Toward a higher codimensional Ueda theory. (ArXiv/1412.2354)

§1. a short review of Ueda's theory.
§2. Main result.
§3. example.
§4. Outline of the prf.

§1 Setting

X : sm cpx mfd.
$\cup$
S : sm cpt Kä ~~mfd~~ hyp. surf.

s.t. $N_{S/X}$: flat ($\overset{\text{i.e.}}{\Longleftrightarrow} N_{S/X} \in H^1(S, U(1))$)

One of the goals of Ueda's theory

… to describe a sufficient cond. for $[S]$ to be flat on $\exists$ nbhd of S in X.

Ueda defined …

◎ type(S, X)

$$:= \max \left\{ n \in \mathbb{Z}_{>0} \cup \{\infty\} \;\middle|\; \begin{array}{l} 0 \le {}^\forall \nu < n, \\ \exists W : \text{a tub. nbhd of } S \text{ in } X \\ \text{s.t.} \\ \mathcal{O}_W(N_{S/W}) \otimes \mathcal{O}_W/I_S^{\nu+1} \\ \qquad \cong \mathcal{O}_W(S) \otimes \mathcal{O}_W/I_S^{\nu+1}. \end{array} \right\}$$

◎ The obstruction class $u_n(S, X) \in H^1(S, N_{S/X}^{-n})$

when type$(S, X) \ge n$.

s.t. $u_n(S, X) = 0 \iff \text{type}(S, X) \ge n+1$. //

Thm 1 (Ueda '83)

Assume $N_{S/X} \in \mathcal{E}_0(S) \cup \mathcal{E}_1(S)$

where $\mathcal{E}_0(S) := \{ E \in \mathrm{Pic}^0(S) \mid {}^\exists \nu \in \mathbb{Z}_{>0}\ E^\nu = \mathbb{1}_S \}$

$\mathcal{E}_1(S) := \left\{ E \in \mathrm{Pic}^0(S) \;\middle|\; \begin{array}{l} \exists \alpha \in \mathbb{R}_{>0}. \text{ s.t. } \forall n \in \mathbb{Z}_{>0} \\ d(\mathbb{1}_S, E^n) \ge (2n)^{-\alpha}. \end{array} \right\}$

d … "invariant dist." (Euclid. dist)

Then type$(S, X) = \infty \Rightarrow [S]$: flat around S. //

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Rmk

① $\mathcal{E}_1(S)$ does not depend on the choice of "d".

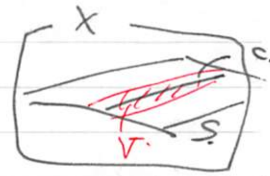
② $\mu(\mathrm{Pic}^0(S) \setminus \mathcal{E}_1(S)) = 0$
← Lebesgue zero.

③ $\mathcal{E}_1(S) = \bigcup_{\#\mathbb{N}}$ (nowhere dense ~~vanish~~ closed subsets of $\mathrm{Pic}^0(S)$) //

§2 main result

$\boxed{\text{Setting}}$ X : cpx mfd.
$\cup$
S : sm hyp. surf of X.
$\cup$
C : sm. cpt Kä. hyp. surf of S.

s.t. $\exists$ a (suff. small tub.) nbhd V of C in S.
s.t. $N_{S/X}|_V$: flat.



$\boxed{\text{Goal}}$ to describe an suff. cond. for $[S]$ to be flat around C.
(i.e. on $\exists W$: nbhd of C in X)

$\mathrm{codim}_C X = 2$

Def. (2-codim'l analogue of "type")

$$\mathrm{type}(C,S,X) := \max\left\{ (n,m) \in \mathbb{Z}^2_{\geq (1,0)} \cup \{\infty\} \;\middle|\; \begin{array}{l} \forall (0,0) \leq (\nu,\mu) < (n,m) \\ \exists W : \text{tub. nbhd of } V \text{ in } X \\ \text{s.t.} \end{array} \right\}$$

(lexicographical order
$(n,m) \geq (\nu,\mu)$
$\overset{\text{def.}}{\Longleftrightarrow}$ $n > \nu$ or $\begin{cases} \nu = n \\ m \geq \mu \end{cases}$

$$\left(\mathcal{O}_W(\widetilde{N_{S/X}|_V}) \otimes \mathcal{O}_W/I_V^{\nu+1}\right)\Big|_V \otimes \mathcal{O}_V/I_C^{\mu+1}$$
$$\cong \left(\mathcal{O}_W(V) \otimes \mathcal{O}_W/I_V^{\nu+1}\right)\Big|_V \otimes \mathcal{O}_V/I_C^{\mu+1} \quad //$$

~~Rmk~~

Rmk We also defined the obstr. class $u_{n,m}(C,S,X) \in H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/X}^{-m})$

when $\mathrm{type}(C,S,X) \geq (n,m)$.

s.t. $\begin{cases} u_{n,m}(C,S,X) = 0 \iff \mathrm{type}(C,S,X) \geq (n, m+1). \\ \text{(Under some assumptions)}, \\ \quad u_{n,m}(C,S,X) = 0 \text{ for } \forall m \geq 0 \\ \quad \iff \mathrm{type}(C,S,X) \geq (n+1, 0) \end{cases}$

~~Thm~~ **Main thm**

Assume (i) $N_{C/S}, N_{S/X}|_C \in \mathcal{E}_0(C)$.

or (ii) $N_{C/S} = N_{S/X}|_C \in \mathcal{E}_1(C)$

or (iii) $N_{S/X}|_C \in \mathcal{E}_0(C)$ and $C \subset V$: exceptional sub. in the sense of Grauert.

