Ueda's classification – type (β)

Setup

- X: complex manifold of $\dim_{\mathbb{C}} = 2$,
- $Y \subset X$: compact (non-singular) Riemann surface holomorphically embedded in X with $c_1(N_{Y/X}) = 0$.

Definition (type (β))

The pair (Y, X) is said to be *of type* (β) if there exists a (non-singular) holomorphic foliation $\mathcal{F}$ defined on a neighborhood of Y which also has Y as a leaf and has $U(1)$*-linear holonomy* along Y
(i.e. the image of the holonomy function $\mathrm{Hol}_{\mathcal{F},Y} : \pi_1(Y, *) \to \mathcal{O}^*_{(\mathbb{C},0)}$ is a subgroup of $U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$).

Observation: (Y, X): of type (β) if Y admits a holomorphic tubular neighborhood ($\Leftarrow$ the fact that $N_{C/X}$ admits $U(1)$-flat connection).

Idea of Ueda's classification theory

Idea

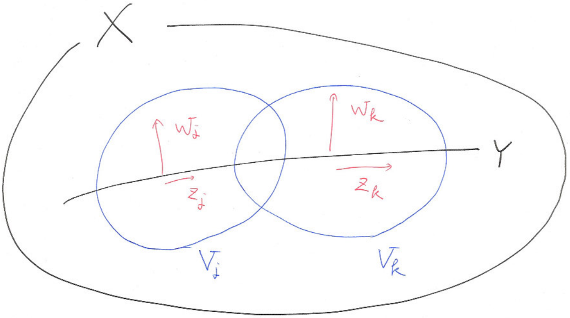
Classify (Y, X) in accordance with the difference from "the case of type (β)" in n-jet sense (along Y, $n \in \mathbb{Z}_{>0}$).

In what follows, we will try to explain Ueda's classification theory in the following steps:

Step 1: Alternative definition of type (β) by using local defining functions
Step 2: The notion "local defining functions of type n"
Step 3: Ueda's obstruction class $u_n(Y, X)$
Step 4: Definition of type (α) and (γ)
Step 5: Ueda's theorems on the classification

Step1: Alternative definition of type (β)

Take an open covering $\{V_j\}$ of a small neighborhood V of Y and a holomorphic coordinates system (z_j, w_j) of each V_j as follows:



- z_j: an extension of a coordinate z_j on $V_j \cap Y$
- w_j: a local defining function of $V_j \cap Y$

Step1: Alternative definition of type (β) (continuation)

From the following, we may assume that there exists $t_{jk} \in U(1)$ such that

$$\left.\frac{w_j}{w_k}\right|_{V_{jk}\cap Y} \equiv t_{jk}$$

holds on each $V_{jk} := V_j \cap V_k$.

Theorem

Let Y be a compact Kähler manifold and N be a line bundle on Y. Assume that $c_1(N) = 0$. Then N is $U(1)$-flat (i.e. the transition functions $\in U(1)$ for a suitable choice of a local trivialization of N).

Therefore, we have the following form of the expansion of the function $t_{jk}w_k|_{V_{jk}}$ by w_j:

$$t_{jk}w_k = w_j + f_{jk}^{(2)}(z_j)\cdot w_j^2 + f_{jk}^{(3)}(z_j)\cdot w_j^3 + f_{jk}^{(4)}(z_j)\cdot w_j^4 + O(w_j^5)$$

Step1: Alternative definition of type (β) (continuation)

Definition (alternative definition of type (β))

The pair (Y, X) is said to be *of type* (β) if

$$t_{jk}w_k = w_j$$

holds on each V_{jk} by choosing w_j's appropriately.

c.f. $\mathcal{F} :=$ "$\{w_j \equiv (\text{constant})\}$"

Definition (type (β), repeated)

The pair (Y, X) is said to be *of type* (β) if there exists a (non-singular) holomorphic foliation $\mathcal{F}$ defined on a neighborhood of Y which also has Y as a leaf and has $U(1)$-*linear holonomy* along Y.

Step2: Local defining functions of type n

$\{(V_j, (z_j, w_j))\}$: as above ($t_{jk}w_k = w_j + f_{jk}^{(2)}(z_j) \cdot w_j^2 + \cdots$).

