Non-Kummer K3 surfaces with Levi-flat hypersurfaces.

9:30 – 10:30.

cut

Def X : cpt cpx surf.

X : K3 $\underset{\text{def}}{\Longleftrightarrow}$ $\pi_1(X) = 0$, $\exists \sigma$: nowhere vanishing hol. 2-form on X.

Def $\underset{(C^\infty)\text{ real hypersurf.}}{H} \subset \underset{\text{cpx mfd}}{X}$: (C^∞) Levi-flat $\underset{\text{def.}}{\Longleftrightarrow}$ $\exists \mathcal{F}$: C^∞ foliation on H of $\mathrm{codim}_{\mathbb{R}} = 1$, each leaf is a hol'ly immersed cpx submfd. of X.

Thm 1 $\exists X$: K3, not a Kummer surf,

s.t. $\exists \{H_t\}_{t \in I}$ ($I \subset \mathbb{R}$: interval) : C^∞ family of Levi-flat hyp. surfaces of X.

$\forall t$, $H_t \approx_{\text{diffeo}} S^1 \times S^1 \times S^1$,

each leaf of H_t is $\begin{cases} \text{dense in } H_t. \\ \text{bihol. to } \mathbb{C} \text{ or } \mathbb{C}^*. \end{cases}$ //

Cor 2 $\exists X$; K3, not a Kummer surf,

$\exists f : \mathbb{C} \to X$: hol,

s.t. $\begin{cases} \overline{f(\mathbb{C})}^{\text{Euc}} \text{ ; real hyp. surf of } \mathrm{codim}_{\mathbb{R}} = 1, \subsetneq X. \\ \overline{f(\mathbb{C})}^{\text{Zar}} = X \end{cases}$ //

⊙ We will construct such X by patching two open complex surfaces M and M' holomorphically.

⊙ $M = (\exists 9\text{pts b-up of } \mathbb{P}^2) \setminus (\text{a nbhd of an ellipt. curve})$

$M' = (\exists 9\text{pts } \quad '' \quad) \setminus (\quad '' \quad)$.

Thm 3 $\exists \mathcal{X} \xrightarrow{\pi} \underset{\text{18 dim'l cpx mfd.}}{B}$: deformation family of K3 surfaces

s.t. $\begin{cases} \dim B = 18, \\ X_t \text{ : as in Thm 1, Cor 2 } (X_t := \pi^{-1}(t)) \text{ for } \forall t \in B. \\ \text{The Kodaira-Spencer map } \rho_{KS,\pi} : T_B \to R^1\pi_* T_{\mathcal{X}/B} \text{ is inj.} \end{cases}$ //

§1. Motivation from Arnold's and Ueda's results.

§2. Construction of a K3 surface.

§3. (Outline of) Prf of Thm 1, ~~Cor 2~~, ~~Thm 3~~ Thm 3.

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§1. Interest ⋯ cpx analytic str. of a nbhd of a cpt cpx curve C embedded in a cpx surf S. with $(C^2)=0$

cut

① $(C^2)=\deg N_{C/S}<0 \underset{[\text{H.Grauert '62}]}{\Longrightarrow} \exists$ str. psd convex nbhds sys. of C.

② $(C^2)>0 \underset{[\text{O.Suzuki '75}]}{\Longrightarrow} \exists$ str. psd concave nbhds sys. of C.

When $(C^2)=0$, ⋯

Thm 4 (Arnol'd '76)

C: ellipt. curve and $N_{C/S}\in \mathrm{Pic}^0(C)$: Diophantine.

i.e. $-\log d(\mathbb{1}_C, N_{C/S}^{\otimes n}) = O(\log n)$ as $n\to\infty$

$\Rightarrow \exists V$: a nbhd of C in S,

$\exists \mathcal{V}$: a nbhd of (0-section) $\subset N_{C/S}$.

s.t.

$$\begin{array}{ccc} V & \underset{\text{bihol}}{\cong} & \mathcal{V} \\ \cup & & \cup \\ C & \cong & \text{0-sect'n} \end{array}$$

//.

([T. Ueda '83] ⋯ Classification of (C,S) with $(C^2)=0$.)

Ueda's classification (Outline)

(C,S) with $(C^2)=0$.

