On minimal singular metrics of certain class of line bdls whose section ring is not fin. gen.

X : sm. proj $/\mathbb{C}$

L : line bdl $/X$

$\boxed{\text{Assume}}$ (※) $\begin{cases} L : \text{nef} \\ R(X,L) := \bigoplus_{m\geq 0} H^0(X, mL) \ : \text{not fin. gen} \end{cases}$ //

$\underline{\underline{Q}}$ Where, how "minimal singular metrics" of L diverges?

eg. of (※) ... Zariski's e.g.

$C \subset \mathbb{P}^2$; sm. ellipt. curve.

$p_1, \cdots, p_{12} \in C$: general.

$\pi : X \longrightarrow \mathbb{P}^2$: b-up at $\{p_1, \cdots, p_{12}\}$

$L := \pi^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}(D)$ $\quad (D := (\pi^{-1})_* C)$

$\rightsquigarrow$ L : nef, big,

$\begin{cases} \forall m \geq 1, \ Bs|mL| \supset D \\ \qquad |mL - D| : \text{free} \end{cases}$ //

① One main conclusion of Main thm is ....

Thm 1 (X, D, L) ; as above (i.e. Zariski's eg.)

$\rightsquigarrow \exists h$: conti. Herm. metric of L s.t. $\sqrt{-1}\,\Theta_h \geq 0$ //

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§1. preliminaries, motivations

§2. Main thm.

§3. prf.

<u>§1</u> (X, L) ... $\left(\begin{array}{l} X : \text{sm. proj} / \mathbb{C} \\ L : \text{line bdl} / X \end{array}\right)$.

<u>Def</u> h : sing. Herm. metric of L

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists} h_{\infty}$: sm. Herm. metric of L.

${}^{\exists}\chi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$; L^1_{loc}.

$h = h_{\infty} \cdot e^{-\chi}$ //

<u>fact</u> L : psd. eff $\Leftrightarrow$ ${}^{\exists} h$: s.H.m of L
s.t. $\sqrt{-1}\Theta_h \geq 0$ //

$$\sqrt{-1}\Theta_h := \sqrt{-1}\Theta_{h_\infty} + \underbrace{dd^c}_{\sqrt{-1}\partial\bar\partial}\chi$$

$$\| $$

$dd^c\varphi$ (where $h = e^{-\varphi}$ local weight.)

<u>Def</u> (D.P.S)

$h_{min,L} = e^{-\varphi_{min,L}}$: s.H.m. of L.

$h_{min,L}$: min. sing. metric

$\underset{\text{def}}{\Longleftrightarrow}$ (i) $dd^c\varphi_{min,L} \geq 0$

(ii) ${}^{\forall} h = e^{-\varphi}$: s.H.m of L s.t. $dd^c\varphi \geq 0$

$h_{min,L} \leq_{sing} h$

(i.e. ${}^{\exists} C > 0$.
$h_{min,L} \leq C \cdot h$). //

Thm (D.P.S)

L: psd. eff $\Longrightarrow$ $\exists\, h_{\min,L}$: min. sing. metric of L //.

(∵) fix $h_\infty = e^{-\varphi_\infty}$: sm. Herm. metric of L.

"the equilibrium metric"

$$(h_\infty)_e = e^{-(\varphi_\infty)_e} \text{ is a min. sing. metric}$$

$$(\varphi_\infty)_e(x) := \varphi_\infty(x) + \sup\left\{ \chi(x) \;\middle|\; \begin{array}{l} \chi : X \to [-\infty, 0] \\ dd^c(\varphi_\infty + \chi) \geq 0 \end{array} \right\} \quad (x \in X)$$

//.

Known results

◎ L: ample $\underset{\text{Kodaira}}{\Longrightarrow}$ L: s.p. (i.e. $\exists$ sm. $h_{\min,L}$)

◎ L: s.a $\Longrightarrow$ L: s.p.

◎ L: nef, big $\underset{\text{Boucksom}}{\Longrightarrow}$ $\forall x \in X$, $\nu(\varphi_{\min,L}, x) = 0$

$$\nu(\varphi_{\min,L}, x) := \liminf_{z \to x} \frac{\varphi_{\min,L}(z)}{\log|z-x|^2}$$

(c.f. f: hol. func $\Longrightarrow$ $\nu(\log|f|^2, x) = \mathrm{mult}_x f$)

more generally,

Thm (Boucksom)

L: big $\Longrightarrow$ $\mathbb{B}_-(L) = \{ x \in X \mid \nu(\varphi_{\min,L}, x) > 0 \}$.

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Rmk L ! nef big $\not\Rightarrow$ $\{\varphi_{\min, L} = -\infty\} = \phi$

$\llcorner \exists$ counter e.g. ([B.E.G.Z])

C.f Thm (Zariski)

L : nef. big

$\rightsquigarrow$ L : s.a $\Leftrightarrow$ $R(X, L)$: fin. gen. //

§2 Main thm

Thm 2 (Main thm) X : sm. proj

$\cup$

D : sm. hyp. surf.

