# Chapter 2: Linear Time Series (TS) Models

Financial TS: collection of a financial measurement over time

Example: log return $r_t$

Data: $\{r_1, r_2, \cdots, r_T\}$ ($T$ data points)

Purpose: What information contained in $\{r_t\}$?

Basic concepts

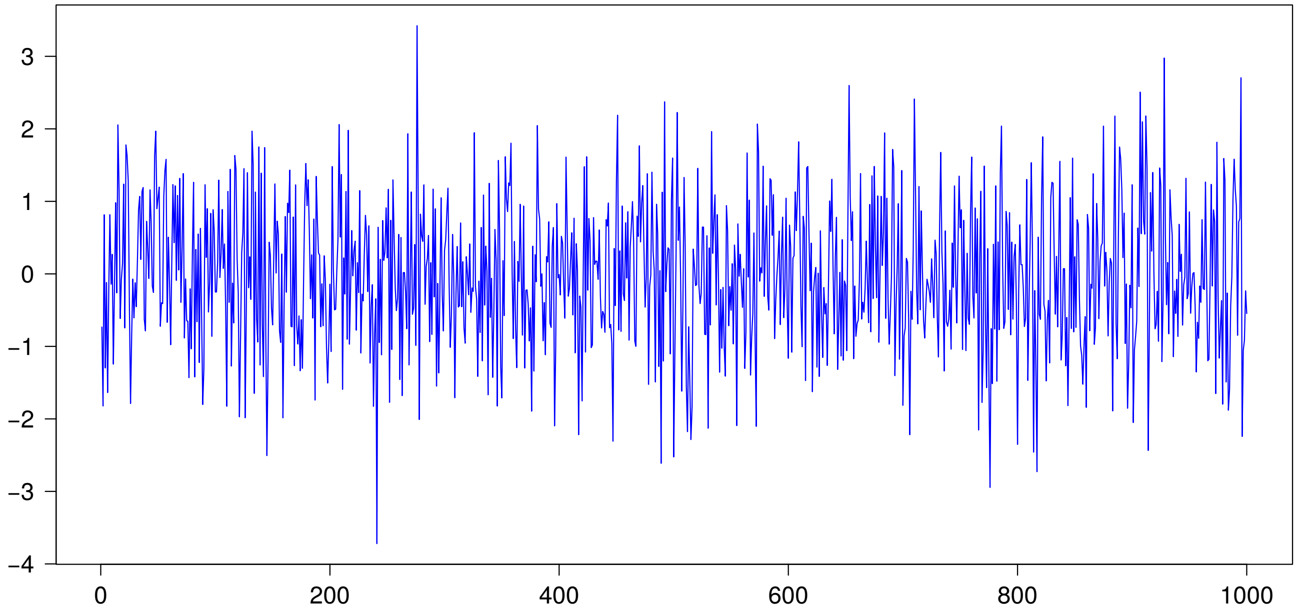
**White noises**:

Let $\{\varepsilon_t\}$ be a time series. If

$$E\varepsilon_t = 0 \ and \ \mathrm{cov}(\varepsilon_t, \varepsilon_s) = \begin{cases} \sigma^2, & t = s, \\ 0, & t \neq s, \end{cases}$$

for any $s, t \in \mathcal{T}$, then $\{\varepsilon_t\}$ is called a **white noise**, denoted by $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$.

# A path of a white noise



**Measure of dependence**

**Mean function**:

$$\mu_t = E(r_t) = \int_{-\infty}^{\infty} y f_t(y) dy.$$

We introduce autocovariances to measure the linear dependence between two observations at any two time points.

**Autocovariance function**:

$$\gamma_{s,t} = Cov(r_s, r_t) = E[(r_s - \mu_s)(r_t - \mu_t)]$$

for any $s$ and $t$.

Note that $\gamma_{s,t} = \gamma_{t,s}$, and $\gamma_{t,t}$ is the variance of $r_t$.

**Autocorrelation function**:

$$\rho_{s,t} = corr(r_s, r_t) = \frac{\gamma_{s,t}}{\sqrt{\gamma_{s,s}\gamma_{t,t}}}.$$

**Weak Stationarity**: first 2 moments are time-invariant, i.e.,

(1) **Mean (or expectation) of returns**:

$$\mu = E(r_t);$$

(2) **Variance (variability) of returns**:

$$\sigma^2 = Var(r_t) = E[(r_t - \mu)^2];$$

(3). Lag$-k$ autocovariance:

$$\gamma_k = Cov(r_t, r_{t-k}) = E[(r_t - \mu)(r_{t-k} - \mu)].$$

We say $\{r_t\}$ is **(second order) weakly stationary**.

