X: cpx mfd. (proj) $L \to X$: h.l. line bdl. (nef)

Our interest: The semi-positivity of L.

Def L: semi-positive (s.p.) $\overset{\text{def}}{\Longleftrightarrow}$ $\exists h$: C^∞ Herm. metric on L s.t. $\sqrt{-1}\Theta_h \geq 0$.

Known: L: s.p. $\overset{\text{clear}}{\Longrightarrow}$ L: nef, $\not\Longleftarrow$ $\exists$ counter e.g. (Demailly - Peternell - Schneider, K-)

e.g. Fix $C_0 \subset \mathbb{P}^2$: sm. ellipt. curve.
$\{p_j\}_{j=1}^9 \subset C_0$: 9 pts, diff. from each other.
$X := Bl_{\{p_j\}^9}\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$,
$C := (\pi^{-1})_* C_0 \rightsquigarrow K_X^{-1} = \mathcal{O}_X(C)$: nef.

Thm (Ueda, Brunella) (X, C): as above.
$\rightsquigarrow K_X^{-1}$: semipositive if $N_{C/X} \in \mathcal{E}_0(C) \cup \mathcal{E}_1(C)$,
where $\left(\cong \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - \cdots - p_9)\right)$
$\mathcal{E}_0(C) := \{F \in \mathrm{Pic}^0(C) \mid \exists n \geq 1 \text{ s.t. } F^n = \mathbb{1}_C\}$
$\mathcal{E}_1(C) := \{F \in \mathrm{Pic}^0(C) \mid -\log d(\mathbb{1}_C, F^n) = O(\log n) \text{ as } n \to \infty\}$. //

Rmk
- It is clear that K_X^{-1}: semi-ample $\Leftrightarrow N_{C/X} \in \mathcal{E}_0(C)$.
- $\forall F \in \mathrm{Pic}^0(C)$, $\exists \{p_j\}^9$ s.t. $N_{C/X} = F$.
- $\mathcal{E}_1(C) \subset \mathrm{Pic}^0(C)$: full-measure, however $\mathcal{E}_1(C) = \bigcup_{\nu=1}^{\infty} \exists S_\nu$ ($S_\nu \subset \mathrm{Pic}^0(C)$: nowhere dense closed sub)

Question (Demailly ...)
$\exists ? \{p_j\}^9 \subset C_0$ s.t. K_X^{-1}: not semi-positive?

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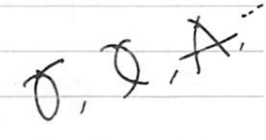
§ Main results

X : cpx surf. (possibly open)

$\cup$

C : nodal cpt curve. (cpt)

$L := \mathcal{O}_X(C)$. $(\rightsquigarrow L : \text{nef})$

$C =$ 

Assume
- $N_{C/X} := L|_C$: top. triv.
- C is a cycle of $\mathbb{P}^1$'s. (for simplicity)

i.e. { (the normalization of C) $= \coprod_{\text{fin}} \mathbb{P}^1$, and the dual graph of C is a cycle graph

Thm A $N_{C/X} \in \mathcal{E}_1(C) \Rightarrow L$: Semi-positive

Thm B $N_{C/X} \in \mathrm{Pic}^0(C) \setminus P_0(C) \Rightarrow L$: not semi-positive.

$P_0(C) = \mathrm{Image}\,(H^1(C, U(1)) \to H^1(C, \mathcal{O}_C^*))$

e.g. when C = a rat'l curve with a node,

$$\rightsquigarrow \begin{array}{ccl} \mathrm{Pic}^0(C) & \cong & \mathbb{C}^* . \\ \cup & & \cup \\ P_0(C) & \cong & U(1) \\ \cup & & \\ \mathcal{E}_0(C) & = & \{ e^{2\pi\sqrt{-1}\alpha} \mid \alpha : \text{rat'l} \} \\ \mathcal{E}_1(C) & = & \{ e^{2\pi\sqrt{-1}\alpha} \mid \alpha : \text{"Diophantine"} \} \end{array}$$

$\Uparrow$

α : alg. irrat'l

Application;

Cor A′ "Thm (Ueda, Brunella)" also holds for nodal C. $(\Leftarrow$ Thm A$)$.

