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2018/12/14(金) 14:00〜15:00

@1号館A棟2階 A1.

Arnol'd's type thm on a nbhd of a cycle of rat'l curves.

Goal of this talk:

- To give a "rigidity"-type result (uniformization) for the cpx str. of a (small) nbhd of a curve $\underbrace{C}_{}\hookrightarrow \underbrace{\text{surf.}}_{S}$

when C : a $\overset{\text{cycle of}}{\text{rat'l curves}}$

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$= \gamma, \chi, A, \cdots$.

§1. previous results.
§2. Main result
§3. Outline of the prf

§1. $\boxed{C: \text{cpt cpx curve (reduced)}, \overset{\text{hol'ly}}{\hookrightarrow} S: \text{non-sing. cpx surf.}}$

Thm 1 (Grauert '62)

$(C^2) := \deg N_{C/S} < 0 \Rightarrow C$ can be "contracted".

$\underbrace{(C^2)}_{C:\text{sm,}} < \min\{0, 4-4\underbrace{g(C)}_{\text{genus}}\} \Rightarrow C$ admits a hol. tub. nbhd.

($^\exists V$: nbhd of C in S, $\exists V'$: nbhd of the zero-sect'n in $N_{C/S}$ s.t. $V \cong V'$, $V \overset{\text{bihol}}{\cup} \hookrightarrow$ zero-sect'n.)

Thm 2 (Arnol'd '76)

C: sm. ellipt. curve s.t. $N_{C/S} \in \mathrm{Pic}^0(C)$: Dioph ($\exists A>0, \exists\alpha>0$ s.t. $\mathrm{dist}_{Euc}(\mathbb{1}_C, N^{\otimes n}) \geq A\cdot n^{-\alpha}$ for $n \geq 1$.)

$\Rightarrow C$ admits a hol. tub. nbhd //

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$\nexists$ such V in general (ellipt. fibr, Serre's e.g. Ueda's e.g. ...)

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{ "generalizatn"

Thm 3 (Ueda '83)

C: cpt, non-sing, $(C^2)=0$, $N_{C/S} \in \mathrm{Pic}^0(C)$: torsion or Dioph.

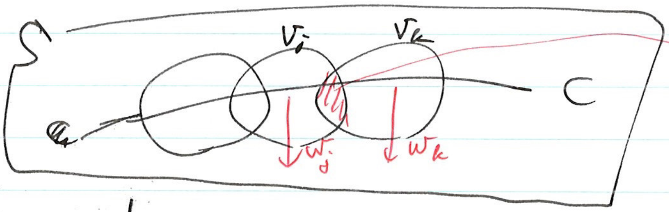
Assume $\forall n \geq 1$, "$u_n(C,S)$" $= 0 \in H^1(C, N_{C/S}^{-n})$ holds.

Ueda's obstruction class

Then (*) $\exists \{V_j\}$: open cov. of a nbhd of C.

$\exists w_j : V_j \to \mathbb{C}$: hol. def. func. of $V_j \cap C$

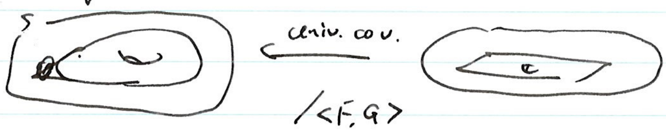
s.t. $\exists t_{jk} \in U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$, $w_j = t_{jk} \cdot w_k$ on V_{jk}. //



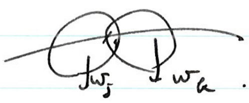
$t_{jk} \cdot w_k = w_j + \cdots\cdots$

(c.f. top. triv. hol. l.b / cpt Kähler $\Rightarrow$ $U(1)$-flat.)

c.f. Siegel's linearization thm, ('42). $f(w) = \lambda \cdot w + h.o.t.$

① 

, "linearizatn" of F and G $\rightsquigarrow$ Thm 2.

② 

$w_k = t_{jk}^{-1} w_j + h.o.t.$ $\overset{\text{linealize}}{\rightsquigarrow}$ Thm 3.

※ [Ueda '91], [K-'17 Indiana] : singular analogue of Thm 3.

for e.g., (*) holds when $\left\{ \begin{array}{l} C : \text{cycle of rat'l curves} \\ N_{C/S} \in U(1) \subset \mathbb{C}^* \cong \mathrm{Pic}^0(C) \\ \quad : \text{Dioph.} \\ (\text{c.f. } H^1(C, N_{C/S}^{-n}) = 0 \text{ in this case.}) \end{array} \right.$

$\boxed{\text{Thm 4}}$ $\mathcal{O}_S(C)|_C$.