Serial correlation or Autocorrelation function (ACF):

$$\rho_k = \frac{cov(r_t, r_{t-k})}{var(r_t)}.$$

Note: $\rho_0 = 1$ and $\rho_k = \rho_{-k}$ for $k \neq 0$. Why?

Existence of ACFs implies that the return is predictable, indicating market inefficiency.

What does weak stationarity mean in practice?

Past: time plot of $\{r_t\}$ varies around a fixed level within a finite range!

Future: the first 2 moments of future $r_t$ are the same as those of the data so that meaningful inferences can be made.

**Strict Stationarity**:

Strict: distributions are time-invariant, i.e.,

$$\begin{aligned} &P(r_{t_1} \le z_1, \cdots, r_{t_n} \le z_n) \\ &\quad = P(r_{t_1+k} \le z_1, \cdots, r_{t_n+k} \le z_n), \end{aligned}$$

for $\forall t_1, \cdots, t_n, k$ and $(z_1, \cdots, z_n)$ and $n$.

We say: $\{r_t\}$ is a **strictly stationary TS**.

**Relations between weak stationarity and strict stationarity:**

**Example: (WS but not SS)** $\{X_t\}$ with $X_{2t-1} = \varepsilon$ and $X_{2t} = \eta$, where $\varepsilon \sim \mathcal{N}(0, 1)$, $\eta \sim \mathcal{U}(-\sqrt{3}, \sqrt{3})$, and $\varepsilon$ and $\eta$ are independent. Then $\{X_t\}$ is WS, not SS.

**Solution**. Clearly, $EX_t = 0$, $EX_t^2 = 1 < \infty$, and

$$\gamma_{t,t+s} = \begin{cases} \text{cov}(X_t, X_t) = 1, & \text{if } s \text{ is even,} \\ \\ \text{cov}(\varepsilon, \eta) = 0, & \text{if } s \text{ is odd.} \end{cases}$$

Hence, $\gamma_{t,t+s}$ is independent of $t$ and only depends on $s$. Thus, $\{X_t\}$ is WS. However, $\{X_t\}$ is not SS since $\varepsilon$ and $\eta$ do not have the same distribution functions.

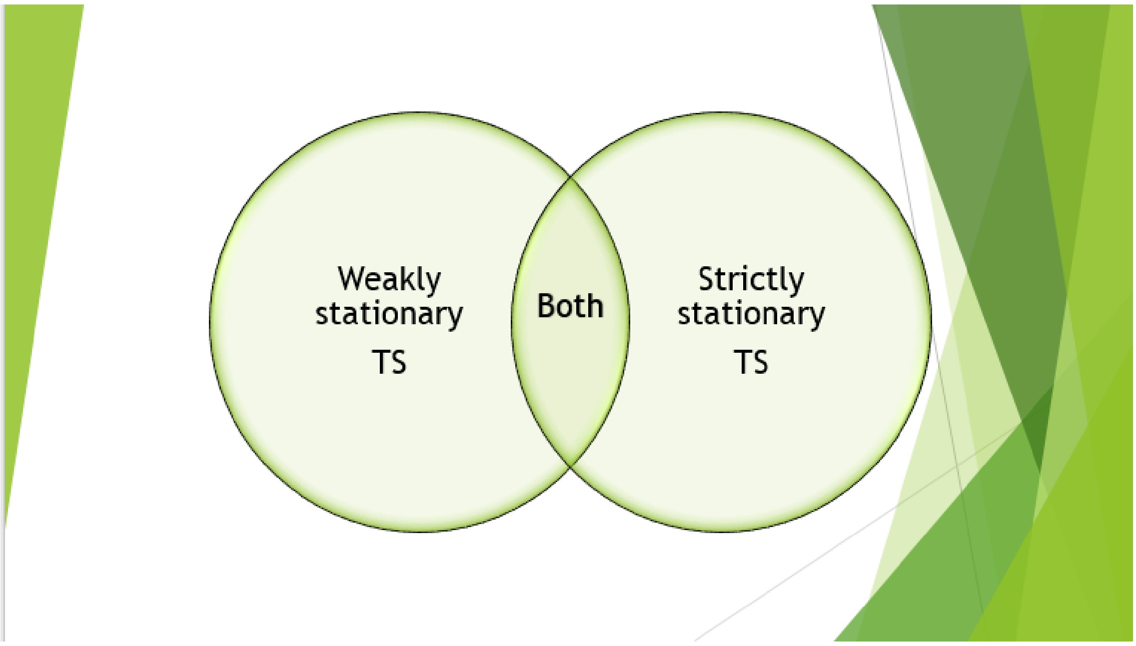
**Example: (SS but not WS)** A TS $\{\varepsilon_t : t \in T\}$ is i.i.d. standard Cauchy distribution with density $f(x) = \frac{1}{\pi(1+x^2)}$. Then, $\{\varepsilon_t\}$ is SS, not WS in that $E\varepsilon_t^2 = \infty$.

