

# Gluing construction of non-projective K3 surfaces and holomorphic tubular neighborhoods of elliptic curves

Takayuki Koike (joint w/Takato Uehara)

Department of Mathematics, Graduate School of Science, Osaka City University

September 4, 2018

- 1 Main results
- 2 Motivation from neighborhoods theories
- 3 Construction of the K3 surface  $X$
- 4 22 generators of  $H_2(X, \mathbb{Z})$  and the period
- 5 Towards the “moduli space” of K3 surfaces constructed by our method

## 1 Main results

2 Motivation from neighborhoods theories

3 Construction of the K3 surface  $X$

4 22 generators of  $H_2(X, \mathbb{Z})$  and the period

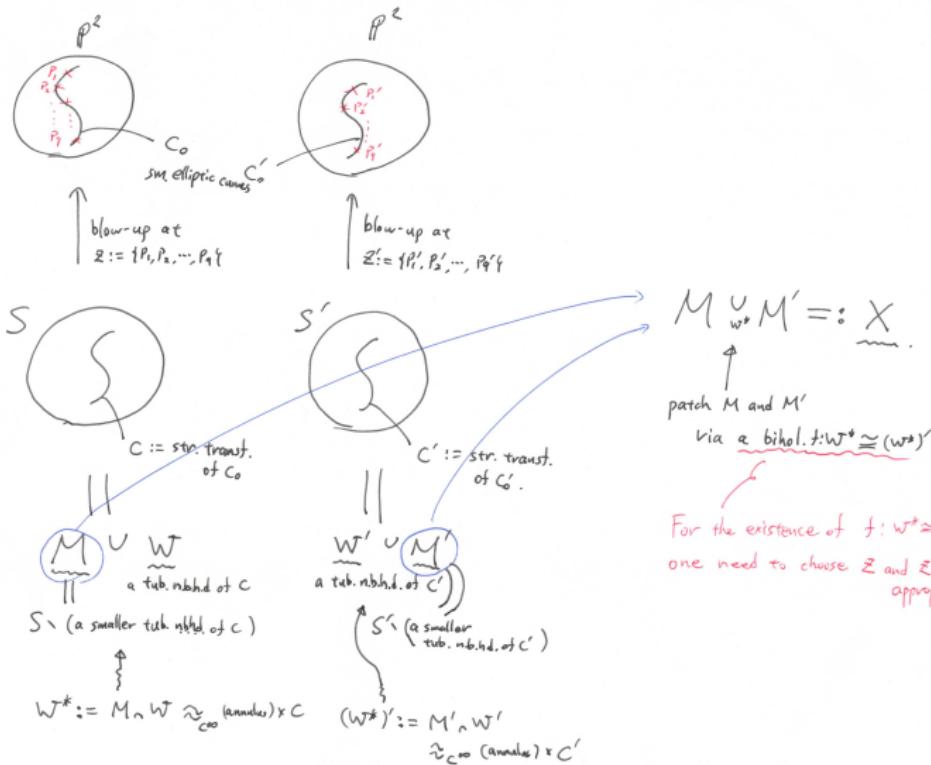
5 Towards the “moduli space” of K3 surfaces constructed by our method

- based on [T. Koike, Complex K3 surfaces containing Levi-flat hypersurfaces, arXiv:1703.03663] + some recent progress  
(j.w/ T. Uehara)

## Goal of this talk:

To construct a (non-projective, non-Kummer) K3 surface  $X$  containing a real 1-parameter family of Levi-flat hypersurfaces by holomorphically patching two open complex surfaces, say  $M$  and  $M'$

- $M$  ( $M'$ ) is the complement of a (appropriate) tubular neighborhood of an elliptic curve in the blow-up  $S$  ( $S'$ ) of the projective plane  $\mathbb{P}^2$  at (appropriate) nine points.
- Neither  $S$  nor  $S'$  admit elliptic fibration structure (nine points are “general”)
- In order to patch  $M$  and  $M'$  holomorphically, we need to take “nice neighborhood”. For this purpose, we need to choose nine points carefully.
- Use a technique from Complex Dynamics for taking “nice neighborhood” (Arnol'd's theorem).



## Remarks, Known results

- For the case where  $S$  and  $S'$  are elliptic, a similar construction of K3 surfaces is known. The resulting K3 surfaces are also elliptic.
- M. Doi showed that a similar construction of K3 surfaces is possible for general  $Z$  and  $Z'$  if one admit (slight) deformations of the complex structures of  $M$  and  $M'$ .  
(Doi, Mamoru, Gluing construction of compact complex surfaces with trivial canonical bundle. J. Math. Soc. Japan 61 (2009), no. 3, 853–884)
- The idea to use Arnol'd-type theorem for patching two open manifolds is also used by H. Tsuji in order to study complex structures on  $S^3 \times S^3$ .  
(H. Tsuji, Complex structures on  $S^3 \times S^3$ , Tohoku Math. J. (2) Volume 36, Number 3 (1984), 351–376)

## Remarks on our construction of K3 surfaces

- In our construction, “the tab for gluing”  $W^* := M \cap M'$  is an open submanifold of  $X$  which admits an annulus bundle structure over an elliptic curve.
- $\exists \Phi: W^* \rightarrow I$  ( $I \subset \mathbb{R}$ : an interval): pluriharmonic .
- $H_t := \Phi^{-1}(t)$  is a compact Levi-flat hypersurface of  $W^*(\subset X)$  which is diffeomorphic to  $S^1 \times S^1 \times S^1$  for each  $t \in I$ .
- For each  $t \in I$ , any leaf of  $H_t$  is biholomorphic to  $\mathbb{C}$  or  $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ , and is dense in  $H_t$ .

