Arnol'd's type thms on a nbhd of a curve. and gluing constr. of K3 surfaces

14:45-15:45,

Goal --- to give a "uniformizat'n" result for the cpx str. of a small nbhd of a cycle of $\mathbb{P}^1$'s.

<u>or</u>, to give "Arnol'd's type thm" for ——— " ——— .

Motivat'n

... "Gluing constr. of K3 surf" by using $Bl_Z\mathbb{P}^2$ ($\#Z = 9$, $Z \subset \mathbb{P}^2$), when the curve $C_0 \subset \mathbb{P}^2$ with $\overset{\deg=3}{C_0} \supset Z$: <u>nodal</u> //

Schedule
- §1 Main result.
- §2. ~~Known~~ Previous results, ~~examples~~.
- §3. Outline of prf.
- ~~§4. $Bl_{9\text{pts}}\mathbb{P}^2$ - case.~~

§1.

S : cpx surf (non-sing)

$\cup$

C : a cycle of $\mathbb{P}^1$'s.

i.e. C : cpt cpx reduced subvar, at most nodal (S.N.C), s.t.
- $\widetilde{C} = \bigsqcup_{\text{finite}} \mathbb{P}^1$, where $i : \widetilde{C} \to C$: normalizat'n.
- dual graph$(\widetilde{C})$ = or or or ..... (cycle graph)

<u>Assume</u> $N_{C/S} := [C]|_C$: top. triv.

<u>Obs</u>

$\mathrm{Pic}^0(C) := \{$top. triv. hol. line blls$\}$

$L \in \mathrm{Pic}^0(C) \underset{\text{naturally}}{\cong} H^1(C, \mathbb{C}^*) \cong \mathbb{C}^*$

$L \longmapsto t(L) \in \mathbb{C}^*$.

Def $L \in \mathrm{Pic}^0(C)$: Unitary-flat $\underset{\text{def.}}{\iff} |t(L)| = 1$. (i.e. $t(L) \in U(1) \subset \mathbb{C}^*$)

" : Dioph. $\underset{\text{def}}{\iff} t(L) = e^{2\pi\sqrt{-1}\cdot\theta}$ for $\exists\theta \in \mathbb{R}\setminus\mathbb{Q}$: Dioph. irrat'l number //

No.

Date

~~Main thm ($k = {}^{\forall}\delta$).~~
~~$(S, C), (S', C')$: as above,~~
~~$N_{C/S} \in \mathrm{Pic}^0(C)$: Dioph. $\Rightarrow$ $\exists V$:~~

~~Main thm ($k = {}^{\forall}s$).~~

C : a cycle of $\mathbb{P}$'s.
S, S' ; sm. cpx surf,
$i : C \longrightarrow S$
$i' : C \longrightarrow S'$: hol. emb's s.t. $\left\{ \begin{array}{l} i^* N_{i(C)/S} \cong (i')^* N_{i'(C)/S'}, \\ \text{top. triv.} \end{array} \right.$

Assume $t(i^* N_{i(C)/S}) = t((i')^* N_{i'(C)/S}) = e^{2\pi\sqrt{-1}\,\exists\underset{\in \mathbb{R}\setminus\mathbb{Q};\ \text{Dioph.}}{\theta}}$

Then $\exists V$; a nbhd of $i(C)$ in S,
$\exists V'$: — " — $i'(C)$ in S',
s.t. $V \underset{\text{bihol}}{\cong} V'$ $(*)$

$$\begin{array}{ccc} V & \cong & V' \\ \cup & & \cup \\ i(C) & \xrightarrow[i' \circ i^{-1}]{\cong} & i'(C) \end{array}$$

//.

Cf Arnold's thm -- C: sm. ellipt. $N_{C/S}$: Dioph $\Rightarrow$ $\underbrace{\exists V : \text{"hol. tub. nbhd" of } C \text{ in } S.}$

$\Updownarrow$

regarded as "standard model" for a nbhd of C. ~~&~~

$(*)$ holds for
$S' := N_{C/S}$,
$i' : C \longrightarrow N_{C/S}$: "zero-sect"

Q What is "the standard model for a nbhd of C"
when C : a cycle of $\mathbb{P}$'s. ?

