

5

Phase transitions on complex networks

Statistical mechanics has long studied how microscopic interaction rules between the elements of a system at equilibrium are translated into its macroscopic properties. In particular, many efforts have been devoted to the understanding of the phase transition phenomenon: as an external parameter (for example the temperature) is varied, a change occurs in the macroscopic behavior of the system under study. For example, a liquid can be transformed into a solid or a gas when pressure or temperature are changed. Another important example of phase transition is by the appearance in various metallic materials of a macroscopic magnetization below a critical temperature. Such spontaneous manifestations of order have been widely studied and constitute an important paradigm of the emergence of global cooperative behavior from purely local rules.

In this chapter, we first recall some generalities and definitions concerning phase transitions and the emergence of cooperative phenomena. For this purpose we introduce the paradigmatic Ising model, which is a cornerstone of the statistical physics approach to complex systems and, as we will see in other chapters, is at the basis of several models used in contexts far from physics such as social sciences or epidemics. After a brief survey of the main properties of the Ising model, we show how the usual scenarios for the emergence of a global behavior are affected by the fact that the interactions between the microscopic elements define a complex network.

5.1 Phase transitions and the Ising model

Phase transitions refer to the change between various states of matter, such as the solid, liquid, and gaseous states. The transition from one state to the other depends upon the value of a set of external parameters such as temperature, pressure, and density characterizing the thermodynamic state of the system. A classic example is provided by liquid water at atmospheric pressure which becomes solid ice at

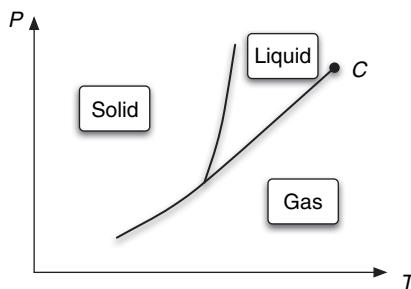


Fig. 5.1. Qualitative phase diagram of a material which, depending on the imposed pressure P and temperature T , can be in the solid, liquid or gaseous states. Note that the line separating liquid from gas ends at the critical point C .

precisely 0°C and boils at 100°C , entering the gaseous state. For each value of the pressure P , the transition occurs at a specific temperature. In Figure 5.1 we draw schematically the corresponding *phase diagram* of the system which shows the coexistence curve between the two phases and for which values of the control parameter(s) each phase is obtained. Generally, phase transitions involve latent heat: the system absorbs or releases energy when changing state, and a discontinuity appears in the parameter describing the state of the system in the two phases, such as the density in the case of liquid–gas transitions. At some special points of the phase diagram, however, the transition is continuous and the system is characterized by a change of state with no latent heat.

The paradigmatic example of continuous phase transitions is provided by the ferromagnetic phase transition in materials such as iron, nickel, or cobalt, where the magnetization increases continuously from zero as the temperature is lowered below the Curie temperature. From a microscopic point of view, the spin of the electrons of each atom in the material, combined with its orbital angular momentum, results in a magnetic dipole moment and creates a magnetic field. We can then depict each atom as carrying a permanent dipole often called simply “spin”. In paramagnetic materials, the magnetization is zero in the absence of any external magnetic field h , while the spins align in the direction of any non-zero value of h . In contrast, for ferromagnetic materials, neighboring spins tend to align in the same direction even for $h = 0$, as this reduces the configuration energy of neighboring atoms.¹ Below the transition temperature, this creates large domains of atoms with aligned spins, endowing the material with a spontaneous magnetization M . As the temperature increases the thermal fluctuations counteract the alignment, inducing spins to flip because of the increasing

¹ It is important to stress that the lower energy for paired spins is a purely quantum mechanical effect deriving from the spin-statistics theorem.

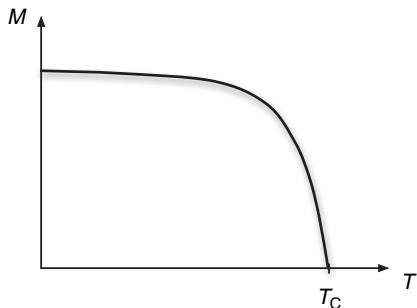


Fig. 5.2. Qualitative evolution of the macroscopic spontaneous magnetization M of a ferromagnet as a function of the temperature T . The magnetization M distinguishes between a ferromagnetic phase at $T < T_c$ and a paramagnetic phase at $T > T_c$.

energy available. When the temperature hits a critical value T_c (called the Curie temperature) the thermal fluctuations destroy the magnetic ordering and the system can no longer keep the spontaneous magnetization, settling in a paramagnetic phase. This change is continuous as more and more spins tend to escape the aligned state when temperature increases, as sketched in Figure 5.2. At the critical point there is a balance of spins in any direction and there is no net magnetization. The striking difference between the existence of a preferred alignment direction determined by the magnetization, and the disordered state with no favored direction, is a typical example of a *spontaneous symmetry breaking*. The magnetization defines the *order parameter*, which quantifies the degree of order in the system.

In order to have a microscopic description of the ferromagnetic phase transition, the most important ingredient is the interaction between spins, which tends to align them. So it seems natural, instead of considering the whole complexity of the atoms, to replace each one of them by a spin. These spins are placed on the sites of the crystalline lattice which describes the arrangement of atoms in the material. In most cases regular lattices (linear, square, or cubic) are considered. Moreover, the interaction between spins is short-ranged, so that, as a first approximation, only couplings between neighboring spins are retained. A further simplification resides in replacing the spin vectors σ_i by numbers σ_i which can take only two values, +1 or -1. This is equivalent to the idea that spins can take only two possible orientations (up for $\sigma_i = +1$, down for $\sigma_i = -1$).² In this simplified description of

² More realistic models such as the Heisenberg model or the XY model consider σ_i as matrices if quantum effects are taken into account or, more simply, three-dimensional or two-dimensional vectors. In particular, if the phase transition occurs at large enough temperature, the quantum effects can be completely neglected.

the ferromagnetic material, the Ising model is defined by the Hamiltonian which associates to each configuration of spins the energy

$$H = - \sum_{i \neq j} J_{ij} \sigma_i \sigma_j, \quad (5.1)$$

where J_{ij} represents the energy reduction if spins are aligned and is either $J > 0$ if a coupling exists between i and j (i.e. if i and j are neighbors on the lattice), or 0 otherwise. Thanks to its simplicity, the Ising model has been extensively studied either when the elements (spins) are located on the sites of a D -dimensional regular lattice, or when they are all connected to each other – $J_{ij} = 1$ for all (i, j) . The first case corresponds to situations close to real magnetic materials; in the second case, on the other hand, each spin is under the equal influence (or “mean field”) of all the others, which allows fluctuations to be neglected and analytical solutions or approximations to be obtained.

