京大代数セミナー用原稿

No. 1.

On a higher codimensional analogue of Ueda theory and its applications.

2016/7/15.

10:30-12:00

◎ Ueda theory ⋯ a nbhd theory of $\underset{\sim}{Y} \subset \underset{\sim}{X}$

Y: cpt smooth curve, X: smooth surface.

with $N_{Y/X}$: unitary flat

(i.e. $N_{Y/X} \in H^1(Y, U(1))$, $\overset{\text{fact}}{\Longleftarrow}$ $(Y^2) = 0$).

[Ueda '83]

(c.f. $(Y^2) < 0$ ⋯ H. Grauert '62, $(Y^2) > 0$ ⋯ O. Suzuki '75)

◎ We can apply Ueda theory to "the semi-positivity problem".

(Def $\underset{\text{hol line bdl}}{L}$: semi-positive $\overset{\text{def}}{\Longleftrightarrow}$ $\exists h$: C^∞ Herm metric on L s.t. $\sqrt{-1}\Theta_h \geq 0$)

semi-positive $\overset{\text{known}}{\Longrightarrow}$ L : nef.

e.g. (~~Arnol'd-Ueda-Brunella~~)

Fix $C \subset \mathbb{P}^2$; sm. ellipt. curve.

Take $p_1, \cdots, p_9 \in C$: 9 pts.

$X := Bl_{\{p_j\}_{j=1}^9} \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$, $Y := (\pi^{-1})_* C$. $\rightsquigarrow (Y^2) = 0$.

$\to K_X^{-1} \underset{\mathcal{O}_X(Y)}{=}$: nef.

Fact (Arnol'd-Ueda-Brunella)

$N_{Y/X} (\cong \mathcal{O}_{\mathbb{P}^2}(3)|_C \otimes \mathcal{O}_C(-p_1 - p_2 - \cdots - p_9)) \in \mathrm{Pic}^0(Y)$;

"torsion" or "Diophantine" $\Longrightarrow$ $K_X^{-1} (= \mathcal{O}_X(Y))$: semi-positive //

(c.f. torsion $\Leftrightarrow$ K_X^{-1} : semi-ample).

$\boxed{\text{Goal}}$:

◎ Generalize Ueda theory to ~~the case codim Y ≥ 1~~ higher (co-)dimensional cases.

◎ Apply it to the semi-positivity problem.

Schedule

§1. A short review of Ueda theory.

§2. Main result.

§3. Application

§4. Outline of the prf.

10:45 ↑

a short
§1 Review of [Ueda '83]

$\underset{\text{cpt sm. curve}}{\underbrace{Y}} \subset \underset{\text{sm. surf.}}{\underbrace{X}}$ with ~~...~~ $N_{Y/X} \in H^1(Y, U(1))$.

Ueda classified (Y, X) into the following two types;

$$\begin{cases} \text{"type}(Y,X) = \infty\text{"} \overset{\text{def}}{\iff} \mathcal{O}_X(Y) \otimes \mathcal{O}_X/I_Y^n \cong \widetilde{N_{Y/X}} \otimes \mathcal{O}_V/I_Y^n \text{ for } {}^\forall n \geq 1 \\ \text{"type}(Y,X) < \infty\text{"} \overset{\text{def}}{\iff} \text{otherwise} \end{cases}$$

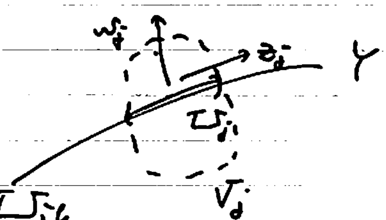
I_Y: def. ideal sheaf of Y.

More Precisely;

by using the loc. notations as below, ...

flat line bdl / ${}^\exists V$: a tub. nbhd of $Y \subset X$ s.t. $H^1(V, U(1)) \underset{\text{natural}}{\cong} H^1(Y, U(1))$
$\widetilde{N_{Y/X}} \longmapsto N_{Y/X}$.

