

Gluing construction of non-projective K3 surfaces and holomorphic tubular neighborhoods of elliptic curves

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- 2 Motivation from neighborhoods theories
- 3 Construction of the K3 surface X
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- 5 Towards the “moduli space” of K3 surfaces constructed by our method

1 Main results

2 Motivation from neighborhoods theories

3 Construction of the K3 surface X

4 22 generators of $H_2(X, \mathbb{Z})$ and the period

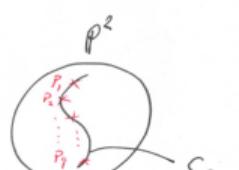
5 Towards the “moduli space” of K3 surfaces constructed by our method

- based on [T. Koike, Complex K3 surfaces containing Levi-flat hypersurfaces, arXiv:1703.03663] + some recent progress (j.w/ T. Uehara)

Goal of this talk:

To construct a (non-projective, non-Kummer) K3 surface X containing a real 1-parameter family of Levi-flat hypersurfaces by holomorphically patching two open complex surfaces, say M and M'

- M (M') is the complement of a (appropriate) tubular neighborhood of an elliptic curve in the blow-up S (S') of the projective plane \mathbb{P}^2 at (appropriate) nine points.
- Neither S nor S' admit elliptic fibration structure (nine points are “general”)
- In order to patch M and M' holomorphically, we need to take “nice neighborhood”. For this purpose, we need to choose nine points carefully.
- Use a technique from Complex Dynamics for taking “nice neighborhood” (Arnol'd's theorem).



blow-up at
 $Z := \{P_1, P_2, \dots, P_q\}$



blow-up at
 $Z' := \{P'_1, P'_2, \dots, P'_{q'}\}$



$C :=$ str. transf.
of C_0

$M \cup \tilde{W}$
a tub. nbhd. of C

$S \setminus (\text{a smaller tub. nbhd. of } C)$

$$W^* := M \cap W \cong_{\infty} (\text{annulus}) \times C$$



$C' :=$ str. transf.
of C'_0 .

$\tilde{W}' \cup M$
a tub. nbhd. of C'

$S' \setminus (\text{a smaller tub. nbhd. of } C')$

$$(W^*)' := M' \cap W'$$

$$M \overset{w^*}{\cup} M' =: X$$

patch M and M'

via a bihol. $f: W^* \cong (W^*)'$

For the existence of $f: W^* \cong (W^*)'$,
one need to choose Z and Z'
appropriately.

Remarks, Known results

- For the case where S and S' are elliptic, a similar construction of K3 surfaces is known. The resulting K3 surfaces are also elliptic.
- M. Doi showed that a similar construction of K3 surfaces is possible for general Z and Z' if one admit (slight) deformations of the complex structures of M and M' .
(Doi, Mamoru, Gluing construction of compact complex surfaces with trivial canonical bundle. J. Math. Soc. Japan 61 (2009), no. 3, 853–884)
- The idea to use Arnol'd-type theorem for patching two open manifolds is also used by H. Tsuji in order to study complex structures on $S^3 \times S^3$.
(H. Tsuji, Complex structures on $S^3 \times S^3$, Tohoku Math. J. (2) Volume 36, Number 3 (1984), 351–376)

Remarks on our construction of K3 surfaces

- In our construction, “the tab for gluing” $W^* := M \cap M'$ is an open submanifold of X which admits an annulus bundle structure over an elliptic curve.
- $\exists \Phi: W^* \rightarrow I$ ($I \subset \mathbb{R}$: an interval): pluriharmonic .
- $H_t := \Phi^{-1}(t)$ is a compact Levi-flat hypersurface of $W^*(\subset X)$ which is diffeomorphic to $S^1 \times S^1 \times S^1$ for each $t \in I$.
- For each $t \in I$, any leaf of H_t is biholomorphic to \mathbb{C} or $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and is dense in H_t .

Main results

We constructed K3 surfaces in such a manner with independent (at least) 18 parameters. As an conclusion of the construction, we have the following:

Theorem (K-, arXiv:1703.03663)

There exists a deformation $\pi: \mathcal{X} \rightarrow B$ of K3 surfaces over an 18 dimensional complex manifold B with injective Kodaira-Spencer map such that each fiber $X_b := \pi^{-1}(b)$ admits a holomorphic map $F_b: \mathbb{C} \rightarrow X_b$ with the following property: The Euclidean closure of $F_b(\mathbb{C})$ is a real analytic compact hypersurface of X_b . Especially, $F_b(\mathbb{C})$ is Zariski dense whereas it is not Euclidean dense. X_b is non-Kummer for general $b \in B$

“ F_b ” can be constructed by considering the immersion of a leaf of H_t into $W^* \subset X$ for each $X = X_b$.

