名大. 解析幾何学セミナー 99元数理科学棟 309. 15:00〜16:30.

ネフ直線束の半正値性と周辺のベースローカスの近傍の関連について

X : sm. proj.

L : hol. l.b. $/X$.

Def L : nef.

$\underset{\text{def}}{\iff} \forall C \subset X$: curve, $L.C \geq 0$ $\left(:= \int_C c_1(L)\right)$ //.

Def L : s.p.

$\underset{\text{def}}{\iff} \exists h$: sm. Herm. metric on L

s.t. $\sqrt{-1}\Theta_h \geq 0$ //.

すぐに分かること L : s.p. $\Longrightarrow$ L : nef.

fact L : nef $\not\Longrightarrow$ L : s.p.

… [DPS], C_0 : sm. ellipt. curve

E : rank 2- vect. bdl $/C_0$

s.t. $0 \to \mathcal{O}_{C_0} \to E \to \mathcal{O}_{C_0} \to E$: not splitting.

$\rightsquigarrow X := \mathbb{P}(E)$; ruled surf.

$\cup$

C : "section" $\rightsquigarrow \mathcal{O}_X(C)$; nef, not s.p.

Q Assume ① $L = \underset{\text{glob.gen.}/X}{\exists A} \otimes \mathcal{O}_X(\underset{\text{sm. hyp.surf of } X}{\exists C})$, ② $L.C = 0$.

$\rightsquigarrow$ When does L admit sm. Herm. metric with s.p. curvature?

☆ **Rmk** $L|_{X-C} \cong A|_{X-C}$: s.p. , $L|_C$: flat ($\Rightarrow$ s.p) //.

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Main results

ArXiv/1312.6402.

Thm A

X : sm. proj
$\cup$
C : sm. hyp.plane.
L : hol. l.b $/X$

Assume

- $L = {}^{\exists}\underset{\text{s.p.}}{\underset{\sim}{A}} \otimes \mathcal{O}_X(C)$, $\quad L.C = 0$.
- $N_{C/X}^{-1}$; ample, $\quad N_{C/X}^{-1} \otimes K_C^{-1}$: nef. big.
- C has a hol. tub nbhd $(*)$

Then L : s.p. //

$\left(* \underset{\text{def}}{\Longleftrightarrow} \begin{array}{l} {}^{\exists}U \text{ : (tub) nbhd of } C \text{ in } X \\ {}^{\exists}U' \text{ : nbhd of (0-section) in } N_{C/X} \\ \text{s.t. } U \cong U' \end{array} \right)$

Thm B (on preparation)

X : sm. proj. surf.
$\cup$
C : emb. sm. curve. s.t. $(C^2) \leq 0$.
L : hol. l.b. $/X$

Assume $\quad L = {}^{\exists}\underset{\text{glob.gen}}{\underset{\sim}{A}} \otimes \mathcal{O}_X(C)$, $\quad L.C = 0$.

Then $u_1(C, X) \neq 0 \Rightarrow L$: not s.p. //

Def of "$u_1(C,X)$"

$I_C \subset \mathcal{O}_X$: def. ideal of C.

$\rightsquigarrow$ $0 \to I_C/I_C^2 \to \mathcal{O}_X/I_C^2 \to \mathcal{O}_X/I_C \to 0$: ex.

$\rightsquigarrow_{|_C}$ $0 \to N_{C/X}^{-1} \to \underset{\substack{\parallel \\ \mathcal{O}_X/I_C^2|_C}}{E} \to \mathcal{O}_C \to 0.$: ex. ---(#)

$\mathcal{O}_X/I_C^2|_C$: rank 2 - vect. bdl.

($\rightsquigarrow$ $0 \to \mathcal{O}_C \to E \otimes N_{C/X}^{-1} \to N_{C/X} \to 0$: ex

$\rightsquigarrow$ $0 \to N_{C/X}^{-1} \to E^* \otimes N_{C/X} \to \mathcal{O}_C \to 0$: ex.)

$\rightsquigarrow$ $u_1(C,X) :=$ ext. class of (#)

$\in \mathrm{Ext}^1(\mathcal{O}_C, N_{C/X}^{-1})$

$= H^1(C, N_{C/X}^{-1})$

Rmk When $(C^2) = 0$,

this definition coincides with

the def. of "1-st Ueda class"

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Here, $u_n(C,X) \in H^1(C, N_{C/X}^{-n})$: n-th Ueda class.

