8
Explaining Complex Distributions with Simple Models

... all models are approximations. Essentially, all models are wrong, but some are useful. However, the approximate nature of the model must always be borne in mind ...

– George Edward Pelham Box

8.1
Kinetic Theory of Gases

The *ideal gas* is a very important subject of study in statistical physics. It constitutes a system where the interaction between the particles or molecules is very weak or negligible. Alternatively, the gas is particularly rarified so that the particles are typically large distances apart, such that the interaction is very small. A typical molecule has its center of mass at $\mathbf{r}$, and let $\mathbf{p}$ denote the momentum of its center of mass. In the absence of external force fields, the energy E of the molecule is given by

$$E = \frac{\mathbf{p}^2}{2m} + E_{\text{int}} \,, \tag{8.1}$$

where the first term on the right is the kinetic energy of the center of mass, while the second term arises out of the asymmetry (polyatomic molecule) of the molecule, and is simply the internal energy of rotation or vibration with respect to the molecular center of mass. The dilute gas assumption allows us to neglect any potential energy of interaction with other molecules, and thus, E is independent of $\mathbf{r}$. The dilute gas approximation also allows us to treat the translational degrees of freedom classically, while the internal degrees of freedom are usually dealt with using quantum mechanics.

8.1.1
Derivation of Maxwell–Boltzmann Distribution

As for the statistical description, we restrict ourselves to a hard-sphere gas with no intermolecular interactions. The easiest way to realize things is to estimate the

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number of collisions that a sample of gas exerts on a planar boundary surface, which is arbitrarily oriented perpendicular to the x-axis. Thus, the collisions per unit area, or pressure is

$$\frac{N}{A} = \rho_N \Delta t \sum_{v_x>0} v_x f(v_x) , \tag{8.2}$$

ρ_N being the number density of particles, Δt is the collision time and $f(v_x)$ is the distribution of the velocity component along x axis. Let m be the mass of each molecule. Now, the change in momentum suffered by the molecules by hitting the wall is given by

$$\Delta p = N 2 m v_x = 2 m \rho_N A \Delta t \sum_{v_x>0} v_x^2 f(v_x) , \tag{8.3}$$

and, hence, the pressure is given by

$$P = \frac{\Delta p}{A \Delta t} = 2 m \rho_N \sum_{v_x>0} v_x^2 f(v_x) . \tag{8.4}$$

For a stationary gas, we recognize that the sum of all velocities above zero is equal to half the sum of all the velocities from negative to positive infinity:

$$P = 2 m \rho_N \sum_{v_x>0} v_x^2 f(v_x) = 2 m \rho_N \frac{1}{2} \sum_{v_x} v_x^2 f(v_x) . \tag{8.5}$$

The above can be rewritten in terms of macroscopic quantities using the ideal gas law

$$\frac{nRT}{V} = m \rho_N \sum_{v_x} v_x^2 f(v_x) = m \rho_N \langle v_x^2 \rangle , \tag{8.6}$$

which generalizes into

$$\langle v_x^2 \rangle = \frac{RT}{M} , \tag{8.7}$$

where R is the ideal gas constant and M is the molar weight of the gas. For an ideal gas in three dimensions, $\langle v^2 \rangle = 3 \langle v_x^2 \rangle = 3RT/M = 3k_B T/m$, k_B being the Boltzmann constant. Hence the expression for mean kinetic energy of the ideal gas would look like

$$\langle E \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{3 k_B T}{2} . \tag{8.8}$$

The distribution function $g(v)$ of velocities would be given by

$$g(v) dv = f(v_x) f(v_y) f(v_z) dv_x dv_y dv_z . \tag{8.9}$$

and the partial derivative of g with respect to one of the axes looks like

$$\frac{\partial g}{\partial v_x} = \frac{dg}{dv} \frac{\partial v}{\partial v_x} = \frac{dg}{dv} \frac{\partial}{\partial v_x} \sqrt{v_x^2 + v_y^2 + v_z^2} = \frac{dg}{dv} \frac{v_x}{v} . \tag{8.10}$$

Rearranging

$$\frac{1}{v} \frac{dg}{dv} = \frac{1}{v_x} \frac{\partial g}{\partial v_x} = \frac{f(v_y) f(v_z)}{v_x} \frac{\partial f(v_x)}{\partial v_x} , \tag{8.11}$$

what follows from the equivalence of the directions is

$$\frac{f(v_y) f(v_z)}{v_x} \frac{\partial f(v_x)}{\partial v_x} = \frac{f(v_z) f(v_x)}{v_y} \frac{\partial f(v_y)}{\partial v_y} = \frac{f(v_x) f(v_y)}{v_z} \frac{\partial f(v_z)}{\partial v_z} . \tag{8.12}$$