- "finite type case", or "type(α)"
  ⋯ the case where "$(C\text{-nbhd in } S) \underset{\text{formally along } C}{\not\cong} \left(\begin{smallmatrix}\text{0-sect'n}\\ \text{-nbhd in } N_{C/S}\end{smallmatrix}\right)$"
  (i.e. $\exists$ N-jet level)
- "infinite type case"
  ⋯ the case where "$(\text{—''—}) \underset{\text{formally along } C}{\cong} (\text{—''—})$"
  - $N_{C/S}$: torsion in $\mathrm{Pic}^0(C)$ =: type (β')
  - $N_{C/S}$: non-tor
    - C admits a psd flat nbhds sys. =: type(β'')
    - the other =: type(γ).

30分

§1 ~~A~~. cpx analytic str. of a nbhd of $\underset{\text{cpx curve}}{\underset{\sim}{C}} \subset \underset{\text{cpx surf.}}{\underset{\sim}{S}}$ (sm) (cpt)

with $(C^2) = \deg N_{C/S} = 0$.

Thm 4 (Arnol'd, '76)

C: ellipt. curve, $N_{C/S} \in \mathrm{Pic}^0(C)$: Diophantine

(i.e. $-\log d(\mathbb{1}_C, N_{C/S}^n) = O(\log n)$ as $n \to \infty$)

$\Rightarrow$ $\exists\, V$: a nbhd of C in S

$\exists\, \widetilde{V}$: a nbhd of 0-sect'n $\subset N_{C/S}$.

s.t.
$$\begin{array}{ccc} V & \underset{\text{bihol}}{\cong} & \widetilde{V} \\ \cup & & \cup \\ C & \cong & \text{0-sect'n}. \end{array}$$

~~¶~~ [Ueda, '83] … classification of (C, S) with $(C^2) = 0$.

◎ Summary of the classificat'n and thms in [Ueda '83].

$\boxed{(C, S) \text{ with } (C^2) = 0}$

- $\begin{pmatrix} C\text{-nbhd} \\ \subset S \end{pmatrix} \underset{\text{formally along } C}{\cong} \begin{pmatrix} \text{(0-sect'n)-nbhd} \\ \subset N_{C/S} \end{pmatrix}$
  - $N_{C/S}$: non-tor
    - the other. (type γ)
      - Thm $\exists$ example of type γ
    - C admits ~~a~~ psd flat nbhds syss. (type β'')
      - Thm. $N_{C/S}$: Dioph. ⇒ (type β'')
  - $N_{C/S}$: tor in $\mathrm{Pic}^0(C)$ (type β')
    - Thm ⇒ C admits ~~a~~ psd flat nbhd syss.
- $(\;\;\;\;) \underset{\text{formally along } C}{\not\cong} (\;\;\;\;)$ (type α)
  - Thm ⇒ C admits str. psd concave nbhd sys. (+ …)

Rmks
① In this Ueda's example, S : open!

Example 6 (Arnol'd - Ueda - Brunella)

$C_0 \subset \mathbb{P}^2$: sm. ellipt. curve. $Z = \{p_1, p_2, \cdots, p_9\} \subset C_0$: 9 pts. 30分 55分

$S := Bl_Z \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$, $C := (\pi^{-1})_* C_0$

$\rightsquigarrow N_{C/S} \cong \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \cdots - p_9) \in \mathrm{Pic}^0(C)$

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Rmk in Example 6,
$N_{C/S}$: tor $\Leftrightarrow$ (C,S) : type (β').
$N_{C/S}$: Dioph $\underset{\text{Arnol'd-Ueda}}{\Longrightarrow}$: type (β''), $\exists V$ as in Thm 4.

Q In e.g. 6, $\exists ?$ Z s.t. ~~N_C~~ (C,S) : type (γ) ?

Q In e.g. 6, when $N_{C/S}$: Dioph, what is the max'l V as in Thm 4?

40分けない

§2. Construct'n of X

Fix $\begin{cases} C_0, C_0' \subset \mathbb{P}^2 : \text{sm. ellipt. curves s.t. } \exists g : C_0' \xrightarrow{\simeq} C_0. \\ L \to C_0 : \text{Dioph l.b.} \end{cases}$

Take $\cdot$ $Z = \{P_1, P_2, \dots, P_9\} \subset C_0$ s.t. $N_{C/S} \cong L$,
where $S := Bl_Z \mathbb{P}^2 \supset C :=$ str. transf. of C_0,
$\cdot$ $Z' = \{P_1', P_2', \dots, P_9'\} \subset C_0'$ s.t. $N_{C'/S'} \cong L^{-1}$, where —.