Rmk In the setting of thm B,

$(C^2)=0 \Longrightarrow \exists V$: nbhd of C in X, $L = \mathcal{O}_V(C)$

Thm B' $\underset{\text{sm. pt. surf}}{V} \supset \underset{\text{curve sm. cpt}}{C}$ $\quad$ $(C^2)=0$.

$\exists n < \infty \quad u_n(C,X) \neq 0 \Longrightarrow \mathcal{O}_V(C)$: not s.p //.

◎ §1. applications.
§2. prf.

§1

§1-1. Applications of Thm A

Cor A' X : sm. proj. surf.
$\cup$
C : sm. curve, genus $= g$.
L : hol. l.b $/X$.

Assume $\begin{cases} \cdot\ L = {}^\exists \underset{\text{s.p.}}{A} \otimes \mathcal{O}_X(C),\ L.C=0 \\ \circ\ (C^2) < \min\{0, 4-4g\} \end{cases}$

Then L : s.p //.

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Cor A′ follows from Thm A and ...

Thm (Grauert. '62)

X : surf.

$\cup$

C : cpt curve, genus $= g$.

$(C^2) < \min\{0, 4-4g\} \Rightarrow C$ has a hol. tub. nbhd //.

$\boxed{\text{eg}}$ (Zariski's ex).

$C_0 \subset \mathbb{P}^2$; sm. ellipt. curve.

Ψ

$p_1, \cdots, p_{12}$: general.

$X := Bl_{\{p_j\}} \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$.

$\cup$

$C := (\pi^{-1})_* C_0$.

$L := \underbrace{\pi^* \mathcal{O}_{\mathbb{P}^2}(1)}_{\text{glob. gen.}} \otimes \mathcal{O}_X(C)$

$\rightsquigarrow L.C = 3 + (3^2 - 12) = 0$.

fact L is nef, not semi-ample //.

$\boxed{\text{Cor}}$. L : s.p. //.

§ 1-2 Applications of Thm B.

① ... a generalization of [DPS] example.
~~② ... on Fujino's question on. ArXiv/0705.1199v4.~~

Ex. C_0 : sm. curve,
$N \to C_0$: hol. l.b. s.t. N^{-1} : glob. gen.
$E \to C_0$: hol. rank 2 - vect. bdl.
appears in an ex. seq.

$$0 \to N^{-1} \to E \to \mathcal{O}_{C_0} \to 0 \quad \cdots (\#)$$

$X := \mathbb{P}(E) \xrightarrow{\pi} C_0$.
$\cup$
C : "section". $\underset{\pi|_C}{\overset{\cong}{\to}} C_0$.

$L := \underbrace{\pi^* N^{-1}}_{\text{glob. gen.}} \otimes \mathcal{O}_X(C)$

$\rightsquigarrow$
- $N_{C/X} \cong \pi^* N|_C$.
- $L.C = 0$.
- $u_1(C, X) = $ ext. class of $(\#)$
$\in H^1(C_0, N^{-1}) \underset{\pi|_C^*}{\cong} H^1(C, N_{C/X}^{-1})$.

Thm B $\rightsquigarrow$ $\boxed{\text{Cor}}$

L : s.p $\iff$ $(\#)$: splits //

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§2. prf.

§2-1 Prf of Thm A

for simplicity, we prove it when (X, C, L) is Zariski's ex.

$$\left[\begin{array}{l} X \supset C. \quad L: \text{l.b.}/X. \\ \cdot L = \underset{\text{s.p.}}{\underline{A}} \otimes \mathcal{O}_X(C), \quad L.C = 0. \\ \cdot C: \text{ellipt. curve}, \quad \text{has a hol. tub. nbhd } V. \\ \cdot N_{C/X}: \text{negative}. \end{array}\right.$$

idea of the prf

$f_C \in H^0(X, \mathcal{O}_X(C))$,

$h_A = e^{-\varphi_A}$; sm. Herm. metric of A

s.t. $\sqrt{-1}\partial\bar{\partial}\varphi_A \geq 0$.

$\rightsquigarrow h_L := |f_C|^{-2} \cdot h_A$: Singular Herm. metric of L

s.t. $\sqrt{-1}\,\Theta_L$

$= \sqrt{-1}\,\partial\bar{\partial}(\log|f_C|^2 + \varphi_A)$

≥ 0.