Cor B′. $\exists \{p_j\}_{j=1}^{9} \subset \mathbb{P}^2$ s.t. K_X^{-1} : nef, however not S.P. $(X := \mathrm{Bl}_{\{p_j\}}\mathbb{P}^2)$

prf of "Thm B ⇒ Cor B′":

Fix $C_0 \subset \mathbb{P}^2$: nodal cubic
and take $\{P_i\}^9 \subset (C_0)_{reg}$: general.
$\xrightarrow{\text{Thm B}}$ $K_X^{-1} = \mathcal{O}_X(C)$: not s.p. //

Rmk C : non-singular ⇒ $P_0(C) = \mathrm{Pic}^0(C)$ //.

§. Outline of the prf

... Run "nodal analogue of Ueda theory" to investigate the nbhd str. of C.

Outline of the prf of Thm A

... We use Thm C (K-, "nodal analogue of Ueda's thm)

$X \supset C$, X : surf, C : nodal cpt curve.

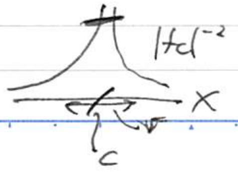
Assume
① "(C,X) : of type ∞ in the sense of Ueda".
② $N_{C/X} \in \mathcal{E}_0(C) \cup \mathcal{E}_1(C)$
③ $i^* N_{C/X} \in \mathcal{E}_0(\widetilde{C})$ ($\widetilde{C} \xrightarrow{i} C$: normalization)
④ $H^1(C, \mathbb{C}(N_{C/X}^{-n})) = 0$ for $\forall n$.

Then
$\exists V$: a nbhd of C in X
s.t. $\mathcal{O}_V(C) \in H^1(V, U(1))$ //

∵ ① ⇐ $H^1(C, N_{C/X}^{-n}) = 0$ for $\forall n$ ⇐ C : a cycle of $\mathbb{P}^1$'s.
③, ④ ⇐

$\xrightarrow{\text{Thm C}}$ $\exists V$, $\exists h_V$: C^∞ Herm. metric on $L|_V = \mathcal{O}_V(C)$
s.t. $\sqrt{-1}\,\Theta_{h_V} \equiv 0$.

⇝ Consider $h := \mathrm{Reg\,min}.\{M \cdot h_V, |f_C|^{-2}\}$
where $f_C \in H^0(X, \mathcal{O}_X(C))$: the canonical section, $M \gg 1$. //



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Outline of the prf of Thm B

... We use:

Thm D (K–, a generalization of Thm 1.2 in [Ueda '91])

$X \supset C$, X: surf., C: nodal curve.

Assume
① (C, X): "of type ζ in the sense of Ueda"
② the dual graph of C is a cycle graph.
③ $N_{C/X} \in \mathrm{Pic}^0(C) \setminus P_0(C)$

Then $0 < {}^\forall \lambda < 1$,

${}^\forall \Phi$: $X \setminus C$ (psh on) with $\Phi(p) = O((-\log d(p, C))^{2\lambda})$ as $p \to C$,

${}^\exists V$: a nbhd of C in X

s.t. $\Phi|_{V \setminus C} \equiv (\text{const})$ //

@ Take $h \left(\underset{\text{locally}}{=} e^{-\varphi}\right)$ (φ psh): sing. Herm. metric on L, minimal singular one.

Set $\Phi := -\log |f_C|_h^2 \cdot \left(\underset{\text{locally}}{=} -\log |f_C|^2 + \varphi\right)$

: psh on $X \setminus C$, $\Phi(p) \sim -\log d(p, C)$

$\underset{\text{Thm D}}{\rightsquigarrow}$ ${}^\exists V$, $\Phi|_{V \setminus C} \equiv \text{const}$)

Thus $|f_C|^{-2}$ has min. sing. among s.H.m's on L with s.p. curvature,

($\rightsquigarrow$ L: not s.p.) //