## Main results

We constructed K3 surfaces in such a manner with independent (at least) 18 parameters. As an conclusion of the construction, we have the following:

Theorem (K-, arXiv:1703.03663)

*There exists a deformation  $\pi: \mathcal{X} \rightarrow B$  of K3 surfaces over an 18 dimensional complex manifold  $B$  with injective Kodaira-Spencer map such that each fiber  $X_b := \pi^{-1}(b)$  admits a holomorphic map  $F_b: \mathbb{C} \rightarrow X_b$  with the following property: The Euclidean closure of  $F_b(\mathbb{C})$  is a real analytic compact hypersurface of  $X_b$ . Especially,  $F_b(\mathbb{C})$  is Zariski dense whereas it is not Euclidean dense.  $X_b$  is non-Kummer for general  $b \in B$*

“ $F_b$ ” can be constructed by considering the immersion of a leaf of  $H_t$  into  $W^* \subset X$  for each  $X = X_b$ .

We also described 22 generators of the K3 lattice and could concretely compute the integrals of the nowhere vanishing holomorphic 2-form  $\sigma$  along 20 of them (joint w/Takato Uehara). As a result, we could improve the previous theorem.

### Theorem (K-, Uehara (in progress))

*There exists a deformation  $\pi: \mathcal{X} \rightarrow B$  of K3 surfaces over a 19 dimensional complex manifold  $B$  with injective Kodaira-Spencer map such that each fiber  $X_b := \pi^{-1}(b)$  admits a holomorphic map  $F_b: \mathbb{C} \rightarrow X_b$  with the following property: The Euclidean closure of  $F_b(\mathbb{C})$  is a real analytic compact hypersurface of  $X_b$ . Especially,  $F_b(\mathbb{C})$  is Zariski dense whereas it is not Euclidean dense.  $X_b$  is non-Kummer and non-projective for general  $b \in B$ .*

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Let

- $S$  be a non-singular complex surface, and
- $C$  be a compact complex curve embedded in  $S$  with  
 $(C^2) := \deg N_{C/S} = c_1(N_{C/S}) = 0$ .

There exists a (small) neighborhood  $W$  of  $C$  in  $S$  which is diffeomorphic to a neighborhood of the zero section in  $N_{C/S}$  (tubular neighborhood theorem).

Our original interest:

What kind of complex analytic structure does  $W$  have?

Remark

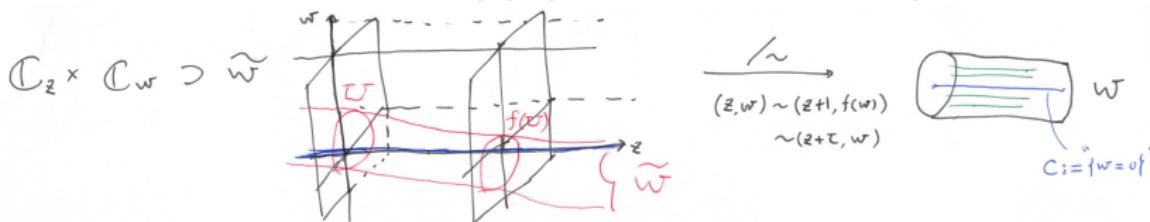
*In general,  $\exists W$  which is biholomorphic to a neighborhood of the zero section in  $N_{C/S}$ .*

## Ueda's example

Fix  $\tau \in \mathbb{C}$  with  $\operatorname{Im} \tau > 0$  and  $f(w) = a_1 w + a_2 w^2 + \dots \in \mathcal{O}_{\mathbb{C}, 0}$  ( $|a_1| = 1$ ). Take a neighborhood  $\widetilde{W}$  of  $\mathbb{C} \times \{0\}$  in  $\mathbb{C} \times \mathbb{C}$ .

$$W := \widetilde{W} / \sim, (z, w) \sim (z+1, f(w)) \sim (z+\tau, w)$$

$C \subset W$ : the image of  $\mathbb{C} \times \{0\}$  (smooth elliptic curve)



Fact [K-, N. Ogawa, arXiv:1808.10219]

$C$  admits a *holomorphic tubular neighborhood* (i.e.  $\exists W$  which is biholomorphic to a neighborhood of the zero section)  
iff  $f$  is *linearizable* around the origin (i.e.  $\exists \varphi \in \mathcal{O}_{\mathbb{C}, 0}$  such that  $\varphi(0) = 0$  and  $\varphi^{-1} \circ f \circ \varphi(w) = a_1 \cdot w$ )

Our main interest is in the following example:

Take a smooth elliptic curve  $C_0 \subset \mathbb{P}^2$  and nine points

$$Z := \{p_1, p_2, \dots, p_9\} \subset C_0.$$

- $S := \text{Bl}_Z \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$ : blow-up at  $Z$

- $C := \pi_*^{-1} C_0$ : the strict transform of  $C_0$

Note that  $N_{C/S} \cong \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \dots - p_9)$ . When  $Z$  is special,  $S$  is an elliptic surface ( $N_{C/S} \in \text{Pic}^0(C)$  is torsion in this case). We are interested in the case where  $Z$  is general.

