Gluing construction of K3 surfaces and Arnol'd's type thm on a nbhd of a curve

(10:30-12:00)

§1. Gluing construction of a K3 surface. X. 1C

§2. Degree of freedom and Period. map.

(§3. Examples and Problems).

j.w/ T.Uehara.

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§1

⊙ $C_0 \subset \mathbb{P}^2$: sm. curve of $\deg = 3$.
$\cup$
$Z := \{P_1, \cdots P_9\}$: nine pts.

$S := Bl_Z \mathbb{P}^2 \xrightarrow[\pi]{} \mathbb{P}^2$; b-up at Z.
$\cup$
$C := (\pi^{-1})_* C_0$; str. transf of C_0.

⊙ (C_0', Z', S', C') ; another model.

Known (K3 surf) $\approx_{C^\infty}$ $S \underset{C=C'}{\#} S'$ "Connected sum"

(if $C \underset{g}{\cong} C'$, $N_{C/S} \cong g^* N_{C'/S'}^{-1}$).

In [K-'17, Arxiv/1703.03663], we carried out this constr. in "holomorphic manner".

In other words, we explicitly described a "smoothing" of the ~~nodal~~ singular K3 surf $S \underset{C \simeq C'}{\cup} S'$.
([~~Fried~~ Friedman'83]) N.C.

when $\begin{cases} C \underset{g}{\cong} C', N_{C/S} \cong g^* N_{C'/S'}^{-1}, \\ N_{C/S} \in Pic^0(C) ; \text{Dioph.} \end{cases}$ //

i.e $\exists A, \alpha > 0$, $\forall n \in \mathbb{Z}_{>0}$, $\mathrm{dist}(\mathbb{1}_C, N_{C/S}^{\otimes n}) \geq A \cdot n^{-\alpha}$.

by using

Thm (Arnol'd. '76) S'' : cpx surf, non-sing
$\cup$
C'' : sm. ellipt. curve, s.t. $N_{C''/S''} \in Pic^0(C'')$; Dioph
$\Rightarrow \exists V''$: $\boxed{\text{hol. tub. nbhd}}$ of C'' in S''. //

(i.e. $\exists \hat{V} \subset N_{C''/S''}$: nbhd of the zero-sec'n s.t. $\underset{\text{the}\,-}{\hat{V}} \cong \underset{=C''}{V''}$) 1D //

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$\exists$ counter e.g. when S'': non-cpt (Ueda).

$\underline{\underline{Q}}$ $\exists$? counter example (S'', C'') for Arnold's thm when $\begin{cases} S'' : \text{cpt}, \\ N_{C''/S''} : \text{non-tor}, \end{cases}$?

$\boxed{\text{Goal of this talk}}$

··· Investigate $\mathcal{E} := \left\{ \begin{array}{l} \text{marked K3 surf.} \\ \text{which can be constructed by the manner as above} \end{array} \right\}$

$\subset \left(\begin{array}{c} \text{marked} \\ \text{K3 moduli} \end{array} \right)$. //

Outline of "gluing construction"

Assume $\begin{cases} \exists g : C \xrightarrow{\sim} C', \\ N_{C/S} \cong g^* N_{C'/S'}, \\ N_{C/S} ; \text{Dioph.} \quad (\Rightarrow N_{C'/S'} : \text{Dioph}). \end{cases}$

Arnold's thm $\rightsquigarrow$ $\exists$ W : a nbhd of C in S.
$\quad = W_{(R)}$ $(\exists R > 0)$

s.t. $W_{(R)} \cong \{ \xi \in N_{C/S} \mid |\xi|_h < R \}$.

where h : flat metric of $N_{C/S}$.

(Known N : top. triv. hol. l.b. / cpt Kähler C.
$\Rightarrow N \in H^1(C, U(1))$, $U(1) := \{ t \in \mathbb{C} \mid |t| = 1 \}$)

By scaling, w.m.a. $R > 1$.

