Singular holomorphic foliations by curves whose canonical rings have infinitely generated section rings

Takayuki KOIKE (the University of Tokyo)

We are interested in
minimal singular metrics h_{min} of **Zariski-type line bundles**.

X : smooth projective variety /C
L : pseudo-effective line bundle /X

h_{min} is a Hermitian metric on L admitting the value ∞ and satisfying the L^2 condition, with semi-positive curvature. h_{min} has the smallest singularities among such metrics.

- h_{min} : exists $\underset{[DPS]}{\Longleftrightarrow}$ L : pseudo-effective.
- L : semi ample ⇒ ∃ smooth h_{min}.
- L : big $\underset{[Bou]}{\Longrightarrow}$ $B_{-}(L) = \{x \in X \mid \nu(h_{min}, x) > 0\}$.

Where ν is Lelong number, an analytic counterpart of the multiplicity. **h_{min} can take ∞ even on a point x with $\nu(h_{min}, x) = 0$.**

The following question is closely related to some important problems about the structure of **section rings** of **nef** line bundles.

Question 1 When L is nef and big, can we take a continuous h_{min} ?

In order to answer this question, we study **Zariski-type line bundles**.

We say L is **of Zariski-type** w.r.t. $D \subset X$ when $D \subset Bs(mL)$ and $|mL - D|$: free for all $m \in$ Z

- In this case, L is **nef** and the section ring of L is **infinitely generated**.
- Zariski constructed such an example by blowing up 12 points of projective plane.

We are trying to
construct h_{min} expricitly on such L by using the theory of
hyperbolic foliations by curves.

According to the theory of Brunella, Lins Neto, and so on, **the leafwise Poincaré metric** of a singular hyperbolic foliation F by curves **glues up to define a singular metric h_F of K_F.**

Question 2 For a Zariski-type big line bundle L on X, are there $(\tilde{X}, \tilde{L})$ and m satisfying the following conditions?

- $\tilde{X} \supset X$ and $\tilde{L}|_X$=L .
- There exists a singular hyperbolic foliation F by curves with $K_F = m\tilde{L}$.

Question 3 In this case, does $h_F|_X = {h_{min}}^m$ hold?

Example

E : smooth elliptic curve,
$p, q \in E$: general points
$X := \mathbb{P}(\mathcal{O}(-q) \oplus \mathcal{O}(p))$
$\pi\colon X \to E$: canonical map
$L := \mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_E(p)$
$C := \mathbb{P}(\mathcal{O}(-q)) \subset X$

Then **(L, C) is of Zariski-type**. Now let us set
$\tilde{X} := \mathbb{P}(\mathcal{O}(-q) \oplus \mathcal{O}(p) \oplus \mathcal{O}(-5p) \oplus \mathcal{O}(-p-q))$
$\tilde{\pi}\colon X \to E$
$\tilde{L} := \mathcal{O}_{\tilde{X}}(1) \otimes \pi^*\mathcal{O}_E(p)$
$D_0 := \mathbb{P}(\mathcal{O}(p) \oplus \mathcal{O}(-5q) \oplus \mathcal{O}(-p-q))$
$D_1 := \mathbb{P}(\mathcal{O}(-q) \oplus \mathcal{O}(-5q) \oplus \mathcal{O}(-p-q))$
$D_2 := \mathbb{P}(\mathcal{O}(-q) \oplus \mathcal{O}(p) \oplus \mathcal{O}(-p-q))$
$D_3 := \mathbb{P}(\mathcal{O}(-q) \oplus \mathcal{O}(p) \oplus \mathcal{O}(-5q))$

and set "canonical local coordinate" (x, y, z, t) on $X - D_0$ where t is a coordinate on E and D_1=$\{x = 0\}$, D_2=$\{y = 0\}$, D_3=$\{z = 0\}$ holds.
Then **$X = \{y = z = 0\}$ and $\tilde{L}|_X = L$** holds.

We can construct F **with K_F=$3\tilde{L}$**, whose leaves can be written in the form
$\{t = t_0, x = x_0, f(t_0)y + g(t_0)z^4 = c\}_{t_0, x_0, c}$
with $f \in H^0(E, 4p - 3q)$ and $g \in H^0(E, q)$.
From this construction, we can show that
we can take a continuous h_{min}.