### Theorem (Brunella, 2010)

Assume that  $S \setminus C$  has no compact complex curve.

$K_S^{-1}$  admits a  $C^\infty$  Hermitian metric with semi-positive curvature iff  $C$  has a pseudoflat neighborhoods system

(i.e.  $\exists$  fundamental system of neighbourhoods  $\{W_\varepsilon\}_{\varepsilon > 0}$  of  $C$  such that  $\partial W_\varepsilon$  is Levi-flat).

Note that  $C$  has a pseudoflat neighborhoods system if  $C$  admits a holomorphic tubular neighborhood.

## Question

*When does  $C$  admit a holomorphic tubular neighborhood?*

### Theorem (Arnol'd (1976))

*Assume  $C$  is a smooth elliptic curve and  $N_{C/S} \in \text{Pic}^0(C)$  is Diophantine (i.e.  $\exists A, \alpha > 0$  such that  $\text{dist}(\mathbb{I}_C, N_{C/S}^{\otimes n}) \geq A \cdot n^{-\alpha}$  for  $\forall n > 0$ , where  $\text{dist}$  is the Euclidean distance of  $\text{Pic}^0(C)$ ). Then  $C$  admits a holomorphic tubular neighborhood.*

For the previous Ueda's example, this theorem can be directly deduced from Siegel's linearization theorem.

## Question

*For the previous example ( $C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S$ ), does  $C$  admit a holomorphic tubular neighborhood when  $N_{C/S} \in \text{Pic}^0(C)$  is not Diophantine?*

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Let  $(C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S)$  and  $(C'_0, Z' = \{p'_1, p'_2, \dots, p'_9\}, C', S')$  be as in the previous section.

### Assumption

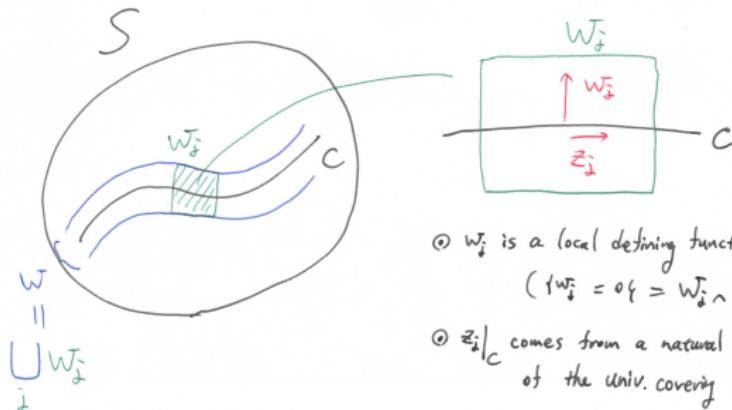
- $\exists g: C \cong C'$
- $N_{C/S} = g^* N_{C'/S'}^{-1}$
- $N_{C/S} \in \text{Pic}^0(C)$  is Diophantine

Then, it follows from Arnol'd's theorem that there exist holomorphic tubular neighborhoods  $W \subset S$  of  $C$  and  $W' \subset S'$  of  $C'$ .

Take a local charts systems  $\{(W_j, (z_j, w_j))\}$  of  $W$  and  $\{(W'_j, (z'_j, w'_j))\}$  of  $W'$  such that

$$\begin{cases} z_j = z_k + A_{jk} \\ w_j = t_{jk} \cdot w_k \end{cases}, \quad \begin{cases} z'_j = z'_k + A_{jk} \\ w'_j = t_{jk}^{-1} \cdot w'_k \end{cases}$$

for some constants  $A_{jk} \in \mathbb{C}$  and  $t_{jk} \in U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$  on  $W_{jk} := W_j \cap W_k$  and  $W'_{jk} := W'_j \cap W'_k$  as follows:



①  $w'_j$  is a local defining function of  $C$   
 $(\{w'_j = 0\} = W'_j \cap C)$

②  $z'_j|_C$  comes from a natural coordinate  
 of the univ. covering  $\mathbb{C}$  of  $C$ .

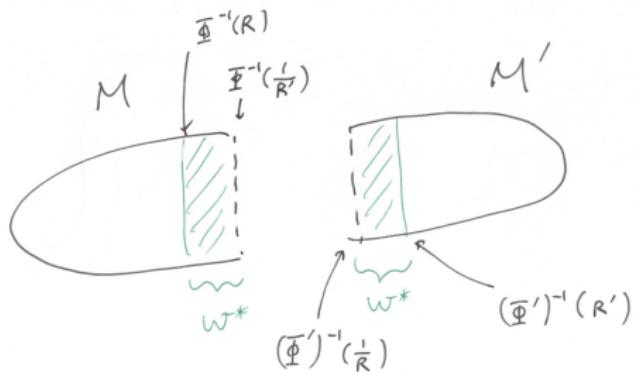
- $\Phi: W \rightarrow \mathbb{R}: (z_j, w_j) \mapsto |w_j|$ : globally defined on  $W$ .
- $\Phi': W' \rightarrow \mathbb{R}: (z'_j, w'_j) \mapsto |w'_j|$ : globally defined on  $W'$ .
- By scaling, we may assume that  
 $\Phi^{-1}([0, R]) \Subset W, (\Phi')^{-1}([0, R']) \Subset W'$  ( $R, R' > 1$ ).
- Replace  $W$  with  $\Phi^{-1}([0, R])$  and  $W'$  with  $(\Phi')^{-1}([0, R'])$ .