We also described 22 generators of the K3 lattice and could concretely compute the integrals of the nowhere vanishing holomorphic 2-form σ along 20 of them (joint w/Takato Uehara). As a result, we could improve the previous theorem.

Theorem (K-, T. Uehara (in progress))

There exists a deformation $\pi: \mathcal{X} \rightarrow B$ of K3 surfaces over a 19 dimensional complex manifold B with injective Kodaira-Spencer map such that each fiber $X_b := \pi^{-1}(b)$ admits a holomorphic map $F_b: \mathbb{C} \rightarrow X_b$ with the following property: The Euclidean closure of $F_b(\mathbb{C})$ is a real analytic compact hypersurface of X_b . Especially, $F_b(\mathbb{C})$ is Zariski dense whereas it is not Euclidean dense. X_b is non-Kummer and non-projective for general $b \in B$.

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Let

- S be a non-singular complex surface, and
- C be a compact complex curve embedded in S with
 $(C^2) := \deg N_{C/S} = c_1(N_{C/S}) = 0$.

There exists a (small) neighborhood W of C in S which is diffeomorphic to a neighborhood of the zero section in $N_{C/S}$ (tubular neighborhood theorem).

Our original interest:

What kind of complex analytic structure does W have?

Remark

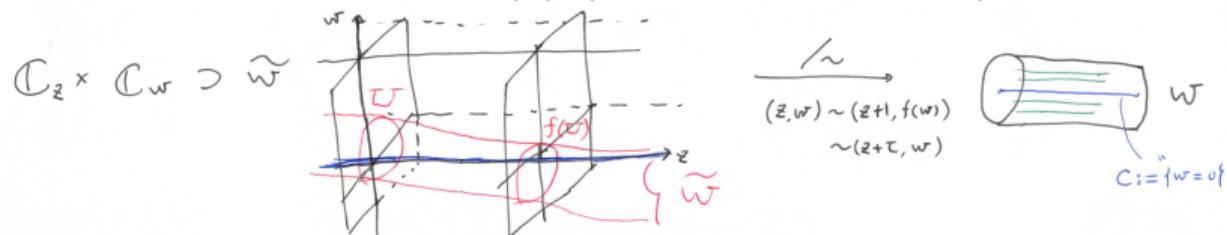
In general, $\exists W$ which is biholomorphic to a neighborhood of the zero section in $N_{C/S}$.

Ueda's example

Fix $\tau \in \mathbb{C}$ with $\text{Im } \tau > 0$ and $f(w) = a_1 w + a_2 w^2 + \dots \in \mathcal{O}_{\mathbb{C}, 0}$ ($|a_1| = 1$). Take a neighborhood \widetilde{W} of $\mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}$.

$$W := \widetilde{W} / \sim, (z, w) \sim (z+1, f(w)) \sim (z+\tau, w)$$

$C \subset W$: the image of $\mathbb{C} \times \{0\}$ (smooth elliptic curve)



Fact [K-, N. Ogawa, arXiv:1808.10219]

C admits a *holomorphic tubular neighborhood* (i.e. $\exists W$ which is biholomorphic to a neighborhood of the zero section)
iff f is *linearizable* around the origin.

Our main interest is in the following example:

Take a smooth elliptic curve $C_0 \subset \mathbb{P}^2$ and nine points $Z := \{p_1, p_2, \dots, p_9\} \subset C_0$.

- $S := \text{Bl}_Z \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$: blow-up at Z
- $C := \pi_*^{-1} C_0$: the strict transform of C_0

Note that $N_{C/S} \cong \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \dots - p_9)$. When Z is special, S is an elliptic surface ($N_{C/S} \in \text{Pic}^0(C)$ is torsion in this case). We are interested in the case where Z is general.

Theorem (Brunella, 2010)

Assume that $S \setminus C$ has no compact complex curve.

K_S^{-1} admits a C^∞ Hermitian metric with semi-positive curvature

iff C has a pseudoflat neighborhoods system

(i.e. \exists fundamental system of neighbourhoods $\{W_\varepsilon\}_{\varepsilon > 0}$ of C such that ∂W_ε is Levi-flat).

Note that C has a pseudoflat neighborhoods system if C admits a hol. tub. n.b.h.d.

Question

When does C admit a holomorphic tubular neighborhood?

Theorem (Arnol'd (1976))

Assume C is a smooth elliptic curve and $N_{C/S} \in \text{Pic}^0(C)$ is Diophantine (i.e. $\exists A, \alpha > 0$ such that $\text{dist}(\mathbb{I}_C, N_{C/S}^{\otimes n}) \geq A \cdot n^{-\alpha}$ for $\forall n > 0$, where dist is the Euclidean distance of $\text{Pic}^0(C)$).

Then C admits a holomorphic tubular neighborhood.

For the previous Ueda's example, this theorem can be directly deduced from Siegel's linearization theorem.

