No. 1.
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On the minimality of canonically attached s.H.m
on certain nef l.b.,

X: sm. proj. var $/\mathbb{C}$,
L: (hol) line bdl $/X$.

<u>Def</u> L: nef
$\underset{\text{def}}{\Longleftrightarrow}$ $^{\forall} C \subset X$: alg. curve, $\underset{\substack{\| \\ c_1(L).C = \int_C \sqrt{-1}\Theta_L}}{L.C} \geq 0$.

<u>Q</u> Where does a <u>minimal singular metric</u> of L have singularities ? when L is nef ?

<u>Def</u> h: sing. Herm. metric on L (<u>sHm</u>)
$\underset{\text{def}}{\Longleftrightarrow}$ $^{\exists} h_\infty$: sm. Herm. metric on L.
$^{\exists} \chi : X \longrightarrow [-\infty, \infty)$: L^1_{loc}
s.t. $h = h_\infty \cdot e^{-\chi}$.

⊙ $h \underset{\text{local}}{=} e^{-\varphi}$ ← the local weight of h.
$\rightsquigarrow \sqrt{-1}\Theta_h = \sqrt{-1}\partial\bar{\partial}\varphi$.

<u>Def</u> h: <u>a min. sing. metric</u> on L
$\underset{\text{def}}{\Longleftrightarrow}$ ∘ $\forall$ local weight φ of h is psh. (i.e. $\sqrt{-1}\Theta_h \geq 0$)
∘ $^{\forall} h'$: a s.H.m of L with sing. curvature, $^{\forall} x \in X$,
$\forall$ local weight φ' of h' around x
$^{\exists} C \in \mathbb{R}$ s.t. $\varphi' \leq \varphi + C$ around x //

<u>Fact</u> L: nef $\Rightarrow$ $^{\exists}$ min. sing. metric on L //.

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Known.

◎ L: ample $\overset{\text{Kodaira's emb. thm}}{\iff}$ L: positive $\Rightarrow$ ${}^{\forall}$ min. sing. metric has no singularity.

◎ L: ~~ ~~ semi-ample $\Longrightarrow$ L: semi-positive. (i.e. $\exists h$: sm. metric on L s.t. $\sqrt{-1}\Theta_h \geq 0$.)
(i.e. $m>1$, $|L^{\otimes m}|$: basepoint free)

$\Longrightarrow$ ${}^{\forall}$ min. sing metric has no sing.

◎ L: semi-positive $\Longrightarrow$ L: nef.

◎ L: ~~ ~~ nef. $\not\Longrightarrow$ L: semi-positive

e.g. (Demailly - Peternell - Schneider)

C_0: sm. ellipt. curve

E: rank 2 - vect. bdl $/C_0$.

s.t. $0 \to \mathcal{O}_{C_0} \to E \longrightarrow \mathcal{O}_{C_0} \longrightarrow 0$: ex. non-splitting.

$X := \mathbb{P}(E) \xrightarrow{\pi} C_0$.

$L := \mathcal{O}_X(C)$, where $C :=$ the section of π.

$\rightsquigarrow L$: nef, ~~ ~~

h: s.H.m. on L with s.p. curvature

$\Longrightarrow h = M \cdot |f_C|^{-2}$ for $\exists M>0$, $\exists f_C \in H^0(X, \mathcal{O}_X(C))$: can. section //

Main results

Thm 1. X: cpx sm. surf.
$\cup$
C: sm. curve $(C^2)=0$. $\rightsquigarrow L := \mathcal{O}_X(C)$: nef.

Assume (C, X): of finite type in the sense of Ueda $\cdots(*)$

Then ~~L: not~~ ~~$\forall h$: s.H.m~~.

$|f_C|^{-2}$ is a min. sing. metric of L,
where $f_C \in H^0(X, \mathcal{O}_X(C))$: can. section //

◎ Def (*) $\underset{\text{def}}{\iff}$ $\exists n \in \mathbb{N}_{\geq 1}$ s.t. for suff. small ~~tub~~ nbhd V of C in X,

$$\begin{cases} \widetilde{N_{C/V}} \otimes \mathcal{O}_V(-C) \otimes \mathcal{O}_V/\mathcal{O}_V(-nC) \not\cong \mathcal{O}_V/\mathcal{O}_V(-nC), \\ \text{where } \widetilde{N_{C/V}} \text{ is the flat ext. of } N_{C/V} \text{ to } V \end{cases}$$ //.

Thm2 "Example 5.9" in [O.Fujino '13] "A transcendental approach to Kollár's inj. thm II".
is ~~str. nef~~, strictly nef
but not semi-positive //

(L: str. nef $\underset{\text{def}}{\iff}$ $\forall C$: alg. curve in X, $L.C \gneq 0$).

§1. some examples and prf of Thm1
§2. ~~prf~~ Thm2.