Dividing all by g one obtains

$$\frac{1}{f(v_x) v_x} \frac{\partial f(v_x)}{\partial v_x} = \frac{1}{f(v_y) v_y} \frac{\partial f(v_y)}{\partial v_y} = \frac{1}{f(v_z) v_z} \frac{\partial f(v_z)}{\partial v_z} = -k , \tag{8.13}$$

where k is some constant independent of the coordinates. Now we have a first-order differential equation for the velocity coordinates, and integrating one obtains $f(v_x) = B \exp[-k v_x^2/2]$. To determine the value of the integrating constant B, we note that for any probability distribution, the integral of the distribution over the entire region of space it encompasses must be unity. Thus,

$$1 = \int_{-\infty}^{\infty} f(v_x) dv_x = B \int_{-\infty}^{\infty} \exp[-k v_x^2/2] dv_x , \tag{8.14}$$

which yields $B = \sqrt{\frac{k}{2\pi}}$, and thus $f(v_x) = \sqrt{\frac{k}{2\pi}} \exp[-k v_x^2/2]$. Thus, (8.9) can be rewritten as

$$g(v) = \left(\frac{k}{2\pi} \right)^{3/2} \exp[-k v_x^2/2] \exp[-k v_y^2/2] \exp[-k v_z^2/2] . \tag{8.15}$$

The mean kinetic energy is thus

$$\langle E \rangle = \frac{3m}{2} \left(\frac{k}{2\pi} \right)^{3/2} \int_{-\infty}^{\infty} v_x^2 \exp[-k v_x^2/2] dv_x \times \int_{-\infty}^{\infty} \exp[-k v_y^2/2] dv_y \int_{-\infty}^{\infty} \exp[-k v_z^2/2] dv_z , \tag{8.16}$$

which simplifies to

$$\langle E \rangle = \frac{3m}{2} \left(\frac{k}{2\pi} \right)^{3/2} \frac{2\pi}{k} \int_{-\infty}^{\infty} v_x^2 \exp[-k v_x^2/2] dv_x = \frac{3m}{2k} . \tag{8.17}$$

Equating that with the mean value of kinetic energy, we get $k = \frac{m}{k_B T}$. Thus,

$$f(v_x) = \left(\frac{m}{2\pi k_B T}\right)^{1/2} \exp\left[-\frac{m v_x^2}{2 k_B T}\right]. \tag{8.18}$$

The distribution of molecular speeds is then given by

$$g(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left[-\frac{m v^2}{2 k_B T}\right], \tag{8.19}$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$. Subsequently, one can also find the distribution function for energy $E = \frac{1}{2} m v^2$ as

$$h(E) = 2\pi \left(\frac{1}{\pi k_B T}\right)^{3/2} E^{1/2} \exp\left(-\frac{E}{k_B T}\right). \tag{8.20}$$

Boltzmann and Gibbs

Ludwig Boltzmann (1844–1906) was an Austrian physicist famous for his contributions to the foundation of statistical mechanics and statistical thermodynamics. He was one of the most important supporter of atomic theory during the time that model was still highly controversial. Boltzmann's most important scientific contributions were in kinetic theory. He is known for the Maxwell–Boltzmann distribution for molecular speeds in a gas. He developed an equation to describe the dynamics of an ideal gas, now known as the Boltzmann equation. Another big contribution came as a logarithmic relation between entropy and probability. In the Maxwell–Boltzmann statistics and the Boltzmann distribution over energies lie the foundations of classical statistical mechanics, and provide a remarkable insight into the meaning of temperature.

Josiah Willard Gibbs (1839–1903) was an American theoretical physicist, chemist, and mathematician. He is responsible for the theoretical foundation of chemical thermodynamics and physical chemistry. In mathematics, he invented vector analysis, and his most important contributions are in conceptualizing chemical potential and free energy. The Gibbsian ensemble sets a foundation in statistical mechanics, while the phase rule is an essential part of thermodynamics. His work and legacy influenced as well as laid the foundation for later important contributions, not only in physics and chemistry, but also in economics. Apart from Irving Fisher, who was strongly influenced by his work, Nobel laureate Paul Samuelson acknowledged the influence of classical thermodynamics developed by Gibbs.