$\{U_j\}$: open cov. of Y.
V_j: a nbhd of U_j in X_j
w_j ... $U_j = \{w_j = 0\} \subset V_j$,
s.t. $dw_j = {}^\exists t_{jk} \cdot dw_k$ on U_{jk}.
$t_{jk} \in U(1)$



$$t_{jk} w_k = w_j + {}^\exists f_{kj,2}(z_j) \cdot w_j^2 + \cancel{{}^\exists f_{kj,3}(z_j) \cdot w_j^3} + \cdots$$

$\rightsquigarrow$ **Def** $u_1(Y,X) := [\{(U_{jk}, f_{kj,2})\}] \underset{\text{fact}}{\in} H^1(Y, N_{Y/X}^{-1})$.

Obs. When $u_1(Y,X) = 0$, then we can take $\{w_j\}$ s.t. $t_{jk} w_k = w_j + O(w_j^3)$
$= w_j + {}^\exists f_{kj,3}(z_j) \cdot w_j^3 + \cdots$

$\rightsquigarrow$ **Def** $u_2(Y,X) := [\{(U_{jk}, f_{kj,3})\}] \underset{\text{fact}}{\in} H^1(Y, N_{Y/X}^{-2})$

Fact ① when $u_1 = u_2 = \cdots = u_{n-1} = 0$, "$\underset{H^1(Y, N^{-n})}{u_n} = 0$" does not depend on $\{w_j\}$

② Either $\begin{cases} {}^\exists n \geq 1,\ u_n \neq 0 \longleftrightarrow \text{type}(Y,X) := n \\ \text{or } {}^\forall n \geq 1,\ u_n = 0 \longleftrightarrow \text{type}(Y,X) := \infty \end{cases}$

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① ~~tr~~ (finite type case ··· skip)

② infinite type case ...

Thm (Ueda)

X: cpx mfd $\quad Y \subset X$: cpt. sm. hypersurf. with $N_{Y/X} \in H^1(Y, X)$

Assume $\begin{cases} \mathrm{type}(Y,X) = \infty, \\ N_{Y/X} \in H^1(Y, U(1)) \text{ : "torsion" or "Diophantine".} \end{cases}$

Then (*) We can take $\{w_j\}$ s.t.

$$w_j = t_{jk} \cdot w_k \quad \text{on } V_{jk}$$

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Rmk (*) $\iff$ ~~$\mathcal{O}_V(Y)$~~ $\exists \underset{\text{nbhd of } Y}{V}$, $\mathcal{O}_V(Y)$: unitary flat. $\iff \exists \mathcal{F}$: non-sing hol. foliation of codim $= 1$

s.t. $\begin{cases} Y \text{ is a leaf of } \mathcal{F}, \\ \text{Holonomy } Hol_{\mathcal{F},Y} \\ \text{coincides the monodromy} \\ \rho_{N_{Y/X}} \text{ of } (N_{Y/X}, \nabla_{flat}) \end{cases}$

$\mathcal{F} \underset{\text{locally}}{=} \{w_j = \text{const}\}$

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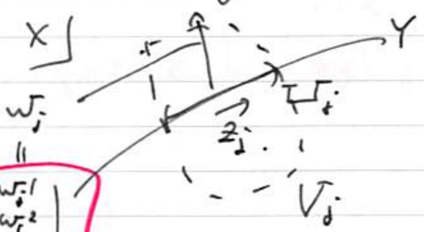
§2 Main results

① ~~We generalized~~

$\underset{\text{cpt cpx mfd}}{Y} \hookrightarrow X$: cpx mfd. $\left[\begin{array}{l} N_{Y/X} \in H^1(Y, \underset{\text{unitary flat.}}{U(r)}) \\ r := \mathrm{codim}_X Y \geq 1 \end{array} \right.$ //

We generalized Ueda's obstr. classes by considering



$$\underset{\text{unitary matrix.}}{T_{jk}} \cdot w_k \underset{\text{on } V_{jk}}{=} w_j + \sum_{\substack{|\alpha| \geq 2 \\ \uparrow}} f_{kj,\alpha}(z_j) \cdot w_j^{\alpha}$$

multi index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_r)$, $|\alpha| := \sum \alpha_\lambda$

$w_j = \begin{pmatrix} w_j^1 \\ w_j^2 \\ \vdots \\ w_j^r \end{pmatrix}$, $\quad f_{kj,\alpha} = \begin{pmatrix} f^1_{kj,\alpha} \\ \vdots \\ f^r_{kj,\alpha} \end{pmatrix}$.

