MSDM5004 Spring 2024

Project 2

Due: 12 May 23:59

(a) Consider the composite mid-point rule:

$$\int_a^b f(x)dx \approx h\left[f_{1/2} + f_{3/2} + \cdots + f_{N-1/2}\right]$$

where

$$h = \frac{b-a}{N}$$

$$f_{k+1/2} = f\left(a + \left(k + \frac{1}{2}\right)h\right), \qquad k = 0, 1, 2, \cdots, N-1.$$

Show that the leading-order error is

$$\epsilon = \mathrm{c}h^2$$

for some constant c.

(b) Consider the integral

$$I = \int_{-\infty}^{\infty} e^{-x^4} dx.$$

Perform a change of variable $x \to z$ to turn the integral into one over a finite interval $[a, b]$:

$$I = \int_a^b g(z)dz.$$

State the new integrand $g(z)$ and the new limits a and b clearly. You should choose your new variable z such that $g(a)$ and $g(b)$ are both finite. Evaluate $g(a)$ and $g(b)$. If any one of them is in an indeterminate form (i.e., $\frac{0}{0}$, $0 \times \infty$, etc.), then evaluate the limits $\lim_{x \to a^+} g(x)$ and/or $\lim_{x \to b^-} g(x)$. You are required to obtain the analytic values of the limits (i.e., not numerically). You may, but are not required, to derive the limits by hand.

(c) Evaluate the integral in (b) by composite mid-point rule and tripling. You are required to include error estimation in your code so that your code can terminate when the relative error $\epsilon \leq 10^{-6}$ is achieved.

(d) Evaluate the integral in (b) by Romberg integration using mid-point rule and tripling instead of trapezoidal rule and doubling. You are required to include error estimation in your code so that your code can terminate when the relative error $\epsilon \leq 10^{-6}$ is achieved.

(e) Evaluate the integral in (b) by composite trapezoidal rule and doubling. You are required to include error estimation in your code so that your code can terminate when the relative error $\epsilon \leq 10^{-6}$ is achieved.

(f) Evaluate the integral in (b) by Gaussian quadrature. You are required to include error estimation in your code so that your code can terminate when the relative error $\epsilon \leq 10^{-6}$ is achieved.

(g) Now consider the integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x_1^4+s_2^4+x_3^4+x_4^4)} \frac{1}{1+x_1^2+x_2^2+x_3^2+x_4^2} dx_1 dx_2 dx_3 dx_4.$$

Sample 10^6 points with distribution

$$w(x_1, x_2, x_3, x_4) \propto e^{-(x_1^4+s_2^4+x_3^4+x_4^4)}$$

and evaluate the integral by Monte-Carlo integration. You are required to include error estimation in your code.