Question

For the previous example ($C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S$), does C admit a holomorphic tubular neighborhood when $N_{C/S} \in \text{Pic}^0(C)$ is not Diophantine?

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Let $(C_0, Z = \{p_1, p_2, \dots, p_9\}, C, S)$ and $(C'_0, Z' = \{p'_1, p'_2, \dots, p'_9\}, C', S')$ be as in the previous section.

Assumption

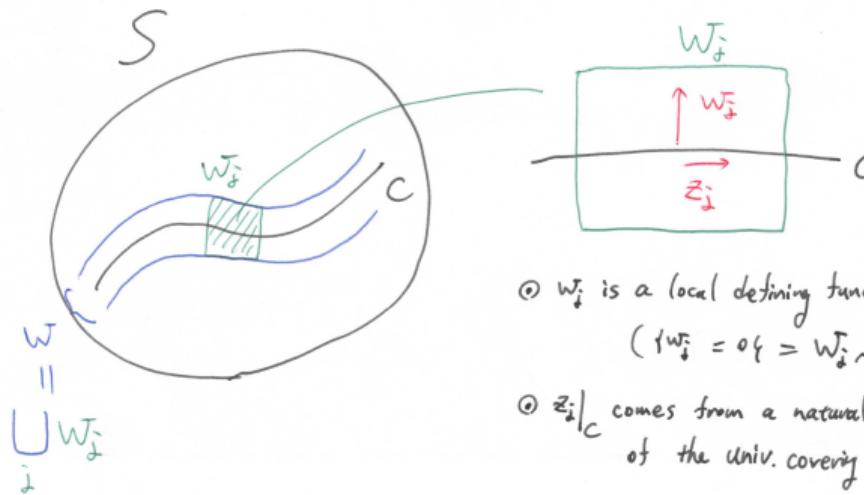
- $\exists g: C \cong C'$
- $N_{C/S} = g^* N_{C'/S'}^{-1}$
- $N_{C/S} \in \text{Pic}^0(C)$ is Diophantine

Then, it follows from Arnol'd's theorem that there exist holomorphic tubular neighborhoods $W \subset S$ of C and $W' \subset S'$ of C' .

Take a local charts systems $\{(W_j, (z_j, w_j))\}$ of W and $\{(W'_j, (z'_j, w'_j))\}$ of W' such that

$$\begin{cases} z_j = z_k + A_{jk} \\ w_j = t_{jk} \cdot w_k \end{cases}, \quad \begin{cases} z'_j = z'_k + A_{jk} \\ w'_j = t_{jk}^{-1} \cdot w'_k \end{cases}$$

for some constants $A_{jk} \in \mathbb{C}$ and $t_{jk} \in U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$ on $W_{jk} := W_j \cap W_k$ and $W'_{jk} := W'_j \cap W'_k$ as follows:



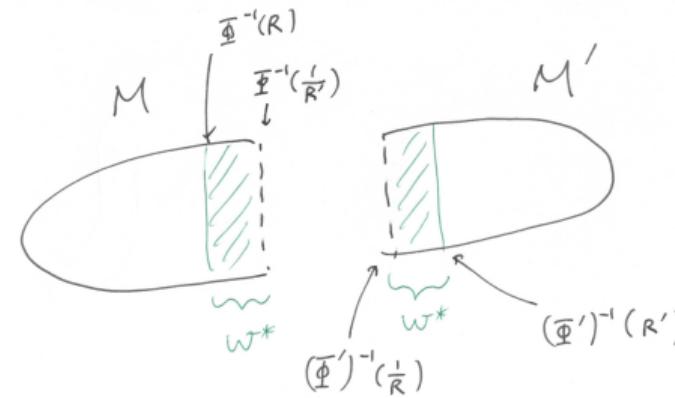
- ① w_j is a local defining function of C
($\{w_j = 0\} = W_j \cap C$)
- ② $z_j|_C$ comes from a natural coordinate
of the univ. covering \mathbb{C} of C .

- $\Phi: W \rightarrow \mathbb{R}: (z_j, w_j) \mapsto |w_j|$: globally defined on W .
- $\Phi': W' \rightarrow \mathbb{R}: (z'_j, w'_j) \mapsto |w'_j|$: globally defined on W' .
- By scaling, we may assume that $\Phi^{-1}([0, R]) \Subset W, (\Phi')^{-1}([0, R']) \Subset W'$ ($R, R' > 1$).
- Replace W with $\Phi^{-1}([0, R])$ and W' with $(\Phi')^{-1}([0, R'])$.

