CORRECTION TO "TOWARD A HIGHER CODIMENSIONAL UEDA THEORY"

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ABSTRACT. Recently, we found that the main theorem [K, Theorem 1] (T. KOIKE, Toward a higher codimensional Ueda theory, Math. Z., Volume 281, Issue 3 (2015), 967–991) was not correct. We have to add some assumptions in [K, Theorem 1]. The main application [K, Corollary 1] needs no correction.

1. CORRECTED FORM OF [K, Theorem 1]

Corrected form of [K, Theorem 1] is the following:

THEOREM 1.1. *Let X be a complex manifold, S a smooth hypersurface of X, and C be a smooth compact hypersurface of S such that $N_{S/X}|_V$ is flat, where V is a sufficiently small neighborhood of C in S. Assume one of the following two conditions holds: (i) $N_{C/S} \in \mathcal{E}_0(C)$, $N_{S/X}|_C \in \mathcal{E}_0(C)$, (ii) $N_{C/S}$ and $N_{S/X}|_C$ are isomorphic to each other and they are elements of $\mathcal{E}_1(C)$. Further assume that $u_{n,m}(C, S, X; \{w_j\}) = 0$ holds for all $n \geq 1, m \geq 0$ and for all system $\{w_j\}$ of order (n, m), and that there exists a system of local defining functions of C in V of extension type infinity. Then there exists a neighborhood W of C in X such that $\mathcal{O}_X(S)|_W$ is flat. Moreover, there exists a smooth hypersurface Y of W which intersects S transversally along C.*

In the above statement, we removed the case (iii) from [K, Theorem 1] and added the assumption on the existence of a system of local defining functions of C in V *of extension type infinity*, which is the notion we posed in [KO]. As a result, we could also add the conclusion on the existence of the transversal Y to [K, Theorem 1]. For the proof of Theorem 1.1, see [KO, §3.4].

Let us explain some terms in Theorem 1.1. We say the line bundle L on a manifold M is *flat* if the transition functions are chosen as constant functions valued in $U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$ (i.e. $L \in H^1(M, U(1))$). We denote by $\mathcal{E}_0(C)$ the set of all flat line bundles F such that there exists a positive integer n with $F^n = \mathbb{I}$, where $\mathbb{I}$ is the holomorphically trivial line bundle. We denote by $\mathcal{E}_1(C)$ the set of all flat line bundles F which satisfies the condition $|\log d(\mathbb{I}, F^n)| = O(\log n)$ as $n \to \infty$, where d is an invariant distance of the Picard group ($\mathcal{E}_1(C)$ does not depend on the choice of d, see [U, §4.1]). Let (C, S, X) be as in Theorem 1.1. In [K, §3.1], we defined the obstruction class $u_{n,m}(C, S, X) = u_{n,m}(C, S, X; \{w_j\}) \in H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m})$ for each $n \geq 1, m \geq 0$ and for each *system* $\{w_j\}$ *of order* (n, m). We here explain the meaning of our new assumption "there exists a system of local defining functions of C in V of extension type infinity". Let V be a sufficiently small tubular neighborhood of C in S and W be a sufficiently small tubular neighborhood of C in X such that $W \cap S = V$. Take a sufficiently fine open covering

$\{U_j\}$ of C, $\{V_j\}$ of V, and $\{W_j\}$ of W such that $V_j = W_j \cap S$, $U_j = V_j \cap C$, and $U_{jk} := U_j \cap U_k = \emptyset$ iff $W_{jk} := W_j \cap W_k = \emptyset$. Extend a coordinates system x_j of U_j to W_j. Let y_j be a defining function of U_j in V_j and w_j a defining function of V_j in W_j. As both $N_{S/X}$ and $N_{C/S}$ are flat in our settings, we may assume that $t_{jk}w_k = w_j + O(w_j^2)$ holds on W_{jk} and $s_{jk}y_k = y_j + O(y_j^2)$ holds on V_{jk} for some constants $t_{jk}, s_{jk} \in U(1)$. The assumption "there exists a system of local defining functions of C in V of extension type infinity" means that we can choose such $\{y_j\}$ with the following two additional properties: (a) $s_{jk}y_k = y_j$ holds on V_{jk} for each j and k, and (b) $\{y_j\}$ is *of extension type infinity* in the sense of [KO, Definition 3.2]: i.e. the class $v_{n,m}(C,S,X;\{z_j\}) \in H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m+1})$ is equal to zero for each $n \geq 1, m \geq 0$ and for *any* type (n,m) extension $\{z_j\}$ of $\{y_j\}$ (the class $v_{n,m}(C,S,X;\{z_j\})$ is the obstruction class we posed in [KO]).

