On some analogues of Ueda theory and their applications.

<u>Goal</u>: Pose analogues of <u>Ueda theory</u> in two manners; — [Ueda '83], [Ueda '91]

(1) Singular Ueda theory. — Arxiv 1507.00109.
(2) Codim-2 Ueda theory. — Arxiv 1412.2354, to appear in Math. Z.

<u>Ueda theory</u>; --- Studying cpx analytic properties of $\underset{\sim}{C} \subset \underset{\sim}{X}$ (a nbhd of; cpx mfd)

$\boxed{\text{s.t. } c_1(N_{C/X})=0}$ where
$\parallel$
$[C]|_C$

- C: sm cpt kä hyp. surf ([Ueda '83])
- C: a rat'l curve with a node ([Ueda '91]) and $\dim X = 2$.

<u>Schedule</u>

§1. Notations and a quick review for Ueda theory
§2. Singular Ueda theory.
§3. Codim-2 Ueda theory.
~~§4. Applications to (non-)semi-positivity of nef line bdls.~~

§1 <u>Notations</u>;

Y: cpx mfd, (var) <u>smooth</u> (kä) or <u>curve with only nodes</u> (reduced).

$$\mathcal{P}(Y) := \{\text{top. triv. hol. line bdls}/Y\}/\sim_{hol.}$$
$$\cup$$
$$\mathcal{P}_0(Y) := \{\text{flat l.b.s}/Y\}/\sim_{hol}$$
$$\cup \quad \underset{\text{fact}}{\equiv} \{\text{flat l.bs}/Y\}/\sim_{flat} := H^1(Y, U(1))$$
$$(U(1) = \{z\in\mathbb{C} \mid |z|=1\})$$

$$\mathcal{E}_0(Y) := \{L\in \mathcal{P}_0(S) \mid \exists n\geq 1,\ L^n = \mathbb{1}_Y\}$$
$$\mathcal{E}_1(Y) := \{L\in \mathcal{P}_0(S) \mid \log \underset{\text{Euclidean dist. on } \mathcal{P}_0(Y) \text{ (inv.)}}{d}(\mathbb{1}_Y, L^n) = O(\log n) \text{ as } n\to\infty\}$$

<u>Rmk</u>

- Y: smooth $\Rightarrow \mathcal{P}(Y) = \mathcal{P}_0(Y)$
- Y: with only nodes $\Rightarrow \mathcal{P}(Y) = \mathrm{Im}(H^1(Y,\mathbb{C}^*) \to H^1(Y, \mathcal{O}_Y^*))$
- $\mathcal{E}_1(Y) = \bigcup_{\nu=0}^{\infty} \exists \underset{\sim}{F_\nu}$ ← nowhere dense closed subset of $\mathcal{P}_0(Y)$, $\underset{\text{Lebesgue measure}}{\mu}(\mathcal{P}_0(Y)\setminus\mathcal{E}_1(Y))=0$

<u>e.g.</u> C: a rat'l curve with a node.

$$\Rightarrow \begin{cases} \mathcal{P}(C) \cong \mathbb{C}^* \\ \mathcal{P}_0(C) \cong U(1) \\ \mathcal{E}_0(C) \text{ --- pts with rat'l angle.} \end{cases}$$

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Let $S \subset \underset{\sim}{X}_{\text{cpx mfd}}$: sm cpt kä hyp. with $c_1(N_{S/X}) = 0$.

Def. (Ueda '83) ◎ V : a nbhd of S in X (small tub.), $\rightsquigarrow H^1(S, U(1)) \cong H^1(V, U(1))$, $N_{S/X} \longmapsto \underset{\sim}{\tilde{N}}$.

$$\mathrm{type}(S, X) := \max\left\{ n \in \mathbb{N}_{\geq 1} \;\middle|\; {}^{\forall}\nu \leq n,\ \mathcal{O}_V(S) \otimes \mathcal{O}_V/\mathcal{O}_V(-(\nu+1) C) \cong \mathcal{O}_V(\tilde{N}) \otimes \mathcal{O}_V/\mathcal{O}_V(-(\nu+1) C) \right\}$$ //

Thm A (Ueda '83)

Let $\underset{\sim}{S} \subset \underset{\sim}{X}_{\text{cpx mfd.}}$ (sm. cpt kä hyp. surf.) be s.t. $c_1(N_{S/X}) = 0$, $\mathrm{type}(S, X) = \infty$.

