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Date 2017・2・6

Non-Kummer K3 surface with Levi-flats. 13:00〜14:30

<u>Main result</u> ... We construct a K3 surface by patching two open surfaces M and M'.

cpx

" M = (a rat'l surface S) \ a nbhd of (an ellipt. curve $C \subset S$) "

M' = S' \ (a nbhd of C')

○ non algebro-geometrical construction.

○ Can regard M and M' as a open sub mfds of a K3 surface. (cpx)

⇝ <u>Thm 1</u> $\exists X$: K3 surface ($/\mathbb{C}$, maybe non-proj) not a Kummer surface.

s.t. i) $\exists f: \mathbb{C} \longrightarrow X$: hol. s.t. f: inj. (immersion)

$\overline{f(\mathbb{C})}^{\text{Euc. op}} \subset X$: C^∞ closed sub mfd of $\dim_{\mathbb{R}}=3$, Levi-flat hyp. surf, $\approx S^1 \times S^1 \times S^1$.

$\left(\overline{f(\mathbb{C})}^{\text{Zar. op}} = X, \ \overline{f(\mathbb{C})}^{\text{Euc. op}} \neq X \right)$

ii) $\exists \{M_t\}_{t\in I}$ ($\exists I \subset \mathbb{R}$: interval)

: C^∞ family of C^∞ Levi-flat hypersurfaces of X.

s.t. $\forall M_t \approx S^1 \times S^1 \times S^1$ (C^∞ diffeo),

each leaf of $\forall M_t$ is dense in M_t //

§1 Construction.

~~§2~~. (Relationship with [Tsuji '84], [Doi 69].)

§2. prf of <u>Thm 1</u>

~~§4~~ §3 Moduli dimension of our K3 surfaces.

§1 Outline :

① $Z := \{p_1, \cdots, p_9\} \subset \mathbb{P}^2$: nine points (suitable). ($p_j \neq p_k$ if $j \neq k$)

$S := Bl_Z \mathbb{P}^2 \longrightarrow \mathbb{P}^2$: b-up at Z.
$\cup$
$C :=$ the str. transf. of the ellipt. curve $C_0 \subset \mathbb{P}^2$ s.t. $C_0 \supset Z$.
(C_0 : unique if Z : general)

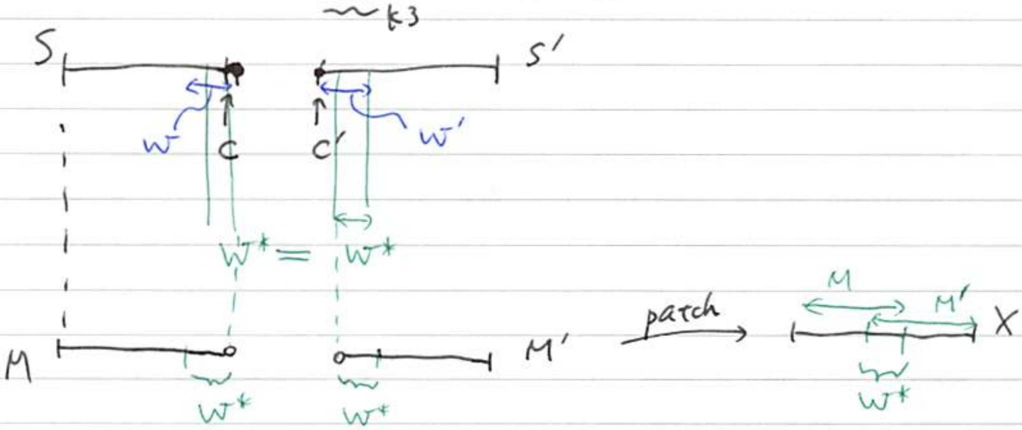
W: a suitable tubular nbhd of C in S.

~~Construct~~
Take (Z', S', C', W') in the similar way.

② Choose Z, Z', W, W' so that ...

