Minimal singular metrics of some line bdls with infinitely generated section rings.

X : sm proj $/\mathbb{C}$

L : line bdl $/X$.

<u>Q</u> Where, how min sing. metrics of L. diverges when $R(X, L) := \bigoplus_{m \geq 0} H^0(X, mL)$

; infin. gen. ?

<u>Case1</u> when (X, L) admits no Zariski decomp. in the sense of Nakayama, birationally,

<u>Case2</u> $\exists D \subset X$: sm. hypersurf ~~of X~~

$\begin{cases} m \geq 1 & |mL| \supset D \\ & |mL - D| : \text{free} \end{cases}$ $(\Rightarrow L : \text{nef})$ //

<u>Goal</u>

<u>Thm A</u> X : sm proj toric bdl / sm ab. var.

L : big l.b. $/X$.

$\rightsquigarrow$ I described the singularity of min. sing. metrics of L. only by using the combinatorial information of $\Omega(L)$ /

<u>Thm B</u> $C \subset \mathbb{P}^2$: sm ellipt. curve.

$P_1, P_2, \cdots, P_{12} \in C$: gen. $\pi : X \to \mathbb{P}^2$: b-up center $= \{P_j\}$

$H := \pi^*(\text{line})$, $D := (\pi^{-1})_* C$.

$L := \mathcal{O}(H + D)$ has a continuous Herm. metric //

§1. motivation.
§2. case 1, Thm A.
§3. case 2, Thm B.

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§1.

X : sm proj.
L : line bdl. $/X$.

Def h : sing. Herm. metric of L.
$\underset{\text{def}}{\iff}$ $\exists h_\infty$: sm Herm metric of L
$\exists \chi : X \to \mathbb{R} \cup \{-\infty\}$; L^1_{loc}.
$h = h_\infty e^{-\chi}$. //.

fact
L : psd. eff. $\iff$ $\exists h$ = sing. Herm. metric of L
s.t. $\sqrt{-1}\Theta_h \geq 0$. //

$$\underset{(h \underset{loc}{=} e^{-\varphi})}{dd^c \varphi} = \sqrt{-1}\Theta_{h_\infty} + \underbrace{dd^c}_{\sqrt{-1}\partial\bar\partial} \chi.$$

Def (D.P.S)
$h_{\min,L} = e^{-\varphi_{\min,L}}$; sing. Herm. metric. of L

$h_{\min,L}$: min. sing. metric
$\underset{\text{def}}{\iff}$
(i) $dd^c \varphi_{\min,L} \geq 0$.
(ii) $\forall \underset{e^{-\varphi}}{h}$: sing. Herm. metric s.t. $dd^c\varphi \geq 0$, $\exists C > 0$. $h_{\min,L} \leq C \cdot h$. //.

Thm (D.P.S)

L : psd. eff. $\Rightarrow$ $\exists h_{min,L}$: min. sig. metric of L //

$\because$ fix h_∞ : sm Herm. metric of L.
$\quad\quad\quad$ $\| \; e^{-\varphi_\infty}$

$\rightsquigarrow$ the equilibrium metric

$(h_\infty)_e = e^{-(\varphi_\infty)_e}$; a min sig. metric of L.

$$(\varphi_\infty)_e(x) = \varphi_\infty(x) + \sup\left\{ \chi(x) \,\middle|\, \begin{array}{l} \chi : X \to [0,\infty], \\ dd^c(\chi+\varphi_\infty) \geq 0 \end{array} \right\}$$
$(x \in X)$

//

◎ known result

◦ L : ample $\xrightarrow[\text{Kodaira's emb. thm.}]{}$ $\exists$ sm. $h_{min,L}$.

//

◎ L : s.a. $\Longrightarrow$ —— // ——

//

◎ L : big nef $\Longrightarrow$ $\forall x \in X$ $\underbrace{\nu}_{\text{Lelong number}}(\varphi_{min,L}, x) = 0$.
$\quad\quad \|$
$\liminf_{z \to x} \frac{\varphi_{min,L}(z)}{\log|z-x|^2}$

Thm (Boucksom)

L : big.

