No. 1.

Date 2015 · 8 · 13

On some analogues of Ueda theory and their applications.

<u>Goal</u> ... Pose some analogues of Ueda theory and apply it to the study of singular Herm. metrics (s.H.m) of line bdls / proj. mfds:

<u>Ueda theory</u> ... studying a cpx analytic properties of a nbhd of $S \subset \underset{\text{cpx mfd (surface)}}{X}$ with $c_1(N_{S/X})=0$, where S: { · a sm. cpt Kä hyp. surf. of X. ← Ueda '83. ; ◦ ~~S: a rat'l curve with a node, and $\dim X = 2$.~~ ← Ueda '91. }

<u>two analogues</u>

{ "higher codim'l Ueda theory" ; "singular Ueda theory". } → [K-'14]: "codim-2 Ueda theory". → apply → — " — $Bl_{\{p_j\}_{j=1}^8}\mathbb{P}^3$

↓

$\boxed{\text{(a) [K-'15]: "nodal Ueda theory"} \xrightarrow{\text{apply}} \text{the anti-canonical bdl of } Bl_{\{p_j\}_{j=1}^9}\mathbb{P}^2.}$

<u>Schedule</u>

§1. Notations, Set-up.

§2. Main results.

§3. Application.

$\gamma, \mathfrak{X}, \cancel{\mathfrak{X}}$ ...

§1.

C: a cpt curve with only nodes.

$P(C) := \{\text{topologically triv. line bdls} / C\} / \cong_{hol}$

$\cup$

$P_0(C) := \{\text{flat line bdls} / C\} / \cong_{hol}$

$\underset{\text{fact}}{=} \{\text{flat line bdls} / C\} / \cong_{flat} := H^1(C, \underset{\{z \in \mathbb{C} \mid |z|=1\}}{U(1)})$

$P_0(C) \supset \mathcal{E}_0(C) := \{ L \in P_0(C) \mid {}^\forall n \geq 1 \ L^n = \mathbb{1}_C \}.$

$\mathcal{E}_1(C) := \{ L \in P_0(C) \mid \log \underset{\text{Euclidean dist. on } P_0(C)}{d}(\mathbb{1}_C, L^n) = O(\log n) \text{ as } n \to \infty \}.$

<u>Rmk</u> ① C: sm. $\Longrightarrow$ $P(C) = P_0(C)$ //.

② $P(C) = \mathrm{Image}\,(H^1(C, \mathbb{C}^*) \longrightarrow \underline{H^1(C, \mathcal{O}_C^*)})$

$\rightsquigarrow$ ${}^\forall L \in P(C)$, L admits "flat conn" ∂.

③ $\underline{\mathcal{E}_1(C)} = \bigcup_{\nu=1}^{\infty} {}^\exists F_\nu$ $\leftarrow$ nowhere vanishing closed subset of $P_0(C)$

④ $\mu_{\text{Lebesgue}}(P_0(C) \setminus \underline{\mathcal{E}_1(C)}) = 0.$

//

<u>e.g.</u> C: a rat'l curve with a node α

$\rightsquigarrow$ $\begin{cases} P(C) \cong \mathbb{C}^*. \\ P_0(C) \cong U(1). \end{cases}$ //

<u>Set-up</u>. X: sm surface

$\cup$

C: a cpt curve with only nodes.

s.t. $N_{C/X} := [C]|_C \in P(C)$.

① Fix. V: a small nbhd of C. s.t. $C \underset{\text{homotopic}}{\simeq} V$.

$$\begin{array}{ccc} H^1(C, \mathbb{C}^*) & \cong & H^1(V, \mathbb{C}^*) \\ \downarrow & \alpha & \downarrow \\ H^1(C, \mathcal{O}_C^*) & & H^1(V, \mathcal{O}_V^*) \\ \ni N_{C/X} & & N_\alpha \end{array}$$

<u>Def</u> ([Ueda '83].) (a generalization of "type" in.)

$$\mathrm{type}(C, X) := \max \left\{ n \in \mathbb{Z}_{\geq 1} \;\middle|\; \begin{array}{l} 0 \leq {}^\forall \nu \lneq n. \\ \mathcal{O}_V([C]) \otimes \mathcal{O}_V / I_C^{\nu+1} \cong \mathcal{O}_V(N_\alpha) \otimes \mathcal{O}_V / I_C^{\nu+1} \end{array} \right\}$$

$I_C = \mathcal{O}_V(-C)$ //

Rmk ∘ type(C,X) does not depend on "α".

∘ type$(C,X) = \infty \iff [C] \underset{\text{formally}}{\cong} N_\alpha$ along C.

—.

§2. Main results

(C,X) : as above.