The above Hamiltonian tells us that at $T = 0$ the system tends to the minimum energy that is achieved when all the spins are aligned (all spins in the +1 state or in the -1 state). To see what is happening to the system when the temperature is increasing it is possible to simulate the model on the computer by using Monte Carlo methods. These mimic the temperature-induced fluctuations by flipping spins according to transition probabilities that bring the system to the Boltzmann equilibrium (see Chapter 4 and Yeomans [1992]). The spin system is put in equilibrium with a thermostat (the external world) which imposes a certain temperature T quantifying the amount of thermal agitation which favors disordered configurations. The most important question then concerns the existence of a phase transition, i.e. of values of the control parameter compatible with either ordered or disordered behaviors. The order parameter is given by the average magnetization $M = \langle \sum_i \sigma_i \rangle / N$, where N is the number of spins in the system. For a one-dimensional chain of Ising spins, it can be shown that, owing to the short range of the interactions, the ordered configuration is obtained only when the temperature is rigorously 0 and no phase transition at a finite temperature is observed (Huang, 1987). The situation is radically different in two dimensions. Figure 5.3 shows two typical snapshots of $N = 100^2$ Ising spins at equilibrium on a square lattice, at two different temperatures. At low enough temperature, a symmetry breaking between the two possible states +1 and -1 takes place, while at high temperature the system is in disordered configurations with a globally vanishing magnetization. In the thermodynamic limit of an infinite N , the competition between thermal agitation and energy minimization leads, as previously described, to a phase transition at a critical value of the control parameter $T = T_c$, between a high-temperature paramagnetic phase ($T > T_c$) with $M = 0$ and a low-temperature ferromagnetic ordered phase with a spontaneous magnetization

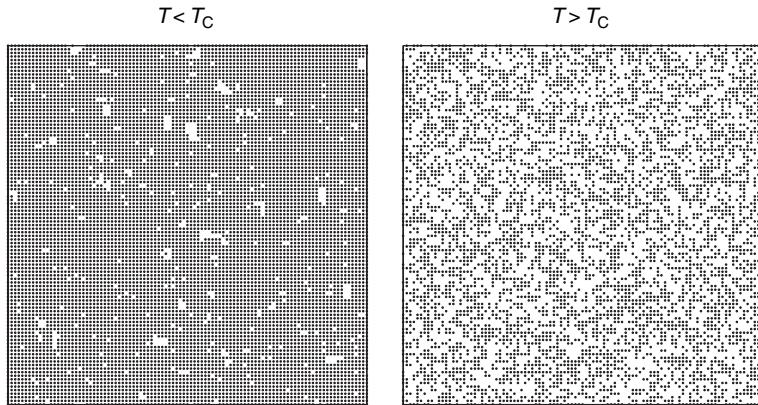


Fig. 5.3. Typical snapshots of a two-dimensional Ising system, below and above the critical temperature. Here the Ising spins are situated on the sites of a square lattice of linear size $L = 100$. Black dots represent spins $+1$, while empty spaces are left where spins take the value -1 . At low temperature, a spontaneous symmetry breaking (here in favor of the $+1$ phase) is observed.

$M \neq 0$ (Figure 5.2). This was shown rigorously by Onsager in 1944 (Onsager, 1944), who completely characterized the corresponding *critical behavior* giving the evolution of the quantities of interest close to the transition.

It is important to note that, while the Ising model was originally defined in the purely physical context of magnetism, the concept of phase transitions is of course much wider and as mentioned earlier has far-reaching applications. In fact, the Ising model itself lies at the basis of many models of social behavior and opinion dynamics, as will be seen in Chapter 10. Thanks to the simplicity of its definition, it has also become the paradigm of phase transitions and of collective phenomena.

5.2 Equilibrium statistical physics of critical phenomena

For an analytical insight into phase transitions we have to rely on the equilibrium statistical physics framework, which considers that the system can be in any possible microscopic configuration. The Hamiltonian then associates an energy $H(\sigma)$ to each configuration σ . As discussed in Chapter 4, at equilibrium the probability for the system to be in each configuration is given by its Boltzmann weight

$$P(\sigma) = \frac{1}{Z} \exp[-\beta H(\sigma)] \quad (5.2)$$

where $\beta = 1/(k_B T)$ is the inverse temperature (k_B is the Boltzmann constant), and $Z = \sum_{\sigma} \exp(-\beta H(\sigma))$ is the partition function ensuring normalization. At low temperature, β is large and therefore low-energy configurations are favored. At high temperature, the Boltzmann probability distribution (5.2) becomes more and

more uniform. Aligned spin configurations are therefore statistically favored in the low-temperature regime. Above the critical temperature, disordered configurations take over and the spontaneous magnetization is zero. The various properties of a model, defined by its Hamiltonian, are obtained through the calculation of (5.2). For example, given an observable O whose value depends on the configuration of the system, the average observed value of O at temperature T is given by

$$\langle O \rangle = \sum_{\sigma} O(\sigma) P(\sigma) = \frac{1}{Z} \sum_{\sigma} O(\sigma) \exp[-\beta H(\sigma)]. \quad (5.3)$$

The analytical solution of the partition function and the exact calculation for the order parameter are possible only in very special cases. In particular, at the transition point, the partition function is singular and the study of the critical properties has been one of the great challenges of statistical physics, culminating in the development of the renormalization group method (see, for instance, Ma, 2000).

The importance of critical phase transitions lies in the emergence at the critical point of cooperative phenomena and *critical behavior*. Indeed, close to the transition point, for T close to T_c , the thermodynamic functions have a singular behavior that can be understood by considering what is happening at the microscopic scale. By definition, the correlation between two spins is $\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$, and the correlation function $G(r)$ is the average of such correlations for pairs of spins situated at distance r . The correlation function measures the fluctuations of the spins away from their mean values, and vanishes for $r \rightarrow \infty$ at both low and high temperature where spins are either all aligned or fluctuate independently, respectively. In particular, at high temperature the correlation function decays with a correlation length ξ that can be considered as an estimate of the typical size of domains of parallel spins. As $T \rightarrow T_c$, long-range order develops in the system and the correlation length diverges: $\xi \rightarrow \infty$. More precisely, ξ increases and diverges as $|(T - T_c)/T_c|^{-\nu}$ and exactly at T_c no characteristic length is preferred: domains of all sizes can be found, corresponding to the phenomenon of *scale invariance* at criticality. Owing to the scale invariance at T_c , the ratio $G(r_1)/G(r_2)$ is necessarily a function only of r_1/r_2 , say $\phi(r_2/r_1)$. Such identity, which can be rewritten as $G(r/s) = \phi(s)G(r)$, has for consequences $G(r/(s_1s_2)) = \phi(s_1s_2)G(r) = \phi(s_1)\phi(s_2)G(r)$, which implies that ϕ is a power law and we have therefore at the critical point

$$G(r) \sim r^{-\lambda}, \quad (5.4)$$

where λ is an exponent to be determined. For temperatures close to but different from T_c , one assumes that only one characteristic scale is present, namely ξ . The

crucial *scaling assumption* then consists of writing that $G(r)$ rescaled by its value at the critical point is in fact only a function of r/ξ :

$$G(r, T) = r^{-\lambda} g(r/\xi(T)). \quad (5.5)$$

The quantity $g(r/\xi)$ tends to the constant $g(0)$ when ξ tends to infinity, recovering the power-law behavior $G(r) \sim r^{-\lambda}$. When the system is not exactly at the critical point, this behavior is modified in a way that depends only on $\xi \simeq |(T - T_c)/T_c|^{-\nu}$, i.e. on the distance of the control parameter from the critical point.