$\mathbb{B}_-(L)$
$= \{ x \in X \mid \nu(\varphi_{min,L}, x) > 0 \}$.

$\left(\begin{array}{l} \text{c.f.} \; f : \text{hol. func.} \\ \nu(\log|f|^2, x) = \mathrm{mult}_x f. \end{array} \right)$

//.

Rmk

L : big, nef $\not\Rightarrow$ $\{\varphi_{\min,L} = -\infty\} \neq \emptyset$. //

[B.E.G.Z] dim = 3

Thm (B.E.G.Z) L : big.

Siu type metrics are minimal singular metrics

$\Leftrightarrow$ $R(X, L)$: fin. gen. //

$\begin{pmatrix} \text{the information} \\ \text{of } R(X,L) \end{pmatrix} = \begin{pmatrix} \text{the information} \\ \text{of the singularity of } h_{\min,L} \end{pmatrix}$ //

§2. case 1, Thm A

X : sm. proj. L : big $/X$.

the condition $\varnothing$

$\Updownarrow$ def

(X, L) admits no Z.D. in the sense of Nakayama, birationally.

$\Updownarrow$

$^\forall f : \tilde{X} \longrightarrow X$: modification

$f^*L \otimes \mathcal{O}(-\sum_{\substack{\Gamma \subset \tilde{X} \\ \text{prime div}}} \nu_\Gamma \cdot \Gamma)$: not nef

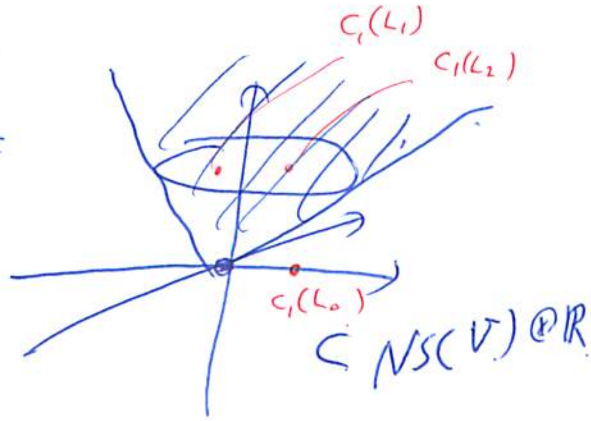
w/. $\nu_\Gamma := \nu(\underbrace{\varphi_{\min,L}}_{f^*}, \Gamma)$ $\quad \varphi_{\min, f^*L}$ //

Example (Nakayama's example)

$$V = \left(\mathbb{C}/\mathbb{Z}+(\pi+\sqrt{-1})\mathbb{Z}\right) \times \left(\mathbb{C}/\mathbb{Z}+(\pi+\sqrt{-1})\mathbb{Z}\right)$$

$\rightsquigarrow$ rank $NS(V) = 3$

$Nef(V) = P.E(V) =$



fix $L_0, L_1, L_2 \to V$

s.t. $\begin{cases} L_0 : \text{not psd.eff} \\ L_1, L_2 : \text{ample} \end{cases}$

$\not\leftrightsquigarrow$

$X := \mathbb{P}(L_0 \oplus L_1 \oplus L_2) \xrightarrow{\pi} V$, $L := \mathcal{O}_{X/V}(1)$

fact. (Nakayama)

(X, L) satisfies $(\ast)$

//

Thm A (for Nakayama's example)

(X, L) : Nakayama's example

$$\square(L) := \left\{ (\alpha, \beta) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \le \alpha, \beta, \quad \alpha+\beta \le 1, \\ (1-\alpha-\beta)L_0 + \alpha L_1 + \beta L_2 : \text{nef} \end{array} \right\}$$

loc. coord of X

x : loc coord of V.

$(z^1, z^2, x) := [\, S_0^*(x) + z^1 S_1^*(x) + z^2 S_2^*(x) \,] \in X$.

w/ S_j^* : loc triv. of L_j^{-1}

$\exists\, h_{min,L}$: min. sing. metric of L.

$$h_{min,L} = e^{-\varphi_{min,L}}$$

s.t. $\varphi_{min,L}(z^1, z^2, x) = \log \max_{(\alpha,\beta)\in\square(L)} |z^1|^{2\alpha} |z^2|^{2\beta}$

$+ (\text{conti. func})$ //

Rmk $\{\varphi_{min,L} = -\infty\} = \{\nu(\varphi_{min,L}, -) > 0\}$

$= \{z^1 = z^2 = 0\} = \mathbb{P}(L_0) \subset X$.