Define $M \subset S$ and $M' \subset S'$ by

$$M := S \setminus \Phi^{-1} \left(\left[0, \frac{1}{R'} \right] \right), \quad M' := S' \setminus (\Phi')^{-1} \left(\left[0, \frac{1}{R} \right] \right).$$

Identify $W \cap M = \Phi^{-1}((1/R', R))$ and $W' \cap M' = (\Phi')^{-1}((1/R, R'))$ by the isomorphism $f: \Phi^{-1}((1/R', R)) \rightarrow (\Phi')^{-1}((1/R, R')) : (z_j, w_j) \mapsto \left(g(z_j), \frac{1}{w_j}\right)$ and denote it by W^* .



$X := M \cup_{W^*} M'$: a compact complex manifold obtained by patching M and M' via f .

Observation

W^* admits a natural foliation \mathcal{F} whose leaves are locally defined by “ $\{w_j = \text{constant}\}$ ”. As each leaf is biholomorphic to \mathbb{C} or \mathbb{C}^* , we have a holomorphic map $F: \mathbb{C} \rightarrow W^* \subset X$ as in Main Theorem.

It is easily observed that $\pi_1(X) = 0$. Therefore, for proving that X is a K3 surface, it is sufficient to show the following:

Proposition

There exists a nowhere vanishing holomorphic 2-form σ on X such that

$$\sigma|_{W^* \cap W_j} = \frac{dz_j \wedge dw_j}{w_j}$$

holds on each $W^ \cap W_j \subset W^* \subset X$.*

Key Lemma

$$H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}.$$

Key Lemma is shown by pulling back a holomorphic function on W^* by $F: \mathbb{C} \rightarrow W^*$ and considering the Maximum principle. \square

Proof of Proposition: As $K_S = -C$, there exists a meromorphic 2-form η on S with $\text{div}(\eta) = -C$. It follows from Key Lemma that the function

$$\frac{\eta|_{W^*}}{\left(\frac{dz_j \wedge dw_j}{w_j} \right)}$$

is a constant map. Thus we may assume that $\eta|_{W^*} = \frac{dz_j \wedge dw_j}{w_j}$. Similarly, one can show the existence of a meromorphic 2-form η' on S' with $\text{div}(\eta') = -C'$ such that

$$\eta'|_{W^*} = \frac{dz'_j \wedge dw'_j}{w'_j}.$$

σ is obtained by patching $\eta|_M$ and $-\eta'|_{M'}$.

Remark

The construction in this section also makes sense even after one generalizes “the materials” as follows:

- S : compact complex manifold of any dimension s.t. $\exists C \in |K_S^{-1}|$: compact complex torus.
- S' : compact complex manifold of any dimension s.t. $\exists C' \in |K_{S'}^{-1}|$: compact complex torus.
- Assume $\exists g: C \cong C'$,
- $N_{C/S} = g^* N_{C'/S'}^{-1}$, and
- $N_{C/S} \in \text{Pic}^0(C)$ is Diophantine

The resulting manifold is a compact Calabi-Yau or HyperKähler.

Question

∃? nice example of such (S, C, S', C') in higher dimension?

“Degrees of freedom” in our construction

- Choice of C_0, C'_0 , and a Diophantine line bundle L on C_0 (**dimension=1** because of $C_0 \cong C'_0$ and Dioph. condition).
- Choice of points $p_1, p_2, \dots, p_8 \in C_0$ (**dimension=8**).
- Choice of points $p'_1, p'_2, \dots, p'_8 \in C'_0$ (**dimension=8**).
- Points $p_9 \in C_0$ and $p'_9 \in C'_0$ are automatically decided by the condition $N_{C/S} = g^* N_{C'/S'}^{-1} = L$ (**dimension=0**).
- Choice of an isomorphism $g: C \cong C'$ (**dimension=1**).
- Choice of (the “scaling” of) the coordinates w_j 's and w'_j 's (R, R', \dots , **dimension=1**)

Remark

Independence of these 19 parameters (in the sense of Kodaira–Spencer’s local deformation theory) is non-trivial.

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In this section, we give 22 cycles

$$A_{\alpha,\beta}, A_{\beta,\gamma}, A_{\gamma,\alpha},$$

$$B_\alpha, B_\beta, B_\gamma,$$

$$C_{1,2}, C_{2,3}, \dots, C_{7,8} \text{ and } C_{6,7,8},$$

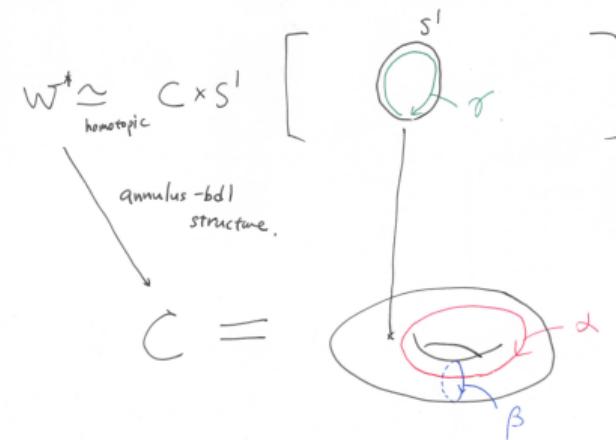
$$C'_{1,2}, C'_{2,3}, \dots, C'_{7,8} \text{ and } C'_{6,7,8}$$

which generates $H_2(X, \mathbb{Z})$, and compute the integration of the nowhere vanishing 2-form σ along these.