The system is therefore controlled by the characteristic length divergence and it is possible to express the singular behavior of any thermodynamic function $\langle O \rangle$ (assuming its limit exists) as a function of the adimensional deviation from the critical point $t = |(T - T_c)/T_c|$ as

$$\langle O \rangle = A|t|^\mu(1 + B|t|^{\mu_1} + \dots), \quad (5.6)$$

where μ is the critical exponent characterizing the scaling of the associated thermodynamic quantity. The typical example is the average magnetization, which is found to follow the law

$$M \approx (T_c - T)^\beta, \quad (5.7)$$

where the value of β depends on the Euclidean dimension of the lattice. Power-law behaviors and critical exponents are also observed for other quantities, such as the magnetic susceptibility or response function $\chi \sim t^{-\gamma}$ which gives the variation dM/dh of the magnetization with respect to a small applied magnetic field h that tends to align the spins.

The interest in critical exponents is also related to the *universality* displayed by many models which, despite having different definitions, share the same sets of critical exponents. The scale invariance allows us to understand intuitively that, close to the transition, which is a large-scale cooperative phenomenon, the overall macroscopic behavior of the system should not depend on its very detailed features. For T close to T_c , the critical behavior will be determined by the symmetries of the microscopic degrees of freedom and the embedding Euclidean space, but, for example, not by the shape of the corresponding regular lattice. In general, critical exponents depend on the dimensionality of the system and the symmetry of the order parameter and not on the detailed form of the microscopic interactions. This is a major conceptual point that helps in understanding why the critical behavior of a fluid system can be the same as that of a ferromagnetic material, despite the difference in the physical interactions at play.

At this point, let us insist on the fact that universality does not mean that the systems under study are equivalent. Universality concerns only the occurrence of a common behavior in the critical region, near a phase transition phenomenon. The

universal quantities are exponents, scaling behaviors, and scaling and correlation functions which describe the large-scale characteristics. At the local microscopic level, each system is still described by its own properties and details. The term universality thus refers to the possible similarities in the large-scale behavior of otherwise very different systems, and challenges us in searching for similarities in the underlying physical mechanisms producing these features.

5.2.1 Mean-field theory of phase transitions

The analytical calculation of the critical exponents and the singular behavior in critical phenomena has represented one of the toughest challenges tackled in statistical physics and culminated in the development of the renormalization group analysis (see, for instance, Ma, 2000). On the other hand, it is possible to grasp the basic physics of critical phenomena without getting into the complications of the renormalization group technique by using the most widely used approximation method: the mean-field theory. In mean field, one considers that each spin is under the equal influence of all the others. The contribution of a given spin σ_i to the total energy of the system, $-\sigma_i \sum_j J_{ij} \sigma_j$, is thus replaced by $-J\sigma_i \sum_j \langle \sigma_j \rangle$ where the sum runs over the neighbors of i , i.e. by $-J\langle k \rangle M \sigma_i$ where $\langle k \rangle$ is the average number of neighbors of each spin and M is the average magnetization. The self-consistency of the approximation imposes $M = \langle \sigma_i \rangle$, so that, by applying the usual statistical equilibrium mechanics to the spin σ_i at equilibrium at temperature T , one obtains³

$$\begin{aligned} M &= \frac{1}{Z} \sum_{\sigma_i=\pm 1} \sigma_i \exp\left(\frac{J\langle k \rangle M}{T} \sigma_i\right) \\ &= \frac{\exp\left(\frac{J\langle k \rangle}{T} M\right) - \exp\left(-\frac{J\langle k \rangle}{T} M\right)}{\exp\left(\frac{J\langle k \rangle}{T} M\right) + \exp\left(-\frac{J\langle k \rangle}{T} M\right)} \\ &= \tanh\left(\frac{J\langle k \rangle}{T} M\right). \end{aligned} \quad (5.8)$$

This self-consistent equation has the form $M = F(M)$, where F has some important properties. First of all, one notes that $F(0) = 0$ which means that $M = 0$ is always a solution of this equation. Moreover, $\lim_{M \rightarrow \infty} F(M) = 1$, F increases strictly (the first derivative $F'(M)$ is positive) and is concave (the second derivative $F''(M)$ is negative), for $M > 0$. This allows the use of the graphical method sketched in Figure 5.4 to find the non-zero solution of the self-consistent equation.

³ In the following for the sake of notation simplicity we will consider that the Boltzmann constant is unitary ($k_B = 1$). The complete expression can be recovered by simple dimensional arguments.

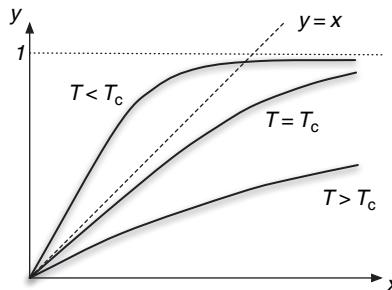


Fig. 5.4. Schematic representation of Equation (5.8). The dashed line represents $y = x$, continuous lines $y = F(x)$ for various values of T . A solution $M > 0$ exists if and only if the slope of the curve $y = F(x)$ in $x = 0$ is larger than 1.

The self-consistent solution for the magnetization is indeed given by the intersection of the curve $y = F(x)$ with the straight line $y = x$, which exists only if the derivative of F at $x = 0$ is larger than 1. This condition translates as

$$\frac{d}{dM} \left[\tanh \left(\frac{J\langle k \rangle}{T} M \right) \right] \Big|_{M=0} > 1. \quad (5.9)$$

This condition is satisfied if and only if T is smaller than a critical temperature T_c such that

$$\frac{J\langle k \rangle}{T_c} = 1. \quad (5.10)$$

Equation (5.8) is symmetric with respect to the change $M \rightarrow -M$, since the original Hamiltonian is itself symmetric if all spins are changed into their opposite ($\sigma_i \rightarrow -\sigma_i$) and therefore, as T decreases below $T_c = J\langle k \rangle$, two opposite non-zero solutions appear. It can be shown that these non-zero solutions are stable while the solution $M = 0$ is unstable for $T < T_c$. By expanding the self-consistent equation around the critical point T_c it is also possible to derive the behavior of the order parameter, leading to $M \approx (T_c - T)^{1/2}$, i.e. $\beta = 1/2$. Analogously the value of all critical exponents can be computed in the mean-field approximation.

While extremely simple, the mean-field theory is by nature only an approximation to the real solution. The main issue is that by definition the large fluctuations present at the critical point are not included in the theory, which is based on a homogeneous description of the system. Therefore, the mean-field results can only be valid when the fluctuations are not important. At the critical point, fluctuations are relevant and it is possible to show that the mean-field approximation is valid only above the so-called upper critical Euclidean dimension which is $D_c = 4$ for the Ising model. While this is a problem in most systems, we know that networks can be generally mapped into graphs that can be considered by definition as

infinite-dimensional objects. In this perspective, mean-field techniques are particularly useful in calculating the behavior of critical phenomena in complex networks, as we will see in various chapters of this book.

