MSDM 5058 Information Science
Assignment 4 (due 20th April, 2023)

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(1) **Stationary ARMA processes**

The following time series is an AR(2) process, given by

$$(1 - 1.1B + 0.18B^2)Y_t = \varepsilon_t$$
$$E[\varepsilon_t, \varepsilon_\tau] = \begin{cases} 1\ , & \text{for } t = \tau, \\ 0\ , & \text{otherwise.} \end{cases}$$

(a) Is it covariance stationary? Why? (*Refer to p.26 of the lecture on Time Series in Complex Systems*).

(b) If so, calculate its auto-covariance functions. (*Refer to p.36 of the lecture on Time Series in Complex Systems*).

Solution:

(a) Yes, the process is covariance-stationary. Since

$$(1 - 1.1c + 0.18c^2) = (1 - 0.9c)(1 - 0.2c)$$

Both roots of the above equation are outside the unit circle.

(b) The auto-covariance functions of this AR(2) process is

$$\gamma_0 = 1.1\gamma_1 - 0.18\gamma_2 + 1 \qquad (1)$$
$$\gamma_1 = 1.1\gamma_0 - 0.18\gamma_1 \qquad (2)$$
$$\gamma_s = 1.1\gamma_{s-1} - 0.18\gamma_{s-2} \qquad (3)$$

Therefore,

$$\gamma_2 = 1.1\gamma_1 - 0.18\gamma_0 \qquad (4)$$

From (1) and (4), we get

$$[1 - 0.18^2]\gamma_0 = 1.1(1 - 0.18)\gamma_1 + 1 \qquad (5)$$

From (2) and (5), we get

$$\gamma_0 = 7.889;\ \gamma_1 = 7.355$$

Substituting back into (4), we get $\gamma_2 = 6.670$. One can obtain other γ from (3).

(2) **Linear Regression Models**

You have a dataset and you want to perform a unit root test on this dataset. You run the following regression, where the numbers in parenthesis are the standard deviations (σ). Assume that the dataset has a large enough number of data points, i.e., $N \rightarrow \infty$.

$$\Delta X_t = 0.51(0.29) - 0.11(0.04)X_{t-1} - 0.39(0.09)\Delta X_{t-1} - 0.32(0.09)\Delta X_{t-2} - 0.22(0.10)\Delta X_{t-3} + 0.07(0.11)\Delta X_{t-4}\,. \qquad (1)$$

(a) What is the t-statistic on the lagged term X_{t-1}. Compare this to the ADF statistic, at what significance level can you reject the null hypothesis? (*Refer to the table at the end of this problem set.*)

(b) Now you want to forecast ΔX_t by using the following three regression models,

$$\Delta X_t = 0.002(0.014) - 0.31(0.10)\Delta X_{t-1}\,, \qquad (2)$$

$$\Delta X_t = 0.021(0.158) - 0.46(0.10)\Delta X_{t-1} - 0.39(0.11)\Delta X_{t-2} - 0.25(0.08)\Delta X_{t-3} + 0.03(0.07)\Delta X_{t-4}\,, \qquad (3)$$

$$\Delta X_t = 1.279(0.57) - 0.51(0.10)\Delta X_{t-1} - 0.44(0.11)\Delta X_{t-2} - 0.30(0.09)\Delta X_{t-3} + 0.02(0.08)\Delta X_{t-4} - 0.16(0.07)Y_{t-1}. \qquad (4)$$

where *Y* is an independent variable. At what significance level can you reject the null hypothesis for *Y*? (*Compare this with a t distribution table*).

(c) In addition, you have the following information on X_t for the next 5 time steps.

t	Y_t	X_t	ΔX_t
1	7.7	0.8	0.0
2	7.9	4.3	3.5
3	7.7	2.9	-1.4
4	7.0	1.3	-1.5
5	6.8	2.1	0.8

For each of the three models, calculate the predicted values for $\Delta X_{t=5}$. What is the forecast error in each of the three models?

Solution:

(a) The *t*-statistic of the lagged X_{t-1} is -2.75. Since the regression model in (1) has a constant, so we use the model in the table that has a constant but no time trend. The critical values for the ADF statistic is -2.57 and -2.86 at the 10% and 5% significance level respectively. Therefore, one can reject the null hypothesis at the 10% level but not the 5% level.

(b) The *t*-statistic of *Y* is -1.6/0.7 ~ -2.29. Compare this with the *t* distribution table, the values (for 2-sided) are -1.96 and -2.33 at 5% and 2% significance levels. Therefore, one can reject the null hypothesis at the 5% level but not the 2% level.

