Contents

(All variables are real and one-dimensional unless otherwise specified.)

1. Background

After learning its formulation in last tutorial, we are going to study some numerical examples of decision theory.

In the simplest case, we are given two hypotheses, viz. the null hypothesis H_0 and the alternative hypothesis H_1. We observe some one-dimensional evidence $Z \in \mathbb{R}$ and want to accordingly decide which hypothesis is more trustworthy. In general, it costs us c_{ij} to believe H_i when H_j is true, and we wish to minimize the expected cost per decision.

1.1 Example: a nuclear accident

For example, let us consider a nuclear accident. After an earthquake, the government suspects that a nuclear plant is leaking radioactive substances and polluting the surrounding environment. Hence, the government decides whether

- H_0: the nuclear plant is safe, or
- H_1: the nuclear plant is leaking radioactive substances.

On one hand, the government is not rewarded for making a correct decision, so $c_{00} = c_{11} = 0$. On the other hand, the government will be accused by either the nuclear industry or environmental NGOs for making a wrong decision; assume their equal cost, i.e. $c_{01} = c_{10} = 1$.

Now the government measures the radioactivity of soil near the plant and gets a reading $Z = 110$ units. Meanwhile, the government's prior beliefs are $p = 0.8$ towards H_0 and $q = 1 - p = 0.2$ towards H_1.

1.2 Definitions

We must first clarify the definitions of various terms. Let $P(H_i \mid H_j)$ represent the probability that the the government believes H_i when H_j is true.

- **Probability of a correct decision:**
  $P_{\text{CD}} = pP(H_0 \mid H_0) + qP(H_1 \mid H_1)$
  The probability that the government believes when the same hypothesis H_i is true.

- **Probability of a wrong decision:**
  $P_{\text{WD}} = pP(H_1 \mid H_0) + qP(H_0 \mid H_1)$
  The probability that the government believes when the other hypothesis H_{1-i} is true.

Each decision results in one of the following four scenarios.

- **Detecting something correctly with probability** $P_{\text{D}} = P(H_1 \mid H_1)$
  The probability that the government believes H_1 given that H_1 is true.

- **Detecting nothing correctly with probability** $P_{\mathrm{N}} = P(H_0 \mid H_0)$
  The probability that the government believes given that H_0 is true.

- **False alarm with probability** $P_{\mathrm{FA}} = P(H_1 \mid H_0)$
  The probability that the government believes given that H_0 is true.
- **Miss with probability** $P_{\mathrm{M}} = P(H_0 \mid H_1)$
  The probability that the government believes given that H_1 is true.

Obviously, $P_{\mathrm{D}} + P_{\mathrm{M}} = 1$ and $P_{\mathrm{N}} + P_{\mathrm{FA}} = 1$. Then, we may write down the expected cost per decision C.

$$
\begin{aligned}
C &= c_{00} p P_{\mathrm{N}} + c_{01} q P_{\mathrm{M}} + c_{10} p P_{\mathrm{FA}} + c_{11} q P_{\mathrm{D}} \\
&= q P_{\mathrm{M}} + p P_{\mathrm{FA}} \qquad \left(\because \begin{cases} c_{00} = c_{11} = 0 \\ c_{01} = c_{10} = 1 \end{cases} \right) \\
&= P_{\mathrm{WD}}
\end{aligned}
$$

2. Triangular distributions

Experiments in laboratories find that the radioactivity reading Z always has a symmetric, i.e. isosceles, triangular distribution $f_Z(z)$ with a base length $w = 80$ units. If the plant is safe, $f_Z(z \mid H_0)$ is centred at $z = k_0 = 90$ units. If the plant is leaking, $f_Z(z \mid H_1)$ is centred at $z = k_1 = 110$ units.

2.1 Deriving a triangular distribution

A general triangular distribution $f_\Delta(\delta)$ has a shape of triangle whose base aligns with the horizontal axis. We can specify any triangular distribution with three parameters including its lower limit a, its upper limit b, and its mode c (Fig. 1). The inequality $a < c < b$ must be satisfied.

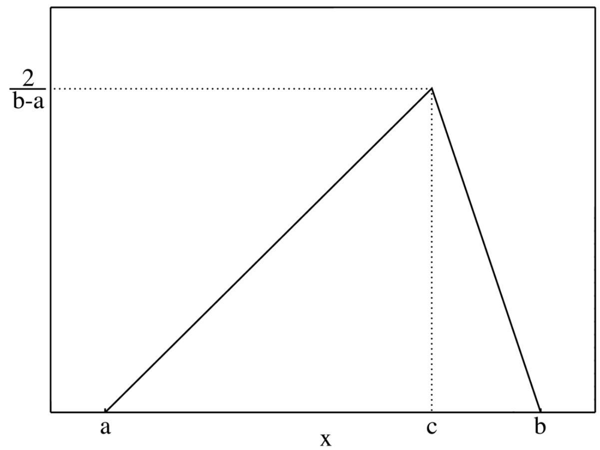

Fig. 1 A general triangular distribution $f_\Delta(\delta)$. Geometrically, a, b, and c are the horizontal coordinates of the triangles' bottom-left, bottom-right, and top vertices. Retrieved from https://upload.wikimedia.org/wikipedia/commons/4/45/Triangular_distribution_PMF.png.

