京都多変数函数論セミナー

2016/5/17(火), 13:30〜 @109.

正則直線束の半正値性と法束が平坦なコンパクト部分多様体の近傍について.

X : cpx mfd, projective (カンタンのため).
↑
L : hol. line bdl.

Def L : semi-positive (s.p.)

$\overset{\text{def}}{\iff}$ $^{\exists}h$: C^∞-Herm. metric with s.p. curvature. $\sqrt{-1}\Theta_h \geq 0$. //

交点数理論的な判定条件 :

L : s.p. $\Longrightarrow$ L : nef.

$\Updownarrow$ def

$^{\forall}C \subset X$: cpt curve,

$(L.C) := \int_C \sqrt{-1}\Theta_h \geq 0$ //

Question 1

Lを nef line bdl とする.

このとき, いつ Lは s.p. となるか? //

Rmk

◎ L : ample $\overset{\text{小平の埋め込み定理}}{\Longleftrightarrow}$ L : positive (とくに s.p.)

中井-Moishezon 判定法

$^{\forall}Z \subset X$: cpt analytic sub. dim = $d \geq 1$,

$(L^d.Z) := \int_Z c_1(L)^d > 0$ //

◎ L: semi-ample (s.a) $\Longrightarrow$ L: semi-positive.

$\Updownarrow$ def

$m \gg 1$ で $\Phi_{|L^m|} : X \longrightarrow \mathbb{P}^N$ 射になる. (i.e. $H^0(X, L^m)$ の元が共通ゼロをもたない). //

◎ まとめると.

ample (positive) $\Longrightarrow$ s.a $\Longrightarrow$ s.p. $\Longrightarrow$ nef.

◎ 一方. L が nef でも s.p. とは限らない.

Example 2 (Demailly-Peternell-Schneider '94)

C: sm. ellipt. curve,

$0 \to \underset{\text{hol. triv. line bdl}/C}{\mathbb{1}_C} \longrightarrow E \longrightarrow \mathbb{1}_C \longrightarrow 0$: non-splitting.

$X := \mathbb{P}(E) \xrightarrow{\pi} C$.

$\cup$

$Y := (\text{the sect'n of } \pi)$

$L := [Y] \rightsquigarrow (L.Y) = (Y.Y) = \deg \underset{\mathbb{1}_Y}{N_{Y/X}} = 0 \geq 0$

$\rightsquigarrow L$: nef

fact h: singular Herm. metric on L, ← i.e. $\exists h_\infty$: C^∞-Herm. metric on L $\exists \chi : X \to \mathbb{R}: L^1_{loc}$. $h = e^{-\chi} \cdot h_\infty$.

$\sqrt{-1}\,\Theta_h \geq 0 \Longrightarrow \exists A > 0$, $h = A \cdot |f_Y|^{-2}$ //

$\left(f_Y \in H^0(X, [Y]) : \text{can. sect'n}\right)$

$\left(\rightsquigarrow \sqrt{-1}\,\Theta_h := \sqrt{-1}\,\partial\bar\partial \chi + \sqrt{-1}\,\Theta_{h_\infty}\right)$

又は. local weight が psh

as current. //

$\left(\sqrt{-1}\,\Theta_{|f_Y|^{-2}} = +\sqrt{-1}\,\partial\bar\partial \log |f_Y|^{+2} \underset{\text{formula}}{\overset{\text{Poincaré-Lelong}}{=}} [Y] \geq 0\right)$

Example 1 で, nef line bdl $L \to X$ は

$Y \subset X$ について. $\begin{cases} L|_{X \setminus Y} : \text{s.p.} & (\Leftarrow |f|^{-2} ; C^\infty \text{ on } X \setminus Y) \\ L|_Y = N_{Y/X} = \mathbb{1}_Y : \text{s.p.} \end{cases}$

本当は SB を考えるべきだが カンタンのため.

しかし. L: not s.p.

Question 3 一般に nef line bdl $L \to X$ として.

$Y = \mathrm{Bs}|L| := \{f = 0 \mid f \in H^0(X, L)\}$

が (たとえば) curve であるときに.

L の (non-)semi-positivity はどのように判定できるか? //

Rmk Question 3 の設定では, $f_1, \cdots, f_N$: basis of $H^0(X, L)$

として $h_{\text{Bergman}, L} := \left(\sum_{j=1}^{N} |f_j|^2\right)^{-1}$ は L の sing. Herm.

metric, with s.p. curvature, $X \setminus Y$ 上 C^∞.

とくに $L|_{X \setminus Y}$: s.p. また, L: nef より

$\deg L|_Y \geq 0 \rightsquigarrow L|_Y$: s.p //

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§1. Question 3 v.s. the nbhd str. of Y.