[$\exists\, W^*$: open cpx surf. $\approx_{C^\infty}$ $C \times$ (annulus)
s.t. W^* can be regarded as a open cpx submfd of both W and W'.

③ $M := S \setminus \overline{(W - W^*)}$
$M' := S' \setminus \overline{(W' \setminus W^*)}$
$\underbrace{X}_{K3} := M \cup_{W^*} M'$.



How to choose Z, Z', W, W'

↑

Thm 2 (Arnol'd) Y: cpx surf.
$\cup$
E: sm. ellipt. curve s.t. $\begin{cases} \deg N_{E/Y} = 0. \\ -\log d(\mathbb{1}_E, N_{E/Y}^n) = O(\log n) \ (n \to \infty) \\ \text{"Diophantine".} \end{cases}$

$\rightsquigarrow \exists\, W$: a tub. nbhd of $E \subset Y$
s.t. $W \cong_{\text{bihol}} \exists$ nbhd of (0-sect'n) $\subset N_{E/Y}$ "

Rmk Thm 2 is an analogue of Siegel's linearization thm.

$$\left[\begin{array}{l} f : (0\text{-nbhd of } \mathbb{C}) \longrightarrow \mathbb{C} \\ \qquad\qquad z \longmapsto a_1 z + a_2 z^2 + a_3 z^3 + \cdots \\ \rightsquigarrow |a_1| = 1 \text{ and } a_1 : \text{diophantine} \\ \quad \Rightarrow f : \text{lin'ble (i.e. } \exists g : \text{hol, } g(0) = 0, \\ \qquad\qquad\qquad g \circ f \circ g^{-1} = id) \end{array}\right.$$

$\left(\begin{array}{l} E \rightsquigarrow C \\ Y \mapsto S. \end{array}\right.$

Obs ⓐ $\deg N_{E/Y} = 0 \rightsquigarrow \exists \{U_j\}$: open cov. of C,

$\exists\, t_{jk} \in U(1)$ (i.e. $|t_{jk}| = 1$)

s.t. $N_{C/S} = [\{(U_{jk}, t_{jk})\}] \left(\in H^1(\{U_j\}, U(1))\right)$

ⓑ Thm 2 $\rightsquigarrow$ ($N_{C/S}$: Dioph $\Rightarrow$) $\exists\, W_j$: open cov. of W as in Thm 2.

s.t. $\exists$ coord : (z_j, w_j) with $(z_j, w_j) = (z_j + {}^{\exists}A_{jk}, t_{jk} w_k)$ on W_{jk}.

coord of U_j "fiber coord".

Construction :

Step ① ; i) Fix $C_0 \subset \mathbb{P}^2$: sm. ellipt. curve.

ii) Take 8 pts $p_1, p_2, \cdots, p_8 \in C_0$.

$p_1', p_2', \cdots, p_8' \in C_0$.

iii) Fix $L_0 \to C_0$: Dioph. flat. line bdl.

iv) Take $p_9, p_9' \in C_0$ s.t.

$$\left\{\begin{array}{l} \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \cdots - p_9) = L_0 \\ \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1' - \cdots - p_9') = L_0^{-1}. \end{array}\right.$$

$\rightsquigarrow Z := \{p_1, \cdots, p_9\}$, $Z' := \{p_1', \cdots, p_9'\}$,

$S := Bl_Z \mathbb{P}^2$, $S' := Bl_{Z'} \mathbb{P}^2$,

$\cup \qquad\qquad \cup$

$C :=$ str. transf of C_0 $C' :=$ — " —.

Step ② i) Take nbhds $C \subset W \subset S$, $C' \subset W' \subset S'$ as in Thm 2.

~~Step ③~~ ii) We may assume ...