The distribution is defined to be nonzero for $\delta \in (a, b)$ but zero otherwise. The normalization condition requires that $\int_{-\infty}^{\infty} f_\Delta(\delta)\mathrm{d}\delta = 1$, but we do not need integration here because the distribution is triangular. Elementary geometry tells us that the area under an h-unit tall distribution is $\frac{1}{2}(b-a)\,h$, so the normalization condition is reduced to $\frac{1}{2}(b-a)\,h = 1$ and yields $h = \frac{2}{b-a}$.

As we now know the top vertex's coordinates (c, h), we may write down the formulae of the two slanted sides in the point-slope form, i.e.

$\frac{f_\Delta - 0}{\delta - a} = \frac{h - 0}{c - a}$ and $\frac{f_\Delta - 0}{\delta - b} = \frac{h - 0}{c - b}$. After rearranging, we will get

$$f_\Delta(\delta; a, b, c) = \begin{cases} \frac{h}{c-a}(\delta - a) & (a \le \delta \le c) \\ \frac{h}{c-b}(\delta - b) & (c \le \delta \le b) \\ 0 & (\text{otherwise}) \end{cases}.$$

Symmetric triangular distribution. In the example, the triangular distribution $f_Z(z)$ is symmetric. The additional constraint reduces one degree of freedom, so we only need two parameters, viz. its base length w and its centre k. In terms of the general formula, $a \equiv k - v$, $b \equiv k + v$ and $c \equiv k$, where $v \equiv \frac{w}{2}$; therefore,

$$f_Z(z; w, k) = \begin{cases} \frac{1}{v^2}(v + z - k) & (k - v \le z \le k) \\ \frac{1}{v^2}(v - z + k) & (k \le z \le k + v) \\ 0 & (\text{otherwise}) \end{cases}.$$

2.2 Bayes detector

The Bayes detector's criterion is

$$\Lambda(z) \equiv \frac{f_Z(z \mid H_1)}{f_Z(z \mid H_0)} \underset{H_1}{\overset{H_0}{\lessgtr}} \frac{c_{10} - c_{00}}{c_{01} - c_{11}} \frac{p}{q} \equiv \eta,$$

where $\underset{H_1}{\overset{H_0}{\lessgtr}}$ sloppily means "prefers H_0 if less than but H_1 if greater than". The critical value z^* turns the inequality into an equality and solves $\Lambda(z^*) = \eta$. In the example, $\eta = \dfrac{1-0}{1-0}\dfrac{0.8}{0.2} = 4$, so the government needs to solve $\Lambda(z^*) = \dfrac{f_Z(z^*; 80, 110)}{f_Z(z^*; 80, 90)} = 4$ for z^*. As the form of $\Lambda(z)$ changes depending on the region of z^*, we have to consider the possible cases separately.

- If $z^* \in [k_1 - v, k_0] = [70, 90]$, $\Lambda(z^*) = \dfrac{z^* - 70}{z^* - 50} = 4$ yields $z^* \approx 43$ (rejected).
- If $z^* \in [k_0, k_1] = [90, 110]$, $\Lambda(z^*) = \dfrac{z^* - 70}{130 - z^*} = 0.25$ yields $z^* = 118$ (rejected).
- If $z^* \in [k_1, k_0 + v] = [110, 130]$, $\Lambda(z^*) = \dfrac{150 - z^*}{130 - z^*} = 0.25$ yields $z^* \approx 123$.
- For other z^*, $\Lambda(z^*)$ is zero, infinite, or undefined, yielding no solutions.

Only the third case is self-consistent, so $z^* \approx 123$ and reduces our condition to $z^* \underset{H_1}{\overset{H_0}{\lessgtr}} 123$. As the current reading is $Z = 110 < 123$, the government should believe H_0 and that the nuclear plant is safe.

Then we can calculate $(P_{\text{D}}, P_{\text{N}}, P_{\text{FA}}, P_{\text{M}})$ and thus $(P_{\text{CD}}, P_{\text{WD}})$ with the area formula of triangle—you should **draw out everything on a graph** to actually see why and how this works. For example, the probability of detection P_{D} is the area of "the part of $f_Z(z \mid H_1)$ right to $z = z^*$", which is the triangle formed with vertices

- $(k_1 + v, 0) = (150, 0)$,
- $(z^*, 0) \approx (123, 0)$, and
- $[z^*, f_Z(z^* \mid H_1)] = [z^*, f_Z(z^*; 80, 110)] \approx (123, 0.0167)$.

This means that $P_{\mathrm{D}} \approx \frac{1}{2}(150 - 123) \times 0.0167 \approx 0.225$. (The exact value is $\frac{2}{9}$.)

