# Toward a higher codimensional Ueda theory

§1. a review of Ueda's theory
§2. Main result.
§3. ~~applications~~ example
§4 def. of "$u_{n,\nu}$"
§5. prf.

$X$ : cpx mfd.
$L$ : hol. line bdl.

<u>Def</u> $L$ : flat $\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ $\exists\{U_j\}$ : open cov. of $X$.
$\exists\{(U_j, t_j)\}$ : loc. triv. of $L$
s.t. $t_j/t_k \equiv \exists t_{jk} \in U(1)$ //.

§1. a review of.
<u>Ueda's theory</u>

… <u>Setting</u> $X$ : sm. cpx mfd.
$\cup$
$S$ : sm. cpt Kä hypsurf.
s.t. $N_{S/X}$ : flat ($\underset{\text{fact.}}{\Longleftrightarrow}$ $N_{S/X}$ : top. triv)

<u>Interest</u> … When does the line bdl $[S]$.
admit a flat structure on a nbhd of $S$ in $X$?

Ueda defined the obstruction classes $\{u_n(S, X)\}$

s.t. … ◎ $u_1(S, X) \in H^1(S, N_{S/X}^{-1})$ : well-defined.

◎ $u_n(S, X) \in H^1(S, N_{S/X}^{-n})$
: well-defined $\Longleftrightarrow$ $u_{n-1}(S, X) = 0$
well-def.

◎ type$(S, X) = n \in \mathbb{N}$.
($\underset{\text{def}}{\Longleftrightarrow}$ $u_\nu(S, X)$ : well-def, $= 0$. for $\nu < n$
and $u_n(S, X) \neq 0$)

$\Longleftrightarrow$ $\exists V$ : tub. nbhd of $S$ in $X$.
$\mathcal{O}_V(\widetilde{N_{S/X}}) \otimes \mathcal{O}_V/\mathcal{O}_V(-\nu S) \cong \mathcal{O}_V(S) \otimes \mathcal{O}_V/\mathcal{O}_V(-\nu S)$
for $1 \leq \nu \leq n$.
and — // — $\not\cong$ — // — for $\nu = n+1$.

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◎ $\mathrm{type}(S, X) = \infty$ $\left( \overset{\Longleftrightarrow}{\text{def.}}\ \forall n \geq 1,\ u_n(S, X) : \text{well-def}, = 0 \right)$

$\Longleftrightarrow$ ~~$V$: tub. nbhd.~~
$\forall n \geq 1$, $\exists V$: tub. nbhd of $S$ in $X$.

$$\mathcal{O}_V(\widetilde{N_{S/X}}) \otimes \mathcal{O}_V/\mathcal{O}_V(-nS) \cong \mathcal{O}_V(S) \otimes \mathcal{O}_V/\mathcal{O}_V(-nS),$$ //.

Thm 1. (Ueda '83)

$X$ : cpx mfd
$\cup$
$S$ : sm. cpt Kä. hyp. surf of $X$.

Assume $N_{S/X} \in \mathcal{E}_0(S) \cup \mathcal{E}_1(S)$

where $\mathcal{E}_0(S) := \{ E \in \mathrm{Pic}^0(S) \mid \exists n \in \mathbb{Z}_{>0},\ E^n = \mathbb{1}_S \}$

$$\mathcal{E}_1(S) := \left\{ E \in \mathrm{Pic}^0(S) \,\middle|\, \begin{array}{l} \exists \alpha \in \mathbb{R}_{>0} \\ \text{s.t. } \forall n \in \mathbb{Z}_{>0}, \\ \underset{\substack{\uparrow \\ \text{Euclid. dist. on } \mathrm{Pic}^0(S)}}{d}(\mathbb{1}_S, E^n) \geq (2n)^{-\alpha} \end{array} \right\}$$

Then $\mathrm{type}(S, X) = \infty$
~~Then~~ $\Rightarrow$ $\exists V$ : a nbhd of $S$ in $X$
s.t. $[S]$ : flat on $V$

//.

Rmk ◎ $\mathcal{E}_1(S)$ does not depend on the choice of "$d$".

