On some analogues of Ueda theory and their applications

a theory on the nbhd str. of $S \subset X$.

(S: cpx submfd of $\dim_{\mathbb{C}} = d-1$, X: cpx mfd of $\dim_{\mathbb{C}} = d$.)

s.t. $c_1(N_{S/X}) = 0$.

Rmk (S-nbhd) $\not\cong_{\text{bihol}}$ ((0-sectn)-nbhd in $N_{S/X}$)

e.g. 1 $T = \mathbb{C}/\Lambda$ ← lattice, $\cong \mathbb{Z}\oplus\mathbb{Z}$ ($\subset \mathbb{C}$.)

$\rightsquigarrow 0 \to \underbrace{\mathbb{1}_T}_{\text{hol. triv. line bdl}} \longrightarrow \overset{\exists!}{\underbrace{E}_{\text{rank 2-vect. bdl. (hol.)}}} \longrightarrow \mathbb{1}_T \to 0$. : ex, non-splitting.

$X_T := \mathbb{P}(E) \xrightarrow{\pi} T$,

$\cup$

$S_T :=$ the sectn of π.

Known ; $\begin{cases} \cdot\ N_{S_T/X_T} = \mathbb{1}_{S_T}. \\ \cdot\ \forall V: S_T\text{-nbhd},\ \nexists f: V \to \mathbb{C};\ \text{hol. def. func. of } S_T \end{cases}$ ($\{f=0\} = S_T$)

Thm (Ueda '83) (S, X); as above,

Assume ; $N_{S/X}^{\otimes n} = \mathbb{1}_S$ for $\exists n \geq 1$, or $\{N_{S/X}^{n}\}_n \subset \mathrm{Pic}^0(S)$ enjoys a Siegel-type cond.

Then $[S]$: flat around S (∗) $\iff$ (S, X) : of infin. type.

(later) we'll explain.

Rmk when $N_{S/X} = \mathbb{1}_S$,

(∗) $\overset{\text{"def"}}{\iff}$ $\exists V : S$-nbhd $\exists f : V \to \mathbb{C}$: hol. def. func of S

Goal of this talk ; … "codim-2 analogue of Ueda's thm".

$X^d \supset S^{d-1} \supset C^{d-2}$, Assuming $\cdot$ C: cpt Kä

$\cdot$ $\exists U$: C-nbhd in S

(non-cpt)

s.t. $N_{S/X}|_U$: flat ~~………~~

Give a criterion for (∗) ~~…~~ around C.

No.

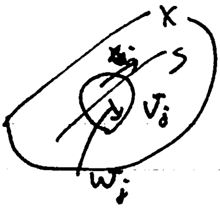
Date

§1. "of infinite type". Main result. [K-'15]

§2. Application to the Levi-flat realization problem (j.w.w. N. Ogawa)
(a local criterion for non-realizability of Levi-flat mfds)

$$\boxed{\text{In this talk, we'll assume. } \quad N_{S/X} = \mathbb{1}_S \quad \text{for simplicity.}}$$

§1. $S^{d-1} \subset X^d$, $N_{S/X} = \mathbb{1}_S$. $\{V_j\}$: covering of S ($\overset{\text{open}}{\subset} X$)



$\rightsquigarrow$ $\exists$ system $\{(V_j, w_j)\}$ of local def. functions w_j of S

s.t. $dw_j|_{V_{jk}\cap S} = dw_k|_{V_{jk}\cap S}$.

$$w_k \underset{\text{Taylor exp. on } V_{jk}}{=} w_j + f^{(2)}_{jk}(x_j)\cdot w_j^2 + f^{(3)}_{jk}(x_j)\cdot w_j^3 + \cdots$$

Assume $f^{(\nu)}_{jk} \equiv 0$ for ${}^\forall j,k$, ${}^\forall \nu \leq n^{\in \mathbb{Z}_{>0}}$

$$\rightsquigarrow \quad w_k \underset{\text{on } V_{jk}}{=} w_j + f^{(n+1)}_{jk}\cdot w_j^{n+1} + O(w_j^{n+2})$$