8.1.2 Maxwell–Boltzmann Distribution in *D* Dimensions

Here we show that for integer or half-integer values of the parameter n the gamma distribution

$$\gamma_n(\xi) = \Gamma(n)^{-1} \xi^{n-1} \exp(-\xi). \tag{8.21}$$

$\Gamma(n)$ is the Gamma function which represents the distribution of the rescaled kinetic energy ξ ($\xi = E/k_B T$) for a classical mechanical system in $D = 2n$ dimensions. T represents the absolute temperature of the system and k_B is the Boltzmann constant.

The normalized probability distribution in momentum space is simply $f(\mathbf{P}) = \prod_i (2\pi m k_B T)^{-D/2} \exp(-\mathbf{p}_i{}^2/2 m k_B T)$, where $\mathbf{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_N\}$ represents the momentum vectors of the N particles. Thus, since the kinetic energy distribution factorizes as a sum of single particle contributions, the probability density factorizes as a product of single particle densities, each one of the form

$$f(\mathbf{p}) = \frac{1}{(2\pi m k_B T)^{D/2}} \exp\left(-\frac{\mathbf{p}^2}{2 m k_B T}\right), \tag{8.22}$$

where $\mathbf{p} = (p_1, \ldots, p_D)$ is the momentum of a generic particle. It is convenient to introduce the momentum modulus p of a particle in D dimensions

$$p^2 \equiv \mathbf{p}^2 = \sum_{k=1}^{D} p_k^2, \tag{8.23}$$

where the p_ks are the Cartesian components, since the distribution (8.22) depends only on $p \equiv \sqrt{\mathbf{p}^2}$. One can then integrate the distribution over the $D-1$ angular variables to obtain the momentum modulus distribution function, with the help of the formula for the surface of a hypersphere of radius p in D dimensions

$$S_D(p) = \frac{2\pi^{D/2}}{\Gamma(D/2)} p^{D-1}. \tag{8.24}$$

One obtains

$$f(p) = S_D(p) f(\mathbf{p}) = \frac{2}{\Gamma(D/2)(2 m k_B T)^{D/2}} p^{D-1} \exp\left(-\frac{p^2}{2 m k_B T}\right). \tag{8.25}$$

The corresponding distribution for the kinetic energy $E = p^2/2m$ is therefore

$$f(E) = \left[\frac{dp}{dE} f(p)\right]_{p=\sqrt{2mE}} = \frac{1}{\Gamma(D/2) k_B T} \left(\frac{E}{k_B T}\right)^{D/2-1} \exp\left(-\frac{E}{k_B T}\right). \tag{8.26}$$

Comparison with the gamma distribution (8.21) shows that the Maxwell–Boltzmann kinetic energy distribution in D dimensions can be expressed as

$$f(E) = (k_B T)^{-1} \gamma_{D/2}(E/k_B T). \tag{8.27}$$

The distribution for the rescaled kinetic energy,

$$\xi = E / k_B T \,, \tag{8.28}$$

is just the gamma distribution of order $D/2$

$$f(\xi) = \left[\frac{dE}{d\xi} f(E)\right]_{E=\xi k_B T} = \frac{1}{\Gamma(D/2)} \xi^{D/2-1} \exp(-\xi) \equiv \gamma_{D/2}(\xi) \,. \tag{8.29}$$

8.2 The Asset Exchange Model

In principle, any model of the distribution of wealth [1] defined by a process of exchange of assets could be called an "asset exchange model". Generally, the economic activity in these models involve interaction between two individuals resulting in the redistribution of their assets. Of course, "individual" here could mean a single person or a conglomerate such as a company. "Asset" generally refers to anything that contributes to the overall wealth of the economy, and could be cash or any other physical asset.

A model for evolution of wealth distribution in an economically interacting population was introduced by Ispolatov *et al.* [2], where specific amounts of assets are exchanged between two individuals. Two different cases were studied: a "random" exchange where either of the individuals are likely to gain, and a "greedy" exchange, where the richer individual always gains. Again, two types of exchanges were considered: the "additive" exchange where a fixed amount of asset transfers from an individual to the other, and a "multiplicative" exchange, where the amount of asset exchanged is a finite fraction of the wealth of one of the individuals. This work reports a variety of distribution for the asset, depending on the exact trading rules. For the additive type, a random exchange produces a Gaussian distribution of wealth, while greedy exchange produces a Fermi-like scaled wealth distribution. For the multiplicative exchanges, the random exchange produces a steady state, while the greedy exchange produces a continuously evolving power law wealth distribution. Studies [7] have also shown that if the amount of money transfered is a small fixed quantity, or a fixed fraction of the trading pair's average wealth, or a random fraction of the average wealth of the entire population, and even in cases of debt or bankruptcy, the equilibrium distribution is exponential.

