Gluing construction of non-projective K3 surfaces and holomorphic tubular neighborhoods of elliptic curves

Takayuki Koike (joint w/Takato Uehara)

Department of Mathematics, Graduate School of Science, Osaka City University

December 20, 2018

- based on [T. Koike, Complex K3 surfaces containing Levi-flat hypersurfaces, arXiv:1703.03663] + some recent progress (j.w/ T. Uehara)

Goal of this talk:

To construct a (non-projective, non-Kummer) K3 surface X containing a real 1-parameter family of Levi-flat hypersurfaces by holomorphically patching two open complex surfaces, say M and M'

- M (M') is the complement of a (appropriate) tubular neighborhood of an elliptic curve in the blow-up S (S') of the projective plane $\mathbb{P}^2$ at (appropriate) nine points.
- Neither S nor S' admit elliptic fibration structure (nine points are "general")
- In order to patch M and M' holomorphically, we need to take "nice neighborhood". For this purpose, we need to choose nine points carefully.
- Use a technique from Complex Dynamics for taking "nice neighborhood" (Arnol'd's theorem).

Remarks, Known results

- For the case where S and S' are elliptic, a similar construction of K3 surfaces is known. The resulting K3 surfaces are also elliptic.
- M. Doi showed that a similar construction of K3 surfaces is possible for general Z and Z' if one admit (slight) deformations of the complex structures of M and M'.
  (Doi, Mamoru, Gluing construction of compact complex surfaces with trivial canonical bundle. J. Math. Soc. Japan 61 (2009), no. 3, 853–884)
- The idea to use Arnol'd-type theorem for patching two open manifolds is also used by H. Tsuji in order to study complex structures on $S^3 \times S^3$.
  (H. Tsuji, Complex structures on $S^3 \times S^3$, Tohoku Math. J. (2) Volume 36, Number 3 (1984), 351–376)

Remarks on our construction of K3 surfaces

- In our construction, "the tab for gluing" $W^* := M \cap M'$ is an open submanifold of X which admits an annulus bundle structure over an elliptic curve.
- $\exists \Phi \colon W^* \to I$ ($I \subset \mathbb{R}$: an interval): pluriharmonic .
- $H_t := \Phi^{-1}(t)$ is a compact Levi-flat hypersurface of $W^* (\subset X)$ which is diffeomorphic to $S^1 \times S^1 \times S^1$ for each $t \in I$.
- For each $t \in I$, any leaf of H_t is biholomorphic to $\mathbb{C}$ or $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and is dense in H_t.

Main results

We constructed K3 surfaces in such a manner with independent (at least) 19 parameters. As a result, we showed for example:

Theorem (K-, T. Uehara)

There exists a deformation $\pi\colon \mathcal{X} \to B$ of K3 surfaces over a 19 dimensional complex manifold B with injective Kodaira-Spencer map such that each fiber $X_b := \pi^{-1}(b)$ admits a holomorphic map $F_b\colon \mathbb{C} \to X_b$ with the following property: The Euclidean closure of $F_b(\mathbb{C})$ is a real analytic compact hypersurface of X_b. Especially, $F_b(\mathbb{C})$ is Zariski dense whereas it is not Euclidean dense. X_b is non-Kummer and non-projective for general $b \in B$.

"F_b" can be constructed by considering the immersion of a leaf of H_t into $W^* \subset X$ for each $X = X_b$.

Let

- S be a non-singular complex surface, and
- C be a compact complex curve embedded in S with $(C^2) := \deg N_{C/S} = c_1(N_{C/S}) = 0$.

There exists a (small) neighborhood W of C in S which is *diffeomorphic* to a neighborhood of the zero section in $N_{C/S}$ (tubular neighborhood theorem).

Our original interest:

What kind of complex analytic structure does W have?