Thm 1 Assume ① type$(C,X) = \infty$

② $N_{C/X} \in \mathcal{E}_0(C) \cup \mathcal{E}_1(C)$

③ $i^*N_{C/X} \in \mathcal{E}_0(\tilde{C})$, where $\tilde{C} \xrightarrow{i} C$: normalization of C.

④ $H^1(C, \mathcal{O}(N_{C/X}^{-n})) = 0$ for $\forall n \in \mathbb{Z}$.

Then

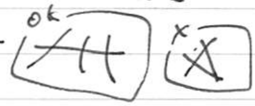
$[C]$: flat on a nbhd of C. //

(i.e. $[C] \cong N_\alpha$ around C)

Cor ①~④ $\Rightarrow$ ⓐ C admits a psd flat nbhd system.

ⓑ $[C]$: semi-positive //.

Thm 2 Assume C : tree and type$(C,X) = n < \infty$.



Then

ⓒ $\exists$ a str. psd concave nbhd of C in X.

ⓓ $\forall \lambda \in (0,1) \subset \mathbb{R}$. $\forall \Psi$: psh on $V \setminus C$

with $\Psi(p) = o(\underline{\mathrm{dist}}(p, C)^{-\lambda n})$ as $p \to C$.

$\exists M \in \mathbb{R}$ s.t. $\Psi \equiv M$ around C.

ⓔ ~~$|f_C|$~~$^{-2}$ $f_C \in H^0(X, [C])$: canonical one.

$\rightarrow |f_C|^{-2}$: a min. sing. metric of $[C]$.

(i.e. ↑ has min. sing. among $\{h : \text{s.Hm. on } [C] \mid \sqrt{-1}\Theta_h \geq 0\}$)

($\Rightarrow [C]$: not s.p.) //

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$\gamma\,\delta\,\varepsilon\cdots$

Thm 3 <u>Assume</u> C: "cycle", $N_{C/X} \in \mathcal{P}(C) \setminus \mathcal{P}_0(C)$, $\mathrm{type}(C, X) \geq 4$.

<u>Then</u> (c') —— " ——

(d') "(D)" holds for $\forall \Psi$ s.t. $\Psi(p) = o((-\log \mathrm{dist}(p, C))^{2\lambda})$

(e') —— " ——.

Rmk ◎ Ueda '83. : ~~Thm 1 and Cor 5(?)~~
(a), (c), (d) holds if C: sm, ①, ② holds.

◎ Ueda '91 : (c') (d') holds for C: a rat'l curve with a node.
s.t. $N_{C/X} \notin \mathcal{P}_0(C)$ ($\Downarrow$ type $= \infty$)

◎ (e), (e), (e') $\Longleftarrow$ (a), (c), (d), (c'), (d') + arguments in [K-'13], [K-'14].

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§3 Application

$\{P_j\}_{j=1}^{9} \subset \mathbb{P}^2$: 9 pts.

<u>Take</u> $C_0 \subset \mathbb{P}^2$: a curve with $\deg = 3$.
s.t. $C_0 \supset \{P_j\}_{j=1}^{9}$.

◎ By applying Thm 1, 2, 3, we can determine a min. sing. metric of K_X^{-1} ($X = Bl_{\{P_j\}}\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$),

<u>except</u> the case where C_0: with only nodes
and $N_{C/X} \in \mathcal{P}_0(C) \setminus (\mathcal{E}_0(C) \cup \mathcal{E}_1(C))$
($C := (\pi^{-1})_* C_0$) $\mathcal{O}_{\mathbb{P}^2}(1)|_{C_0} \otimes \mathcal{O}_{C_0}(-P_1 - P_2 - \cdots - P_9)$.

<u>e.g.</u>

◎ C_0: sm. $\rightsquigarrow$ $\mathrm{type}(C, X) = \infty$.
if $N_{C/X} \in \mathcal{E}_0 \cup \mathcal{E}_1 \Rightarrow K_X^{-1}$: s.p (Ueda, Brunella, Neeman)
$\| [C]$
(generalize to the case of $C_0 =$ ◎, $\gamma, \delta, \varepsilon$)

◎ C_0; with some nodes.

$\underset{\text{Thm 1, 3}}{\rightsquigarrow}$ $N \in \mathcal{E}_0 \cup \mathcal{E}_1 \Rightarrow K_X^{-1}$: s.p.
$N \notin \mathcal{P}_0(C) \Rightarrow K_X^{-1}$: not s.p. and ν(min. sing. metric, —) $\neq 0$.

Cor $\exists\{P_j\}_{j=1}^{9} \subset \mathbb{P}^2$ <u>s.t.</u> K_X^{-1} : nef, however it is not s.p.