$\circ$ $C' \subset W'_{(R')} \subset S'$: defined in the same manner. $(R' > 1)$

$$M := S \setminus \overline{W_{(\frac{1}{R'})}} \supset W_{(R)} \setminus \overline{W_{(1/R')}}$$

$$\cong \{ \xi \in N_{C/S} \mid \tfrac{1}{R'} < |\xi|_h < R \}$$

$$\cong \text{ via } g$$

$$\{ \xi' \in N_{C'/S'} \mid \tfrac{1}{R} < |\xi'|_{h'} < R' \}$$

$$M' := S' \setminus \overline{W'_{(1/R)}} \supset W'_{(R')} \setminus \overline{W'_{(1/R)}} =: W^*$$

$$X := M \cup_{W^*} M'.$$

Fact

Lem $H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}$.

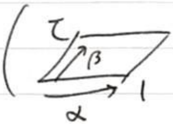
$\rightsquigarrow$ $\exists \eta^{(\prime)}$: mero. 2-form on $S^{(\prime)}$ with $\operatorname{div}(\eta^{(\prime)}) = -C^{(\prime)}$,
″

s.t. $\eta^{(\prime)}|_W = \dfrac{dz \wedge dw}{w}$, (c.f. $K_S = -C$)

where $W \cong \mathbb{C}_z \times \{w \in \mathbb{C} \mid |w| < R\} / \sim$,
$W_{(R)}$

$(z, w) \sim (z+1, e^{2\pi\sqrt{-1}a_\alpha} \cdot w)$

$\sim (z+\tau, e^{2\pi\sqrt{-1}a_\beta} \cdot w)$

( , a_α, a_β; "monodromy" of $N_{C/S}$ along α, β, respectively.)

$\rightsquigarrow$ $\eta|_M$ and $-\eta'|_{M'}$ glue up to define a ~~m~~ nowhere-vanishing hol. 2-form σ on X. ①

Fact $X \cong_{C^\infty} (K3)$ ②

①+② $\Rightarrow$ (X, σ): K3 surf //.

§2. Degrees of freedom and Period map

hol. dim.

① Choice of the pair (C_0, L) of $\begin{cases} C_0: \text{sm. ellipt. curve} \\ L \to C_0: \text{Dioph. l.b.} \end{cases}$ (1)

② Choice of $p_1, p_2, \cdots, p_8 \in C_0$ (8)

③ Choice of C_0', $g: C_0 \xrightarrow{\cong} C_0'$ (1)

④ Choice of $p_1', \cdots, p_8' \in C_0'$. (8)

⑤ Choice of p_9 s.t. $\mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 \cdots - p_9) \cong L$. 0

⑥ ― ″ ― p_9' s.t. $\mathcal{O}_{\mathbb{P}^2(3)}|_{C_0'} \otimes \mathcal{O}_{C_0'}(-p_1' \cdots - p_9') \cong \underset{g^*}{\widehat{L^{-1}}}$. 0

⑦ Scaling of the "fiber coord" w and the depth of the tab for gluing. (1)
(R, R′, rotation of "w")

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$(A^2)=0,\ (B^2)=-2,\ (A.B)=1$

Thm (K-, Uehara)

$\Lambda_{K3} := \langle A_{\alpha\beta}, B_{\beta}\rangle \oplus \langle A_{\beta\gamma}, B_{\alpha}\rangle \oplus \langle A_{\gamma\alpha}, B_{\beta}\rangle \oplus E_8(-1) \oplus E_8(-1)$: K3 lattice,

$\mathcal{D} := \left\{ [x] \in \mathbb{P}(\Lambda_{K3}\otimes\mathbb{C}) \;\middle|\; \begin{array}{l} (x.x)=0 \\ (x.\bar{x})>0 \end{array} \right\}$: Period dom.

$(p,q)\in\mathbb{R}^2$: "Diophantine pair" i.e. $\exists\alpha, A>0$, $\min\{dist(np,\mathbb{Z}), dist(nq,\mathbb{Z})\} \geq A\cdot n^{-\alpha}$.