Def $\underbrace{u_n(S,X)}_{\text{Ueda's obstr. class}} := [\{(V_{jk}\cap S, f^{(n+1)}_{jk})\}] \underset{\text{[illegible]}}{\in} H^1(S, N_{S/X}^{-n})$ $\big(N_{S/X}^{-n} = \mathbb{1}_S\big)$

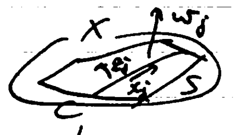
Fact ~~Known~~

$u_n(S,X) = 0 \iff \exists \{(V_j, \widehat{w}_j)\}$: new system s.t. $\widehat{w}_k \underset{\text{on } V_{jk}}{=} \widehat{w}_j + O(w_j^{n+2})$ //

Def (S,X): of infin. type

$\underset{\text{def.}}{\iff}$ ${}^\forall n$. $u_n(S,X) = 0$ "

⊙ Codim 2-analogue



$$\cdots \quad w_k \underset{\text{on } V_{jk}}{=} w_j + \underbrace{f^{(2)}_{jk}(x_j, z_j)}\cdot w_j^2 + O(w_j^3).$$

$$\hat{f}^{(2)}_{jk}(x_j, z_j) \underset{\text{on } V_{jk}\cap S}{=} \sum_{m=0}^{\infty} \hat{g}^{(2,m)}_{jk}(x_j)\cdot z_j^m.$$

Assume $\begin{cases} f_{jk}^{(\nu)} \equiv 0 & \text{for } \forall j,k, \ \forall \nu \leq n, \\ g_{jk}^{(n,\mu)} \equiv 0 & \text{for } \quad \text{''}, \ \forall \mu \lneq m \end{cases}$

Def $u_{n,m}(C,S,X) := [\{(V_{jk} \cap C,\ g_{jk}^{(n,m)})\}] \underset{\text{fact}}{\in} H^1(S, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m})$

Rmk
$u_{n,m}(C,S,X) = 0 \iff \exists \{(V_j, \widetilde{w}_j)\}$: new system.
s.t. ~~$\widetilde{w}_k = \widetilde{w}_j + (f_{jk}^1$~~
"$\widetilde{f}_{jk}^{(n+1)}$" $= O(z_j^{m+1})$.

Main result ([K-'15])

(**Setting** : $\underset{\text{cpt Käh}}{C^{d-2}} \subset S^{d-1} \subset X^d$
$N_{S/X}|_{(C\text{-nbhd})} = \mathbb{1}_{(C\text{-nbhd})}$ ← for simplicity.)

Assume $\exists \ell \geq 1$ s.t. $N_{C/S}^{\otimes \ell} = \mathbb{1}_C$.
or C: excep. subvar. of S in the sense of Grauert.
($\Leftrightarrow \deg N_{C/S} < 0$ if $\dim C = 1$.)

Then (1) for each $n \geq 1$,
$\forall m \geq 0,\ u_{n,m}(C,S,X) = 0$
$\iff \exists V$: C-nbhd in X s.t. $u_n(S\cap V, V) = 0$.

(2) $\forall n,m,\ u_{nm}(C,S,X) = 0$
$\iff \exists V$: C-nbhd in X
$\exists f: V \to \mathbb{C}$: 'hol. def. func of $V \cap S$'

§2. Application to the Levi-flat realization problem

Thm (Barrett['90] - Inaba['92]) Della Sala ('14) (--- a local criterion for Non-realizability) ~~[illegible]~~

$\nexists C^\infty$ Levi-flat realization of $(S^3, \mathcal{F}_{\text{Reeb}})$
i.e. $\forall X$: cpx mfd of $\dim_{\mathbb{C}} X = 2$,
$\nexists i : (S^3, \mathcal{F}_{\text{Reeb}}) \hookrightarrow X$: leafwise hol, C^∞ emb. //

No.

