Linearisation of Laser Equations for Lyapunov Exponent Calculation

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1 Class C Equations

$$\dot{E}_x = -E_x - \Delta_p E_y - P_y + f_x(T), \tag{1}$$

$$\dot{E}_y = -E_y + \Delta_p E_x + P_x + f_y(T), \tag{2}$$

$$\frac{1}{\gamma_{pc}} \dot{P}_x = -P_x + \Delta_p P_y + E_y N, \tag{3}$$

$$\frac{1}{\gamma_{pc}} \dot{P}_y = -P_y - \Delta_p P_x - E_x N, \tag{4}$$

$$\frac{1}{\gamma_{nc}} \dot{N} = \Lambda - N + \big(P_y E_x - E_y P_x\big). \tag{5}$$

where the forcing terms are,

$$f_x(T) = \alpha \cos\phi \sin\omega\phi \; \delta(t-T) \tag{6}$$
$$f_y(T) = \alpha \sin\phi \sin\omega\phi \; \delta(t-T) \tag{7}$$

with $\alpha =$ "Kick Amplitude", $\phi = \tan^{-1}\left(\frac{E_y}{E_x}\right)$, and $\omega =$ "Number of petals".

1.1 Auxiliary Calculation

Want $\partial_{E_x} f_x(T)$, $\partial_{E_y} f_x(T)$, $\partial_{E_x} f_y(T)$, and $\partial_{E_y} f_y(T)$.

$$\begin{aligned}
\partial_{E_x} f_x(T) &= \alpha \partial_{E_x}\big(\cos\phi\big)\sin\omega\phi + \alpha\partial_{E_x}\big(\sin\omega\phi\big)\cos\phi && (8)\\
&= \big(\alpha\omega\cos\phi\cos\omega\phi - \alpha\sin\phi\sin\omega\phi\big)\partial_{E_x}\phi && (9)\\
&= \left(\frac{-E_y}{E_x^2+E_y^2}\right)\big(\alpha\omega\cos\phi\cos\omega\phi - \alpha\sin\phi\sin\omega\phi\big)\delta(t-T) && (10)
\end{aligned}$$

$$\begin{aligned}
\partial_{E_y} f_x(T) &= \alpha \partial_{E_y}\big(\cos\phi\big)\sin\omega\phi + \alpha\partial_{E_y}\big(\sin\omega\phi\big)\cos\phi && (11)\\
&= \big(\alpha\omega\cos\phi\cos\omega\phi - \alpha\sin\phi\sin\omega\phi\big)\partial_{E_y}\phi && (12)\\
&= \left(\frac{E_x}{E_x^2+E_y^2}\right)\big(\alpha\omega\cos\phi\cos\omega\phi - \alpha\sin\phi\sin\omega\phi\big)\delta(t-T) && (13)
\end{aligned}$$

$$\partial_{E_x} f_y(T) = \alpha \partial_{E_x} \big(\sin\phi \big) \sin\omega\phi + \alpha \partial_{E_x} \big(\sin\omega\phi \big) \cos\phi \tag{14}$$

$$= \big(\alpha\omega \sin\phi \cos\omega\phi + \alpha \cos\phi \sin\omega\phi \big) \partial_{E_x} \phi \tag{15}$$

$$= \left(\frac{-E_y}{E_x^2 + E_y^2} \right) \big(\alpha\omega \sin\phi \cos\omega\phi + \alpha \cos\phi \sin\omega\phi \big) \delta(t - T) \tag{16}$$

$$\partial_{E_y} f_y(T) = \alpha \partial_{E_y} \big(\sin\phi \big) \sin\omega\phi + \alpha \partial_{E_y} \big(\sin\omega\phi \big) \cos\phi \tag{17}$$

$$= \big(\alpha\omega \sin\phi \cos\omega\phi + \alpha \cos\phi \sin\omega\phi \big) \partial_{E_y} \phi \tag{18}$$

$$= \left(\frac{E_x}{E_x^2 + E_y^2} \right) \big(\alpha\omega \sin\phi \cos\omega\phi + \alpha \cos\phi \sin\omega\phi \big) \delta(t - T) \tag{19}$$

1.2 Jacobian

Now we get the Jacobian for the Class C laser including the terms that belong to the forcing/kicking,

$$\mathbf{Df} = \begin{pmatrix} -1 + \partial_{E_x} f_x(T) & -\Delta_p + \partial_{E_y} f_x(T) & 0 & -1 & 0 \\ \Delta_p + \partial_{E_x} f_y(T) & -1 + \partial_{E_y} f_y(T) & 1 & 0 & 0 \\ 0 & \gamma_{pc} N & -\gamma_{pc} & \gamma_{pc}\Delta_p & \gamma_{pc} E_y \\ -\gamma_{pc} N & 0 & -\gamma_{pc}\Delta_p & -\gamma_{pc} & -\gamma_{pc} E_x \\ \gamma_{nc} P_y & -\gamma_{nc} P_x & -\gamma_{nc} E_y & \gamma_{nc} E_x & -\gamma_{nc} \end{pmatrix} \tag{20}$$

2 Class B

We have a simular senario for the Class B system.

$$\dot{E}_x = \left(\frac{N}{1 + \Delta_p^2} - 1 \right) \big(E_x + \Delta_p E_y \big) + f_x(T), \tag{21}$$

$$\dot{E}_y = \left(\frac{N}{1 + \Delta_p^2} - 1 \right) \big(E_y - \Delta_p E_x \big) + f_y(T),, \tag{22}$$

$$\frac{1}{\gamma_{nc}} \dot{N} = \Lambda - N - \frac{|E|^2 N}{1 + \Delta_p^2}, \tag{23}$$

where $|E|^2 := Ex^2 + Ey^2$, and $f_x(T)$ and $f_y(T)$ are defined as above.

2.1 Jacobian

If we define:

$$\alpha := \frac{N}{1+\Delta_p^2} - 1$$

$$\mathbf{Df} = \begin{pmatrix} \alpha + \partial_{E_x} f_x(T) & \alpha\Delta_p + \partial_{E_y} f_x(T) & \frac{E_x + \Delta_p E_y}{1+\Delta_p^2} \\ -\alpha\Delta_p + \partial_{E_x} f_y(T) & \alpha + \partial_{E_y} f_y(T) & \frac{E_y - \Delta_p E_x}{1+\Delta_p^2} \\ \frac{-2\gamma_{nc} E_x N}{1+\Delta_p^2} & \frac{-2\gamma_{nc} E_y N}{1+\Delta_p^2} & \gamma_{nc}\Big(-1 - \frac{|E|^2}{1+\Delta_p^2}\Big) \end{pmatrix}. \tag{24}$$