談話会.

§1. Introduction (semi-positivity of hol. line bdls).
§2. "nbhds theories" and its application.
§3. Other applications...

§1.

X : Projective complex mfd.
$\uparrow$
L : hol. line bdl.

Thm (Kodaira's embedding thm, Nakai-Moishezon criterion)
T.F.A.E.
(1) L : ample.
(2) $\forall Z \subset X$: d-dim'l subvar, $\int_Z c_1(L)^d > 0$.
(3) $\exists h$: C^∞ Herm. metric s.t. $\sqrt{-1}\,\Theta_h > 0$. " L : positive "

Q Analogue for "semi-" case?

Known
◎ L : semi-ample $\Rightarrow$ L : semi-positive (s.p.) $\Rightarrow$ L : nef.

$\left[\exists h : C^\infty \text{ Herm. metric s.t. } \sqrt{-1}\,\Theta_h \geq 0\right.$ "

$\left[\forall C \subset X : \text{curve}, \int_C c_1(L) \geq 0.\right.$ "

◎ $\not\Leftarrow$ $\not\Leftarrow$ (∃ counter examples).

c.f... **Conj** (Abundance conj)
When, $L = K_X := \det T_X^*$, L : nef $\Rightarrow$ L : semi-ample. "

Problem semi-positivity criterion of L when L : nef? "

$\Downarrow$

Typical case; When X : proj. surf $\leftarrow$ mfd with $\dim_{\mathbb{C}} = 2$,
$\cup$
C : cpt curve (Riem surf),
$(C^2) := \deg N_{C/X} = 0$
$L = L_C :=$ the l.b. corresp. to the divisor C "
$\Longrightarrow$ L_C : nef. ($\rightsquigarrow$ Q when L_C : s.p.?)

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§2. "nbhds theories"

$\boxed{X^{②} \supset \underset{\sim}{C^{①}} \text{ cpt. sm.}}$

Q What kinds of cpx str. does a tub. nbhd of C have?

Rmk $\nexists$ hol. version of "tub. nbhd thm".

Grauert '62 ... the case of $(C^2) < 0$

($\Rightarrow C$ admits str. psd convex nbhds sys)

Suzuki '75 ... the case of $(C^2) > 0$

($\Rightarrow C$ admits str. psd concave nbhds sys)

Ueda '83 ... —— " —— $(C^2) = 0$.

Ueda classified (C^1, X^2) into three cases (α, β, γ) by using "Ueda class". "Ueda theory".

§2.1 When (C, X) : of type (α)

... the case where the cpx str. of a nbhd of C is "completely different" from that of the zero-section $C\ N_{C/X}$.

e.g. (Serre's e.g.)

$E := \mathbb{C}^3/\sim \quad ((z, \xi, \eta) \sim (z+1, \xi, \eta+\xi) \sim (z+\tau, \xi, \eta))$

$\downarrow Pr.$

$C_0 := \mathbb{C}/\sim \quad (z \sim z+1 \sim z+\tau) \qquad \tau \in \mathbb{H}.$

$X = \mathbb{P}(E), \supset C := \mathbb{P}(\mathbb{1}_C) \qquad (\mathbb{1}_C \hookrightarrow E)$ $\quad (z, \eta) \mapsto (z, 0, \eta)$

$L = L_C \qquad (L_C^{\otimes 2} = K_C^{-1} := \Lambda^2 T_X)$ $\Rightarrow (C, X)$: of type (α).

$X \setminus C \cong_{hol} \mathbb{C}^* \times \mathbb{C}^*$: Stein!

Thm (Demailly-Peternell-Schneider '94) "For Serre's e.g. L_C : nef, but not s.p."

the first example of L : nef, not s.p.

Thm (K— '14, '15)

(C, X): of type (α) $\Longrightarrow$ $L = L_C$: nef, but not s.p. //

§2.2. type (β)

--- the case where the cpx str. of a nbhd of C is similar to that of the zero-section $\subset N_{C/X}$.

e.g. $X = \underset{\text{ellipt. curve}}{C_0} \times \mathbb{P}^1 \supset C := C_0 \times \{0\}$. $\rightsquigarrow$ type (β).