We here remark that, as we will see later, [K, Corollary 1] needs no correction. It is because the example of general 8 points blow-up of $\mathbb{P}^3$ automatically satisfies this condition.

2. Details of the mistakes

There are following three mistakes in [K]: one is on the well-definedness of the obstruction classes, another one is in the statement of [K, Lemma 1], and the other one is in a Taylor expansion in Lemma 6 and Lemma 7. In this section, we explain the details of these mistakes.

2.1. Mistake on the well-definedness of the obstruction classes.

The first one is on the well-definedness of the obstruction classes. In [K, Proposition 3], we stated that the (n,m)-th Ueda class $u_{n,m}(C,S,X)$ of the triple (C,S,X) is independent of the choice of a system $\{w_j\}$ of order (n,m) up to non-zero constant multiples. However, we found a critical mistake in the proof. Thus, now we should denote the obstruction class by $u_{n,m}(C,S,X;\{w_j\})$. See also [KO, §2.2.2, §3.2.2].

2.2. Mistake in the statement of [K, Lemma 1].

We also found a mistake in [K, Lemma 1], which is an open analogue of [KS, Lemma 2]. The corrected form of [K, Lemma 1] should be stated as follows:

Lemma 2.1 (Corrected form of [K, Lemma 1]). *Let C be a compact complex manifold embedded in a complex manifold S Fix a sufficiently small connected neighborhood V of C in S and a sufficiently fine open covering $\{V_j\}$ of V which consists of a finite number of open sets. Fix also a relatively compact open domain $V_0 \subset V$ which contains C. For each flat line bundle E on V, there exists a positive constant $K = K(E)$ such that, for each 1-cocycle $\alpha = \{(V_{jk}, \alpha_{jk})\}$ of E which can be realized as the coboundary of some 0-cochain, there exists a 0-cochain $\beta = \{(V_j \cap V_0, \beta_j)\}$ of E such that $\alpha|_{V_0}$ is equal to the coboundary $\delta(\beta)$ of β and the inequality*

$$\max_j \sup_{V_0 \cap V_j} |\beta_j| \leq K \cdot \max_{jk} \sup_{V_0 \cap V_{jk}} |\alpha_{jk}|$$

holds.

This mistake is critical for proving [K, Theorem 1] for the case (iii), which is why we had to remove this case.

2.3. **Mistake in a Taylor expansion in Lemma 6 and Lemma 7.** Here we explain the mistake under the configuration of Lemma 7. Lemma 7 is the lemma for defining the system of functions $\{G_j^{(n,m)}\}$ inductively: i.e. assuming that $\{G_j^{(\nu,\mu)}\}$ has already defined for each $(\nu,\mu)<(n,m)$, we are stating how to define $\{G_j^{(n,m)}\}$ in this lemma. For the definition of $\{G_j^{(n,m)}\}$, we regard $G_j^{(\nu,\mu)}z_j^\mu$ as the function defined on W_j which does not depend on the variable w_j and considered the expansion

$$\begin{aligned} G_j^{(\nu,\mu)}(x_j)\cdot z_j^\mu &= G_j^{(\nu,\mu)}(x_j(x_k,z_k,w_k))\cdot z_j(x_k,z_k,w_k)^\mu \\ &= G_j^{(\nu,\mu)}(x_j(x_k,0,0))\cdot s_{jk}^\mu z_k^\mu \\ &\quad +\sum_{q=1}^\infty G_{jk,0,q}^{(\nu,\mu)}(x_k)\cdot z_k^{\mu+q}+\sum_{p=1}^\infty\sum_{q=0}^\infty G_{jk,p,q}^{(\nu,\mu)}(x_k)\cdot z_k^{\mu+q}w_k^p \end{aligned}$$

on W_{jk}, in which we made a mistake. This expansion should be

$$\begin{aligned} G_j^{(\nu,\mu)}(x_j)\cdot z_j^\mu &= G_j^{(\nu,\mu)}(x_j(x_k,z_k,w_k))\cdot z_j(x_k,z_k,w_k)^\mu \\ &= G_j^{(\nu,\mu)}(x_j(x_k,0,0))\cdot s_{jk}^\mu z_k^\mu \\ &\quad +\sum_{q=1}^\infty G_{jk,0,q}^{(\nu,\mu)}(x_k)\cdot z_k^{\mu+q}+\sum_{p=1}^\infty\sum_{q=0}^\infty G_{jk,p,q}^{(\nu,\mu)}(x_k)\cdot z_k^{q}w_k^p. \end{aligned}$$