$\subset \mathbb{B}_-(L)$ //

Cor $\forall x \in \mathbb{P}(L_0)$, $\forall f_1, \cdots, f_N \in \mathcal{O}_{X,x}$. $\forall c > 0$.

$\varphi_{min,L} \neq \underbrace{c \log \Sigma |f_j|^2}_{\text{corresp. to } \mathfrak{a}^c} + O(1)$ around x //

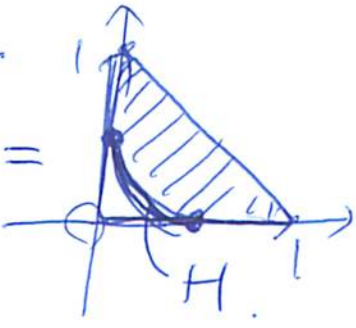
where $\mathfrak{a} := \langle f_j \rangle \subset \mathcal{O}_{X,x}$.

Cor $\forall c > 0$. $x \in \mathbb{P}(L_0)$

$\mathcal{J}(h^c_{min,L})_x \ (:= \{f \in \mathcal{O}_{X,x} \mid |f|^2 e^{-c\varphi_{min,L}} : ~~\text{loc. bdd}~~ \text{loc integrable}\})$

$= \langle (z^1)^p (z^2)^q \mid (p,q) \in$  $\rangle$

where

$\square(L) =$ 

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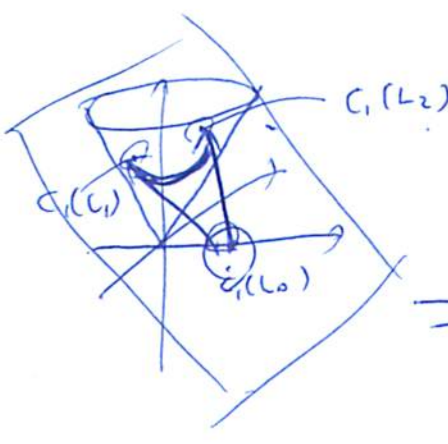


$= \mathbb{R}^2$

Outline of prf of thm A. ← for Nakayama's example.

construct such $h_{min,L} = e^{-\varphi_{min,L}}$.

$\boxed{\text{idea}}$ regard $\square(L)$ as a set of sing Her metrics //.

$m := (\alpha, \beta) \in \square(L)$

$X \supset \mathcal{O}(\mathbb{P}(L_1 \oplus L_2)) \otimes \pi^* L_0 = L$.

$\varphi_m(z^1, z^2, x)$

$:= (1-\alpha-\beta)\left(\log|1|^2 + \pi^*(\text{log weight of the standard sm Herm metric of } L_0)\right)$

$+ \alpha \left(\log|z^1|^2 + \pi^*(\quad '' \quad L_1)\right)$

$+ \beta \left(\log|z^2|^2 + \pi^*(\quad '' \quad L_2)\right)$

$\rightsquigarrow h_m := e^{-\varphi_m}$: sing. Herm. metric of L.

$dd^c \varphi_m \geq 0$.

$h_{min,L} := \min_{m \in \square(L)} h_m$.

; min sing. metric.

($\Leftarrow$ approxim. thm by Demailly) //

§3 case 2, Thm B

Zariski's example

$C \subset \mathbb{P}^2$: sm. ellipt.

$p_1, \cdots p_{12} \in C$: gen. pts.

$\pi : X \longrightarrow \mathbb{P}^2$: bup. center $= \{p_i\}_{i=1}^{12}$

$H := \pi^*(\text{line})$

$D := (\pi^{-1})_* C$. $\quad L := \mathcal{O}(D+H)$

$\rightsquigarrow$ fact $m \geq 1$.

$|mL| \supset D$.

