Date 2017 · 5 · 22.

Complex K3 surfaces containing Levi-flat hyp.surfaces

may possibly be non-proj.

(complex) X : K3 $\iff$ X : cpt cpx surf,
$\pi_1(X) = 0$, $\exists \sigma$: nowhere vanishing hol. 2-form on X.

Def $H \subset \underset{\text{cpx mfd}}{X}$: (C^ω-) real hyp. surf.

H : Levi-flat $\underset{\text{def}}{\iff}$ $\exists \mathcal{F}$: (C^ω-) foliation on H of real codim $=1$,
each leaf of $\mathcal{F}$ is a hol'ly immersed cpx submfd of X.

Thm 1 $\exists X$: K3, not a Kummer surf.
s.t. $\exists \{H_t\}_{t \in I}$ ($I \subset \mathbb{R}$: interval)
; C^ω-family of Levi-flat hyp. surfaces of X
s.t. $\forall t$, $H_t \approx_{\text{diffeo}} S^1 \times S^1 \times S^1$,
each leaf of H_t is $\begin{cases} \text{dense in } H_t. \\ \text{bihol. to } \mathbb{C} \text{ or } \mathbb{C}^* \end{cases}$ //

$\rightsquigarrow$ **Cor 2** $\exists X$: K3, not a Kummer surf.
$\exists f : \mathbb{C} \to X$: hol. immersion.
s.t. $\begin{cases} \overline{f(\mathbb{C})}^{\text{Euc}} \text{ ; real hyp. surf of codim}_{\mathbb{R}} = 1, \subsetneq X. \\ \overline{f(\mathbb{C})}^{\text{Zar}} = X \end{cases}$ //

① We will construct such X
by patching two open cpx surfaces M and M'

① $M = (\exists 9\text{pts b-up of } \mathbb{P}^2) \setminus (\text{a nbhd of an ellipt. curve})$
$M' =$ ———— // ————

Thm 3. $\exists \mathcal{X} \xrightarrow{\pi} \underset{18\text{ dim'l cpx mfd}}{B}$: Deformation family of K3 surfaces
s.t. $\begin{cases} \dim B = 18. \\ X_t := \pi^{-1}(t) : \text{as in Thm 1, Cor 2. for } \forall t \in B. \\ \text{The Kodaira-Spencer map } \rho_{KS,\pi} : T_B \to R^1\pi_* T_{\mathcal{X}/B} : \text{inj} \end{cases}$ //

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§1 Motivation from Arnol'd's and Ueda's thms.
§2. Construction of ~~a K3 surface~~ X.
§3. Outline of ~~Thm~~ the prfs of Thms.

§1 S: cpx surf.
$\cup$
C: cpt curve, $(C^2) = \deg N_{C/S} = 0$. $\rightsquigarrow$ $\mathcal{O}_S(C)$: nef l.b./S.

interest;

① Is $\mathcal{O}_S(C)$ semi-positive? (i.e. $\exists$? h: C^∞ Herm. metric on $\mathcal{O}_S(C)$ with semi-posit. curvature)
② What kind of nbhds systems does C have?

Thm 4 (Arnol'd '76)

C: ellipt. curve, $N_{C/S} \in \mathrm{Pic}^0(C)$; Diophantine
(i.e. $-\log d(\mathcal{O}_C, N_{C/S}^n) = O(\log n)$ as $n \to \infty$) d: inv. dist.

$\Longrightarrow$ $\exists W$: a nbhd of C in S,
$\exists \hat{W}$: a nbhd of 0-sect'n $\subset N_{C/S}$
s.t. $W \cong_{\mathrm{biho}} \hat{W}$
$\cup \quad\quad \cup$
$C = $ 0-sect'n //

Outline of prf:

For simplicity Assume that $\exists W$: c-nbhd of $C \subset S$
s.t. $V = \widetilde{V}/\langle F \rangle$, $\widetilde{V} \xrightarrow{/\langle F\rangle} V$: covering.
$\cup \quad \cup$
$\widetilde{C} \longrightarrow C$;
$\cong$
$\mathbb{C}^*$. $F: \widetilde{V} \to \widetilde{V}$: isom,
$F|_{\widetilde{C}} = $ "$\times \lambda$" $(|\lambda| \neq 1)$.