Hayes [3] proposed two models of asset exchange in a closed, nonevolving economy based on simple exchange rules: *yard-sale* (YS) and *theft-and-fraud* (TF). In the YS model, the amount of wealth exchanged is a finite fraction of that of the poorer trader, and the resultant distribution corresponds to a monopoly, where all the wealth accumulates with one trader, as reported in an earlier study [4]. In the TF model, the trading pair randomly chooses the loser, and the amount of wealth exchanged is a random fraction of the donor. Thus the rich trader has more to lose while the poor trader has more to gain. The resultant equilibrium distribution is exponential.

Some extensions of these models [5, 6] produce distributions which are of considerable interest. In an asymmetric exchange model [6], the wealth dynamics are defined by

$$w_i(t+1) = \begin{cases} w_i(t) + \epsilon \left(1 - \tau \left[1 - \frac{w_i(t)}{w_j(t)}\right]\right) w_j(t) \,, & \text{if} \quad w_i(t) \le w_j(t) \\ w_i(t) + \epsilon w_j(t) \,, & \text{otherwise} \,, \end{cases} \tag{8.30}$$

where ϵ is a random number between 0 and 1. Here, $\tau = 0$ corresponds to the random exchange model [7] while $\tau = 1$ corresponds to the minimum exchange model [3, 4]. In general, the relation between agents is asymmetric and the richer agent dictates the terms of the trade. τ is known as the "thrift" parameter, and it measures the degree to which the richer agent is able to use its power. If one considers a uniform distribution of τ among agents between 0 and 1, one observes a power-law distribution for larger wealth, with Pareto exponent 1.5 [6].

M.N. Saha and S.N. Bose

Meghnad Saha (1893–1956) was an Indian astrophysicist. His best-known work concerned the thermal ionization of elements, and it led him to formulate what came to be known as the Saha equation. This equation is one of the basic tools for interpretation of the spectra of stars in astrophysics. By studying the spectra of various stars, one can find their temperature and subsequently, using Saha's equation, determine the ionization state of the various elements making up the star. M. N. Saha and B. N. Srivastava's book "A treatise on heat", India Press, Allahabad (1931), is the first textbook describing a possible application of kinetic theory of gas to income and wealth distribution in a society.

Satyendra Nath Bose (1894–1974) was an Indian physicist, specializing in mathematical physics. He is best known for his work on quantum mechanics in the early 1920s. His derivation of Planck's radiation law without referring to classical physics provided the foundation for the Bose–Einstein statistics and subsequently to the theory of the Bose–Einstein condensate. Particles obeying Bose–Einstein statistics are named boson, after him. The importance of his work is several fold, and has laid the foundation for several Nobel prize winning works.

8.3 Gas-Like Models

In 1960, Mandelbrot wrote, "There is a great temptation to consider the exchanges of money which occur in economic interaction as analogous to the exchanges of energy which occur in physical shocks between molecules. In the loosest possible terms, both kinds of interactions *should* lead to *similar* states of equilibrium. That is, one *should* be able to explain the law of income distribution by a model similar to that used in statistical thermodynamics: many authors have done so explicitly, and all the others of whom we know have done so implicitly." [8]. Unfortunately Mandelbrot did not provide any reference to this material!

The study of pairwise money transfer and the resulting statistical distribution of money has almost no counterpart in modern economics. Econophysicists initiated a new direction here. The search theory of money [9] is somewhat related, but this work was largely influenced by [10] studying the dynamics of money. A probability distribution of money among the agents was only recently obtained numerically within the search theoretical approach [11].

In analogy to two-particle collisions with a resulting change in their individual kinetic energy (or momenta), income exchange models may be based on two-agent interactions. Here two randomly selected agents exchange money by some predefined mechanism. Assuming the exchange process does not depend on previous exchanges, the dynamics follows a Markovian process

$$\begin{pmatrix} m_i(t+1) \\ m_j(t+1) \end{pmatrix} = \mathcal{M} \begin{pmatrix} m_i(t) \\ m_j(t) \end{pmatrix} \qquad (8.31)$$

where $m_i(t)$ is the income of agent i at time t and the collision matrix $\mathcal{M}$ defines the exchange mechanism.

