1 (20 points) Consider the problem of solving the equation $f(x) = 0$, where

$$f(x) = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

Write down the iteration algorithm of Newton's method, then perform 2 iterations with the starting point $x^{(0)} = 1$.

2 (15 points) Find the east square polynomial of degree 1 for the data in the table.

x_i	2	3	4
y_i	3	4	15

3 (15 points) For the linear advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0,$$

consider the numerical scheme

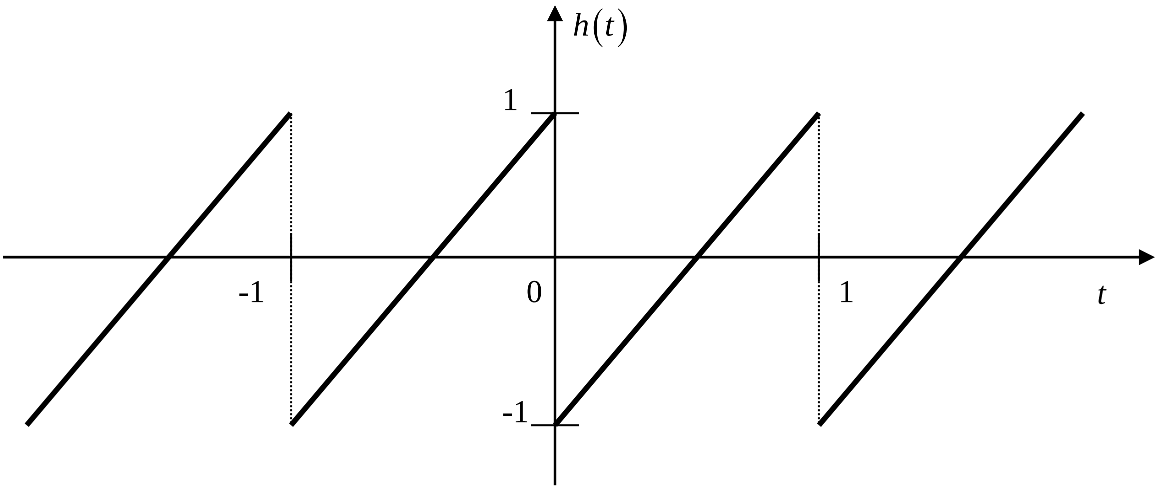
$$\frac{U_j^{n+1} - \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right)}{\Delta t} + \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0.$$

If $\mu = \Delta t / \Delta x$ is constant, show that the scheme is first order in both t and x.

4. The signal $h(t)$ as shown below, with unit amplitude and period, has a Fourier series expansion given by

$$h(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{in\pi} e^{-in2\pi f_0 t} ,$$

where $f_0 = \frac{1}{T} = 1\text{Hz}$.



The signal is passed through a frequency filter such that frequencies $|f| > 3.5$ are blocked. The signal is then sampled with a sampling time $\Delta = 0.2$ s and the total sampling data is 1000 such that the total sampling time is 200 s.

(a) What frequencies (positive and negative) of the original signal are not blocked by the filter?

Hereafter, the term “signal” refers to the signal after passing through the filter.

(b) What is the critical sampling time Δ below which one can determine the signal?

(c) With the sampling time $\Delta = 0.2$ s, what is the Nyquist critical frequency, f_c?

(d) Denoting the Fourier transform of the signal by $H(f)$, state why one only needs to plot the positive frequency part of the power spectrum, $|H(f)|^2$?

No mathematical proof is required.

(e) At what frequencies (we only consider the positive part) shall we observe the effect of aliasing? What are the original frequencies (can be negative) that are spuriously folded to these frequencies?

