No. 1

Date 2015 · 6 · 20

On the semi-positivity of a nef l.b.s and the nbhd of the stable base loci.

(In, prf of Thm 1,3 のぞいて [illegible])

X : ~~[illegible]~~ (proj) cpx mfd.

$\cup$

S : hyp. surf. s.t. $N_{S/X} := \mathcal{O}_X(S)|_S$: top. triv. $c_1(N_{S/X}) = 0$.

$\rightsquigarrow$ $L := \mathcal{O}_X(S)$: nef.

Q When ~~does~~ is L semi-positive ?

(i.e. When does L admit a C^∞ Herm. metric with s.p. curvature)

Rmk L : semi-ample $\Rightarrow$ L : ~~semi~~ s.p. $\Rightarrow$ L : nef.
$\not\Leftarrow$ $\not\Leftarrow$

§1. Main results.
§2. applications.
(§3. Outline of prf).

§1. Notations.

$P(S) := \{\text{top. triv. hol. line bdls}/S\}/\sim_{hol}$

$\cup$ cpx var.

$P_0(S) := \{\text{flat l.bs}/S\}/\sim_{hol}$

$\cup$ $\underset{\text{fact}}{=}$ $\{\text{flat l.bs}\}/\sim_{flat} := H^1(S, U(1))$

$(U(1) := \{z \in \mathbb{C} \mid |z| = 1\})$

$\mathcal{E}_0(S) := \{L \in P_0(S) \mid {}^{\exists} n \geq 1 \ L^n = \mathcal{O}_S\}$

$\mathcal{E}_1(S) := \{L \in P_0(S) \mid \log d(\mathcal{O}_S, L^n) = O(\log n) \ n \to \infty\}$

d : Eucli. dist. on $P_0(S)$

Rmk
- S : non-sing $\Rightarrow$ $P(S) = P_0(S)$
- S : with only nodes $\Rightarrow$ $P(S) \cong \mathrm{Im}(H^1(S, \mathbb{C}^*) \to H^1(S, \mathcal{O}_S^*))$
- $\mathcal{E}_1(S) = \bigcup_{\nu=0}^{\infty} F_\nu$ $\leftarrow$ nowhere dense. cl. subset of $P_0(S)$. $\mu(P_0(S) \setminus \mathcal{E}_1(S)) = 0$ (μ : Lebesgue measure.)
- C : a rat'l curve with a node. $\Rightarrow$ $P(C) \cong \mathbb{C}^*$, $P_0(C) \cong U(1)$

Thm 1 X : cpx mfd of $\dim = 2$.

$\cup$

C : cpt curve with only nodes s.t. $N_{C/X}$: top. triv.

Assume $\begin{cases} i^* N_{C/X} \in \mathcal{E}_0(\tilde{C}) & (\tilde{C} \xrightarrow{i} C : \text{normalization}) \\ h^1(C, N_{C/X}^{-n}) = h^1(C, \mathbb{C}(N_{C/X}^{-n})) = 0 & \text{for } \forall n \geq 1. \end{cases}$

Then

$N_{C/X} \in \mathcal{E}_0(C) \cup \mathcal{E}_1(C) \Longrightarrow \mathcal{O}_X(C)$: semi-positive //.

Thm 2 X : cpx mfd of $\dim = 2$

$\cup$

C : cpt curve with only nodes s.t. $N_{C/X}$: top. triv.

Assume $\begin{cases} \cdot \text{ the dual graph of } C \text{ is a cycle graph.} \\ \cdot\ h^1(C, N_{C/X}^{-n}) = 0 \text{ for } ~~\forall n\geq 1~~\ n = 1, 2, 3, 4. \end{cases}$

$\gamma \rightsquigarrow 0$

$1.$

Then $N_{C/X} \in \mathcal{P}(C) \setminus \mathcal{P}_0(C) \Rightarrow \mathcal{O}_X(C)$: not s.p. //.

Thm 3. X : cpx mfd.