(c) Substituting the values into the three regression models:

1st model: $\Delta X_5 = 0.002 - 0.31(-1.5) = 0.467$

2nd model:

$$\Delta X_5 = 0.021 - 0.46(-1.5) - 0.39(-1.4) - 0.25(3.5) + 0.03(0.0) = 0.382$$

3rd model:

$$\Delta X_5 = 1.279 - 0.51(-1.5) - 0.44(-1.4) - 0.30(3.5) + 0.02(0) - 0.16(7) = 0.49$$

The forecast value of ΔX_5 is 0.8. Therefore, the forecast errors of the three models are 0.333, 0.418 and 0.31, respectively.

(3) **Fourier Transform of Impulses**

In this question, you will learn about the Fourier Transform of impulses. Let us begin by considering a Gaussian impulse. (*Use the notations of Fourier Transform in the lecture, p.57.*)

(a) Given the following Gaussian impulse, $f(t) = ce^{-b(t-t_0)^2}$, where *t* is the time variable. What is its Fourier Transform? What is the Fourier Transform of this impulse look like?

(b) A Dirac delta function δ(*x*) has the property that,

$$\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases}$$

with the additional feature that

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \quad .$$

Let us approximate a Dirac delta impulse $\delta(t - t_0)$ by using the Gaussian impulse above, where $c = \frac{1}{\sqrt{2\pi\sigma^2}}$ and $b = \frac{1}{2\sigma^2}$, so that $\int_{-\infty}^{\infty} ce^{-b(t-t_0)^2} dt = 1$. The Dirac delta impulse can be approximated by letting $\sigma \to 0$. From the result of (a), what is the Fourier Transform of a Dirac delta impulse. Comment on your result.

(c) Next, we consider some impulses that have temporal power law decay. Let us begin by considering a step function as follows.

$$f(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

If you naively perform the Fourier Transform, you will get something that is undefined. Instead, let us begin by considering the following function,

$$f(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

Fourier Transform the above function and let $\alpha \to 0$ to get the Fourier Transform of the Dirac delta impulse. What is its Fourier Transform now? (***Note that the discontinuity at t = 0 will give a delta function in* $f(\omega)$.) What is its power spectral distribution function? Comment on your result.

(d) Now, let us add another step function, but this time is a negative step function, so that the step function now becomes,

$$f(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}.$$

Again use the approximate as in c), what do you get for the Fourier Transform of this new step function?

(e) Use the result in d) to obtain the Fourier Transform of $f(t) = 1/t$, $|t|$ and $1/t^2$, $-\infty \leq t \leq \infty$. (*Hint: For f(t) = 1/t, start from its inverse Fourier Transform and redefine the parameters in the integral,* $t \to -\omega, \omega \to t'$.)

Solution:

(a) The Fourier Transform $f(\omega)$ of the Gaussian impulse is,

$$f(\omega) = \int_{-\infty}^{\infty} c e^{-b(t-t_0)^2} e^{-i\omega t} dt$$

For the exponent, we complete the square as follows,

$$b(t-t_0)^2 + i\omega t = b\left[t^2 - 2t_0 t + t_0{}^2 + \frac{i\omega t}{b}\right]$$

$$= b\left[\left(t - \left(t_0 - \frac{i\omega}{2b}\right)\right)^2 + \left(\frac{\omega}{2b}\right)^2 + \frac{it_0\omega}{b}\right]$$

Hence,

$$f(\omega) = ce^{-b\left[\left(\frac{\omega}{2b}\right)^2 + \frac{it_0\omega}{b}\right]} \int_{-\infty}^{\infty} e^{-b\left(t-\left(t_0-\frac{i\omega}{2b}\right)\right)^2} dt$$

Define $t - \left(t_0 - \frac{i\omega}{2b}\right) = t'$, and recall $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, we get

$$f(\omega) = c\sqrt{\frac{\pi}{b}} e^{-b\left[\left(\frac{\omega}{2b}\right)^2 + \frac{it_0\omega}{b}\right]}.$$

(b) By taking the limit of the result in (a), we get $f(\omega) = e^{-it_0\omega}$. The Fourier Transform of the Dirac delta impulse has a magnitude which is equal to 1 and a complex phase $t_0\omega$.