Definition (Local defining functions of type n)

$\{w_j\}$ is said to be *of type* n if, for any $\nu \leq n$, it holds that $f_{jk}^{(\nu)} \equiv 0$ for each j, k.

i.e.

$$t_{jk}w_k = w_j + f_{jk}^{(n+1)}(z_j) \cdot w_j^{n+1} + f_{jk}^{(n+2)}(z_j) \cdot w_j^{n+2} + \cdots$$

holds for $\{w_j\}$ of type n.

$\exists\{w_j\}$ of type $n \Leftrightarrow$ "(Y, X) seems to be type (β) in n-jet along Y"

Note: Our $\{w_j\}$ is always at least of type 1.

Step3: Ueda's obstruction class $u_n(Y, X)$

Assume that $\exists\{w_j\}$ of type n. One can deduce from

$$t_{jk}w_k = w_j + f_{jk}^{(n+1)}(z_j) \cdot w_j^{n+1} + f_{jk}^{(n+2)}(z_j) \cdot w_j^{n+2} + \cdots .$$

that

$$\left(\frac{1}{t_{jk}w_k}\right)^n = \left(\frac{1}{w_j}\right)^n \cdot (1 - f_{jk}^{(n+1)}(z_j) \cdot w_j^n + O(w_j^{n+1}))^n.$$

Therefore,

$$f_{jk}^{(n+1)}|_{V_j \cap Y} = \frac{1}{n}\left[\frac{1}{w_j^n} - t_{jk}^{-n}\frac{1}{w_k^n}\right]\Bigg|_{V_j \cap Y}.$$

From the calculation above, we have that

Prop.

$\left\{\left(V_j \cap Y, f_{jk}^{(n+1)}\right)\right\}$ *satisfies the* 1*-cocycle condition as sections of* $N_{Y/X}^{-n}$.

Step3: Ueda's obstruction class $u_n(Y, X)$ (continuation)

Definition

$u_n(Y, X) := \left[\left\{\left(V_j \cap Y, f_{jk}^{(n+1)}\right)\right\}\right] \in H^1(Y, N_{Y/X}^{-n})$: n*-th Ueda class.*

Here we denote by $H^1(Y, N_{Y/X}^{-n})$ the 1-st Čech cohomology group $\check{H}^1(Y, \mathcal{O}_Y(N_{Y/X}^{-n}))$ of the sheaf of holomorphic sections of $N_{Y/X}^{-n}$.

Key Prop.

(1) *When* $\exists\{w_j\}$ *of type* n*, the condition "*$u_n(Y, X) = 0$*" does not depend on the choice of* $\{w_j\}$ *of type* n*.*
(2) *Assume that* $\exists\{w_j\}$ *of type* n*. Then* $u_n(Y, X) = 0$ *iff* $\exists\{w_j\}$ *of type* $n + 1$*.*

Step4: Definition of type (α) and (γ)

By the Key Proposition in the previous page, only one of the following holds:

- $\exists n \in \mathbb{Z}_{>0}$ s.t. $\exists \{w_j\}$ of type n and $u_n(Y, X) \neq 0$.
- $\forall n \in \mathbb{Z}_{>0}$, $\exists \{w_j\}$ of type n and $u_n(Y, X) = 0$.

In the former case, (Y, X) is said to be *of finite type* (or more precisely, *of type* n).
In the latter case, (Y, X) is said to be *of infinite type*.

Note: (Y, X) of type $(\beta) \Rightarrow$ of infinite type.

Definition

(Y, X) is said to be *of type* (α) if it is of finite type.
(Y, X) is said to be *of type* (γ) if it is of infinite type however it is not of type (β).