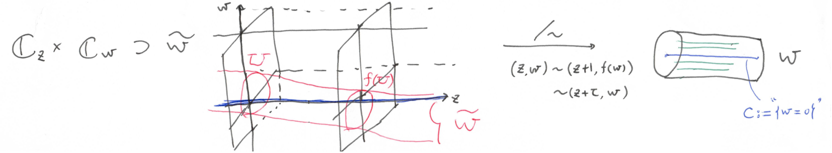
Remark

In general, $\nexists W$ which is biholomorphic to a neighborhood of the zero section in $N_{C/S}$.

Ueda's example

Fix $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ and $f(w) = a_1 w + a_2 w^2 + \cdots \in \mathcal{O}_{\mathbb{C},0}$ $(|a_1| = 1)$. Take a neighborhood $\widetilde{W}$ of $\mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}$.

$W := \widetilde{W}/\sim$, $(z, w) \sim (z+1, f(w)) \sim (z+\tau, w)$
$C \subset W$: the image of $\mathbb{C} \times \{0\}$ (smooth elliptic curve)



Fact [K-, N. Ogawa, arXiv:1808.10219]

C admits a *holomorphic tubular neighborhood* (i.e. $\exists W$ which is biholomorphic to a neighborhood of the zero section)
iff f is *linearizable* around the origin.

Our main interest is in the following example:
Take a smooth elliptic curve $C_0 \subset \mathbb{P}^2$ and nine points $Z := \{p_1, p_2, \dots, p_9\} \subset C_0$.

- $S := \mathrm{Bl}_Z\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$: blow-up at Z
- $C := \pi_*^{-1}C_0$: the strict transform of C_0

Note that $N_{C/S} \cong \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \cdots - p_9)$. When Z is special, S is an elliptic surface ($N_{C/S} \in \mathrm{Pic}^0(C)$ is torsion in this case). We are interested in the case where Z is general.

Theorem (Brunella, 2010)

Assume that $S \setminus C$ has no compact complex curve.
K_S^{-1} admits a C^∞ Hermitian metric with semi-positive curvature
iff C has a pseudoflat neighborhoods system
(i.e. $\exists$ fundamental system of neighbourhoods $\{W_\varepsilon\}_{\varepsilon>0}$ of C such that ∂W_ε is Levi-flat).

Note that C has a pseudoflat neighborhoods system if C admits a hol. tub. n.b.h.d.

Question

When does C admit a holomorphic tubular neighborhood?

Theorem (Arnol'd (1976))

Assume C is a smooth elliptic curve and $N_{C/S} \in \mathrm{Pic}^0(C)$ is Diophantine (i.e. $\exists A, \alpha > 0$ such that $\mathrm{dist}(\mathbb{I}_C, N_{C/S}^{\otimes n}) \geq A \cdot n^{-\alpha}$ for $\forall n > 0$, where dist is the Euclidean distance of $\mathrm{Pic}^0(C)$).
Then C admits a holomorphic tubular neighborhood.

For the previous Ueda's example, this theorem can be directly deduced from Siegel's linearization theorem.

Question

For the previous example $(C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S)$, does C admit a holomorphic tubular neighborhood when $N_{C/S} \in \mathrm{Pic}^0(C)$ is not Diophantine?

Let $(C_0, Z = \{p_1, p_2, \ldots, p_9\}, C, S)$ and $(C'_0, Z' = \{p'_1, p'_2, \ldots, p'_9\}, C', S')$ be as in the previous section.

Assumption

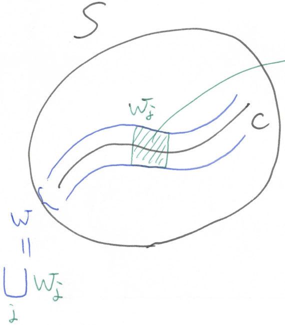
- $\exists g\colon C \cong C'$
- $N_{C/S} = g^* N_{C'/S'}^{-1}$
- $N_{C/S} \in \mathrm{Pic}^0(C)$ is Diophantine

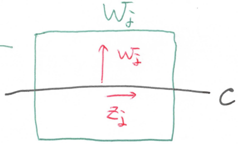
Then, it follows from Arnol'd's theorem that there exist holomorphic tubular neighborhoods $W \subset S$ of C and $W' \subset S'$ of C'.