☆ modify h_L around C !

~~ten~~ V': nbhd of (0-section) in $N_{C/X}$ (negative)

s.t. $V \cong V'$.

we may assume V ; 1-convex.

$\rightsquigarrow$ $\boxed{\text{Lem}}$ $L|_V \cong p^*(L|_C)$

w/ $p: N_{C/X} \to C$ s.t. induce.

//.

(∵) V : 1-convex, max. cpt set $= C$.

$$(\mathcal{O}_V(-C)\otimes K_V^{-1})|_C = K_C^{-1} \ \text{: s.p.}$$

Ohsawa vanishing thm.
$\rightsquigarrow$ $\forall q>0,\ H^q(V, \mathcal{O}_V(-C)) = 0$.

@ $0 \to \mathcal{O}_V(-C) \to \mathcal{O}_V \to \mathcal{O}_V/\mathcal{O}_V(-C) \to 0$: ex.

$$\rightsquigarrow \underbrace{H^1(V,\mathcal{O}_V(-C))}_{=0} \to H^1(V,\mathcal{O}_V) \to H^1(C,\mathcal{O}_C) \to \underbrace{H^2(V,\mathcal{O}_V(-C))}_{=0}.$$

$$\rightsquigarrow H^1(V,\mathcal{O}_V) \underset{p^*}{\cong} H^1(C,\mathcal{O}_C)$$

$$\rightsquigarrow \begin{array}{ccccc} H^1(V,\mathcal{O}_V) & \longrightarrow & H^1(V,\mathcal{O}_V^*) & \longrightarrow & H^2(V,\mathbb{Z}) \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle p^*} & & \\ H^1(C,\mathcal{O}_C) & \longrightarrow & H^1(C,\mathcal{O}_C^*) & & \end{array}$$

$L|_V \mapsfrom \Omega$, $L|_V \mapsto 0$, $L|_V \mapsto L|_C$.

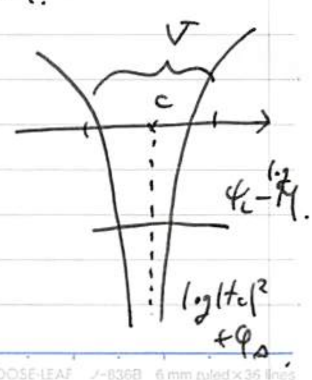
//.

@ $L|_C$: flat.

$\overset{\rightsquigarrow}{\text{Len}}$ $\exists\, g_L$: sm. Herm. metric of $L|_V$, $g_L = e^{-\varphi_L}$ s.t. $\partial\bar\partial\varphi_L \geq 0$.

$$\rightsquigarrow e^{-\max\{\varphi_L - \log M,\ \log|t_C|^2 + \varphi_A\}} = \min\{M g_L, |t_C|^{-2} h_A\} \quad (M \gg 1)$$

//.



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§2-2 Prf of Thm B'

for simplicity, we only prove Thm B';

X: surf.

$\cup$ cpt

C: curve, s.t. $\begin{cases} (C^2)=0. \\ \exists n.\ u_n(C,X)\neq 0. \end{cases}$

(we'll prove that. L: not s.p).

by contrad.

$\cdots$ Assume $\exists h_L = e^{-\varphi_L}$: sm. metric of. L s.t. $\partial\bar\partial\varphi_L \geq 0$.

Thm (Ueda)

$\underset{\text{surf}}{X} \supset \underset{\text{curve}}{C}$; $(C^2)=0$, $\exists n$, $u_n(C,X)\neq 0$

Then $\forall V$: C-nbhd, $\forall 0<a<n$.

$\forall \Psi: V\setminus C \longrightarrow \mathbb{R}$: psh.

s.t. $\Psi(p) = o\left(\dfrac{1}{\mathrm{dist}(p,C)^a}\right)$ as $p\to C$,

Ψ; $\geq$ const around C //.

☆ if $h_L = e^{-\varphi_L}$ exists, we can construct such Ψ. by

$$\Psi := -\log|f_C|^2_{h_L} = -\log|f_C|^2 + \varphi_L.$$ //.

$\boxed{\text{Rmk}}$ $\rightsquigarrow (C^2)=0 \Rightarrow |f_C|^{-2}$: sp. sing. Herm. metric with min sing //.