5. Suppose we can generate a Gaussian deviate x with zero mean and unit s.d., i.e.,

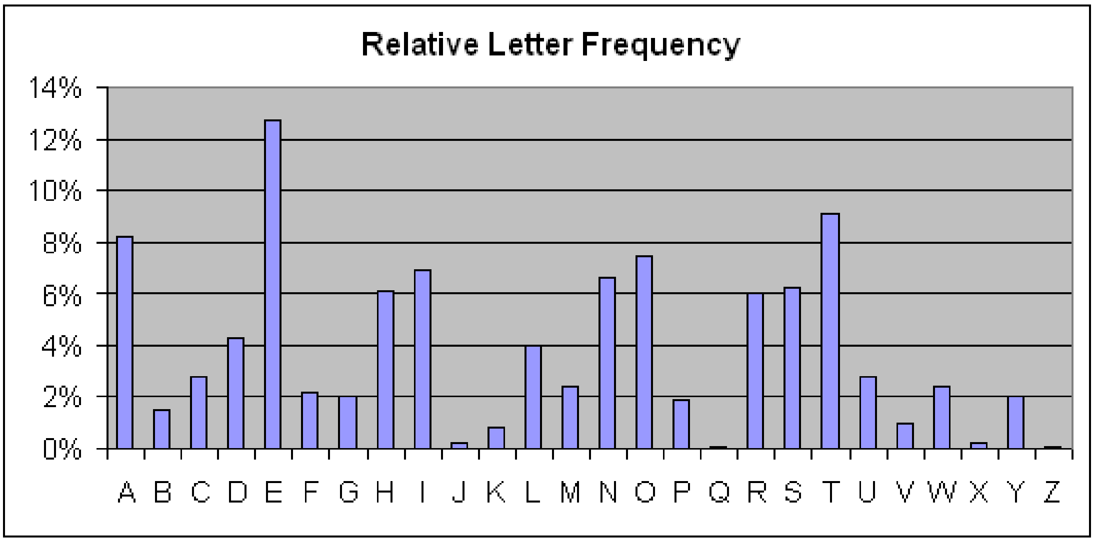
$$P(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right].$$

Define $\xi = m + \rho x$. Show that

$$P(\xi) = \frac{1}{\sqrt{2\pi}\rho} \exp\left[-\frac{(\xi - m)^2}{2\rho^2}\right],$$

which means that ξ is a Gaussian deviate with mean m and s.d. ρ.

6. The figure below shows the relatively frequencies of different alphabets appearing in the literature. It is shown that the letter “e” is the most frequently appeared letter with a relative frequency of 13%. The data is based on a very large sample such that the frequencies here can be considered exact. Let us assume a model that what letter appears next in a sentence is a random process, and the probability of finding an “e” is 0.13.



So counting the number of "e"s can in fact be represented by a random variable $x = 0,1$. If the letter is "e", then $x = 1$, otherwise, $x = 0$. From the statistics shown above, we know that $p(1) = 0.13$, $p(0) = 0.87$.

(a) Suppose you can generate a uniform deviate in $[0,1)$, suggest how you are going to mimic the random process governed by $p(1) = 0.13$, $p(0) = 0.87$.

(b) Evaluate the mean of *x*, denoted by μ, from first principle.

(c) Evaluate $\mathrm{var}(x)$ and hence the s.d. of *x* from first principle.

Now assume we don't know $p(x)$ and are going to estimate it by experiments. Suppose we choose a sentence or paragraph with *N* characters. Then the relative frequency can be estimated by the random variable

$$X = \frac{1}{N}\left(x_1 + x_2 + \cdots + x_N\right),$$

where $x_i = 0,1$ is the random variable corresponding to the i^{th} character, which is governed by $p(x)$.

(d) What kind of probability distribution function governs *X*, when $N \to \infty$. Write down the mean and the s.d. under this limit.

When *N* is large but not infinite, we know that the s.d. is small and hence it is very unlikely that *X* will be different from μ by too much. Therefore, we use *X* as an estimator of the actual mean:

$$\mu \approx X \pm \sqrt{\frac{\langle x^2 \rangle - X^2}{N-1}}.$$

Consider the following sentence from the lyrics of a song chosen arbitrarily:

Love is one big illusion I should try to forget

(e) Evaluate X from the above sentence.

(f) Evaluate $\langle x^2 \rangle$ using the above sentence.

(g) Evaluate the error estimation $\sqrt{\dfrac{\langle x^2 \rangle - X^2}{N-1}}$.

7. Consider the PDF $Q(x) = Cx^2(x-1)^2,\quad x \in (-1,1)$.

(a) Find C.

We shall sample $Q(x)$ using the rejection method.

(b) Choose a comparison function of the form $f(x) = A(x-1)^2$, where A is a constant. Find the value of A so that f is a valid comparison function and at the same time minimizes the rejection rate. What is this minimum rejection rate?

(c) Write down the procedure of your code to sample $Q(x)$ using the rejection method.