In the following sense, these 22 cycles can be regarded as a “marking” of X :

- $H_2(X, \mathbb{Z}) = \langle A_{\alpha,\beta}, B_\gamma \rangle \oplus \langle A_{\beta,\gamma}, B_\alpha \rangle \oplus \langle A_{\gamma,\alpha}, B_\beta \rangle \oplus \langle C_\bullet \rangle \oplus \langle C'_\bullet \rangle$.
- $\langle A_{\alpha,\beta}, B_\gamma \rangle \cong \langle A_{\beta,\gamma}, B_\alpha \rangle \cong \langle A_{\gamma,\alpha}, B_\beta \rangle \cong U$,
- $\langle C_\bullet \rangle \cong \langle C'_\bullet \rangle \cong E_8(-1)$.

Let α, β and γ be loops in W^* defined as follows:



- $A_{\alpha,\beta} := \alpha \times \beta \subset W^* \subset X$
- $A_{\beta,\gamma} := \beta \times \gamma \subset W^* \subset X$
- $A_{\gamma,\alpha} := \gamma \times \alpha \subset W^* \subset X$

As $A_{\alpha,\beta}, A_{\beta,\gamma}$ and $A_{\gamma,\alpha}$ are included in W^* and $\sigma|_{W^*} = \frac{dz_j \wedge dw_j}{w_j}$, one can explicitly compute the integrals.

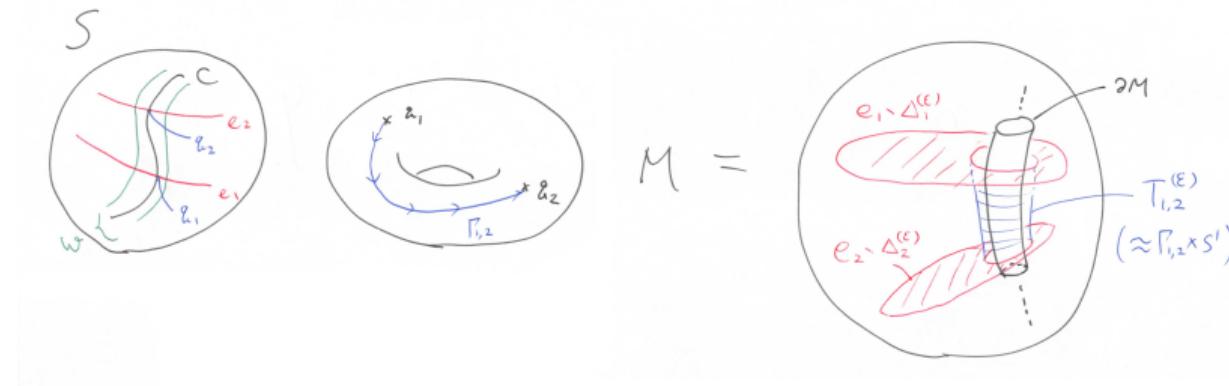
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\alpha,\beta}} \sigma = a_\beta - \tau \cdot a_\alpha,$
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\beta,\gamma}} \sigma = \tau,$
- $\frac{1}{2\pi\sqrt{-1}} \int_{A_{\gamma,\alpha}} \sigma = 1,$

where

- τ is a complex number with $\text{Im}\tau > 0$ such that $C \cong \mathbb{C}/\langle 1, \tau \rangle$,
- a_α (resp. a_β) is a real number such that the monodromy of the flat line bundle $N_{C/S}$ along the loop α (resp. β) is $\exp(2\pi\sqrt{-1} \cdot a_\alpha)$ (resp. $\exp(2\pi\sqrt{-1} \cdot a_\beta)$).

Let e_ν (resp. e'_ν) be the exceptional divisor corresponding to the point p_ν (p'_ν) in S (resp. S'). Denote by h (resp. h') the preimage of a hyperplane in S (resp. S').

$C_{1,2} \subset M \subset X$ is defined by $C_{1,2} := (e_1 \setminus \Delta_1^{(\varepsilon)}) \cup T_{1,2}^{(\varepsilon)} \cup (e_2 \setminus \Delta_2^{(\varepsilon)})$.



Note that $C_{1,2} \sim e_1 - e_2$ holds when we regard $C_{1,2} \subset M$ as a cycle of S .

Similarly, we define

- $C_{2,3}, C_{3,4}, \dots, C_{7,8}$, and $C_{6,7,8} \subset M$ ($C_{6,7,8} \sim -h + e_6 + e_7 + e_8$ as a cycle of S),
- $C'_{1,2}, C'_{2,3}, \dots, C'_{7,8}$, and $C'_{6,7,8} \subset M'$.

