§4.

Outline of prf (ii)の方)

{ Outline is the same as the argument in [Ueda '83]
idea comes from "Siegel's linearization technique" in cpx Dyn.

Consider a coord. ~~transf~~ change $w_j \rightsquigarrow u_j$

by ~~Tjk~~. $w_j = u_j + \sum_{|\alpha| \geq 2} F_{j,\alpha}(z_j) \cdot u_j^\alpha$ for "nice" $F_{j,\alpha}$'s. ---(\#\#)

such that (formally) $T_{jk} \cdot u_k = u_j$ holds.

$$T_{jk} \cdot w_k = \underbrace{T_{jk} \cdot u_k}_{= u_j} + \underbrace{\sum_{|\alpha| \geq 2} T_{jk} \cdot (F_{k,\alpha} \cdot u_k^\alpha)}_{=}$$

(\#) $= w_j + \sum_{|\alpha| \geq 2} f_{kj,\alpha}(z_j) \cdot w_j^\alpha$ ----- exp. by u_j ①

$= (u_j + \sum F_{j,\alpha} \cdot u_j^\alpha) + \sum_\beta f_{kj,\alpha} \cdot (u_j + \sum F_{j,\alpha} u_j^\alpha)^\beta$ .... exp. by u_j ②

by comparing the coeff. of u_j^α's of ① and ② + "of intin. type".

$\to$ $F_{j,\alpha}$: (~~uniquely~~) determined. (uniquely for dioph. case)

Convergence of (\#\#) $\Leftarrow$ { "torsion" (Kodaira–Spencer's lem.) or "Dioph". (Ueda's lemma).

$\to$ by implicit func. thm, one can solve (\#\#) to have u_j.

$\rightsquigarrow \mathcal{F} = \{u_j = \text{const}\}$ //

Cor

X^n : cpx mfd,

$\uparrow$

L : hol. l.b. s.t. $\exists D_1, D_2, \cdots, D_{n-1} \in |L|$ with $C := \bigcap_{\lambda=1}^{n-1} D_\lambda$: sm. ellipt. curve.

Assm { $L|_C$: Dioph.
$\{D_\lambda\}$ intersects transv'ly along C.

Then L : semi-positive //

e.g.

(V^n, F) : del Pezzo mfd of deg $=1$. (V : proj. dim $= n$, F : ample l.b/V with { $K_V^{-1} \cong F^{n-1}$, $(F^n) = 1$.)

$C \subset V$: intersection of general $(n-1)$ elem of $|F|$.

$z \in C$: ~~"generic"~~ s.t. $F|_C \otimes \mathcal{O}_C(-z)$: tor or Dioph. $\Rightarrow K^{-1}_{Bl_z V}$: semi-positive //

2018. 11/22 16:30〜18:00. 東工大 キカセミナー.
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On a higher codimensional analogue of Ueda theory and its applications

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X : cpx mfd (maybe non-cpt)
$\cup$
Y : $\underset{\text{cpx}}{\widehat{\text{cpt}}}$ sub mfd (hol'ly embedded into X) s.t. $N_{Y/X}$: "flat" in some sense.

Q. cpx analytic structure of a $\underset{\text{small, tubular}}{\text{nbhd}}$ V of Y in X ?

∄ hol. version of "tub. nbhds sys" c.f. deformation theory ...

$\updownarrow$

(psh-)"convexity". $\boxed{\text{or}}$

Q what kind of p.s.h. functions (plurisubharmonic) are there on V or $V \setminus Y$ (if $\operatorname{codim}_{\mathbb{C}} Y = 1$) ?

[upper semi-conti. function with "$\sqrt{-1}\partial\bar\partial \varphi \geq 0$"]

§1. Motivation from semi-positivity problem for a nef line bdl.
§2. ~~Ueda theory~~ known results. for the case $\begin{cases} \dim X = 2 \\ \dim Y = 1 \end{cases}$. Main thm "§4"
§3. Ueda class $u_n(Y, X)$ and Ueda type $\operatorname{type}(Y, X)$.
§4. Outline of prf of Main thm. + Applicary
~~(§5. Application and example.)~~

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§1.

