Higher codimensional Ueda theory for a compact submanifold with unitary flat normal bundle (arXiv:1606.01837)

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1. Configurations and our main interest

X: complex manifold,
$Y \subset X$: compact complex submanifold of codimension $r \geq 1$ with *unitary flat* normal bundle.
i.e. $N_{Y/X} \in \mathrm{Image}\,(H^1(Y, U(r)) \to H^1(Y, GL_r(\mathcal{O}_Y))) =: \mathcal{P}_r(Y)$.

Main interest

Compare a neighborhood of Y in X and a neighborhood of the zero section in $N_{Y/X}$. □

Main interest in local description:
Let $\{U_j\}$ be an open covering of Y and $\{V_j\}$ be a neighborhood of U_j in X. *Is there a defining functions system* $w_j = (w_j^1, w_j^2, \dots, w_j^r)$ *of* U_j *in* V_j *with* $w_j = T_{jk} w_k$*?*
(i.e. $w_j^\lambda = (T_{jk} w_k)^\lambda := \sum_{\mu=1}^r (T_{jk})_\mu^\lambda \cdot w_k^\mu$ holds on each V_{jk}, where $T_{jk} \in U(r)$ is the transition matrix of $N_{Y/X}$)

2. Obstruction classes and the type of the pair

As $T_{jk} \in U(r)$ is the transition matrix of $N_{Y/X}$, one can take w_j with $dw_j = T_{jk} dw_k$ on each U_{jk}: i.e.

$$(T_{jk} w_k)^\lambda = w_j^\lambda + O(|w_j|^2) = w_j^\lambda + \textstyle\sum_{|\alpha|\geq 2} f_{kj,\alpha}^\lambda(z_j) \cdot w_j^\alpha$$

for some $f_{kj,\alpha}^\lambda$ ($\alpha \in \mathbb{Z}_{\geq 0}^r$, $|\alpha| := \sum_{\lambda=1}^r \alpha_\lambda$, $w_j^\alpha := \prod_{\lambda=1}^r (w_j^\lambda)^{\alpha_\lambda}$).

$$u_1(Y,X) := [\{(U_{jk}, \textstyle\sum_{|\alpha|=2}\sum_{\lambda=1}^r f_{kj,\alpha}^\lambda \cdot (\partial/\partial w_j^\lambda) \otimes (dw_j)^\alpha)\}] \in H^1(Y, N_{Y/X} \otimes S^2 N_{Y/X}^*).$$

If $u_1(Y,X) = 0$, then one can take $\{w_j\}$ such that

$$(T_{jk} w_k)^\lambda = w_j^\lambda + O(|w_j|^3) = w_j^\lambda + \textstyle\sum_{|\alpha|\geq 3} f_{kj,\alpha}^\lambda(z_j) \cdot w_j^\alpha.$$

$$u_2(Y,X) := [\{(U_{jk}, \textstyle\sum_{|\alpha|=3}\sum_{\lambda=1}^r f_{kj,\alpha}^\lambda \cdot (\partial/\partial w_j^\lambda) \otimes (dw_j)^\alpha)\}] \in H^1(Y, N_{Y/X} \otimes S^3 N_{Y/X}^*).$$

If $u_2(Y,X) = 0, \dots$

Properties of $u_n(Y,X)$ and Definition of the type

- $u_n(Y,X) \in H^1(Y, N_{Y/X} \otimes S^{n+1} N_{Y/X}^*)$.
- "$u_n(Y,X) = 0$" does not depend on the choice of $\{w_j\}$.
- $\mathrm{type}\,(Y,X) := \max\{n \mid u_\nu(Y,X) = 0\ \forall \nu < n\} \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ □

3. Main result

$\mathcal{E}_0^{(r)}(Y) := \{E \in \mathcal{P}_r(Y) \mid \#(\mathrm{Image}\,\rho_E) < \infty\}$, where $\rho_E \colon \pi_1(Y,*) \to U(r)$ is the monodromy of E.

$$\mathcal{E}_1^{(r)}(Y) := \bigcup_{\pi\colon \widetilde{Y} \to Y:\text{finite normal covering}} \left\{E \in \mathcal{P}_r(Y) \,\middle|\, \pi^*E \in \mathcal{S}_A^{(r)}(\widetilde{Y}) \text{ for some } A > 0\right\}, \text{ where}$$

$$\mathcal{S}_A^{(r)}(\widetilde{Y}) := \left\{\bigoplus_{\lambda=1}^r L_\lambda \,\middle|\, \begin{array}{l} L_\lambda \in \mathcal{P}_1(\widetilde{Y}),\ d\left(\mathbb{I}_{\widetilde{Y}}^{(1)}, \bigotimes_{\lambda=1}^r L_\lambda^{a_\lambda}\right) \geq \frac{1}{(2|a|)^A} \\ \text{for } a = (a_\lambda)_\lambda \in \mathbb{Z}^r \text{ with } |a| \geq 1 \end{array}\right\}.$$