L : psd. eff. l.b $/X$.

$\boxed{\text{Assume}}$

- $(*)$ D has a hol. tub. nbhd.
- $L \otimes \mathcal{O}_X(-D)$: s.p,

Then

- $h_{\min, L}|_D \not\equiv -\infty \Leftrightarrow L|_D$: psd. eff,
- In this case,

$h_{\min, L}|_D$: min sig. metric of $L|_D$ //

Rmk $(*)$ $\overset{\text{def.}}{\Longleftrightarrow}$ $\exists\, V$: a nbhd of D in X.

$\exists\, V'$: a nbhd of 0-section in $N_{D/X}$.

s.t. $V \underset{\text{bihol.}}{\cong} V'$ //

Rmk When $(L \otimes \mathcal{O}_X(-D))|_D$: ample,

we can concretely write down a min. sig metric of L. //

prf of Thm 2 $\Rightarrow$ Thm 1

... We use...

Thm (Grauert)

X : 2-dim cpx mfd.
$\cup$
D : sm. hyp. surf. cpt, genus $= g$.

$(D^2) < \min\{0, 4-4g\}$

$\Longrightarrow$ $(*)$ holds for (X, D) //.

⊙ In Zariski's e.g.,

$$\begin{cases} g(D) = 1. \\ (D^2) = 3^2 - 12 = -3. \\ L \otimes \mathcal{O}(-D) = \pi^* \mathcal{O}_{\mathbb{P}^2}(1) \end{cases}$$

//.

§3. prf of Main thm

X : sm. proj L : psd. eff l.b $/X$.
$\cup$
D : sm. hyp. surf.

for simplicity

s.t. $\left\{ \begin{array}{l} A := L \otimes \mathcal{O}(-D) \ : \text{s.p}, \ \underline{A|_D : \text{ample}} \\ \exists\, U : \text{a nbhd of } D \text{ in } X \\ \exists\, U' : \text{a nbhd of 0-section in } N_{D/X}. \\ \quad \text{s.t. } U' \underset{i}{\overset{\approx}{\Rightarrow}} U. \end{array} \right.$

✡ We may assume

$L|_D$: psd. eff.

We regard U' as a nbhd of

$$\begin{pmatrix} D' := \mathbb{P}(L|_D) & \text{in} & X' := \mathbb{P}(A|_D \oplus L|_D) \\ \wr\Vert & & \wr\Vert \\ \mathbb{P}(\mathcal{O}_D) & \subset & \mathbb{P}(\underbrace{(A\otimes L^{-1})|_D}_{\substack{\mathcal{O}(-D)|_D \\ \Vert \\ N_{D/X}^{-1}}} \oplus \mathcal{O}_D) \\ \Vert & \nearrow & \\ \text{0-section} \subset N_{D/X} & & \end{pmatrix}$$

Claim 1

$I := \{ h : \text{s.H.m of } L|_U \mid \sqrt{-1}\Theta_h \geq 0,\ \cancel{\text{loc. bdd on } U-D}\ 0 < h \lesssim_{sig} |f_D|^{-2} \text{ on } U \}$ (def. func of D)

$I' := \{ h' : \text{s.H.m of } L'|_{U'} \mid \sqrt{-1}\Theta_{h'} \geq 0,\ \cancel{\text{loc. bdd on } U'-D'}\ 0 < h' \lesssim_{sig} |f_{D'}|^{-2} \}$

$L' = \mathcal{O}_{\mathbb{P}(A|_D \oplus L|_D)}(1)$

$\rightsquigarrow$ (1) $I, I' \neq \phi$

(2) $(I, \lesssim_{sig}) \underset{1:1}{\overset{\exists}{\rightleftarrows}} (I', \lesssim_{sig})$ (i.e. $\sim_{sig}$ $\cancel{\text{up to equiv}}$) (i.e. $\cancel{h \sim h}$, $\gtrsim_{s.g.}$ $\lesssim_{sig}$)

(3) $\{ h : \text{s.H.m of } L \mid \sqrt{-1}\Theta_h \geq 0,\ \cancel{\text{loc. bdd on } X-D}\ h|_U \lesssim_{sig} |f_D|^{-2} \cancel{\text{around } D} \}$

$\xrightarrow[\text{restr.}]{} I$.