Date

Rmk
- $\mathcal{F}_{\text{Reeb}}$ has a C^∞ flat holonomy along T (Contracting; Torus leaf)
- Ueda's thm is ~~needed~~ used in the prf.

✗ holonomy-flatness v.s. Ueda thry (Barrett '90)

X ; cpx mfd of $\dim_{\mathbb{C}} X = d$.

$\cup$

$(M, \mathcal{F}_M)$; Levi-flat hyp. surf, $\dim_{\mathbb{R}} M = 2d-1$.

$\Downarrow$

L ; leaf. $(\dim_{\mathbb{C}} L = d-1)$ s.t. $L \subset M$: sub mfd.

Fix $\{V_j\}$: open cov. of L.

Thm (Barrett) ... ~~T.F.A.E.~~ $\forall n \geq 1$,

c.f. Barrett Fornaess '88

$①_n$ $\exists \{(V_j \cap M, g_j)\}$; sys. of. $g_j : V_j \cap M \to \mathbb{R}$; leaf wise const.

$g_j|_{V_j \cap L} = 0$, $g_j'|_{V_j \cap L} \neq 0$.

s.t. $g_k \underset{\text{on } V_{jk} \cap M}{=} g_j + O(g_j^{n+1})$

$\Longleftrightarrow$

$②_n$ $\exists \{(V_j, h_j)\}$; sys. of. $h_j : V_j \to \mathbb{C}$: h.l def. fnc. of $L \cap V_j$.

s.t. $\begin{cases} \mathrm{Im}(h_j|_M) = o(|h_j|^n) \\ h_k \underset{\text{on } V_{jk}}{=} h_j + O(h_j^{n+1}) \end{cases}$

Obs ⟹ holonomy-flatness $\Rightarrow$ $\forall n$, $①_n$ holds $\Longrightarrow$ $①_1$ holds

$\Downarrow$ Thm $\qquad$ $\Downarrow$ Thm

$\forall n$, $②_n$ holds $\qquad$ $②_1$ holds

$\Downarrow$ $\qquad$ $\Downarrow$

(L, X) : of infin. type. $\qquad$ $N_{L/X} = \mathbb{1}_L$. //

//

◎ Thm (Barrett-Inaba) is shown by using

Obs + Ueda's thm + existence of an annulus leaf accumulating to T

◎ We ~~can~~ can deduce from Obs + (Main result) [K-'15] + ~~(+)~~ the follwg:

Thm (K–, Ogawa) $(M, \mathcal{F}_M)$: Levi-flat mfd of $\dim_{\mathbb{R}} = 5$.

Assume $\exists (N^3, \mathcal{F}_N)$: the holonomy along T is C^∞ flat. contracting
$\quad \overset{\Psi}{T}$: torus leaf

$\exists (N^3, \mathcal{F}_N) \hookrightarrow (M, \mathcal{F}_M)$: C^2-emb. transv. to $\mathcal{F}_N$, leafwise hol

Then $(M, \mathcal{F}_M)$: NOT C^∞ Levi-flat realizable //

e.g. $(\mathbb{R}^5, \mathcal{F}_{Reeb})$ $(= \mathbb{C}^2 \times \mathbb{R} - 0/\sim)$: Reeb component, (堀内さんが主にdefしたもの)

$\quad \cup$
$\quad L$: "bounding leaf", $\approx S^3 \times S^1$: Hopf mfd.
$\quad \cup$
$\quad T$: cpx torus.

[Della-Sala] .... $\forall (M^5, \mathcal{F}_M)$: $(\mathbb{R}^5, \mathcal{F}_{Reeb})$ の L-nbhd を含むものは cpt. NOT Levi-flat realizable

Thm $\Longrightarrow \forall (\ 〃\ )$: $(\ 〃\ )$ T-nbhd 〃

$\Uparrow$
$(T\text{-nbhd}) \subset L$ ... non compact!

No.

Date