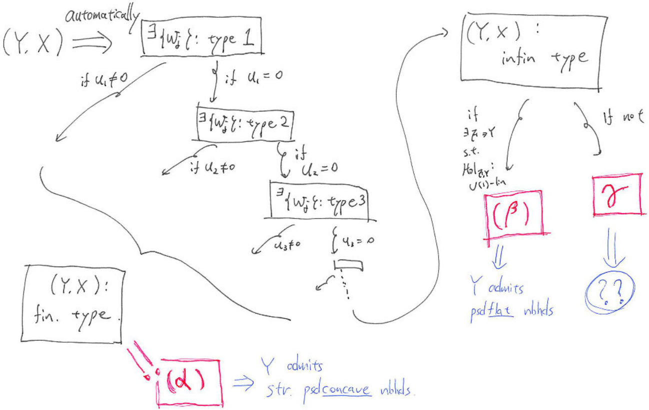
Step5: Ueda's theorems on the classification

Theorem (Ueda '83)

(1) $N_{Y/X} \in \mathrm{Pic}^0(C)$: torsion $\Rightarrow$ (Y, X): of type (α) or (β).
(2) $N_{Y/X} \in \mathrm{Pic}^0(C)$: Diophantine (see below) $\Rightarrow$ (Y, X): of type (β).
(3) (Y, X): of type (α) $\Rightarrow$ there exists a $\mathbb{R}$-valued function Φ on a neighborhood V of Y s.t. $\Phi|_{V\setminus Y}$: s.p.s.h, $\Phi(p) \to +\infty$ as $p \to Y$. Especially, Y has a str. pseudoconcave neighborhoods system in this case.
(4) $\exists$ an example (Y, X) of type (γ).

Observation: When (Y, X): of type (β), then there exists a $\mathbb{R}$-valued function Φ on a neighborhood V of Y s.t. $\Phi|_{V\setminus Y}$: pluriharmonic, $\Phi(p) \to +\infty$ as $p \to Y$ ($\Leftarrow \Phi(z_j, w_j) := \log|w_j|$). Especially, Y has a psudoflat neighborhoods system in this case.

Summary


automatically
(Y, X) ⟹
∃{W_j}: type 1
if u_1≠0
if u_1=0
∃{W_j}: type 2
if u_2≠0
if u_2=0
∃{W_j}: type3
u_3≠0
u_3=0
(Y, X):
fin. type
(α)
⟹ Y admits
str. psd concave nbhds.
(Y, X):
infin type
if
∃Z ⊃ Y
s.t.
Hol_{Z,Y}:
U(1)-lin
If not
(β)
γ
Y admits
psd flat nbhds
??

複素近傍の分類理論とLevi葉層

(1日目)

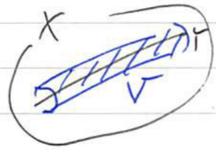
X: cpx mfd (may be open) ← surface ($\dim_{\mathbb{C}} X = 2$)

$\cup$

Y: cpx submfd. cpt ← curve ($\dim_{\mathbb{C}} Y = 1$, cpt Riemann Surface)

"for simplicity".

Q What kind of cpx str does a $\underbrace{\text{(suff. small) nbhd}}_{V}$ of Y have?
(tubular)



Topologically,

⊙ $V \approx$ (a nbhd of 0-sect'n in $N_{Y/X}$)
(tub. nbhd thm)
↖ normal bdl
(holomorphic! ← (1,0)-part of $N^{C^\infty}_{Y/X} \otimes \mathbb{C}$)

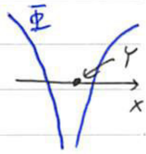
⊙ Classification ⋯ $(Y^2) := \deg N_{Y/X}$ (or $c_1(N_{Y/X})$) $\boxed{>0}$, $=0$, $\boxed{<0}$

10分

$\boxed{\text{When } (Y^2) > 0 \text{ or } < 0}$, we can choose V s.t. ∂V admits Contact structure.

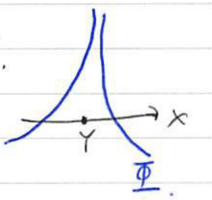
$\Uparrow$

∵ If $(Y^2) < 0$, $\exists \Phi$: (a nbhd of Y in X) $\longrightarrow \mathbb{R} \cup \{-\infty\}$
s.t. · Φ: C^2 on (nbhd of Y) $\setminus Y$
[Grauert '62] $\Phi \equiv -\infty$ along Y.
$dd^c\Phi := \sqrt{-1}\partial\bar{\partial}\Phi > 0$ on (nbhd of Y) $\setminus Y$. "psh function"
"str. psh function".



→ $V := \{\Phi < -M\}$ for $M << 0$.