We'll give "the standard model". in §3.

Rmk Ueda gave an example (C, S) s.t. $\left\{ \begin{array}{l} C : \text{ellipt. curve}. \\ N_{C/S} \in \mathrm{Pic}^0(C) \setminus \{\text{Dioph}\}, \\ (*) : \text{does not hold}. \end{array} \right.$

(S (open) admits a hol. foliation $\mathcal{F}$ with
$C \in \mathcal{F}$, $\mathrm{Hol}_{\mathcal{F}, C}[\gamma_j](w) \in \mathcal{O}_{\mathbb{C},0}$: has $0 \in \mathbb{C}$ as a cremer fixed p.t.)
$j = 1, 2$ γ_1, γ_2.

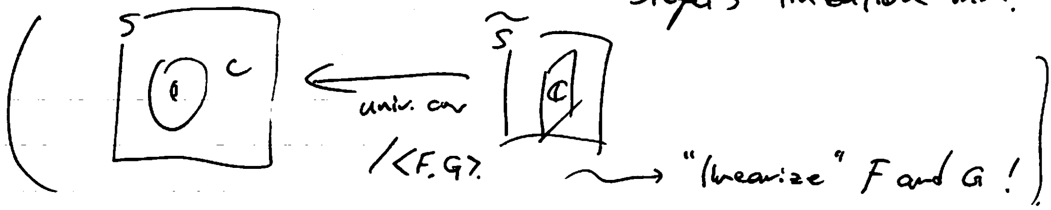
Similarly, $\exists$ an example (C, S) s.t. $\left\{ \begin{array}{l} C : \text{cycle of } \mathbb{P}\text{'s} \\ N_{C/S} \in \cancel{\mathrm{Pic}} \text{ ; unitary flat, not Dioph,} \\ \text{~~(*) does not hold~~} \end{array} \right.$

§2. previous known results

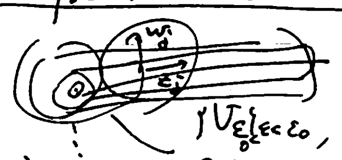
Thm (Grauert '62) — dim = 2 and 1, for simplicity.

S: cpx surf.
C: cpt analytic sub $\subset S$, sm (for simplicity).
$\rightsquigarrow$ $\deg N_{C/S} < 0 \Rightarrow$ C can be contracted,
$< \min\{0, 4-4\underbrace{g}_{\text{genus}}(C)\} \Rightarrow$ C admits a hol. tub. nbhd.

'76 ··· Arnol'd's thm.
Arnol'd applied linearizing technique in the pf. of Siegel's linearization thm.



'83. Ueda generalized Arnol'd's thm.
- classification of (C,S) with $N_{C/S}\in \mathrm{Pic}^0(C)$, when C: sm. curve cpt.
- Gave a sufficient cond ("Dioph" + $\underbrace{u}_{\text{Ueda's obstr. class}}=0$) for C to admit a psd-flat nbhds system



$$\left(\begin{array}{l} W_k(w_j, z_j) = \exists\, t_{kj}\cdot w_j \quad (t_{kj}\in U(1)) \\ z_k(w_j, z_j) = ?? \end{array}\right)$$

'91 Ueda studied a nbhd of a rat'l curve with a node C when $N_{C/S}\in \mathrm{Pic}^0(C)\setminus\{\text{unitary flat}\}$.
'17 K—, nodal (weak) analogue of [Ueda '83].

Thm (K—'17). S: sm. surf
C: a cycle of $\mathbb{P}^1$'s,
$N_{C/S}$: Dioph $\Rightarrow$ C admits a psd-flat nbhds system.

No.

Date

singular analogue.

Rmk Ueda '83 $\longrightarrow$ K-'17.

C: sm. "$w_j = t_{jk} w_k$"

C: cycle of $\mathbb{P}^1$'s "$w_j = t_{jk} w_k$".

