[\rho^2(x^2+y^2) - 2\rho xy \right] \right\} \right] dxdy$

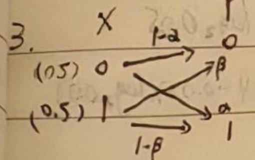
$= \iint \left[-\frac{1}{2\sigma^2(1-\rho^2)}\left[\rho^2(x^2+y^2) - 2\rho xy\right] - \ln\sqrt{1-\rho^2} \right] \cdot p(x,y)\, dxdy$.

Let $u = \frac{x}{\sqrt{1-\rho^2}}$, $v = \frac{y}{\sqrt{1-\rho^2}} \Rightarrow \iint \underbrace{\left[-\frac{1}{2\sigma^2}(\rho^2u^2 + \rho^2v^2 - 2\rho uv) - \ln\sqrt{1-\rho^2} \right]}_{} \cdot \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2}(t_1^2+t_2^2)\right) dt_1 dt_2$.

$t_1 = u - \rho v$, $t_2 = \sqrt{1-\rho^2} \cdot v \Rightarrow \underbrace{-\frac{1}{2\sigma^2}(\rho^2 t_1^2 - \rho^2 t_2^2 - 2\rho\sqrt{1-\rho^2}\, t_1 t_2) - \ln\sqrt{1-\rho^2}}_{}$

$= \iint \left[-\frac{1}{2\sigma^2} \cdot (\rho t_1 + (1-\sqrt{1-\rho^2})t_2) \cdot (\rho t_1 - (1+\sqrt{1-\rho^2})t_2) - \ln\sqrt{1-\rho^2} \right] \cdot \frac{1}{2\pi\sigma^2} \cdot \exp\left(-\frac{1}{2}(t_1^2+t_2^2)\right) dt_1 dt_2$

3.



(a). $P(X=0)=0.5=P(X=1)$

$H(X) = -\frac{1}{2}\cdot\log\frac{1}{2}\times 2 = \log 2 = 1.$ (suppose $\log_2 2$)

(b). $P(Y=0)=\frac{1}{2}(1-\alpha+\beta)$; $P(Y=1)=\frac{1}{2}(1-\beta+\alpha)$

$S = \sum_i -P_i \log(P_i)$

$$H(Y) = -\frac{1}{2}(1-\alpha+\beta)\log\frac{1}{2}(1-\alpha+\beta) - \frac{1}{2}(1-\beta+\alpha)\log\frac{1}{2}(1-\beta+\alpha)$$

$$= [\log(1-\alpha+\beta)-1]\cdot\left[-\frac{1}{2}(1-\alpha+\beta)\right] + [\log(1-\beta+\alpha)-1]\left[-\frac{1}{2}(1-\beta+\alpha)\right]$$

$$= 1-\frac{1}{2}\left(\log(1-\alpha+\beta)\cdot(1-\alpha+\beta) + \log(1-\beta+\alpha)\cdot(1-\beta+\alpha)\right)$$

(c). $\begin{cases} P(X=0,Y=0)=\frac{1-\alpha}{2} \\ P(X=0,Y=1)=\frac{\alpha}{2} \\ P(X=1,Y=0)=\frac{\beta}{2} \\ P(X=1,Y=1)=\frac{1-\beta}{2} \end{cases}$

$$H(X,Y) = -\sum\sum p(x,y)\cdot\log(p(x,y))$$

$$= -\frac{1-\alpha}{2}\log(1-\alpha) + \frac{1-\alpha}{2} - \frac{\alpha}{2}\cdot\log\alpha + \frac{\alpha}{2}$$

$$-\frac{\beta}{2}\log\beta + \frac{\beta}{2} - \frac{1-\beta}{2}\log(1-\beta) + \frac{1-\beta}{2}$$

$$= -\frac{1-\alpha}{2}\log(1-\alpha) - \frac{1-\beta}{2}\log(1-\beta) - \frac{\alpha}{2}\log\alpha - \frac{\beta}{2}\log\beta + 1$$