If $(Y^2) > 0$, $\exists \Phi$: (a nbhd of Y) $\longrightarrow \mathbb{R} \cup \{+\infty\}$
s.t. Φ: str. psh on (Y-nbhd) $\setminus Y$,
[Suzuki '75] $\Phi \equiv +\infty$ along Y.



→ $V := \{\Phi > M\}$ for $M >> 0$.

20分 → **Q** How about the case of $(Y^2) = 0$?

20分

Eg. 1 $Y :=$ cpt Riem. surf, $X := Y \times \mathbb{P}^1$ ($\mathbb{P}^1 = \mathbb{CP}^1$)

Regard $Y \subset X$ via $Y \cong Y \times \{0\} \subset X$.

$\rightarrow$ $N_{Y/X} \underset{\mathrm{hol}}{\cong} \mathbb{1}_Y$, $(Y^2) = 0$.

$\rightsquigarrow$ $\Phi : X = Y_z \times \mathbb{P}^1_w \longrightarrow \mathbb{R} \cup \{\pm\infty\}$
$(z, w) \longmapsto \log|w|$;

$\rightsquigarrow$ Both Φ and $-\Phi$: psh. (~~not~~ str. psh) $\rightsquigarrow$ $\boxed{\partial V : \text{Levi-flat}}$

Φ : pluriharmonic.

$\partial V = \{\Phi = M\}$ for some M

Eg. 2 (Serre's e.g.)

$Y := \mathbb{C}/\langle 1, \tau \rangle$: an ellipt. curve (torus) ($\tau \in \mathbb{H}$)

$\uparrow$ affine bdl

z
$\uparrow$
(z, w)

$Z := \mathbb{C}^2/\sim$, for $(y, \zeta) \in \mathbb{C}^2$, $(y, \zeta) \sim (y+1, \zeta+1)$
$\sim (y+\tau, \zeta+\bar{\tau})$

$X := Z \cup$ "∞-sect'n" $\cong Y$ $\rightsquigarrow$ Regard $Y \subset X$.

$\rightsquigarrow$ $N_{Y/X} \underset{\mathrm{hol}}{\cong} \mathbb{1}_Y$, $(Y^2) = 0$.

Consider $\Phi : X \longrightarrow \mathbb{R}$ induced by $(y, \zeta) \longmapsto |\zeta - \bar{y}|^2$.

$\rightsquigarrow$ Φ : well-def, str. psh around $Y = \{\zeta = \infty\}$,
$\Phi|_Y \equiv +\infty$.

$\rightsquigarrow$ $\boxed{\text{Similar to the case of } (Y^2) > 0}$

Eg 3 (Ueda's e.g) [Ueda'83]

$\exists (Y, X)$ s.t. $\forall V$: nbhd of Y, with $(Y^2) = 0$

$\nexists \Phi : V \to \mathbb{R} \cup \{-\infty\}$ with $\{\Phi = -\infty\} = Y$, Φ : str. psh.
simple application of [Grauert'62]

$\nexists \Phi : V \to \mathbb{R} \cup \{+\infty\}$ conti. with $\{\Phi = +\infty\} = Y$, Φ : psh on $V \setminus Y$.
[K-'15] arXiv:1510.02287

40分

Tommorrow: we'll treat the case where

$\begin{cases} \exists V : Y\text{-nbhd} \subset X, \\ \exists \mathcal{F} : \text{hol. foliation (non-singular) with } Y \in \mathcal{F} \text{ on } V. \end{cases}$

$(\Rightarrow (Y^2)=0)$.

Rmk
- It is the case for e.g. 1~3.
- $\not\exists$ counter-example for the existence of $(V, \mathcal{F})$ if $(Y^2)=0$.

→ ⊙ Explain Ueda's classification theory [Ueda'83] of (Y, X) under this config.

⊙ Explain the relationship between the holonomy of $\mathcal{F}$ and Ueda theory. //

50分 ——

1回目の資料の案

- ⊙ psh の def をのっけとく.
- ⊙ Ex 1~3 をかいとく.
- ⊙ $Bl_9\mathbb{P}^2$ の例を Ex.4 として書いておく ← (時間あまったら これをせつめい)
- ⊙ あと. Ex 3 のしょうさいも書いとく.