- $\varnothing$ type (β) $\Rightarrow$ $L = L_C$: s.p. (simple obs. by def. of "(β)")
- $\varnothing$ $\nexists$ "cpx dynamical" criterion for type (β)-ness.

e.g. $C_0 \subset \mathbb{P}^2$: sm. ellipt. curve.

$p_1, p_2, \dots, p_9 \in C_0$: nine points.

$X := \mathrm{Bl}_{\{p_1, \dots, p_9\}} \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$

$\cup$

$C := (\pi^{-1})_* C_0$: str. transf. of C_0.

$\rightsquigarrow$ $L = L_C \cong K_X^{-1}$, $N_{C/S} \cong \pi^* \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - \cdots - p_9) \in \mathrm{Pic}^0(C_0)$

Thm (Arnol'd - Ueda - Brunella)

$N_{C/X} \in \mathrm{Pic}^0(C_0)$: Dioph. $\leftarrow$ i.e. $\exists \alpha > 0, \exists A > 0$ s.t. $\mathrm{dist}_{\mathrm{Pic}(C_0)}(N_{C/S}^{\otimes n}, \mathbb{1}_C) \geq A \cdot n^{-\alpha}$ //

$\Rightarrow$ (C, X): of type (β)

$\Rightarrow$ L: semi-positive //

Rmk $\mathcal{C} := \{N \in \mathrm{Pic}^0(C_0) \mid N: \text{Dioph}\}$.

$\rightsquigarrow$ $\mathcal{C}$: full-measure, however $\mathcal{C} = \bigsqcup_{\#\mathbb{Z}}$ (nowhere dense closed sub of Pic^0) //

Q What kinds of cpx str does a nbhd of C in X have when $N_{C/S} \notin \mathcal{C}$?? $\leftarrow$ "β or γ"

open. Is $L = L_C$: s.p. when —"— ? //

Cf Thm (K—)

For this, "C": singular

$\exists \{p_1, \dots, p_9\} \subset \mathbb{P}^2$ s.t. K_X^{-1}: nef, but not s.p. when $X = \mathrm{Bl}_{\{p_1, \dots, p_9\}} \mathbb{P}^2$. //

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§2.3. of type (γ)

… the case "where (the cpx str. of) a nbhd of C is "formally" similar to that (0-section) $C N_{C/S}$ but not similar to ――――――― ".

… "mysterious case".

Thm (Ueda) … $\exists$ example of (C, X) : of type (γ). (X: open surf)

Thm (K–, K– and N. Ogawa) $L := L_C$: "nef" but not s.p. for (C, X) : Ueda's example".

Q $\exists$? example of (C, X) with X: cpt?

§3. Other applications

- (Non-) embeddability criterion for Levi-flat mfds
- Gluing construction of K3 surfaces.

$C_0 \subset \mathbb{P}^2$: sm. ellipt. curve.
$\ni p_1, \cdots, p_9$
$S := Bl_{\{p_j\}} \mathbb{P}^2$
$\cup$
C : str. transf

/

$C_0' \subset \mathbb{P}^2$: sm. ellipt. curve
$\ni p_1', \cdots, p_9'$.
$S' := Bl_{\{p_j'\}} \mathbb{P}^2$,
$\cup$
C' : str transf

"

Thm (K–)

Assume $C_0 \underset{g}{\cong} C_0'$, $g^* N_{C'/S'} \cong N_{C/S}$, $N_{C/S}$: Dioph.

Then One can glue $S \setminus$ (a nbhd of C) and $S' \setminus$ (a nbhd of C') holomorphically.

The resulting cpt surf X is K3. "

Cor $\exists X$: K3 s.t. $\exists f : \mathbb{C} \longrightarrow X$: hol inj. immersion with $\overline{f(\mathbb{C})}^{\mathrm{Euc}} \approx_{C^\omega} S^1 \times S^1 \times S^1$, $\overline{f(\mathbb{C})}^{\mathrm{Zar}} = X$. //

Thm (K–, T. Uehara)

For general choice of the parameters, $\rho(X) = 0$ (Picard number) $\rightsquigarrow X$: non-Kummer non-proj.

($\dim_{\mathbb{C}} = 19$, "indep".)