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On the minimality of can. attached sing. Herm. metrics on certain nef l.b.

X : sm. proj. var $/\mathbb{C}$.

L : hol. line bdl $/X$.

Q When does a nef line bdl admit a sm. Herm. metric with semi-positive curvature?

Def L : semi-positive (s.p.)

$\overset{\text{def}}{\Longleftrightarrow}$ $^\exists h$: sm. Herm metric on L s.t. $\sqrt{-1}\Theta_h \geq 0$.

$\left(\sqrt{-1}\Theta_h \underset{\text{locally}}{=\!=} \sqrt{-1}\partial\bar\partial\varphi \text{, where } h \underset{\text{locally}}{=\!=} e^{-\underset{\text{"local weight function"}}{\varphi}}\right)$

$\Longleftrightarrow$ $^\exists h$: sm. Herm. metric on L s.t. $\underbrace{\varphi}_{\text{(the local weight)}}$: psh for each locus. //

Known

◎ L : ample $\underset{\text{Kodaira's emb. thm}}{\Longleftrightarrow}$ L : positive. $\Rightarrow$ L : s.p.

◎ L : s.a. $\Rightarrow$ L : s.p //.

◎ L : s.p $\Longrightarrow$ L : nef $\left(L.\underset{\text{curve}}{\underset{\sim}{C}} = \int_C \sqrt{-1}\Theta_h \right)$

◎ L : nef $\not\Longrightarrow$ L : s.p.

e.g. (Demailly - Peternell - Schneider)

s.t.
- C_0 : sm. elliptic curve.
- E : rank 2 - vect. bdl $/C_0$
- $0 \to \mathcal{O}_{C_0} \to E \to \mathcal{O}_{C_0} \to 0$: ex. non-splitting.
- $X := \mathbb{P}(E) \xrightarrow{\pi} C_0$, $C :=$ the section of π.
- $L := \mathcal{O}_X(C)$: nef, but not s.p. //

◎ (K−). $L = A \otimes \mathcal{O}_X(D)$ for $\begin{cases} {}^{\exists}A : \text{s.p. line bdl}/X, \\ D \subset X : \text{sm. hyp.surf.} \end{cases}$

$: \cancel{\text{nef}}\ c_1(L|D) = 0.$

<u>Assume</u> $\begin{cases} N_{D/X}^{-1} : \text{ample}, \ N_{D/X}^{-1} \otimes K_D^{-1} ; \text{nef, big}, \\ {}^{\exists}V : \text{a nbhd of } D \text{ in } X \\ {}^{\exists}V' : \text{a nbhd of the 0-section in } N_{D/X} \\ \text{s.t. } V \cong V' \text{ (bihol)}. \end{cases}$

for example, $\begin{cases} X = \text{surface}, \\ (D^2) < \min\{0, 4 - 4g(D)\}. \end{cases}$

<u>Then</u> L : s.p //.

<u>Q</u> How about the case where
the cpx str. of a nbhd of D in X
is a non-triv. one?

$\boxed{\text{thm 1}}$ X : cpx sm <u>surf</u>.
$\cup$
C : sm. curve, $(C^2) = 0$.
$L = \mathcal{O}_X(C)$. ($\leadsto L$: nef)
<u>Assume</u> (C, X) : <u>of finite type in the sense of Ueda.</u> (*)
<u>Then</u> L : not s.p. //.

(*) $\underset{\text{def}}{\Longleftrightarrow}$ $\exists n \in \mathbb{N}_{\geq 1}$,
s.t. $\begin{cases} \widetilde{N_{C/V}} \otimes \mathcal{O}_V(-C) \otimes \mathcal{O}_V/\mathcal{O}_V(-nC) \not\cong \mathcal{O}_V/\mathcal{O}_V(-nC) \\ \quad \text{for a tub. nbhd } V \text{ of } C \text{ in } X, \\ \text{where } \widetilde{N_{C/V}} \text{ is the flat ext. of } N_{C/V} \text{ //.} \end{cases}$

As an application of $\boxed{\text{Thm 1}}$, we'll show; (**)

$\boxed{\text{Thm 2}}$ "Example 5.9" in [Fujino '13] — "A transcendental approach to Kollár's inj. thm II" J. Reine Angew. Math.
is str. nef, but <u>not s.p</u> //.