Define  $M \subset S$  and  $M' \subset S'$  by

$$M := S \setminus \Phi^{-1}\left(\left[0, \frac{1}{R'}\right]\right), \quad M' := S' \setminus (\Phi')^{-1}\left(\left[0, \frac{1}{R}\right]\right).$$

Identify  $W \cap M = \Phi^{-1}((1/R', R))$  and  
 $W' \cap M' = (\Phi')^{-1}((1/R, R'))$  by the isomorphism

$f: \Phi^{-1}((1/R', R)) \rightarrow (\Phi')^{-1}((1/R, R')) : (z_j, w_j) \mapsto \left(g(z_j), \frac{1}{w_j}\right)$   
and denote it by  $W^*$ .



$X := M \cup_{W^*} M'$ : a compact complex manifold obtained by  
patching  $M$  and  $M'$  via  $f$ .

## Observation

$W^*$  admits a natural foliation  $\mathcal{F}$  whose leaves are locally defined by “ $\{w_j = \text{constant}\}$ ”. As each leaf is biholomorphic to  $\mathbb{C}$  or  $\mathbb{C}^*$ , we have a holomorphic map  $F: \mathbb{C} \rightarrow W^* \subset X$  as in Main Theorem.

It is easily observed that  $\pi_1(X) = 0$ . Therefore, for proving that  $X$  is a K3 surface, it is sufficient to show the following:

## Proposition

There exists a nowhere vanishing holomorphic 2-form  $\sigma$  on  $X$  such that

$$\sigma|_{W^* \cap W_j} = \frac{dz_j \wedge dw_j}{w_j}$$

holds on each  $W^* \cap W_j \subset W^* \subset X$ .

## Key Lemma

$$H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}.$$

Key Lemma is shown by pulling back a holomorphic function on  $W^*$  by  $F: \mathbb{C} \rightarrow W^*$  and considering the Maximum principle.  $\square$

**Proof of Proposition:** As  $K_S = -C$ , there exists a meromorphic 2-form  $\eta$  on  $S$  with  $\text{div}(\eta) = -C$ . It follows from Key Lemma that the function

$$\frac{\eta|_{W^*}}{\left( \frac{dz_j \wedge dw_j}{w_j} \right)}$$

is a constant map. Thus we may assume that  $\eta|_{W^*} = \frac{dz_j \wedge dw_j}{w_j}$ .

Similarly, one can show the existence of a meromorphic 2-form  $\eta'$  on  $S'$  with  $\text{div}(\eta') = -C'$  such that  $\eta'|_{W^*} = \frac{dz'_j \wedge dw'_j}{w'_j}$ .

$\sigma$  is obtained by patching  $\eta|_M$  and  $-\eta'|_{M'}$ .  $\square$

## Remark

*The construction in this section also makes sense even after one generalize “the materials” as follows:*

- $S$ : compact complex manifold of any dimension s.t.  
 $\exists C \in |K_S^{-1}|$ : compact complex torus.
- $S'$ : compact complex manifold of any dimension s.t.  
 $\exists C' \in |K_{S'}^{-1}|$ : compact complex torus.
- Assume  $\exists g: C \cong C'$ ,
- $N_{C/S} = g^* N_{C'/S'}^{-1}$ , and
- $N_{C/S} \in \text{Pic}^0(C)$  is Diophantine

*The resulting manifold is a compact Calabi-Yau or HyperKähler.*

## Question

$\exists?$  nice example of such  $(S, C, S', C')$  in higher dimension?

## “Degrees of freedom” in our construction

- Choice of  $C_0, C'_0$ , and a Diophantine line bundle  $L$  on  $C_0$  (**dimension=1** because of  $C_0 \cong C'_0$  and Dioph. condition).
- Choice of points  $p_1, p_2, \dots, p_8 \in C_0$  (**dimension=8**).
- Choice of points  $p'_1, p'_2, \dots, p'_8 \in C'_0$  (**dimension=8**).
- Points  $p_9 \in C_0$  and  $p'_9 \in C'_0$  are automatically decided by the condition  $N_{C/S} = g^* N_{C'/S'}^{-1} = L$  (**dimension=0**).
- Choice of an isomorphism  $g: C \cong C'$  (**dimension=1**).
- Choice of (the “scaling” of) the coordinates  $w_j$ 's and  $w'_j$ 's ( $R, R' \dots$ , **dimension=1**)

### Remark

*Independence of these 19 parameters (in the sense of Kodaira–Spencer's local deformation theory) is non-trivial.*