$|mL - D|$: free,

L : nef big, not s.p. $R(X, L)$: not ~~fin~~ fin. gen //.

Thm B (X, L) : as above

$\rightsquigarrow \exists h_{min,L} = e^{-\varphi_{min,L}}$; conti //

Outline of prf of thm B

$\boxed{\text{idea}}$ use. simple cpx analytic structure of. $D \subset X$ and reduce to ~~the~~ case. (X', L') ~~when~~
$\left\{ \begin{array}{l} \text{relative } \mathcal{O}(1) \\ \mathbb{P}^1\text{-bdl}/D. \end{array} \right.$

Thm (Grauert, '62)

X : 2-dim cpx mfd.

$\cup$

D : sm hypersurf., cpt, genus $= g$.

$(D^2) < \min\{0, 4-4g\}$

$\Rightarrow$ $\exists\, V$: tub. nbhd. of D in X

$\exists\, V'$: tub. nbhd of (0-section) in $N_{D/X}$

$V \underset{\text{bihol.}}{\cong} V'$

Cor (X, D, L) : Zariski's example. $\leftarrow (D^2) = -3$ ($\shortparallel$ $3^2 - 12$, $\shortparallel$ $4 \cdot$)

D has a hol. tub. nbhd.

prf. of thm B

fix. $V \underset{i}{\cong} V'$ as above.

$A := L \otimes \mathcal{O}(-D)$ $/X$

regard V' as a nbhd of

$D' := \mathbb{P}(L|_D)$ in $X' := \mathbb{P}(L|_D \oplus A|_D)$

$$= \mathbb{P}(\mathcal{O}_D \oplus \underbrace{(A \otimes L^{-1})}_{= \mathcal{O}(-D)}|_D) \qquad \underbrace{\mathcal{O}(-D)}_{= N^*_{D/X}}$$

$L' := \mathcal{O}_{X'/D}(1)$

$(\cancel{X', L', D'})$

Fact

$h_{min,L'} = e^{-\varphi_{min,L'}}$: min. sing. metric of L'.

s.t. $\varphi_{min,L'}$: conti //.

◎ $L'|_{U'} \cong L|_U$ (C^∞-line bdl) induces ~~Connect~~ $\rightsquigarrow \exists\, e^{-\psi_{L'}}$; sing. Herm. metric of $L|_U$

$$\begin{array}{ccc} L'|_{U'} & \cong & L|_U \\ \downarrow & & \downarrow \\ U' & \xleftarrow[i]{\sim} & U \\ \cup & & \cup \\ D' & \cong & D \end{array}$$

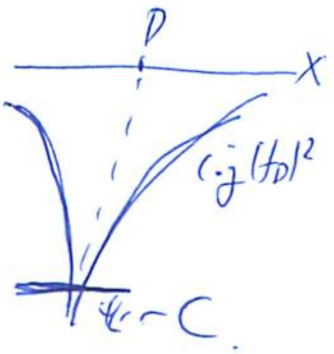
s.t.

$\psi_{L'} = (\varphi_{min,L'}) + \left(\begin{smallmatrix}\text{sm.} \\ \text{harm. function}\end{smallmatrix}\right)$

defining func of D $\in H^0(X, D)$; loc weight of sm. Herm metric of A, $dd^c_x \varphi_A \geq 0$

$$\rightsquigarrow \varphi_{min,L} := \begin{cases} \log|f_D|^2 + \varphi_A & \text{(outside of } D) \\ \max\{\ \ ''\ \ , \psi_{L'} - C\} & \text{(around } D) \end{cases}$$

$C \gg 1$



$(\rightsquigarrow dd^c \varphi_{min,L} \geq 0.)$ //

Thm B'

X : sm proj. var.

$\cup$

D : sm hyp. surf.

L : psd. eff lin bdl $/X$.

Assume $\begin{cases} L \otimes \mathcal{O}(-D) : \text{s.a} \\ D \text{ has a hol. tub. nbhd} \end{cases}$

Then ~~$\varphi_{min,L}$~~ $h_{min,L}|_D \not\equiv \infty \iff L|_D$: psd. eff.

in this case, $\varphi_{min,L}|_D$; a min. sing. metric of $L|_D$ //