$$w_j^{\alpha} := \prod_{\lambda=1}^{r} (w_j^{\lambda})^{\alpha_\lambda}$$

Def

$$u_1(Y,X) := \left[\left\{ \left(U_{jk}, \sum_{\lambda, |\alpha|=2} f^{\lambda}_{kj,\alpha}(z_j) \frac{\partial}{\partial w_j^{\lambda}} \otimes (dw_j)^{\alpha} \right) \right\} \right] \underset{\text{Fact.}}{\in} H^1(Y, N_{Y/X} \otimes S^2 N^*_{Y/X}).$$

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Fact

∘ When $u_1 = u_2 = \cdots = u_{n-1} = 0$,

we can define $u_n(Y, X) \in H^1(Y, N_{Y/X} \otimes S^{n+1} N^*_{Y/X})$,

∘ "$u_n = 0$". ~~which~~ does not depend on $\{w_j\}$.

∘ Either • $\exists n \geq 1$, $u_n \neq 0$ ← type$(Y, X) := n$

or • $\forall n \geq 1$, $u_n = 0$ ← type$(Y, X) := \infty$.

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Main thm

$Y \subset X$: as above.

Assume type$(Y, X) = \infty$,

$N_{Y/X}$: "torsion" or "Diophantine".

Then

(i) $\exists V$: a nbhd of Y in X

$\exists \mathcal{F}$: non-sing hol. foliation on V of codim $= r$,

s.t. $\begin{cases} Y \text{ is a leaf of } \mathcal{F}, \\ Hol_{\mathcal{F}, Y} = \rho_{N_{Y/X}}. \end{cases}$

local description ;

"$w_j = T_{jk} \cdot w_k$" on V_{jk}

(**)

(ii) $\forall S$: sm. hyp. surf $\subset V$.

with $\begin{cases} Y \subset S, \\ N_{Y/S} \in H^1(Y, U(r-1)) \end{cases}$

$\to \exists V' = V'_S$: Y-nbhd. $\subset X$.

$\exists \mathcal{G}_S$: sm. hol. foliation of codim $= 1$ on V'

s.t. $\begin{cases} S' := S \cap V' \text{ is a leaf of } \mathcal{G}_S, \\ Hol_{\mathcal{G}_S, S'} : \pi_1(S', *) \to U(1) : \text{linear} \\ \mathcal{G}_S : \mathcal{F}\text{-inv.} \left(\text{i.e. } \forall L \in \mathcal{G}_S, \forall L' \in \mathcal{F}, \ L \supset L' \text{ or } L \cap L' = \emptyset \right) \end{cases}$

hol. imm.

"we can take $\{\hat{w}_j\}$ s.t. (**) and $S' = \{\hat{w}_j^{(r)} = 0\}$

and $\hat{w}_j = {}^{\exists}F_j(w_j)$ "

($F_j \in \mathbb{C}\{x^1, \cdots, x^r\}$)

Def

• $N_{Y/X}$: torsion $\overset{\text{def}}{\iff}$ $\#\,\mathrm{Image}\,(\rho_{N_{Y/X}} : \pi_1(Y, *) \to U(r)) < \infty$.

• : Dioph. $\overset{\text{def}}{\iff}$ $\exists N_1, \cdots, N_r \in H^1(Y, U(1))$, $\exists A > 0$.

s.t. $N_{Y/X} = N_1 \oplus \cdots \oplus N_r$,

$\forall \alpha = (\alpha_1, \cdots, \alpha_r) \in \mathbb{Z}^r$,

$|\alpha| \geq 1 \Rightarrow d\left(\mathcal{O}_Y, \bigotimes_{\lambda=1}^{r} N_\lambda^{\alpha_\lambda}\right) \geq \dfrac{1}{(2|\alpha|)^A}$

"inv. dist" on $H^1(Y, U(1))$

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Obs ◎ $\forall n \geq 1$, $H^1(Y, N_{Y/X} \otimes S^{n+1} N^*_{Y/X}) = 0 \Rightarrow \mathrm{type}(Y, X) = \infty$.