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In this section, we give 22 cycles

$$A_{\alpha,\beta}, A_{\beta,\gamma}, A_{\gamma,\alpha},$$

$$B_\alpha, B_\beta, B_\gamma,$$

$$C_{1,2}, C_{2,3}, \dots, C_{7,8} \text{ and } C_{6,7,8},$$

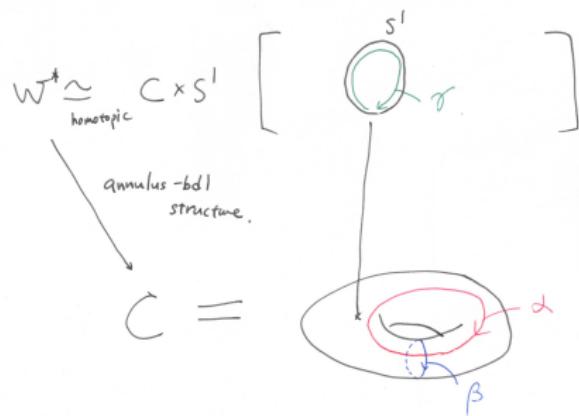
$$C'_{1,2}, C'_{2,3}, \dots, C'_{7,8} \text{ and } C'_{6,7,8}$$

which generates  $H_2(X, \mathbb{Z})$ , and compute the integration of the nowhere vanishing 2-form  $\sigma$  along these.

In the following sense, these 22 cycles can be regarded as a “marking” of  $X$ :

- $H_2(X, \mathbb{Z}) = \langle A_{\alpha,\beta}, B_\gamma \rangle \oplus \langle A_{\beta,\gamma}, B_\alpha \rangle \oplus \langle A_{\gamma,\alpha}, B_\beta \rangle \oplus \langle C_\bullet \rangle \oplus \langle C'_\bullet \rangle$ .
- $\langle A_{\alpha,\beta}, B_\gamma \rangle \cong \langle A_{\beta,\gamma}, B_\alpha \rangle \cong \langle A_{\gamma,\alpha}, B_\beta \rangle \cong U$ ,
- $\langle C_\bullet \rangle \cong \langle C'_\bullet \rangle \cong E_8(-1)$ .

Let  $\alpha, \beta$  and  $\gamma$  be loops in  $W^*$  defined as follows:



- $A_{\alpha, \beta} := \alpha \times \beta \subset W^* \subset X$
- $A_{\beta, \gamma} := \beta \times \gamma \subset W^* \subset X$
- $A_{\gamma, \alpha} := \gamma \times \alpha \subset W^* \subset X$

As  $A_{\alpha,\beta}, A_{\beta,\gamma}$  and  $A_{\gamma,\alpha}$  are included in  $W^*$  and  $\sigma|_{W^*} = \frac{dz_j \wedge dw_j}{w_j}$ , one can explicitly compute the integrals.

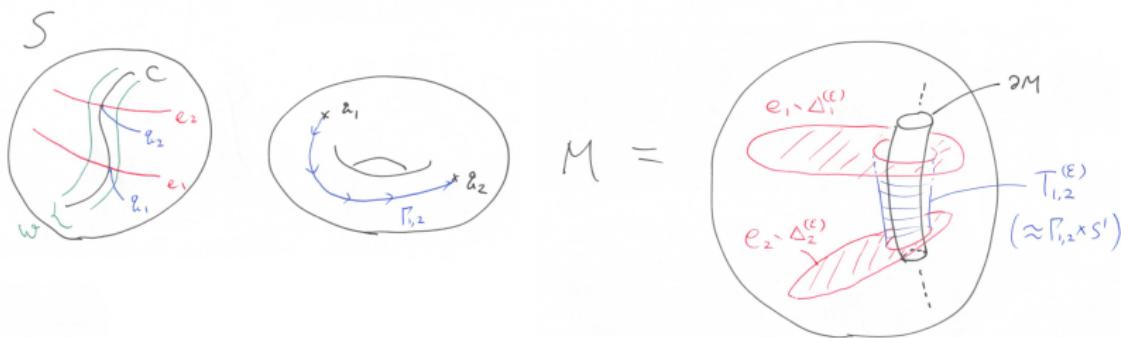
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\alpha,\beta}} \sigma = a_\beta - \tau \cdot a_\alpha,$
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\beta,\gamma}} \sigma = \tau,$
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\gamma,\alpha}} \sigma = 1,$

where

- $\tau$  is a complex number with  $\text{Im}\tau > 0$  such that  $C \cong \mathbb{C}/\langle 1, \tau \rangle$ ,
- $a_\alpha$  (resp.  $a_\beta$ ) is a real number such that the monodromy of the flat line bundle  $N_{C/S}$  along the loop  $\alpha$  (resp.  $\beta$ ) is  $\exp(2\pi\sqrt{-1} \cdot a_\alpha)$  (resp.  $\exp(2\pi\sqrt{-1} \cdot a_\beta)$ ).

Let  $e_\nu$  (resp.  $e'_\nu$ ) be the exceptional divisor corresponding to the point  $p_\nu$  ( $p'_\nu$ ) in  $S$  (resp.  $S'$ ). Denote by  $h$  (resp.  $h'$ ) the preimage of a hyperplane in  $S$  (resp.  $S'$ ).

$C_{1,2} \subset M \subset X$  is defined by  $C_{1,2} := (e_1 \setminus \Delta_1^{(\varepsilon)}) \cup T_{1,2}^{(\varepsilon)} \cup (e_2 \setminus \Delta_2^{(\varepsilon)})$ .