Take a local charts systems $\{(W_j, (z_j, w_j))\}$ of W and $\{(W'_j, (z'_j, w'_j))\}$ of W' such that

$$\begin{cases} z_j = z_k + A_{jk} \\ w_j = t_{jk} \cdot w_k \end{cases}, \quad \begin{cases} z'_j = z'_k + A_{jk} \\ w'_j = t_{jk}^{-1} \cdot w'_k \end{cases}$$

for some constants $A_{jk} \in \mathbb{C}$ and $t_{jk} \in U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$ on $W_{jk} := W_j \cap W_k$ and $W'_{jk} := W'_j \cap W'_k$ as follows:





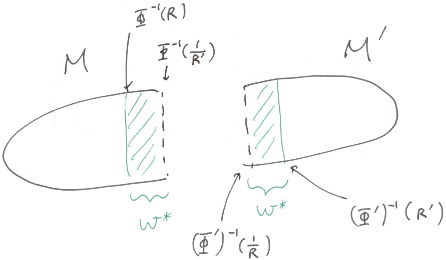
- w_j is a local defining function of C
  ($\{w_j = 0\} = W_j \cap C$)
- $z_j|_C$ comes from a natural coordinate of the univ. covering $\mathbb{C}$ of C.

- $\Phi\colon W \to \mathbb{R}$: $(z_j, w_j) \mapsto |w_j|$: globally defined on W.
- $\Phi'\colon W' \to \mathbb{R}$: $(z'_j, w'_j) \mapsto |w'_j|$: globally defined on W'.
- By scaling, we may assume that $\Phi^{-1}([0,R]) \Subset W, (\Phi')^{-1}([0,R']) \Subset W'$ $(R, R' > 1)$.
- Replace W with $\Phi^{-1}([0,R))$ and W' with $(\Phi')^{-1}([0,R'))$.

Define $M \subset S$ and $M' \subset S'$ by

$$M := S \setminus \Phi^{-1}\left(\left[0, \frac{1}{R'}\right]\right), \quad M' := S' \setminus (\Phi')^{-1}\left(\left[0, \frac{1}{R}\right]\right).$$

Identify $W \cap M = \Phi^{-1}((1/R', R))$ and $W' \cap M' = (\Phi')^{-1}((1/R, R'))$ by the isomorphism $f\colon \Phi^{-1}((1/R', R)) \to (\Phi')^{-1}((1/R, R')) : (z_j, w_j) \mapsto \left(g(z_j), \frac{1}{w_j}\right)$ and denote it by W^*.



$X := M \cup_{W^*} M'$: a compact complex manifold obtained by patching M and M' via f.

Observation

W^* admits a natural foliation $\mathcal{F}$ whose leaves are locally defined by "$\{w_j =$ constant$\}$". As each leaf is biholomorphic to $\mathbb{C}$ or $\mathbb{C}^*$, we have a holomorphic map $F\colon \mathbb{C} \to W^* \subset X$ as in Main Theorem.

It is easily observed that $\pi_1(X) = 0$. Therefore, for proving that X is a K3 surface, it is sufficient to show the following:

Proposition

There exists a nowhere vanishing holomorphic 2-form σ on X such that

$$\sigma|_{W^* \cap W_j} = \frac{dz_j \wedge dw_j}{w_j}$$

holds on each $W^* \cap W_j \subset W^* \subset X$.

Key Lemma

$H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}$.

