1.

$y \backslash x$	1	2	3	4	$P(y)$
1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{4}$
2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{4}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$
4	$\frac{1}{4}$	0	0	0	$\frac{1}{4}$
$P(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	

$H[X,Y] = -\sum_x\sum_y P(x,y)\cdot\log P(x,y)$

$H[X|Y] = -\sum_x\sum_y P(x,y)\cdot\log P(x|y)$

$= \sum_y P(y)\cdot H[X|Y=y]$

$I[X:Y] = \sum_x\sum_y P(x,y)\cdot\log\frac{P(x,y)}{P(x)P(y)}$

(a). ↗

(b). $H[X,Y] = 2\times\frac{3}{8} + 6\times\frac{4}{16} + 1\times\frac{2}{4} + 4\times\frac{5}{32} = \frac{27}{8}$

$= \underset{\frac{7}{4}}{H[X]} + \underset{\frac{13}{8}}{H[Y|X]}$

$= \underset{2}{H[Y]} + \underset{\frac{11}{8}}{H[X|Y]}$

(c). $H[X] = \frac{1}{2}+\frac{1}{2}+\frac{3}{4} = \frac{7}{4}$

$H[Y] = 4\times\frac{2}{4} = 2$

(d). $H[X|Y=1] = \frac{1}{2}+\frac{1}{2}+\frac{3}{8}+\frac{3}{8} = \frac{7}{4}$ $\xrightarrow{y=1,\ =H[X]}$

$H[X|Y=2] = \frac{1}{2}+\frac{1}{2}+\frac{3}{8}+\frac{3}{8} = \frac{7}{4}$ $\xrightarrow{y=2,\ =H[X]}$

$H[X|Y=3] = 4\times\frac{2}{4} = 2$ $\xrightarrow{y=3,\ >H[X]}$

$H[X|Y=4] = 0$ $\xrightarrow{y=4,\ <H[X]}$

(e). $y=1,2,\ =H[X]$

$y=3,\ >H[X]$

$y=4,\ <H[X]$

(f).

$H[X|Y] = \frac{1}{4}\times\left(\frac{7}{4}\times2+2+0\right) = \frac{11}{8}$

Similarly, $H[Y|X=1] = \frac{1}{2}+\frac{3}{8}+\frac{3}{8}+\frac{1}{2} = \frac{7}{4}$

$H[Y|X=2] = \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$

$H[Y|X=3] = H[Y|X=4] = \frac{3}{2}$

$\Rightarrow H[Y|X] = \frac{1}{2}\times\frac{7}{4}+\frac{1}{4}\times\frac{3}{2}+2\times\frac{1}{8}\times\frac{3}{2}$

$= \frac{13}{8}$

(g). $I[X:Y] = 0\times4+\left(-\frac{1}{16}+\frac{1}{8}+0\times2\right)$

$+\left(-\frac{1}{16}+0+2\times\frac{1}{16}\right)+\frac{1}{4}$

$= \frac{1}{8}+\frac{1}{4} = \frac{3}{8}$

$= H[X]-H[X|Y] = \frac{14}{8}-\frac{11}{8} = \frac{3}{8}$

$= H[Y]-H[Y|X] = 2-\frac{13}{8} = \frac{3}{8}$

Verified. ✓

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2). (a) (i). $H = -\sum_{i=1}^{6} p_i \log p_i$, $L(p, \lambda_0) = H + \lambda_0(\sum_1^6 p_i - 1)$

$\Rightarrow$ best $p_i^* = e^{\lambda_0 - 1}$. combining with normalization $\Rightarrow p_i = \frac{1}{n}$.