§1. Some examples and prf of <u>Thm 1</u>.
§2. <u>Thm 2</u>.

<u>§1</u>

<u>Cor 1</u>. C_0 : sm. curve.
E : a rank 2-vect. bdl $/C_0$.
F : a flat line bdl $/C_0$
s.t. $0 \to F \to E \to \mathcal{O}_{C_0} \to 0$: ex. ·····(*).
$X := \mathbb{P}(E) \xrightarrow{\pi} C_0$, $C :=$ the section of π.
$\rightsquigarrow \underbrace{\mathcal{O}_X(C)}_{\text{nef}}$: s.p. $\Longleftrightarrow$ (*) splits //.

<u>Cor 2</u>. C_0 : a sm. curve of genus 2.
$C_0 \hookrightarrow Y$: the Jacobian of C_0.
$p, q \in C_0$: conjugate to each other by the ~~hyp~~ hyp. ellipt. involution.
$X :=$ the b-up of Y at $\{p, q\}$.
$X \supset C :=$ the strict transf. of C_0.
$\rightsquigarrow \mathcal{O}_X(C)$: nef, but not s.p. //.

<u>prf of Cor 1, 2</u>

Neeman showed that <u>the type $= 1$</u> in the sense of Ueda for these situations //.
i.e. (**) holds for $n = 1$.

<u>prf of Thm 1</u>

··· a simple application of Ueda's thm.

$\underset{\text{sm. curve}}{C} \subset \underset{\text{sm. surf}}{X}$.

<u>Assume</u> (C, X) is of type $\underline{n} < \infty$.

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<u>i.e</u> $\widetilde{N_{C/V}} \otimes \mathcal{O}_V/\mathcal{O}_V(-\nu C) \cong \mathcal{O}_V(C) \otimes \mathcal{O}_V/\mathcal{O}_V(-\nu C)$

holds for $\nu = 1, 2, \cdots, n-1$,

and $\text{———}''\text{———} \not\cong \text{———}''\text{———}$ for $\nu = n$ //

→ <u>Thm (Ueda)</u> '83. "On the neighborhood of a cpt cpx curve with top. triv. normal bdl", Math. Kyoto. Univ.

$\forall a \in (0, n) \subset \mathbb{R}$,

$\forall V$: a nbhd of C in X.

$\forall \Psi$: ~~...~~ a psh function on $V \setminus C$. (i.e. "$\sqrt{-1}\partial\bar\partial\Psi \geq 0$")

s.t. $\Psi(p) = o(\mathrm{dist}(p, C)^{-a})$ as $p \to C$.

<u>Then</u> $\exists V_0$: a nbhd of C in V

<u>s.t.</u> $\Psi \equiv \mathrm{const}$ on V_0 //

From now on, we'll show that

$h = {}^{\exists}(\mathrm{const}) \cdot |f_C|^{-2}$ holds around C

for $\forall h$: singular Hermitian metric on $\mathcal{O}_X(C)$ with s.p. curvature

where $f_C \in H^0(C, \mathcal{O}_X(C))$; canonical section.

<u>Def</u> L : a line bdl $/X$.

h : sing. Herm. metric on L

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists h_\infty$: sm. Herm. metric on L.

$\exists \chi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$: locally L^1

s.t. $h = h_\infty \cdot e^{-\chi}$.