Key Lemma is shown by pulling back a holomorphic function on W^* by $F\colon \mathbb{C} \to W^*$ and considering the Maximum principle. $\square$

$\boxed{\text{Proof of Proposition:}}$ As $K_S = -C$, there exists a meromorphic 2-form η on S with $\mathrm{div}(\eta) = -C$. It follows from Key Lemma that the function

$$\frac{\eta|_{W^*}}{\left(\frac{dz_j \wedge dw_j}{w_j}\right)}$$

is a constant map. Thus we may assume that $\eta|_{W^*} = \frac{dz_j \wedge dw_j}{w_j}$. Similarly, one can show the existence of a meromorphic 2-form η' on S' with $\mathrm{div}(\eta') = -C'$ such that $\eta'|_{W^*} = \frac{dz'_j \wedge dw'_j}{w'_j}$.
σ is obtained by patching $\eta|_M$ and $-\eta'|_{M'}$. $\square$

Remark

The construction in this section also makes sense even after one generalize "the materials" as follows:

- S: *compact complex manifold of any dimension s.t.* $\exists C \in |K_S^{-1}|$: *compact complex torus.*
- S': *compact complex manifold of any dimension s.t.* $\exists C' \in |K_{S'}^{-1}|$: *compact complex torus.*
- *Assume* $\exists g \colon C \cong C'$,
- $N_{C/S} = g^* N_{C'/S'}^{-1}$, *and*
- $N_{C/S} \in \mathrm{Pic}^0(C)$ *is Diophantine*

The resulting manifold is a compact Calabi-Yau or HyperKähler.

Question

$\exists$? *nice example of such* (S, C, S', C') *in higher dimension?*

"Degrees of freedom" in our construction

- Choice of C_0, C_0', and a Diophantine line bundle L on C_0 (dimension$=1$ because of $C_0 \cong C_0'$ and Dioph. condition).
- Choice of points $p_1, p_2, \ldots, p_8 \in C_0$ (dimension$=8$).
- Choice of points $p_1', p_2', \ldots, p_8{}' \in C_0'$ (dimension$=8$).
- Points $p_9 \in C_0$ and $p_9' \in C_0'$ are automatically decided by the condition $N_{C/S} = g^* N_{C'/S'}^{-1} = L$ (dimension$=0$).
- Choice of an isomorphism $g\colon C \cong C'$ (dimension$=1$).
- Choice of (the "scaling" of) the coordinates w_j's and w_j''s (R, R'..., dimension$=1$)

Remark

Independence of these 19 parameters (in the sense of Kodaira–Spencer's local deformation theory) is non-trivial.

In this section, we give 22 cycles

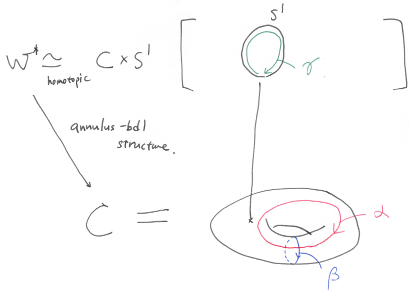
$A_{\alpha,\beta}, A_{\beta,\gamma}, A_{\gamma,\alpha}$,
$B_\alpha, B_\beta, B_\gamma$,
$C_{1,2}, C_{2,3}, \dots, C_{7,8}$ and $C_{6,7,8}$,
$C'_{1,2}, C'_{2,3}, \dots, C'_{7,8}$ and $C'_{6,7,8}$

which generates $H_2(X, \mathbb{Z})$, and compute the integration of the nowhere vanishing 2-form σ along these.

In the following sense, these 22 cycles can be regarded as a "marking" of X:

- $H_2(X, \mathbb{Z}) = \langle A_{\alpha,\beta}, B_\gamma \rangle \oplus \langle A_{\beta,\gamma}, B_\alpha \rangle \oplus \langle A_{\gamma,\alpha}, B_\beta \rangle \oplus \langle C_\bullet \rangle \oplus \langle C'_\bullet \rangle$.
- $\langle A_{\alpha,\beta}, B_\gamma \rangle \cong \langle A_{\beta,\gamma}, B_\alpha \rangle \cong \langle A_{\gamma,\alpha}, B_\beta \rangle \cong U$,
- $\langle C_\bullet \rangle \cong \langle C'_\bullet \rangle \cong E_8(-1)$.