$\Uparrow$ (for e.g.) Y: sm. ellipc. curve and $N_{Y/X}$: Diophantine ⋯ (∗∗∗)

Rmk ◎ Arnold '76 ⋯ "Mainthm" (i) for (∗∗∗). (and $r=1$?)

◎ Ueda '83 ⋯ —〃— for $r=1$.

◎ (K–'15), [K–, N. Ogawa '15] ⋯ $r=2$.

§3 Application

X: cpx mfd of $\dim = n$.

$\uparrow$

L: hol. line bdl, $D_1, \cdots, D_{n-1} \in |L|$.

$Y := \bigcap_{\lambda=1}^{n-1} D_\lambda$

Obs ◎ $\exists$ sing. Herm. metric h on L s.t. $\begin{cases} \sqrt{-1}\Theta_h \geq 0 \\ \{h = \infty\} = Y, \\ h|_{X \setminus Y} : C^\infty. \end{cases}$ ("Bergman type metric")

◎ If $\exists V$: Y-nbhd s.t. $L|_V$: unitary flat,

Then it is known that L is semi-positive

($\Leftarrow$ "regularized min. construction")

⤳ **Cor** (Main thm (ii)). Assume $\begin{cases} D_1, \cdots, D_{n-1} \text{ intersects transversally along } Y. \\ L|_Y \in H^1(Y, U(1)), \text{: "Diophantine".} \end{cases}$

Then L: semi-positive //

Example $(V^n, \underset{\text{ample}}{F})$: sm. del Pezzo mfd of $\deg = 1$.

(i.e. $F^{n-1} = K_V^{-1}$, $(F^n) = 1$)

Take $p \in V$: "general". (~~so that $F \otimes \mathcal{O}$~~

Then $K^{-1}_{Bl_p V}$: (nef, not semi-ample, however) semi-positive //

c.f. $n=2$ ⋯ $Bl_{9\,pts}\mathbb{P}^2$.

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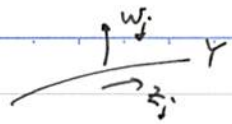
§4. Outline of prf



(i) Fix $\{w_j\}$ as in §2. $\left(T_{jk} \cdot w_k = w_j + \sum_{|\alpha| \geq 2} f_{kj\alpha} \cdot w_j^{\alpha} \right)$

c.f. [Ueda '83]

① Solve a "schröder type functional equat'n"

$$W_j = v_j + \sum_{|\alpha| \geq 2} \underbrace{F_{j,\alpha}(z_j)}_{= \begin{pmatrix} F^1_{j,\alpha} \\ \vdots \\ F^r_{j,\alpha} \end{pmatrix}} \cdot v_j^{\alpha} \quad \cdots (\star).$$

to construct $\{v_j\}$

with $\begin{cases} v_j : \text{def. func of } U_j \text{ in } V_j \\ v_j = T_{jk} \cdot v_k \quad \text{on } V_{jk}. \end{cases}$

② Existence of "nice" $\{F_{j,\alpha}\}_{|\alpha| = n+1} \Leftarrow u_n = 0$

$(\Leftarrow \underset{= \infty}{\text{type}(Y, X)})$

③ Convergence of $X \longmapsto X + \sum_{\alpha} \| F_{j,\alpha} \|_{L^\infty} \cdot X^{\alpha}$

$\Leftarrow$ "torsion" $N_{Y/X}$: or "Proph."

$X \supset$ hyp.surf. $S \supset Y$, $N_{Y/S}$: unitary ~~flat~~

(ii) $N_{Y/S}$: unitary flat. $\rightsquigarrow$ $N_{Y/X}$, $N_{S/X}|_Y \to 0$: split.

⊙ $\rightsquigarrow$ One can take an initial system $\{w_j\}$ so that

$\begin{cases} \{w_j^1 = 0\} = S \cap V_j \quad \text{in } V_j. \\ T_{jk} = \begin{pmatrix} U(1) & 0 \\ 0 & U(r-1) \end{pmatrix} \end{cases}$

↓ solve func. eq. $(\star)$

Fact the solut'n $\{v_j\}$ satisfies $\{v_j^1 = 0\} = S \cap V_j$

$(+ \ T_{jk} v_k^0 = v_j^0)$

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