Even after this correction, the inductive definition of $\{G_j^{(n,m)}\}$ can be executed just as in Lemma 7. However, the norm estimate problem occurs in this case so that we can not show the convergence of the functional equation (8) in [K, §4]. To avoid this difficulty, we have to refine not only the system $\{w_j\}$, but also the extension $\{z_j\}$ of $\{y_j\}$ by using a suitable functional equation at the same time (with fixing only $\{x_j\}$ and $\{y_j\}$), see [KO, §3.4] for the details.

We here remark that, by the same reason, we also have to correct [K, Proposition 4, Lemma 5].

3. Proof of [K, Corollary 1]

Here we prove the following:

Corollary 3.1 (=[K, Corollary 1]). *Let $C_0\subset\mathbb{P}^3$ be a complete intersection of two quadric surfaces of $\mathbb{P}^3$ and let $p_1,p_2,\dots,p_8\in C_0$ be 8 points different from each other. Assume $\mathscr{O}_{\mathbb{P}^3}(-2)|_{C_0}\otimes\mathscr{O}_{C_0}(p_1+p_2+\cdots+p_8)\in\mathscr{E}_1(C_0)$. Then the anti-canonical bundle of the blow-up of $\mathbb{P}^3$ at $\{p_j\}_{j=1}^8$ is not semi-ample, however admits a smooth Hermitian metric with semi-positive curvature.*

Proof of Corollary 3.1. We use the notations in [K, §5.2]. We apply Theorem 1.1 to the triple (C,S_0,X). We here remark that the existence of the transversal Y is clear in this example (consider $Y:=S_\infty$).

All we have to do here is to check the added condition "there exists a system of local defining functions of C in V of extension type infinity". Let $\{s_{jk}\}$ and $\{y_j\}$ be as in §1 here. As $u_n(C, S_0) \in H^1(C, N^{-n}) = 0$ for each $n \geq 1$, we can conclude from [U, Theorem 3] that we may assume the condition (a) $s_{jk}y_k = y_j$ holds on V_{jk} for each j and k. We will check the condition (b) the class $v_{n,m}(C, S, X; \{z_j\}) \in H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m+1})$ is equal to zero for each $n \geq 1, m \geq 0$ and for *any* type (n, m) extension $\{z_j\}$ of $\{y_j\}$. First we will check the case where $(n, m) = (1, 0)$. Note that the class $v_{1,0}(C, S, X; \{z_j\})$ does not depend on the choice of an extension $\{z_j\}$ of $\{y_j\}$ (nor a system $\{w_j\}$). It can be shown by just the same (and much more simple) argument as in [KO, §3.2.2]. Thus, it is sufficient to show that $v_{1,0}(C, S, X; \{z_j\}) = 0$ for a suitably fixed extension $\{z_j\}$ of $\{y_j\}$. For this purpose, let us fix an extension z_j of y_j such that z_j is a defining function of $W_j \cap S_\infty$. Let

$$s_{jk}z_k = z_j + p_{jk}^{(1)}(x_j, z_j) \cdot w_j + O(w_j^2)$$

be the expansion in w_j and

$$p_{jk}^{(1)}(x_j, y_j) = q_{jk}^{(1,0)}(x_j) + O(y_j)$$

be the expansion of $p_{jk}^{(1)}|_{V_{jk}}$ in y_j for each j and k. As $s_{jk}z_k/z_j$ is holomorphic around $W_{jk} \cap S_\infty$, we obtain that $p_{jk}^{(1)}(x_j, z_j)$ can be divided by z_j. Therefore we obtain that $v_{1,0}(C, S, X; \{z_j\}) = [\{q_{jk}^{(1,0)}\}] \equiv [\{0\}] = 0$. Next we will check the case where $(n, m) > (1, 0)$. In this case, as N is non-torsion and $n+m-1 > 0$, we obtain that $H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m+1}) = H^1(C, N^{-n-m+1}) = 0$ holds, which proves the assertion. $\square$

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