Note that  $C_{1,2} \sim e_1 - e_2$  holds when we regard  $C_{1,2} \subset M$  as a cycle of  $S$ .

Similarly, we define

- $C_{2,3}, C_{3,4}, \dots, C_{7,8}$ , and  $C_{6,7,8} \subset M$   
 $(C_{6,7,8} \sim -h + e_6 + e_7 + e_8$  as a cycle of  $S$ ),
- $C'_{1,2}, C'_{2,3}, \dots, C'_{7,8}$ , and  $C'_{6,7,8} \subset M'$ .

As  $C_\bullet \setminus W^*$  (resp.  $C'_\bullet \setminus W^*$ ) is an analytic subset of  $M \setminus W^*$  (resp.  $M' \setminus W^*$ ), we have that

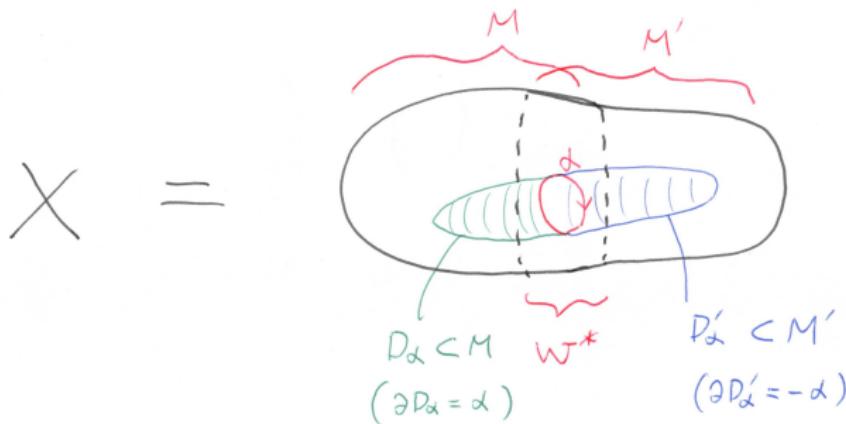
$$\int_{C_\bullet} \sigma = \int_{C_\bullet \cap W^*} \frac{dz_j \wedge dw_j}{w_j}, \quad \int_{C'_\bullet} \sigma = \int_{C'_\bullet \cap W^*} \frac{dz'_j \wedge dw'_j}{w'_j}.$$

By using this description, we can calculate the integrals.

Denote by  $q_0$  a inflection point of  $C$ ,  $q'_0$  a inflection point of  $C'$ ,  $q_\nu$  the intersection point  $C \cap e_\nu$ , and by  $q'_\nu$  the intersection point  $C' \cap e'_\nu$ . Then we have

- $\frac{1}{2\pi\sqrt{-1}} \int_{C_{\nu,\nu+1}} \sigma = \int_{q_\nu}^{q_{\nu+1}} dz_j \quad (1 \leq \nu \leq 7),$
- $\frac{1}{2\pi\sqrt{-1}} \int_{C_{6,7,8}} \sigma = \int_{q_0}^{q_6} dz_j + \int_{q_0}^{q_7} dz_j + \int_{q_0}^{q_8} dz_j,$
- $\frac{1}{2\pi\sqrt{-1}} \int_{C'_{\nu,\nu+1}} \sigma = \int_{q'_\nu}^{q'_{\nu+1}} dz'_j \quad (1 \leq \nu \leq 7),$
- $\frac{1}{2\pi\sqrt{-1}} \int_{C'_{6,7,8}} \sigma = \int_{q'_0}^{q'_6} dz'_j + \int_{q'_0}^{q'_7} dz'_j + \int_{q'_0}^{q'_8} dz'_j.$

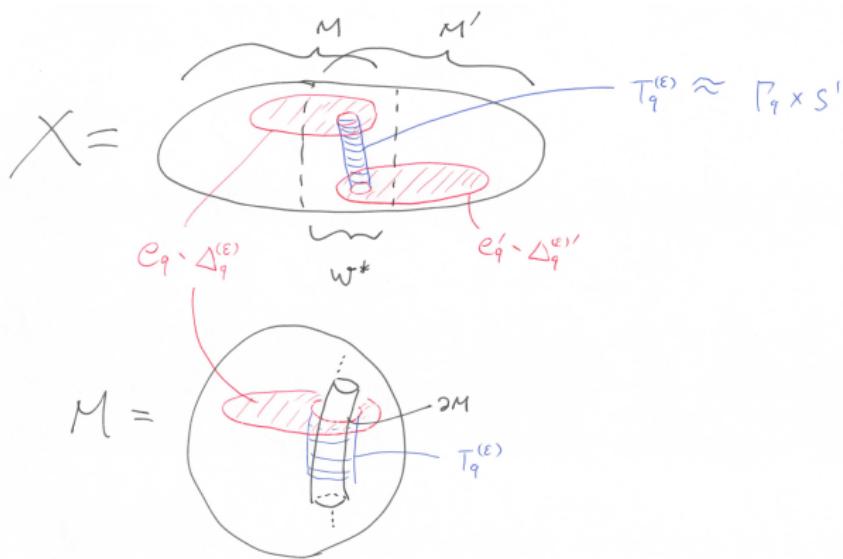
$B_\alpha$  is defined as follows by using the fact that  
 $\pi_1(M) = \pi_1(M') = 0$ .