Let α, β and γ be loops in W^* defined as follows:



- $A_{\alpha,\beta} := \alpha \times \beta \subset W^* \subset X$
- $A_{\beta,\gamma} := \beta \times \gamma \subset W^* \subset X$
- $A_{\gamma,\alpha} := \gamma \times \alpha \subset W^* \subset X$

As $A_{\alpha,\beta}, A_{\beta,\gamma}$ and $A_{\gamma,\alpha}$ are included in W^* and $\sigma|_{W^*} = \frac{dz_j \wedge dw_j}{w_j}$, one can explicitly compute the integrals.

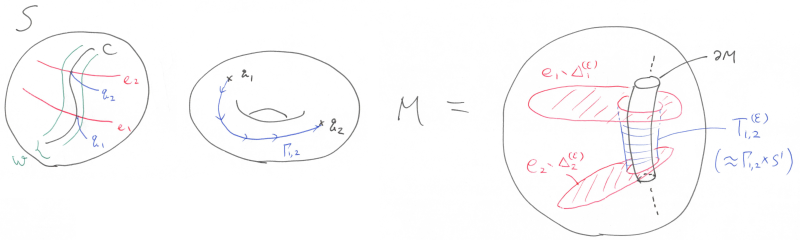
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\alpha,\beta}} \sigma = a_\beta - \tau \cdot a_\alpha$,
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\beta,\gamma}} \sigma = \tau$,
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\gamma,\alpha}} \sigma = 1$,

where

- τ is a complex number with $\mathrm{Im}\tau > 0$ such that $C \cong \mathbb{C}/\langle 1, \tau \rangle$,
- a_α (resp. a_β) is a real number such that the monodromy of the flat line bundle $N_{C/S}$ along the loop α (resp. β) is $\exp(2\pi\sqrt{-1} \cdot a_\alpha)$ (resp. $\exp(2\pi\sqrt{-1} \cdot a_\beta)$).

Let e_ν (resp. e'_ν) be the exceptional divisor corresponding to the point p_ν (p'_ν) in S (resp. S'). Denote by h (resp. h') the preimage of a hyperplane in S (resp. S').

$C_{1,2} \subset M \subset X$ is defined by $C_{1,2} := (e_1 \setminus \Delta_1^{(\varepsilon)}) \cup T_{1,2}^{(\varepsilon)} \cup (e_2 \setminus \Delta_2^{(\varepsilon)})$.



Note that $C_{1,2} \sim e_1 - e_2$ holds when we regard $C_{1,2} \subset M$ as a cycle of S.

Similarly, we define

- $C_{2,3}, C_{3,4}, \cdots, C_{7,8}$, and $C_{6,7,8} \subset M$ ($C_{6,7,8} \sim -h + e_6 + e_7 + e_8$ as a cycle of S),
- $C'_{1,2}, C'_{2,3}, \cdots, C'_{7,8}$, and $C'_{6,7,8} \subset M'$.

As $C_\bullet \setminus W^*$ (resp. $C'_\bullet \setminus W^*$) is an analytic subset of $M \setminus W^*$ (resp. $M' \setminus W^*$), we have that

$$\int_{C_\bullet} \sigma = \int_{C_\bullet \cap W^*} \frac{dz_j \wedge dw_j}{w_j}, \quad \int_{C'_\bullet} \sigma = \int_{C'_\bullet \cap W^*} \frac{dz'_j \wedge dw'_j}{w'_j}.$$

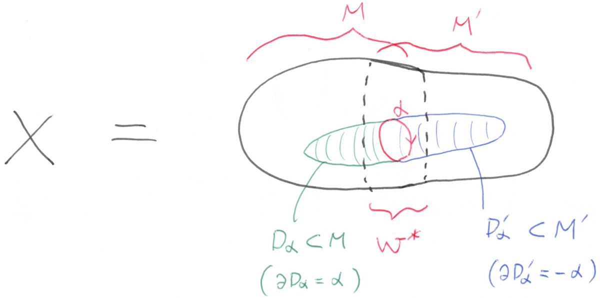
By using this description, we can calculate the integrals.