$\Longleftrightarrow$ h is a "metric" s.t. the local weight func φ of h is L^1_{loc} for each locus //

<u>Rmk</u> $|f_C|^{-2}$: sing. Herm. metric with the local weight $\varphi = \log|f_C|^2$ psh

$\left(\rightsquigarrow \sqrt{-1}\,\Theta_{|f_C|^{-2}} \geq 0 \right).$

Assume h : a sing. Herm. metric with s.p. curvature.

$\Psi := -\log |f_C|^2_h.$

$\rightsquigarrow \Psi \underset{\text{locally}}{=\!=} -\log \left(|f_C|^2 \cdot e^{-\varphi} \right)$ (φ: the local weight of h.)

$= \underbrace{\varphi}_{\text{psh on } V} - \underbrace{\log |f_C|^2}_{\text{harmonic on } V \setminus C}. = o(\mathrm{dist}(p, C)^{-\frac{n}{2}})$

$\xrightarrow{\text{Ueda's thm.}}$ ${}^\exists V_0$: a nbhd of C in X, ${}^\exists M$: const.

$\Psi \equiv M$ on V_0. $\rightsquigarrow h|_{V_0} = e^{-\varphi}|_{V_0} = e^{-M} \cdot |f_C|^{-2}$ //.

Cor. $|f_C|^{-2}$ is a sing. Herm. metric on $\mathcal{O}_X(C)$ with s.p. curvature. with minimal singularity. //

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§2. Def. L : str. nef $\overset{\text{def.}}{\Longleftrightarrow}$ ${}^\forall \underset{\text{curve}}{C \subset X}$, $L.C > 0$ //

Q1. str. nef $\overset{?}{\Longrightarrow}$ s.e.

No ... e.g. (Mumford)

$\tilde{C}$: a sm. cpt curve of genus $= g > 1$

fact ‖ ${}^\exists F$: rank 2-vect. bdl $/\tilde{C}$.
s.t. $\deg(F) = 0$, $S^m F$: stable for ${}^\forall m \geq 1$.

$\rightsquigarrow Y := \mathbb{P}(F)$, $L_Y := \mathcal{O}_{\mathbb{P}(F)}(1)$.

$\rightsquigarrow L_Y$: str. nef, but not s.p //.

Q2. str. nef $\overset{?}{\Longrightarrow}$ s.p.

Claim the above L_Y is s.p //.

prf [Narasimhan - Seshadri] $\rightsquigarrow$ F : flat.

i.e $\exists h_F$: sm. metric on F

s.t $\exists \{U_j\}_j$: open cov of $\tilde{C}$.

$\exists (s_j, t_j)$: loc. frame of F on U_j

s.t. $|s_j|^2_{h_F} \equiv |t_j|^2_{h_F} \equiv 1$, $(s_j, t_j)_{h_F} \equiv 0$ //

$\rightsquigarrow$ h_{L_Y} := the fiberwise F.S. metric assoc. to h_F.

We use $(w, x) := [w \cdot s_j^*(x) + t_j^*(x)] \in \mathbb{P}(F) = Y$ as a loc coord Y

$\rightsquigarrow$ the local weight φ_{L_Y} of h_{L_Y} is;

$$\varphi_{L_Y} = \log(1 + |w|^2) \ : \text{psh} \quad //$$

e.g. (= Example 5.9 in [Fujino '13])

$\tilde{C}$, $Y = \mathbb{P}(F)$, L_Y : as above.

$0 \to \mathcal{O}_{\tilde{C}} \to E \to \mathcal{O}_{\tilde{C}} \to 0$: ex, non-splitting.

$\tilde{X} := \mathbb{P}(E) \xrightarrow{\pi} \tilde{C}$.

$\cup$

$\tilde{D}$:= the section of π.

$$\begin{tikzcd} \tilde{Y} := \tilde{X} \times_{\tilde{C}} Y \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\ulcorner"] & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "\pi'"'] & \tilde{C} \end{tikzcd}$$

$$\tilde{L} := \mathcal{O}_{\tilde{Y}}(\tilde{D} \times_{\tilde{C}} Y) \otimes p_2^* L_Y$$

$\rightsquigarrow$ $\tilde{L}$: str. nef, but not s.a //

Q3 ([Fujino '13 Question 5.(0)])
Is $\widetilde{L}$ s.p?

