# Non-projective K3 surfaces containing Levi-flat hypersurfaces

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## K3 surfaces:

Compact complex manifold $X$ of complex dimension $2$ (i.e. compact complex surface) is said to be K3 surface if

- $X$ is simply connected, and
- the canonical line bundle $K_X := \Lambda^2 T_X^*$ is holomorphically trivial.

- Any K3 surface is a Kähler surface (Siu).
- $X, X'$: K3 surfaces $\Rightarrow X \approx_{\mathrm{diffeo}} X'$ (Kodaira, ...)
- $H^0(X, \mathbb{Z}) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z}$
- $H^1(X, \mathbb{Z}) \cong H^3(X, \mathbb{Z}) \cong 0$
- $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$
- $\exists$ (coarse) moduli space $\mathcal{M}$ of (marked) K3 surfaces with $\dim \mathcal{M} = 20$.

## Examples of K3 surfaces:

- A smooth quartic $X \subset \mathbb{P}^3$.
- Kummer surface (non-singular surface obtained by blowing-up 16 singular points of $A/\iota$, where $A = \mathbb{C}^2/\Gamma$ is an abelian surface and $\iota$ is an involution of $A$).

## Question

*Let $X$ be a K3 surface (which is not a Kummer surface).*

- *$\exists$? $H \subset X$: Levi-flat hypersurface?*
- *$\exists$? $F\colon \mathbb{C} \to X$: non-constant holomorphic map?*

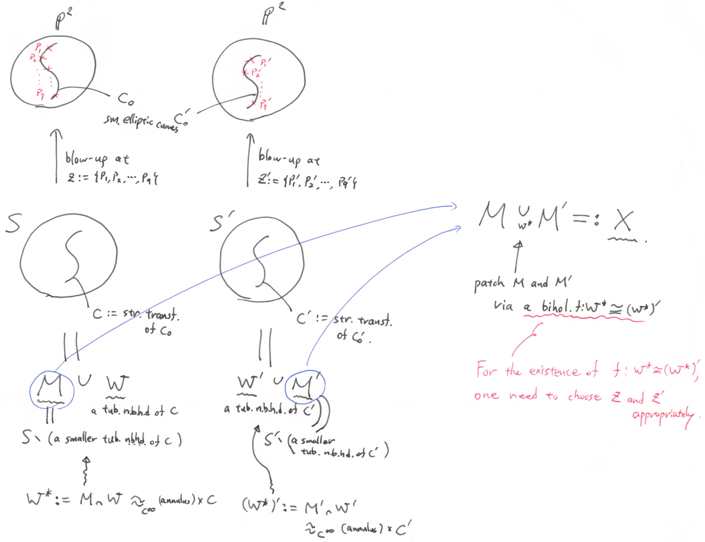
$H \subset X$: $C^\omega$ Levi-flat hypersurface if $H$ is locally defined by $\operatorname{Im} z_n = 0$ for some holomorphic local coordinates $(z_1, z_2, \ldots, z_n)$.

Known examples of K3 surfaces except Kummer surfaces are too abstract to answer these questions.

## Goal of this talk:

To construct a K3 surface $X$ containing a real $1$-parameter family of Levi-flat hypersurfaces by holomorphically patching two open complex surfaces, say $M$ and $M'$

- $M$ $(M')$ is the complement of a (appropriate) tubular neighborhood of an elliptic curve in the blow-up $S$ $(S')$ of the projective plane $\mathbb{P}^2$ at (appropriate) nine points.
- Neither $S$ nor $S'$ admit elliptic fibration structure (nine points are "general")
- In order to patch $M$ and $M'$ holomorphically, we need to take "nice neighborhood". For this purpose, we need to choose nine points carefully.
- Use a technique from Complex Dynamics for taking "nice neighborhood" (Arnol'd's theorem).