$\cup$

S : sm hyp. surf of X.

s.t. $\begin{cases} \cdot\ \underbrace{\bigcap_{m \geq 1} |L^{\otimes m}|}_{= C} \text{ is a sm cpt Kä hyp. surf of } S, \text{ where } L = \mathcal{O}_X(S) \\ \cdot\ N_{S/X} : \text{flat around } C \end{cases}$

Assume (i) $N_{C/S} \in \mathcal{E}_0(C)$, $N_{S/X}|_C \in \mathcal{E}_0(C)$

or (ii) $N_{C/S} \cong N_{S/X}|_C \in \mathcal{E}_1(C)$

or (iii) $N_{S/X}|_C \in \mathcal{E}_0(C)$, $C \subset S$: exceptional.

Then

$h^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m}) = 0$ for $\forall n \geq 1$, $\forall m \geq 0$.

$\Longrightarrow L$: s.p. //

Rmk (X, C) : as in Thm 2, $N_{C/X} \notin \mathcal{P}_0(C)$

$\rightsquigarrow |f_C|^{-2}$ has minimal sing. among singular Hermitian metrics of $\mathcal{O}_X(C)$ with s.p. curvature,

where $f_C \in H^0(X, \mathcal{O}_X(C))$: canonical //

Cor ($\Leftarrow$ Thm 1, 2)

$\underset{\text{sm. surf}}{X} \supset \underset{\text{a cycle of } ~~\text{a nodal curve}~~ \text{ rat'l curves}}{C} \Longrightarrow \begin{cases} \circ\ N_{C/X} \in \mathcal{E}_1(C) \Rightarrow \mathcal{O}_X(C) : \text{s.p.} \\ \circ\ N_{C/X} \in \mathcal{P}(C) \setminus \mathcal{P}_0(C) \Rightarrow \mathcal{O}_X(C) : \text{not s.p.} \text{ //} \end{cases}$

Rmk ◎ Thm 1 is a singular version of the following:

Thm 4 (⇐ [Ueda'83] + [K-'13]) (cf. [Brunella'10])

X : ~~sm.~~ cpx mfd. $\supset S$: sm hyp. surf. of X with $c_1(N_{S/X})=0$

Assume $H^1(S, N_{S/X}^{-n})=0$ for $\forall n \geq 1$

Then $N_{S/X} \in \mathcal{E}_0(S) \cup \mathcal{E}_1(S) \Rightarrow \mathcal{O}_X(S)$: s.p //

◎ Thm 3 is a codim-2-analogue of Thm 4.

◎ Thm 2 is a generalization of the following:

Thm 5 (⇐ [Ueda'91] + [K-'14])

X : sm surf. $\supset C$: a rat'l curve with a node

$N_{C/X} \in \mathcal{P}(C) \setminus \mathcal{P}_0(C) \Rightarrow \mathcal{O}_X(C)$: not s.p //

§2. application.

Known (⇐ Thm 4, [Brunella '10])

$C_0 \subset \mathbb{P}^2$: sm ellipt. curve, $\{P_i\}_{i=1}^9 \subset C_0$.

$X \xrightarrow{\pi} \mathbb{P}^2$: blow-up at $\{P_i\}_{i=1}^9$

$\cup \qquad \qquad \cup$

$C \Longrightarrow := (\pi^{-1})_* C_0$.

$\rightsquigarrow N_{C/X} \in (\underbrace{\mathcal{E}_0(C)}_{\text{simple}}) \cup \mathcal{E}_1(C) \Longrightarrow K_X^{-1}$: s.p.

Q (Demailly ...)

Is there a configuration $\{P_i\}^9 \subset C_0$ s.t. K_X^{-1} : not s.p.

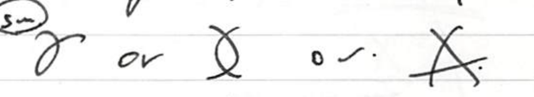
application ① ... a singular analogue of the above.

② ... answers to (an analogue of) the above question when C_0 : sing.

③ ... 3-dim'l analog of the above.

Application ①, ②

$C_0 \subset \mathbb{P}^2$: ~~sing~~ curve of deg $= 3$, with only nodes.

($C_0 =$ (sm) )

$\{p_i\}_{i=1}^{9} \subset C_0 \setminus (C_0)_{sing}$. $\qquad X \xrightarrow{\pi} \mathbb{P}^2$ b-up at $\{p_i\}_i$.