◎ $\mathrm{Pic}^0(S) \setminus \mathcal{E}_1(S)$ : Lebesgue measure zero.

◎ $\mathcal{E}_1(S) = \bigsqcup_{\# \mathbb{N}} (\text{nowhere dense closed subset of } \mathrm{Pic}^0(S).)$

//.

## §2 Main result

.. ~~we defined~~

--- ~~Setting~~ $X$ : cpx mfd.
$\cup$
$S$ : sm. hyp. surf of $X$
$\cup$
$C$ : sm. cpt Kä hypsurf of $S$.

s.t. $N_{S/X}$ : flat on a nbhd of $C$ in $S$. ($V$)

$\rightsquigarrow$ we defined
new obstruction classes $\{u_{n,m}(C,S,X)\}$.
for $n \geq 1$, $m \geq 0$. $\in H^1(C, N_{S/X}^{-n} \otimes N_{C/S}^{-m})$

and showed...

**Thm 2**

Assume (i) $N_{C/S} \in \mathcal{E}_0(C)$ and $N_{S/X}|_C \in \mathcal{E}_0(C)$

or (ii) $N_{C/S} = N_{S/X}|_C \in \mathcal{E}_1(C)$

or (iii) $N_{S/X}|_C \in \mathcal{E}_0(C)$ and.
$^\exists V'$ : a str. 1-convex nbhd of $C$ in $S$
s.t. $C$ is the maximal cpt analytic sub. of $V'$

Then
$\forall n, m,\ u_{n,m}(C,S,X) = 0 \Longrightarrow$ $^\exists W$ : a nbhd of $C$ in $X$ s.t. $[S]|_W$ : flat //

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## §3 example

$\mathbb{P}^3 \ni P_1, P_2, \cdots, P_8$ : general 8 points.

$\rightsquigarrow \exists \{Q_\lambda\}_{\lambda \in \mathbb{P}^1}$ : 1-dim family of quad surf of $\mathbb{P}^3$.

We may assume that $Q_0, Q_\infty$ : sm, $Q_0 \pitchfork Q_\infty$.

$\rightsquigarrow \begin{cases} C_0 := Q_0 \cap Q_\infty \text{ : sm. ellipt. curve.,} \\ \mathcal{O}_{Q_0}(C_0) = \mathcal{O}_{Q_0}(K_{Q_0}^{-1}) \end{cases}$

$\odot$ $X := \mathrm{Bl}_{\{P_i\}} \mathbb{P}^3 \xrightarrow{\pi} \mathbb{P}^3$, $S_\lambda := (\pi^{-1})_* Q_\lambda$.

$\rightsquigarrow \mathcal{O}_X(K_X^{-1}) = \mathcal{O}_X(2S_0) = \mathcal{O}_X(2S_\infty)$

$N_{S_0/X} = \mathcal{O}_{S_0}(C)$, $(C := (\pi^{-1})_* C_0)$

$N := N_{S_0/X}|_C = N_{C/S_0} \cong \mathcal{O}_{\mathbb{P}^3}(2)|_{C_0} \otimes \mathcal{O}_{C_0}(-P_1 - P_2 \cdots - P_8)$

Fact $K_X^{-1}$ : semi-ample $\iff N \in \mathcal{E}_0(C_0)$.

$N \notin \mathcal{E}_0(C) \implies \mathrm{Bs}|K_X^{-m}| = C$ for $\forall m \geq 1$. //

Cor 3 ($\Leftarrow$ Thm 2 (ii))

$N \in \mathcal{E}_0(C) \cup \mathcal{E}_1(C)$

$\implies \exists W$ : a nbhd of $C$ in $X$

s.t. $K_X^{-1}|_W$ : flat //

prf … $H^1(X, N^{-n} \otimes N^{-m}) = 0$ for $\forall n \geq 1, \forall m \geq 0$.

Cor 4 $N \in \mathcal{E}_0(C) \cup \mathcal{E}_1(C)$

$\implies K_X^{-1}$ admits a $C^\infty$ metric with semi-positive curvature

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## §4 definition of "$u_{n,m}(C,S,X)$".