Similarly, we define  $B_\beta$ .

At this moment, we do not know how to calculate the integrals  
 $\int_{B_\alpha} \sigma$  and  $\int_{B_\beta} \sigma$ .

$B_\gamma$  is defined by  $B_\gamma := (e_9 \setminus \Delta_9^{(\varepsilon)}) \cup T_9^{(\varepsilon)} \cup (e'_9 \setminus \Delta_9^{(\varepsilon)'}).$



By the same argument as in the  $C_\bullet$  case, we have that

$$\frac{1}{2\pi\sqrt{-1}} \int_{B_\gamma} \sigma = \int_{g(q_9)}^{q'_9} dz'_j.$$

lattice	cycle	$\frac{1}{2\pi\sqrt{-1}} \int \sigma$	corresponding parameter
U	$A_{\beta,\gamma}$	$\tau$	choice of $C_0$ (and $C'_0$ )
	$B_\alpha$	???	choice of $w_j$ 's ( $R, R', \dots$ )
U	$A_{\gamma,\alpha}$	1	—
	$B_\beta$	???	choice of $w_j$ 's ( $R, R', \dots$ )
$E_8(-1)$	$C_{1,2}$	" $p_2 - p_1$ " in $C$	choice of $p_2 - p_1$
	$C_{2,3}$	" $p_3 - p_2$ " in $C$	choice of $p_3 - p_2$
	$\vdots$	$\vdots$	$\vdots$
	$C_{7,8}$	" $p_8 - p_7$ " in $C$	choice of $p_8 - p_7$
	$C_{6,7,8}$	" $p_6 + p_7 + p_8$ " in $C$	choice of $p_6 + p_7 + p_8$
	$C'_{1,2}$	" $p'_2 - p'_1$ " in $C'$	choice of $p'_2 - p'_1$
	$C'_{2,3}$	" $p'_3 - p'_2$ " in $C'$	choice of $p'_3 - p'_2$
	$\vdots$	$\vdots$	$\vdots$
$E_8(-1)$	$C'_{7,8}$	" $p'_8 - p'_7$ " in $C'$	choice of $p'_8 - p'_7$
	$C'_{6,7,8}$	" $p'_6 + p'_7 + p'_8$ " in $C$	choice of $p'_6 + p'_7 + p'_8$
	$A_{\alpha,\beta}$	$a_\beta - \tau \cdot a_\alpha$	choice of $p_9$ and $p'_9$ (i.e. $N_{C/S}$ and $N_{C'/S'}$ )
U	$B_\gamma$	" $p'_9 - g(p_9)$ "	choice of $g: C \cong C'$

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- Fix a Diophantine pair  $(p, q) \in \mathbb{R}^2$ .  
(i.e.  $-\log \text{dist}((np, nq), \mathbb{Z}^2) = O(\log n)$  as  $n \rightarrow \infty$ )
- We first consider the subspace  $\Xi_{(p,q)}$  of the period domain  $\mathcal{D}_{\text{period}}$  which is defined by considering all the K3 surfaces constructed by our method with

$$N_{C/S} \mapsto [p + q\tau] \in \mathbb{C}/\langle 1, \tau \rangle$$

by the isomorphism  $\text{Pic}^0(C) \cong \mathbb{C}/\langle 1, \tau \rangle$  ( $\tau$  moves).

- $\dim \Xi_{(p,q)} = 19$ , with coordinates system

$$(\tau, p_1, p_2, \dots, p_8, p'_1, p'_2, \dots, p'_8, s, x),$$

where  $(s, x)$  are the parameter for the gluing.

- $s$  is defined by changing  $g: C \rightarrow C'$  with the composition  $g \circ P_s$ , where  $P_s$  is the automorphism of  $C$  defined by “ $z \mapsto z + s$ ”.

## Lemma

$$\Xi_{(p,q)} \subset v_{(p,q)}^\perp, \text{ where } v_{(p,q)} := A_{\alpha\beta} + p \cdot A_{\beta\gamma} - q \cdot A_{\gamma\alpha}$$

lattice	cycle	$\frac{1}{2\pi\sqrt{-1}} \int \sigma$	corresponding parameter
U	$A_{\beta,\gamma}$	$\tau$	choice of $C_0$ (and $C'_0$ )
	$B_\alpha$	$??? =: x - 2\tau$	choice of $w_j$ 's ( $R, R', \dots$ )
U	$A_{\gamma,\alpha}$	1	—
	$B_\beta$	$??? =: y - 2$	choice of $w_j$ 's ( $R, R', \dots$ )
$E_8(-1)$	$C_{1,2}$	“ $p_2 - p_1$ ” in $C$	choice of $p_2 - p_1$
	$C_{2,3}$	“ $p_3 - p_2$ ” in $C$	choice of $p_3 - p_2$
	$\vdots$	$\vdots$	$\vdots$
	$C_{7,8}$	“ $p_8 - p_7$ ” in $C$	choice of $p_8 - p_7$
	$C_{6,7,8}$	“ $p_6 + p_7 + p_8$ ” in $C$	choice of $p_6 + p_7 + p_8$
	$C'_{1,2}$	“ $p'_2 - p'_1$ ” in $C'$	choice of $p'_2 - p'_1$
	$C'_{2,3}$	“ $p'_3 - p'_2$ ” in $C'$	choice of $p'_3 - p'_2$
$E_8(-1)$	$\vdots$	$\vdots$	$\vdots$
	$C'_{7,8}$	“ $p'_8 - p'_7$ ” in $C'$	choice of $p'_8 - p'_7$
	$C'_{6,7,8}$	“ $p'_6 + p'_7 + p'_8$ ” in $C$	choice of $p'_6 + p'_7 + p'_8$
	$A_{\alpha,\beta}$	$a_\beta - \tau \cdot a_\alpha$	choice of $p_9$ and $p'_9$ (i.e. $N_{C/S}$ and $N_{C'/S'}$ )
U	$B_\gamma$	$“p'_9 - g(p_9)” =: s$	choice of $g: C \cong C'$