As $C_\bullet \setminus W^*$ (resp. $C'_\bullet \setminus W^*$) is an analytic subset of $M \setminus W^*$ (resp. $M' \setminus W^*$), we have that

$$\int_{C_\bullet} \sigma = \int_{C_\bullet \cap W^*} \frac{dz_j \wedge dw_j}{w_j}, \quad \int_{C'_\bullet} \sigma = \int_{C'_\bullet \cap W^*} \frac{dz'_j \wedge dw'_j}{w'_j}.$$

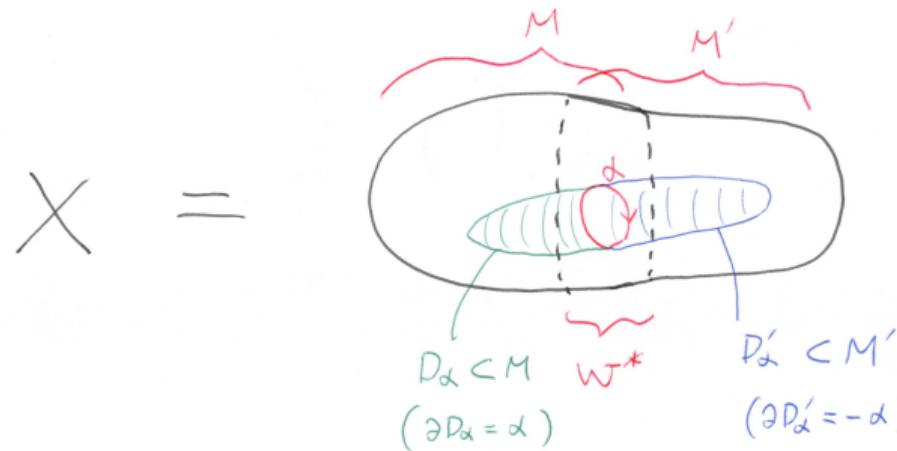
By using this description, we can calculate the integrals.

Denote by q_0 a inflection point of C , q'_0 a inflection point of C' , q_ν the intersection point $C \cap e_\nu$, and by q'_ν the intersection point $C' \cap e'_\nu$. Then we have

- $\frac{1}{2\pi\sqrt{-1}} \int_{C_{\nu,\nu+1}} \sigma = \int_{q_\nu}^{q_{\nu+1}} dz_j \quad (1 \leq \nu \leq 7),$
- $\frac{1}{2\pi\sqrt{-1}} \int_{C_{6,7,8}} \sigma = \int_{q_0}^{q_6} dz_j + \int_{q_0}^{q_7} dz_j + \int_{q_0}^{q_8} dz_j,$
- $\frac{1}{2\pi\sqrt{-1}} \int_{C'_{\nu,\nu+1}} \sigma = \int_{q'_\nu}^{q'_{\nu+1}} dz'_j \quad (1 \leq \nu \leq 7),$
- $\frac{1}{2\pi\sqrt{-1}} \int_{C'_{6,7,8}} \sigma = \int_{q'_0}^{q'_6} dz'_j + \int_{q'_0}^{q'_7} dz'_j + \int_{q'_0}^{q'_8} dz'_j.$

In what follows, we simply denote $\int_{q_\nu}^{q_{\nu+1}} dz_j$ by “ $q_{\nu+1} - q_\nu$ ”, for example.

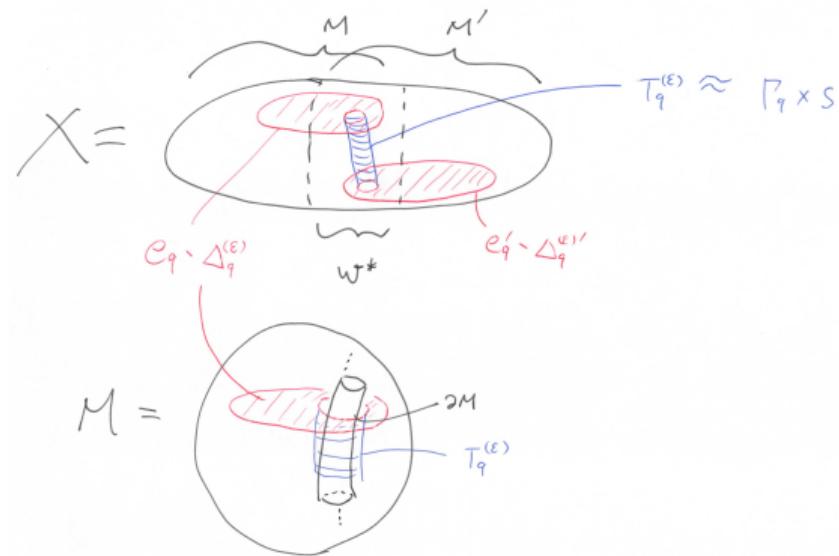
B_α is defined as follows by using the fact that $\pi_1(M) = \pi_1(M') = 0$.