$W = \bigsqcup_j W_j$, $W_j = \{(z_j, w_j) \mid z_j \in U_j,\ |w_j| < {}^{\exists}R\}$ $(R > 1)$

$W' = \bigsqcup_j W_j'$, $W_j' = \{(z_j, w_j') \mid z_j \in U_j,\ |w_j'| < {}^{\exists}R'\}$ $(R' > 1)$

$$\left(\begin{array}{l} (z_j, w_j) = (z_j + A_{jk}, t_{jk} w_k) \\ (z_j, w_j') = (z_j + A_{jk}, t_{jk}^{-1} \cdot w_k) \end{array}\right)$$

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iii) $W^* := \bigsqcup_j W_j^*$, $W_j^* := \{(z_j, w_j) \in W_j \mid \frac{1}{R'} < |w_j| < R\}$.

<u>Regard</u> $W^* \subset W$ by $W_j^* \xhookrightarrow[\text{natural}]{} W_j$.

<u>Regard</u> $W^* \subset W'$ by $W_j^* \longrightarrow W_j'$, $(z_j, w_j) \longmapsto (z_j, \underset{w_j'}{\underbrace{w_j^{-1}}})$.

<u>Step③</u> $X := M \cup_{W^*} M'$ //.

$M = S \setminus \overline{(W \setminus W^*)}$, $W \setminus W^* = $ "$|w_j| \le \frac{1}{R'}$".

<u>Obs 3</u>

- X: cpt cpx surf.
- $X \supset M \supset W^*$, $X \supset M' \supset W^*$; open cpx submfds.
- $t \in (\frac{1}{R'}, R) \subset \mathbb{R} \rightsquigarrow M_t := \bigsqcup_j \{(z_j, w_j) \in W_j^* \mid |w_j| = t\}$ $\subset W^* \subset X$. $M_t \subset X$: Levi-flat.
- $M_t \longrightarrow C_0$; S^1-bdl, $(z_j, w_j) \longmapsto z_j$. $\boxed{\text{The foliation of } M_t \underset{\text{locally}}{=} \text{"}\{w_j = \text{const}\}\text{"}}$. ~~leaf of M_t~~
- $\rho: \pi_1(C_0, *) \to U(1)$; rep., corresp. to $L_0 \in \mathrm{Pic}^0(C_0)$

  $\alpha, \beta \in \pi_1(C_0,*)$: generator $\rightsquigarrow$ α, β: non-torsion $\Rightarrow$ each leaf of $M_t \cong \mathbb{C}$

  α: torsion, β: non-tor $\Rightarrow$ — " — $\cong \mathbb{C}^*$.

  dense!

<u>Prop 4</u> $H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}$ //

<u>prf</u> $F: W^* \to \mathbb{C}$: hol.

Take $t \in (\frac{1}{R'}, R)$ $\rightsquigarrow$ $^\exists B := \max_{x \in M_t} |F(x)| = F(^\exists x_t)$. $(x_t \in M_t)$.

Take a leaf $\mathcal{L}_t \subset M_t$ with $\mathcal{L}_t \ni x_t$.

Maximum principle for $F|_{\mathcal{L}_t}$ $\rightsquigarrow$ $F|_{\mathcal{L}_t} \equiv$ const $=: A$.

$\mathcal{L} \subset M_t$: dense $\rightsquigarrow$ $F|_{M_t} \equiv$ ~~const~~ A.

$\rightsquigarrow \underbrace{\{x \in W^* \mid F(x) = A\}}_{\text{analytic sub of } W^*} \supset \underbrace{M_t}_{\mathrm{Codim}_{\mathbb{R}} = 1}$ $\rightsquigarrow$ $F \equiv A$ //

$C \rightsquigarrow$ Hopf surf.
$S \rightsquigarrow$ Hopf 3-fold
$X \underset{\approx}{\rightsquigarrow} S^3 \times S^3$.

→ Topologically the same constr.

ここで [Tsuji] [Poi]. に言及

Do not need ~~not to~~ ~~the~~ choice of the z, z' as in Step①
Need a deformation of cpx str. of M, M'.

§2. Prf of Thm 1.