$\qquad\qquad \cup$

$\qquad\qquad C := (\pi^{-1})_* C_0$

$$N_{C/X} \cong \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \cdots - p_9)$$

Cor $\Longrightarrow$

① $N_{C/X} \in \mathcal{E}_0(C_\varphi) \cup \mathcal{E}_1(C_\varphi) \Rightarrow K_X^{-1}$: s.p.

② $N_{C/X} \in \mathcal{P}(C) \setminus \mathcal{P}(C) \Rightarrow K_X^{-1}$: not s.p.

Application ③

$\{p_j\}_{j=1}^{8} \subset \mathbb{P}^3$: general.

$\rightsquigarrow \exists\, Q_0, Q_1$: two quadric ~~two~~ surfaces of $\mathbb{P}^3$

s.t. · Q_0, Q_1 intersect each other along $C_0 := Q_0 \cap Q_1$ (transversally)

$\qquad\qquad \subset$ sm. ellipt. curve

· $\{p_j\}_{j=1}^{8} \subset C_0$.

$$\begin{array}{ccccc} X & \supset & S & \supset & C \\ \downarrow \pi & & \| & & \| \\ \mathbb{P}^3 & & (\pi^{-1})_* Q_0 & & (\pi^{-1})_* C_0 \end{array}$$

π : b-up at $\{p_j\}$.

$$N := N_{C/S} = N_{S/X}|_C \cong \mathcal{O}_{\mathbb{P}^3}(2)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - \cdots - p_8)$$

Thm 3 $\rightsquigarrow N \in \underbrace{\mathcal{E}_0(C_A)}_{\text{simple case}} \cup \mathcal{E}_1(C_{A}) \Longrightarrow K_X^{-1}$: s.p. $\left(\overset{\|}{\mathcal{O}_X(2S)}\right)$ //

c.f. Q (Totaro's question)

$\exists ?\ (\underset{\dim X \geq 3}{X}, \underset{\text{nef}}{L})$ s.t. $\#\{C' \underset{\text{cpt curve}}{\subset} X \mid L.C' = 0\} = \#\mathbb{Z}$.

[Lesieur Ottem [4]] $\{p_j\}_{j=1}^{8} \subset \mathbb{P}^3$: very general.

$\Longrightarrow (X, K_X^{-1})$ satisfies $\#\{C' \mid K_X^{-1}.C' = 0\} = \#\mathbb{Z}$ //

Application ③ $\rightsquigarrow$ $\exists (X, L)$ s.t. $\left| \begin{array}{l} L : \text{s.p.} \\ \#\{C' \mid L.C' = 0\} = \#\mathbb{Z}. \end{array} \right.$

dim = 3. $L = K_X^{-1}$, nef.

§3 Outline of prf.

prf of Thm. 2

for (X, C) as in Thm 2,

we constructed "Singular analogue of Ueda theory"

and showed that, ...

Claim (C, X) : as in Thm 2.

$\Psi : X \setminus C \to \mathbb{R}$: psh.

Assume $\Psi(p) = o((-\log d(p, C))^{2\lambda})$ as $p \to C$

Then Ψ : const. around C. $(\exists\, 0 < \lambda < 1)$

① Let h be a s.H.m. of $\mathcal{O}_X(C)$ with s.p. curvature.

$\rightsquigarrow$ $\Psi := (-\log |f_C|_h)|_{X \setminus C}$: psh, $= O(\log |f_C|)$

$\rightsquigarrow$ $h = |f_C|^{-2} \cdot {}^{\exists}M$ around C "$\lambda = \frac{1}{2}$" //

prf of Thm 1, 3

codim 2-analogue of Ueda theory [K-, '14] $\rightsquigarrow$ $L := \mathcal{O}_X(S)$ is flat

sing- ———— " $\rightsquigarrow$ $L := \mathcal{O}_X(C)$ around C

for $\left| \begin{array}{l} (C, S, X) \text{ as in Thm 3} \\ (\Sigma, X) \text{ as in Thm 1.} \end{array} \right.$

$\rightsquigarrow$ We can construct a C^∞ Herm. metric on L with s.p. curvature

by gluing up $\left| \begin{array}{l} \text{flat metric on } L \text{ around } C \\ \underline{\text{and}} \text{ s.H.m. defined by the sections of } L. \end{array} \right.$

(it has sing. along C) by using "Regularized min" //

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