Similarly, we define B_β .

At this moment, we do not know how to calculate the integrals $\int_{B_\alpha} \sigma$ and $\int_{B_\beta} \sigma$.

B_γ is defined by $B_\gamma := (e_9 \setminus \Delta_9^{(\varepsilon)}) \cup T_9^{(\varepsilon)} \cup (e'_9 \setminus \Delta_9^{(\varepsilon)'}).$



By the same argument as in the C_\bullet case, we have that

$$\frac{1}{2\pi\sqrt{-1}} \int_{B_\gamma} \sigma = \int_{g(q_9)}^{q'_9} dz'_j \quad (= "p'_9 - g(p_9)".)$$

lattice	cycle	$\frac{1}{2\pi\sqrt{-1}} \int \sigma$	corresponding parameter
U	$A_{\beta,\gamma}$	τ	choice of C_0 (and C'_0)
	B_α	???	choice of w_j 's (R, R', \dots)
U	$A_{\gamma,\alpha}$	1	—
	B_β	???	choice of w_j 's (R, R', \dots)
$E_8(-1)$	$C_{1,2}$	" $p_2 - p_1$ " in C	choice of $p_2 - p_1$
	$C_{2,3}$	" $p_3 - p_2$ " in C	choice of $p_3 - p_2$
	\vdots	\vdots	\vdots
	$C_{7,8}$	" $p_8 - p_7$ " in C	choice of $p_8 - p_7$
	$C_{6,7,8}$	" $p_6 + p_7 + p_8$ " in C	choice of $p_6 + p_7 + p_8$
	$C'_{1,2}$	" $p'_2 - p'_1$ " in C'	choice of $p'_2 - p'_1$
	$C'_{2,3}$	" $p'_3 - p'_2$ " in C'	choice of $p'_3 - p'_2$
	\vdots	\vdots	\vdots
$E_8(-1)$	$C'_{7,8}$	" $p'_8 - p'_7$ " in C'	choice of $p'_8 - p'_7$
	$C'_{6,7,8}$	" $p'_6 + p'_7 + p'_8$ " in C	choice of $p'_6 + p'_7 + p'_8$
	$A_{\alpha,\beta}$	$a_\beta - \tau \cdot a_\alpha$	choice of p_9 and p'_9 (i.e. $N_{C/S}$ and $N_{C'/S'}$)
U	B_γ	" $p'_9 - g(p_9)$ "	choice of $g: C \cong C'$

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- Fix a Diophantine pair $(p, q) \in \mathbb{R}^2$.
(i.e. $-\log \text{dist}((np, nq), \mathbb{Z}^2) = O(\log n)$ as $n \rightarrow \infty$)
- We first consider the subspace $\Xi_{(p,q)}$ of the period domain $\mathcal{D}_{\text{period}}$ which is defined by considering all the K3 surfaces constructed by our method with

$$N_{C/S} \mapsto [p + q\tau] \in \mathbb{C}/\langle 1, \tau \rangle$$

by the isomorphism $\text{Pic}^0(C) \cong \mathbb{C}/\langle 1, \tau \rangle$ (τ moves).

- $\dim \Xi_{(p,q)} = 19$, with coordinates system

$$(\tau, p_1, p_2, \dots, p_8, p'_1, p'_2, \dots, p'_8, s, x),$$

where (s, x) are the parameter for the gluing.

- s is defined by changing $g: C \rightarrow C'$ with the composition $g \circ P_s$, where P_s is the automorphism of C defined by “ $z \mapsto z + s$ ”.

Lemma

$$\Xi_{(p,q)} \subset v_{(p,q)}^\perp, \text{ where } v_{(p,q)} := A_{\alpha\beta} + p \cdot A_{\beta\gamma} - q \cdot A_{\gamma\alpha}$$