In this class of models, one considers a closed economic system where total money M and total number of agents N is fixed. This corresponds to a situation where no production or migration occurs and the only economic activity is confined to trading. Each agent i, individual or corporate, possesses money $m_i(t)$ at time t. In any trading, a pair of traders i and j exchange their money [2, 7, 12, 13], such that their total money is (locally) conserved and none end up with negative money ($m_i(t) \geq 0$, i.e., debt not allowed)

$$m_i(t+1) = m_i(t) + \Delta m\,; \quad m_j(t+1) = m_j(t) - \Delta m\,. \qquad (8.32)$$

Following local conservation

$$m_i(t) + m_j(t) = m_i(t+1) + m_j(t+1)\,, \qquad (8.33)$$

time (t) changes by one unit after each trading. The simplest model considers a random fraction of total money to be shared [7]

$$\Delta m = \epsilon_{ij}[m_i(t) + m_j(t)] - m_i(t)\,, \qquad (8.34)$$

where ϵ_{ij} is a random fraction ($0 \leq \epsilon_{ij} \leq 1$) changing with time or trading. The steady-state ($t \to \infty$) distribution of money is a Gibbs distribution

$$P(m) = (1/T)\exp(-m/T)\,; \quad T = M/N\,. \qquad (8.35)$$

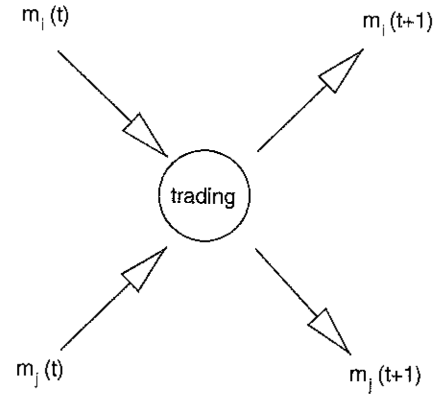

Figure 8.1 Schematic diagram of the trading process. Agents i and j redistribute their money in the market: $m_i(t)$ and $m_j(t)$, their respective money before trading, changes over to $m_i(t+1)$ and $m_j(t+1)$ after trading.

Hence, no matter how uniform or justified the initial distribution is, the eventual steady state corresponds to the Gibbs distribution where most of the people have very little money. This follows from the conservation of money and additivity of entropy

$$P(m_1)P(m_2) = P(m_1 + m_2)\,. \qquad (8.36)$$

This steady state result is quite robust and realistic as well. In fact, several variations of the trading, and of the "lattice" (on which the agents can be put and each agent trades with its "lattice neighbors" only), whether compact, fractal or small-world like [14], leaves the distribution unchanged. Some other variations like random sharing of an amount $2m_2$ only (not of $m_1 + m_2$) when $m_1 > m_2$ (trading at the level of lower economic class in the trade), lead to an even more drastic situation: all the money in the market drifts to one agent and the rest become truly paupers [3, 4].

If one allows for debt in such simple models, things look very different. From the point of view of individuals, debt can be viewed as negative money. As an agent borrows money from a bank, its cash balance M increases, but at the expense of cash obligation or debt D which is a negative money. Thus the total money of the agent $M_b = M - D$ remains the same. Thus if the boundary condition $m_i \geq 0$ is relaxed, $P(m)$ never stabilizes and keeps spreading in a Gaussian manner towards $m = +\infty$ and $m = -\infty$. Total money is conserved, and some agents become richer at the expense of others going into debt, so that $M = M_b + D$.

Xi *et al.* [15] imposed a constraint on the total debt of all agents in the system. Banks set aside a fraction R of the money deposited into bank accounts, whereas the remaining $1 - R$ can be loaned further. If the initial amount of money in the system is M_b, then, with repeated loans and borrowing, the total amount of positive money available to the agents increases to $M = M_b/R$, where $1/R$ is called the money-multiplier, and this extra money comes from the increase of the total debt in the system. The maximal total debt is $D = M_b/R - M_b$ and is limited by the factor R. For maximal debt, the total amounts of positive (M_b/R) and negative ($M_b(1-R)/R$) money circulate among the agents in the system. The distributions of positive and negative money are exponential with two different money temperatures $T_+ = M_b/RN$ and $T_- = M_b(1-R)/RN$, as confirmed by computer simulations [15]. Similar results were also observed elsewhere [16].

