# Complex K3 surfaces containing Levi-flat hypersurfaces.

$X$ : K3 $\overset{\text{}}{\underset{\text{def}}{\Leftrightarrow}}$ $X$ : non-sing complex surf, cpt, (may possibly be non-proj.)
s.t. $\pi_1(X) = 0$, $\exists\,\sigma$ : nowhere vanishing hol 2-form on $X$.

E.g. Kummer surf.

Def $H \subset \underset{\sim}{X}_{\text{cpx mfd}}$ : ($C^\omega$-) real hyp. surf.
$H$ : Levi-flat $\underset{\text{def}}{\Leftrightarrow}$ $\exists\,\mathcal{F}$ : ($C^\omega$-) foliation on $H$ of $\mathrm{codim}_{\mathbb{R}} = 1$
s.t. each leaf of $\mathcal{F}$ is a holomorphically immersed cpx submfd of $X$.

" $X \setminus H = (\overset{\text{weakly}}{\text{psd conv}}) \sqcup (\overset{\text{weakly}}{\text{psd conv}})$ "

Thm 1 $\exists\, X$ : K3, not a Kummer surf. [(sm. surf.) $\xrightarrow[\text{resolution of sing.}]{}$ ${}^{\exists}$ab. surf / involution]
s.t. $\exists\,\{H_t\}_{t\in I}$ ($I \subset \mathbb{R}$ : interval)
(*) ; $C^\omega$-family of Levi-flat hyp. surfaces of $X$.
s.t. $\forall t$, $H_t \approx_{\text{diffeo}} S^1\times S^1\times S^1$
each leaf of $H_t$ is $\begin{cases}\text{dense in } H_t \\ \text{bihol. to } \mathbb{C} \text{ or } \mathbb{C}^* \end{cases}$ //

Cor 2 $\exists\, X$ : K3, not a Kummer surf.
s.t. $\exists\, f : \mathbb{C} \to X$ : hol. immersion
(**) s.t. $\begin{cases} \overline{f(\mathbb{C})}^{\text{Euc}} : \text{real hyp.surf of } \mathrm{codim}_{\mathbb{R}} = 1, \subsetneq X. \\ \overline{f(\mathbb{C})}^{\text{Zar}} = X \end{cases}$ //

⊙ We will construct such $X$ by patching two open cpx surfaces $M$ and $M'$.

⊙ $M, M' = ({}^{\forall}9\text{pts b-up of } \mathbb{P}^2) \setminus ({}^{\exists}\text{a nbhd of an ellipt curve})$

Thm 3 $\exists\, \mathcal{X} \xrightarrow{\pi} \underset{\sim}{B}$ : proper hol submersion
18 dim'l cpx mfd s.t. $\begin{cases} \forall t \in B,\ X_t := \pi^{-1}(t) \text{ ; K3 s.t. (*), (**)}, \\ \text{The Kodaira-Spencer mp is inj} \end{cases}$ //

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§1. Motivation from "nbhd theories".

§2. Construction of $X$.

§3. Outline of the prfs //

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## §1

**Motivation**: nbhd structures of $\underset{\text{cpt cpx curve.}}{\underset{\sim}{C}} \subset \underset{\text{cpx surf}}{\underset{\sim}{S}}$

$\underset{W}{\underset{\|}{\mathrm{Q}}}$ $\exists ?\ \phi : (\text{a nbhd of } C \text{ in } S) \to \mathbb{R} \cup \{-\infty\}$ ; $\begin{cases} (\text{non-sing}) \\ \text{conti.} : C^2 \text{ on } W \setminus C \\ \phi|_C \equiv -\infty \end{cases}$.

s.t. $\underset{\text{"cpx Hessian"}}{\sqrt{-1}\partial\bar\partial\phi}$ $\begin{cases} \geq 0 \\ \text{or} = 0 \\ \text{or} \leq 0 \end{cases}$ on $\underset{?}{\underline{(\text{a nbhd of } C) \setminus C}}$

⊙ $(C^2) := \deg N_{C/S} < 0 \Rightarrow \exists \phi$ with $\sqrt{-1}\partial\bar\partial\phi \gneq 0$ (H. Grauert '62)

⊙ $(C^2) > 0 \Rightarrow \exists \phi$ with $\sqrt{-1}\partial\bar\partial\phi \lneq 0$ (O. Suzuki '75)

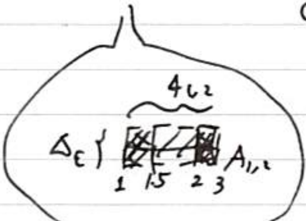
$\rightsquigarrow$ We are interested in the case $(C^2) = 0$. (c.f. Ueda theory, T. Ueda '83)