**Example: (WS & SS)** A TS $\{y_t : t \in \mathcal{T}\}$ is i.i.d. with $y_t \sim \mathcal{N}(0,1)$. Then $\{y_t : t \in \mathcal{T}\}$ is both WS and SS.

**Example: (neither SS nor WS)** A TS $\{\varepsilon, \eta, \xi_3, \xi_4, ...\}$, where $\varepsilon \sim \mathcal{N}(0,1)$, $\eta \sim t_1$, and $\{\xi_j : j = 3, 4, ...\}$ is i.i.d. $\sim \mathcal{E}(1)$. All of them are independent. Then it is neither SS nor WS.

**Summary:**

- A strict stationary time series with finite second moments is weakly stationary;
- Weakly stationary does not imply strictly stationary.




From now on, the term “stationary” means “ second-order weakly stationary”.

**Property of ACVF and ACF for a stationary process**

For a WS time series $\{r_t : t \in \mathcal{T}\}$, $\gamma_k$ and $\rho_k$ have the following properties.

(1) $\gamma_0 = \text{var}(r_t)$; $\quad \rho_0 = 1$.

(2) $|\gamma_k| \le \gamma_0$; $\quad |\rho_k| \le 1$.

(3) $\gamma_k = \gamma_{-k}$; $\quad \rho_k = \rho_{-k}$.

When $\rho_k = 0$ for all $k \neq 0$, $\{r_t\}$ is a sequence of **white noises**.

Sample mean and sample variance are used to estimate the mean and variance of returns:

$$\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t \ \ and \ \hat{\sigma}_r^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2.$$

Test $H_0 : \mu = 0$ vs $H_a : \mu \neq 0$. Compute

$$t = \frac{\sqrt{T}\bar{r}}{\hat{\sigma}_r}.$$

Compare $t$ ratio with $N(0,1)$ dist.

**Decision rule**: Reject $H_0$ of zero mean if $|t| > Z_{\alpha/2}$ or $p-$value is less than $\alpha$

Sample autocorrelation function (ACF)

$$\widehat{\rho}_k = \frac{\sum_{t=1}^{T-k}(r_t - \bar{r})(r_{t+k} - \bar{r})}{\sum_{t=1}^{T}(r_t - \bar{r})^2},$$

where $\bar{r}$ is the sample mean and $T$ is the sample size.

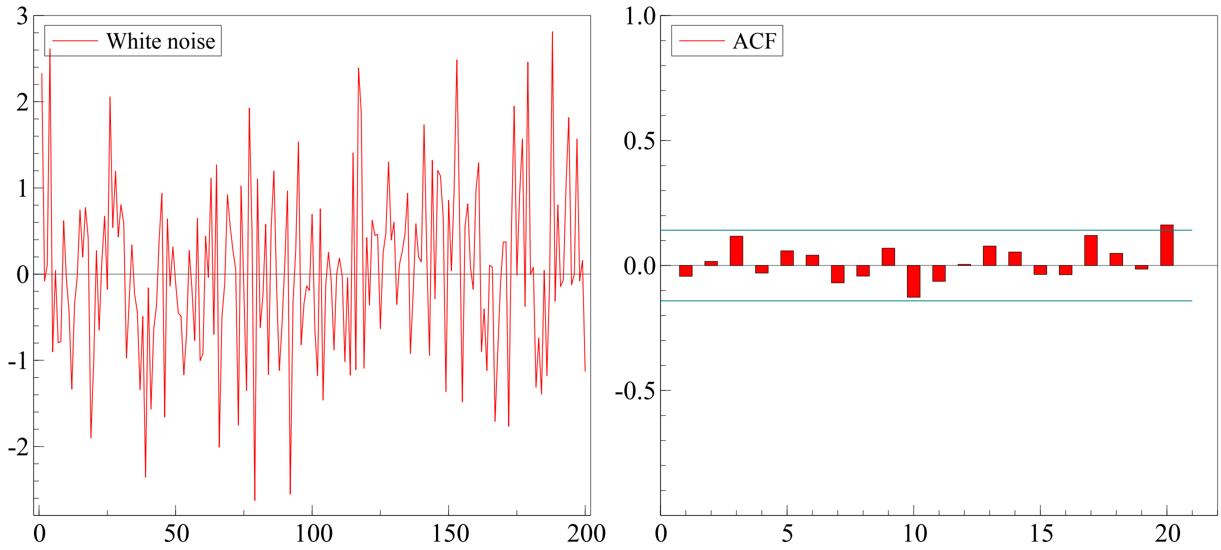
Test zero serial correlations (market efficiency)