Set

- $x := 2\tau + \frac{1}{2\pi\sqrt{-1}} \int_{B_\alpha} \sigma$  (a coordinate)
- $y := 2 + \frac{1}{2\pi\sqrt{-1}} \int_{B_\beta} \sigma$  ("dummy")

The relation between  $x$  and  $y$  can be obtained by the relation

$$0 = (\sigma, \sigma) = 2\tau x + 2y + (\exists \text{quadric function of } \tau, q - p\tau, p_1 - p_2, \dots)$$

The (normalized) volume is:

$$\text{vol}(\sigma) := (\sigma, \bar{\sigma}) = 2\text{Re}(\bar{\tau}x) + 2\text{Re}(y) + (\exists \text{quadric function of } q - p\tau, \dots)$$

- $\text{vol}(\sigma)$  depends linearly on the coordinate  $x$  if one fix the other 18 coordinates.

Via the Poincare duality,

$$\begin{aligned}
 \frac{1}{2\pi\sqrt{-1}}\sigma = & (2(q-p\tau) + s)A_{\alpha\beta} + xA_{\beta\gamma} + yA_{\gamma\alpha} \\
 & + \tau B_\alpha + B_\beta + (q-p\tau)B_\gamma \\
 & + \sum_{j=1}^7 \exists c_{j,j+1} C_{j,j+1} + \exists c_{6,7,8} C_{6,7,8} \\
 & + \sum_{j=1}^7 \exists c'_{j,j+1} C'_{j,j+1} + \exists c'_{6,7,8} C'_{6,7,8}
 \end{aligned}$$

$c_\bullet$ 's depends only on  $(p_1, p_2, \dots, p_8)$ , and  $c'_\bullet$ 's depends only on  $(p'_1, p'_2, \dots, p'_8)$ .

## Theorem (K-, T. Uehara)

$\exists \widehat{V}_{(p,q)} \geq 0$  depending only on  $(p, q)$  such that,

$$\begin{aligned} \forall \widehat{\sigma} = & a_{\alpha\beta}A_{\alpha\beta} + a_{\beta\gamma}A_{\beta\gamma} + a_{\gamma\alpha}A_{\gamma\alpha} \\ & + b_\alpha B_\alpha + b_\beta B_\beta + b_\gamma B_\gamma \\ & + \sum_{j=1}^7 c_{j,j+1} C_{j,j+1} + c_{6,7,8} C_{6,7,8} \quad \in \mathcal{D}_{\text{Period}}, \\ & + \sum_{j=1}^7 c'_{j,j+1} C'_{j,j+1} + c'_{6,7,8} C'_{6,7,8} \end{aligned}$$

it holds that  $\widehat{\sigma} \in \Xi_{(p,q)}$  iff the following holds:

- $b_\beta \neq 0$  (set  $b_\beta = 1$  by “normalizing”),
- $\operatorname{Im} b_\alpha \neq 0$ ,
- The normalized volume  $\operatorname{vol}(\widehat{\sigma})$  is larger than  $\widehat{V}_{(p,q)}$ .

### Remark

$$\widehat{V}_{(p,q)} = V_{(p,q)} + V'_{(p,q)}, \text{ where}$$

- $V_{(p,q)} = \text{vol}_\eta(S \setminus (\text{“the maximal hol. tub. n.b.h.d.” of } C))$ ,
- $V'_{(p,q)} = \text{vol}_{\eta'}(S' \setminus (\text{“the maximal hol. tub. n.b.h.d.” of } C'))$

### Remark

*It is observed by a standard argument that a general member of  $\Xi_{(p,q)}$  corresponds to a K3 surface  $X$  with Picard number = 0, which means that  $X$  is non-Kummer and non-projective.*

### Question

- How large can one take a hol. tub. n.b.h.d. of  $C$  in  $S$ ?
- Is  $V_{(p,q)}$  equal to zero? or  $> 0$ ?
- How does  $V_{(p,q)}$  depend on the Diophantine pair  $(p, q)$ ?

To investigate a Kähler-geometric approach to this kinds of problems, now I'm also interested in the following:

### Question

*How is the relation between a Ricci-Flat Kähler metric on  $X$  and some “canonical metric” on  $S$  or  $S \setminus C$  (complete Ricci-Flat Kähler)?*