On a nbhd of a torus leaf of a certain class of hol. fol's on a cpx surfaces.

Date 2015 . 10 . 28. (14:30 - 15:30)

X: sm. surf.
$\cup$
C: cpt curve s.t. $\deg N_{C/X} = 0$.

interest: (non-) semipositivity criterion of $\mathcal{O}_X(C)$

Def $\mathcal{O}_X(C)$: semi-positive $\underset{\text{def}}{\iff}$ $\exists C^\infty$ Herm. metric h on $[C]$ s.t. $\sqrt{-1}\Theta_h \geq 0$.

Goal of this talk; to give a cpx dynamical criterion when X admits a "nice" foliation, and C is a leaf.

Main application

Thm A $\exists\, \mathcal{X} \xrightarrow{\varpi} \Omega$: hol. submersion from 3-dim'l cpx mfd $\mathcal{X}$ to a nbhd of $U(1) := \{\tau \in \mathbb{C} \mid |\tau| = 1\}$ in $\mathbb{C}$.
$\cup$
$\exists\, \mathcal{C}$: divisor of $\mathcal{X}$ s.t. , $C_\tau := X_\tau \cap \mathcal{C}$; sm. ellipt. curve ($X_\tau := \varpi^{-1}(\tau)$)

s.t. (1) $\mathcal{O}_{X_\tau}(C_\tau)$: semi-positive for $\begin{cases} \forall \tau \in \Omega \setminus U(1) \\ \text{almost every } \tau \in U(1) \end{cases}$

(2) $\#\{\tau \in U(1) \mid \mathcal{O}_{X_\tau}(C_\tau) : \text{not semi-positive}\} \geq \# 2^{\aleph}$. (Cardinality)

Schedule
§1 Main result (= Thm B), proof. "Thm B $\Rightarrow$ Thm A".
§2 Thm B v.s. Ueda theory
§3. prf of Thm B

---

§1 Fix
- C : sm. ellipt. curve.
- γ_1, γ_2 : generators of $\pi_1(C, *)$
- $f_0 \in \mathcal{O}_{\mathbb{C},0}$ s.t. $f(0) = 0$, $f'(0) \neq 0$
$\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxx} \overset{\shortparallel}{t}$.

$\rightsquigarrow \exists\, X = X_{(C,\gamma_1,\gamma_2,f_0)}$, $\exists\, i = i_{(C,\gamma_1,\gamma_2,f_0)} : C \hookrightarrow X$ embedding

s.t. ...

(1) $\exists \pi : X \twoheadrightarrow C$: hol. submersion s.t. $\pi|_{i(C)} = id_{i(C)}$

(2) $\exists \mathcal{F}$: non-sing hol. foliation on X.

s.t. , C is a leaf of $\mathcal{F}$

, $Hol_C : \pi_1(C, *) \longrightarrow \mathcal{O}_{\mathbb{C},0}$ is the map defined by

$\gamma_1 \longmapsto$ ~~$(z \mapsto z)$~~ $(x \mapsto x)$

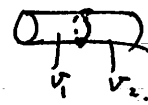
$\gamma_2 \longmapsto$ ~~$(z \mapsto f(z))$~~ $(x \mapsto f(x))$

holonomy of $\mathcal{F}$ along C.

Construction of (X, i)

① $C =$ 

$=: U_1$, $=: U_2$

$\rightsquigarrow U_1 \cap U_2 = V_1 \cup V_2$. 

② fix $0 < r \ll R \ll 1$.

③ $X_{(C,\gamma_1,\gamma_2,t_0)} := (\underbrace{U_1 \times \Delta_R}_{=:X_1}) \cup (\underbrace{U_2 \times \Delta_r}_{=:X_2}) / \sim$

gluing function : $(X_2 \supset)\ V_1 \times \Delta_r \longrightarrow X_1$, $(z, x) \longmapsto (z, x)$

$(X_2 \supset)\ V_2 \times \Delta_r \longrightarrow X_1$, $(z, x) \longmapsto (z, f(x))$

④ $Pr_1 : X_j \longrightarrow U_j$ glues up to define $\pi : X \to C$.

⑤ submfds $U_j \times \{0\} \subset X_j$ ⟶ $i(C) \subset X$.