Assume $N_{S/X} \in \mathcal{E}_0(S) \cup \mathcal{E}_1(S)$

Then $[S]|_V \cong \tilde{N}$ holds on V (by shrinking V) //.

Thm B (Ueda '83 + '91)

Let $C \subset \underset{\sim}{X}_{\text{cpx mfd}}$ ($\dim X = 2$) be a cpt curve s.t. $c_1(N_{C/X}) = 0$.

Rmk: $\mathrm{type}(C, X) = \infty$ in this case. $\Uparrow$ $H^1(C, N_{C/X}^{-n}) = 0$.

Assume C : sm, and $\mathrm{type}(C, X) = n < \infty$ (resp. C : a rat'l curve with a node.) $N_{C/X} \notin \mathcal{P}_0(C)$

Then ◎ $\exists$ a str. psd concave nbhd of C in X.

◎ ${}^{\forall}a \in (0, n) \subset \mathbb{R}$, ${}^{\forall}\Psi$: psh. on $V \setminus C$ (V nbhd of C in X)
(resp. ${}^{\forall}a \in (0, 1) \subset \mathbb{R}$)

$\Psi(p) = o(\mathrm{dist}(p, C)^{-a})$ (resp. $o((-\log \mathrm{dist}(p, C))^{2a})$)

$\Longrightarrow \Psi \equiv \mathrm{const.}$ around C. //

§2 Singular Ueda theory. --- $\boxed{\text{Setting}}$ X : sm. surf $\cup$ C : cpt curve with only nodes.

$\boxed{\text{Main results}}$:

Thm 2.1 Assume.
- $\mathrm{type}(C, X) = \infty$.
- $N_{C/X} \in \mathcal{E}_0(C) \cup \mathcal{E}_1(C)$
- $i^* N_{C/X} \in \mathcal{E}_0(\tilde{C})$, where $i : \tilde{C} \to C$: normalization
- $H^1(C, \mathcal{O}(N_{C/X}^{-n})) = 0$ for ${}^{\forall}n \in \mathbb{Z}$.

Automatically holds if C : cycle of rat'l curves. $\gamma, \delta, \alpha, \cdots$

Then $[C]$: flat around C (i.e. "$[C]|_V \cong \tilde{N}$")

Thm 2.2 "Thm B" holds also for ....

- C with tree dual graph, Ψ with $\Psi(p) = O(\mathrm{dist}(p, C)^{-a})$ $(0 < a < \mathrm{type}(C, X))$
- C with cycle dual graph, $N_{C/X} \in \mathcal{P}(C) \setminus \mathcal{P}_0(C)$, and $\mathrm{type}(C, X) \geq 4$, Ψ with $\Psi(p) = o((-\log \mathrm{dist}(p, C))^{-2a})$ $(0 < a < 1)$

Application

Cor 2.3. $C \subset X$ sm. surf., $(C^2) = 0$.
C: cycle of rat'l curves.

(1) $N_{C/X} \in \mathcal{E}_1(C) \Rightarrow$
- C has a hess-flat nbhd system.
- $[C]$ admits C^∞ Herm. metric with s.p. curvature.

(2) $N_{C/X} \notin \mathcal{P}_0(C) \Rightarrow |f_C|^{-2}$: sing. Herm. metric on $[C]$ with s.p. curv., with min. sing.

($f_C \in H^0(X, [C])$: can. section) //

Rmk
(1) $\Leftarrow$ (Thm A) arguments in [Brunella '10], or [K-'13].
(2) $\Leftarrow$ Thm B + arguments in [K-'14].

E.g. $\{P_j\}_{j=1}^{9} \subset \mathbb{P}^2$: 9 points, $P_j \neq P_k$ for $\forall j \neq k$.

$$\uparrow \pi$$

$$X := Bl_{\{P_j\}_{j=1}^9} \mathbb{P}^2.$$

Take $C_0 \subset \mathbb{P}^2$: curve of deg $= 3$ s.t. $\{P_j\} \subset C_0$.
$C := (\pi^{-1})_* C_0$.