(d). $I(X;Y) = H(X)+H(Y)-H(X,Y).$

$$= 1+\frac{1}{2}\left[(1-\alpha)\log\frac{1-\alpha}{1-\alpha+\beta} + \beta\cdot\log\frac{\beta}{1-\alpha+\beta} + (1-\beta)\cdot\log\frac{1-\beta}{1-\beta+\alpha} + \alpha\cdot\log\frac{\alpha}{1-\beta+\alpha}\right]$$

(e). $C = \max_{P(x)} I(X:Y).$

so $\alpha=\beta=0$, $C_{max}=1.$

(f). $\alpha=\beta=\frac{1}{2}$, $C_{min}=0$

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4. (a). $H(X) = -0.01 \times \log_2 0.01 - 0.24 \times \log_2 0.24 - 0.05 \times \log_2 0.05$

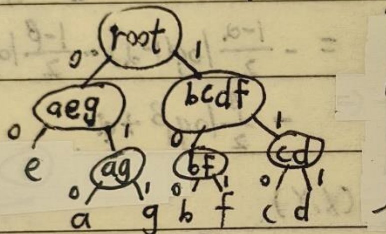
$-0.2 \times \log_2 0.2 - 0.47 \times \log_2 0.47 - 0.01 \times \log_2 0.01 - 0.02 \times \log_2 0.02$

$\approx 1.9323264.$

(b). Shannon:

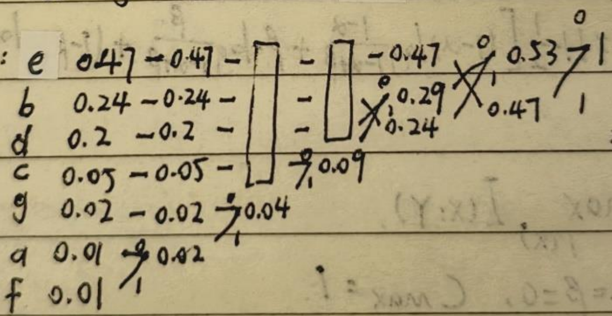
$$\left.\begin{aligned} L(a) = L(f) &= \lceil -\log_2 0.01 \rceil = 7 \\ L(b) &= \lceil -\log_2 0.24 \rceil = 3 \\ L(c) &= \lceil -\log_2 0.05 \rceil = 5 \\ L(d) &= \lceil -\log_2 0.2 \rceil = 3 \\ L(e) &= \lceil -\log_2 0.47 \rceil = 2 \\ L(g) &= \lceil -\log_2 0.02 \rceil = 6. \end{aligned}\right\} \begin{aligned} E &= \Sigma L \cdot p \\ &= 0.14 + 0.72 + 0.25 \\ &\quad + 0.6 + 0.94 + 0.12 \\ &= 2.77 \end{aligned}$$

Fano:



$$\left.\begin{aligned} &L(e) = 2 \\ &L(\text{other}) = 3 \end{aligned}\right\} \begin{aligned} E &= 0.47 \times 2 \\ &\quad + 0.53 \times 3 \\ &= 2.53 \end{aligned}$$

Huffman:



$$\Rightarrow \begin{cases} L(e) = 1 \\ L(b) = 2 \\ L(d) = 3 \\ L(c) = 4 \\ L(g) = 5 \\ L(a) = L(f) = 6 \end{cases}$$

$\Rightarrow E = 0.47 + 0.48 + 0.6 + 0.2 + 0.1 + 0.12 = 1.97$

(c). as in (b):

	e	b	d	c	g	a	f
Shannon	00	100	101	11100	111110	1111110	1111111
Fano	00	100	111	110	011	010	101
Huffman	1	01	000	0010	00110	001110	001111

(d). Huffman is the optimal code. $E(L) - H(X) \approx 0.0376736.$