Then $\mathrm{type}(C,S,X) = \infty$

$\implies [S]$: flat around C //.

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§3. example

$P_1, P_2, \cdots P_8 \in \mathbb{P}^3$: general 8 pts.

$\rightsquigarrow \exists \{Q_q\}_{q \in \mathbb{P}^1}$: 1-dim. family of quad. surf. of $\mathbb{P}^3$.

we may assume $\begin{cases} Q_0, Q_\infty : \text{sm.} \\ Q_0 \pitchfork Q_\infty \end{cases}$

$\rightsquigarrow \begin{cases} C_0 := Q_0 \cap Q_\infty : \text{sm. ellipt. curve} \\ \mathcal{O}_{Q_0}(C_0) \cong \mathcal{O}_{Q_0}(K_{Q_0}^{-1}) \end{cases}$

$X := Bl_{\{P_i\}} \mathbb{P}^3 \xrightarrow{\pi} \mathbb{P}^3$.

$S_q := (\pi^{-1})_* Q_q$.

$C := (\pi^{-1})_* C_0$.

$(X \supset \begin{matrix} S_0 \\ S_\infty \end{matrix} \supset C)$ S

Fact

- $K_X^{-1} = [2S_0]$
- $N_{S_0/X} = [C] \rightsquigarrow N_{S_0/X}|_C = N_{C/S_0} =: N$.
- $N \cong \mathcal{O}_{\mathbb{P}^3}(2)|_{C_0} \otimes \mathcal{O}_{C_0}(-P_1 - P_2 - \cdots - P_8)$ //

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@ Thm 1 ⇝ $N \in \mathcal{E}_1(C)$ ($\cup\ \mathcal{E}_0(C)$) $\Rightarrow$ $[C]$: flat on a nbhd of C in S_0.

<u>Cor 3</u> (Thm 2. (ii) + $H^1(C, N^{-(n+m)}) = 0$ if $N \notin \mathcal{E}_0(C)$).

$N \in \mathcal{E}_1(C)$ ($\cup\ \mathcal{E}_0(C)$)

$\Rightarrow$ $K_X^{-1}|_{S_0}$: flat around C. //

<u>Cor 4</u>

$N \in \mathcal{E}_1(C) \Rightarrow K_X^{-1}$ is <u>not s.a.</u>

but it admit a C^∞ metric with semi-positive curv.

(c.f. [Lesieutre, Otten] arkiv 1410.4467) //

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§4. Outline of the prf of Thm 2.

$\{U_j\}$: suff. fine open cov. of C.

$\{V_j\}$: open cov. of V s.t. $\begin{cases} V_{jk} \neq \emptyset \Leftrightarrow U_{jk} \neq \emptyset \\ U_j = V_j \cap C \end{cases}$

$\{W_j\}$: open cov of a suff. small tub. nbhd W of V in S s.t. $\begin{cases} W_{jk} \neq \emptyset \Leftrightarrow V_{jk} \neq \emptyset \\ V_j = W_j \cap S \end{cases}$

⇝ $N_{S/X}|_V = [\{(V_{jk}, {}^\exists t_{jk})\}] \in H^1(V, \mathcal{O}_V^*)$, $t_{jk} \in U(1)$

${}^\exists w_j$: def. func of V_j in W_j s.t. $\left.\frac{w_j}{w_k}\right|_{W_{jk}} \equiv t_{jk}$.

Fix (x_j, z_j, w_j) : coord of W_j s.t. $\{z_j = 0\} \cap V_j = U_j$.

⇝ <u>for j,k s.t. $W_{jk} \neq \emptyset$</u>

$$t_{jk} W_k \underset{\text{expand on } W_{jk}}{=} w_j + \sum_{\nu=2}^{\infty} \sum_{\mu=0}^{\infty} {}^\exists g_{jk}^{(\nu,\mu)}(x_j) \cdot z_j^{\mu} \cdot w_j^{\nu}$$

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(Rmk. type$(C,S,X) \geq (n,m)$

$\rightsquigarrow$ We can choose $\{W_j\}$ s.t.

$$(\nu,\mu) < (n,m) \Rightarrow g_{jk}^{(\nu+1,\mu)} \equiv 0.$$

in this case,

$$U_{n,m}(C,S,X) := [\{(U_{jk}, g_{jk}^{(n+1,m)})\}]$$

<u>Strategy of the prf for (i), (ii)</u>

... Choose nice $\{G_j^{(n,m)}\}$ s.t.

the solution u_j of the functional e.q.

$$\text{"} W_j = u_j + \sum_{n=2}^{\infty}\sum_{m=0}^{\infty} G_{jk}^{(n,m)}(x_i)\cdot z_j^m u_j^n \text{"} \cdots (*)$$

Satisfies $u_j = t_{jk}\cdot u_k$ for $\forall j,k$.

<u>by the computation</u>

We can show that the cond. for $\{G_j^{(n,m)}\}$ is:

$$\delta\{(U_j, G_j^{(n,m)})\}$$

$$= \{(U_{jk}, g_{jk}^{(n,m)} + \exists \text{ hol. func on } U_{jk}.)\} \cdots (**)$$

uniquely determined from $\{G_j^{(\nu,\mu)}\}_{(\nu,\mu)<(n,m)}$

$\in$ $U_{n,m}(C,S,X)$
Fact

<u>Thus</u> if type$(C,S,X) = \infty$,
we can inductively define $\{G_j^{(n,m)}\}$
s.t. $(**)$ holds.

$\odot$ The conditions (i), (ii) is needed.

for the convergence of $(*)$ //.

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