Denote by q_0 a inflection point of C, q_0' a inflection point of C', q_ν the intersection point $C \cap e_\nu$, and by q_ν' the intersection point $C' \cap e_\nu'$. Then we have

- $\frac{1}{2\pi\sqrt{-1}} \int_{C_{\nu,\nu+1}} \sigma = \int_{q_\nu}^{q_{\nu+1}} dz_j \ (1 \leq \nu \leq 7)$,
- $\frac{1}{2\pi\sqrt{-1}} \int_{C_{6,7,8}} \sigma = \int_{q_0}^{q_6} dz_j + \int_{q_0}^{q_7} dz_j + \int_{q_0}^{q_8} dz_j$,
- $\frac{1}{2\pi\sqrt{-1}} \int_{C'_{\nu,\nu+1}} \sigma = \int_{q'_\nu}^{q'_{\nu+1}} dz'_j \ (1 \leq \nu \leq 7)$,
- $\frac{1}{2\pi\sqrt{-1}} \int_{C'_{6,7,8}} \sigma = \int_{q'_0}^{q'_6} dz'_j + \int_{q'_0}^{q'_7} dz'_j + \int_{q'_0}^{q'_8} dz'_j$.

In what follows, we simply denote $\int_{q_\nu}^{q_{\nu+1}} dz_j$ by "$q_{\nu+1} - q_\nu$", for example.

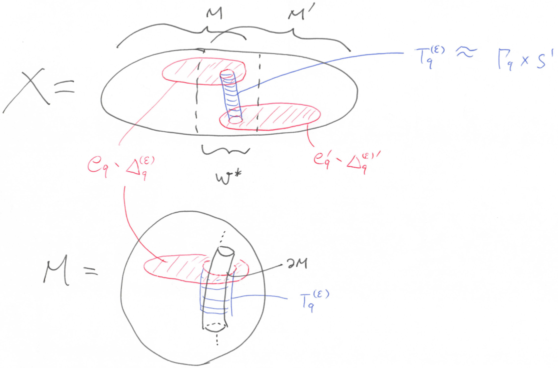
B_α is defined as follows by using the fact that $\pi_1(M)=\pi_1(M')=0$.



Similarly, we define B_β.

At this moment, we do not know how to calculate the integrals $\int_{B_\alpha}\sigma$ and $\int_{B_\beta}\sigma$.

B_γ is defined by $B_\gamma := (e_9 \setminus \Delta_9^{(\varepsilon)}) \cup T_9^{(\varepsilon)} \cup (e_9' \setminus \Delta_9^{(\varepsilon)}{}')$.



By the same argument as in the $C_\bullet$ case, we have that

$$\frac{1}{2\pi\sqrt{-1}}\int_{B_\gamma}\sigma = \int_{g(q_9)}^{q_9'} dz_j' \;(= \text{“}p_9' - g(p_9)\text{”}).$$

lattice	cycle	$\frac{1}{2\pi\sqrt{-1}}\int\sigma$	corresponding parameter
U	$A_{\beta,\gamma}$	τ	choice of C_0 (and C_0')
	B_α	???	choice of w_j's (R, R',...)
U	$A_{\gamma,\alpha}$	1	—
	B_β	???	choice of w_j's (R, R',...)
$E_8(-1)$	$C_{1,2}$	"$p_2 - p_1$" in C	choice of $p_2 - p_1$
	$C_{2,3}$	"$p_3 - p_2$" in C	choice of $p_3 - p_2$
	$\vdots$	$\vdots$	$\vdots$
	$C_{7,8}$	"$p_8 - p_7$" in C	choice of $p_8 - p_7$
	$C_{6,7,8}$	"$p_6 + p_7 + p_8$" in C	choice of $p_6 + p_7 + p_8$
$E_8(-1)$	$C_{1,2}'$	"$p_2' - p_1'$" in C'	choice of $p_2' - p_1'$
	$C_{2,3}'$	"$p_3' - p_2'$" in C'	choice of $p_3' - p_2'$
	$\vdots$	$\vdots$	$\vdots$
	$C_{7,8}'$	"$p_8' - p_7'$" in C'	choice of $p_8' - p_7'$
	$C_{6,7,8}'$	"$p_6' + p_7' + p_8'$" in C	choice of $p_6' + p_7' + p_8'$
U	$A_{\alpha,\beta}$	$a_\beta - \tau \cdot a_\alpha$	choice of p_9 and p_9' (i.e. $N_{C/S}$ and $N_{C'/S'}$)
	B_γ	"$p_9' - g(p_9)$"	choice of $g\colon C \cong C'$