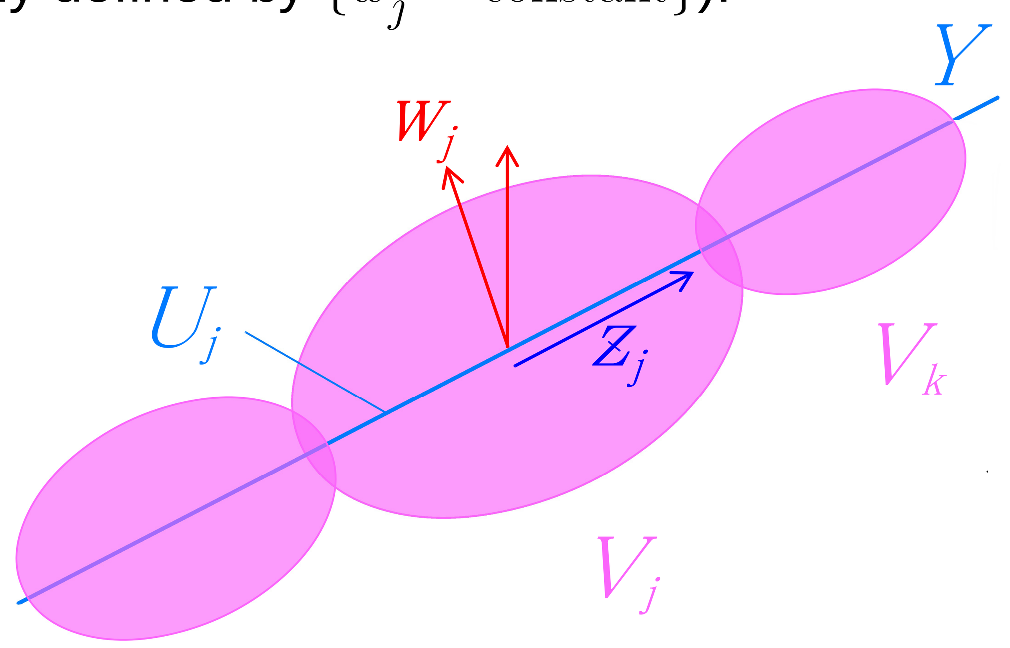
Main Theorem

Assume $\mathrm{type}\,(Y,X) = \infty$ *and* $N_{Y/X} \in \mathcal{E}_0^{(r)}(Y) \cup \mathcal{E}_1^{(r)}(Y)$. *Then the following holds:*
(i) *There exists a non-singular holomorphic foliation* $\mathcal{F}$ *of codimension* r *on some neighborhood* V *of* Y *which includes* Y *as a leaf with* $\mathrm{Hol}_{\mathcal{F},Y} = \rho_{N_{Y/X}}$.
(ii) *For each hypersurface* S *such that* $Y \subset S$ *and* $N_{Y/S}$ *is unitary flat, there exists a non-singular holomorphic foliation* $\mathcal{G}_S$ *of codimension* 1 *on* V *with the following properties by shrinking* V *if necessary:* $\mathcal{G}_S$ *includes* $S \cap V$ *as a leaf with* $U(1)$*-linear holonomy, and each leaf of* $\mathcal{F}$ *is holomorphically immersed into a leaf of* $\mathcal{G}_S$.
□

The assertions (i) and (ii) in local description:
(i) means that one can take $\{w_j\}$ with $w_j = T_{jk} w_k$ holds on each V_{jk} (The leaves of $\mathcal{F}$ is locally defined by $\{w_j = \mathrm{constant}\}$).
(ii) means that there exists a convergent power series $F^\lambda \in \mathbb{C}\{X^1, X^2, \dots, X^r\}$ such that, by setting

$$\widehat{w}^\lambda := F^\lambda(w_j^1, w_j^2, \dots, w_j^r),$$

$\{\widehat{w}_j\}$ is a new defining functions system with $w_j = \widehat{T}_{jk} w_k$ holds on each V_{jk} for some $\widehat{T}_{jk} \in U(r)$ and $\{\widehat{w}_j^1 = 0\} = S \cap V_j$ (The leaves of $\mathcal{G}_S$ is locally defined by $\{w_j^1 = \mathrm{constant}\}$).



4. History

Arnol'd : The case where Y is an elliptic curve [A].
Ueda : The case where $r = 1$ [U].
K-, Ogawa : The case where $r = 2$ [K], [KO].

5. Application

Theorem (ii) can be applied to the "semi-positivity problem" on a nef line bundle, since the assertion (ii) implies the unitary flatness of the line bundle $[S]$ on V. For example:

Corollary (Application to the semi-positivity problem)

Let X *be a complex manifold of dimension* n *and* L *be a holomorphic line bundle on* X. *Take* $D_1, D_2, \dots, D_{n-1} \in |L|$. *Assume that* $C := \bigcap_{\lambda=1}^{n-1} D_\lambda$ *is a smooth elliptic curve,* $L|_C \in \mathcal{E}_1^{(1)}(C)$*, and* $\{D_\lambda\}_{\lambda=1}^{n-1}$ *intersects transversally along* C. *Then* L *is semi-positive (i.e.* L *admits a* C^∞ *Hermitian metric with semi-positive curvature).* □

Note that L as in Corollary has C as a stable base locus: $C = SB(L) := \bigcap_{m\geq 1} \mathrm{Bs}\,|L^m|$.

Example

Let (V, L) *be a del Pezzo manifold of degree* 1 *(i.e.* V *is a projective manifold of dimension* n *and* L *is an ample line bundle on* V *with* $K_V^{-1} \cong L^{n-1}$ *and the self-intersection number* (L^n) *is equal to* 1*), and* $C \subset V$ *be an intersection of general* $n-1$ *elements of* $|L|$. *For each point* $q \in C$ *with* $L|_C \otimes [-q] \in \mathcal{E}_1^{(0)}(C) \cup \mathcal{E}_1^{(1)}(C)$*, the anti-canonical bundle of the blow-up of* V *at* q *is semi-positive.* □

This example can be regarded as a generalization of the known semi-positivity criterion for the anti-canonical bundle of the blow-up of $\mathbb{P}^2$ at 9 points [A], [U], [B].

References

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