$\rightsquigarrow \exists\, C = C_{(p,q)} > 0$: const.

s.t. $\exists\ \mathcal{X} \xrightarrow[\pi]{} \left\{ [x]\in\mathcal{D} \;\middle|\; \begin{array}{l} (x.A_{\alpha\beta} + p\cdot A_{\beta\gamma} - q\cdot A_{\gamma\alpha}) = 0 \\ (x.\bar{x}) > C_{(p,q)} \end{array} \right\}$

$\underset{B_{(p,q)}}{\overset{\|}{}}$

: proper hol. surj, ~~with~~ "deform. of K3 surfs",

s.t. Period map $B_{(p,q)} \to \mathcal{D}$ coincides with the natural ~~inj~~ inclusion.

$\forall t\in B_{(p,q)}$, $\pi^{-1}(t) =: X_t$: K3 surface constructed by "gluing constr" //

Cor $\exists X$: K3, with $\rho(X)=0$ ($\Rightarrow X$: non-proj, non-Kummer).

s.t. $\exists F : \mathbb{C}\to X$: hol. inj.

s.t. $\overline{F(\mathbb{C})}^{\,\mathrm{Euc}} \underset{C^\omega}{\approx} S^1\times S^1\times S^1 \subset X$: "Levi-flat hyp. surf",

$\overline{F(\mathbb{C})}^{\,\mathrm{Zar}} = X$ "

Outline of the pf. of "Thm $\Rightarrow$ Cor"

- (p,q) : general, $t\in B_{(p,q)}$: general $\Rightarrow \rho(\underset{X_t}{\overset{\|}{X}}) = 0$.
- $W^* \cong \{\xi\in N_{C/S} \mid \frac{1}{R'} < |\xi|_h < R\}$ admits a hol. foliation induced by the flat conn. $\underset{X}{\overset{\cap}{}}$

Φ each leaf : $\mathbb{C}$ when (p,q) : general. //

Outline of prf. of "Thm"

for some X obtained by gluing (S,C) and (S',C')

Step 1 Concrete description of the marking $A_{\{\}} \subset H_2(X,\mathbb{C})$.

Step 2 — " — calculation of $\int_{\underset{A_{\{\}}}{\uparrow}} \hat{\sigma}$'s, where $\hat{\sigma} := \frac{1}{2\pi\sqrt{-1}}\sigma$.

it turns out that
$\rightsquigarrow$ $\mathcal{E} := \left\{ [\sigma] \in \mathcal{D} \;\middle|\; \begin{array}{l} (X,\sigma) : \text{obtained by "gluing construction"} \\ \text{with } p = a_\alpha,\ q = a_\beta \end{array} \right\}$ — monodromy of $N_{C/S}$.

Satisfies $\begin{cases} \mathcal{E} \neq \phi. \\ \mathcal{E} \subset B_{(p,\varepsilon)} \end{cases}$

Step 3 Show openness and closedness of $\mathcal{E} \subset \underbrace{B_{(p,\varepsilon)}}_{\text{conn.}}$ //

$\boxed{\text{Step 1}}$ and 2 ① A's

take $\tau \in \mathbb{H}$ with $\underset{\underset{S}{\cap}}{C} \cong C_0 \cong \underset{\underset{S'}{\cap}}{C'} \cong C_0' \cong \mathbb{C}/\langle 1,\tau\rangle$

$W^* \longrightarrow C$: annulus bdl.
$\{w \in \mathbb{C} \mid \frac{1}{R} < |w| < R\}$.

$W^* \underset{\text{homotopic}}{\simeq} \underbrace{S^1 \times S^1}_{\substack{\wr\wr \\ C}} \times S^1$. α,β,γ ; corresponding generator (loop) of $\pi_1(W^*)$

Rmk
holonomy of the foliation of W^* ; $\alpha \longmapsto e^{2\pi\sqrt{-1}a_\alpha}$ rotation along γ
$\beta \longmapsto e^{2\pi\sqrt{-1}a_\beta}$ — " — //

$(a_\alpha, a_\beta) \in \mathbb{R}^2$; "Dioph. pair".

$A_{\alpha\beta} := \alpha\times\beta$
$A_{\beta\gamma} := \beta\times\gamma$
$A_{\gamma\alpha} := \gamma\times\alpha$.

