On some analogues of Ueda theory and ~~their~~ applications.

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$\boxed{13{:}30 - 14{:}10}$

X : (proj) cpx mfd. $\boxed{\dim_{\mathbb{C}} X = 2}$

L : hol. line bdl $/X$. : nef.

Def.
- L : nef $\overset{\text{def}}{\iff}$ $\forall C \subset X$ cpt curve ($\mathbb{C}$px), $L.C = \int_C c_1(L) \geq 0$.
- L : semi-positive (s.p.) $\overset{\text{def}}{\iff}$ $\exists C^\infty$ Herm. metric on L s.t. $\sqrt{-1}\,\Theta_h \geq 0$ "
  (Θ_h ← Chern curvature.)

Known : L : s.p. $\Longrightarrow$ L : nef. (the converse $\not\Longleftarrow$ ← (Demailly–Peternell–Schneider '94))

Main interest .. When is L (non-) s.p. ?

especially when $L = [{}^\exists C]$ for ${}^\exists C \subset X$: cpt. curve
(i.e. $\exists f_C \in H^0(X, L)$; "can. section" s.t. $\operatorname{div}(f_C) = C$ ")

Most interesting case ; ..... when $N_{C/X}$ $(= L|_C)$: topologically trivial.
($\Downarrow$ L : nef) "

Strategy.

"Ueda theory" [Ueda '83] : **Data** $\begin{cases} N_{C/X} \in \mathrm{Pic}^0(C) \\ \text{"}u_n(C,X)\text{"} \in H^1(C, N_{C/X}^{-n}) \end{cases}$ (type(C,X)) $\longrightarrow$ cpx analytic structure of a nbhd of C $\longrightarrow$ (non-)semi positivity of $[C]$

generalize it!!

([Brunella '10] [K— '13] [K— '14])

Schedule
- §1 Notation, Set-up.
- §2 Main results.
- §3 Application.
- (§4. higher dim'l case)

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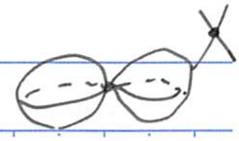
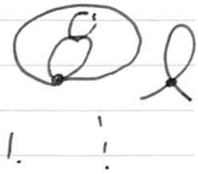


§1. C: a cpt curve with only nodes.



$P(C) := \{\text{topologically triv. hol. line bdls}/C\}/\cong_{hol}$.

$\cup$

$P_0(C) := \{\text{flat line bdls}/C\}/\cong_{hol}$.

$\underset{\text{fact}}{=} \{\text{flat} \text{ —"— } /C\}/\cong_{flat} := H^1(C, U(1))$

$(U(1) = \{z \in \mathbb{C} \mid |z| = 1\})$

$P_0(C) \supset \mathcal{E}_0(C) := \{L \in P_0(C) \mid \exists n \geq 1,\ L^n = \mathbb{1}_C\}$

$\mathcal{E}_1(C) := \{L \in P_0(C) \mid \log d(\mathbb{1}_C, L^n) = O(\log n) \text{ as } n \to \infty\}$

Rmk. ⓪ C: non-sing $\Rightarrow P(C) = P_0(C)$.

① $P(C) = \mathrm{Image}(H^1(C, \mathbb{C}^*) \to H^1(C, \mathcal{O}_C^*))$

$\rightsquigarrow \forall L \in P(C)$, L admits "flat connection ∂"

② $\mathcal{E}_1(C) = \bigcup_{\nu=1}^{\infty} \exists F_\nu \leftarrow$ nowhere dense closed sub. of $P_0(C)$

③ $\mu_{\text{Lebesgue}}(P_0(C) \setminus \mathcal{E}_1(C)) = 0$ "

e.g. C: a rat'l curve with a node 

$\rightsquigarrow P(C) \cong \mathbb{C}^*$, $P_0(C) \cong U(1)$.

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Set-up: X: sm. surface, $\supset C$: cpt curve with only nodes.

s.t. $N_{C/X} := [C]|_C \in P(C)$.

① Fix V: a nbhd of C in X. deformation retr. of C.

$$\begin{tikzcd} \exists\alpha \arrow[d] \arrow[rr, bend left] & H^1(C, \mathbb{C}^*) \arrow[r, "\cong", "\text{natural}"'] \arrow[d] & H^1(V, \mathbb{C}^*) \arrow[d] \\ N_{C/X} & H^1(C, \mathcal{O}_C^*) \ni N_{C/X} & H^1(V, \mathcal{O}_V^*) \ni N_\alpha \end{tikzcd}$$

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(a generalization of "type" in [Ueda '83])

Def $\mathrm{type}(C,X) := \max\left\{ n \in \mathbb{Z}_{\geq 1} \;\middle|\; {}^{\forall}\nu < n,\ \mathcal{O}_V(C) \otimes \mathcal{O}_V/I_C^{\nu+1} \cong \mathcal{O}_V(N_\alpha) \otimes \mathcal{O}_V/I_C^{\nu+1} \right\}$

$(I_C := \mathcal{O}_V(-C))$ //

Rmk ∘ $\mathrm{type}(C,X) = \infty \iff [C], N_\alpha$: formally isomorphic along C.

