Cost model for simple obstacle avoidance

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1 Sphere-sphere collision

1.1 Function definition

Cost function

$$c(\boldsymbol{q}) = a(\boldsymbol{r}(\boldsymbol{q})) \tag{1}$$

Residual Function

$$\boldsymbol{r}(\boldsymbol{q}) = \boldsymbol{p} - \boldsymbol{p}_{\text{ref}} \tag{2}$$

Activation Function

$$d = s(\boldsymbol{r}^\top \boldsymbol{r}) \tag{3}$$

s($\cdot$) = square root function.

If $d < \epsilon$:

$$a(\boldsymbol{r}) = \frac{1}{2}(d - \epsilon)^2, \tag{4}$$

else:

$$a(\boldsymbol{r}) = 0 \tag{5}$$

1.2 Computing the Derivatives

Objective: to calculate

$$\frac{\partial c}{\partial \boldsymbol{q}}, \frac{\partial^2 c}{\partial \boldsymbol{q}^2} \tag{6}$$

by chain rule.

Derivatives of the residual function

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}} = \boldsymbol{J} \tag{7}$$

$$\frac{\partial^2 \boldsymbol{r}}{\partial \boldsymbol{q}^2} = \boldsymbol{0} \tag{8}$$

Note: Eq. 8 is an approximations (ignoring the derivative of the Jacobian).

Derivatives of the activation function

if $d < \epsilon$:

$$\frac{\partial a}{\partial \boldsymbol{r}} = \frac{d-\epsilon}{d}\boldsymbol{r} \tag{9}$$

$$\frac{\partial^2 a}{\partial \boldsymbol{r}^2} = \frac{d-\epsilon}{d}\boldsymbol{I} + \frac{\epsilon}{d^3}\boldsymbol{r}\boldsymbol{r}^\top \tag{10}$$

else:

$$\frac{\partial a}{\partial r} = 0 \tag{11}$$

$$\frac{\partial^2 a}{\partial r^2} = 0 \tag{12}$$

Derivatives of the cost function

First derivative:

$$\frac{\partial c}{\partial \boldsymbol{q}} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}}^\top \frac{\partial a}{\partial \boldsymbol{r}} \tag{13}$$

$$\frac{\partial c}{\partial \boldsymbol{q}} = \frac{d-\epsilon}{d}\boldsymbol{J}^\top \boldsymbol{r} \tag{14}$$

Second derivative:

$$\frac{\partial^2 c}{\partial \boldsymbol{q}^2} = \frac{\partial}{\partial q}(\frac{\partial c}{\partial \boldsymbol{q}}) \tag{15}$$

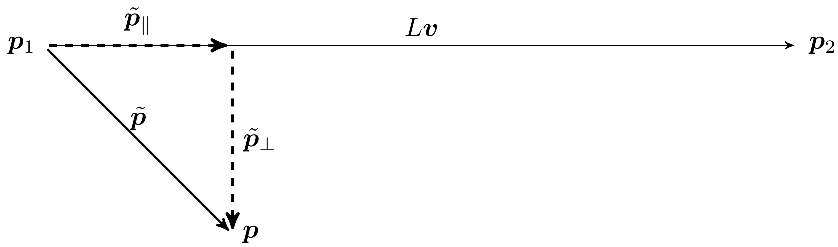
$$\frac{\partial^2 c}{\partial \boldsymbol{q}^2} = \frac{\partial}{\partial q}(\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}}^\top \frac{\partial a}{\partial \boldsymbol{r}}) = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}}^\top \frac{\partial^2 a}{\partial \boldsymbol{r}^2}\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}} + \frac{\partial a}{\partial \boldsymbol{r}}\frac{\partial^2 \boldsymbol{r}}{\partial \boldsymbol{q}^2} \tag{16}$$

$$\frac{\partial^2 c}{\partial \boldsymbol{q}^2} = \boldsymbol{J}^\top(\frac{d-\epsilon}{d}\boldsymbol{I} + \frac{\epsilon}{d^3}\boldsymbol{r}\boldsymbol{r}^\top)\boldsymbol{J} \tag{17}$$

$$\frac{\partial^2 c}{\partial \boldsymbol{q}^2} = \frac{d-\epsilon}{d}\boldsymbol{J}^\top\boldsymbol{J} + \frac{\epsilon}{d^3}\boldsymbol{J}^\top\boldsymbol{r}\boldsymbol{r}^\top\boldsymbol{J} \tag{18}$$

2 Sphere-Capsule Collision

Main idea: calculate the distance between a point (the center of sphere) and a line segment (the center line of a capsule).



