Partially non-stationary Vector Time Series

1. Cointegration

Definition 1. (Engle and Granger 1987): For a series y_t with no deterministic components, let

$$(1-B)^d y_t = z_t.$$

If z_t is a stationary ARMA process, we said that $y_t \sim I(d)$.

Definition 2. (Engle and Granger 1987): If all elements of the vector $\mathbf{y}_t$ are $I(d)$ and there exists a vector $\beta \neq 0$ such that

$$\beta' \mathbf{y}_t \sim I(d-b)$$

for some $b > 0$, the vector process is said to be cointegrated $CI(d, b)$ and β is called cointegrating vector. .

Example. The bivariate system:

$$\begin{aligned} y_{1t} &= \gamma y_{2t} + \varepsilon_{1t} \\ y_{2t} &= y_{2,t-1} + \varepsilon_{2t} \end{aligned}$$

where $\gamma \neq 0$, ε_{1t} and ε_{2t} being uncorrelated white noise processes.

$\triangle y_{2t} = \varepsilon_{2t}$, where $\triangle \equiv 1 - B$.

$$\triangle y_{1t} = \gamma \triangle y_{2t} + \triangle \varepsilon_{1t} = \gamma \varepsilon_{2t} + \varepsilon_{1t} - \varepsilon_{1,t-1}.$$

Thus, both y_{1t} and y_{2t} are $I(1)$ processes, but the linear combination $y_{1t} - \gamma y_{2t}$ is stationary.

Hence $\mathbf{y}_t = (y_{1t}, y_{2t})'$ is cointegrated with a cointegrating vector $\beta = (1, -\gamma)'$.

In general, if the vector process $\mathbf{y}_t$ has k components, then there can be more than one cointegrating vector β'.

It is assumed that there are r linearly independent cointegrating vectors with $r < k$, which make the $k \times r$ matrix β.

The rank of matrix β is r, which is called the cointegration rank of $\mathbf{y}_t$.

2. Vector Error Correction Modeling

Assume y_t is the vector AR(p) model:

$$\mathbf{y}_t = \sum_{i=1}^{p} \Phi_i \mathbf{y}_{t-i} + \varepsilon_t$$

or

$$\Phi(B)\mathbf{y}_t = \varepsilon_t$$

where the initial values, $\mathbf{y}_{-p+1}, ..., \mathbf{y}_0$, are fixed and $\varepsilon_t \sim N(0, \Sigma)$.

Assume $|\Phi(z)| = |I_k - \sum_{i=1}^{p} \Phi_i z^i| = 0$ has $d < k$ unit root and the remaining roots outside the unit circle.

The rank of $\Phi(1) = I_k - \sum_{i=1}^{p} \Phi_i$ is r and $r = k - d$

We can decompose $\Phi(1)$ as

$$\alpha\beta' = -\Phi(1) = -\mathbf{I}_k + \Phi_1 + ... + \Phi_p.$$

$\Phi(B)$ can be re-expressed as

$$\Phi(B) = \Phi^*(B)(1 - B) + \Phi(1)B,$$

where $\Phi^*(B) = \mathbf{I}_k - \sum_{i=1}^{p-1} \Phi_i^* B^i$ with $\Phi_i^* = -\sum_{j=i+1}^{p} \Phi_j$.

The so-called vector error correction model is

$$\Phi^*(B)(1-B)\mathbf{y}_t = \alpha\beta'\mathbf{y}_{t-1} + \varepsilon_t$$

or

$$\triangle \mathbf{y}_t = \alpha\beta'\mathbf{y}_{t-1} + \sum_{i=1}^{p-1} \Phi_i^* \triangle \mathbf{y}_{t-i} + \varepsilon_t,$$

[Granger representation theorem (Engle and Granger 1987)].

$\beta'\mathbf{y}_{t-1}$ is stationary and β is cointegrating matrix.

One motivation for the **VECM**(p) form is to consider the relation $\beta'\mathbf{y}_t = \mathbf{c}$ as defining the underlying economic relations and assume that the agents react to the disequilibrium error $\beta'\mathbf{y}_t - \mathbf{c}$ through the adjustment coefficient α to restore equilibrium; that is, they satisfy the economic relations. The cointegrating vector, β is sometimes called the **long-run parameters**.

3. Test for the Cointegration The cointegration rank test determines the linearly independent columns of Π. Johansen (1988, 1995a) and Johansen and Juselius (1990) proposed the cointegration rank test by using the reduced rank regression.