lattice	cycle	$\frac{1}{2\pi\sqrt{-1}} \int \sigma$	corresponding parameter
U	$A_{\beta,\gamma}$	τ	choice of C_0 (and C'_0)
	B_α	$??? =: x - 2\tau$	choice of w_j 's (R, R', \dots)
$E_8(-1)$	$A_{\gamma,\alpha}$	1	—
	B_β	$??? =: y - 2$	choice of w_j 's (R, R', \dots)
$E_8(-1)$	$C_{1,2}$	" $p_2 - p_1$ " in C	choice of $p_2 - p_1$
	$C_{2,3}$	" $p_3 - p_2$ " in C	choice of $p_3 - p_2$
	\vdots	\vdots	\vdots
	$C_{7,8}$	" $p_8 - p_7$ " in C	choice of $p_8 - p_7$
	$C_{6,7,8}$	" $p_6 + p_7 + p_8$ " in C	choice of $p_6 + p_7 + p_8$
	$C'_{1,2}$	" $p'_2 - p'_1$ " in C'	choice of $p'_2 - p'_1$
	$C'_{2,3}$	" $p'_3 - p'_2$ " in C'	choice of $p'_3 - p'_2$
	\vdots	\vdots	\vdots
U	$A_{\alpha,\beta}$	$a_\beta - \tau \cdot a_\alpha$	choice of p_9 and p'_9 (i.e. $N_{C/S}$ and $N_{C'/S'}$)
	B_γ	$"p'_9 - g(p_9)" =: s$	choice of $g: C \cong C'$

Set

- $x := 2\tau + \frac{1}{2\pi\sqrt{-1}} \int_{B_\alpha} \sigma$ (a coordinate)
- $y := 2 + \frac{1}{2\pi\sqrt{-1}} \int_{B_\beta} \sigma$ ("dummy")

The relation between x and y can be obtained by the relation

$$0 = (\sigma, \sigma) = 2\tau x + 2y + (\exists \text{quadric function of } \tau, q - p\tau, p_1 - p_2, \dots)$$

The (normalized) volume is:

$$\text{vol}(\sigma) := (\sigma, \bar{\sigma}) = 2\text{Re}(\bar{\tau}x) + 2\text{Re}(y) + (\exists \text{quadric function of } \tau, q - p\tau, p_1 - p_2, \dots)$$

- $\text{vol}(\sigma)$ depends linearly on the coordinate x if one fix the other 18 coordinates.

Via the Poincare duality,

$$\begin{aligned}
 \frac{1}{2\pi\sqrt{-1}}\sigma = & (2(q-p\tau) + s)A_{\alpha\beta} + xA_{\beta\gamma} + yA_{\gamma\alpha} \\
 & + \tau B_\alpha + B_\beta + (q-p\tau)B_\gamma \\
 & + \sum_{j=1}^7 \exists c_{j,j+1} C_{j,j+1} + \exists c_{6,7,8} C_{6,7,8} \\
 & + \sum_{j=1}^7 \exists c'_{j,j+1} C'_{j,j+1} + \exists c'_{6,7,8} C'_{6,7,8}
 \end{aligned}$$

c_\bullet 's depends only on (p_1, p_2, \dots, p_8) , and c'_\bullet 's depends only on $(p'_1, p'_2, \dots, p'_8)$.

Theorem (K-, T. Uehara)

$\exists \widehat{V}_{(p,q)} \geq 0$ depending only on (p, q) such that,

$$\begin{aligned} \forall \quad \widehat{\sigma} = & \quad a_{\alpha\beta}A_{\alpha\beta} + a_{\beta\gamma}A_{\beta\gamma} + a_{\gamma\alpha}A_{\gamma\alpha} + b_\alpha B_\alpha + b_\beta B_\beta + b_\gamma B_\gamma \\ & + \sum_{j=1}^7 c_{j,j+1}C_{j,j+1} + c_{6,7,8}C_{6,7,8} + \sum_{j=1}^7 c'_{j,j+1}C'_{j,j+1} + c'_{6,7,8}C'_{6,7,8} \in \mathcal{D}_{\text{Period}}, \end{aligned}$$

it holds that $\widehat{\sigma} \in \Xi_{(p,q)}$ iff the following holds:

- $b_\beta \neq 0$ (set $b_\beta = 1$ by "normalizing"),
- $\text{Im } b_\alpha \neq 0$,
- The normalized volume $\text{vol}(\widehat{\sigma})$ is larger than $\widehat{V}_{(p,q)}$.

Remark

$\widehat{V}_{(p,q)} = V_{(p,q)} + V'_{(p,q)}$, where

- $V_{(p,q)} = \text{vol}_\eta(S \setminus (\text{"the maximal hol. tub. n.b.h.d." of } C))$,
- $V'_{(p,q)} = \text{vol}_{\eta'}(S' \setminus (\text{"the maximal hol. tub. n.b.h.d." of } C'))$

Remark

It is observed by a standard argument that a general member of $\Xi_{(p,q)}$ corresponds to a K3 surface X with Picard number = 0, which means that X is non-Kummer and non-projective.

Question

- How large can one take a hol. tub. n.b.h.d. of C in S ?
- Is $V_{(p,q)}$ equal to zero? or > 0 ?
- How does $V_{(p,q)}$ depend on the Diophantine pair (p,q) ?

To investigate a Kähler-geometric approach to this kinds of problems, now I'm also interested in the following:

Question

How is the relation between a Ricci-Flat Kähler metric on X and some “canonical metric” on S or $S \setminus C$ (complete Ricci-Flat Kähler)?