§1. Cor1 (: a generalization of D.P.S. exmple)
C_0: sm. curve.
E : a rank 2 - vect. bdl $/C_0$
s.t. $\exists F$: a flat line bdl $/C_0$
s.t. $0 \to F \to E \to \mathcal{O}_{C_0} \to 0$: ex. ---(**).
$X := \mathbb{P}(E) \xrightarrow{\pi} C_0$, C := the section of π.
$\rightsquigarrow$ $\underset{\text{nef}}{\mathcal{O}_X(C)}$: semi-positive $\iff$ (**) splits,
if (**) does not split, then
$|f_C|^{-2}$ is a min. sing. metric on $\mathcal{O}_X(C)$ //.

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Cor2. C_0 : a sm. curve of genus $=2$,
$C_0 \hookrightarrow Y$: the Jacobian of C_0.
$\overset{\psi}{p}, q$: conjugate to each other by the hyp. ellipt. involution.
$X :=$ the b-up of Y at $\{p, q\}$.
$\check{C} :=$ the str. transf. of C_0.
Then $|f_C|^{-2}$ is a min. sing. metric on $\mathcal{O}_X(C)$ //.

prf of Cor 1, 2.
Neeman showed that (C, X) is of type 1 $(\tilde{N} \otimes \mathcal{O}(-C) \otimes \mathcal{O}_V/\mathcal{O}(-2C) \not\cong \mathcal{O}_V/\mathcal{O}(-2C))$ for these examples. //

prf of Thm 1
... a simple application of Ueda's thm.
$\underset{\text{sm. cpt curve}}{\underbrace{C}} \subset \underset{\text{sm. surf.}}{X}$
Assume (C, X) is of type $n < \infty$.
i.e. $\tilde{N}_{C/V} \otimes \mathcal{O}_V/\mathcal{O}_V(-\nu C) \cong \mathcal{O}_V(C) \otimes \mathcal{O}_V/\mathcal{O}_V(-\nu C)$ for $\nu = 2, 3, \cdots, n-1$,
$\qquad\qquad\qquad\qquad \not\cong \qquad\qquad$ for $\nu = n$ //.

Thm (Ueda)
$\forall a \in (0, n) \subset \mathbb{R}$,
$\forall V$: a nbhd of C in X.
$\forall \Psi$: a psh func. on $V \setminus C$.
s.t. $\Psi(p) = o(\mathrm{dist}(p, C)^{-a})$ as $p \to C$.
Then $\exists V_0$: a nbhd of C in V,
s.t. $\Psi|_{V_0} \equiv {}^{\exists}\mathrm{const}$ //.

~~proof of.~~

Let h be a s.H.m. on $\mathcal{O}_X(C)$ with s.p. curvature.

$\Psi := -\log|f_C|_h^2$ $\quad$ ($f_C \in H^0(X, \mathcal{O}_X(C))$: can. section)

$\underset{\text{local.}}{=}$ $-\log|\underbrace{f_C}_{\text{hol. def. func of } C.}|^2 + \varphi$ $\quad$ $(h = e^{-\varphi})$

; psh on $X \setminus C$, $= o(\mathrm{dist}(P, C)^{-\frac{1}{2}})$

Ueda's thm. $\rightsquigarrow$ $\exists V_0$: a nbhd of C in X. $\exists M$; const.

$\Psi \equiv M$.

$\underline{\text{i.e}}$ $|f_C|_h^2 = e^{-M} \rightsquigarrow h = e^{-M} \cdot |f_C|^{-2}$ //.

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§2.

$\underline{Q1}$. L : str. nef $\overset{?}{\Rightarrow}$ L : semi-ample ?

$\underline{\text{No}}$ ... $\underline{\text{e.g. (Mumford)}}$

$\tilde{C}$: a sm. cpt curve of genus $= g > 1$.

$\underline{\text{fact}}$ $\left(\begin{array}{l} \exists F : \text{rank 2-vect. bdl} / \tilde{C}. \\ \text{s.t. } \deg(F) = 0, \ S^m F : \text{stable for } {}^{\forall} m \geq 1 \end{array} \right.$

$\rightsquigarrow Y := \mathbb{P}(F)$, $L_Y := \mathcal{O}_{\mathbb{P}(F)}(1)$

$\rightsquigarrow L_Y$: str. nef. but not semi-ample. //

$\underline{Q2}$ str. nef $\overset{?}{\Rightarrow}$ ~~s.p.~~ semi-positive ?

<u>Claim</u> the above L_Y is s.p //.

<u>prf</u> [Narasimhan - Seshadri] $\rightsquigarrow$ F : flat.

<u>i.e</u> $\exists h_F$: sm. metric on F

<u>s.t</u> $\exists \{U_j\}_j$: open cov of $\tilde{C}$.

$\exists (s_j, t_j)$: loc. frame of F on U_j

s.t. $|s_j|^2_{h_F} \equiv |t_j|^2_{h_F} \equiv 1$, $(s_j, t_j)_{h_F} \equiv 0$ //

$\rightsquigarrow$ $h_{L_Y} :=$ the fiberwise F.S. metric assoc. to h_F.