5.3 The Ising model in complex networks

The Ising model (5.1) considers an ensemble of variables that interact pairwise. The couplings J_{ij} can therefore be seen as the presence ($J_{ij} > 0$) or the absence ($J_{ij} = 0$) of a link between i and j , defining the topology of the *interaction* network. The recent discovery of the relevance of network representations in many areas of science has stimulated the study of models in which the spins interacting via Equation (5.1) are located on the nodes of a network whose topology can be more complex than the previously described cases. A first motivation for such studies stems from the fundamental interest in understanding how the topology of the interactions change the nature of the phase transition. Moreover, the Ising variables can represent not only magnetic spins, but also, for example, opinions in the field of social sciences (see Chapter 10). In this case, complex networks are the natural framework for representing social interactions, and understanding the emergence of collective behavior of variables defined on a network becomes a natural question. In this chapter, we will show how the different characteristics of complex networks determine various behaviors for the phase transition, depending on the structure of the network.

5.3.1 Small-world networks

A first property of complex networks that may influence the emergence of collective phenomena is the small-world behavior. In this case, the distances between the nodes of the network are small and the diameter scales as the logarithm of the network size N . This strikingly contrasts with finite-dimensional lattices where for a lattice of N sites in D dimensions, the average distance between nodes is of order $N^{1/D} \gg \log N$. For a mean-field topology, on the contrary, the distance between two nodes is by definition 1. Small-world networks therefore appear as an intermediate situation between these two well-studied cases. The case of the Ising model on the small-world networks of Watts and Strogatz (1998) is particularly interesting in this respect: their construction allows continuous interpolation from a one-dimensional lattice topology, for which no phase transition is observed, to an almost random graph, for which a finite T_c exists on random graphs, with a mean-field behavior for the transition (Kanter and Sompolinsky, 1987).

Let us consider a system of N Ising spins $\sigma_i = \pm 1$, $i = 1, \dots, N$, with Hamiltonian

$$H(\sigma) = -J \sum_{i=1}^N \sigma_i \sum_{j=1}^m \sigma_{\mu(i,j)} \quad (5.11)$$

where periodic boundary conditions are assumed (i.e. $\sigma_{N+j} = \sigma_j$), and where the independently and identically distributed numbers $\mu(i, j)$ are drawn from the probability distribution

$$P(\mu(i, j)) = (1-p)\delta_{\mu(i,j), i+j} + \frac{p}{N} \sum_{l=1}^N \delta_{\mu(i,j), l}. \quad (5.12)$$

For $p = 0$, this corresponds to a pure one-dimensional Ising model where every site is connected to its $2m$ nearest neighbors by ferromagnetic bonds of strength 1. At finite p , this corresponds to spins sitting on a Watts–Strogatz network in which each link of the one-dimensional chain has been rewired with probability p , yielding typically pmN long-range connections. For $p = 1$, the one-dimensional structure is completely replaced by randomly rewired links.

Barrat and Weigt (2000) have studied this model using the replica formalism. The obtained equations allow unveiling the behavior of the system. At high temperature, a stable paramagnetic phase is obtained for any p . The stability of this phase can be investigated at small p by means of a first order perturbation in p , i.e. an expansion around the topology of the one-dimensional model. The interesting observation is that the expansion contains powers of $p\xi_0$, where $\xi_0 \sim \exp[Jm(m+1)/T]$ is the correlation length of the one-dimensional system and implies that the first order approximation breaks down when $p\xi_0$ becomes of order 1. This condition is met either by increasing p or by decreasing T . At fixed temperature, there is therefore a crossover from a weakly perturbed one-dimensional system for $p \ll p_{\text{cr}}(T)$, with

$$p_{\text{cr}}(T) \propto \exp[-Jm(m+1)/T], \quad (5.13)$$

to a regime dominated by the network disorder for larger p . Conversely, at fixed and finite p , lowering the temperature until $T < T_{\text{cr}}$ with

$$T_{\text{cr}} \propto -Jm(m+1)/\log(p) \quad (5.14)$$

brings the system into a regime in which the network disorder dominates the system behavior, i.e. a mean-field regime in which a phase transition is expected. Different analytical treatments can also be applied. Viana Lopes *et al.* (2004) use combinatorial methods to show that the transition temperature in a modified Watts–Strogatz model (with deterministic shortcuts) behaves as $1/\ln p$, and

that the critical behavior is of mean-field type. Another approach is provided by Nikoletopoulos et al. (2004), who solve the problem using the replica formalism within a replica symmetric approximation.

It is worth explaining the physical meaning of the crossover between the one-dimensional and the mean-field behavior, since it is general and does not depend on the precise model defined on the small-world network. Similar arguments can in fact be used in the study of different dynamical models defined on small-world networks, such as the Voter model (Castellano, Vilone and Vespignani, 2003) or the Naming Game (Dall'Asta *et al.*, 2006b). In the pure one-dimensional model ($p = 0$), domains of correlated spins have the typical size ξ_0 which, for the Ising model, corresponds to the typical size of a domain of spins having the same value. The average number of long-range links arriving in such a domain is, by definition of the Watts–Strogatz model, of order $p\xi_0$. It is then intuitively clear that the behavior of the system will be very different for $p\xi_0 \ll 1$ and $p\xi_0 \gg 1$. In the first case, the interactions will be dominated by the one-dimensional structure, and in the second case, the numerous long-range connections will lead to a mean-field-like regime. This argument can be rephrased in terms of a competition between the size ξ_0 of correlated regions and the average distance $N^* \sim 1/p$ between two long-range links. At fixed network disorder p , and as the temperature T is lowered, ξ_0 increases: this implies that a change in the system behavior is expected at the crossover temperature T_{cr} such that $p\xi_0(T_{\text{cr}}) \sim 1$.

Gitterman (2000) studies a slightly different construction of the network, obtained by addition of long-range links to a one-dimensional network. While a finite p as N goes to infinity yields a finite *density* of long-range links, this study considers a finite *number* of these links (which would correspond in the original formulation to rewiring probabilities $p = \mathcal{O}(1/N)$). In this case, a phase transition appears as soon as p is larger than a certain f_c/N , with f_c finite, i.e. as soon as a sufficiently large number of long-range links is present in the network.

Various Monte Carlo simulations of Ising spins on Watts–Strogatz networks have confirmed the above picture and the existence of a phase transition of mean-field character for any finite p (Barrat and Weigt, 2000; Pekalski, 2001; Hong, Kim and Choi, 2002b). Numerical studies have also considered the Ising model on Watts–Strogatz networks constructed starting from a D -dimensional network. For $D = 2$ or 3 , a phase transition is present even in the ordered case (at $p = 0$). For any finite rewiring probability p , the character of this transition changes to mean-field, as indicated by the measure of the critical exponents (Herrero, 2002; Svenson and Johnston, 2002).