- **Case 1**. There is no separate drift in the **VECM**(p) form.

$$\triangle \mathbf{y}_t = \alpha\beta' \mathbf{y}_{t-1} + \sum_{i=1}^{p-1} \Phi_i^* \triangle \mathbf{y}_{t-i} + \varepsilon_t.$$

Let $\Psi = [\Phi_1^*, ..., \Phi_{p-1}^*]$. Then the **VECM**($p$) form is rewritten in these variables as

$$Z_{0t} = \alpha\beta' Z_{1t} + \Psi Z_{2t} + \varepsilon,$$

where

$$\begin{aligned} Z_{0t} &= \triangle \mathbf{y}_t \\ Z_{1t} &= \mathbf{y}_{t-1} \\ Z_{2t} &= [\triangle \mathbf{y}'_{t-1}, ..., \triangle \mathbf{y}'_{t-p+1}]' \end{aligned}$$

The log-likelihood function is given by

$$\ell = -\frac{kT}{2} \log 2\pi - \frac{T}{2} \log |\Sigma|$$

$$-\frac{1}{2}\sum_{t=1}^{T}(Z_{0t}-\alpha\beta' Z_{1t}-\Psi Z_{2t})'\Sigma^{-1}(Z_{0t}-\alpha\beta' Z_{1t}-\Psi Z_{2t}).$$

The maximizer of the preceding log-likelihood function is called the MLE of VECM.

Let

$$M_{i2}=\frac{1}{T}\sum_{t=1}^{T} Z_{it}Z_{2t} \ and \ M_{22}=\frac{1}{T}\sum_{t=1}^{T} Z_{2t}Z_{2t}$$

Define

$$\begin{aligned} R_{it} &= Z_{it}-M_{i2}M_{22}^{-1}Z_{2t}, \\ S_{ij} &= \frac{1}{T}\sum_{t=1}^{T} R_{it}R_{jt}^{'}, \quad i,j=0,1. \end{aligned}$$

The maximum likelihood estimator for β is obtained from the eigenvectors that correspond to the r largest eigenvalues of the following equation:

$$|\lambda S_{11}-S_{10}S_{00}^{-1}S_{01}|=0.$$

Johansen (1988): two test statistics for the null hypothesis that there are at most r cointegrating vectors

$$H_0:\lambda_i=0 \quad \text{for } i=r+1,...,k.$$

Trace test

The trace statistic for H_0:

$$\lambda_{trace} = -T \sum_{i=r+1}^{k} \log(1 - \lambda_i) \longrightarrow_L$$
$$tr\left\{ \int_0^1 (dW)\widetilde{W}' \left(\int_0^1 \widetilde{W}\widetilde{W}' dr \right)^{-1} \int_0^1 \widetilde{W}(dW)' \right\},$$

where $tr(A)$ is the trace of a matrix A, W is the $k - r$ dimensional BM, and $\widetilde{W}$ is the BM itself, or their modification.

Maximum Eigenvalue Test

The maximum eigenvalue statistic for H_0

$$\lambda_{\max} = -T \log(1 - \lambda_{r+1}) \longrightarrow_L$$
$$\max\left\{ \int_0^1 (dW)\widetilde{W}' \left(\int_0^1 \widetilde{W}\widetilde{W}' dr \right)^{-1} \int_0^1 \widetilde{W}(dW)' \right\},$$

where $\max(A)$ is the maximum eigenvalue of a matrix A.

General Models:

You can consider a vector error correction model with a deterministic term. The deterministic term D_t can contain a constant, a linear trend, and seasonal dummy variables. Exogenous variables can also be included in the model.

$$\triangle \mathbf{y}_t = \Pi \mathbf{y}_{t-1} + \sum_{i=1}^{p-1} \Phi_i^* \triangle \mathbf{y}_{t-i} + AD_t + \varepsilon_t,$$

where $\Pi = \alpha\beta'$.

- **Case 2**. There is no separate drift in the **VECM**(p) form, but a constant enters only via the error correction term.

$$\triangle \mathbf{y}_t = \alpha(\beta', \beta_0)(\mathbf{y}_{t-1}', 1)' + \sum_{i=1}^{p-1} \Phi_i^* \triangle \mathbf{y}_{t-i} + \varepsilon_t.$$

- **Case 3**. There is a separate drift and no separate linear trend in the **VECM**(p) form.

$$\triangle \mathbf{y}_t = \alpha\beta' \mathbf{y}_{t-1} + \sum_{i=1}^{p-1} \Phi_i^* \triangle \mathbf{y}_{t-i} + \delta_0 + \varepsilon_t.$$

- **Case 4**. There is a separate drift and no separate linear trend in the **VECM**(p) form, but a linear trend enters only via the error correction term.

$$\triangle \mathbf{y}_t = \alpha(\beta', \beta_1)(\mathbf{y}'_{t-1}, t)' + \sum_{i=1}^{p-1} \Phi_i^* \triangle \mathbf{y}_{t-i} + \delta_0 + \varepsilon_t.$$

- **Case 5**. There is a separate linear trend in the **VECM**(p) form.

$$\triangle \mathbf{y}_t = \alpha\beta'\mathbf{y}_{t-1} + \sum_{i=1}^{p-1} \Phi_i^* \triangle \mathbf{y}_{t-i} + \delta_0 + \delta_1 t + \varepsilon_t.$$

Trace test and **Maximum Eigenvalue Test** are similar to the case 1, and their limiting distributions are still a functional of BM, but are more complicated.