X : projective,
$L \to X$: nef line bdl. (i.e. $(L.C) = \int_C c_1(L) \geq 0$ for $^\forall C \subset X$ cpt curve)

Q When is L semi-positive ?
i.e. when does L admit a C^∞ Herm. metric with $\sqrt{-1}\Theta_h \geq 0$?

Chern curvature; $\sqrt{-1}\Theta_h \underset{\text{locally}}{=} \sqrt{-1}\partial\bar\partial\varphi$ for $h \underset{\text{locally}}{=} e^{-\varphi}$

c.f. L : ample $\overset{\text{Kodaira emb.}}{\Longleftrightarrow}$ L : positive $\overset{\text{Nakai-Moishezon Criterion.}}{\Longleftrightarrow}$ $^\forall Z \subset X$: d-dim'l cpt sub. var, $(L^d . Z) := \int_{Z_{\text{reg}}} c_1(L)^d > 0$
($^\exists h$: C^∞ Herm. metric with $\sqrt{-1}\Theta_h > 0$)

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example (Serre's e.g.)

C: ellipt. curve, $0 \to \underset{\text{hol'ly triv. l.b. /C}}{\mathbb{1}_C} \to E \to \mathbb{1}_C \to 0$: exact, non-splitting.

$X := \mathbb{P}(E) \supset Y := \mathbb{P}(\mathbb{1}_C)$. ($\rightsquigarrow N_{Y/X} = \mathbb{1}_Y$).

$L := \mathcal{O}_X(Y) \rightsquigarrow L$: nef.

Thm (Demailly-Peternell-Schneider)

$\{$sing. Herm. metric h on L with $\sqrt{-1}\Theta_h \geq 0\} \overset{(*)}{=} \{a \cdot |f_Y|^{-2}\}_{a \in \mathbb{R}_{>0}}$,
where $f_Y \in H^0(X, \mathcal{O}_X(Y))$: canonical sect'n.

especially, L: not semi-positive

Obs

① (Y, X): "of infinite type" in Ueda theory.
Serre's e.g.

② [K-'14]: X: dim $= 2$, $Y \subset X$: dim $= 1$, Assume $N_{Y/X}$: top. triv, (Y, X): of fin. type. Then (*) holds.

③ [Brunella '10] $X = Bl_Z \mathbb{P}^2$ for $Z \subset \mathbb{P}^2$: very general nine pts. $Z = \{p_1, \dots, p_9\}$

$\rightsquigarrow \exists! \, Y \in |K_X^{-1}|$: sm. ellipt. curve.

$\rightsquigarrow K_X^{-1}$ (nef): semi-positive $\iff$ Y admits psd flat nbhds system
(⇒ "of fin. type")

In general ⇒ $\mathcal{O}_X(Y)$: not semi-posi.

Q (nbhd str. of Y) v.s. (semi-positivity of L)
~~when~~ for $Y := SB(L) = \bigcap_{m=1}^{\infty} \mathrm{Base}|L^m|$?
typically, $L|_Y$ or $N_{Y/X}$: "flat".

§2 … for $\dim X = 2$

$c_1(N_{Y/X}) = (Y^2) \gtreqless 0$.

Background

[H. Grauert '62] $(Y^2) < 0 \Rightarrow$ $\exists V$: a nbhd of Y, $\exists \psi: V \to \mathbb{R} \cup \{-\infty\}$, s.t. ψ: psh, $\psi|_{V \setminus Y}$: s.psh, $\{\psi = -\infty\} = Y$.
"psd convex nbhds sys. str."

[O. Suzuki '75] $(Y^2) > 0 \Rightarrow$ $\exists V$: a nbhd of Y. (**) $\exists \psi: V \setminus Y \to \mathbb{R}$ s.t. $\psi(p) \to +\infty$ as $p \to Y$, ψ: s.p.s.h.
"str. psd concave nbhds sys."