$\mathbb{P}^2$
$p_1$ $p_2$ $p_9$
$C_0$
sm. elliptic curves
$C_0'$
$\mathbb{P}^2$
$p_1'$ $p_2'$ $p_9'$
blow-up at $Z := \{p_1, p_2, \cdots, p_9\}$
blow-up at $Z' := \{p_1', p_2', \cdots, p_9'\}$
$S$
$S'$
$C :=$ str. transf. of $C_0$
$C' :=$ str. transf. of $C_0'$.
$M \cup W$
a tub. nbhd of $C$
$S \setminus$ (a smaller tub. nbhd of $C$)
$W^* := M \cap W \approx_{C^\infty}$ (annulus) $\times C$
$W' \cup M'$
a tub. nbhd. of $C'$
$S' \setminus$ (a smaller tub. nbhd. of $C'$)
$(W^*)' := M' \cap W'$
$\approx_{C^\infty}$ (annulus) $\times C'$
$M \cup_{W^*} M' =: X$.
patch $M$ and $M'$
via a bihol. $f: W^* \cong (W^*)'$
For the existence of $f: W^* \cong (W^*)'$,
one need to choose $Z$ and $Z'$
appropriately.

# Remarks, Known results

- For the case where $S$ and $S'$ are elliptic, a similar construction of K3 surfaces is known. The resulting K3 surfaces are also elliptic.
- M. Doi showed that a similar construction of K3 surfaces is possible for general $Z$ and $Z'$ if one admit (slight) deformations of the complex structures of $M$ and $M'$.

  (Doi, Mamoru, Gluing construction of compact complex surfaces with trivial canonical bundle. J. Math. Soc. Japan 61 (2009), no. 3, 853–884)
- The idea to use Arnol'd-type theorem for patching two open manifolds is also used by H. Tsuji in order to study complex structures on $S^3 \times S^3$.

  (H. Tsuji, Complex structures on $S^3 \times S^3$, Tohoku Math. J. (2) Volume 36, Number 3 (1984), 351–376)

# Remarks on our construction of K3 surfaces

- In our construction, “the tab for gluing” $W^* := M \cap M'$ is an open submanifold of $X$ which admits an annulus bundle structure over an elliptic curve.
- $\exists \Phi\colon W^* \to I$ ($I \subset \mathbb{R}$: an interval): pluriharmonic .
- $H_t := \Phi^{-1}(t)$ is a compact Levi-flat hypersurface of $W^*(\subset X)$ which is diffeomorphic to $S^1 \times S^1 \times S^1$ for each $t \in I$.
- For each $t \in I$, any leaf of $H_t$ is biholomorphic to $\mathbb{C}$ or $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and is dense in $H_t$.

# Main results

We constructed K3 surfaces in such a manner with independent (at least) $18$ parameters. As an conclusion of the construction, we have the following:

**Theorem (K-, arXiv:1703.03663)**

*There exists a deformation* $\pi\colon \mathcal{X} \to B$ *of K3 surfaces over an* $18$ *dimensional complex manifold* $B$ *with injective Kodaira-Spencer map such that each fiber* $X_b := \pi^{-1}(b)$ *admits a holomorphic map* $F_b\colon \mathbb{C} \to X_b$ *with the following property: The Euclidean closure of* $F_b(\mathbb{C})$ *is a real analytic compact hypersurface of* $X_b$*. Especially,* $F_b(\mathbb{C})$ *is Zariski dense whereas it is not Euclidean dense.* $X_b$ *is non-Kummer for general* $b \in B$

"$F_b$" can be constructed by considering the immersion of a leaf of $H_t$ into $W^* \subset X$ for each $X = X_b$.

Last June, we described $22$ generators of the K3 lattice and could concretely compute the integrals of the nowhere vanishing holomorphic $2$-form $\sigma$ along $20$ of them (joint w/Takato Uehara). As a result, we could improve the previous theorem.