$\{ h' : \text{''} \ L' \mid \sqrt{-1}\Theta_{h'} \geq 0,\ \cancel{\text{loc. bdd on } X'-D'}\ h'|_{U'} \lesssim_{sig} |f_{D'}|^{-2} \}$

$\xrightarrow[\text{restr.}]{} I'$. //

Claim 2 fix $\begin{cases} e^{-\varphi_A} : \text{sm. Herm. metric of } A|_D,\ dd^c\varphi_A > 0. \\ e^{-\varphi_L} : \text{sm. Herm. metric of } L|_D \end{cases}$

· loc. coord of X

$\cdots (z, x) := [z s_A^*(x) + s_L^*(x)]$

$\begin{pmatrix} s_A^* : \text{loc. triv of } A|_D^{-1} \\ s_L^* : \quad \text{''} \quad L|_D^{-1}. \end{pmatrix}$

$\rightsquigarrow \tilde{\varphi}(z, x) := \log \max_{t\in[0,1]} |z|^{2t} e^{(t\varphi_A + (1-t)\varphi_L)}(x)$

$\rightsquigarrow e^{-\tilde{\varphi}}$: min. sig. metric of L' //.

◎ Claim 1, 2 $\Rightarrow$ Main Thm

~~(pf. of)~~ claim 1 $\rightsquigarrow$ $\{h_{min,L} = \infty\} \subset D$,

"Sing. of $h_{min,L}$" = "sing of $h_{min,L'}$"

$\rightsquigarrow$ Without loss of generality, we may assume

$$\begin{cases} X = X', \\ L = L' \end{cases}$$

min. sing. metric of L'

Claim 2 $\rightsquigarrow$ $\tilde{\varphi}(0, x) = \log \max_t |z|^{2t}$ —— ⓪.

$\{z = 0\} = D'$

$$\underset{t=0}{=} (\varphi_L)_e(x)$$

; min. sing. metric of $L|_D$ //

◎ prf of claim 1

(1) $\overset{\text{fix}}{=}$ $t_D \in H^0(X, D)$

($e^{-\varphi_A}$; sm. Herm. metric of A, $dd^c \varphi_A \geq 0$.

$\rightsquigarrow [e^{-(\varphi_A + \log|t_D|^2)}] \in I$ //

(2) We will construct $I' \xrightarrow{\Phi} I$.

$$\begin{array}{ccccc}
L|_U & \searrow & & \swarrow & i^*L'|_{U'} \\
\cup & & U = i^{-1}(U') & & \cup \\
L|_D & & \hookuparrow & & i^*L'|_{D'} \\
& \searrow & D = i^{-1}(D') & \swarrow &
\end{array}$$

homotopic.

$\rightsquigarrow$ $L|_U \underset{C^\infty}{\cong} i^*L'|_{U'}$.

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fix $\left| \begin{array}{l} \cdot\ h' = e^{-\varphi'} : \text{s.H.m of } L'|_{V'} \\ \qquad \text{s.t. } dd^c\varphi' \geq 0, \\ \qquad\quad \varphi' : \cancel{\text{loc. bdd on } \overline{V'} \setminus D'} \\ \qquad\qquad \geq \log|f_D|^2 \\ \cdot\ f \in \Gamma_{C^\infty}(V, \underbrace{Hom(L|_V, i^*L'|_{V'})}) \end{array} \right.$

$$\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad V \times \mathbb{C} \quad (\text{as } C^\infty\text{-l.b})$$

$$\rightsquigarrow f : X \to \mathbb{C}^\times : C^\infty.$$

s, s' : loc. triv of $L|_V$, $L'|_{V'}$

s.t. $|s'|^2_{h'} = e^{-\varphi'}$

$$\rightsquigarrow |s|^2_{\Phi'(h'),x} := \frac{|s|^2_{i^*h',x}}{|f|^2(x)} = \frac{|f(x)\cdot s(x)|^2_{h',x}}{|f(x)|^2}$$

$$= \frac{\cancel{|f(x)|^2} \left|\frac{s}{s'}\right|^2 |s'|^2_{h',x}}{\cancel{|f(x)|^2}}$$

$$= \left|\frac{s(x)}{s'(x)}\right|^2 e^{-\varphi'}$$

$\rightsquigarrow$ loc. weight of $\Phi'(h')$

$$= \varphi' - \log\left|\frac{s}{s'}\right|$$

sm. harmonic

$\rightsquigarrow$ we define

$$I' \longrightarrow I$$
$$[h'] \longmapsto [\Phi'(h')]$$

//

(3) fix $\left| \begin{array}{l} \cdot\ e^{-\varphi} : \text{s.H.m of } L|_V \\ \qquad \text{s.t. } dd^c\varphi \geq 0, \\ \qquad\quad \infty > \varphi' \geq \log|f_D|^2 \text{ on } \overline{V} \\ \cdot\ D \subset V \Subset U \end{array} \right.$

$$\rightsquigarrow \overline{\varphi} := \begin{cases} \log|f_D|^2 + \varphi_A & (\text{in } X \setminus V) \\ \max\{\log|f_D|^2 + \varphi_A, \varphi - C\psi\} & (\text{in } U) \end{cases} \quad (C \gg 1)$$ //