$\rightsquigarrow$ We can determine a minimal singular metric of K_X^{-1} except the case where C_0 : with only nodes and $N_{C/X} \in \mathcal{P}_0(C) \setminus (\mathcal{E}_0(C) \cup \mathcal{E}_1(C))$ //

Rmk
- [Ueda '83], [Brunella '10] ... K_X^{-1} : s.p. if C_0 : smooth, $N_{C/X} \in \mathcal{E}_0 \cup \mathcal{E}_1$.
- Demailly's question. Assume C_0 : smooth. How does [~~minl. sing metric~~ semi positivity of K_X^{-1}] depend on $N_{C/X} \in \mathcal{P}(C) \setminus \mathcal{P}_0(C)$?
- Cor 2.3.(2) $\Rightarrow$ [C_0 : cycle of rat'l curves, $N_{C/X} \notin \mathcal{P}_0(C) \Rightarrow K_X^{-1}$: not s.p.].

Q.E.D.

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§3. Codim-2 Ueda theory

Setting ; $\underset{\text{sm. cpt Kä hyp. surf of } S}{C} \subset \underset{\text{sm. hyp surf of } X}{S} \subset \underset{\text{cpx mfd}}{X}$ s.t. $N_{S/X}$: flat around C

codim 2

◎ W : a (small tub.) nbhd of C in X,

$\widetilde{N}$: flat ext. of $N_{S/X}|_C$ to W.

Main interest ; When does $\widetilde{N} \cong [S]|_W$ hold ?

Def type(C,S,X) :=

$$\max\left\{ \underset{\in \mathbb{Z}^2}{(n,m) \geq (1,0)} \;\middle|\; \begin{array}{l} {}^\forall(\nu,\mu) \lneq (n,m), \\ \left[\mathcal{O}_W([S]) \otimes \mathcal{O}_W/\mathcal{O}_W(-(\nu+1)S')\right]\big|_{S'} \otimes \mathcal{O}_C/\mathcal{O}_C(-(\mu+1)C) \cong \left[\mathcal{O}_W(\widetilde{N})\otimes\cdots\right]\big| \end{array} \right\}$$

$S' := S \cap W$

$(n,m) \geq (n',m')$ $\overset{\text{def}}{\iff}$ $n \gneq n'$ or $n = n'$ and $m \geq m'$.

Thm 3.1 (C,S,X) : as above, type$(C,S,X) = \infty$.

Assume (1) $N_{C/S} \in \mathcal{E}_0(C)$ and $N_{S/X}|_C \in \mathcal{E}_0(C)$

or (2) $N_{C/S}$, $N_{S/X}|_C \in \mathcal{E}_1(C)$, $N_{C/X} \cong N_{S/X}|_C$.

or (3) $N_{S/X}|_C \in \mathcal{E}_0(C)$, $C \subset S$: ~~nega~~ exceptional in the sense of Grauert.

Then $\widetilde{N} \cong [S]|_W$ holds (by shrinking W)

Application

$Q_0, Q_1 \subset \mathbb{P}^3$: general quadric surfaces.

$\rightsquigarrow$ $C_0 := Q_0 \cap Q_1$: sm. ellipt. curve,

$p_1, \cdots, p_8$: general 8 points.

$\rightsquigarrow$ $X := Bl_{\{p_j\}_{j=1}^8} \mathbb{P}^3 \xrightarrow{\pi} \mathbb{P}^3$. $C := (\pi^{-1})_* C_0$.

$S := (\pi^{-1})_* Q_0$

◎ $N_{C/S} \in \mathcal{E}_0(C) \iff K_X^{-1}$: semiample ($\implies K_X^{-1}$: semi-positive)

◎ $N_{C/S} \in \mathcal{E}_1(C) \implies$ ◎. Apply Thm A to (C,S) $\rightsquigarrow$ $\underset{N_{S/X}}{[C]}$: flat around C.

◎ Apply Thm 3.1 (2) to (C,S,X)

$\rightsquigarrow$ $\underset{K_X^{-1}}{[2 \cdot S]}$: flat around C

$\xrightarrow{\text{arguments in [K-, '13]}}$ K_X^{-1} : semipositive //.