§2. 上田理論とその一般化に向けて.

§1. 以下(簡単のため), 主に $\dim X = 2$ で,
かつ $Y := Bs|L|$ が non-singular curve なる
場合を考える.

目標 … Y-nbhd の構造から, L の (non-) semi-positivity
を判定する.

c.f. Example (後述, $\mathbb{P}^2$ の 9点 b-up, Arnol'd - Ueda - Brunella)

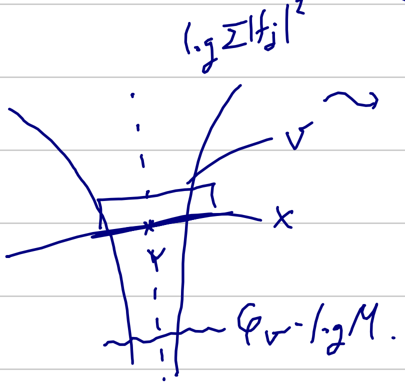
Idea ① … "(Regularized) minimum construction"
Idea ② … L の singular Hermitian metric から $X \setminus Y$ 上 psh を構成.

Idea ① について

Assume $\exists V$: Y- nbhd $\subset X$.
$\exists h_V$: C^∞ Herm. metric on $L|_V$ with s.p. curvature.

idea : $h := \min\{M \cdot h_V,\ h_{Bergman, L}\}$ を考える. $(M \gg 1)$.

$$h \overset{\text{localに}}{=} e^{-\varphi} \text{ とする}, \quad M \cdot h_V \overset{\text{localに}}{=} e^{+\log M - \underset{psh}{\varphi_V}}, \quad h_{Bergman, L} = e^{-\underbrace{\log \sum_{j=1}^{N} |f_j|^2}_{\text{psh, } Y \text{でのみ} -\infty}}$$



$\rightsquigarrow \varphi = \max\{-\log M + \varphi_V,\ \log \Sigma |f_j|^2\}$,
: $M \gg 1$ で ∂V - nbhd では $\underbrace{\text{こちら}}$ が max を実現.

$\rightsquigarrow h$ は bdd Herm. metric on L.

③ $\sqrt{-1}\,\Theta_h = \sqrt{-1}\,\partial\bar\partial\,\underset{\text{psh.}}{\varphi} \geq 0 \rightsquigarrow h$: semi-positive curvature をもつ.

④ "max" のかわりに "Regularized max" を用いれば,
h は C^∞ Herm. metric with s.p. curvature, とできる.

まとめると, $\exists V$; Y-nbhd, $\exists h_V$; $L|_V$ の C^∞ Herm. metric with s.p. curv.
$\Rightarrow L$: s.p. //

Example 4 (Arnol'd['77] – Ueda['83] – Brunella['10])

Fix $C \subset \mathbb{P}^2$; sm. curve of deg $=3$ (ellipt. curve).
$\{P_1, P_2, \cdots, P_9\}$; 9 pts. ($\{P_1,\dots,P_9\} \subset C$)

$X := Bl_{\{P_1,\cdots,P_9\}}\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$.

$Y := (\pi^{-1})_* C \subset X$. $\rightsquigarrow (Y.Y) = 3^2 - 9 = 0$.

$L := [Y]$ …… nef.

Known
- $N_{Y/X} \cong \mathcal{O}_{\mathbb{P}^2}(3)|_C \otimes [-P_1 - P_2 - \cdots - P_9]$
- $N_{Y/X} \in \mathrm{Pic}^0(Y)$: torsion $\Rightarrow L$: semi-ample,
- $N_{Y/X}$: non-torsion $\Rightarrow L$: <u>not</u> s.a.

$\Leftarrow$ (Simple computation)

Thm (Brunella '10)

$X \setminus Y$ が cpt curve を含まないとき,
L : s.p. $\Longleftrightarrow$ Y は X 中 psdflat nbhd をもつ //

$\Uparrow$

$\exists V$: Y-nbhd. $L = [Y]$ は V 上 "$U(1)$-flat".

$\Uparrow$ [Arnol'd-Ueda.]

$\{P_1, \cdots, P_9\} \subset C$: "general". //

idea ① は
このステップの
一般化とみなせる.

"Zariski's example"

Example 5. (K—'15, Ann. Inst. Fourier (Grenoble))

Fix $C \subset \mathbb{P}^2$: sm. curve of deg $= 3$.

$\{p_1, \cdots, p_{12}\}$: general.

$X := Bl_{\{p_1, \cdots, p_{12}\}} \mathbb{P}^2 \xrightarrow{\pi} C$.