$F(z,w) = (\lambda z + O(w),\ t \cdot w + O(w^2))$, $t \in U(1)$, dioph.

Enough to show:

$\exists \phi(z,w) = (z + O(w),\ w + O(w^2))$: $\widetilde{V} \to \widetilde{V}$ ($\phi_1(z,w)$, $\phi_2(z,w)$)
s.t. $F \circ \phi(z,w) = (\lambda \cdot \phi_1(z,w),\ t \cdot \phi_2(z,w))$.

$\Uparrow$
∘ formally construct ϕ by solving "Schröder eq".
∘ Show the convergence by using Dioph. cond. (cf. Siegel's linearization thm).

⊙ [Ueda '83] : Classification of (C,S) with $(C^2)=0$.

e.g. Serre's e.g.

$\boxed{(C,S)}$ or

$\left(\begin{smallmatrix}C\text{-nbhd}\\ C\,S\end{smallmatrix}\right) \not\cong_{\text{formally along } C} \left(\begin{smallmatrix}0\text{-sectn}\\ \text{nbhd of } N_{C/S}\end{smallmatrix}\right)$ type α $\overset{\text{fact}}{\Longrightarrow}$ $\begin{cases} C \text{ admits a psd-concave nbhds sys.} \\ \mathcal{O}_S(C) : \text{not semi-positive} \end{cases}$ (Ser)

$(\quad) \cong_{\text{formally along } C} (\text{—})$ → C admits psd-flat nbhds sys. type β $\overset{\text{fact}}{\Downarrow}$ $\mathcal{O}_S(C)$: semi-positive; e.g. (C,S) as in Thm 4

or → the other type γ. ← "mysterious case"

⊙ Ueda constructed an example of (C,S) : type γ //.
open!

Example 5. $C_0 \subset \mathbb{P}^2$: sm. ellipt. curve.
$Z := \{P_1, P_2, \cdots, P_9\} \subset C_0$: nine pts "general".
$S := Bl_Z \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$,
$C := (\pi^{-1})_* C_0$.
$\rightsquigarrow N_{C/S} \cong \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-P_1-P_2-\cdots-P_9) \in \mathrm{Pic}^0(C)$.

⊙ $N_{C/S}$: torsion $\Longleftrightarrow$ $\exists\, S \longrightarrow \mathbb{P}^1$: ellipt. fibr. $\rightsquigarrow$ (C,S) : of type β.
$C \longrightarrow *$

⊙ Z : "general" $\rightsquigarrow$ (C,S) : type β $\overset{\text{Brunella}}{\Longleftrightarrow}$ $\mathcal{O}_S(C)$: semi-positive.
~~[Arnold-Ueda-Brunella]~~
$\Uparrow$ Arnold-Ueda.
"a.e." Z (Lebesgue measure)

Q $\exists ?\, Z$ s.t. (C,S) of type γ in Example 5 ?

§2. Construction of a K3 X.

⊙ (C,S,Z), (C',S',Z') ⋯ as in Example 5,
s.t. $C \cong_{\text{bihol}} C'$, $N_{C/S} \cong N_{C'/S'}^{-1}$: Diophantine.

⊙ $W^{(\prime)} \subset S^{(\prime)}$: $C^{(\prime)}$-nbhd : as in Thm 4.

Coordinates sys. of $W^{(\prime)}$;

$W^{(\prime)} = \bigcup_j W_j^{(\prime)}$, $(W_j, (z_j, w_j))$ ← loc. def. func of C.
"coord of $W_j \cap C$"
s.t. $z_j^{(\prime)} = z_{jk}^{(\prime)} + {}^{\exists}A_{jk}$ on $W_{jk}^{(\prime)}$,
$w_j = {}^{\exists}t_{jk} \cdot w_k$ on W_{jk}, $w_j' = t_{jk}^{-1} \cdot w_k'$ on W_{jk}' "
$\;\; t_{jk} \in U(1)$

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<u>Obs 6.</u> ⊙ $\Phi^{(\prime)} : W^{(\prime)} \longrightarrow \mathbb{R}_{\geq 0}$

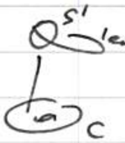
$(z_j^{(\prime)}, w_j^{(\prime)}) \longmapsto |w_j^{(\prime)}|$: well-def (c.f. flat metric on $N_{C/S}$)

⊙ Without loss of generality, we may assume

$W^{(\prime)} = \Phi^{-1}([0, R^{(\prime)}))$, $R, R' > 1$.