Fix $\{U_j\}$ : suff. fine open cov. of $C$

$\{V_j\}$ : suff. small nbhd of $U_j$ in $S$.

s.t. $V := \bigcup V_j$ : tub. nbhd of $C$ in $S$.

$\{W_j\}$ : suff. small nbhd of $V_j$ in $X$

s.t. $W := \bigcup W_j$ : tub. nbhd of $V$ in $X$.

$\rightsquigarrow$ $N_{S/X} = [\{(U_{jk}, \exists t_{jk})\}] \in H^1(S, \mathcal{O}_S^*)$, $t_{jk} \in U(1)$

$\exists w_j$ : ~~loc.~~ def. func of $V_j$ in $W_j$ s.t. $\left.\frac{w_j}{w_k}\right|_{V_{jk}} \equiv t_{jk}$.

Fix $z_j$ : def. func. of $U_j$ in $V_j$ ~~s.t. $\frac{z_j}{z_k}|_{U_{jk}} \equiv s_{jk} \in U(1)$~~.

$$t_{jk} \cdot w_k = w_j + \sum_{n=0}^{\infty} f_{jk}^{(n+1)}(\underset{\text{coord of } U_j}{x_j}, z_j) \cdot w_j^{n+1},$$

$$f_{jk}^{(n+1)}(x_j, z_j) = \sum_{m=0}^{\infty} g_{jk}^{(n+1,m)}(x_j) \cdot z_j^m.$$

**Def** $\{(V_j, w_j)\}$ : of order $(n,m)$

$\overset{\Longleftrightarrow}{\text{def}}$ $\nu \lneq n \Longrightarrow f_{jk}^{(\nu+1)} \equiv 0$,

$\mu < m \Longrightarrow g_{jk}^{(n+1,\mu)} \equiv 0$. for $\forall j,k$.

~~Lem~~ **Fact** $\{(V_j, w_j)\}$ : of order $(n,m)$

$\Longrightarrow \{(U_{jk}, g_{jk}^{(n+1,m)})\} \in Z^1(\{U_j\}, \mathcal{O}(N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m}))$

$\rightsquigarrow$ **Def** $u_{n,m}(C,S,X)$ : well-def $\underset{\text{def}}{\Longleftrightarrow}$ $\exists\{(V_j, w_j)\}$ : of order $(n,m)$.

In this case,

$$u_{n,m} := [\{(U_{jk}, g_{jk}^{(n+1,m)})\}] \quad //$$

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## §5. Outline of the prf --- for (i), (ii).

~~Goal~~ Strategy --- Define a functional equation.

$$\underset{\sim\sim}{w_j} = u_j + \sum_{\nu=2}^{\infty} F_j^{(\nu)}(x_j, z_j) \cdot u_j^{\nu} \quad \cdots (*)$$

$\{(V_i, w_i)\}$ : of order $(1,0)$

s.t. the solution $u_j = \widetilde{w}_j$ satisfies that
$$\widetilde{w}_j = t_{jk} \cdot \widetilde{w}_k \quad \text{for } \forall j, k.$$

Q What is the condition for $\{F_j^{(\nu)}\}$ ?
$$= \sum_{\mu=0}^{\infty} G_j^{(\nu,\mu)}(x_j) \cdot z_j^{\mu}.$$

--- the answer is :
$$\delta\{(\mathcal{U}_j, G_j^{(\nu,\mu)})\}$$
$$= \{(\mathcal{U}_{jk}, g_{jk}^{(\nu,\mu)} + \underbrace{\exists \text{ function on } \mathcal{U}_{jk}}_{\text{uniquely defined from } \{G_j^{(\nu',\lambda)}\}})\}$$

$\nu' < \nu$ or ($\nu' = \nu$ and $\lambda < \mu$)

$$\underset{\text{face}}{\in} \underset{\sim\sim\sim}{U_{\nu,\mu}(C, S, \lambda)}.$$

Thus we can formally define $\{F_j^{(\nu)}\}$
if $U_{n,m}(C, S, X) = 0$ for $\forall n, m$.

Rmk conditions (i), (ii) is needed
for the convergence of $(*)$ //