8.3.1
Model with Uniform Savings

In any trading, savings come naturally [17]. A saving propensity factor λ was therefore introduced in the random exchange model [13] (see [7] for a model without savings), where each trader at time t saves a fraction λ of its money $m_i(t)$ and trades randomly with the rest

$$m_i(t+1) = \lambda m_i(t) + \epsilon_{ij}\left[(1-\lambda)(m_i(t) + m_j(t))\right], \tag{8.37}$$

$$m_j(t+1) = \lambda m_j(t) + (1-\epsilon_{ij})\left[(1-\lambda)(m_i(t) + m_j(t))\right], \tag{8.38}$$

where

$$\Delta m = (1-\lambda)\left[\epsilon_{ij}\{m_i(t) + m_j(t)\} - m_i(t)\right], \tag{8.39}$$

and ϵ_{ij} is a random fraction, coming from the stochastic nature of the trading.

The market (noninteracting at $\lambda = 0$ and 1) becomes "interacting" for any nonvanishing $\lambda (< 1)$: For fixed λ (same for all agents), the steady state distribution $P(m)$ of money is exponentially decaying on both sides with the most-probable money per agent shifting away from $m = 0$ (for $\lambda = 0$) to M/N as $\lambda \to 1$ (Figure 8.2). This self-organizing feature of the market, induced by sheer self-interest of saving by each agent without any global perspective, is quite significant as the fraction of paupers decrease with saving fraction λ and most people end up with some finite fraction of the average money in the market (for $\lambda \to 1$, the socialists' dream is achieved with just people's self-interest of saving!). Interestingly, self-organization also occurs in such market models when there is restriction in the commodity market [25]. Although this fixed saving propensity does not give the Pareto-like power-law distribution, the Markovian nature of the scattering or trading processes (8.36) is effectively lost. Indirectly through λ, the agents get to know (start interacting with) each other and the system cooperatively self-organizes towards a most-probable distribution ($m_p \neq 0$) (see Figure 8.2).

There have been a few attempts to analytically formulate this problem [18] but no analytic expression has yet been found. It has also been claimed through heuristic arguments (based on numerical results) that the distribution is a close approximate form of the gamma distribution [19]

$$P(m) = \frac{n^n}{\Gamma(n)} m^{n-1} \exp(-nm)\,, \tag{8.40}$$

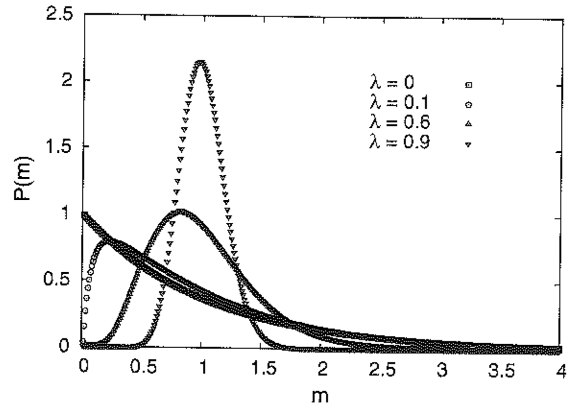

Figure 8.2 Steady state money distribution $P(m)$ for the model with uniform savings. The data shown are for different values of λ: 0, 0.1, 0.6, 0.9 for a system size $N = 100$. All datasets shown are for average money per agent $M/N = 1$.

where $\Gamma(n)$ is the gamma function whose argument n is related to the savings factor λ as

$$n = 1 + \frac{3\lambda}{1-\lambda}\,. \tag{8.41}$$

This result has also been supported by numerical results in [20]. However, a later study [21] analyzed the moments, and found that moments up to the third order agree with those obtained from the form of (8.41), and discrepancies start from the fourth order onwards. Hence, the actual form of the distribution for this model still remains to be found.

It seems that a very similar model was proposed by Angle [22, 23] several years back in sociology journals. Angle's one parameter inequality process model (OPIP) is described by the equations:

$$\begin{aligned} m_i(t+1) &= m_i(t) + \mathcal{D}_t w m_j(t) - (1-\mathcal{D}_t) w m_i(t) \\ m_j(t+1) &= m_i(t) + (1-\mathcal{D}_t) w m_i(t) - \mathcal{D}_t w m_j(t)\,, \end{aligned} \tag{8.42}$$

where w is a fixed fraction and $\mathcal{D}_t$ takes value 0 or 1 randomly. The numerical simulation results of OPIP fit well with gamma distributions.