$\cup$

$Y := (\pi^{-1})_* C$, $\rightsquigarrow (Y.Y) = 3^2 - 12 = -3$.

$L := \pi^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes [Y] \rightsquigarrow (L.Y) = 0$, L: nef,

(X, Y, L) : "Zariski's example".

Known ... $\forall m \geq 1$, $Bs|L^m| = Y$. $\rightsquigarrow$ L : not semi-ample. //

Idea ① + $\underbrace{\text{Grauert's thm + Rossi's thm}}_{(\Rightarrow \exists V : Y\text{-nbhd}, \exists h_V, \cdots)}$ (on the nbhd str. of negative subvars)

$\Rightarrow$ L : semi-positive

//

Idea ② ... $L = [Y]$ のとき.

Take
- $f_Y \in H^0(X, [Y])$: can. section.
- h : singular Hermitian metric with s.p. curvature
  (∵ L : nef 故ら必ず存在)

$\rightsquigarrow \Phi : X \longrightarrow \mathbb{R}$

$x \longmapsto -\log|f_Y|_h^2$ を考える.

$\rightsquigarrow$ $\Phi(x) \underset{\text{locally}}{=\!=} \underbrace{-\log|f_Y(x)|^2}_{X \setminus Y\text{では pluriharmonic.}} + \underbrace{\varphi}_{h\text{の local weight.} \leftarrow \underline{\text{psh}}}$.

$\rightsquigarrow$ Φ は $X \setminus Y$ 上 psh, Yでは ($+\infty$ 向きには)
高々 $|\log \underline{d(x, Y)}|$ で発散.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ↑ local Euclidean dist.

とくに h : C^∞ なら,

$$\underbrace{\Phi(x) \sim -\log d(x, Y) \quad \text{as } x \to Y}_{(*)}$$

//

応用

Thm 6 (K—, '15, Kyoto. J. Math, Idea ② + [Ueda '83, Thm 2])

X : sm. surface,
$\cup$
Y ; sm. cpt curve.
s.t. $c_1(N_{Y/X}) = 0$, (Y, X) : "of finite type"
in the sense of Ueda.
$\Longrightarrow$ $L := [Y]$: not s.p.

//

(§2で後述,
[Ueda '83, Thm 2]
$\rightsquigarrow$ このときは (*) のような
$X \setminus Y$ 上の psh func. はない!)

Rmk Example 2
(D.P.S - e.g.) は
of finite type.
$\rightsquigarrow$ Thm 6 は, Example 2 の一般化とみなせる //

§2. 上田理論とその一般化に向けて.

[Ueda '83]

<u>Setting</u> X : sm. cpx mfd.

$\cup$

Y : cpt cpx mfd, sm. $\mathrm{codim}_X Y = 1$.

<u>s.t.</u> $N_{Y/X}$: unitary flat. line bdl //

<u>Def</u> $E \longrightarrow Y$: hol. vect. bdl of rank $= r$.

E : unitary flat $\overset{\text{def}}{\iff}$ $E \in \mathrm{Image}\left(H^1(Y, U(r)) \to H^1(Y, GL_r(\mathcal{O}_Y))\right)$

<u>i.e.</u> 適切に loc. triv. をとることで, trans. matrix を $U(r)$-valued loc. const. function としてとれる //.

<u>Rmk</u> $\{E : \text{unitary flat v.b. of rank } r/_Y\} \ni E$

$\downarrow \uparrow$ 1:1

$\{\rho : \pi_1(Y, *) \longrightarrow U(r) : \text{g.p. hom}\} / \sim \ \ni \rho$

$\left(\rho \sim \rho' \overset{\text{def}}{\iff} \exists A \in GL_r(\mathbb{C}) \quad \rho' = A \cdot \rho \cdot A^{-1}\right)$

$r=1$ では $= H^1(Y, U(1))$ //

$E \mapsto$ monodromy ρ_E (!!)

$E_\rho := \tilde{Y} \times \mathbb{C}^r / \sim_\rho \ \leftarrow \rho$

$\tilde{Y}$: univ. cov,

$(z, w^1, w^2, \cdots, w^r) \sim_\rho (\gamma z, \rho(\gamma)\cdot(w^1, \cdots, w^r))$

$(\gamma \in \pi_1(Y, *))$

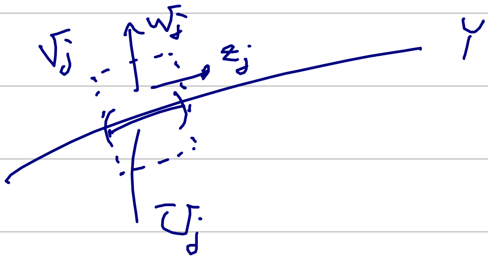
Ueda's obstruction classes

(Y, X) : as above, $\{U_j\}$: open cov. of Y,

V_j ; U_j-nbhd $\subset X$.

z_j : coord. of U_j,

w_j : def. func of U_j in V_j



$N_{Y/X}$: unitary-flat

$\rightsquigarrow$ $V_{jk} := V_j \cap V_k$ 上 $dw_j|_{U_{jk}} = {}^{\exists}t_{jk} \cdot dw_k|_{U_{jk}}$ とできる.