**Theorem (K-, Uehara (in progress))**

*There exists a deformation $\pi\colon \mathcal{X} \to B$ of K3 surfaces over a $19$ dimensional complex manifold $B$ with injective Kodaira-Spencer map such that each fiber $X_b := \pi^{-1}(b)$ admits a holomorphic map $F_b\colon \mathbb{C} \to X_b$ with the following property: The Euclidean closure of $F_b(\mathbb{C})$ is a real analytic compact hypersurface of $X_b$. Especially, $F_b(\mathbb{C})$ is Zariski dense whereas it is not Euclidean dense. $X_b$ is non-Kummer and non-projective for general $b \in B$.*

Let

- $S$ be a non-singular complex surface, and
- $C$ be a compact complex curve embedded in $S$ with $(C^2) := \deg N_{C/S} = c_1(N_{C/S}) = 0$.

There exists a (small) neighborhood $W$ of $C$ in $S$ which is *diffeomorphic* to a neighborhood of the zero section in $N_{C/S}$ (tubular neighborhood theorem).

**Our original interest:**

What kind of complex analytic structure does $W$ have?

**Remark**

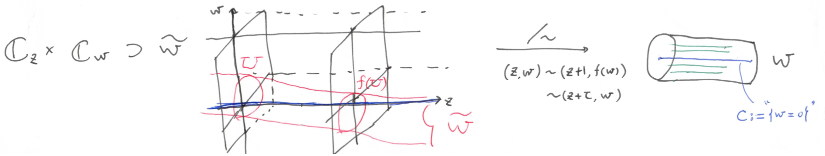
*In general, $\nexists W$ which is biholomorphic to a neighborhood of the zero section in $N_{C/S}$.*

# Ueda's example

Fix $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ and $f(w) = a_1 w + a_2 w^2 + \cdots \in \mathcal{O}_{\mathbb{C},0}$ $(|a_1| = 1)$. Take a neighborhood $\widetilde{W}$ of $\mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}$.

$W := \widetilde{W}/\sim$, $(z, w) \sim (z+1, f(w)) \sim (z+\tau, w)$

$C \subset W$: the image of $\mathbb{C} \times \{0\}$ (smooth elliptic curve)




## Fact (K-)

$C$ admits a *holomorphic tubular neighborhood* (i.e. $\exists W$ which is biholomorphic to a neighborhood of the zero section)
<u>iff</u> $f$ is *linearizable* around the origin (i.e. $\exists \varphi \in \mathcal{O}_{\mathbb{C},0}$ such that $\varphi(0) = 0$ and $\varphi^{-1} \circ f \circ \varphi(w) = a_1 \cdot w$)

Our main interest is in the following example:
Take a smooth elliptic curve $C_0 \subset \mathbb{P}^2$ and nine points $Z := \{p_1, p_2, \dots, p_9\} \subset C_0$.

- $S := \mathrm{Bl}_Z\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$: blow-up at $Z$
- $C := \pi_*^{-1}C_0$: the strict transform of $C_0$

Note that $N_{C/S} \cong \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \cdots - p_9)$. When $Z$ is special, $S$ is an elliptic surface ($N_{C/S} \in \mathrm{Pic}^0(C)$ is torsion in this case). We are interested in the case where $Z$ is general.

**Theorem (Brunella, 2010)**

*Assume that $S \setminus C$ has no compact complex curve.*
*$K_S^{-1}$ admits a $C^\infty$ Hermitian metric with semi-positive curvature iff $C$ has a pseudoflat neighborhoods system (i.e. $\exists$ fundamental system of neighbourhoods $\{W_\varepsilon\}_{\varepsilon>0}$ of $C$ such that $\partial W_\varepsilon$ is Levi-flat).*

Note that $C$ has a pseudoflat neighborhoods system if $C$ admits a holomorphic tubular neighborhood.

## Question

*When does $C$ admit a holomorphic tubular neighborhood?*

## Theorem (Arnol'd (1976))

*Assume $C$ is a smooth elliptic curve and $N_{C/S} \in \mathrm{Pic}^0(C)$ is Diophantine (i.e. $\exists A, \alpha > 0$ such that $\mathrm{dist}(\mathbb{I}_C, N_{C/S}^{\otimes n}) \geq A \cdot n^{-\alpha}$ for $\forall n > 0$, where $\mathrm{dist}$ is the Euclidean distance of $\mathrm{Pic}^0(C)$). Then $C$ admits a holomorphic tubular neighborhood.*

For the previous Ueda's example, this theorem can be directly deduced from Siegel's linearization theorem.