⑥ foliations "$\{U_j \times \{const\}\}$" of X_j ⟶ $\mathcal{F}$ //

Rmk ① $(X_{(C,\gamma_1,\gamma_2,t)}, i_{(C,\gamma_1,\gamma_2,t)})$! unique up to "shrinking"

② $i(C)$: leaf, $\mathcal{F}$: non-sing $\xrightarrow{\text{C.S. index}} \deg N_{i(C)/X} = 0$ //

Thm B $(X, C) := (X_{(C,\gamma_1,\gamma_2,f)}, i(C)_{(C,\gamma_1,\gamma_2,f)})$

$\tau := f'(0)$

(1) 0: repelling, attracting, or Siegel fixed point of f
($|\tau|>1$, $|\tau|<1$, $|\tau|=1$, f: linearizable.)

$\Longrightarrow$ $\begin{cases} C \text{ admits a psd-flat nbhd system} \\ \mathcal{O}_X(C) : \text{semi-positive.} \end{cases}$

(2) 0: rationally indifferent fixed p.t. of f and $f^n \neq id$ for $\forall n$.
$\left(\begin{array}{l}|\tau|=1, \\ \arg(\tau) \in \mathbb{Q}\pi.\end{array}\right)$

$\Longrightarrow$ $\begin{cases} C: \text{admits a str. psd concave nbhd system} \\ \mathcal{O}_X(C) : \text{not semi-positive.} \end{cases}$

(3) Assume "cond $(*)$": $^\forall D$: 0-nbhd in $\mathbb{C}$, $^\exists$periodic cycle. $\{f(\eta), f^2(\eta), \cdots, f^m(\eta) = \eta\} \subset D \setminus \{0\}$.

Then $\mathcal{O}_X(C)$: not semi-positive.

Rmk
- $A := \{\tau \in U(1) \mid \log|1-\tau^\ell| = O(\log \ell) \text{ as } \ell \to \infty\}$
- $B_d := \{\tau \in U(1) \mid ~~for~~ {}^\exists A>1 \text{ s.t. } \liminf_{\ell\to\infty} A^\ell |1-\tau^\ell|^{\frac{1}{d+1}} = 0\}$ $(d \geq 1)$

$\rightsquigarrow$ A: full-measure in $U(1)$, $\# B_d \geq \# \mathbb{Z}$?

$f'(0) = \tau \in A \Rightarrow$ 0 is a Siegel fixed p.t. (Siegel.)

$f'(0) \in B \Rightarrow$ condition $(*)$ holds (Ueda, Cremer)

prf of (Thm B $\Rightarrow$ Thm A) ; Consider $f_\tau(x) := \tau\cdot x + x^2$ //.

§2 Thm B v.s. Ueda theory

$C \hookrightarrow X$ (cpt curve, sm. ; sm. surf. ; $\deg N_{C/X} = 0$.) $\rightsquigarrow$ $^\exists \tilde{N}$: flat l.b. def'd on a suff. small nbhd V of C in X. s.t. $\tilde{N}|_C = N_{C/X}$.

Ueda's classification of (C, X)

type (α) $\overset{\text{def}}{\iff}$ $\mathcal{O}_V(C) \not\cong_{\text{formally}} \mathcal{O}_V(\tilde{N})$ along C,

(β) $\overset{\text{def}}{\iff}$ $\mathcal{O}_V(C) \cong \mathcal{O}_V(\tilde{N})$ for $^\exists V$.

(γ) $\overset{\text{def}}{\iff}$ $\mathcal{O}_V(C) \cong_{\text{formally}} \mathcal{O}_V(\tilde{N})$ along C, however $^\forall V$, $\mathcal{O}_V(C) \not\cong \mathcal{O}_V(\tilde{N})$

No.

Date

KAM. theory

Kolmogorov Arnold Moser

◎ $(X, C) = (X_{(C,\gamma_1,\gamma_2,f)},\ i_{(C,\gamma_1,\gamma_2,f)}(C))$,
$\tau = f'(0) \in U(1)$.

⇝ (1) 0 is a Siegel fixed p.t. $\Rightarrow$ (C, X): of type (β)

(2) 0 is an indifferent fixed p.t. $\Rightarrow$ (C, X): — " — (α)

(3) cond. (*) holds $\Rightarrow$ (C, X): — " — (γ) (Ueda) //.