$\xrightarrow{\text{concrete calculation}}$

$\int_{A_{\alpha\beta}} \hat{\sigma} = a_\beta - \tau\cdot a_\alpha$

$\int_{A_{\beta\gamma}} \hat{\sigma} = \tau$

$\int_{A_{\gamma\alpha}} \hat{\sigma} = 1.$ //

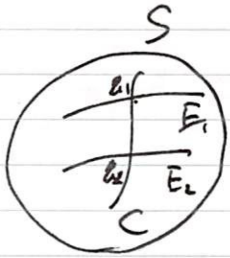
JB

"$\underline{E_8(-1)}$". $q_\nu :=$ (Excep. div. corresp. to P_ν) $\cap$ C

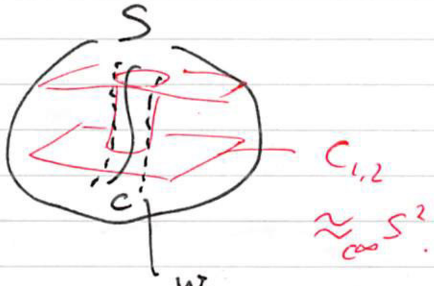
$$\text{(Excep. div. corresp. to } P_\nu) = E_\nu .$$

$C_{1,2} := E_1 - E_2 \;(\in H_2(S, \mathbb{Z}))$

$\in \underline{H_2(M, \mathbb{Z})}$.



$\curvearrowright$ Fix a path $P_{1,2}$ ~~with~~ $\subset C$ from q_1 to q_2



$W(\varepsilon)$

Consider $\underline{\varepsilon \to R}$

$\underline{\text{Rigard}}$ $C_{1,2} \in H_2(X, \mathbb{Z})$.

$\underline{\text{Similarly}}$ Define $C_{2,3}, C_{3,4}, \cdots, C_{7,8}, \underline{C_{6,7,8}}$,

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \wr$

$C'_{1,2}, C'_{2,3}, \cdots, C'_{6,7}, C'_{7,8}, C'_{6,7,8} \quad\; -H + E_6 + E_7 + E_8$.

① $\langle C_\cdot \rangle \cong \langle C'_\cdot \rangle \cong E_8(-1)$ (as lattice).

② $$\int_{C_{1,2}} \hat{\sigma} = \cancel{\int \hat{\sigma}} + \int \hat{\sigma} + \cancel{\int \hat{\sigma}}$$

$\quad\quad\quad\quad \downarrow \varepsilon \searrow 0$

$$\int_{P_{1,2}} dz . \quad (C = \mathbb{C}_z / \langle 1, c \rangle)$$

$\underline{\text{Similarly}}$,
$$\int_{C_{p,p+1}} \hat{\sigma} = \int_{P_{p,p+1}} dz$$
$$\int_{C_{6,7,8}} \hat{\sigma} = \int_{P_6 + P_7 + P_8} dz \quad \left(P_\nu : \text{path from } \underline{\underline{q_0}} \text{ to } q_\nu \right)$$

$\uparrow$ fixed inflection p.t.

$$\int_{C'_{p,p+1}} \hat{\sigma} = \int_{P'_{p,p+1}} \bar{g}^* dz$$
$$\int_{C'_{6,7,8}} \hat{\sigma} = \int_{P'_6 + P'_7 + P'_8} \bar{g}^* dz .$$

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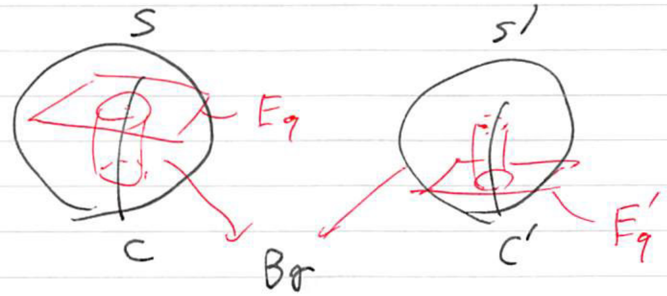
[B.] ② $\pi_1(M^{(\prime)}, *) = 0$.