## Question

*For the previous example $(C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S)$, does $C$ admit a holomorphic tubular neighborhood when $N_{C/S} \in \mathrm{Pic}^0(C)$ is not Diophantine?*

Let $(C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S)$ and $(C_0', Z' = \{p_1', p_2', \dots, p_9'\}, C', S')$ be as in the previous section.

## Assumption

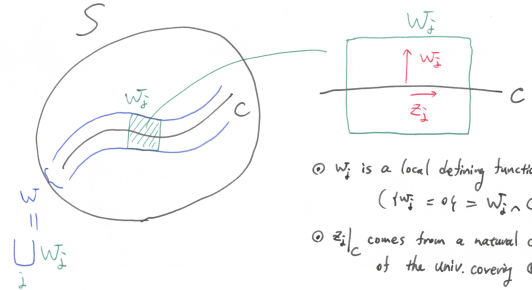
- $\exists g \colon C \cong C'$
- $N_{C/S} = g^* N_{C'/S'}^{-1}$
- $N_{C/S} \in \mathrm{Pic}^0(C)$ is Diophantine

Then, it follows from Arnol'd's theorem that there exist holomorphic tubular neighborhoods $W \subset S$ of $C$ and $W' \subset S'$ of $C'$.

Take a local charts systems $\{(W_j, (z_j, w_j))\}$ of $W$ and $\{(W'_j, (z'_j, w'_j))\}$ of $W'$ such that

$$\begin{cases} z_j = z_k + A_{jk} \\ w_j = t_{jk} \cdot w_k \end{cases}, \quad \begin{cases} z'_j = z'_k + A_{jk} \\ w'_j = t_{jk}^{-1} \cdot w'_k \end{cases}$$

for some constants $A_{jk} \in \mathbb{C}$ and $t_{jk} \in U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$ on $W_{jk} := W_j \cap W_k$ and $W'_{jk} := W'_j \cap W'_k$ as follows:




⊙ $w_j$ is a local defining function of $C$
( $\{w_j = 0\} = W_j \cap C$ )

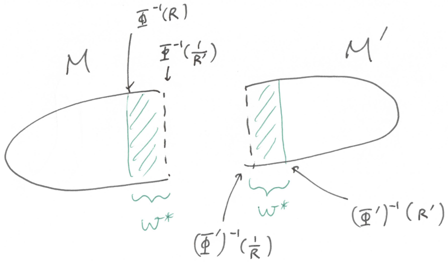
⊙ $z_j|_C$ comes from a natural coordinate of the univ. covering $\mathbb{C}$ of $C$.

- $\Phi \colon W \to \mathbb{R}$: $(z_j, w_j) \mapsto |w_j|$: globally defined on $W$.
- $\Phi' \colon W' \to \mathbb{R}$: $(z'_j, w'_j) \mapsto |w'_j|$: globally defined on $W'$.
- By scaling, we may assume that $\Phi^{-1}([0, R]) \Subset W, (\Phi')^{-1}([0, R']) \Subset W'$ $(R, R' > 1)$.
- Replace $W$ with $\Phi^{-1}([0, R))$ and $W'$ with $(\Phi')^{-1}([0, R'))$.

Define $M \subset S$ and $M' \subset S'$ by

$$M := S \setminus \Phi^{-1}\left(\left[0, \frac{1}{R'}\right]\right), \quad M' := S' \setminus (\Phi')^{-1}\left(\left[0, \frac{1}{R}\right]\right).$$

Identify $W \cap M = \Phi^{-1}((1/R', R))$ and $W' \cap M' = (\Phi')^{-1}((1/R, R'))$ by the isomorphism $f\colon \Phi^{-1}((1/R', R)) \to (\Phi')^{-1}((1/R, R')) : (z_j, w_j) \mapsto \left(g(z_j), \frac{1}{w_j}\right)$ and denote it by $W^*$.




