EXERCISE 1 – Computational Linear Algebra, Spring'22

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1 ▶ Quiz

a) Let $A, B \in \mathbb{C}^{n \times n}$.

Do AB and BA always have the same eigenvalues? [**Yes/No**]

b) Assume that $A, B \in \mathbb{R}^{n \times n}$ are symmetric matrices.

Does AB always have real eigenvalues? [**Yes/No**]

What if A is symmetric and B is symmetric positive definite? [**Yes/No**]

c) The matrix T from a Schur form of $A \in \mathbb{C}^{n \times n}$ is diagonal if and only if A is Hermitian. [**True/False**]

2 ▶ Guitar String

Consider the wave equation for the linearized model of a guitar string as explained in the lecture (see also Section 1.2 in the lecture notes). The partial differential equation is given by

$$\frac{\partial^2}{\partial t^2} u(x,t) = \frac{T}{\mu} \frac{\partial^2}{\partial x^2} u(x,t), \quad u(0,t) = u(L,t) = 0, \tag{1}$$

where T denotes the tension and μ the weight of the string per meter. Both are assumed to be constant in our model.

Separation of variables, $u(x,t) = v(t)w(x)$, leads to differential equations for $v(t)$ and $w(x)$, depending on an additional parameter λ (the eigenvalue). For $v(t)$, the solution at a specific λ is

$$v(t) = a\cos(\sqrt{\lambda}t) + b\sin(\sqrt{\lambda}t),$$

where a and b are still to be determined. This corresponds to the frequency $\omega = \sqrt{\lambda}/2\pi$. It remains to find the parameters λ for which the differential equation

$$-\frac{T}{\mu} w''(x) = \lambda w(x), \quad w(0) = w(L) = 0, \tag{2}$$

has a nontrivial solution.

a) Implement the finite difference discretization explained in the lecture (see also Section 1.2.2 in the lecture notes) to solve (2). Define $N + 2$ equidistant nodes $0 = x_0 < \cdots < x_{N+1} = L$ on the interval $[0, L]$, and define $f_k = f(x_k)$. Use

$$\frac{\partial^2 f}{\partial x^2}(x_k) \approx \frac{1}{h^2}\Big(f_{k+1} - 2f_k + f_{k-1}\Big),$$

to approximate the second derivative. The resulting system is an eigenvalue problem:

$$Au = \lambda u.$$

You should have a MATLAB function with parameters N, L, T and μ that returns the discretized matrix A in sparse format.

b) Let $L = \mu = 1, T = 2, N = 100$. In MATLAB, use `eigs` to calculate the ground state (the eigenvector corresponding to the smallest eigenvalue), and the next two states (eigenvectors corresponding to 2nd and 3rd smallest eigenvalues), and plot them.

Hint: use `help eigs` for the calling syntax of `eigs`.

c) Let $L = T = \mu = 1$. Compute and plot the accuracy of the ground state for different discretizations, that is, the error in function of N. Compare the analytical expression $\lambda_1 = \pi^2 T/(L^2\mu)$ for the smallest eigenvalue of (2) with the numerical solution obtained by `eigs`. How does the error behave?

d) [**OPT**[1]] The "A"-string of a guitar is tuned to have a fundamental frequency (the value $\omega = \sqrt{\lambda}/2\pi$ related to the smallest eigenvalue λ) of 110 Hz. Determine the tension T that achieves this frequency. For this, derive a method that does *not* need to use the analytical solution but only solves discretized problems with `eigs`.

Use the following constants to get a realistic result: $L = 0.65$ m, $\mu = 4.5 \cdot 10^{-3}$ kg/m. Use $N = 250$.

3 ▶ Companion Matrix

Let $p = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_0 \in \mathbb{C}[z]$ be a polynomial. The *companion matrix* of p is given by

$$A = \begin{pmatrix} 0 & & & -\alpha_0 \\ 1 & \ddots & & -\alpha_1 \\ & \ddots & 0 & \vdots \\ & & 1 & -\alpha_{n-1} \end{pmatrix} \in \mathbb{C}^{n,n}.$$

Show that the roots of p coincide with the eigenvalues of A.

4 ▶ Matrix Exponential

The *matrix exponential* of a square matrix $X \in \mathbb{C}^{n\times n}$ is defined as

$$\exp(X) = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}X^k.$$

This series converges for all X.

a) Let D be a diagonal matrix. What is $\exp(D)$?

b) Suppose X is diagonalizable. How can you compute $\exp(X)$ using a)?

Hint: Consider first the computation of X^k for an integer k.

c) Implement a method in MATLAB to compute $\exp(A)$ for diagonalizable matrices. Compute the relative error of your method for the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Use `expm` in MATLAB as the exact answer.

d) Do the same as in c) but now for the matrices

$$A(\alpha) = \begin{bmatrix} 1+\alpha & 1 \\ 0 & 1-\alpha \end{bmatrix},$$

where α is a parameter. Plot the relative error for $\alpha =$ `logspace(-16,2,100)`. Try to explain what you see.

Hint: Plot and relate the error with the condition number of the matrix of eigenvectors Q.

[1] Questions with **OPT** are considered optional and can be skipped.