**E.g.** $\Delta_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$. $\quad P(w) = \lambda \cdot w + \cdots + w^d$

$A_{1,3} := \{w \in \mathbb{C} \mid 1 < |w| < 3\}$. $\quad \cdots$ poly. with $\begin{cases} P'(0) = \lambda \in U(1) := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}. \\ P(0) = 0. \end{cases}$

$S := \Delta_\varepsilon \times A_{1,2} / \sim$, $\quad (\varepsilon \ll 1)$

$(z, w) \sim (z', w') \iff \begin{cases} z' = 2 \cdot z \\ w' = P(w). \end{cases}$




$C := \{0\} \times A_{1,2} / \sim \quad \subset S$ : ellipt. curve

| | $\lambda^n = 1$ for $\exists n$, $P(w) = \lambda w + w^d$ | $\lambda^n \neq 1$ for $\forall n$, $P$ : lin'ble. $P(w) = \lambda \cdot w$ | $\lambda^n \neq 1$ for $\forall n$, $P$ : non-lin'ble. |
|---|---|---|---|
| $N_{C/S} \in \mathrm{Pic}^0(C)$ | torsion. | non-tor. | non-torsion. |
| cpt curve on $S \setminus C$ | $\nexists$ cpt curve. | $\nexists$ cpt curve. | $\exists$ cpt curve (small) if $P$ has periodic cycle (c.f. [Ueda '83]) |
| $\phi$ | $\exists \phi$ with $\sqrt{-1}\partial\bar\partial\phi \lneq 0$. "type $\alpha$" | $\exists \phi$ with $\sqrt{-1}\partial\bar\partial\phi \equiv 0$. "type $\beta$" $\phi(z,w) := \log|w|$. $\rightsquigarrow \{\phi = \text{const}\}$ : Levi-flat! | $\nexists \phi$ with neither $\sqrt{-1}\partial\bar\partial\phi$ $\leq 0$ nor $\equiv 0$ nor $\geq 0$. "type $\gamma$" |

$((U(1)), d)$

$-\log d(1, \lambda^n) = O(\log n)$ as $n \to \infty$

//.

**Rmk**. $\lambda$ : Diophantine $\Rightarrow$ $P$ : lin'ble. $\Rightarrow$ $(C, S)$ : type $\beta$ in the above e.g.

$\{$ generalization $\cdots$ $\begin{cases} \text{Arnol'd '76}, \\ \text{Ueda '83.} \end{cases}$

**Thm 4** (Arnol'd '76)

$C$: ellipt. curve, $N_{C/S} \in \mathrm{Pic}^0(C)$ : Dioph. (d: invariant dist.)

$\hat{S}$: cpx surf. $\left(\text{i.e. } -\log d(\mathbb{1}_C, N_{C/S}^n) = O(\log n) \text{ as } n \to \infty\right)$

$\Longrightarrow$ $\exists\, W$ ; a nbhd of $C$ in $S$,

$\exists\, \hat{W}$ : a nbhd of $\hat{C} := (0\text{-section}) \subset N_{C/S}$

s.t. $\begin{array}{ccc} W & \cong_{\text{bihol}} & \hat{W} \\ \cup & \circlearrowleft & \cup \\ C & = & \hat{C} \end{array}$ // $(\Rightarrow (C,S)$: type $\beta)$

**Q** How about the case where $S$: Proj. surf ?

**Example 5** $C_0 \subset \mathbb{P}^2$ : sm. ellipt. curve (curve of degree 3)

$Z := \{p_1, p_2, \cdots, p_9\} \subset C_0$ : nine pts, "general".

$S := Bl_Z \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$ ; b-up at $Z$.

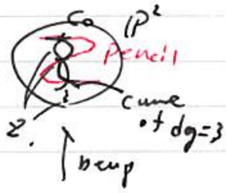
$C := (\pi^{-1})_* C_0$ ; str. transf. of $C_0$.

$\rightsquigarrow N_{C/S} \cong \mathcal{O}_{\mathbb{P}^2}(C_0)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \cdots - p_9) \in \mathrm{Pic}^0(C_0)$ //

if $Z \subset C_0$ : "special". ...

⊙ $N_{C/S}$ : torsion. $\Longleftrightarrow$ $S$ admits an elliptic fibration str. ($\rightsquigarrow$ "type $\beta$")

⊙ Arnol'd's thm $\Longrightarrow$ $(C,S)$: of type $\beta$ if $N_{C/S}$ : Dioph.