Obs3 $\rightsquigarrow$ enough to show that X is $\begin{cases} \text{a K3 surf.} \\ \text{not a Kummer surf} \end{cases}$

X: K3 $\Longleftarrow$ Prop 5 + L $\quad \exists \sigma$: glob. hol $(2,0)$-form on X, nowhere vanishing,

$$\sigma|_{W_j^*} = dz_j \wedge \frac{dw_j}{w_j}. \quad //$$

Prop 6

$$H_q(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & q=4 \\ 0 & 3 \\ \mathbb{Z}^{22} & 2 \\ 0 & 1 \\ \mathbb{Z} & 0 \end{cases} \Longleftarrow \text{Mayer-Vietoris seq.} \left(X \supset M, M' \supset W^*, \; S \supset M \supset W \supset W^* \right) \quad //$$

X: non Kummer $\underset{\text{Face}}{\Longleftarrow}$ $\exists\, \mathcal{X} \xrightarrow{\pi} B$: deformation family of K3 surfaces, $X_t \mapsto t$

s.t. each fiber X_t ~~$\cong$~~ as in §1. (can be constructed by the manner.)

$\dim B \geq 5$

~~B~~ $T_B \xrightarrow[\text{K.S. map}]{} R^1\pi_* T_{\mathcal{X}/B}$: inj.

We'll explain in §3.

Here we show Prop 5.

prf of Prop 5 :

$K_S = -C \rightsquigarrow \exists \eta$: mero. $(2,0)$-form on S s.t. $\mathrm{div}(\eta) = -C$.

$K_{S'} = -C' \rightsquigarrow \exists \eta'$: —— // —— on S' s.t. $\mathrm{div}(\eta') = -C'$.

$$F_j := \frac{\eta|_{W_j^*}}{dz_j \wedge \frac{dw_j}{w_j}} : W_j^* \longrightarrow \mathbb{C} \quad \text{: hol. nowhere vanishing.}$$

Patch

$\exists F : W^* \to \mathbb{C}$ s.t. $F|_{W_j^*} = F_j$.

Prop 4.

$F \equiv \exists A \in \mathbb{C}^*$. → We may assume $A \equiv 1$.

i.e. $\eta|_{W_j^*} = dz_j \wedge \frac{dw_j}{w_j}$.

Similarly, w.m.a. $\eta'|_{W_j^*} = dz_j \wedge \frac{dw_j'}{w_j'}$

$\rightsquigarrow \sigma := \{ (M, \eta|_M), (M', \eta'|_{M'}) \}$ //

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§3 Moduli, deformation.

$\underline{\underline{Q}}$ $\dim \{ X \in (K3 \text{ moduli})^{20} \mid X$ can be constructed in the manner as in §1 $\} = ?$

① Parameters ;

	?	"dimension" (expected)	we showed ...
· choice of $C_0 \subset \mathbb{P}^2$.			
· choice of $P_1, P_2, \cdots, P_8$		8	16 dim'l (indep)
· — " — $P_1', \cdots, P_8'$.		8	
· choice of $L_0 . (\rightsquigarrow P_9, P_9')$		? (1?)	
· Patching parameters for "$M \cup_W M'$" ($R, R', \cdots$)		2 ($z_j \cdot w_j$)	$\underline{\underline{(21)}}$

☆ Here, we'll explain how to construct the deformation family of K3 surfaces corresp. to the change of the choice of $P_1, \cdots, P_8$.

· Fix C_0, L_0, Z'.

· Fix $P_1, \cdots, P_8 \subset C_0$, U_λ : a suff. small nbhd of P_λ in C_0 $(\lambda = 1, 2, \cdots, 8)$.

$\cdot B := U_1 \times U_2 \times \cdots \times U_8$

$t = (z_1, z_2, \cdots, z_8) \rightsquigarrow z(t) \in C_0$; the pt defined by $\mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-z_1 - z_2 - \cdots - z_8 - z(t)) \cong L_0$.

· $\mathcal{S} \to B$: a deformation family s.t. $S_t \cong Bl_{(z_1, z_2, \cdots, z_8, z(t))} \mathbb{P}^2$.

