On a neighborhood of an elliptic curve and a gluing construction of K3 surfaces

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- S: a complex surface (non-singular)
- $C \subset S$: holomorphically embedded compact curve such that $N_{C/S} := j^*[C]$: topologically trivial

($j : C \to S$: emb., $[C]$: hol. line bundle which corresp. to the divisor C)

Our main interest:

Complex analytic structure of a (small/tubular) neighborhood W of C in S? What kinds of psh functions are there on W or $W \setminus C$?

c.f. Arnol'd's theorem on the "linearizability" of a neighborhood, Ueda's Classification theory.

Important Remark

Remark

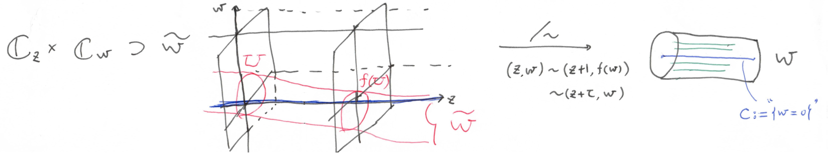
In general, $\nexists W$ which is biholomorphic to a neighborhood of the zero section in $N_{C/S}$ ("holomorphic tubular neighborhood" does not exist in general).

For example, "holomorphic tubular neighborhood" does not exist if

- S admits an elliptic fibration $\pi\colon S \to B$ over a curve B, C is a general fiber, and the Kodaira-Spencer map is injective at the point $\pi(C)$.
- S is a ruled surface over an elliptic curve and C is a section with holomorphically trivial $N_{C/S}$ such that $S \setminus C \cong \mathbb{C}^* \times \mathbb{C}^*$ (Serre's example).

Ueda's example

Fix $\tau \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$ and $f(w) = a_1 w + a_2 w^2 + \cdots \in \mathcal{O}_{\mathbb{C},0}$ ($|a_1| = 1$). Take a neighborhood $\widetilde{W}$ of $\mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}$.
$W := \widetilde{W}/\sim$, $(z, w) \sim (z + 1, f(w)) \sim (z + \tau, w)$
$C \subset W$: the image of $\mathbb{C} \times \{0\}$ (smooth elliptic curve)



Fact [K-, N. Ogawa, arXiv:1808.10219]

C admits a *holomorphic tubular neighborhood* (i.e. $\exists W$ which is biholomorphic to a neighborhood of the zero section)
<u>iff</u> f is *linearizable* around the origin.

Let C be a smooth compact curve. Then $N_{C/S}$ is Hermitian flat.

For the existence of a n.b.h.d. W of C in S s.t. $[C]|_W$ is Hermitian flat:
(Note that $\exists$ such W if C admits a hol. tub. n.b.h.d.)

	Formally, ...		cpt curves on $(\mathrm{nbhd}) \setminus C$
Fibration e.g.	"$\exists W$" formally	$\exists W$	many
Serre's e.g.	"$\nexists W$" formally	$\nexists W$	$\nexists$
$\mathrm{U}_{f:\mathrm{lin},a_1:\mathrm{tor}}$	"$\exists W$" formally	$\exists W$	many
$\mathrm{U}_{f:\mathrm{lin},a_1:\mathrm{non-tor}}$	"$\exists W$" formally	$\exists W$	$\nexists$
$\mathrm{U}_{f:\mathrm{non-lin},a_1:\mathrm{tor}}$	"$\nexists W$" formally	$\nexists W$	$\nexists$
$\mathrm{U}_{f:\mathrm{non-lin},a_1:\mathrm{non-tor}}$	"$\exists W$" formally	$\nexists W$	???

c.f. Ueda's theorems (1983)

I've worked around a generalization of Ueda-type theorems to higher (co-) dimensional and singular cases.

However, in this talk, we will give two kinds of applications of (Arnol'd- and) Ueda-type theorems on the complex structure of a neighborhood of C (a curve in a surface with $c_1(N_{C/S}) = 0$):

(i) a study on non-projective and non-Kummer K3 surfaces

(ii) a study on (non-) existence of a smooth Hermitian metric on a nef line bundle over a projective manifold with semi-positive curvature

Goal of this section:

Gluing construction of non-projective and non-Kummer K3 surfaces.

We will construct a K3 surface X by holomorphically patching two open complex surfaces, say M and M'.