(ii). $L(p, \lambda_0, \lambda_1) = H + \lambda_0(\sum_1^6 p_i - 1) + \lambda_1(\sum_1^6 i p_i - \frac{25}{6})$

$\frac{\partial L}{\partial p_i} = 0 \Rightarrow p_i^* = e^{\lambda_0 - 1 + \lambda_1 \cdot i}$.

combining with $\sum_1^6 p_i = 1$ and $\sum_1^6 i p_i = \frac{25}{6}$, use fsolve function

$\Rightarrow \lambda_0 \approx -1.70$; $\lambda_1 \approx 0.23634$

$[p_1, p_2, p_3, p_4, p_5, p_6] \approx [0.0852, 0.10792, 0.13669, 0.173134, 0.219293, 0.27776]$

(b). (i). $H = -\sum_{i=1}^{n} p_i \log p_i$. subject to $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} p_i \varepsilon_i = \bar{\varepsilon}$

$L(p, \lambda_0, \lambda_1) = H + \lambda_0 \cdot (\sum_{i=1}^{n} p_i - 1) + \lambda_1 (\sum_{i=1}^{n} p_i \varepsilon_i - \bar{\varepsilon})$

$\frac{\partial L}{\partial p_i} = 0 \Rightarrow p_i^* = e^{\lambda_0 - 1 + \lambda_1 \cdot \varepsilon_i}$ $(*)$

with $\sum_{i=1}^{n} p_i = 1 \Rightarrow e^{\lambda_0 - 1} = \frac{1}{\sum e^{\lambda_1 \varepsilon_i}}$

$\hookrightarrow p_i^* = \frac{e^{\lambda_1 \varepsilon_i}}{\sum_{i=1}^{n} e^{\varepsilon_i \lambda_1}}$

Let $\lambda_1 = -1/kT \Rightarrow p_i^* = \frac{e^{-\frac{\varepsilon_i}{kT}}}{\sum_{i=1}^{n} e^{-\frac{\varepsilon_i}{kT}}}$

(ii) hence with $\sum_{i=1}^{n} p_i \varepsilon_i = \bar{\varepsilon}$

$\Rightarrow$ the mean energy is given by: $\frac{\sum_{i=1}^{n} \varepsilon_i e^{-\frac{\varepsilon_i}{kT}}}{\sum_{i=1}^{n} e^{-\frac{\varepsilon_i}{kT}}} = \bar{\varepsilon}$.

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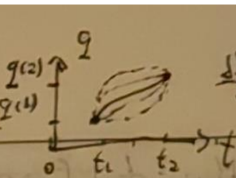


$\frac{\delta \int \rho\, dt}{\delta q} = 0$ at $q(t)$

(3). (a). $V = (\dot{P}, \dot{Q}) = (\dot{p}_\alpha, \dot{q}_\alpha)$

$\dot{p}_\alpha = \frac{d}{dt} p_\alpha$, $\dot{q}_\alpha = \frac{d}{dt} q_\alpha$

$\because f(\rho) = f[\rho(\underset{\text{position}}{q_\alpha}, p_\alpha, t)]$, $\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_\alpha (\dot{p}_\alpha \frac{\partial}{\partial p_\alpha} + \dot{q}_\alpha \frac{\partial}{\partial q_\alpha})$

$\therefore \frac{\partial f(\rho)}{\partial t} = -\nabla\cdot[f(\rho)\cdot V] = \frac{\partial f}{\partial \rho} \cdot \frac{\partial \rho}{\partial t}$.

$\because \delta \rho(q_\alpha, p_\alpha, t)$

$= \sum_\alpha \left[\frac{\partial \rho}{\partial q_\alpha}\delta q_\alpha + \frac{\partial \rho}{\partial p_\alpha}\delta p_\alpha + 0\right] = \sum_\alpha \left[\frac{\partial \rho}{\partial q_\alpha}\delta q_\alpha + \frac{\partial \rho}{\partial p_\alpha}\delta p_\alpha\right]$

$$\left(\begin{aligned} &= \sum_\alpha \left[\frac{\partial \rho}{\partial q_\alpha}\delta q_\alpha + \frac{\partial \rho}{\partial p_\alpha}\frac{d}{dt}(\delta q_\alpha)\right] \\ &= \sum_\alpha \left[\frac{\partial \rho}{\partial q_\alpha}\delta q_\alpha + 0 - \frac{d}{dt}\frac{\partial \rho}{\partial p_\alpha}(\delta q_\alpha)\right] \\ &= \sum_\alpha \left[\left(\frac{\partial \rho}{\partial q_\alpha} - \frac{d}{dt}\frac{\partial \rho}{\partial p_\alpha}\right)\cdot \delta q_\alpha\right] \end{aligned}\right)$$

Note that $\delta q_\alpha = d\delta p_\alpha / dt$.