§3. prf of ThmB

(1) ... skip, (2) $\Leftarrow$ (" type(α) $\Rightarrow$ $\mathcal{O}_X(C)$: not. s.p. " [K-'14])

(3) ... Assuming $\exists h$: C^∞ herm. metric on $[C]$ with s.p-curv. and considering $\psi := -\log|f_C|_h$ $(f_C \in H^0(X, \mathcal{O}_X(C)))$,
It is suff. to show :-

Claim $\forall W \Subset X$, tub nbhd of C in X,
$\forall \psi : W \to (-\infty, +\infty]$: conti. func.
$\psi|_{W \setminus C}$ psh $\Rightarrow$ ψ: bdd from above around C //.
(§3.(1).)

◎ $M := \sup_{\partial W} \psi$.

◎ Cond. (*) ⇝ $\exists \{C_n\}$: a seq. of cpt leaves s.t. $C_n \to C$.

◎ fix $p \in C$, and regard $\pi^{-1}(p)$ as a domain in $\mathbb{C}$.
⇝ ~~we may assume that~~ $C_n \cap \pi^{-1}(p)$: ~~repelling~~ cycle of f. for ~~∞ infin. many n~~

⇝ for infin. many n,
◎ $\exists P_n$: a leaf of $\mathcal{F}$ s.t. $\left\{ \begin{array}{l} P_n \cap W \cong \Delta^* \\ \partial W \cap P_n \cong \partial\Delta \\ \overline{P_n} \supset C_n \end{array} \right.$ (j_n)

◎ $\psi|_{C_n}$: bdd from above ⇝ psh func $\psi \circ j_n$: on Δ^* extends to define a psh func on Δ.

◎ max'l principle ⇝ $\psi \circ j_n \leq \sup_{\partial\Delta} \psi \circ j_n \leq M$. ⇝ $\psi|_{C_n} \leq M$.

◎ $\left.\begin{array}{l}\psi: \text{conti.} \\ C_n \to C\end{array}\right)$ ⇝ $\psi|_C \leq M$ //.

On a nbhd of a torus leaf of a certain class of hol. fol's on a cpx surfaces.

14:30 − 15:30

X : sm. surf.
$\cup$
C : cpt curve. s.t. $\deg N_{C/X} = 0$.

Main interest ; (non) semi-positivity criterion of $\mathcal{O}_X(C)$

Def L : hol. line bdl $/X$

$\mathcal{O}_X(L)$: semi-positive $\overset{\text{def}}{\iff}$ $\exists$ C^∞ Herm. metric h on L s.t. $\sqrt{-1}\,\Theta_h \geq 0$.

Goal of this talk ; give a cpx dynamical criterion when X admits a "nice" foliation and C is an invariant sm ellipt. curve.

Main application ··· **Thm A** $\exists$ $\mathcal{X} \xrightarrow{\pi} \Omega$: hol submersion from 3-dim'l cpx mfd $\mathcal{X}$ to a nbhd Ω of $U(1) := \{\tau \in \mathbb{C} \mid |\tau| = 1\}$ in $\mathbb{C}$

$\cup$

$\exists$ $\mathcal{C}$: divisor of $\mathcal{X}$

s.t, denoting $X_\tau := \pi^{-1}(\tau)$ and $C_\tau := X_\tau \cap \mathcal{C}$, (: cpt curve)

(1) $\mathcal{O}_{X_\tau}(C_\tau)$: semi-positive for $\forall \tau \in \Omega \setminus U(1)$ and for almost every $\tau \in U(1)$ in the sense of Lebesgue measure

(2) $\#\{\tau \in U(1) \mid \mathcal{O}_{X_\tau}(C_\tau)$: not semi-positive$\} \geq \# 2^{\mathbb{Z}}$ //

Schedule
§1. Main result. (= Thm B) and prf of Thm A.
§2. Main result v.s. Ueda theory.
§3. Prf of Thm B

§1.

Fix • C: sm. ellipt. curve, $\gamma_1, \gamma_2 \in \pi_1(C, *)$: generators

• $f \in \mathcal{O}_{\mathbb{C},0}$ s.t. $f(0) = 0$ and $\tau := f'(0) \neq 0$.