$t_{jk} \in U(1)$

$\rightsquigarrow$ $t_{jk} \cdot w_k = w_j + O(w_j^2)$

Def $\{w_j\}$: of type n $(n \geq 1)$

$\overset{\text{def}}{\Longleftrightarrow}$ $t_{jk} \cdot w_k = w_j + O(w_j^{n+1})$

$= w_j + f_{kj, n+1}(z_j) \cdot w_j^{n+1} + \cdots\cdots$

Def $u_n(Y, X) := [\{(U_{jk}, f_{kj,n+1})\}] \in H^1(Y, N_{Y/X}^{-n})$

: "n-th obstr. class"

Fact 7

① $u_n(Y,X)$は, type n の system $\{w_j\}$ のえらび方に, up to "$\times$ ($\mathbb{C}^*$の元)" でしか依存しない.

② $u_n(Y,X) = 0 \Longleftrightarrow {}^{\exists}\{w_j\}$: system of type $n+1$

$\rightsquigarrow$ ${}^{\exists}n$, type n の system が存在し, $u_n(Y,X) \neq 0$

or

${}^{\forall}n \geq 1$, type n の system が存在し, $u_n(Y,X) = 0$

type$(Y,X) := n$.

type$(Y,X) := \infty$

[Ueda '83, Thm 1.2]: Xが surface, Yが cpt curve で

$\text{type}(Y,X) < \infty$ のとき.

Yは str. psd concave nbhd sys. をもつ,

+ $X \setminus Y$上の psh func の Yでの発散の仕方について. //

[〃, Thm 3]: $X^{n+1} \supset \underset{\text{cpt}}{Y^n}$, $\text{type}(Y,X) = \infty$ のとき,

$N_{Y/X}$ が torsion か, 又は Diophantine-type condition

"$\exists A>0,\ \forall m \geq 1,\ d(\mathbb{1}_Y, N_{Y/X}^m) \geq (2m)^{-A}$" をみたす

$\Rightarrow$ $\exists$ system $\{w_j\}$ s.t. $w_j = t_{jk} \cdot w_k$ on V_{jk} $\cdots(*)$ //

invariant dist. on $H^1(Y, U(1))$

Rmk

$(*) \iff V := \bigsqcup_j V_j$ 上 $[Y]$: unitary flat.

$\iff \exists \mathcal{F}$: non-sing. hol. foliation s.t. $\begin{cases} Y: \text{a leaf of } \mathcal{F}, \\ \mathrm{Hol}_{\mathcal{F},Y} = \rho_{N_{Y/X}}. \end{cases}$ //

Rmk [Arnol'd '76] ⋯ Yが ellipt. curve のとき.

($\text{codim}_X Y$ は一般?)

model example

とくに $\exists V$: Y-nbhd $\exists \mathcal{F}$: V上 hol. foliation s.t. Y: a leaf of $\mathcal{F}$.

$\rho := \mathrm{Hol}_{\mathcal{F},Y}$; $\left\| \begin{array}{l} \rho(\alpha)(w) = w \\ \rho(\beta)(w) = {}^{\exists}t \cdot w + O(w^2) \\ \quad\quad\quad\quad\quad\ \ \ t \in U(1) \end{array} \right.$

Y

$\Downarrow$ Siegel's iterat'n thm.

tが Dioph. cond. をみたせば: wをとりかえて

$\rho(\beta)(w) = t \cdot w$ へ. //

Problem ◎ Y が singular なときは? ⇝ [Ueda '91], [K– '16]
to appear in Indiana U. Math. J.