⊙ $H_t := \Phi^{-1}(t)$ $(t \in [0,R))$: Levi-flat $\subset W$.

$(\simeq S^1 \times S^1 \times S^1)$



leaf : "w_j = const" $\cdots \cong \mathbb{C}$ or $\mathbb{C}^*$

dense $\subset H_t$.

<u>Construction of X</u>

$M := S \setminus \Phi^{-1}([0, \frac{1}{R'}]) \subset S$.

$M' := S' \setminus (\Phi')^{-1}([0, \frac{1}{R}]) \subset S'$.

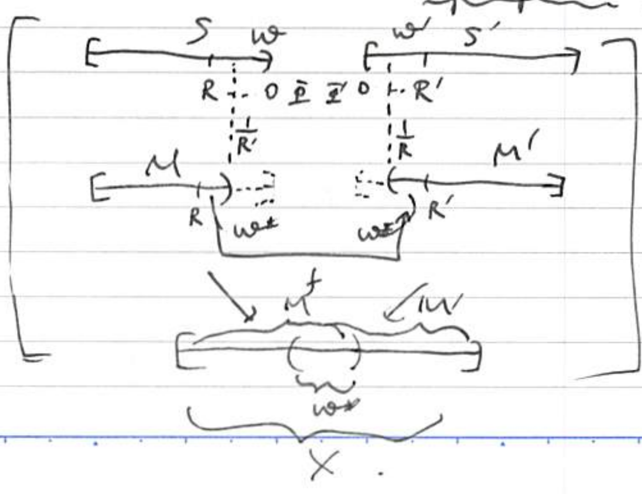
$M \supset W^* := \Phi^{-1}((\frac{1}{R'}, R))$.

$M' \supset (\Phi')^{-1}((\frac{1}{R}, R'))$ ← ~~regard~~ identify it ~~as~~ with W^* via ...

$$\left[\begin{array}{ccc} f : \Phi^{-1}((\frac{1}{R'}, R)) & \longrightarrow & (\Phi')^{-1}((\frac{1}{R}, R')) \quad \text{-- isom} \\ \cup & & \cup \\ W_j \cap \{\frac{1}{R'} < |w_j| < R\} & & W_j' \cap \{\frac{1}{R} < |w_j'| < R'\} \\ (z_j, w_j) & \longmapsto & (z_j, \frac{1}{w_j}) \end{array}\right.$$ //.

⇝ $X := M \cup_{W^*} M'$ //

--- cpt cpx surf.



※ Simple calculation by using Mayer-Vietoris seq

⇝ $H_q(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (q=4) \\ 0 & 3 \\ \mathbb{Z}^{22} & 2 \\ 0 & 1 \\ \mathbb{Z} & 0. \end{cases}$

Rmk / ⊙ [Doi '09] .. Topologically the same constr. of K3.
(need to deform the cpx structures of M and M')

⊙ [Tsuji '84] ... Constr. of $(S^3 \times S^3, {}^{\exists}J)$ (cpx str.)
by using Arnol'd - type thm.
("M" = Hopf 3-fold
"C" = Hopf surf.)

§3. prfs.

Fact 7. X: cpt cpx surf,
$H_k(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k=4 \\ 0 & 3 \\ \mathbb{Z}^{22} & 2 \\ 0 & 1 \\ \mathbb{Z} & 0 \end{cases}$
and
$^{\exists}\sigma$: nowhere van. glob. hol 2-form
$\to$ X: K3.

Fact 8. $\mathcal{X} \xrightarrow{\pi} B$: Deformation family of K3 surfaces,
$\begin{cases} \dim B \geq 5. \\ \text{The Kodaira–Spencer map is inj.} \end{cases}$
$\Rightarrow$ $^{\exists}t \in B$, $X_t := \pi^{-1}(t)$: not a Kummer surf "

By Obs 6 + Fact 7 + Fact 8,
all we have to do is:

$\begin{cases} \cdot \text{ construct } \sigma \text{ on } X \stackrel{\text{as in §2}}{=} M \cup_{W^*} M'. \\ \cdot \text{ "count" degrees of freedom in the construction in §2.} \end{cases}$

Lem 9 $H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}$.