[T. Ueda '83] Case $(Y^2) = 0$.
… posed "ueda class" $u_n(Y, X) \in H^1(Y, N_{Y/X}^{-n})$. for $n \geq 1$.
$\rightsquigarrow (Y, X)$: of fin. type if $\exists n$. $u_n \neq 0$
of infin. type if $\forall n$. $u_n = 0$.

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~~Thm~~ Heuristically,

flat line bdl /V s.t. $\tilde{N}|_Y = N_{Y/X}$

~~$u_n \neq 0$~~ $u_1 = 0 \iff \tilde{N}$ and $\mathcal{O}_X(Y)$ coincide in 1-jet alg Y.

$\downarrow$

u_2: defined,

$u_2 = 0 \iff$ —— // —— in 2-jet //

$\downarrow$

u_3: defined

$\vdots$

[Ueda '83]. ∘ (Y,X): of fin. type $\Rightarrow$ (**) holds, + on the growth of Ψ.

∘ (Y,X): of infin type + $N_{Y/X} \in \mathrm{Pic}^0(Y)$: torsion elem or Dioph. elem $\Rightarrow$ $\exists V$: nbhd of Y, $\exists \Psi: V \setminus Y \to \mathbb{R}$ s.t. $\Psi(p) = O(-\log \mathrm{dist}(p, Y))$ as $p \to \infty$, $\Psi|_{V \setminus Y}$: plurihar harm.

i.e. $\exists A, \alpha > 0$ s.t. $\forall n \geq 1$, $\mathrm{dist}(\mathbb{1}_Y, N_{Y/X}^n) \geq A \cdot n^{-\alpha}$

"psd flat nbhds sys". $V \supset \exists \Psi|_{V \setminus Y} \equiv 0$. //

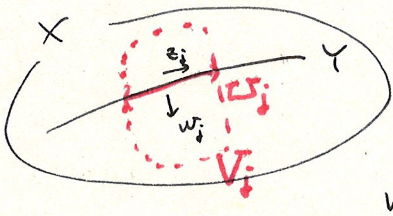
Today's goal: to ~~define~~ generalize this thm. ~~into~~ to the case

where (***) X: n-dim'l cpx mfd $\cup$ Y: cpt sub mfd of codim $= r$ with $N_{Y/X}$: Unitary flat

(i.e. $N_{Y/X} \in \mathrm{Image}(H^1(Y, U(r)) \to H^1(Y, GL(r)))$)

c.f. ~~[Koike '15]~~ [Koike-Ogawa '18]. 今回のは [Koike, Nagoya] to appear.

§3 (Y,X): as (***).



$\{U_j\}$: covering of Y. (finitely many).

~~$\{V_j\}$: ..~~ V_j: nbhd of U_j ($V_j \cap Y = U_j$)

$V = \bigcup_j V_j$.

$w_j = {}^t(w_j^1, w_j^2, \cdots, w_j^r)$ (たて): defining functn

z_j: coord of U_j $\xrightarrow[\text{ext.}]{}$ (z_j, w_j): coord. on V_j.

$e_j = {}^t(e_j^1, e_j^2, \cdots, e_j^r)$ (たて): local frame of $N_{Y/X}$ on U_j.

By Assumption, $\exists T_{jk} \in U(r)$ s.t. $e_j = T_{jk} \cdot e_k$. on $U_{jk} = U_j \cap U_k$.

We may assume $dw_j|_{U_j} = e_j$ ($\Leftarrow$ simple obs).

$$\rightsquigarrow \sum_{\mu=1}^{r} (T_{jk})_\mu^\lambda \cdot w_k^\mu = w_j^\lambda + \sum_{|\alpha| \geq 2} {}^\exists f_{kj,\alpha}^\lambda(z_j) \cdot w_j^\alpha \text{ on } V_{jk}.$$

Taylor exp.