$\Rightarrow \frac{\partial f(\rho)}{\partial t} = -\sum_\alpha \left[\frac{\partial}{\partial q_\alpha}(f(\rho)\cdot\frac{d}{dt}q_\alpha) + \frac{\partial}{\partial p_\alpha}(f(\rho)\cdot\frac{d}{dt}p_\alpha)\right]$

$= -\sum_\alpha \left[\frac{\partial}{\partial p_\alpha}(f(\rho)\cdot\dot{p}_\alpha) + \frac{\partial}{\partial q_\alpha}(f(\rho)\cdot\dot{q}_\alpha)\right]$.

(b). $\int \frac{\partial(f(\rho))}{\partial t} dP dQ$

$= \int -\sum_\alpha \left[\frac{\partial}{\partial p_\alpha}(f(\rho)\dot{p}_\alpha) + \frac{\partial}{\partial q_\alpha}(f(\rho)\cdot\dot{q}_\alpha)\right] dP dQ$.

$\int \frac{\partial f(\rho)}{\partial t} dP dQ = 0 \Leftrightarrow \delta \int_{t_1}^{t_2} \sum_\alpha \left(\frac{\partial \rho}{\partial q_\alpha}\delta q_\alpha + \frac{\partial \rho}{\partial p_\alpha}\delta p_\alpha\right) dt = 0$. $(f(0) = 0)$

(Use Gauss theorem).

$= \left[\sum_\alpha \frac{\partial \rho}{\partial p_\alpha}\delta q_\alpha\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_\alpha \left(\frac{\partial \rho}{\partial q_\alpha} - \frac{d}{dt}\frac{\partial \rho}{\partial p_\alpha}\right)\delta q_\alpha dt = 0$

$\Leftrightarrow \frac{d}{dt}\frac{\partial \rho}{\partial p_\alpha} - \frac{\partial \rho}{\partial q_\alpha} = 0$. which is Lagrange's equation.

(c). $S = -k_B \int \underbrace{\rho \ln \rho}_{f(\rho)} dP dQ$

$\because \int \frac{\partial \rho \ln \rho}{\partial t} dP dQ = 0$

$\therefore \int \rho \ln \rho\, dP dQ = C \Rightarrow S = -k_B \cdot C$ is a constant in time.

actually $S = k_B \cdot \log \Delta\Gamma$, $\Delta\Gamma$ is the number of stationary states with energy distributed between E_0 and $E_0 + \delta E$.

(4). (a). $f'(a) = \frac{1}{a} - 1 \begin{cases} >0, \ f(a)\uparrow, & 0<a\le 1 \\ <0, \ f(a)\downarrow, & a>1 \end{cases}$

$\Rightarrow f(a)_{max} = f(1) = 0-1+1 = 0$, $\quad \ln x \le x-1$,

$\therefore f(a) \le 0$, if and only if $a=1$. with equality

(b). $\sum p \ln\frac{p}{q} = -\sum p\cdot \ln\frac{q}{p} \geq -\sum p\cdot\left(\frac{q}{p}-1\right)$. (Let $a=\frac{q}{p}. >0$),

$= -\sum(q-p) = \sum(p-q) = 0$. if and only if $p=q$ with equality.

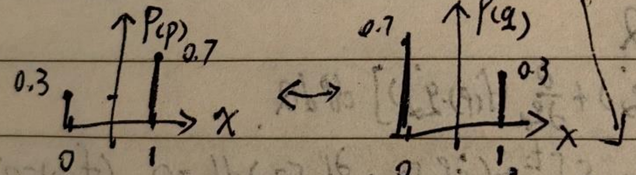
($\sum \approx \int$) $\quad$ ($\sum p = \sum q = 1$)

(c). (i). $D(p\|q) = \sum_x p(x)\log\frac{p(x)}{q(x)}$

$= 0.3\ln\frac{0.3}{0.7} + 0.7\cdot\ln\frac{0.7}{0.3} \approx 0.33892$

$D(q\|p) = 0.7\cdot\ln\frac{0.7}{0.3} + 0.3\cdot\ln\frac{0.3}{0.7} \approx 0.33892$

(ii). In this case, the parameter space is symmetrical.



ignoring the tags (0/1)

loss/probability.