- Fix a Diophantine pair $(p, q) \in \mathbb{R}^2$.
  (i.e. $-\log \mathrm{dist}((np, nq), \mathbb{Z}^2) = O(\log n)$ as $n \to \infty$)
- We first consider the subspace $\Xi_{(p,q)}$ of the period domain $\mathcal{D}_{\mathrm{period}}$ which is defined by considering all the K3 surfaces constructed by our method with

$$N_{C/S} \mapsto [p + q\tau] \in \mathbb{C}/\langle 1, \tau \rangle$$

  by the isomorphism $\mathrm{Pic}^0(C) \cong \mathbb{C}/\langle 1, \tau \rangle$ (τ moves).
- $\dim \Xi_{(p,q)} = 19$, with coordinates system

$$(\tau, p_1, p_2, \ldots, p_8, p_1', p_2', \ldots, p_8', s, x),$$

  where (s, x) are the parameter for the gluing.
- s is defined by changing $g\colon C \to C'$ with the composition $g \circ P_s$, where P_s is the automorphism of C defined by "$z \mapsto z + s$".

Lemma

$\Xi_{(p,q)} \subset v_{(p,q)}^{\perp}$, *where* $v_{(p,q)} := A_{\alpha\beta} + p \cdot A_{\beta\gamma} - q \cdot A_{\gamma\alpha}$

lattice	cycle	$\frac{1}{2\pi\sqrt{-1}} \int \sigma$	corresponding parameter
U	$A_{\beta,\gamma}$	τ	choice of C_0 (and C_0')
	B_α	$??? =: x - 2\tau$	choice of w_j's (R, R',...)
U	$A_{\gamma,\alpha}$	1	—
	B_β	$??? =: y - 2$	choice of w_j's (R, R',...)
$E_8(-1)$	$C_{1,2}$	"$p_2 - p_1$" in C	choice of $p_2 - p_1$
	$C_{2,3}$	"$p_3 - p_2$" in C	choice of $p_3 - p_2$
	$\vdots$	$\vdots$	$\vdots$
	$C_{7,8}$	"$p_8 - p_7$" in C	choice of $p_8 - p_7$
	$C_{6,7,8}$	"$p_6 + p_7 + p_8$" in C	choice of $p_6 + p_7 + p_8$
$E_8(-1)$	$C'_{1,2}$	"$p'_2 - p'_1$" in C'	choice of $p'_2 - p'_1$
	$C'_{2,3}$	"$p'_3 - p'_2$" in C'	choice of $p'_3 - p'_2$
	$\vdots$	$\vdots$	$\vdots$
	$C'_{7,8}$	"$p'_8 - p'_7$" in C'	choice of $p'_8 - p'_7$
	$C'_{6,7,8}$	"$p'_6 + p'_7 + p'_8$" in C	choice of $p'_6 + p'_7 + p'_8$
U	$A_{\alpha,\beta}$	$a_\beta - \tau \cdot a_\alpha$	choice of p_9 and p'_9 (i.e. $N_{C/S}$ and $N_{C'/S'}$)
	B_γ	"$p'_9 - g(p_9)$" $=: s$	choice of $g\colon C \cong C'$

Set

- $x := 2\tau + \frac{1}{2\pi\sqrt{-1}} \int_{B_\alpha} \sigma$ (a coordinate)
- $y := 2 + \frac{1}{2\pi\sqrt{-1}} \int_{B_\beta} \sigma$ ("dummy")