$X := M \cup_{W^*} M'$: a compact complex manifold obtained by patching $M$ and $M'$ via $f$.

## Observation

*$W^*$ admits a natural foliation $\mathcal{F}$ whose leaves are locally defined by "$\{w_j =$ constant$\}$". As each leaf is biholomorphic to $\mathbb{C}$ or $\mathbb{C}^*$, we have a holomorphic map $F\colon \mathbb{C} \to W^* \subset X$ as in Main Theorem.*

It is easily observed that $\pi_1(X) = 0$. Therefore, for proving that $X$ is a K3 surface, it is sufficient to show the following:

## Proposition

*There exists a nowhere vanishing holomorphic $2$-form $\sigma$ on $X$ such that*

$$\sigma|_{W^*\cap W_j} = \frac{dz_j \wedge dw_j}{w_j}$$

*holds on each $W^* \cap W_j \subset W^* \subset X$.*

## Key Lemma

$H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}$.

Key Lemma is shown by pulling back a holomorphic function on $W^*$ by $F\colon \mathbb{C} \to W^*$ and considering the Maximum principle. $\square$

$\boxed{\text{Proof of Proposition:}}$ As $K_S = -C$, there exists a meromorphic 2-form $\eta$ on $S$ with $\mathrm{div}(\eta) = -C$. It follows from Key Lemma that the function

$$\frac{\eta|_{W^*}}{\left(\frac{dz_j \wedge dw_j}{w_j}\right)}$$

is a constant map. Thus we may assume that $\eta|_{W^*} = \frac{dz_j \wedge dw_j}{w_j}$.
Similarly, one can show the existence of a meromorphic 2-form $\eta'$ on $S'$ with $\mathrm{div}(\eta') = -C'$ such that $\eta'|_{W^*} = \frac{dz'_j \wedge dw'_j}{w'_j}$.
$\sigma$ is obtained by patching $\eta|_M$ and $-\eta'|_{M'}$. $\square$

## "Degrees of freedom" in our construction

- Choice of $C_0, C_0'$, and a Diophantine line bundle $L$ on $C_0$ (dimension$=1$ because of $C_0 \cong C_0'$ and Dioph. condition).
- Choice of points $p_1, p_2, \dots, p_8 \in C_0$ (dimension$=8$).
- Choice of points $p_1', p_2', \dots, p_8{}' \in C_0'$ (dimension$=8$).
- Points $p_9 \in C_0$ and $p_9' \in C_0'$ are automatically decided by the condition $N_{C/S} = g^* N_{C'/S'}^{-1} = L$ (dimension$=0$).
- Choice of an isomorphism $g\colon C \cong C'$ (dimension$=1$).
- Choice of (the "scaling" of) the coordinates $w_j$'s and $w_j'$'s ($R, R'$..., dimension$=1$)

### Remark

*Independence of these $19$ parameters (in the sense of Kodaira–Spencer's local deformation theory) is non-trivial.*

In this section, we give $22$ cycles

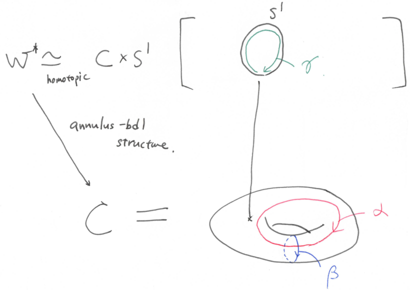
$A_{\alpha,\beta}, A_{\beta,\gamma}, A_{\gamma,\alpha}$,
$B_\alpha, B_\beta, B_\gamma$,
$C_{1,2}, C_{2,3}, \dots, C_{7,8}$ and $C_{6,7,8}$,
$C'_{1,2}, C'_{2,3}, \dots, C'_{7,8}$ and $C'_{6,7,8}$

which generates $H_2(X,\mathbb{Z})$, and compute the integration of the nowhere vanishing $2$-form $\sigma$ along these.