In the gas-like models with uniform savings, the distribution of wealth shows a self-organizing feature. A peaked distribution with a most-probable value indicates an economic scale. Empirical observations in homogeneous groups of individuals as in waged income of factory laborers in the UK and US [24], and data from a population survey in the US among students of different school and colleges produce similar distributions [23]. This is a simple case where a homogeneous population (say, characterized by a unique value of λ) has been identified.

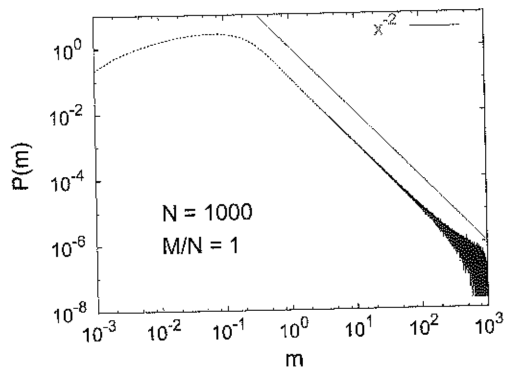

Figure 8.3 Steady-state money distribution $P(m)$ for the distributed λ model with $0 \le \lambda < 1$ for a system of $N = 1000$ agents. The x^{-2} is a guide to the observed power law, with $1 + \nu = 2$. Here, the average money per agent $M/N = 1$.

8.3.2 Model with Distributed Savings

In a real society or economy, the interest of saving varies from person to person, which implies that λ is a very inhomogeneous parameter. To move a step closer to this real situation, one considers the saving factor λ to be widely distributed within the population [26–28]. The evolution of money in such a trading can be written as

$$m_i(t+1) = \lambda_i m_i(t) + \epsilon_{ij}\left[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)\right], \tag{8.43}$$

$$m_j(t+1) = \lambda_j m_j(t) + (1-\epsilon_{ij})\left[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)\right]. \tag{8.44}$$

The trading rules are similar to the previous rules, except that

$$\Delta m = \epsilon_{ij}(1-\lambda_j)m_j(t) - (1-\lambda_i)(1-\epsilon_{ij})m_i(t), \tag{8.45}$$

where λ_i and λ_j are the saving propensities of agents i and j. The agents have fixed (over time) saving propensities, distributed independently, randomly and uniformly (white) within an interval 0 to 1: agent i saves a random fraction λ_i $(0 \le \lambda_i < 1)$ and this λ_i value is quenched for each agent, that is λ_i are independent of trading or t.

The distribution $P(m)$ of money M is found to follow a strict power-law decay, which fits to Pareto law with $\nu = 1.01 \pm 0.02$ (Figure 8.3). This power law is extremely robust. For a distribution

$$\rho(\lambda) \sim |\lambda_0 - \lambda|^{\alpha}, \quad \lambda_0 \neq 1, \quad 0 < \lambda < 1, \tag{8.46}$$

of quenched λ values among the agents, the Pareto law with $\nu = 1$ is universal for all α. The data in Figure 8.3 corresponds to $\lambda_0 = 0$, $\alpha = 0$.

The role of the agents with high saving propensity ($\lambda \to 1$) is crucial: the power law behavior is truly valid up to the asymptotic limit if 1 is included. Indeed, had we

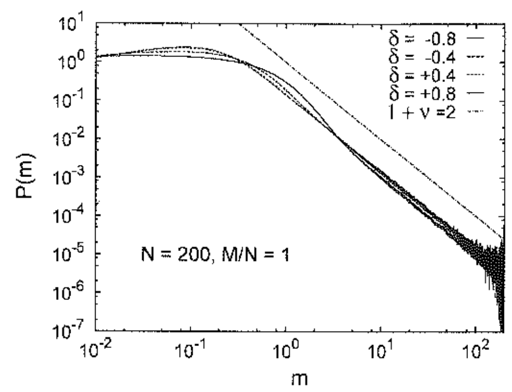

Figure 8.4 Steady-state money distribution $P(m)$ in the model for $N = 200$ agents with λ distributed as $\rho(\lambda) \propto \lambda^{\alpha}$ with different values of α. For all cases, the average money per agent $M/N = 1$.

assumed $\lambda_0 = 1$ in (8.46), the Pareto exponent ν immediately switches over to $\nu = 1 + \alpha$. Of course, $\lambda_0 \neq 1$ in (8.46) leads to the universality of the Pareto distribution with $\nu = 1$ (irrespective of λ_0 and α). Obviously, $P(m) \sim \int_0^1 P_\lambda(m)\rho(\lambda)d\lambda \sim m^{-2}$ for $\rho(\lambda)$ given by (8.46) and $P(m) \sim m^{-(2+\alpha)}$ if $\lambda_0 = 1$ in (8.46) (for large m values).