We use $(w, x) := [w \cdot s_j^*(x) + t_j^*(x)] \in \mathbb{P}(F) = Y$ as a loc. coord Y

$\rightsquigarrow$ the local weight φ_{L_Y} of h_{L_Y} is ;

$$\varphi_{L_Y} = \log(1 + |w|^2) \ : \ \text{psh} \quad //$$

<u>e.g. (= Example 5.9 in [Fujino '13])</u>

$\tilde{C}$, $Y = \mathbb{P}(F)$, L_Y : as above.

$0 \to \mathcal{O}_{\tilde{C}} \to E \to \mathcal{O}_{\tilde{C}} \to 0$: ex, non-splitting.

$\tilde{X} := \mathbb{P}(E) \xrightarrow{\pi} \tilde{C}$.

$\cup$

$\tilde{D} :=$ the section of π.

$$\begin{tikzcd} \tilde{Y} := \tilde{X} \times_{\tilde{C}} Y \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\ulcorner"] & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "\pi'"'] & \tilde{C} \end{tikzcd}$$

$$\tilde{L} := \mathcal{O}_{\tilde{Y}}(\tilde{D} \times_{\tilde{C}} Y) \otimes p_2^* L_Y$$

$\rightsquigarrow$ $\tilde{L}$: str. nef. but not s.a //

Q3 ([Fujino '13 Question 5.(0)])

Is $\tilde{L}$ s.p?

Cor 3 $\tilde{L}$ is not s.p //.

prf of Cor3.

Let $h_{\tilde{L}}$ be a sing. Herm. metric of $\tilde{L}$ with s.p. curvature

Fix a **sm.** Herm. metric h_∞ of $\mathcal{O}_{\tilde{X}}(\tilde{D})$.

$\rightsquigarrow$ $\exists\, \chi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$: L^1_{loc}.

s.t. $h_{\tilde{L}} = (p_1^* h_\infty) \otimes (p_2^* h_{LY}) \cdot e^{-\chi}$.

⊙ local coord. system of $\tilde{Y}$

x ⋯ a loc. coord of $\exists\, U \subset \tilde{C}$ open disc.

(w, x) ⋯ as in e.g. (Mumford).

(z, x) ⋯ a loc. coord system of $\tilde{X}$

(z : a fiber coord. of $\tilde{X} \xrightarrow[\pi]{} \tilde{C}$).

$\rightsquigarrow$ (the local weight of $h_{\tilde{L}}$)

$$= \varphi_\infty(z, x) + \log(1 + |w|^2) + \chi(z, w, x)$$

where $h_\infty \underset{\text{locally}}{=} e^{-\varphi_\infty}$.

⊙ $\boxed{\tilde{\chi}(z, x) := \max_{w_0 \in \pi^{-1}(x)} \chi(z, w_0, x)}$ psh

⊙ $\tilde{Y}|_U := \pi^{-1}(U) \times_U \pi'^{-1}(U)$

$$\cong U_x \times \mathbb{P}^1_z \times \mathbb{P}^1_w \xrightarrow{\chi|_{(\tilde{Y}|_U)}} \mathbb{R} \cup \{-\infty\}.$$

$\rightsquigarrow$ $\varphi_\infty(z, x) + \cancel{\log(1 + |w_0|^2)} + \chi(z, w_0, x)$

: psh for each locus of $U_x \times \mathbb{P}^1_z$.

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$\rightsquigarrow$ $\varphi(z,x) + \widetilde{\chi}(z,x)$: psh on each leaves of $U_x \times \mathbb{P}^1_z$.

$\rightsquigarrow$ $h_\infty \cdot e^{-\widetilde{\chi}}$: Sing. Herm. metric on $\mathcal{O}_{\widetilde{X}}(\widetilde{D})$ with s.p. curvature.

$\overset{\text{Cor 1}}{\rightsquigarrow}$ $(h_\infty \cdot e^{-\widetilde{\chi}})|_{V_0} \geqq \underset{\text{const}}{\underbrace{{}^{\exists}M}} \cdot |f_{\widetilde{D}}|^{-2}$.

for $\begin{cases} {}^{\exists}V_0 : \text{a nbhd of } \widetilde{D} \text{ in } \widetilde{X}. \\ f_{\widetilde{D}} \in H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\widetilde{D})) : \text{can. section.} \end{cases}$

$$\rightsquigarrow p_1^*(M|f_{\widetilde{D}}|^{-2}) \otimes (p_2^* h_{L_Y})$$
$$\leq p_1^*(h_\infty \cdot e^{-\widetilde{\chi}}) \otimes (p_2^* h_{L_Y})$$
$$\leq (p_1^* h_\infty) \otimes (p_2^* h_{L_Y}) \cdot e^{-\chi} = h_{\widetilde{L}}.$$

$\rightsquigarrow$ $h_{\widetilde{L}}$ must have singularities along $p_1^{-1}(\widetilde{D})$ //.

<u>Cor</u>. $(p_1^*|f_{\widetilde{D}}|^{-2}) \otimes (p_2^* h_{L_Y})$:
a Sing. Herm. metric of $\widetilde{L}$
with s.p. curvature.
with minimal sing //.