In the following sense, these $22$ cycles can be regarded as a "marking" of $X$:

- $H_2(X,\mathbb{Z}) = \langle A_{\alpha,\beta}, B_\gamma\rangle \oplus \langle A_{\beta,\gamma}, B_\alpha\rangle \oplus \langle A_{\gamma,\alpha}, B_\beta\rangle \oplus \langle C_\bullet\rangle \oplus \langle C'_\bullet\rangle$.
- $\langle A_{\alpha,\beta}, B_\gamma\rangle \cong \langle A_{\beta,\gamma}, B_\alpha\rangle \cong \langle A_{\gamma,\alpha}, B_\beta\rangle \cong U$,
- $\langle C_\bullet\rangle \cong \langle C'_\bullet\rangle \cong E_8(-1)$.

Let $\alpha, \beta$ and $\gamma$ be loops in $W^*$ defined as follows:




- $A_{\alpha,\beta} := \alpha \times \beta \subset W^* \subset X$
- $A_{\beta,\gamma} := \beta \times \gamma \subset W^* \subset X$
- $A_{\gamma,\alpha} := \gamma \times \alpha \subset W^* \subset X$

As $A_{\alpha,\beta}, A_{\beta,\gamma}$ and $A_{\gamma,\alpha}$ are included in $W^*$ and $\sigma|_{W^*} = \frac{dz_j \wedge dw_j}{w_j}$, one can explicitly compute the integrals.

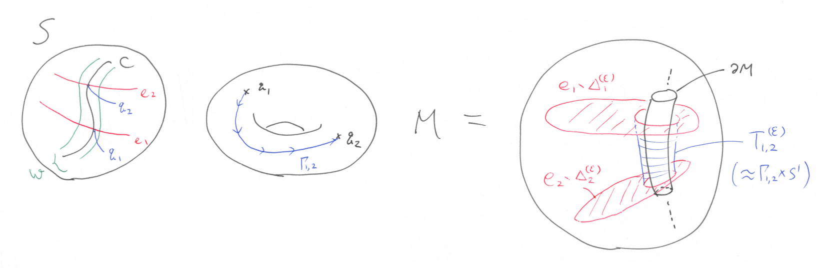
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\alpha,\beta}} \sigma = a_\beta - \tau \cdot a_\alpha$,
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\beta,\gamma}} \sigma = \tau$,
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\gamma,\alpha}} \sigma = 1$,

where

- $\tau$ is a complex number with $\mathrm{Im}\tau > 0$ such that $C \cong \mathbb{C}/\langle 1, \tau \rangle$,
- $a_\alpha$ (resp. $a_\beta$) is a real number such that the monodromy of the flat line bundle $N_{C/S}$ along the loop $\alpha$ (resp. $\beta$) is $\exp(2\pi\sqrt{-1} \cdot a_\alpha)$ (resp. $\exp(2\pi\sqrt{-1} \cdot a_\beta)$).

Let $e_\nu$ (resp. $e'_\nu$) be the exceptional divisor corresponding to the point $p_\nu$ ($p'_\nu$) in $S$ (resp. $S'$). Denote by $h$ (resp. $h'$) the preimage of a hyperplane in $S$ (resp. $S'$).

$C_{1,2} \subset M \subset X$ is defined by $C_{1,2} := (e_1 \setminus \Delta_1^{(\varepsilon)}) \cup T_{1,2}^{(\varepsilon)} \cup (e_2 \setminus \Delta_2^{(\varepsilon)})$.




Note that $C_{1,2} \sim e_1 - e_2$ holds when we regard $C_{1,2} \subset M$ as a cycle of $S$.

Similarly, we define

- $C_{2,3}, C_{3,4}, \cdots, C_{7,8}$, and $C_{6,7,8} \subset M$ ($C_{6,7,8} \sim -h + e_6 + e_7 + e_8$ as a cycle of $S$),
- $C'_{1,2}, C'_{2,3}, \cdots, C'_{7,8}$, and $C'_{6,7,8} \subset M'$.