Finally, another paradigmatic model for the study of phase transitions – the XY model – has also been studied on small-world networks. In this model, spins have a *continuous* degree of freedom and are two-dimensional vectors of fixed length,

determined by their angle θ_i with respect to a reference direction. Their interaction is given by their scalar product, thus yielding the Hamiltonian

$$H = - \sum_{i \neq j} J_{ij} \cos(\theta_i - \theta_j). \quad (5.15)$$

In this case, the order parameter is given by the average value of $\sum \exp(i\theta_j)/N$ (where in this expression i does not design a lattice site but the imaginary square root of -1 : $i^2 = -1$). Numerical simulations of this model on small-world networks show that a phase transition with mean-field exponents is obtained even for very small values of p , suggesting a finite transition temperature for any finite p (Kim *et al.*, 2001; Medvedyeva *et al.*, 2003).

Interestingly, in all these cases, the physical argument taking into account the competition between the correlation length ξ_0 of the ordered system (at $p = 0$) and the average distance N^* between long-range connections also applies. Indeed, for the XY system, $\xi_0 \sim 1/T$ so that $\xi_0 \sim N^*$ implies $T_c \sim p$ at small p . For Watts–Strogatz networks constructed from 2- or 3- dimensional lattices, $\xi_0 \sim |T - T_c^0|^{-\nu}$, and $N^* \sim p^{-1/D}$ so that the shift in the critical temperature is given by $T_c - T_c^0 \sim p^{1/(D\nu)}$, where T_c^0 is the critical temperature for $p = 0$.

5.3.2 Networks with generic degree distributions

Another crucial ingredient of many complex networks, as explained in Chapter 2, lies in their heterogeneous connectivity structure signalled by a broad degree distribution. While Watts–Strogatz networks have bounded connectivity fluctuations, one can expect a heterogeneous structure for the underlying network to have dramatic consequences on the possible phase transitions of spin models defined on such networks.

Following numerical simulations of Aleksiejuk, Holyst and Stauffer (2002), this problem has been addressed analytically almost simultaneously by three different methods, by Bianconi (2002), Leone *et al.* (2002) and Dorogovtsev, Goltsev and Mendes (2002). These studies consider random graphs taken from an ensemble defined by a given degree distribution $P(k)$, i.e. without correlations between the degrees of neighboring sites. In this case, it is possible to write a simple mean-field approach, valid for generic networks (Leone *et al.*, 2002). As explained in Section 5.2.1, in the mean-field approximation each spin σ_i is under the influence of the average of its neighbors and the mean-field Hamiltonian reads as

$$H = -J \langle \sigma \rangle \sum_i k_i \sigma_i, \quad (5.16)$$

where k_i is the degree of node i . The various spins are therefore effectively decoupled. The average $\langle \sigma_i \rangle$ is then computed and has to be self-consistently equal to $\langle \sigma \rangle$. Solving the self-consistency equation allows its value to be obtained.

For spins located on the nodes of generic networks, the global average $M = \langle \sum_i \sigma_i \rangle / N$ mixes nodes of potentially very different degrees, so that it is more convenient to define the *average magnetization of the class of nodes with degree k* ,

$$\langle \sigma \rangle_k = \frac{1}{N_k} \sum_{i/k_i=k} \langle \sigma_i \rangle, \quad (5.17)$$

where N_k is the number of nodes having degree k . We will see in other chapters that this approach, which consists of dividing the nodes according to their degree classes, is very effective and helps in obtaining useful analytical approximations.

We define u as the average magnetization *seen by a node on its nearest neighbors*. It is important to note that this definition is not equivalent to that of the average magnetization M . As detailed in Chapter 1, the probability that any edge is pointing to a nearest neighbor with degree k is $kP(k)/\langle k \rangle$, and therefore the average magnetization seen on any given neighbor is given by

$$u = \sum_k \frac{kP(k)}{\langle k \rangle} \langle \sigma \rangle_k. \quad (5.18)$$

This expression considers the proper average over all possible degrees of the magnetization $\langle \sigma \rangle_k$ of the neighbors and is different from the average net magnetization $M = \sum_k P(k) \langle \sigma \rangle_k$. Nevertheless, the magnetization will be non-zero, indicating a ferromagnetic phase, if and only if $u \neq 0$. The mean-field approach then amounts to writing the Hamiltonian as

$$H = -J \sum_k \sum_{i/k_i=k} \sigma_i k u \quad (5.19)$$

i.e. to consider that a spin σ_i on a node of degree k “feels” a local field ku . The mean-field equation in each class thus reads

$$\langle \sigma \rangle_k = \tanh(\beta Jku). \quad (5.20)$$

The combination of (5.18) and (5.20) yields the following closed equation for u :

$$u = \sum_k \frac{kP(k)}{\langle k \rangle} \tanh(\beta Jku). \quad (5.21)$$

Similar to (5.8), this equation is of the form $u = F(u)$, with $F(0) = 0$ and F increasing and concave, so that a non-zero solution is obtained if and only if the derivative of F in 0 is greater than 1. This condition reads

$$\sum_k \frac{k P(k)}{\langle k \rangle} \beta J k > 1, \quad (5.22)$$

which means that the critical temperature is

$$T_c = J \frac{\langle k^2 \rangle}{\langle k \rangle}. \quad (5.23)$$

This result, valid for any network, relies on a crude mean-field approximation which can in fact be refined. A detailed replica calculation (Leone *et al.*, 2002) yields the result

$$\frac{1}{T_c} = \frac{1}{2J} \ln \left(\frac{\langle k^2 \rangle}{\langle k^2 \rangle - 2\langle k \rangle} \right), \quad (5.24)$$

and also allows an analysis of the critical behavior. In particular, the above expressions indicate that, in heavy-tailed networks where the degree fluctuations $\langle k^2 \rangle$ diverge in the thermodynamic limit $N \rightarrow \infty$ and $\langle k^2 \rangle \gg \langle k \rangle$, the transition temperature T_c is in fact infinite. Let us consider the case of scale-free networks, which display power-law distributions $P(k) \sim k^{-\gamma}$. The second moment $\langle k^2 \rangle$ and thus the critical temperature T_c are finite if and only if $\gamma > 3$. In fact, the transition is of mean-field character if $\gamma \geq 5$, while non-trivial exponents are found for $3 < \gamma < 5$. The dependence of T_c with γ is shown in Figure 5.5: as γ decreases towards 3, T_c diverges, and only the ferromagnetic phase survives for $\gamma < 3$ (Leone *et al.*, 2002). The effect of the heterogeneous network topology on the physics of phase transition is also revealed in the way the system approaches the thermodynamic limit. Phase transitions are rigorously defined only in the thermodynamic limit of an infinite number of spins. At finite but large N , however, a qualitative change of behavior can be observed as the temperature is lowered, with a stronger tendency of the system to display ordered configurations at $T < T_c^{\text{eff}}(N)$. In the usual case of a finite transition temperature T_c , such a change of behavior appears at a temperature $T_c^{\text{eff}}(N)$ which tends to T_c as N grows. In the case of (5.23), on the other hand, the change of behavior appears at higher temperature as the system size increases. In the thermodynamic limit, $T_c^{\text{eff}} \rightarrow \infty$ and the transition disappears, because the hubs polarize all the network and the only possible phase is then an ordered, ferromagnetic phase.