Let:
$\boldsymbol{p}_1, \boldsymbol{p}_2$ = initial and end point of the line
$\boldsymbol{p}$ = the center of a sphere

Define a few quantities:

$$L = ||\boldsymbol{p}_2 - \boldsymbol{p}_1||, \tag{19}$$

$$\boldsymbol{v} = \frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{L}, \tag{20}$$

$$\tilde{\boldsymbol{p}} = \boldsymbol{p}_1 - \boldsymbol{p}, \tag{21}$$

where L and $\boldsymbol{v}$ are the length and the vector of the line from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$, and $\tilde{\boldsymbol{p}}$ is the vector from $\boldsymbol{p}_1$ to $\boldsymbol{p}$.

We can decompose $\tilde{\boldsymbol{p}}$ into two parts: one that is parallel to $\boldsymbol{v}$ and the other one that is perpendicular:

$$\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{p}}_{||} + \tilde{\boldsymbol{p}}_{\perp} \tag{22}$$

The parallel part, $\tilde{\boldsymbol{p}}_{||}$, is simply the projection of $\tilde{\boldsymbol{p}}$ on $\boldsymbol{v}$:

$$\tilde{\boldsymbol{p}}_{||} = (\tilde{\boldsymbol{p}}^{\top}\boldsymbol{v})\boldsymbol{v} \tag{23}$$

The perpendicular part, $\tilde{\boldsymbol{p}}_{\perp}$, corresponds to the "distance vector" whose magnitude corresponds to the nearest distance from the point $\boldsymbol{p}$ to the line.

$$\begin{aligned}
\tilde{\boldsymbol{p}}_{\perp} &= \tilde{\boldsymbol{p}} - \tilde{\boldsymbol{p}}_{||} && (24)\\
&= \tilde{\boldsymbol{p}} - (\tilde{\boldsymbol{p}}^{\top}\boldsymbol{v})\boldsymbol{v} && (25)
\end{aligned}$$

$\tilde{\boldsymbol{p}}_{\perp}$ is a good candidate for the residual function, but we need first to determine if the nearest point along the line is within the line segment or not. If not, the residual should instead be the distance to either $\boldsymbol{p}_1$ or $\boldsymbol{p}_2$.

We can do this by computing the following quantity:

$$t = \tilde{\boldsymbol{p}}^{\top}\boldsymbol{v} \tag{26}$$

If t is negative, the nearest point is before point $\boldsymbol{p}_1$. If it is positive but larger than L, the nearest point is after point $\boldsymbol{p}_2$. Otherwise, it lies within the line segment, and we can use $\tilde{\boldsymbol{p}}_{\perp}$.

2.1 Function definition

Residual Function

Calculate t:

$$t = \tilde{\boldsymbol{p}}^{\top}\boldsymbol{v} \tag{27}$$

If $t < 0$:

$$\boldsymbol{r}(\boldsymbol{q}) = \boldsymbol{p} - \boldsymbol{p}_1 \tag{28}$$

else, if $t > L$:

$$\boldsymbol{r}(\boldsymbol{q}) = \boldsymbol{p} - \boldsymbol{p}_2 \tag{29}$$

else:

$$\begin{aligned} \boldsymbol{r}(\boldsymbol{q}) &= \tilde{\boldsymbol{p}} - \tilde{\boldsymbol{p}}_{\perp} && (30)\\ &= \tilde{\boldsymbol{p}} - (\tilde{\boldsymbol{p}}^{\top}\boldsymbol{v})\boldsymbol{v} && (31) \end{aligned}$$

2.2 Computing the Derivatives

Derivatives of the residual function

If $t < 0$ or $t > L$:

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}} = \boldsymbol{J} \tag{32}$$

else:

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}} = (\boldsymbol{I} - \boldsymbol{v}\boldsymbol{v}^{\top})\boldsymbol{J} \tag{33}$$