Cor 3 $\widetilde{L}$ is not s.p //.

prf of Cor 3.
Let $h_{\widetilde{L}}$ be a sing. Herm. metric of $\widetilde{L}$ with s.p. curvature.
Fix a sm Herm. metric h_∞ of $\mathcal{O}_{\widetilde{X}}(\widetilde{D})$.

$\rightsquigarrow \exists \chi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$: L^1_{loc}.
s.t. $h_{\widetilde{L}} = (p_1^* h_\infty) \otimes (p_2^* h_{L_Y}) \cdot e^{-\chi}$.

◎ local coord. system of $\widetilde{Y}$

x … a loc. coord of $U \subset \widetilde{C}$ open disc.
(w, x) … as in e.g. (Mumford)
(z, x) … a loc. coord system of $\widetilde{X}$
(z: a fiber coord. of $\widetilde{X} \xrightarrow[\pi]{} \widetilde{C}$)

$\rightsquigarrow$ (the local weight of $h_{\widetilde{L}}$)

$$= \varphi_\infty(z, x) + \log(1 + |w|^2) + \chi(z, w, x)$$

where $h_\infty \underset{\text{locally}}{=} e^{-\varphi_\infty}$.

◎ $\boxed{\widetilde{\chi}(z, x) := \max_{w_0 \in \pi'^{-1}(x)} \chi(z, w_0, x)}$ psh

◎ $\widetilde{Y}|_U := \pi^{-1}(U) \times_U \pi'^{-1}(U)$

$$\simeq U_x \times \mathbb{P}^1_z \times \mathbb{P}^1_w \xrightarrow{\chi|_{(\widetilde{Y}|_U)}} \mathbb{R} \cup \{-\infty\}.$$

$$\rightsquigarrow \varphi_\infty(z, x) + \cancel{\log(1 + |w_0|^2)} + \chi(z, w_0, x)$$

: psh for each locus of $U_x \times \mathbb{P}^1_z$.

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$\rightsquigarrow$ $\varphi(z,x) + \widetilde{\chi}(z,x)$: psh on each locus of $U_x \times \mathbb{P}^1_z$.

$\rightsquigarrow$ $h_\infty \cdot e^{-\widetilde{\chi}}$; Sing. Herm. metric on $\mathcal{O}_{\widetilde{X}}(\widetilde{D})$ with s.p. curvature.

$\overset{\text{Cor 1.}}{\rightsquigarrow}$ $(h_\infty \cdot e^{-\widetilde{\chi}})|_{V_0} \geq \underbrace{{}^{\exists}M}_{\text{const}} \cdot |f_{\widetilde{D}}|^{-2}$.

for $\begin{cases} {}^{\exists}V_0 : \text{a nbhd of } \widetilde{D} \text{ in } \widetilde{X}. \\ f_{\widetilde{D}} \in H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\widetilde{D})) : \text{can. section.} \end{cases}$

$\rightsquigarrow p_1^*(M|f_{\widetilde{D}}|^{-2}) \otimes (p_2^* h_{L_Y})$

$\leq p_1^*(h_\infty \cdot e^{-\widetilde{\chi}}) \otimes (p_2^* h_{L_Y})$

$\leq (p_1^* h_\infty) \otimes (p_2^* h_{L_Y}) \cdot e^{-\chi} = h_{\widetilde{L}}$.

$\rightsquigarrow$ $h_{\widetilde{L}}$ must have singularities along $p_1^{-1}(\widetilde{D})$ //.

Cor. $(p_1^* |f_{\widetilde{D}}|^{-2}) \otimes (p_2^* h_{L_Y})$:

a sing. Herm. metric of $\widetilde{L}$ with s.p. curvature. with minimal sing //.