As $C_\bullet \setminus W^*$ (resp. $C'_\bullet \setminus W^*$) is an analytic subset of $M \setminus W^*$ (resp. $M' \setminus W^*$), we have that

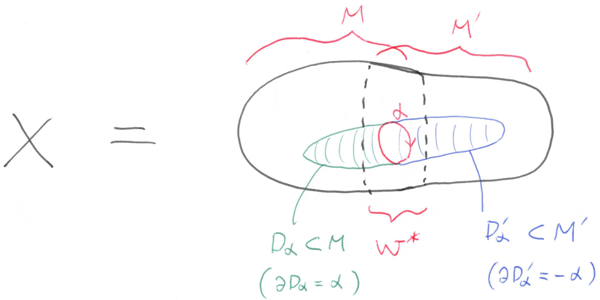
$$\int_{C_\bullet} \sigma = \int_{C_\bullet \cap W^*} \frac{dz_j \wedge dw_j}{w_j}, \quad \int_{C'_\bullet} \sigma = \int_{C'_\bullet \cap W^*} \frac{dz'_j \wedge dw'_j}{w'_j}.$$

By using this description, we can calculate the integrals.

Denote by $q_0$ a inflection point of $C$, $q_0'$ a inflection point of $C'$, $q_\nu$ the intersection point $C \cap e_\nu$, and by $q_\nu'$ the intersection point $C' \cap e_\nu'$. Then we have

- $\frac{1}{2\pi\sqrt{-1}}\int_{C_{\nu,\nu+1}} \sigma = \int_{q_\nu}^{q_{\nu+1}} dz_j$ $(1 \leq \nu \leq 7)$,
- $\frac{1}{2\pi\sqrt{-1}}\int_{C_{6,7,8}} \sigma = \int_{q_0}^{q_6} dz_j + \int_{q_0}^{q_7} dz_j + \int_{q_0}^{q_8} dz_j$,
- $\frac{1}{2\pi\sqrt{-1}}\int_{C'_{\nu,\nu+1}} \sigma = \int_{q'_\nu}^{q'_{\nu+1}} dz'_j$ $(1 \leq \nu \leq 7)$,
- $\frac{1}{2\pi\sqrt{-1}}\int_{C'_{6,7,8}} \sigma = \int_{q'_0}^{q'_6} dz'_j + \int_{q'_0}^{q'_7} dz'_j + \int_{q'_0}^{q'_8} dz'_j$.

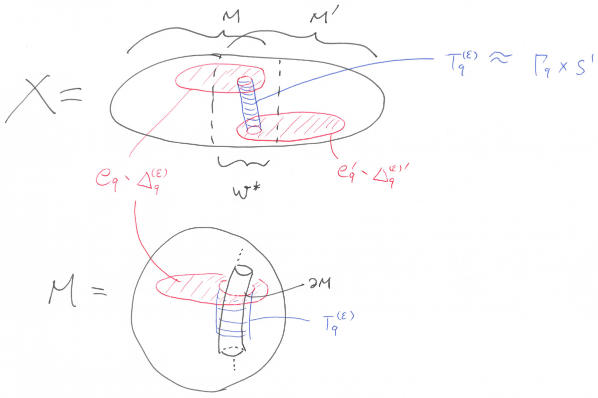
$B_\alpha$ is defined as follows by using the fact that $\pi_1(M) = \pi_1(M') = 0$.




Similarly, we define $B_\beta$.

At this moment, we do not know how to calculate the integrals $\int_{B_\alpha} \sigma$ and $\int_{B_\beta} \sigma$.

$B_\gamma$ is defined by $B_\gamma := (e_9 \setminus \Delta_9^{(\varepsilon)}) \cup T_9^{(\varepsilon)} \cup (e_9' \setminus \Delta_9^{(\varepsilon)}{}')$.