(∵) $F: W^* \to \mathbb{C}$: hol.
Take $t \in (\frac{1}{R}, R)$ $\rightsquigarrow$ $^{\exists}B := \max_{x \in H_t} |F(x)|$
$= |F(x_t)|$ $({}^{\exists}x_t \in H_t)$.
Take a leaf $\mathcal{L}_t \subset H_t$ with $\mathcal{L}_t \ni x_t$.
Maximum principle for $F|_{\mathcal{L}_t} : \mathcal{L}_t \to \mathbb{C}$ $\rightsquigarrow$ $F|_{\mathcal{L}_t} \equiv {}^{\exists}A \in \mathbb{C}$.
$\mathcal{L}_t \subset H_t$: dense $\rightsquigarrow$ $F|_{H_t} \equiv A$.
$\{x \in W^* \mid F(x) = A\}$; analytic sub of W^*, $\supset H_t$ $\rightsquigarrow$ $F \equiv A$ //

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Prop 10. $\exists \sigma$: glob. hol. 2-form on $X = M \cup_{w*} M'$,
s.t. $\begin{cases} \text{nowhere vanishing}, \\ \sigma|_{W_j^*} = dz_j \wedge \dfrac{dw_j}{w_j} \text{ for each } j. \end{cases}$ //

$$\left(\begin{array}{c} \shortparallel \\ d(z_k + A_{jk}) \wedge \dfrac{d(t_{jk} w_k)}{t_{jk} w_k} = dz_k \wedge \dfrac{dw_k}{w_k} \end{array} \right) \text{ on } W_{jk}^*.$$

pf

$K_S = -C \rightsquigarrow \exists \eta$: mero. 2-form. on S
s.t. $\mathrm{div}(\eta) = -C$.

$F_j := \dfrac{\eta|_{W_j^*}}{dz_j \wedge \frac{dw_j}{w_j}} : W_j^* \longrightarrow \mathbb{C}$: hol, nowhere vanishing.

$W_j^* \overset{?}{=} W_j \cap W^*$.

Patch.

$\exists F : W^* \longrightarrow \mathbb{C}$: hol s.t. $F|_{W_j^* \cap W^*} = F_j$.

Lem 9 $\rightsquigarrow$ we may assume $F_j = 1$.

$$\Updownarrow$$

$$\eta|_{W_j^*} = dz_j \wedge \frac{dw_j}{w_j}.$$

Similarly $\exists \eta'$; mero 2-form on S',
s.t. $\begin{cases} \mathrm{div}(\eta') = -C', \\ \eta'|_{W_j'} = +dz_j' \wedge \dfrac{dw_j'}{w_j'}. \end{cases}$

$$\textcircled{\scriptsize{o}}\ f^* dz_j' \wedge \frac{dw_j'}{w_j'} \underset{\uparrow}{=} dz_j \wedge \frac{d(w_j^{-1})}{w_j^{-1}} = -dz_j \wedge \frac{dw_j}{w_j}$$

$f(z_j, w_j) = (z_j, \frac{1}{w_j})$

$\rightsquigarrow \sigma := \{(M, \eta|_M), (M', -\eta'|_{M'})\}$ //

Q $\dim \{ X \in (\text{K3 moduli})^{20} \mid X$ can be constructed in the manner as in §2 $\} = ?$

Fix $C_0 \underset{\text{ellipc.}}{\subset} \mathbb{P}^2$, $L_0 \longrightarrow C_0$: Dioph. l.b.

Parameters:
- Choice of $g : C_0 \underset{\text{isom}}{\cong} C_0'$. — 1-dim'l
- Choice of $P_1, P_2, \cdots, P_8 \in C_0$ — 8-dim'l
  $P_1', \cdots, P_8' \in C_0'$ — 8-dim'l
  $(\rightsquigarrow \overset{\exists!}{P_9}, \overset{\exists!}{P_9'}$ s.t. $N_{C/S} \cong N_{C'/S'}^{-1} \cong L_0)$
- Choice of fiber coord. "w_j". — 1-dim'l. Face $\overset{A_i}{:}$ indep.