- M (M') is the complement of a (appropriate) tubular neighborhood of an elliptic curve in the blow-up S (S') of the projective plane $\mathbb{P}^2$ at (appropriate) nine points.
- Neither S nor S' admit elliptic fibration structure (nine points are "general")
- In order to patch M and M' holomorphically, we need to take "nice neighborhood". For this purpose, we need to choose nine points carefully.
- Use a technique from Complex Dynamics for taking "nice neighborhood" (Arnol'd's theorem).

Remarks, Known results

- For the case where S and S' are elliptic, a similar construction of K3 surfaces is known. The resulting K3 surfaces are also elliptic.
- M. Doi showed that a similar construction of K3 surfaces if one admit (slight) deformations of the complex structures of M and M'.

  (Doi, Mamoru, Gluing construction of compact complex surfaces with trivial canonical bundle. J. Math. Soc. Japan 61 (2009), no. 3, 853–884)
- The idea to use Arnol'd-type theorem for patching two open manifolds is also used by H. Tsuji in order to study complex structures on $S^3 \times S^3$.

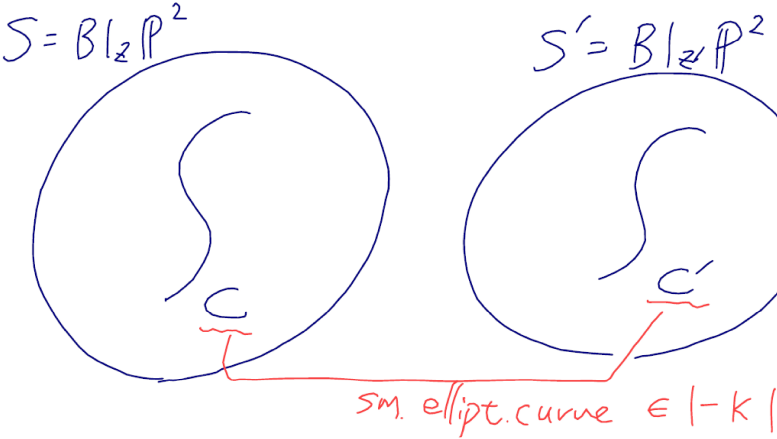
  (H. Tsuji, Complex structures on $S^3 \times S^3$, Tohoku Math. J. (2) Volume 36, Number 3 (1984), 351–376)

Take a smooth elliptic curve $C_0 \subset \mathbb{P}^2$ and nine points $Z := \{p_1, p_2, \dots, p_9\} \subset C_0$.

- $S := \mathrm{Bl}_Z\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$: blow-up at Z
- $C := \pi_*^{-1}C_0$: the strict transform of C_0

Note that $N_{C/S} \cong \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \cdots - p_9)$. When Z is special, S is an elliptic surface ($N_{C/S} \in \mathrm{Pic}^0(C)$ is torsion in this case). We are interested in the case where $\underline{Z \text{ is general}}$.

Let (S', C') be another model which is constructed by another choice of an elliptic curve C_0' and another nine points configuration $Z' := \{p_1', p_2', \dots, p_9'\} \subset C_0'$.


$S = Bl_Z \mathbb{P}^2$
$S' = Bl_{Z'} \mathbb{P}^2$
C
C'
sm. ellipt. curve $\in |-K|$

Assumptions

In what follows, we always assume the following:

Assumption

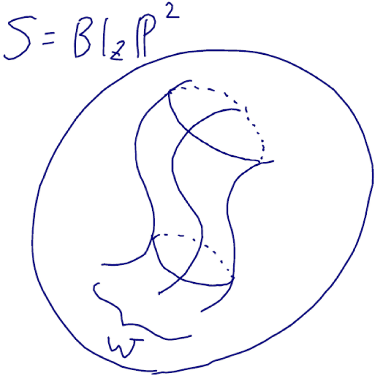
- $\exists g \colon C \cong_{\text{bihol.}} C'$
- $N_{C/S} = g^* N_{C'/S'}^{-1}$
- $N_{C/S} \in \mathrm{Pic}^0(C)$ is Diophantine

$N_{C/S} \in \mathrm{Pic}^0(C)$ is said to be *Diophantine* if $\exists A, \alpha > 0$ such that $\mathrm{dist}(\mathbb{I}_C, N_{C/S}^{\otimes n}) \geq A \cdot n^{-\alpha}$ for $\forall n > 0$.