Arnol'd's thm $\rightsquigarrow$ $\exists V$: C-nbhd $\subset S$ s.t. $V \underset{\phi}{\xleftarrow{\simeq}} \{(z, w) \in L \mid \underbrace{|w|}_{h} < R\}$ $(R>1)$
(z: coord of C, w: fiber coord., h: flat metric on L)

$\exists V'$: C'-nbhd $\subset S'$ s.t. $V' \underset{\phi'}{\xleftarrow{\simeq}} \{(z', w') \in L^{-1} \mid |w'| < R'\}$ $(R'>1)$

$\circ$ $M := S \setminus \phi(\{|w| \le \frac{1}{R'}\})$, $M' := S' \setminus \phi'(\{|w'| \le \frac{1}{R}\})$,

$\rightsquigarrow$ Patch M and M'
by using $M \supset W^* := \phi(\{\frac{1}{R'} < |w| < R\})$
$\|$ identify via $(z, w) \sim (\underbrace{g^{-1}(z)}_{z'}, \underbrace{\frac{1}{w}}_{w'})$
$M' \supset W^* := \phi'(\{\frac{1}{R} < |w'| < R'\})$

$\rightsquigarrow$ $X := M \cup_{W^*} M'$: cpt cpx mfd.

Rmk
[Doi '09] … Topologically the same constr. of K3.
(need to deform the cpx str. of M, M')

[Tsuji '84] … Constr. of $(S^3 \times S^3, J)$
by ~~a similar~~ using Arnol'd-type thm.

~~8~~ 50分 くらい

50分 以内

§3 Outline of prf

$$W^* \begin{matrix}\subset M \subset \\ \subset M' \subset\end{matrix} X$$
$\cup$
$H_t := \phi(\{(z,w) \mid |w|_h = t\})$ $(t \in (\frac{1}{R'}, R))$; Levi-flat hypersurface as in Thm1

$\rightsquigarrow$ Enough to show ;

- X : K3. $\Longleftarrow$ Prop 7 + Prop 8.
- X : non-Kummer. $\Longleftarrow$ Thm 3. (fact)

Prop 7 $H_q(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & q=4 \\ 0 & 3 \\ \mathbb{Z}^{22} & 2 \\ 0 & 1 \\ \mathbb{Z} & 0. \end{cases}$

☺ use Mayer-Vietoris seq. for $W^* \begin{matrix}\subset M \\ \subset M'\end{matrix} \subset X$, $W^* \begin{matrix}\subset V \\ \subset M\end{matrix} \subset S$. ...

prop 8 $\exists \sigma$: nowhere vanishing hol. 2-form on X.

☺ η : mero 2-form on S with $\mathrm{div}(\eta) = -C$.
η' : — // — on S' with $\mathrm{div}(\eta') = -C'$.

◎ consider a hol. func $F := \frac{\eta}{(\phi^{-1})^* dz \wedge \frac{dw}{w}} : W^* \longrightarrow \mathbb{C}^*$.

Fact $H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}$. //

$\rightsquigarrow$ $F \equiv \mathrm{const} \in \mathbb{C}^*$. $\rightsquigarrow$ We may assume $\begin{cases} \eta|_{W^*} = (\phi^{-1})^* dz \wedge \frac{dw}{w}. \\ \eta'|_{W^*} = (\phi'^{-1})^* dz' \wedge \frac{dw'}{w'} \end{cases}$

$\rightsquigarrow$ $\sigma := \begin{cases} \eta & \text{on } M \\ \eta' & \text{on } M' \end{cases}$. //

Thm 3 .... ~~fix~~ Fix C_0, C_0', L

Parameters ; $\left.\begin{matrix} P_1, P_2, \cdots, P_8, \\ P_1', P_2', \cdots, P_8', \\ (z, w) \sim (\underbrace{g^{-1}(z) + t}_{z'}, \underbrace{\frac{s}{w}}_{w'}). \end{matrix}\right\}$ 19.

$t \in \mathbb{C}$, $s \in \mathbb{C}$

M.V-seq.

$$\rho\left(\frac{\partial}{\partial t}\right),\ \rho\left(\frac{\partial}{\partial s}\right)$$

$$H^0(M, T_M) \oplus H^0(M', T_{M'}) \to H^0(W^*, T_{W^*})$$

$$\to H^1(X, T_X) \to H^1(M, T_M) \oplus H^1(M', T_{M'}) \to H^1(W^*, T_{W^*})$$

$$T_B \xrightarrow{\rho_{KS}} H^1(X, T_X)$$

$$\frac{\partial}{\partial p_\nu},\ \frac{\partial}{\partial p_\nu'} \mapsto (11)$$

$M \subset S$