$\rightsquigarrow$ $\exists D_\alpha^{(\prime)} \subset M^{(\prime)}$: $\approx$ disc. s.t. $\partial D_\alpha^{(\prime)} = \alpha$.
(β) (β)
(γ) (γ)

$\rightsquigarrow$ $B_\alpha := D_\alpha \cup_\alpha D'_\alpha$,
$B_\beta := D_\beta \cup_\beta D'_\beta$,

⊙



$\rightsquigarrow$ $\int_{B_\gamma} \hat{\sigma} = \int_{\Gamma_{q,q'}} dz$, where $\Gamma_{q,q'}$: path $\subset C$ from p_q to $g^{-1}(p'_q)$.

Q
$\int_{B_\alpha} \hat{\sigma} = ?$
$\int_{B_\beta} \hat{\sigma} = ?$ "

From ~~those~~ calculation above, we have that

$H^{2,0}(X) \hookrightarrow H^2(X, \mathbb{C}) \overset{\text{dual}}{\cong} H_2(X, \mathbb{C})$

$\sigma \longmapsto a_{\alpha\beta} \cdot A_{\alpha\beta} + a_{\beta\gamma} \cdot A_{\beta\gamma} + a_{\gamma\alpha} A_{\gamma\alpha} + b_\alpha \cdot B_\alpha + b_\beta \cdot B_\beta + b_\gamma \cdot B_\gamma + \sum c_\cdot \cdot C_\cdot + \sum c'_\cdot \cdot C'_\cdot$,

where
$b_\alpha = \int_{A_{\beta\gamma}} \cdot \sigma = \tau$. "choice of C_0, C'_0"
$b_\beta = \int_{A_{\gamma\alpha}} \cdot \sigma = 1$ "normalization"
$b_\gamma = \int_{A_{\alpha\beta}} \cdot \sigma = a_\beta - \tau \cdot a_\alpha$ "choice of $L \in N_{\alpha\beta}$)"
$C_\cdot, C'_\cdot$ ... determined uniquely by the choice of $\{P_1, \dots, P_8$; $P'_1, \dots, P'_8\}$.
$(\gamma \to \alpha)$

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(つづき) $a_{\alpha\beta} = \int_{B_\gamma} \hat{\sigma} + 2\cdot l_\gamma$ ··· determined by the choice of $g: C \overset{\sim}{\rightarrow} C'$ ③ ①

$\rightsquigarrow$ $x := a_{\beta\gamma}$,
$y := a_{\gamma\alpha}$: determined by the ~~stomed~~. ⑦ ①

"Scaling of the fiber coord's w, and the depth ~~for~~ of the tab for glue".

Rmk $\begin{cases} (\sigma.\sigma) = 0 \iff \tau\cdot x + y = (\text{const.}) \\ (\sigma.\bar{\sigma}) > 0 \iff \mathrm{Re}(\bar{\tau}\cdot x) + \mathrm{Re}(y) > (\text{const}). \end{cases}$ ··· (∗)

(Regarding $\begin{cases} x, y: \text{unknown}, \\ \text{the other 20 coefficients: known} \end{cases}$)

Rmk $\sigma = a_{\alpha\beta}\cdot A_{\alpha\beta} + u\cdot B_\gamma + x\cdot A_{\beta\gamma} + \tau B_\alpha + y\cdot A_{\gamma\alpha} + B_\beta + \Sigma c.C. + \Sigma c'.C'$.

$(u := a_\beta - \tau\cdot a_\alpha)$

$\in (A_{\alpha\beta} + a_\alpha\cdot A_{\beta\gamma} + a_\beta\cdot A_{\gamma\alpha})^\perp$.

① Main part of the prf of "Thm", Step 3

··· $\mathcal{E} \subset B_{(p,2)} = B(a_\alpha, a_\beta)$: open, ①closed. ← by "relative version" of Arnold's thm
$\neq \phi$.