- $N_{C/S} \in \mathrm{Pic}^0(C)$ is Diophantine for almost every choice of Z in the sense of Lebesgue measure.
- We will explain why do we need this condition latter.

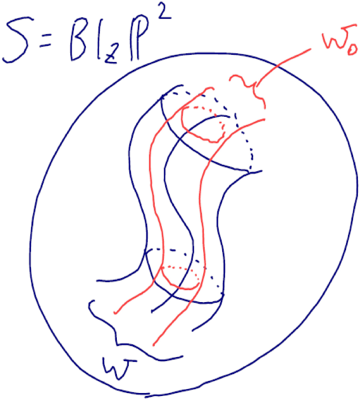
Outline of the construction –Step 1

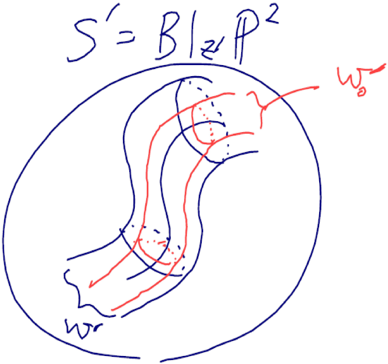
First, we take "nice" neighborhoods W of C in S and W' of C' in S':



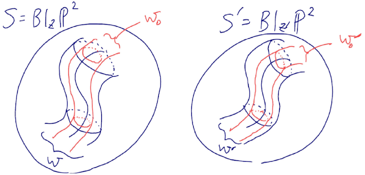
Outline of the construction –Step 2

Next, we take "nice" neighborhoods $W_0 \Subset W$ of C and $W_0' \Subset W'$ of C' appropriately:





Outline of the construction –Step 3



$$M := S \setminus W_0, \quad M' := S' \setminus W_0'$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$W^* := W \setminus W_0 \cong W' \setminus W_0'$$

$$\rightsquigarrow X := M \cup_{W^*} M'$$

Question

How should we choose "nice" neighborhoods W, W_0, W', and W_0' (in order to patch M and M' holomorphically)?

Here we use the following:

Theorem (Arnol'd (1976))

Assume C is a smooth elliptic curve and $N_{C/S} \in \mathrm{Pic}^0(C)$ is Diophantine.
Then C admits a **holomorphic tubular neighborhood** W *(i.e. W can be chosen so that W is biholomorphic to a neighborhood of the zero-section in $N_{C/S}$).*

Arnol'd's theorem is shown by using complex dynamical technique as in the proof of **Siegel's linearization theorem**, which is the reason why Diophantine condition is needed in our assumption.

What follows from Arnol'd's theorem and our assumptions

- "$N_{C/S}$: Diophantine" + Arnol'd's thm
  $\Rightarrow W$, W_0: holomorphic tubular neighborhoods of C
  $\Rightarrow W \backslash W_0 \cong_{\text{bihol.}}$ (an annulus bundle over C)

- "$N_{C/S}$: Diophantine" + "$N_{C/S} = g^* N_{C'/S'}^{-1}$" + Arnol'd's thm
  $\Rightarrow W'$, W_0': holomorphic tubular neighborhoods of C'
  $\Rightarrow W' \backslash W_0' \cong_{\text{bihol.}}$ (an annulus bundle over C')

- "$g \colon C \cong C'$" + "$N_{C/S} = g^* N_{C'/S'}^{-1}$" + observations above
  $\Rightarrow (W^* :=)\ W \backslash W_0 \cong_{\text{bihol.}} W' \backslash W_0'$
  $\Rightarrow$ One can glue M and M' holomorphically by using W^* as a "tab for gluing".

Observation

W^* admits a foliation $\mathcal{F}$ which is naturally defined by considering the flat connection on $N_{C/S}$. Each leaf is biholomorphic to $\mathbb{C}$ or $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

It is easily observed that $\pi_1(X) = 0$. Therefore, for proving that X is a K3 surface, it is sufficient to show the following:

Proposition

There exists a nowhere vanishing holomorphic 2-form σ on X.

