MSDM5004
Numerical Methods and Modeling in Science
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Lecture 13

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2.7. Lax-Wendroff scheme

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Perform Taylor expansion of $u(x_j, t_{n+1})$ at (x_j, t_n).

$$u(x_j, t_{n+1}) = u(x_j, t_n) + \frac{\partial u}{\partial t}(x_j, t_n)\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t_n)(\Delta t)^2 + O((\Delta t)^3).$$

From the PDE,

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = -a\frac{\partial^2 u}{\partial x \partial t} = -a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = -a\frac{\partial}{\partial x}\left(-a\frac{\partial u}{\partial x}\right) = a^2\frac{\partial^2 u}{\partial x^2}$$

Therefore,

$$
\begin{aligned}
u(x_j, t_{n+1}) =& u(x_j, t_n) - a\frac{\partial u}{\partial x}(x_j, t_n)\Delta t + \frac{a^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_n)(\Delta t)^2 + O((\Delta t)^3) \\
\approx& u(x_j, t_n) - a\frac{\partial u}{\partial x}(x_j, t_n)\Delta t + \frac{a^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_n)(\Delta t)^2.
\end{aligned}
$$

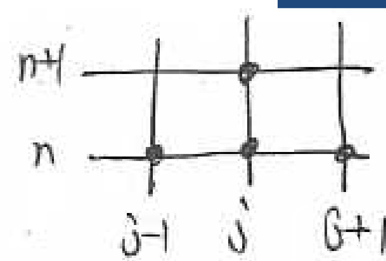
Using the approximation

$$
\frac{\partial u}{\partial x}(x_j, t_n) \approx \frac{u(x_{j+1}, t_n) - u(x_{j-1}, t_n)}{2\Delta x},
$$

$$
\frac{\partial^2 u}{\partial x^2}(x_j, t_n) \approx \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)}{(\Delta x)^2}
$$

We have the Lax-Wendroff scheme

$$
\nu = \frac{\Delta t}{\Delta x}
$$



$$
U_j^{n+1} = U_j^n - \frac{a\nu}{2}(U_{j+1}^n - U_{j-1}^n) + \frac{a^2\nu^2}{2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)
$$

Truncation error

$$
\begin{aligned}
T(x_j, t_n) =& \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t} + \frac{a\nu}{2\Delta t}\left[u(x_{j+1}, t_n) - u(x_{j-1}, t_n)\right] \\
& - \frac{a^2\nu^2}{2\Delta t}\left[u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)\right] \\
=& \left(\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}\right) + \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2}\right) \\
& + \frac{a}{6}\frac{\partial^4 u}{\partial x^4}(\Delta x)^2 + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}(\Delta t)^2 + O((\Delta t)^3) + O((\Delta x)^3) \\
=& \frac{a}{6}\frac{\partial^3 u}{\partial x^3}(\Delta x)^2 + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}(\Delta t)^2 + O((\Delta t)^3) + O((\Delta x)^3).
\end{aligned}
$$

Second order in both *t* and *x*.

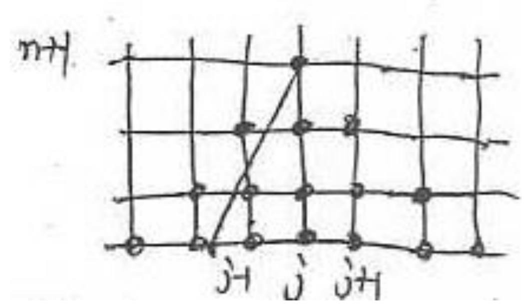
CFL condition

The domain of dependence of $u(x_j, t_{n+1})$ $\{x_j - at_{n+1}\}$.