2日目

Ueda's classification of (Y, X) — [X : cpx surf, $\cup$ Y : cpt cpx curve with $(Y^2)=0$ ($= \deg N_{Y/X}$)]

for simplicity, $\boxed{\text{assume}}$ " $\exists V$: a nbhd of Y in X
$\exists \mathcal{F}$: (non-sing) hol. foliation on V with $Y \in \mathcal{F}$. "

Q [the holonomy of $\mathcal{F}$] v.s. [psh functions on a nbhd of Y].

5分

◎ Observation

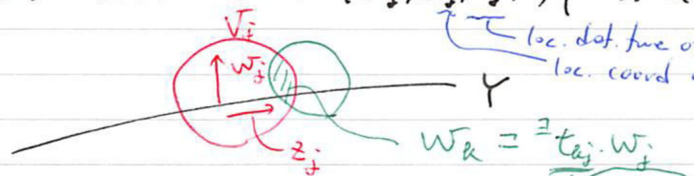
" If the holonomy $\mathrm{Hol}_{\mathcal{F},Y} : \pi_1(Y, *) \longrightarrow \mathcal{O}_{\mathbb{C},0}$ is $U(1)$-linearizable,

(i.e. $\mathrm{Image}(\mathrm{Hol}_{\mathcal{F},Y}) \subset U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$.)

Then $\exists \Phi$: ($^\exists$ a nbhd V of Y) $\longrightarrow \mathbb{R} \cup \{+\infty\}$: psh s.t. $-\Phi$ is also psh (p.h.).

($\Rightarrow \partial V$: Levi-flat) c.f. e.g.1

∵ Consider foliation chart $\{(V_j, (z_j, w_j))\}$ of a nbhd of Y.
(w_j: loc. def. fnc of Y; z_j: loc. coord of Y)

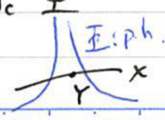


$\rightsquigarrow \Phi(z_j, w_j) := -\log|w_j|$ "

15分

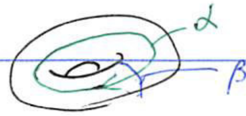
∅ When (*) $\mathrm{Hol}_{\mathcal{F},Y}[\gamma](w) = \underset{\in U(1)}{t_\gamma} \cdot w + \underset{O(w^2)}{(\text{h.o.t.})}$ for $^\forall \gamma \in \pi_1(Y, *)$,

"(h.o.t.)"'s are an obstruction for the existence of pluriharmonic Φ //



→ cpx dynamics for $\mathrm{Hol}_{\mathcal{F},Y}$ v.s. Existence of Levi-flats.
($U(1)$-linearization problem for hol. function with 1-var.)

20分



20分

Assume $\begin{cases} Y: \text{ellipt. curve} \\ (*) \text{ the linear part of } \mathrm{Hol}_{\pi,Y}[\gamma] \text{ is "}\underset{U(1)}{\underset{\Uparrow}{\exists\text{triv}}}\text{" for } \forall\gamma\in\pi_1(Y,*) \end{cases}$ in what follows.

$\rightsquigarrow$

		λ: torsion		λ: non-tor	
		f: lin'ble	f: non-lin'ble	f: lin'ble	f: non-lin'ble
μ: tor	g: lin'ble	I	II	III	$\boxed{\text{IV}}$
	g: non-lin'ble		V	~~VI~~	~~VII~~
μ: non-tor	g: lin'ble			VIII	~~IX~~
	g: non-lin'ble				$\boxed{\text{X}}$

where
$f(w) = \lambda\cdot w + (\text{h.o.t.}) := \mathrm{Hol}_{\pi,Y}[\alpha]$,
$g(w) = \mu\cdot w + (\text{h.o.t.}) := \mathrm{Hol}_{\pi,Y}[\beta]$,
$(\lambda, \mu \in U(1))$

◎ $\nexists (Y, X, V, \mathcal{F})$ of Case VI, VII, IX.

◎ Case I, III, VIII $\Longrightarrow$ $\exists\Phi: V \to \mathbb{R}\cup\{+\infty\}$: p.h., $\{\Phi=\infty\}=Y$ "by observation"

$(\Rightarrow \exists$ Levi-flats$)$.