Outline of the proof: As $K_S = -C$, there exists a meromorphic 2-form η on S with $\mathrm{div}(\eta) = -C$. We can also take a meromorphic 2-form η' on S' with $\mathrm{div}(\eta') = -C'$. σ is obtained by patching $\eta|_M$ and $-\eta'|_{M'}$ after appropriate normalizations. $\square$

Some remark on the construction of the 2-form σ on X

For patching $\eta|_M$ and $-\eta'|_{M'}$ on W^*, we use the following:

Key Lemma

$H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}$.

Key Lemma is shown by considering the restriction of a given holomorphic function on W^* to a leaf of $\mathcal{F}$ and considering the Maximum principle.

By using this Key Lemma, one can describe the 2-form $\sigma|_{W^*}$ very explicitly.
$\Rightarrow$ We could explicitly compute the integrations $\int \sigma$ along 20 2-cycles of 22 appropriately chosen 2-cycles ("marking" of X).

Main results

We constructed K3 surfaces in such a manner with independent (at least) 19 parameters. As an conclusion of the construction, we have the following:

Theorem (K-, T. Uehara)

There exists a deformation $\pi\colon \mathcal{X} \to B$ of K3 surfaces over a (at least) 19 dimensional complex manifold B with injective Kodaira-Spencer map such that each fiber $X_b := \pi^{-1}(b)$ admits a holomorphic map $F_b\colon \mathbb{C} \to X_b$ with the following property: The Euclidean closure of $F_b(\mathbb{C})$ is a real analytic compact hypersurface of X_b. Especially, $F_b(\mathbb{C})$ is Zariski dense whereas it is not Euclidean dense. X_b is non-Kummer and non-projective for general $b \in B$.

"Degrees of freedom" in our construction

- Choice of C_0, C_0', and a Diophantine line bundle L on C_0 (dimension$=1$ because of $C_0 \cong C_0'$ and Dioph. condition).
- Choice of points $p_1, p_2, \dots, p_8 \in C_0$ (dimension$=8$).
- Choice of points $p_1', p_2', \dots, {p_8}' \in C_0'$ (dimension$=8$).
- Points $p_9 \in C_0$ and $p_9' \in C_0'$ are automatically decided by the condition $N_{C/S} = g^* N_{C'/S'}^{-1} = L$ (dimension$=0$).
- Choice of an isomorphism $g \colon C \cong C'$ (dimension$=1$).
- Choice of the "size" of the tab for gluing W^* (dimension$=1$)

lattice	cycle	$\frac{1}{2\pi\sqrt{-1}}\int\sigma$	corresponding parameter
U	$A_{\beta,\gamma}$	τ	choice of C_0 (and C_0')
	B_α	???	choice of w_j's (R, R',...)
U	$A_{\gamma,\alpha}$	1	—
	B_β	???	choice of w_j's (R, R',...)
$E_8(-1)$	$C_{1,2}$	"$p_2 - p_1$" in C	choice of $p_2 - p_1$
	$C_{2,3}$	"$p_3 - p_2$" in C	choice of $p_3 - p_2$
	$\vdots$	$\vdots$	$\vdots$
	$C_{7,8}$	"$p_8 - p_7$" in C	choice of $p_8 - p_7$
	$C_{6,7,8}$	"$p_6 + p_7 + p_8$" in C	choice of $p_6 + p_7 + p_8$
$E_8(-1)$	$C'_{1,2}$	"$p'_2 - p'_1$" in C'	choice of $p'_2 - p'_1$
	$C'_{2,3}$	"$p'_3 - p'_2$" in C'	choice of $p'_3 - p'_2$
	$\vdots$	$\vdots$	$\vdots$
	$C'_{7,8}$	"$p'_8 - p'_7$" in C'	choice of $p'_8 - p'_7$
	$C'_{6,7,8}$	"$p'_6 + p'_7 + p'_8$" in C	choice of $p'_6 + p'_7 + p'_8$
U	$A_{\alpha,\beta}$	$a_\beta - \tau \cdot a_\alpha$	choice of p_9 and p'_9 (i.e. $N_{C/S}$ and $N_{C'/S'}$)
	B_γ	"$p'_9 - g(p_9)$"	choice of $g\colon C \cong C'$

Question

For the previous example $(C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S)$, *does* C *admit a holomorphic tubular neighborhood when* $N_{C/S} \in \mathrm{Pic}^0(C)$ *is not Diophantine?*

- X: projective manifold
- $L \to X$ **nef** line bundle

(i.e. $(L.C) := \int_{C_{\mathrm{reg}}} c_1(L) \geq 0$ for any effective compact curve C)

Main question in this section:

When is L semi-positive?