By the same argument as in the $C_\bullet$ case, we have that

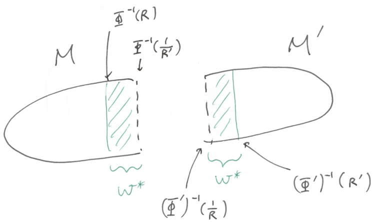
$$\frac{1}{2\pi\sqrt{-1}}\int_{B_\gamma}\sigma = \int_{g(q_9)}^{q_9'} dz_j'.$$

| lattice | cycle | $\frac{1}{2\pi\sqrt{-1}}\int\sigma$ | corresponding parameter |
|---|---|---|---|
| U | $A_{\beta,\gamma}$ | $\tau$ | choice of $C_0$ (and $C_0'$) |
| | $B_\alpha$ | ??? | choice of $w_j$'s ($R$, $R'$,...) |
| U | $A_{\gamma,\alpha}$ | 1 | — |
| | $B_\beta$ | ??? | choice of $w_j$'s ($R$, $R'$,...) |
| $E_8(-1)$ | $C_{1,2}$ | "$p_2 - p_1$" in $C$ | choice of $p_2 - p_1$ |
| | $C_{2,3}$ | "$p_3 - p_2$" in $C$ | choice of $p_3 - p_2$ |
| | $\vdots$ | $\vdots$ | $\vdots$ |
| | $C_{7,8}$ | "$p_8 - p_7$" in $C$ | choice of $p_8 - p_7$ |
| | $C_{6,7,8}$ | "$p_6 + p_7 + p_8$" in $C$ | choice of $p_6 + p_7 + p_8$ |
| $E_8(-1)$ | $C'_{1,2}$ | "$p'_2 - p'_1$" in $C'$ | choice of $p'_2 - p'_1$ |
| | $C'_{2,3}$ | "$p'_3 - p'_2$" in $C'$ | choice of $p'_3 - p'_2$ |
| | $\vdots$ | $\vdots$ | $\vdots$ |
| | $C'_{7,8}$ | "$p'_8 - p'_7$" in $C'$ | choice of $p'_8 - p'_7$ |
| | $C'_{6,7,8}$ | "$p'_6 + p'_7 + p'_8$" in $C$ | choice of $p'_6 + p'_7 + p'_8$ |
| U | $A_{\alpha,\beta}$ | $a_\beta - \tau \cdot a_\alpha$ | choice of $p_9$ and $p'_9$ (i.e. $N_{C/S}$ and $N_{C'/S'}$) |
| | $B_\gamma$ | "$p'_9 - g(p_9)$" | choice of $g\colon C \cong C'$ |

# An interesting direction of the deformation I

A deformation family $\pi_1 \colon \mathcal{X}_1 \to \Delta$ on the disc $\Delta \subset \mathbb{C}$ can be constructed by

- fixing the choice of $C_0, C_0', Z, Z'$, and $g \colon C \cong C'$, and
- only considering the change of the coordinates $w_j$'s $(R,\ R' \ldots)$




- c.f. a degeneration of K3 to a nodal K3 "$S \cup_C S'$".
- We can concretely calculate $\frac{\partial}{\partial b} \int_{B_\alpha} \sigma$ and $\frac{\partial}{\partial b} \int_{B_\beta} \sigma (\neq 0)$
- $\int_{A_\bullet} \sigma$, $\int_{C_\bullet} \sigma$, and $\int_{B_\gamma} \sigma$ are fixed.

# An interesting direction of the deformation II

By general theories (Kodaira-Spencer's local deformation theory, Torelli theorem...), there exists a deformation $\pi_2 \colon \mathcal{X}_2 \to B$ such that $\frac{\partial}{\partial b} \int_{A_{\beta,\gamma}} \sigma \equiv 0$ and $\frac{\partial}{\partial b} \int_{A_{\alpha,\beta}} \sigma \neq 0$ for some direction.

## Question

*Which fiber $X_b := \pi_2^{-1}(b)$ can be constructed in the same manner as in the previous section? (i.e. $\exists$ a subset such as "$W^*$"$\subset X_b$?)*

This question is related to the previous question on the existence of a holomorphic tubular neighborhood of $C$ in $S$.

## Question (repeated)

*For the previous example $(C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S)$, does $C$ admit a holomorphic tubular neighborhood when $N_{C/S} \in \mathrm{Pic}^0(C)$ is not Diophantine?*