2.3 Neyman-Pearson detector

Suppose that the government allows the probability of false alarm P_{FA} to be $\alpha = 0.05$, i.e. making a false alarm once every twenty times. To find out the corresponding z^*, we may follow last subsection's logic and solve

$$\begin{aligned} P_{\mathrm{FA}} &= \frac{1}{2}(k_0 + v - z^*) f_Z(z^* \mid H_0) \\ &= \frac{1}{2}(90 + 40 - z^*) f_Z(z^*; 80, 90) = \alpha \end{aligned}$$

for z^* to get $z^* \approx 117$. Since $Z = 110 < 117$, according to this detector, the government should, again, trust the nuclear plant.

3. General distributions

The Government Laboratory performs new radioactive experiments and finds that it is inaccurate to model $f_Z(z)$ with triangular distributions;

instead, with all the parameters $(w = 80, k_0 = 90, k_1 = 110)$ unchanged, they suggest that

$$\begin{cases} f_Z(z \mid H_0) & = & \dfrac{5}{\sqrt{2\pi}w}\exp\left[-\dfrac{25}{2}\left(\dfrac{z-k_0}{w}\right)^2\right] \\ f_Z(z \mid H_1) & = & \dfrac{2}{w}\exp\left(-\dfrac{4}{w}|z-k_1|\right) \end{cases}$$

and claim these "curved triangles" are better approximations.

3.1 Analysis

In this complicated case, the likelihood function becomes

$$\begin{aligned} \Lambda(z) &= \frac{\dfrac{2}{w}\exp\left(-\dfrac{4}{w}|z-k_1|\right)}{\dfrac{5}{\sqrt{2\pi}w}\exp\left[-\dfrac{25}{2}\left(\dfrac{z-k_0}{w}\right)^2\right]} \\ &= \frac{2}{5}\sqrt{2\pi}\exp\left[\frac{(z-90)^2}{512}-\frac{|z-110|}{20}\right], \end{aligned}$$

which is still analytic but too tedious for solving $\Lambda(z^*) = \eta = 4$. Hence, we may resort to numerical methods and find out two self-consistent solutions, i.e. $z^* \approx 40$ or $z^* \approx 122$. (If you are patient in doing the math, you can get their analytic forms.)

It makes sense to have two solutions because of the presence of z^2 and $|z|$. How do we interpret them then? In fact, **both** solutions are valid and necessary. By testing with numbers like $z \in \{0, 80, 160\}$, you will see that

$\Lambda(z) \begin{cases} < 4 & (40 < z < 122) \\ > 4 & (\text{otherwise}) \end{cases}$. However, if it is somehow known that z almost never drops below 40, we could just simplify our detector as $z \underset{H_1}{\overset{H_0}{\lessgtr}} 122$.

4. Convolution

Although the government finally trusts that the soil is not polluted, environmental NGOs continue suspecting that seawater near the nuclear plant, which gives a reading $Z = 65$ units, has been contaminated.

They proceed to study two hypotheses for the seawater's radioactivity reading Z, viz.

- $H_0 : Z = B$, i.e. Z is only due to radioactivity of background radiation B, and
- $H_1 : Z = B + N$, i.e. Z is due to B and radioactivity of nuclear waste N.

They find that B is normally distributed with mean $\mu_B = 50$ units and variance $\sigma_B^2 = 100$ units, whereas N is normally distributed with mean $\mu_N = 20$ and variance $\sigma_N^2 = 21$ units.

4.1 Addition of random variables

The distribution of H_0 is clearly equivalent to $f_B(b) = \mathcal{N}\left(\mu_B, \sigma_B^2\right)$, so

$$f_Z(z \mid H_0) = \frac{1}{10\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-50}{10}\right)^2\right].$$

But how about the distribution of H_1? It turns out that we need **convolution**. The convolution between functions $g(x)$ and $h(x)$ is defined as

$$g(x) * h(x) \equiv \int_{-\infty}^{\infty} g(\xi) h(x-\xi) \mathrm{d}\xi\,,$$

and the distribution of two random variables' sum is equal to the convolution product of their distributions. Hence for $Z = B + N$,

$$\begin{aligned} f_Z(z \mid H_1) &= f_B(b) * f_N(n) \\ &= \mathcal{N}\left(\mu_B, \sigma_B^2\right) * \mathcal{N}\left(\mu_N, \sigma_N^2\right)\,. \end{aligned}$$

Luckily, we do not need to calculate the integral as it has been proved that the convolution of two normal distributions is another normal distribution.

$$\mathcal{N}\left(\mu_B, \sigma_B^2\right) * \mathcal{N}\left(\mu_N, \sigma_N^2\right) = \mathcal{N}\left(\mu_B + \mu_N, \sigma_B^2 + \sigma_N^2\right)$$

That is, through convolution, two normal distributions' means add up to become the resultant normal distribution's mean, so do their variances. Therefore,

$$f_Z(z \mid H_1) = \frac{1}{11\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-70}{11}\right)^2\right].$$

Exercise. The government has read the environmental NGOs' reports and acknowledged their findings. If the threshold remains unchanged, i.e. $\eta = 4$, should the government prefer H_0 or H_1 for $Z = 65$ units?

Answer. $z \underset{H_1}{\overset{H_0}{\lessgtr}} z^* \approx 67.4$, so the government should prefer H_0. (The other root $z^* \approx -158$ is rejected.)