$\alpha = (\alpha_1, \cdots, \alpha_r)$, $|\alpha| = \sum \alpha_\nu$, $w_j^\alpha = \prod_\nu (w_j^\nu)^{\alpha_\nu}$

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or $\quad T_{jk}\cdot w_k \overset{(\#)}{=} w_j + \sum_{|\alpha|\geq 2} f_{kj,\alpha}(z_j)\cdot w_j^\alpha \qquad \left(f_{kj,\alpha} = {}^t(f^1_{kj,\alpha}, \cdots, f^r_{kj,\alpha}) \right)$ on V_{jk}.

Def Assume $f_{kj,\alpha} \equiv 0$ for α with $|\alpha| \leq n$. ($\{w_j\}$: "of type n").

$\rightsquigarrow u_n(Y,X) := \left[\left\{\left(U_{jk}, \sum_{\lambda=1}^{r} e^*_{j,\lambda} \otimes f^\lambda_{kj,\frac{n+1}{\cancel{2}}} \right)\right\}\right] \in H^1(Y, N_{Y/X} \otimes S^{n+1} N^*_{Y/X})$,

($e^*_{j,\lambda}$: dual of e^λ_j)

where $f_{kj,n} = \left(f^\lambda_{kj,n+1} \right)_\lambda$

$f^\lambda_{kj,n+1} = \sum_{|\alpha|=n+1} f^\lambda_{kj,\alpha}\cdot e_j^\alpha$. //

prop (i) $u_n(Y,X)$ does not depend on the choice of $\{w_j\}$: of type n (with $dw_j = e_j$).

(ii) $u_n(Y,X) = 0 \iff \exists\, \{w_j\}$: of type $n+1$. ($\rightarrow u_{n+1}$: defined)

$\rightsquigarrow$ (only) one of the following holds: //

- $\exists\, n \geq 1$ s.t. $\begin{cases} \forall m < n, & u_m = 0 \\ u_n \neq 0 \end{cases}$ $\quad\leftarrow$ "of finite type".
- $\forall n \geq 1$, $u_n = 0$ $\quad\leftarrow$ "of infinite type" //

MainThm $\rho : \pi_1(Y,*) \longrightarrow U(r)$: repr. corresp. to $N_{Y/X}$.

Assume $\begin{cases} \#(\text{Image}\,\rho) < \infty \text{ or } \left[N_{Y/X} : \text{direct sum } N_1 \oplus \cdots \oplus N_r \text{ with } \exists A, B > 0 \; \text{dist}(\mathbb{1}_Y, \otimes N_\lambda^{\alpha_\lambda}) \geq A\cdot|\alpha|^{-B} \text{ (up to finite cover } \tilde{Y} \to Y) \right] \\ (Y,X): \text{of infinite type}. \end{cases}$

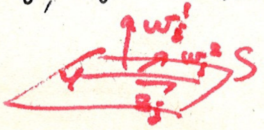
Then (i) $\exists\, V$: nbhd of Y in X,

$\exists$ "!" $\mathcal{F}$: hol. foliation of codim $= r$, s.t. $\begin{cases} Y : \text{a leaf of } \mathcal{F}, \\ \text{Hol}_{\mathcal{F},Y} = \rho. \end{cases}$

for Dioph. case.

(ii) If $\exists\, S$: a hyp. surf $\subset V$ with $Y \subset S$, $N_{Y/S}$: $U(1)$-flat,

by shrinking V, $\exists\, \mathcal{G}$: hol. foliation of codim $= 1$ on V s.t. $\begin{cases} S : \text{a leaf of } \mathcal{G}. \\ \text{Hol}_{\mathcal{G},S} : U(1)\text{-linear}, \\ \text{each leaf of } \mathcal{F} \text{ is hol'ly immersed into a leaf of } \mathcal{G}. \end{cases}$ //



$\rightsquigarrow \varphi := -\log|w_j^1|$: p.h. on $V \setminus S$.

$\varphi(p) = O(-\log\ )$.

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