**Q** $\exists ?\, Z$ s.t. $(C,S)$: of type $\gamma$ in Eg.5 ?


**Q** When $N_{C/S}$ : Dioph,

what is "$S \setminus$ (maximal $W$ as in Thm 4)" in Eg.5 ?

## §2. Construction of $X$

⊙ $(C, S, Z)$, $(C', S', Z')$ ... as in Eg.5.

s.t. $C \cong_{\text{bihol}} C'$, $N_{C/S} \cong N_{C'/S'}^{-1}$ : Dioph //

$\rightsquigarrow$ $\exists\, W^{(\prime)} \subset S^{(\prime)}$ ; $C^{(\prime)}$-nbhd : as in Thm 4.

Coord. of $W^{(\prime)}$

$W^{(\prime)} = \bigsqcup_j W_j^{(\prime)} \ni (\underset{\text{Euc. coord of } C^{(\prime)}}{z_j^{(\prime)}}, \underset{\text{defining func. of } C^{(\prime)}}{w_j^{(\prime)}})$ s.t. $\left\{\begin{array}{l} z_j = z_k + \exists A_{jk} \in \mathbb{C} \\ w_j = \exists t_{jk} \cdot w_k \\ \quad\ \ \, U(1) \end{array}\right.$ $\left\{\begin{array}{l} z_j' = z_k' + A_{jk} \\ w_j' = t_{jk}^{-1} \cdot w_k' \end{array}\right.$ //

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**Obs. 6** ∘ $\Phi^{(\prime)} : W^{(\prime)} \longrightarrow \mathbb{R}_{\geq 0}$. ; well-def.

$$(z_j^{(\prime)}, w_j^{(\prime)}) \longmapsto |w_j^{(\prime)}|. \qquad (\text{c.f. flat metric on } N_{C/S})$$

∘ We may assume $W^{\prime} = \Phi^{(\prime)-1}([0, R^{(\prime)}))$, $R, R' > 1$.

∘ $H_t := \Phi^{-1}(t) \quad (t \in [0, R))$ : Levi-flat $\subset W$.

$C \xleftarrow{S^1\text{-bdl}} \approx S^1 \times S^1 \times S^1$. leaf : "$w_j = \mathrm{const}$". $\cong \mathbb{C}$ or $\mathbb{C}^*$.

**Construct'n of X**

$$M := S \setminus \Phi^{-1}([0, \tfrac{1}{R'}]) \subset S.$$
$$M' := S' \setminus (\Phi')^{-1}([0, \tfrac{1}{R}]) \subset S'.$$
$$\rightsquigarrow M \supset W^* := \Phi^{-1}((\tfrac{1}{R'}, R))$$
$$M' \supset (\Phi')^{-1}((\tfrac{1}{R}, R'))$$ ← identify it with $W^*$ via...

$$\left[\; f : \Phi^{-1}((\tfrac{1}{R'}, R)) \xrightarrow{\cong} (\Phi')^{-1}((\tfrac{1}{R}, R')) \right.$$
$$(z_j, w_j) \longmapsto \left(z_j, \frac{1}{w_j}\right), \quad z_j = z_j' \text{ via } C \underset{\text{isom}}{\cong} C', \quad \frac{1}{w_j} = w_j'.$$

$$\rightsquigarrow X := M \sqcup_{W^*} M'.$$

**Rmk** Simple calculation by using Mayer-Vietoris seq.

$$\rightsquigarrow H_q(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & q = 4 \\ 0 & 3 \\ \mathbb{Z}^{22} & 2 \\ 0 & 1 \\ \mathbb{Z} & 0. \end{cases}$$

//

Rmk ⊙ [Doi '09] .. Topologically the same constr. of K3.
(need to deform the cpx structure of M and M')

⊙ [Tsuji '84] ... Constr. of $(S^3 \times S^3, \overset{\exists}{\underset{\text{cpx str.}}{J}})$
by using Arnol'd-type thm.
("$\mathcal{B}$" = Hopf 3-fold
"$\mathcal{C}$" = Hopf surf.)

## §3. prfs.

**Fact 7.** $X$: cpt cpx surf,

$$H_k(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k=4 \\ 0 & 3 \\ \mathbb{Z}^{22} & 2 \\ 0 & 1 \\ \mathbb{Z} & 0 \end{cases}$$

and

$\exists\, \sigma$: nowhere van. glob. hol 2-form

$\Rightarrow X$: K3.