Definition

L is semi-positive iff $\exists$ C^∞'ly smooth Hermitian metric h on L with semi-positive Chern curvature (i.e. $\sqrt{-1}\Theta_h \geq 0$).

c.f. Kodaira's embedding theorem + Nakai–Moishezon criterion

- L: semi-positive $\implies$ L: nef (easy)
- Serre's example gives a counterexample for the converse implication (Demailly–Peternell–Schneider).

Let

- S: a complex surface (non-singular)
- $C \subset S$: holomorphically embedded compact curve such that $N_{C/S} := j^*[C]$: topologically trivial

as before. Set

- $X := S$, and
- $L := [C]$.

L is nef (at least if $X = S$ is projective).

For example, ...

- When S admits an elliptic fibration $\pi\colon S \to B$ over a curve B and C is a general fiber, $L := [C]$ is **semi-positive** (consider π^*(the Funibi-Study metric)).
- (S, C) is the Serre's example, $L := [C]$ is **not semi-positive** (Demailly–Peternell–Schneider).

Conjecture

$L = [C]$ is semi-positive iff $\exists$ a n.b.h.d. W of C in S s.t. $L|_W$ is Hermitian flat?

c.f. Brunella's technique in his paper in 2010 on $S = \mathrm{Bl}_Z\mathbb{P}^2$.

Definition of "Ueda type" ($(\alpha), (\beta)$ and (γ))

Let C be a smooth compact curve.
For the existence of a n.b.h.d. W of C in S s.t. $[C]|_W$ is Hermitian flat:
(Note that $\exists$ such W if C admits a hol. tub. n.b.h.d.)

	Formally, ...	
type (α)	"$\nexists W$ " formally	$\nexists W$
type (β)	"$\exists W$" formally	$\exists W$
type (γ)	"$\exists W$ " formally	$\nexists W$

- $\{(S, C) \mid \text{type}(\alpha)\} \ni$ Serre's e.g., $\mathrm{U}_{f:\text{non-lin},a_1:\text{tor}}$, $\cdots$
- $\{(S, C) \mid \text{type}(\beta)\} \ni$ fibration e.g., $\mathrm{U}_{f:\text{lin},a_1:\text{tor}}$, , $\mathrm{U}_{f:\text{lin},a_1:\text{non-tor}}$, $\cdots$
- Essentially only the know example of (S, C) of type (γ) is $\mathrm{U}_{f:\text{non-lin},a_1:\text{non-tor}}$.

Main results

Theorem (K-' 14, 15, 17...)

When C is smooth, then

- (S, C): *of type* (α) $\Longrightarrow$ L: **not** *semi-positive.*
- (S, C): *of type* (β) $\Longrightarrow$ L: *semi-positive.*
- (S, C): $\mathrm{U}_{f:\mathrm{non-lin}, a_1:\mathrm{non-tor}}$ $\Longrightarrow$ L: **not** *semi-positive.*

When C is a cycle of rational curves, then

- $N_{C/S}$: *Hermitian flat (i.e.* $\in S^1 \subset \mathbb{C}^* \cong \mathrm{Pic}^0(C)$*)* $\Longrightarrow$ *"similar" to the smooth case.*
- $N_{C/S}$: not *Hermitian flat (i.e.* $\notin S^1 \subset \mathbb{C}^* \cong \mathrm{Pic}^0(C)$*)* $\Longrightarrow$ L: **not** *semi-positive.*

Question

For the previous example $(C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S)$, does K_S^{-1} $(= [C])$ semi-positive when $N_{C/S} \in \mathrm{Pic}^0(C)$ is not Diophantine?

c.f.

Question (repeated)

For the previous example $(C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S)$, does C admit a holomorphic tubular neighborhood when $N_{C/S} \in \mathrm{Pic}^0(C)$ is not Diophantine?

Corollary

There exists a nine points configuration $Z = \{p_1, p_2, \dots, p_9\} \subset \mathbb{P}^2$ such that K_S^{-1} is nef however it is **not** *semi-positive for $S := \mathrm{Bl}_Z\mathbb{P}^2$ (C_0: nodal in this case).*