$S_t \to t$

· $\mathcal{C} \subset \mathcal{S}$: str. transf. of ~~$R B \times E_0$~~ $C_0 \times B$.

Thm 07 ("rel. ver." of Arnol'd's thm)

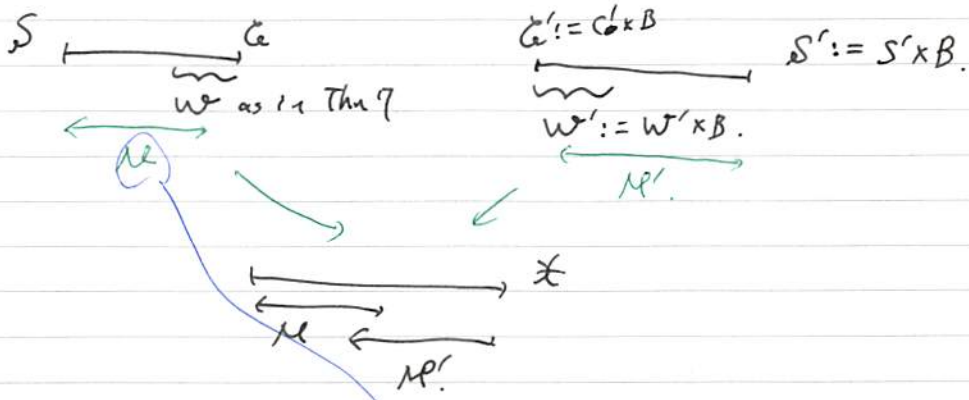
$P_1, P_2, \cdots, P_8$: general

$\Longrightarrow$ (by shrinking B if necessary),

$\exists \mathcal{W}$: $\mathcal{C}$-nbhd $\subset$ ~~$\mathcal{C}$~~ $\mathcal{S}$

s.t. $\mathcal{W} \cong$ a nbhd of (0-section) $\subset N_{\mathcal{C}/\mathcal{S}}$ //

⇝ We can construct $\mathcal{X} \xrightarrow{\pi} B$ by ...



Kodaira-Spencer map :

⓪ Fact $p_1, \dots, p_8$: general $p: \mathcal{S} \to B.$

$\Rightarrow \forall t \in B, \quad \rho_{KS,p} : T_{B,t} \longrightarrow H^1(S_t, T_{S_t})$: inj.

① Mayer-Vietoris seq

$$\Rightarrow H^0(M_t, T_{M_t}) \oplus H^0(W_t, T_{W_t}) \xrightarrow{\alpha} H^0(W_t^*, T_{W_t^*})$$

$M_t = \mathcal{M} \cap X_t.$

$$\xrightarrow{\beta} H^1(S_t, T_{S_t}) \xrightarrow{\gamma} H^1(M_t, T_{M_t}) \oplus H^1(W_t, T_{W_t})$$

Prop 4 $\Rightarrow \alpha$: surj

$\Rightarrow \gamma$: inj.

②
$$\begin{tikzcd}
T_{B,t} \arrow[r, "\rho"] \arrow[dr, "0"'] & H^1(S_t, T_{S_t}) \arrow[d, "\text{restr.}"] \arrow[dl, phantom, "\circlearrowright"] \\
& H^1(W_t, T_{W_t})
\end{tikzcd}$$

(definition of K.S. map)

⇝ $T_{B,t} \longrightarrow H^1(M_t, T_{M_t})$: inj.

⇝
$$\begin{tikzcd}
T_{B,t} \arrow[r, "\rho_{KS,\pi}"] \arrow[dr, "\text{inj}"'] & H^1(X_t, T_{X_t}) \arrow[d, "\text{restr}"] \arrow[dl, phantom, "\circlearrowright"] \\
& H^1(M_t, T_{M_t})
\end{tikzcd}$$

$\Rightarrow \rho_{KS,\pi}$: inj //.

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