… similar to "e.g. 1", called "type (β)" in Ueda's Classification

◎ Case II $\underset{[\text{Ueda'83}]}{\Longrightarrow}$ $\exists\Phi: V \to \mathbb{R}\cup\{+\infty\}$: str. psh. s.t. $\{\Phi=\infty\}=Y$, Φ: C^∞ on $V\setminus Y$.

(we'll explain it later)

… similar to "e.g. 2", called "type (α)" in Ueda's Classification

◎ Case IV, X $\underset{[\text{K-, Ogawa}]}{\Longrightarrow}$ (**) $\boxed{\begin{array}{l} {}^\forall V: \text{a nbhd of } Y, \\ \begin{cases} \nexists\Phi: V \to \mathbb{R}\cup\{-\infty\} \text{ with } \begin{cases} \{\Phi=-\infty\}=Y \\ \Phi: \text{str. psh}, \end{cases} \\ \nexists\Phi: V \to \mathbb{R}\cup\{+\infty\}: \text{conti. s.t. } \begin{cases} \{\Phi=\infty\}=Y \\ \Phi: \text{psh on } V\setminus Y. \end{cases} \end{cases} \end{array}}$

Rmk First example of (Y, X) with the property (**) was constructed by Ueda.

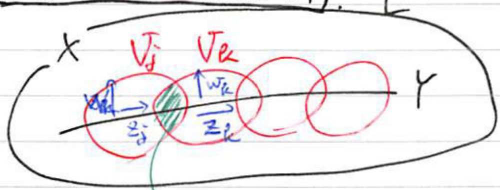
… a generalization of e.g. 3, called "type (γ)" in Ueda's classification

35分

☆ Case V については
意図的にとばし、
時間があれば
あとでホソク?

35分

$\boxed{\text{Ueda's classification}}$. Setting: (Y, X), $(Y^2) = 0$. (without any assumption on $(V, \mathcal{F})$) ".



on V_{jk}, $w_k = t_{kj} \cdot w_j + f^{(2)}_{kj}(z_j) \cdot w_j^2 + \cdots$

<u>fact</u> we may assume $t_{kj} \in U(1)$ for $\forall j, k$.

<u>(Y, X)</u>;

- <u>type (α)</u> $\overset{\text{def}}{\iff}$ $\exists n \in \mathbb{N}$ s.t. $\nexists \{w_j\}$ as above with $f^{(2)}_{kj} \equiv f^{(3)}_{kj} \equiv \cdots \equiv f^{(n)}_{kj} \equiv 0$ for $\forall j, k$. "formally non-linearizable case"
- <u>type (β)</u> $\overset{\text{def}}{\iff}$ $\exists \{w_j\}$ s.t. $w_k = t_{kj} \cdot w_j$ for $\forall j, k$. "linearizable case".
- <u>type (γ)</u> $\overset{\text{def}}{\iff}$ $\forall n \in \mathbb{N}$, $\exists \{w_j\}$ s.t. $f^{(2)}_{kj} \equiv f^{(3)}_{kj} \equiv \cdots \equiv f^{(n)}_{kj} \equiv 0$, for $\forall j, k$, however $\nexists \{w_j\}$ s.t. $w_k = t_{kj} \cdot w_j$ for $\forall j, k$. "formally linearizable, however non-linearizable case".

<u>Thm ([Ueda '83])</u>

(1) (Y, X): type (α) $\Rightarrow$ $\exists V$: a nbhd of Y, $\exists \Phi : V \to \mathbb{R} \cup \{+\infty\}$: conti. s.t. $\begin{cases} \{\Phi = \infty\} = Y, \\ \Phi|_{V \setminus Y} : \text{str. psh.} \end{cases}$

(+ estim. of the singularity of Φ)

(2) $N_{Y/X} \in \mathrm{Pic}^0(Y)$: torsion $\Longrightarrow$ (Y, X): type (α) or type (β).

(3) $N_{Y/X} \in \mathrm{Pic}^0(Y)$: Diophantine $\Longrightarrow$ (Y, X): type (β) c.f. (Siegel's linearization thm).

(4) $\exists (Y, X)$: of type (γ) "

50分

$\boxed{\text{2日目の資料の案}}$ ← 1日目/2日目で表う1一枚でいいど??

- 4×4の表
- (type α, β, γ の def ?)
- 上田の定理.