Improve.

Arnol'd '76 $\longrightarrow$ singular analogue. $\boxed{\text{Main thm}}$

C: sm. ellipc, not only $w_j = t_{jk} w_k$, $z_j = z_k + {}^\exists A_{jk}$

§3. Outline of prf

Step 1 Take a nbhd V of C in S.

$\rightsquigarrow$ Define a class $\alpha(V) \in H^1(V, \mathcal{O}_V)$ which reflects the difference between the cpx strs of V and "the standard model of a nbhd of C".

$\rightsquigarrow$ Want to show: $\alpha(V)|_W = 0 \in H^1(W, \mathcal{O}_W)$ $\underbrace{\text{by shrinking } W\text{: nbhd of } C}$.

$$\Uparrow$$

$$\varinjlim_{W} H^1(W, \mathcal{O}_W) \xrightarrow{\text{restr}} H^1(C, \mathcal{O}_C) : \text{inj.} \quad (\text{since } \alpha(V)|_C = 0 \text{ by assumption})$$

$$\Uparrow$$

$$\varinjlim_{W} H^1(W, \mathcal{O}_W(-C)) = 0.$$

Step 2. ~~Show the vanishing $\varinjlim H^1(W, \mathcal{O}_W(-C)) = 0$.~~

Take $\{(V_{jk}, F_{jk})\} \in \check{Z}^1(\{V_j\}, \mathcal{O}_V(-C))$

$\to$ Construct the "formal primitive" $\{(V_j, F_j)\}$

s.t. $\begin{cases} \delta\{(V_j, F_j)\} = \{(V_{jk}, F_{jk})\}, \\ F_j = \sum_{\nu=1}^{\infty} ({}^\exists\text{hol func}) \cdot \underline{w_j}^\nu \end{cases}$

nice loc. def. func of C

(Here we use [K-'17]).

Step 3 Show the convergence of F_j

//

γ

C : rat'l curve with a node, for simplicity

~~Today, more~~

E.g. (the standard model ~~ov~~ a nbhd of C with $t(N_{C/S}) = \underset{\text{given}}{t}$)

C, $t \in U(1)$: given.

$\uparrow i$

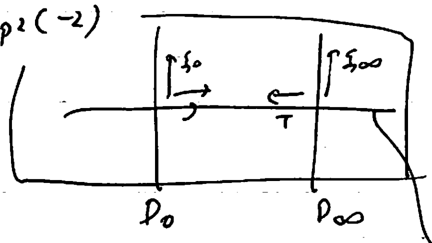
$\tilde{C} = \mathbb{P}^1$, $i(0) = i(\infty) =$ nodal p.t.

$\pi : \underbrace{\mathcal{O}_{\mathbb{P}^1}(-2)}_{\text{fiber coord}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ $\quad s = T^{-1}$; non-homog. coord. of

fiber coord ; ξ_0 : around $D_0 := \{s = 0\}$.

ξ_∞ : around $D_\infty := \{s = \infty\}$.

$\mathcal{O}_{\mathbb{P}^1}(-2)$



$\hat{V}_0^+ := \{\,|s| < \varepsilon,\ |\xi_0| < \varepsilon\}$

$\hat{V}_0^- := \{\,|T| < \varepsilon,\ |\xi_\infty| < \varepsilon\}$

$\hat{V}_1$: small nbhd of $\tilde{C} - (D_0 \cup D_\infty)$ $\quad (0 < \varepsilon \ll 1)$

regard the zero-sect'n as $\tilde{C}$

$\tilde{V} := \hat{V}_0^+ \cup \hat{V}_1 \cup \hat{V}_0^-$

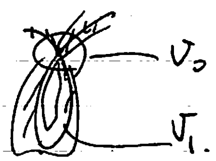
$V := \tilde{V}/\sim$, $\quad \sim$: gen'd by $F : \hat{V}_0^+ \xrightarrow{\cong} \hat{V}_0^-$, $\quad (s, \xi_0) \longmapsto (\underset{=T}{t\xi_0}, \underset{=\xi_\infty}{s})$

Regard the natural proj. $i : \tilde{V} \to V$ as a extension of $i : \tilde{C} \to C$.