Another simple way of obtaining the previous results (5.24) has been proposed by Dorogovtsev *et al.* (2002), for uncorrelated networks: the mere fact that the distribution of the degree of a *neighbor* of a vertex is $k P(k)/\langle k \rangle$ implies that the neighbors of a randomly chosen vertex have an average degree $\langle k^2 \rangle/\langle k \rangle$. For Ising spins

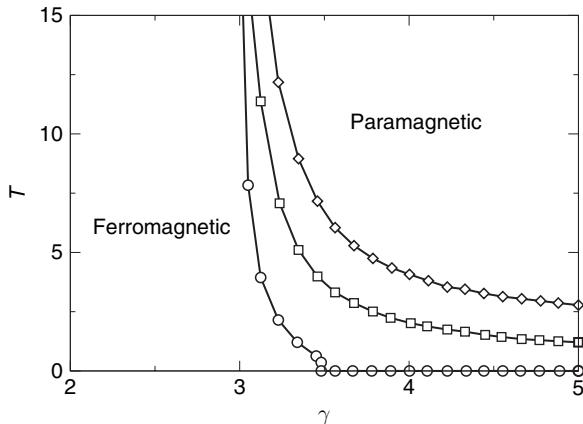


Fig. 5.5. Phase diagram of the Ising model on scale-free graphs with a power law degree distribution $P(k) = ck^{-\gamma}$, $m \leq k < \infty$. The ferromagnetic transition lines depend on the value of m , and are shown for $m = 1$ (circles), $m = 2$ (squares), and $m = 3$ (diamonds). For $m = 1$ and $\gamma > \gamma^* \simeq 3.47875$ the graph does not have a giant component so that $T_c = 0$. Data from Leone *et al.* (2002).

sitting on the nodes of a Cayley tree with coordination number q , the critical temperature is known and equal to $T_c = 2J / \ln[q/(q-2)]$ (Baxter, 1982). Assuming a local tree-like structure for the generic network allows this formula to be used with the substitution $q = \langle k^2 \rangle / \langle k \rangle$, recovering the previous result (5.24). More involved calculations (Dorogovtsev *et al.*, 2002) using recurrence relations show that this relation is in fact exact, still under the assumption of local tree-like structure and of the absence of correlations between the degrees of neighboring nodes.

It is also possible to extend the results for the Ising model to different statistical mechanics spin models. For example, the Potts model is a generalization of the Ising model to spins allowed to take q different values. The energy of a configuration of spins is then given by

$$H = - \sum_{i \neq j} J_{ij} \delta_{\sigma_i, \sigma_j}, \quad (5.25)$$

with couplings J_{ij} equal to $J > 0$ or 0, as in the ferromagnetic Ising case (which corresponds to $q = 2$). While the phase transition is of second order for the Ising model, it becomes of first order for $q \geq 3$, with a hysteresis region in temperature where both ferromagnetic and paramagnetic behaviors are possible. For this different type of transition, Dorogovtsev, Goltsev and Mendes (2004) show that the lower bound of the hysteresis region is given by

$$\frac{J}{T_c} = \ln \left(\frac{\langle k^2 \rangle + (q-2)\langle k \rangle}{\langle k^2 \rangle - 2\langle k \rangle} \right). \quad (5.26)$$

This temperature T_c thus diverges if $\langle k^2 \rangle$ diverges. In the case of scale-free networks with $P(k) \sim k^{-\gamma}$, this means that the first order phase transition survives if and only if $\gamma > 3$. The phenomenology is similar to what happens with the Ising model.

Equations (5.24) and (5.26) help one to grasp what makes phase transitions on heterogeneous networks remarkable. The appearance of the heterogeneity parameter $\kappa = \langle k^2 \rangle / \langle k \rangle$ shows that the topological characteristics of the interaction network strongly influence the behavior of the phase transitions. In particular, the critical temperature of an Ising model defined on a network with arbitrary degree distribution $P(k)$ is finite if and only if the second moment $\langle k^2 \rangle$ of the distribution is finite. In the opposite case, the large degrees of the hubs prevent any thermal fluctuation from destroying the long-range order: only a ferromagnetic phase is allowed. In general, if κ is large but finite, which can occur, for example, in networks for which the broad degree distribution is bounded by some structural cut-off, the phase transition occurs at a temperature that is much larger than in the case of homogeneous networks or finite-dimensional lattices.

5.4 Dynamics of ordering processes

Up to now in this chapter, we have reviewed the equilibrium properties of the Ising model, and shown how these properties are modified when the spins are located on the nodes of a network instead of regular lattices.

Physical systems are, however, not always at equilibrium. In particular, when the temperature is abruptly changed from T_1 to T_2 , the system, if it was at equilibrium at T_1 , is not at equilibrium at T_2 and has to adapt to the new temperature. This adaptation takes a certain time which corresponds to the approach to the new equilibrium state at T_2 . If both T_1 and T_2 correspond to the same equilibrium phase, either disordered or ordered, this adaptation or “transient” is fast since the properties of the system are not strongly changed. If on the other hand this temperature change crosses a phase transition, with $T_1 > T_c$ and $T_2 < T_c$ (“quench” to the ordered phase), the system has to evolve from disordered configurations towards the ordered, low-temperature ones. This *ordering process* has been largely studied for spin models, as a framework for the understanding of the ordering dynamics of magnets, in particular for spins located on regular lattices (see the review by Bray [1994]).

As put forward in Chapter 4, a convenient way to study dynamical processes is to perform numerical Monte Carlo simulations of the model under scrutiny (Krauth, 2006). One then has to specify the probability of transition from one configuration to another. In order to describe the relaxation to equilibrium, these transition probabilities need to satisfy the detailed balance condition. The most

popular choices are the Metropolis algorithm (see Chapter 4) and the so-called Glauber dynamics, in which the transition probability from one configuration σ to another σ' reads

$$W(\sigma \rightarrow \sigma') = \frac{1}{1 + \exp([H(\sigma') - H(\sigma)]/k_B T)}. \quad (5.27)$$

It is straightforward to check that this choice respects the detailed balance condition (4.5) $W(\sigma \rightarrow \sigma')/W(\sigma' \rightarrow \sigma) = \exp([H(\sigma) - H(\sigma')]/k_B T)$ and that the shape of the transition rates as a function of the energy difference is similar to the Metropolis algorithm. More precisely, the Glauber and Metropolis rates become close to each other at small values of the temperature T . In particular, at exactly $T = 0$, the only difference between Metropolis and Glauber transition probabilities occurs for $H(\sigma') = H(\sigma)$, i.e. if the proposed change of configuration does not modify the energy of the system. The probability of accepting this change is then 1 for Metropolis and 1/2 for Glauber dynamics. According to the concept of universality, the main features of the dynamics should not depend on this particular choice.

On regular lattices, spins forming a finite-dimensional lattice in an initially disordered (random) configuration tend to order when they are quenched below the critical temperature. This process, known as *coarsening*, is illustrated in Figure 5.6 for a two-dimensional system of Ising spins quenched at zero temperature (Bray, 1994). Starting from a disordered configuration, domains of aligned spins start to form and grow. The typical size of such domains grows with time: small domains are absorbed by larger ones, and the interfaces between the domains tend to become smoother and to decrease their curvature. For any finite system, at large enough times one domain overcomes the other and all the spins become aligned.