**Fact 8.** $\mathcal{X} \xrightarrow{\pi} B$ : Deformation family of K3 surfaces,

$\begin{cases} \dim B \geq 5. \\ \text{The Kodaira–Spencer map is inj.} \end{cases}$

$\Rightarrow \exists\, t \in B$, $X_t := \pi^{-1}(t)$ : not a Kummer surf //

By Obs 6 + Fact 7 + Fact 8,
all we have to do is:

- construct $\sigma$ on $X \overset{\text{as in §2}}{\Leftarrow} M \cup_{W^*} M'$.
- "count" degrees of freedom in the construction in §2.

**Lem 9** $H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}$.

(∵) $F: W^* \longrightarrow \mathbb{C}$ : hol.

Take $t \in (\frac{1}{R}, R) \rightsquigarrow \exists\, B := \max_{x \in H_t} |F(x)|$

$\quad\quad = |F(x_t)| \quad (\exists x_t \in H_t)$.

Take a leaf $\mathcal{L}_t \subset H_t$ with $\mathcal{L}_t \ni x_t$.

Maximum principle for $F|_{\mathcal{L}_t} : \mathcal{L}_t \longrightarrow \mathbb{C} \rightsquigarrow F|_{\mathcal{L}_t} \equiv \exists A \in \mathbb{C}$.

$\mathcal{L}_t \subset H_t$ : dense $\rightsquigarrow F|_{H_t} \equiv A$.

$\{x \in W^* \mid F(x) \equiv A\}$ ; analytic sub of. $W^*$, $\supset H_t \rightsquigarrow F \equiv A$ //

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**Prop 10.** $\exists \sigma$: glob. hol. 2-form on $X = M \cup_{w*} M'$,

s.t. $\begin{cases} \text{nowhere vanishing}, \\ \sigma|_{W_j^*} = dz_j \wedge \frac{dw_j}{w_j} \text{ for each } j. \end{cases}$ //

$$\left( \overset{\shortparallel}{d(z_k + A_{jk}) \wedge \frac{d(t_{jk}w_k)}{t_{jk}w_k}} = dz_k \wedge \frac{dw_k}{w_k} \right) \text{ on } W_{jk}^*.$$

*pf* $K_S = -C \rightsquigarrow \exists \eta$: mero. 2-form. on $S$ s.t. $\operatorname{div}(\eta) = -C$.

$F_j := \frac{\eta|_{W_j^*}}{dz_j \wedge \frac{dw_j}{w_j}}$ : $W_j^* \longrightarrow \mathbb{C}$ : hol, nowhere vanishing.

$W_j^* = W_j \cap W^*$.

Patch.

$\exists F : W^* \longrightarrow \mathbb{C}$ : hol s.t. $F|_{W_j^* \cap \cancel{W^*}} = F_j$.

~~Prop~~ Lem 9 $\rightsquigarrow$ we may assume $F_j \equiv 1$.

$\Updownarrow$

$\eta|_{W_j^*} = dz_j \wedge \frac{dw_j}{w_j}$.

Similarly $\exists \eta'$; mero 2-form on $S'$,

s.t. $\begin{cases} \operatorname{div}(\eta') = -C', \\ \eta'|_{W_j'^*} = +dz_j' \wedge \frac{dw_j'}{w_j'}. \end{cases}$

$\circledcirc$ $f^* dz_j' \wedge \frac{dw_j'}{w_j'} \underset{\uparrow}{=} dz_j \wedge \frac{d(w_j^{-1})}{w_j^{-1}} = -dz_j \wedge \frac{dw_j}{w_j}$

$f(z_j, w_j) = (z_j, \frac{1}{w_j})$

$\rightsquigarrow \sigma := \{(M, \eta|_M), (M', -\eta'|_{M'})\}$ //

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$\cancel{\mathcal{Q}}$ Q $\dim \{ X \in (\text{K3 moduli})^{20} \mid X \text{ can be constructed in the manner as in §2} \} = ?$

Fix $C_0 \underset{\text{ellipc.}}{\subset} \mathbb{P}^2$, $L_0 \longrightarrow C_0$ : Dioph. l.b.

Parameters : $\begin{cases} \cdot \text{ Choice of } g : C_0 \underset{\text{isom}}{\cong} C_0' & \text{— 1-dim'l} \\ \cdot \text{ Choice of } P_1, P_2, \cdots, P_8 \in C_0 & \text{— 8-dim'l} \\ \quad P_1', \cdots, P_8' \in C_0' & \text{— 8-dim'l} \\ \quad (\rightsquigarrow \exists! P_9, \exists! P_9' \text{ s.t. } N_{C/S} \cong N_{C'/S'}^{-1} \cong L_0) \\ \circ \text{ Choice of "fiber coord." "} w_j \text{"}. & \text{— 1-dim'l. Face } \mathbb{P}^1 \text{ indep.} \end{cases}$