$\to t(N_{C/S}) = t$ //

How to define $\alpha(V)$;

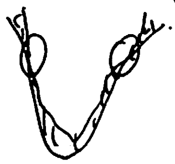
(C, S) ; as in Main thm, $C = \gamma$.



$\leftarrow \hat{V} = \hat{V}_0^+ \cup \hat{V}_1 \cup \hat{V}_0^- \subset$ (univ. cov. of V),

where $\hat{V}_0^\pm$: copies of V_0,

$\hat{V}_1$: a copy of V_1.



Decompose $i^* C = \underset{\substack{\uparrow \\ \text{str. transf. of } C}}{\tilde{C}} + \underset{\hat{V}_0^+}{D_0} + \underset{\hat{V}_0^-}{D_\infty}$ $\quad \xrightarrow{\text{simple computation}} \deg N_{\tilde{C}/\tilde{V}} = -2$.

Grauert's thm. ⇝ $\widetilde{V} \xhookrightarrow{\exists\,\mathrm{emb}} \mathcal{O}_{\mathbb{P}^1}(-2)$.

$\cup \qquad\qquad \cup$

$\widetilde{C}$ $=\!=\!=$ the zero-sect'n.

① $S = T^{-1}$: a non-homog coord. of $\widetilde{C}$.

One can see that

$\exists$ ~~hol~~ mero. ext. of $S\ (\in T^{-1})$ to a nbhd of $\widetilde{C}$ in $\widetilde{V}$

with $D_0 = \{S=0\}$,

$D_\infty = \{T=0\}$. ($\Leftarrow$ Ohsawa's vanishing thm)

② [K-'17] ⇝ $\exists$ defining function w_j of $C\cap V_j$ in V_j.
$(j=0,1)$

s.t. $w_1 = t_I \cdot w_0$ for $\exists\, t_I \in U(1)$, on V^I,

where $V_0 \cap V_1 = \underbrace{V^+ \sqcup V^-}_{\text{conn. comps}}$.

⇝ one have a global def. func. $\widehat{w} : \widehat{V} \to \mathbb{C}$ of i^*C

$\left(\widehat{w} := \begin{cases} t_+ w_0 & \widehat{V}_0^+ \\ w_1 & \widehat{V}_1 \\ t_- w_0 & \widehat{V}_0^- \end{cases} \right)$.

③ $\xi_0 := \dfrac{\widehat{w}}{S}$, $\xi_\infty := \dfrac{\widehat{w}}{T}$.

⇝ "deck trans" $F : \widehat{V}_0^+ \to \widehat{V}_0^-$ (i.e. F : s.t. $i^{-1}(i(p)) = \{p, F(p)\}$)
is written as

$$F(S, \xi_0) = \Big(\underbrace{\frac{t\cdot\xi_0}{G}}_{=T}, \underbrace{G\cdot S}_{=\xi_\infty} \Big)$$

(※ G depends on the choice of how to extend S (T^{-1}))

for $\exists\, G : \widehat{V}_0^+ \to \mathbb{C}^*$.

We may assume ~~$G\equiv 1$~~ $G(0,0)=1$ (by scaling $\widehat{w}$).

⇝ $\alpha(v) := \{ (V^+, \log G), (V^-, 0) \}$.

Rmk

$\alpha = 0 \Longrightarrow$ $\exists\, H_j : \widetilde{V}_j \to \mathbb{C}^*$, $(j=1,2)$ $\Longrightarrow$ by replacing
(by shrinking V)

$\begin{cases} \dfrac{F^*H_2}{H_1} = G \text{ on } \widetilde{V}^+ \cap \widetilde{V}_1 \\ H_1|_C \equiv 1 \end{cases}$ $\begin{cases} S \text{ with } \dfrac{S}{H_1} \\ T \text{ with } H_1 \cdot T \end{cases}$, $G \equiv 1$ //.