Numerical domain of dependence $x_{j-n-1} \leq x \leq x_{j+n+1}$

CFL condition $x_{j-n-1} \leq x_j - at_{n+1} \leq x_{j+n+1}$

$$|a|\nu \leq 1.$$



Fourier analysis of stability

$$U_j^n = [\lambda(k)]^n e^{ikx_j}$$

$$\lambda(k) = 1 - 2a^2\nu^2 \sin^2 \frac{k\Delta x}{2} - ia\nu \sin k\Delta x$$

$$|\lambda(k)|^2 = 1 - 4a^2\nu^2(1 - a^2\nu^2) \sin^4 \frac{k\Delta x}{2}$$

This scheme is stable, i.e., $|\lambda(k)|^2 \leq 1$, if and only if

$$|a|\nu \leq 1.$$

An example

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = 0, \quad x \geq 0,\ t \geq 0$$

where $$a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4}$$

The initial condition is

$$u(x,0) = u_0(x) = \begin{cases} 1 & \text{if } 0.2 \leq x \leq 0.4, \\ 0 & \text{otherwise,} \end{cases}$$

The boundary condition is

$$u(0,t) = 0$$

The exact solution is $$u(x,t) = u_0\left(x - \frac{t}{1+x^2}\right)$$

Using the upwind scheme (here *a>0*)

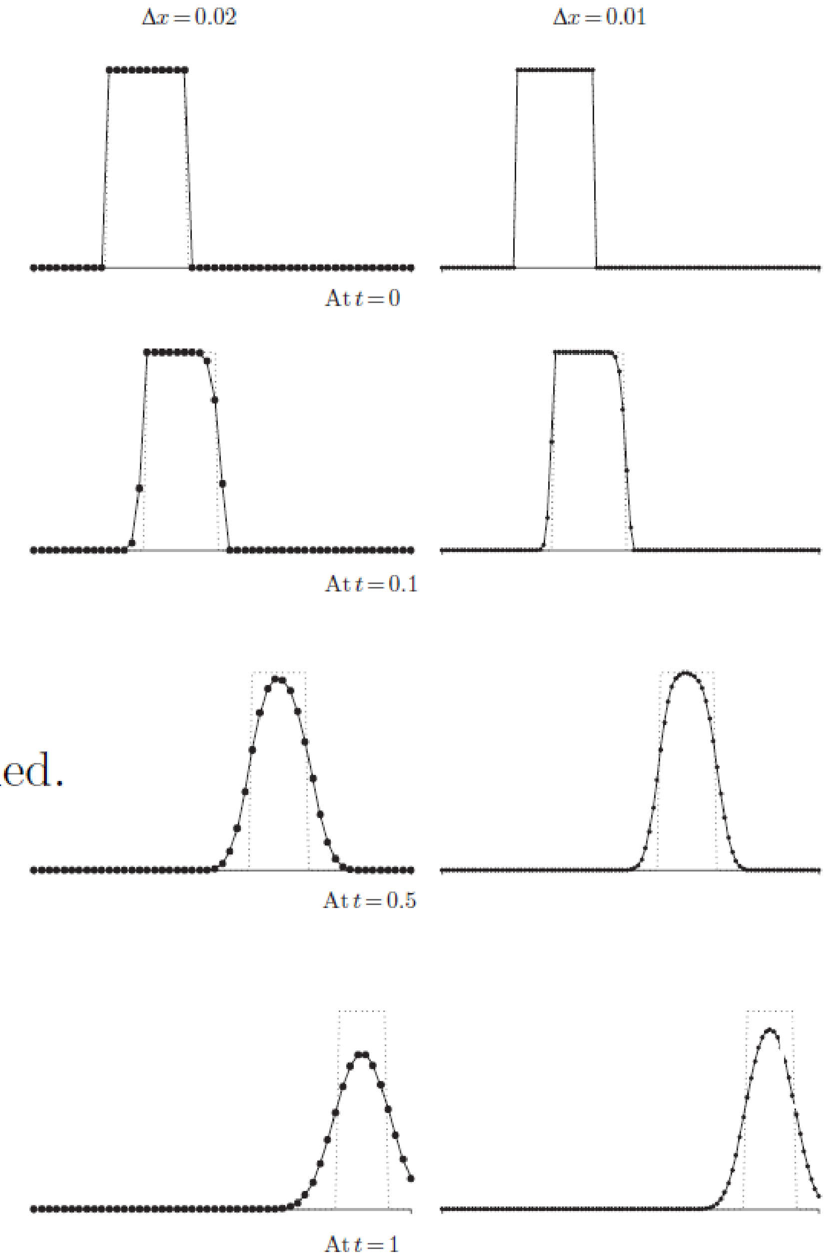
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a(x_j, t_n)\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0$$

$$\Delta t = \Delta x$$

Since $a(x,t) \leq 1$

the CFL stability condition is satisfied.