Interestingly, if quenched at strictly zero temperature, the system may also remain trapped in states that are not fully ordered, but consist for instance in

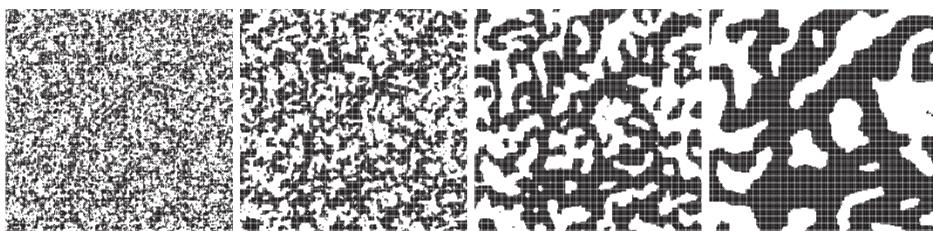


Fig. 5.6. Evolution of the Ising model with Glauber dynamics at zero temperature for spins located on the sites of a two-dimensional square lattice of linear size $L = 200$. Black dots represent spins +1, while empty spaces are left where spins take the value -1. From left to right the system, starting from a disordered configuration, is represented at times $t = 1, 4, 16, 64$.

two dimensions of alternating stripes of + and – spins (Spirin, Krapivsky and Redner, 2001; 2002). In larger dimensions, the probability of reaching a fully ordered configuration decreases and vanishes in the thermodynamic limit. The system is likely to end up in particular configurations in which domains of opposite spins coexist, each spin being aligned with its local field, and therefore unable to evolve, except for some spins called “blinkers” which have an equal number of + and – neighbors and therefore can flip without any energy change.

As a natural complement to the study of the equilibrium properties of spin models on complex networks, it is interesting to investigate how the approach to equilibrium occurs in these cases, and how the ordering process unfolds.⁴ Various efforts have therefore been devoted to the understanding of the prototypical dynamics of the Ising model quenched at zero temperature on complex networks. The main issues lie in the ability of the system to reach the perfectly ordered state corresponding to equilibrium, and the time needed for such ordering. Boyer and Miramontes (2003) have considered the Glauber dynamics at $T = 0$ of Ising spins located on the nodes of a small-world network obtained by adding, with probability p , to each site of a two-dimensional regular lattice, a link connected to a randomly chosen other site. Because of these shortcuts, the distances between nodes are smaller for $p > 0$ than for $p = 0$ and one could expect that this would favor the ordering process. On the contrary, it turns out that the shortcuts lead to an effective “pinning” of the interfaces because some sites have larger degree than others. At short times, domains of aligned spins form and grow according to the usual coarsening phenomenon. The coarsening, however, stops when the domains reach a typical size which depends on the disorder p , and the system remains frozen with coexisting domains of up and down spins.

For Ising spins on completely random networks, the situation bears some similarities. As noted by Svenson (2001), the Glauber dynamics at zero temperature does not always lead the system to full ferromagnetic ordering. In fact, Häggström (2002) has shown analytically that the dynamics always fails to reach the global energy minimum (ordered state) in the thermodynamic limit $N \rightarrow \infty$. Numerical investigations reveal that the system remains trapped in a set of configurations with two highly intertwined domains of opposite spins, having roughly the same size (Castellano *et al.*, 2005). The system is not frozen, and a certain number of spins keep flipping back and forth: these spins have an equal number of up and down neighbors, so that their state does not influence the energy of the system. In this stationary active state, the system wanders forever in an iso-energy set of configurations. For heterogeneous graphs, the global behavior of the Glauber dynamics at

⁴ Note that the study of the dynamics at exactly $T = T_c$ also allows the investigation of the critical properties of a model, see e.g. Medvedyeva *et al.* (2003); Zhu and Zhu (2003).

$T = 0$ is similar to the one exhibited on random graphs (Castellano *et al.*, 2005): with a certain probability, the system reaches a state composed of two domains of opposite magnetization with a large number of interconnections, and a certain number of spins which flip without changing the energy. When the system size grows the probability of reaching this disordered stationary state tends to increase, making full ordering less likely.

A mean-field theory of the zero-temperature Glauber dynamics has been put forward by Zhou and Lipowsky (2005), and further analyzed by Castellano and Pastor-Satorras (2006b). By dividing the nodes of a network into degree classes, it is possible to write evolution equations for the probability of a node of degree k to carry a spin +. For random uncorrelated networks, one can then obtain an estimate of the ordering time as a function of the network size N . For a network with degree distribution $P(k) \sim k^{-\gamma}$, this approach predicts that the ordering time grows logarithmically with N if $\gamma < 5/2$, and tends to a constant as $N \rightarrow \infty$ for $\gamma > 5/2$. These predictions are, however, invalidated by numerical simulations, which point out that the value $5/2$ does not play any particular role. For all the values of γ investigated, the probability for the system to reach full order goes to 0 in the thermodynamic limit. Moreover, the ordering time for the runs that actually order grows as a power law of N (Castellano and Pastor-Satorras, 2006b). The failure of the mean-field analysis can be understood through a careful analysis of the simulations, which shows the breakdown of the basic assumption that the local field seen by a spin is independent of its value. On the contrary, the probability for a + spin to have more + than – neighbors is very large (Castellano and Pastor-Satorras, 2006b), and strong correlations are present. The spins arrange themselves in two giant components of opposite signs which remain of similar size for a long time, competing through an extensive boundary, as shown in Figure 5.7.

Interestingly, although the heterogeneity of the network plays a crucial role in the equilibrium properties of the Ising model, it does not lead to strong differences in the behavior of the ordering Glauber dynamics: homogeneous and heterogeneous topologies lead to qualitatively similar behaviors.

5.5 Phenomenological theory of phase transitions

In the previous sections, we have considered a particular model, namely the Ising model, which represents the most paradigmatic microscopic model allowing for the emergence of collective phenomena through a phase transition. In this framework, the Hamiltonian (5.1) of the system is defined in terms of the microscopic variables, in this case the spins, which interact according to a certain topology. The order parameter is also defined as a function of the microscopic degrees of freedom, and the effects of the interaction topology are studied through

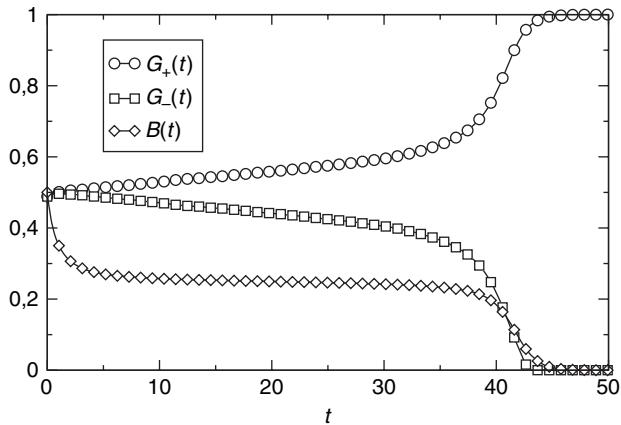


Fig. 5.7. Glauber dynamics at zero temperature for Ising spins located on the nodes of a random network with degree distribution $P(k) \sim k^{-\gamma}$. The figure shows the temporal evolution for one particular run (which finally reaches ordering) of the size, normalized by the total network size N , of the giant components of + and - spins, G_+ and G_- . The evolution of the boundary B , defined as the fraction of edges joining spins of opposite signs, is also shown. Here $N = 10^5$, $\gamma = 4$. Data from Castellano and Pastor-Satorras (2006b).