Individual test: for example,

$H_0 : \rho_1 = 0$ vs $H_a : \rho_1 \neq 0$

$$t = \frac{\widehat{\rho}_1}{\sqrt{1/T}} = \sqrt{T}\widehat{\rho}_1 \sim N(0,1).$$

**Decision rule**: Reject $H_0$ if $|t| > Z_\alpha/2$ or $p-$value less than




White noise series and the *sample autocorrelation function*

Joint test (Ljung-Box statistics):

$H_0: \ \rho_1 = \cdots = \rho_m = 0 \ vs \ H_a: \ \rho_i \neq 0$ for some $i$.

$$Q(m) = T(T+2)\sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{T-k} \sim \chi^2_m.$$

Asym. $\chi^2$ dist with $m$ degrees of freedom.

**Decision rule**: Reject $H_0$ if $Q(m) > \chi^2_m(\alpha)$ or $p$-value is less than $\alpha$.

Sources of serial correlations in financial TS

Bid-ask bounce (ch. 5)

Risk premium, etc. (ch. 3)

Thus, significant sample ACF does not necessarily imply market inefficiency.

Example: $\cdots$

Back-shift (lag) operator: A useful notation in TS analysis.

Definition: $Br_t = r_{t-1}$ or $Lr_t = r_{t-1}$.

$B^2 r_t = B(Br_t) = Br_{t-1} = r_{t-2}$.

$B$ (or $L$) means time shift! $Br_t$ is the value of the series at time $t-1$.

Question: What is $B2$?

What are the important statistics in practice?

Conditional quantities, not unconditional ones.

Available data: $\{r_1, r_2, \cdots, r_{t-1}\} \equiv F_{t-1}$.

The return is decomposed into two parts as

$$\begin{aligned} r_t &= \text{ predictiable part } + \text{ not predic. part} \\ &= \text{function of elements of } F_{t-1} + a_t \\ &\equiv \mu_t + a_t. \end{aligned}$$

Assume that

$$E(a_t|F_{t-1}) = 0$$

since we do not have the information on the $t-$day.

$$E(r_t|F_{t-1}) = \mu_t + E(a_t|F_{t-1}) = \mu_t,$$

and hence

$$r_t = \mu_t + a_t.$$

$\mu_t$ is the best predictor of $r_t$ in mean square error, i.e. for any $g_t \in F_{t-1}$, we have

$$E(r_t - g_t)^2 > E(r_t - \mu_t)^2 \ if \ g_t \neq \mu_t.$$

Math proof:

$$\begin{aligned}
E(r_t - g_t)^2 &= E[r_t - \mu_t - (g_t - \mu_t)]^2 \\
&= E(r_t - \mu_t)^2 + E(\mu_t - g_t)^2 \\
&\quad -2E[(r_t - \mu_t)(g_t - \mu_t)] \\
&= E(r_t - \mu_t)^2 + E(\mu_t - g_t)^2 \\
&> E(r_t - \mu_t)^2 \ if \ g_t \neq \mu_t.
\end{aligned}$$

$\{a_t\}$ is a white noise series, but may not be i.i.d.

Denote $\sigma_t^2 = Var(r_t|F_{t-1})$.

$$\sigma_t^2 = E[(r_t - \mu_t)^2|F_{t-1}] = E[a_t^2|F_{t-1}].$$

Denote $\epsilon_t = a_t/\sigma_t$. Then $\{\epsilon_t\}$ is an uncorrelated sequence with mean zero and variance 1.

$r_t$ can be decomposed as

$$r_t = \mu_t + \sigma_t \epsilon_t.$$

$\sigma_t$: conditional standard deviation (commonly called volatility in finance)

Univariate TS analysis serves two purposes:

1. a model for $\mu_t$

2. understanding models for $\sigma_t^2$: properties, forecasting, etc.

**Linear time series**: $r_t$ is linear if $r_t$ can be written as

$$r_t = \mu + \sum_{i=0}^{\infty} \psi_i a_{t-i},$$

where $\mu$ is a constant, $\psi_0 = 1$ and $\{a_t\}$ is a sequence of white noises

White noise is a uncorrelated time series with zero mean and finite variance. It is not predictable.

Univariate linear time series models

1. autoregressive (AR) models

2. moving-average (MA) models

3. mixed ARMA models

4. seasonal models

5. regression models with time series errors

6. fractionally differenced models (long-memory)