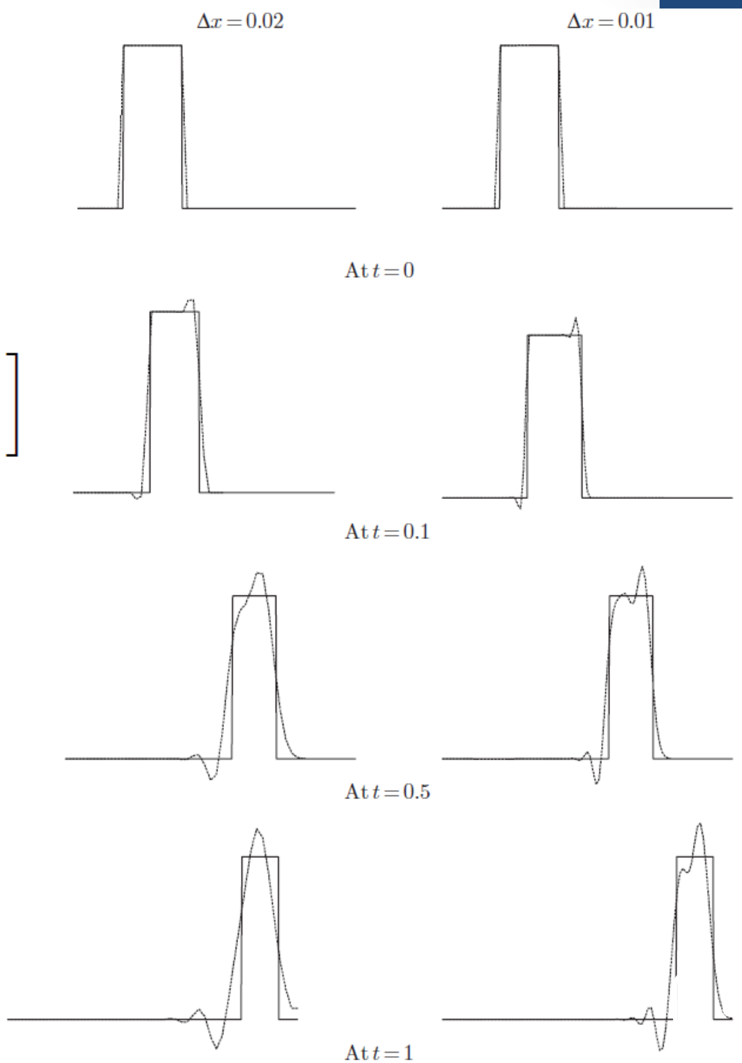
Significant smoothing of the edges of the pulse compared with the exact solution.



Using the Lax-Wendroff scheme

$$
\begin{aligned}
U_j^{n+1} =& U_j^n - \frac{a_j^n \nu}{2}(U_{j+1}^n - U_{j-1}^n) \\
&+ \frac{1}{2}(\Delta t)^2 \left[-\left(\frac{\partial a}{\partial t}\right)_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} \right. \\
&\left. + \frac{a_j^n}{(\Delta x)^2}\left(a_{j+\frac{1}{2}}^n (U_{j+1}^n - U_j^n) - a_{j-\frac{1}{2}}^n (U_j^n - U_{j-1}^n)\right) \right]
\end{aligned}
$$

$$\Delta t = \Delta x$$



- Maintains the height and width of the pulse better than the upwind scheme.
- Generates oscillations

Smooth initial condition

$$u(x,0) = \exp[-10(4x-1)^2].$$

Using the Lax-Wendroff scheme

$$\Delta t = \Delta x$$

Considerably more accurate