$\mathcal{W} := \{ W \subset S : C\text{-nbhd} \mid \exists V: \text{hol. tab. nbhd of } C, \exists r>0, W \Subset V, W \cong \{\xi \in N_{V/S} \mid 0 \le |\xi| < r\} \}$

Known $W_{max} := \bigsqcup \mathcal{W}$: a hol. tab nbhd of C.

$v := \int_{S \setminus W_{max}} \eta \wedge \bar{\eta}$.

v': similarly defined.

~~By a simple argument~~, $\mathcal{E} \cap \{(x,y) \in \mathbb{C}^2 \mid (*) \text{ holds}\}$

$\subset \{(x,y) \in \mathbb{C}^2 \mid (\sigma.\sigma) = 0, (\sigma.\bar{\sigma}) > v + v'\}$

: closed //

Q How "large" is $W_{max} \subset S$??

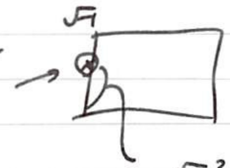
§3 Example ... K3 surf (X,σ) with. $\hat{\sigma} = 2\mu\cdot A_{\alpha\beta} + \mu\cdot B_\gamma + \sqrt{-1}M\cdot A_{\beta\gamma} + \sqrt{-1}\cdot B_\alpha + (M-\mu^2)\cdot A_{\gamma\alpha} + B_\beta$,

where $\begin{cases} \mu := -2^{\frac{2}{3}} \\ M \gg 1 : \text{large posi. real number} \end{cases}$ //

$c_0^{(1)} \equiv 0$, $\tau = \sqrt{-1}$, $a_\alpha = 0$, $a_\beta{}^2 = -2^{\frac{2}{3}}$

① Take $C_0 = C_0' \subset \mathbb{P}^2$ s.t. $C_0 \cong \mathbb{C}/\langle 1, \sqrt{-1}\rangle$. ⇒ decided.

$L \in \mathrm{Pic}^0(C_0) \cong \mathbb{C}/\langle 1,\sqrt{-1}\rangle$; defined by



② ④ $p_1 = p_2 = \cdots = p_8 = p_1' = \cdots = p_8' = [0] \in \mathbb{C}/\langle 1,\sqrt{-1}\rangle$.
inflection p.t. ⇒ $C_r \equiv C_r' \equiv 0$.

③ $j: C_0 \longrightarrow C_0'$; defined by
$z \longmapsto z - 2\cdot\mu$. ⇒ $a_{\alpha\beta}$: decided.

⑦ Take "nice" fiber coordinates "w" of the hol. tub. nbhds,
"nice" $R>1$ $R'>1$ s.t. $(x,y) = (\sqrt{-1}M, M-\mu^2)$.

⇑ "Then".

$$\begin{cases} (\hat{\sigma}.\hat{\sigma}) = 4\cancel{\mu^2} - 2\cancel{\mu^2} + 2\cdot(-1)\cdot M - 2(\sqrt{-1})^2 + 2\cdot(M-\cancel{\mu^2}) - 2 = 0. \\ (\hat{\sigma}.\bar{\hat{\sigma}}) = \mathrm{Re}(\sqrt{-1}\cdot\sqrt{-1}M) + \mathrm{Re}(M-\mu^2) + (\mathrm{const}) > V + V' \end{cases}$$

$= 2M - \mu^2 \gg 0$ //

Rmk (X,σ) : constructed as above.

⊙ $\mathrm{Pic}(X) \cong \mathbb{Z}^{17}$, gen'd by C_i's, C_i''s, and B_γ.

⊙ $\omega_s := 2A_{\gamma\alpha} + B_\beta + s\cdot(2A_{\alpha\beta} + B_\gamma) \in H^2(X,\mathbb{C})$

Simple calculation ... $\exists!\, s>0$ s.t. $\omega.\hat{\sigma} = 0$.

$\rightsquigarrow \omega_s \in H^{1,1}(X,\mathbb{R})$. $\omega :=$ $s = \frac{-1}{2\mu}\cdot(M-\mu^2)$

$\rightsquigarrow \omega^2 > 0$,
$\omega \perp \mathrm{Pic}(X)$. $\rightsquigarrow \omega$: nef class.