Patriarca *et al.* [32] studied the correlation between the saving factor λ and the average money held by an agent whose savings factor is λ. This numerical study revealed that the product of this average money and the unsaved fraction remains constant, or in other words, the quantity $\langle m(\lambda)\rangle(1-\lambda)$ is a constant. This result turns out to be the key to the formulation of a mean field analysis to the model [33, 34].

In a recent mean field approach [33, 34], one can calculate the distribution for the ensemble average of money for the model with distributed savings. It is assumed that the distribution of money of a single agent over time is stationary, which means that the time-averaged value of money of any agent remains unchanged independent of the initial value of money. Taking the ensemble average of all terms on both sides of (8.43), one can write

$$\langle m_i\rangle = \lambda_i\langle m_i\rangle + \langle\epsilon\rangle\left[(1-\lambda_i)\langle m_i\rangle + \left\langle \frac{1}{N}\sum_{j=1}^{N}(1-\lambda_j)m_j\right\rangle\right]. \tag{8.47}$$

The last term on the right is replaced by the average over the agents, where it is assumed that any agent (ith agent here) on the average, interacts with all others in the system, which is the *mean field* approach.

Writing

$$\overline{\langle(1-\lambda)m\rangle} \equiv \left\langle \frac{1}{N}\sum_{j=1}^{N}(1-\lambda_j)m_j\right\rangle \tag{8.48}$$

and since ϵ is assumed to be distributed randomly and uniformly in [0, 1], so that $\langle\epsilon\rangle = 1/2$, (8.47) reduces to

$$(1-\lambda_i)\langle m_i\rangle = \overline{\langle(1-\lambda)m\rangle} \,. \tag{8.49}$$

Since the right side is free of any agent index, it seems that this relation is true for any arbitrary agent, that is $\langle m_i\rangle(1-\lambda)$ = constant, where λ is the saving factor of the ith agent. What follows is, $d\lambda$ = const. $\frac{dm}{m^2}$. An agent with a (characteristic) saving propensity factor (λ) ends up with wealth (m) such that one can in general relate the distributions of the two

$$P(m)dm = \rho(\lambda)d\lambda \,. \tag{8.50}$$

Therefore, the distribution in m is bound to be of the form

$$P(m) \propto \frac{1}{m^2} \,, \tag{8.51}$$

for uniform distribution of savings factor λ, that is $\nu = 1$. This analysis can also explain the nonuniversal behavior of the Pareto exponent ν, that is $\nu = 1 + \alpha$ for $\rho(\lambda) = (1-\lambda)^\alpha$. Thus, this mean field study explains the origin of the universal ($\nu = 1$) and the nonuniversal ($\nu \neq 1$) Pareto exponents in the distributed savings model.

These model income distributions $P(m)$ compare very well with the wealth distributions of various countries: data suggests Gibbs-like distribution in the low-income range (more than 90% of the population) and Pareto-like in the high-income range [29–31] (less than 10% of the population) of various countries. In fact, we compared one model simulation of the market with saving propensity of the agents distributed following (8.46), with $\lambda_0 = 0$ and $\alpha = -0.7$ [26]. The qualitative resemblance of the model income distribution with the real data for Japan and the US in recent years is quite intriguing. In fact, for negative α values in (8.46), the density of traders with low saving propensity is higher and since a $\lambda = 0$ ensemble yields a Gibbs-like income distribution (8.35), we see an initial Gibbs-like distribution which crosses over to Pareto distribution with $\nu = 1.0$ for large m values. The position of the crossover point depends on the value of α. It is important to note that any distribution of λ near $\lambda = 1$, of finite width, eventually gives the Pareto law for large m limit. The same kind of crossover behavior (from Gibbs to Pareto) can also be reproduced in a model market of mixed agents where $\lambda = 0$ for a finite fraction p of the population and λ is distributed uniformly over a finite range near $\lambda = 1$ for the rest $1-p$ fraction of the population.

In recent years, several papers discuss these models at length and provide rigorous analysis of the models and related ones [35–39] using a variety of approaches like Fokker–Planck equations and generalized Boltzmann transport equations. Several issues regarding the structure and dynamics of such models are now known.

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