◎ $\mathrm{Codim}_X Y > 1$ なるときは? ⇝ [K– '15. Math. Z] ← ミスアリ.
$r :=$
[K–, N. Ogawa '16] ← 修正ベツ) $r=2$

‖ 最近, $r := \mathrm{Codim}_X Y$ が一般のときについて,
[Ueda '83, Thm 3] の類似を得た. ← **設定**: $\underset{\text{cpt, sm}}{Y \subset X}$, $N_{Y/X}$: unitary flat

① $\left.\begin{array}{l} t_{jk} \rightsquigarrow T_{jk} \in U(r) \\ w_j \rightsquigarrow (w_j^1, w_j^2, \cdots, w_j^r) \end{array}\right)$ として, 同じく obstr. class

$u_n(Y, X) \in H^1(Y, \underbrace{N_{Y/X} \otimes S^{n+1} N_{Y/X}^*}_{})$ が定まる //

さらに Fact 7 と同様の主張が成立し, type(Y, X) が考えられる.
このとき, [Ueda '83, Thm 3] と同じ方針で次を得た:

Thm 8 $\underset{\sim \text{cpt}}{Y} \subset X \overset{\text{s.t.}}{,}\ N_{Y/X}$: unitary flat. (Y, X) : of infin. type.

Assume $\#\, \mathrm{Image}(\rho_{N_{Y/X}} : \pi_1(Y, *) \to U(r)) < \infty$,

or $\exists\, \pi : \tilde{Y} \to Y$: finite normal covering

s.t. $\pi^* N_{Y/X} = \bigoplus_{\lambda=1}^{r} {}^{\exists}\underset{\sim}{L_\lambda}$ unitary flat line bdl$/\tilde{Y}$,

$\exists A > 0.\ \forall \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_r) \in \mathbb{Z}^r$,

$|\alpha| := \alpha_1 + \cdots + \alpha_r \geq 1$

$\Rightarrow d(\mathbb{1}_{\tilde{Y}}, \bigotimes_{\lambda=1}^{r} L_\lambda^{\alpha_\lambda}) \geq (2|\alpha|)^{-A}$

Then

(i) ${}^{\exists}V$: Y-nbhd, ${}^{\exists}\mathcal{F}$: non-sing. hol. foliatn of codim $= r$.

s.t. $\begin{cases} Y : \text{a leaf of } \mathcal{F} \\ \mathrm{Hol}_{\mathcal{F}, Y} = \rho_{N_{Y/X}} \end{cases}$

(ii) S : sm. hyp.surf. $\subset V$ s.t. $\begin{cases} Y \subset S \\ N_{Y/S} : \text{unitary flat.} \end{cases}$

$\rightsquigarrow$ 必要ならば Vを小さくして,

${}^{\exists}\mathcal{G}_S$: V上 non-sing. hol. foliatn. of codim $= 1$,

s.t. $\begin{cases} S \cap V : \text{a leaf of } \mathcal{G}_S \\ \mathrm{Hol}_{\mathcal{G}_S, S\cap V} = \rho_{\underset{\sim}{N_{S/X}|_Y}} \\ \mathcal{G}_S : \mathcal{F}\text{-inv} \end{cases}$

//

$0 \to N_{Y/S} \to N_{Y/X} \to \underset{\sim}{N_{S/X}|_Y} \to 0$

: split, これも unitary flat //

Rmk Thm 8 (ii) は. とくに S が π-inv. と言っている.

たとえば $X = Y \times \mathbb{C}^r \xrightarrow[\pi := pr_2]{} \mathbb{C}^r$.

$\cup$

$Y = Y \times \{0\}$ のときは, ...

$V = \pi^{-1}(\Omega)$ (Ω : $\mathbb{C}^r$ 中 0-nbhd.)

$\cup$

S : hyp. surf, $N_{Y/S}$! unitary flat とする.

$(\overset{\cup}{Y})$

$[S]|_Y = [S]|_S|_Y = N_{S/X}|_Y$: unitary flat.

$\rightsquigarrow$ $c_1([S]|_Y)$: top. triv.

$\rightsquigarrow$ $c_1(\underbrace{[S]|_{\pi^{-1}(x)}})$: — " — for $\forall x \in \Omega$.

$\|$ もし $\pi^{-1}(x) \not\subset S$ なら.

[$\underbrace{\text{effective divisor}}$] "$\pi^{-1}(x) \cap S$".

$\|$ $\underline{Y\text{: proj. であるから}}$ top. triv. より

0 i.e. $\pi^{-1}(x) \cap S = \phi$. //

応用

Cor 9 $L \longrightarrow \underset{\substack{\text{cpx mfd} \\ \dim = n}}{X}$: line bdl.

Assume $\exists D_1, D_2, \cdots, D_{n-1} \in |L|$

s.t. $\left| \begin{array}{l} C := \bigcap D_\nu \ : \text{sm. ellipt. curve.} \\ D_\nu \text{ は } C \text{ で transv. に交わる}, \\ L|_C \ : \text{Diophantine} \end{array} \right.$

Then L ! s.p //