Multivariate GARCH Modeling

BEKK Representation Engle and Kroner (1995) propose a general multivariate GARCH model and call it a **BEKK** representation.

Let $\mathcal{F}(t-1)$ be the $\sigma-$ field generated by the past values of ε_t, and let H_t be the conditional covariance matrix of the $k \times 1$ random vector ε_t.

Let H_t be measurable with respect to $\mathcal{F}(t-1)$; then the multivariate GARCH model can be written as

$$\begin{aligned} \varepsilon_t|\mathcal{F}(t-1) &\sim N(0, H_t) \\ H_t &= C_0'C_0 + A'\varepsilon_{t-1}\varepsilon_{t-1}'A + G'H_{t-1}G \end{aligned}$$

where C_0, A and G are $k \times k$ parameter matrices with C_0 is an upper triangular matrix.

A bivariate GARCH(1,1) model as follows:

$$\begin{aligned} H_t &= C_0'C_0 + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}' \begin{bmatrix} \varepsilon_{1,t-1}^2 & \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon_{2,t-1}\varepsilon_{1,t-1} & \varepsilon_{2,t-1}^2 \end{bmatrix} \\ &\quad \times \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}' H_{t-1} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \end{aligned}$$

or, representing the univariate model,

$$h_{11,t} = c_1 + a_{11}^2\varepsilon_{1,t-1}^2 + 2a_{11}a_{21}\varepsilon_{1,t-1}\varepsilon_{2,t-1} + a_{21}^2\varepsilon_{2,t-1}^2$$

$$+g_{11}^2 h_{11,t-1} + 2g_{11}g_{21}h_{12,t-1} + g_{21}^2 h_{22,t-1}.$$

SAS statements are:

```
model y1 y2;
garch q=1 p=1 form=bekk;
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Bollerslev, Engle and Wooldridge (1988) and is called the **BEW** representation.

$$\mathbf{vech}(H_t) = \mathbf{c}^* + A^*\mathbf{vech}(\varepsilon_{t-1}\varepsilon_{t-1}') + G^*\mathbf{vech}(H_{t-1}),$$

where $\mathbf{c}^*$ is a $k(k+1)/2$-dimensional vector; A^* and G^* are $k(k+1)/2 \times k(k+1)/2$ matrices.

SAS statement is

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Multivariate GARCH-M model.

VAR-GARCH models,

and VARX-GARCH models.

$$\begin{aligned}
\Phi(B)\mathbf{y}_t &= \delta + \Theta^*(B)\mathbf{x}_t + \Lambda\mathbf{h}_t + \varepsilon_t \\
\varepsilon_t|\mathcal{F}_{t-1} &\sim N(0, H_t) \\
H_t &= C_0'C_0 + A'\varepsilon_{t-1}\varepsilon_{t-1}'A + G'H_{t-1}G,
\end{aligned}$$

where $\mathbf{y}_t$ is a $k \times 1-$vector of endogenous variables; $\mathbf{x}_t$ is an $k \times 1-$ vector of exogenous variables ; $\mathbf{h}_t = \mathbf{vech}(H_t)$ and Λ is a $k \times k(k+1)/2$ parameter matrix.

General GARCH models can be also applied to **BEW** and *diagonal* representations.

Estimation of GARCH Model The log-likelihood function of the multivariate GARCH model is written without a constant term

$$\ell = -\frac{1}{2}\sum_{t=1}^{T}\Big[\log|H_t| - \varepsilon_t' H_t^{-1}\varepsilon_t\Big].$$

Covariance Stationarity Define the multivariate GARCH process as

$$\mathbf{h}_t = \sum_{i=1}^{\infty} G(B)^{i-1}[\mathbf{c} + A(B)\eta_t]$$

where $\mathbf{h}_t = \mathbf{vec}(H_t)$, $\mathbf{c} = \mathbf{vec}(C_0'C_0)$ and $\eta_t = \mathbf{vec}(\varepsilon_t\varepsilon_t')$. This representation is equivalent to a GARCH(p,q) model by the following algebra:

$$\begin{aligned}
\mathbf{h}_t &= \mathbf{c} + A(B)\eta_t + \sum_{i=2}^{\infty} G(B)^{i-1}[\mathbf{c} + A(B)\eta_t] \\
&= \mathbf{c} + A(B)\eta_t + G(B)\sum_{i=1}^{\infty} G(B)^{i-1}[\mathbf{c} + A(B)\eta_t] \\
&= \mathbf{c} + A(B)\eta_t + G(B)\mathbf{h}_t.
\end{aligned}$$

Defining $A(B) = (A \otimes A)' B^i$ and $G(B) = (G \otimes G)' B^i$ gives a **BEKK** representation.

The necessary and sufficient conditions for covariance stationarity of the multivariate GARCH process is that all the eigenvalues of $A(1) + G(1)$ are less than one in modulus.