Thm $\exists X = X_{(C,f)}$: a sm cpx surf.

$\exists i = i_{(C,f)}: C \hookrightarrow X$: embedding

s.t. (1) $\exists \pi: X \to C$: hol. submersion s.t. $\pi|_{i(C)} = id$

(2) $\exists \mathcal{F}$: sm. hol. foliation on X.

s.t. • $i(C)$ is a leaf of $\mathcal{F}$.

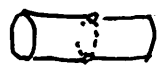
• $Hol_C[\gamma_1](\xi) = \xi$ and

$Hol_C[\gamma_2](\xi) = f(\xi)$ hold,

where $Hol_C: \pi_1(C, *) \to \mathcal{O}_{\mathbb{C},0}$: holonomy along C ($= i(C) \subset X$).

Construction:

◎ $C =$  U_1, U_2,

$U_1 \cap U_2 =$  $V_1 \sqcup V_2$.

◎ $f'(0) \neq 0 \rightsquigarrow$ for suff. small nbhd D_0 of 0 in $\mathbb{C}$,

$f: D_0 \longrightarrow D_1 := f(D_0) \subset \mathbb{C}$: isom.

$D_2 := f|_{D_0}^{-1}(D_0 \cap D_1) \subset D_0$.

$\rightsquigarrow X = X_{(C,f)} := (U_1 \times D_0) \underset{\sim}{\cup} (U_2 \times D_2)$

glueing function: $V_1 \times D_2 \longrightarrow U_1 \times D_0$, $(z, \xi) \mapsto (z, \xi)$

$V_2 \times D_2 \longrightarrow U_1 \times D_0$, $(z, \xi) \mapsto (z, f(\xi))$

◎ Pr_1: $\left.\begin{array}{l} U_1 \times D_0 \to U_1 \\ U_2 \times D_2 \to U_2 \end{array}\right)$: glue up to define π.

◎ foliations $\left.\begin{array}{l} \text{“}\{U_1 \times \{\xi\}\}_{\xi \in D_0}\text{”} \\ \text{“}\{U_2 \times \{\xi\}\}_{\xi \in D_2}\text{”} \end{array}\right) \longrightarrow \mathcal{F}$.

◎ submfds $\left.\begin{array}{l} U_1 \times \{0\} \\ U_2 \times \{0\} \end{array}\right) \longrightarrow i(C)$ //

Rmk

$\{$ $X_{(C,f)}, i_{(C,f)}$: Unique up to shrinking

$C = i(C)$: a leaf $\xRightarrow{\text{C.S. index thm}} \deg N_{C/X} = 0$.

Thm B (Main result)

$(X, C) := (X_{(\tau,f)}, i_{(\tau,f)}(C))$.

$\tau := f'(0)$.

(1) 0: repelling ($|\tau|>1$), attracting ($|\tau|<1$), or Siegel fixed p.t. of f ($|\tau|=1$, f: linearizable at 0)

$\Rightarrow$ C admits a psd flat nbhd system.

$\mathcal{O}_X(C)$: semi-positive.

(2) 0: rationally indifferent fixed p.t. and $f^n \not\equiv id$ for $^\forall n \geq 1$.
(i.e. $|\tau|=1$, $\frac{\arg(\tau)}{\pi} \in \mathbb{Q}$)

$\Rightarrow$ C admits a str. psd concave nbhd system,
$\mathcal{O}_X(C)$: not semi-positive.

(3) $^\forall D$: a nbhd of 0 in $\mathbb{C}$, $^\exists$periodic cycle $\{\eta, f(\eta), f^2(\eta), \dots, \underset{\eta}{\underset{\shortparallel}{f^m(\eta)}}\} \subset D\setminus\{0\}$

Cond.(∗)

$\Rightarrow$ $\mathcal{O}_X(C)$: not semi-positive. //

Rmk

◎ $\begin{cases} |\tau|=1, \\ \log|1-\tau^\ell| = O(\log \ell) \text{ as } \ell\to\infty \end{cases}$ $\Rightarrow$ 0: Siegel fixed p.t.

◎ $\{\tau\in U(1) \mid \log|1-\tau^\ell| = O(\log\ell)\} \subset U(1)$: full-measure.