The relation between x and y can be obtained by the relation

$$0 = (\sigma, \sigma) = 2\tau x + 2y + (\exists \text{quadric function of } \tau, q - p\tau, p_1 - p_2, \ldots)$$

The (normalized) volume is:

$$\mathrm{vol}(\sigma) := (\sigma, \overline{\sigma}) = 2\mathrm{Re}(\overline{\tau} x) + 2\mathrm{Re}(y) + (\exists \text{quadric function of } \tau, q - p\tau, p_1 - p_2, \ldots)$$

- $\mathrm{vol}(\sigma)$ depends linearly on the coordinate x if one fix the other 18 coordinates.

Via the Poincare duality,

$$
\begin{aligned}
\frac{1}{2\pi\sqrt{-1}}\sigma = \quad & (2(q-p\tau)+s)A_{\alpha\beta} + xA_{\beta\gamma} + yA_{\gamma\alpha} \\
+ \quad & \tau B_\alpha + B_\beta + (q-p\tau)B_\gamma \\
+ \quad & \sum_{j=1}^{7} \exists c_{j,j+1}C_{j,j+1} + \exists c_{6,7,8}C_{6,7,8} \\
+ \quad & \sum_{j=1}^{7} \exists c'_{j,j+1}C'_{j,j+1} + \exists c'_{6,7,8}C'_{6,7,8}
\end{aligned}
$$

$c_\bullet$'s depends only on $(p_1, p_2, \ldots, p_8)$, and $c'_\bullet$'s depends only on $(p'_1, p'_2, \ldots, p'_8)$.

Theorem (K-, T. Uehara)

$\exists \widehat{V}_{(p,q)} \geq 0$ *depending only on* (p,q) *such that,*

$$\forall \quad \widehat{\sigma} = \begin{aligned}[t] & a_{\alpha\beta}A_{\alpha\beta} + a_{\beta\gamma}A_{\beta\gamma} + a_{\gamma\alpha}A_{\gamma\alpha} + b_\alpha B_\alpha + b_\beta B_\beta + b_\gamma B_\gamma \\ + \quad & \sum_{j=1}^{7} c_{j,j+1}C_{j,j+1} + c_{6,7,8}C_{6,7,8} + \sum_{j=1}^{7} c'_{j,j+1}C'_{j,j+1} + c'_{6,7,8}C'_{6,7,8} \in \mathcal{D}_{\mathrm{Period}}, \end{aligned}$$

it holds that $\widehat{\sigma} \in \Xi_{(p,q)}$ *iff the following holds:*

- $b_\beta \neq 0$ *(set* $b_\beta = 1$ *by "normalizing"),*
- $\operatorname{Im} b_\alpha \neq 0$,
- *The normalized volume* $\mathrm{vol}(\widehat{\sigma})$ *is larger than* $\widehat{V}_{(p,q)}$.

Remark

$\widehat{V}_{(p,q)} = V_{(p,q)} + V'_{(p,q)}$, *where*

- $V_{(p,q)} = \mathrm{vol}_\eta(S \setminus (\,$*"the maximal hol. tub. n.b.h.d." of* $C))$,
- $V'_{(p,q)} = \mathrm{vol}_{\eta'}(S' \setminus (\,$*"the maximal hol. tub. n.b.h.d." of* $C'))$

Remark

It is observed by a standard argument that a general member of $\Xi_{(p,q)}$ *corresponds to a K3 surface* X *with Picard number* $= 0$*, which means that* X *is non-Kummer and non-projective.*

Question

- *How large can one take a hol. tub. n.b.h.d. of C in S?*
- *Is $V_{(p,q)}$ equal to zero? or > 0?*
- *How does $V_{(p,q)}$ depend on the Diophantine pair (p, q)?*

To investigate a Kähler-geometric approach to this kinds of problems, now I'm also interested in the following:

Question

How is the relation between a Ricci-Flat Kähler metric on X and some "canonical metric" on S or $S \setminus C$ (complete Ricci-Flat Kähler?)?