the solution of the model. It is possible to follow a different path to the understanding of phase transitions which is given by the phenomenological theory of Landau for phase transitions. In this approach, the starting point is the existence of the order parameter x , whose expression in terms of microscopic variables is not specified. The free energy of the system is written as an expansion in powers of x close to the phase transition and the properties of this transition are studied as a function of the expansion parameters. This expansion is justified by the fact that close to the transition the order parameter is supposed to be small as the phase transition separates an ordered phase with $x > 0$ at low temperature from a disordered one with $x = 0$ at high temperature. The thermodynamically stable state is obtained for the minimum of the free energy and this condition allows the calculation of the corresponding value of the order parameter as a function of the temperature (Huang, 1987).

The phenomenological approach has been adapted to the case of a model defined on a network with arbitrary $P(k)$ by Goltsev, Dorogovtsev and Mendes (2003). To take into account the possible heterogeneities of the network, the thermodynamic potential is written as

$$\Phi = -hx + \sum_k P(k)\phi(T, x, kx), \quad (5.28)$$

where h is the field conjugated to x . The term $\phi(T, x, kx)$ is the contribution of vertices with degree k : at a mean-field level, one can consider that a site with degree k receives from its neighbors an effective field kx . The expansion of Φ will therefore be in x and kx , and one can already anticipate that the moments of $P(k)$ will appear in the theory. One also assumes that $\phi(T, x, y)$ can be expanded as

$$\phi(T, x, y) = \sum_{l,m} \phi_{l,m}(T) x^l y^m. \quad (5.29)$$

In the framework of Landau's theory, the potential Φ is moreover directly written as an expansion in powers of the order parameter:

$$\Phi = -hx + f_2 x^2 + f_3 x^3 + f_4 x^4 + \dots, \quad (5.30)$$

where higher order terms are irrelevant and $f_2 = a(T - T_c)$ linearly vanishes at T_c , with $a > 0$. In the absence of an external field ($h = 0$), the minimum of Φ corresponds to the disordered phase $x = 0$ for $T > T_c$ and to an ordered phase with $x > 0$ for $T < T_c$. In the ordered phase, the value of the order parameter is determined by the condition that Φ is minimum, i.e.

$$\frac{d\Phi}{dx} = 0, \quad \frac{d^2\Phi}{dx^2} > 0. \quad (5.31)$$

The stability of the ordered phase near the transition thus implies $f_3 > 0$ or, if $f_3 = 0$, $f_4 > 0$.

The comparison of the two expansions of Φ allows writing, by identifying the coefficients of x^2 ,

$$f_2(T) = \phi_{20}(T) + \phi_{11}(T)\langle k \rangle + \phi_{02}(T)\langle k^2 \rangle, \quad (5.32)$$

which shows, since $f_2 \propto T - T_c$, that the critical temperature T_c depends on $\langle k^2 \rangle$. In particular, for $\phi_{20} = 0$, T_c is a function of the ratio $\langle k^2 \rangle / \langle k \rangle$, as it has indeed appeared in the previous subsections.

A careful analysis of the analytical properties of Φ at $x = 0$ can be carried out through its successive derivatives

$$\Phi^{(n)}(0) = n! \sum_{m=0}^n \phi_{n-m,m}(T) \langle k^m \rangle, \quad (5.33)$$

whose possible divergences are determined by the moments of $P(k)$: if $\langle k^m \rangle$ diverges for m larger than a certain p , then a singularity occurs in the p^{th} derivative of Φ . The appearance of such a singularity allows us to obtain deviations from the standard mean-field results. Goltsev *et al.* (2003) use for definiteness a scale-free distribution $P(k) \sim k^{-\gamma}$, and explore the various possibilities emerging from the

Table 5.1 Various possible critical behavior on random networks with a degree distribution $P(k) \sim k^{-\gamma}$, for the order parameter x and the response function χ_x .

	x	χ_x
$f_3 = 0,$ $f_4 > 0$	$\left\{ \begin{array}{ll} \gamma > 5 & \tau^{1/2} \\ \gamma = 5 & \tau^{1/2}/(\ln \tau^{-1})^{1/2} \\ 3 < \gamma < 5 & \tau^{1/(\gamma-3)} \end{array} \right.$	τ^{-1}
$f_3 > 0$	$\left\{ \begin{array}{ll} \gamma > 4 & \tau \\ \gamma = 4 & \tau/(\ln \tau^{-1}) \\ 3 < \gamma < 4 & \tau^{1/(\gamma-3)} \end{array} \right.$	
arbitrary f_3 and f_4	$\left\{ \begin{array}{ll} \gamma = 3 & e^{-cT} \\ 2 < \gamma < 3 & T^{-1/(3-\gamma)} \end{array} \right.$	T^{-1} T^{-2}

The critical behavior depends on the coefficients f_3 and f_4 , and on the exponent γ . Here $\tau \equiv 1 - T/T_c$, and c is a constant which is determined by the complete form of $P(k)$. If $f_3 < 0$, or if $f_3 = 0$ and $f_4 < 0$, at $\gamma > 3$, the system undergoes a first-order phase transition. From Goltsev *et al.* (2003).

variation of γ and the value of f_3 and f_4 . In particular, the critical behavior of the order parameter x and of the susceptibility $\chi_x = dx/dh$ are summarized in Table 5.1. The main finding is that the usual mean-field behavior is recovered only if $\gamma > 5$ in the case $f_3 = 0$, $f_4 > 0$ or $\gamma > 4$ for $f_3 > 0$. In the other cases, the divergence of moments of $P(k)$ leads to the appearance of a singularity in the expansion of Φ , and thus to an anomalous critical behavior. For $\gamma \leq 3$, the critical temperature is infinite in the thermodynamic limit, so that x and χ_x are not critical, but may be computed as a function of temperature (see Table 5.1).

The above results show that the effect of network connectivity fluctuations on the emergence of collective phenomena from microscopic rules can be generally described using the standard techniques of statistical physics. The randomness and small-world properties of the considered networks generically result in a mean-field character of the transition. For heterogeneous networks, however, a richer scenario emerges, which can be analyzed through mean-field approximate approaches.

In heterogeneous random networks, the local tree-like structure allows the system to be treated through mean-field methods. The vertices with large connectivity, however, induce strong correlations in their neighborhoods. Such hubs are much more numerous in heterogeneous networks than in Erdős–Rényi homogeneous random graphs. The possible divergence of moments of $P(k)$ determines the critical behavior, and the heterogeneity parameter, defined by the ratio of the two first

moments of the degree distribution, appears as a crucial quantity which determines the existence of a phase transition at a finite value of the control parameter. If the degree fluctuations diverge, the hubs polarize the system at any finite temperature, so that only the ordered phase can exist and the high-temperature disordered phase disappears.