$\Uparrow$ $\forall n,\ H^1(C, N_\alpha^{-n}) = 0$

∘ $\mathrm{type}(C,X)$ does not depend on "α" //

§2. Main results --- (C,X) : as above.

Thm 1 Assume.
① $\mathrm{type}(C,X) = \infty$
② $N_{C/X} \in \mathcal{E}_0(C) \cup \mathcal{E}_1(C)$
③ $i^* N_{C/X} \in \mathcal{E}_0(\tilde{C})$ $\quad (\tilde{C} \xrightarrow{i} C$: normalization$)$
④ $H^1(C, \mathbb{C}(N_{C/X}^{-n})) = 0$ for $\forall n$.

($C =$  p! ② $\Rightarrow$ ③, ④)

Then

$[C]$: semi-positive //

Thm 2 ∘ C : tree and $\mathrm{type}(C,X) = n < \infty$.

or

∘ C : cycle and $\begin{cases} N_{C/X} \in \mathcal{P}(C) \setminus \mathcal{P}_0(C) \\ \mathrm{type}(C,X) \geq 4. \end{cases}$

Then $|f_C|^{-2}$: min. singular metric on $[C]$.

$(\Rightarrow [C]$: not s.p.$)$ //

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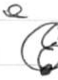
Rmk
- [Ueda '83] + [K-'13] $\Rightarrow$ ~~Thm 1 holds~~ Thm 1 for non-sing C (Assume only ① and ②)
- [Ueda '83] + [K-'14] $\Rightarrow$ Thm 2 "tree case" for ~~++~~.
- [Ueda '91] + [K-'14] $\Rightarrow$ Thm 2 ~~"cycle case"~~ for the case $C =$ 

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c.f. Aubin-Yau thm

K_X^{-1} : s.p. $\Longleftrightarrow$ X : $(C^{(0)})$ Kähler metric with s.p. Ricci curv. の存在.

§3. Application

$\{P_j\}_{j=1}^{9} \subset \mathbb{P}^2$: 9 pts.

$\rightsquigarrow \exists\, C_0 \subset \mathbb{P}^2$: a curve of $\deg = 3$.

s.t. $C_0 \supset \{P_j\}_{j=1}^{9}$.

By applying thm 1, 2,

we can determine a min. sing. metric of K_X^{-1} ($= [C]$, $\Lambda^2 T_X$) $\quad (X := Bl_{\{P_j\}_{j=1}^{9}} \mathbb{P}^2)$

EXCEPT the case where

C_0 : with only nodes.

and $N := N_{C_0/\mathbb{P}^2} \otimes \mathcal{O}_{C_0}(-P_1 - P_2 - \cdots - P_9) \in \mathcal{P}_0(C_0) \setminus (\mathcal{E}_0(C_0) \cup \mathcal{E}_1(C_0))$

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Known C_0 : non-sing $\rightsquigarrow C_0 =$ 

$\rightsquigarrow N \in \mathcal{E}_0 \cup \mathcal{E}_1 \Rightarrow K_X^{-1}$: s.p.

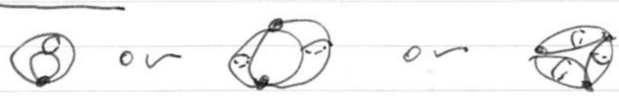
(Brunella, Ueda, Neeman)

Q C_0 : non-sing. : fix.

$\exists ?\ \{P_1, \cdots, P_9\} \subset C_0$ s.t. K_X^{-1} : not s.p. ?

(Demailly)

Cor ($\Leftarrow$ Thm 2)

$C_0 =$  ; fix.

$\cup$

$\{P_1, \cdots, P_9\}$: general.

$\Longrightarrow K_X^{-1}$: not s.p. //

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§4. Thm $\{P_j\}_{j=1}^{8} \subset \mathbb{P}^3$: general. $\rightsquigarrow \exists\, Q_0, Q_1$: quad. s.t. $C = Q_0 \cap Q_1$: sm. ellipt. curve

$(N_{C/Q_0} \otimes \mathcal{O}_C(-P_1 - \cdots - P_8) \in \mathcal{E}_0 \cup \mathcal{E}_1$

$\Longrightarrow K_{Bl\,\mathbb{P}^3}^{-1}$: s.p.