(c) The Fourier Transform of the unit step function by using the exponential decay function is,

$$f(\omega) = \int_0^\infty e^{-\alpha t} e^{-i\omega t} dt \ = \frac{1}{\alpha + i\omega} = \frac{\alpha}{\alpha^2 + \omega^2} - \frac{i\omega}{\alpha^2 + \omega^2}.$$

As $\alpha \to 0$

$$\frac{\alpha}{\alpha^2 + \omega^2} \to \pi\delta(\omega); \ \frac{i\omega}{\alpha^2 + \omega^2} \to \frac{i}{\omega}.$$

Therefore, the Fourier Transform of the unit step takes the form,

$$f(\omega) = \int_0^\infty e^{-i\omega t} dt \ = \pi\delta(\omega) - \frac{i}{\omega}.$$

The power spectral distribution function is equal to $|f(\omega)|^2$, therefore, it has a power law behavior $1/\omega^2$, which is similar to a Brown noise spectrum as discussed in the lecture.

(d) The Fourier Transform now becomes,

$$f(\omega) = \int_0^\infty e^{-\alpha t} e^{-i\omega t} dt - \int_{-\infty}^0 e^{\alpha t} e^{-i\omega t} dt = \frac{1}{\alpha + i\omega} - \frac{1}{\alpha - i\omega} = \frac{-2i\omega}{\alpha^2 + \omega^2} \to \frac{-2i}{\omega}.$$

(e) Instead of calculating the Fourier Transform of $1/t$, which will involve contour integrals, we will use the duality property here. Recall that the inverse Fourier Transform of a function is defined as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty f(\omega) e^{i\omega t} d\omega \ .$$

For the unit step function in d), we can get back by performing the inverse Fourier Transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \left(\frac{-2i}{\omega}\right) e^{i\omega t} d\omega \ .$$

Redefine the parameters in the integral, $t \to -\omega, \omega \to t'$, we get

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left(\frac{-2i}{\omega}\right) e^{i\omega t} d\omega \ = \frac{-i}{\pi} \int_{-\infty}^\infty \left(\frac{1}{t'}\right) e^{-i\omega t'} dt' \ = \ \tilde{f}(\omega).$$

The integral will give a function $\tilde{f}(\omega)$, (refer to the result in (d)) where

$$\tilde{f}(\omega) = \begin{cases} 1, & \omega < 0 \\ -1, & \omega > 0 \end{cases}.$$

Therefore, the Fourier Transform of $f(1/t)$ is, $f(\omega) = -i\pi \ \mathrm{sign}(\omega)$.

Using the result in (d), the Fourier Transform of $|t|$ is,

$$f(\omega) = \int_0^\infty te^{-\alpha t}e^{-i\omega t}dt + \int_{-\infty}^0 te^{\alpha t}e^{-i\omega t}dt$$
$$= -\frac{d}{d\alpha}\int_0^\infty e^{-\alpha t}e^{-i\omega t}dt - \frac{d}{d\alpha}\int_{-\infty}^0 e^{\alpha t}e^{-i\omega t}dt$$
$$= -\frac{d}{d\alpha}\left(\frac{1}{\alpha+i\omega}+\frac{1}{\alpha-i\omega}\right) = -\frac{d}{d\alpha}\left(\frac{2\alpha}{\alpha^2+\omega^2}\right) =\rightarrow \frac{-2}{\omega^2}.$$

Using the result of the Fourier Transform of $|t|$, and again by the duality property, we get, for the Fourier Transform of $1/t^2$, to be $f(\omega) = -\pi|\omega|$, ($|\omega|$ here means taking its absolute value.)

Augmented Fuller-Dickey Table:

Significance level	0.01	0.025	0.05	0.10
Sample Size T	The τ statistic: No Constant or Time Trend ($a_0 = a_2 = 0$)			
25	−2.65	−2.26	−1.95	−1.60
50	−2.62	−2.25	−1.95	−1.61
100	−2.60	−2.24	−1.95	−1.61
250	−2.58	−2.24	−1.95	−1.62
300	−2.58	−2.23	−1.95	−1.62
∞	−2.58	−2.23	−1.95	−1.62
	The τ_μ statistic: Constant but No Time Trend ($a_2 = 0$)			
25	−3.75	−3.33	−2.99	−2.62
50	−3.59	−3.22	−2.93	−2.60
100	−3.50	−3.17	−2.89	−2.59
250	−3.45	−3.14	−2.88	−2.58
500	−3.44	−3.13	−2.87	−2.57
∞	−3.42	−3.12	−2.86	−2.57
	The τ_τ statistic: Constant + Time Trend			
25	−4.38	−3.95	−3.60	−3.24
50	−4.15	−3.80	−3.50	−3.18
100	−4.05	−3.73	−3.45	−3.15
250	−3.99	−3.69	−3.43	−3.13
500	−3.97	−3.67	−3.42	−3.13
∞	−3.96	−3.67	−3.41	−3.12

Source: The table is reproduced from Fuller (1996).