◎ f: poly. of deg $= d$ and $^\exists A>1$, $\underbrace{\liminf_{\ell\to\infty} A^\ell |1-\tau^\ell|^{\frac{1}{d^\ell-1}} = 0}_{(\ast\ast)}$

$\Longrightarrow$ (∗) (Cremer–Ueda)

◎ $^\forall d\geq 1$, $\#\{\tau\in U(1) \mid (\ast\ast)\} \geq 2^{\aleph}$.

prf of Thm B $\Rightarrow$ Thm A ; --- Consider $f_\tau(\xi) := \tau\xi + \xi^2$ //

§2 Thm B v.s. Ueda theory

X: sm. surf, $\supset C$: sm. cpt curve, s.t. $\deg N_{C/X} = 0$

$\rightsquigarrow$ $^{\exists!}\tilde{N}$: flat l.b. / a nbhd of C s.t. $\tilde{N}|_C = N_{C/X}$.

Ueda's classification of (C, X) (V)

No.

Date

◎ ~~type(α)~~ $\overset{\text{def}}{\iff}$ $\mathcal{O}_V(C) \not\cong_{\text{formally}} \mathcal{O}_V(\tilde{N})$ along C.

type(β) $\overset{\text{def}}{\iff}$ $\mathcal{O}_V(C) \cong \mathcal{O}_V(\tilde{N})$

type(γ) $\overset{\text{def}}{\iff}$ $\mathcal{O}_V(C) \cong_{\text{formally}} \mathcal{O}_V(\tilde{N})$ along C. however $\mathcal{O}_V(C) \not\cong \mathcal{O}_V(\tilde{N})$

◎ Assume $\tau = f'(0) \in U(1)$, $X = X_{(\tau,f)}$ $C = i_{(\tau,f)}(C)$

$\rightsquigarrow$ (1) 0 : Siegel fixed pt. $\iff$ (X, C) : of type (β)

(2) 0 : irrationally indifferent, $f^n \neq id$ for $\forall n$ $\iff$ (X, C) : of type (α)

~~0~~ : (3) condition ($*$) $\implies$ (X, C) : of type (γ).

([Ueda]'s prf for the existence of type(γ))

§3. prf of Thm B

(1) … Skip.

(2) $\Leftarrow$ "(C, X) : of type (α) $\implies \mathcal{O}_X(C)$: not semi-positive" [K–, '14].

(3) Assuming $\mathcal{O}_X(C)$ admits C^∞ metric h with sp. curvature and by considering $\psi := -\log|f_C|_h$

f_C : $H^0(X, [C])$: Can. Section,

it is suff. to show…

Claim $\forall W \subset X$: C-nbhd, $\forall \psi : W \to (-\infty, \infty]$; conti. func, $\psi|_{W \setminus C}$: psh $\overset{(SS_{C-1,1})}{\implies}$ ψ : bdd from above around C //.

① $M := \sup_{\partial W} \psi$.

② condition ($*$) $\rightsquigarrow$ $\exists \{C_n\}$: Seq. of cpt leaves of $\mathcal{F}$ s.t. $C_n \to C$.

③ $p \in C$: fix. regard $\pi^{-1}(p) \subset \mathbb{C}$,

$\rightsquigarrow$ $\forall n$. $C_n \cap \pi^{-1}(p)$: subset of the Julia set of f

$\rightsquigarrow$ $\forall n$. $\exists \Gamma_n$: a leaf of $\mathcal{F}$ s.t. $W \cap \Gamma_n \overset{\cong}{\underset{i_n}{\longleftarrow}} \Delta^*$ punctured disc, $i_n(\partial \Delta) = \Gamma_n \cap \partial W$, $\overline{\Gamma_n} \supset C_n$.

④ $\psi|_{C_n}$: bdd from above

$\rightsquigarrow$ $\psi \circ i_n : \Delta^* \to \mathbb{R}$ extends to a psh func. $\Delta \to \mathbb{R}$

$\underset{\text{maxi'l principle}}{\rightsquigarrow}$ $\psi \circ i_n \leq M$. $\rightsquigarrow$ $\psi|_{C_n} \leq M$ $\underset{\text{continuity of } \psi}{\rightsquigarrow}$ $\psi|_C \leq M$ //