X : sm. proj. (surf)

L : hol. l.b. $/X$.

Def
- L : nef. $\overset{\text{def}}{\iff}$ $\forall C \subset X$: curve $L.C \geq 0$.
- L : s.p. $\overset{\text{def}}{\iff}$ ${}^{\exists}$ sm. Herm. metric on L s.t. $\sqrt{-1}\,\Theta_h \geq 0$ //.

◎ Known results

◎ L : s.p. $\Longrightarrow$ L : nef.

◎ L : nef $\not\Longrightarrow$ L : s.p.

e.g. (D.P.S).

◎ Question Assume
$$\begin{cases} ① \ L = {}^{\exists}\underset{\text{s.p}}{A} \otimes \mathcal{O}_X(\overset{\exists}{C}) & C:\text{ sm. hyp. surf of } X \\ ② \ c_1(L|_C) = 0 \end{cases}$$

$\rightarrow$ When does L admit sm. Herm. metric w/ s.p. curvature?

$$\left(\begin{array}{l} ① \Rightarrow L|_{X \setminus C} : \text{s.p.} \\ ② \Rightarrow L|_C : \text{s.p.} \\ ①+② \Rightarrow L : \text{nef} \end{array} \right).$$

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§1. main results.
§2. examples.
§3. Outline of prfs.

§1 Main results

Thm A (ArXiv/1312.6402)

X, C, A, L: as above.
(X: surface.)

Assume $(C^2) < \min\{0, 4-4g\}$,
where g = genus of C.

Then L: s.p //

Thm B (ArXiv/1405.4698)

X, C, A, L: as above.
(X: surface.)

Assume $A = \mathcal{O}_X$, $(C^2) = 0$.

Then $\exists n \in \mathbb{N}_{\geq 1}$ $u_n(C, X) \neq 0$ (or "finite type" in the sense of Ueda.)

$\Longrightarrow$ L: not. s.p. //

Def $u_n(C, X) \neq 0$

$$\overset{\text{def}}{\Longleftrightarrow} \quad \widetilde{N_{C/X}} \otimes \mathcal{O}_X(-C) \otimes \mathcal{O}_X/\mathcal{O}_X(-nC) \not\cong \mathcal{O}_X/\mathcal{O}_X(-nC)$$

$\widetilde{N_{C/X}}$: flat ext. of $N_{C/X}$ to a tub. nbhd. of C in X //

§2. Some examples

$\boxed{\text{e.g. A}}$ … Zariski's e.g.

$\mathbb{P}^2 \supset C_0$: a sm. ellipt. curve.

$\pi \uparrow \qquad \ni P_1, P_2, \cdots, P_{12}$: general 12 pts

X : b-up at $\{P_i\}_{i=1}^{12}$.

$C := (\pi^{-1})_* C_0, \quad A := \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$

$L := A \otimes \mathcal{O}_X(C) \overset{\text{Thm A}}{\Longrightarrow} L$: s.p //.

$\boxed{\text{e.g. B}}$. … a generalization of [D.P.S.] e.g.

C_0 : a sm. curve.

E : a rank 2 - vect. bdl $/C_0$.

F : a line bdl $/C_0$.

s.t. $\begin{cases} c_1(F) = 0. \\ 0 \to F \to E \to \mathcal{O}_{C_0} \to 0 \text{ : ex.} \end{cases}$ --- $(*)$

$X := \mathbb{P}(E) \xrightarrow{\pi} C_0$,

$\cup$

C : the section of π.

$\overset{\text{Thm B}}{\Longrightarrow} \quad \mathcal{O}_X(C)$: s.p $\iff (*)$ splits //.

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§3

Outline of prf. of Thm A

Thm (Grauert)

X : surf. $\supset C$: sm. curve.
$(C^2) < \min\{0, 4-4g\}$
$\Longrightarrow$ $\exists V$: a nbhd of C in X.
$\exists V'$: a nbhd of the 0-section in $N_{C/X}$
s.t. $V \underset{\text{bihol}}{\cong} V'$ //.

① Grauert's thm + a version of Rossi's thm
$\rightsquigarrow$ $L|_V$: s.p. $\backsim h_1$
② Assumption $\rightsquigarrow$ $L|_{X \setminus C}$: s.p. $\backsim h_2$.
③ "regularized min." $\{h_1, h_2\}$ is a sm. Herm. metric on L w/ s.p. curvature //.

Outline of prf of Thm B

Thm (Ueda)

(C, X) : of fin. type, $u_n(C, X) \neq 0$.
$0 < a < n$
V : a nbhd of C in X.
Φ : $V \setminus C \to \mathbb{R}$: psh. s.t.
$\Phi(p) = O\left(\frac{1}{\mathrm{dist}(C,p)^a}\right)$ as $p \to C$

Then Φ : constant around C. //.

① $f_C \in H^0(X, \mathcal{O}_X(C))$: canonical one.
h : singular Herm. metric on $\mathcal{O}_X(C)$ w/ s.p. curv.

Ueda's thm $\rightsquigarrow$ $\Phi := \log |f_C|_h^2$: constant around C
$\rightsquigarrow$ $|f_C|^{-2}$: min. sing. metric of L //.