§2. Main result

Thm 5 (K-'18) ← [Improvement of Thm 4. Sing. analogue of Thm 2]

(C, S) : as above, C: a cycle of rat'l curves.
(C', S') : ——— // ——— .

s.t. $C = C'$, $\quad$ ($t: \mathrm{Pic}^0(C) \xrightarrow{\cong} \mathbb{C}^*$)
$N_{C/S} \cong N_{C'/S'}$,
$\exists \theta \in \mathbb{R} \setminus \mathbb{Q}$: Dioph. s.t. $t(N_{C/S}) = t(N_{C'/S'}) = e^{2\pi\sqrt{-1}\theta}$

Then $\exists V$: a nbhd of C, $\exists V'$: a nbhd of C'

(**) s.t. $V \cong V'$ (bihol)
$\cup \quad \cup$
$C = C'$ //

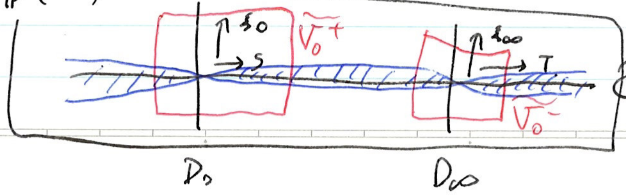
Rmk $\exists$ example of (C, C', S, S') s.t. (**) does not hold
if θ : not Dioph.

E.g. 6 $S = Bl_Z \mathbb{P}^2$ for a suitable nine points $Z = \{P_1, \cdots, P_9\} \subset \gamma \subset \mathbb{P}^2$.

E.g. 7 "standard model of a nbhd C with $t(N_{C/S}) = t$".
$t = e^{2\pi\sqrt{-1}\theta} \in U(1)$, $C = \gamma$ (for simplicity) : given.

$\pi: \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \mathbb{P}^1$
non-homog. coord $s = T^{-1}$.
fiber coord ξ_0 around $\{s=0\} =: D_0$
ξ_∞ — // → $\{s=\infty\} =: D_\infty$.

$\mathcal{O}_{\mathbb{P}^1}(-2)$



$\widetilde{C}$:= zero-sect'n.

$\widetilde{w} := S\cdot\xi_0 = T\cdot\xi_\infty$: glob. $\left(\Leftarrow \mathcal{O}_{\mathbb{P}^1}(-2) \rightsquigarrow \xi_\infty = S^2\cdot\xi_0\right)$.

$\varepsilon > 0$.

$\widetilde{V_0^+} := \{|S| < \varepsilon,\ |\xi_0| < \varepsilon\}$, $\widetilde{V_0^-} := \{|T| < \varepsilon,\ |\xi_\infty| < \varepsilon\}$,

$\widetilde{V_1}$:= a small nbhd of $\widetilde{C} \setminus \{S = 0, \infty\}$.

$\widetilde{V} := \widetilde{V_0^+} \cup \widetilde{V_1} \cup \widetilde{V_0^-}$.

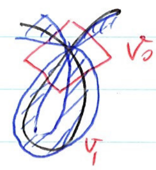
$V := \widetilde{V}/\sim$ $\left(\widetilde{V_0^+} \ni (S, \xi_0) \sim F(S, \xi_0) \in \widetilde{V_0^-},\ F(S,\xi_0) = (\underbrace{t\cdot\xi_0}_{=T}, \underbrace{S}_{=\xi_\infty}) \right)$

$i : \widetilde{V} \to V$: quot.

$C := i(\widetilde{C})$, $(\rightsquigarrow t(N_{C/S}) = t)$. //

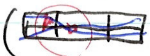
Obs in E.g. 7, Let $V_0 := i(\widetilde{V_0^+}) = i(\widetilde{V_0^-})$

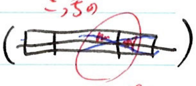
$V_1 := i(\widetilde{V_1})$



$\rightsquigarrow w_0 := S\cdot\xi_0$ on V_0,

$w_1 := S\cdot\xi_0$ on V_1.

$\rightsquigarrow w_1 =$ $1\cdot w_0$ on $V^+ := i($  $)$

$t\cdot w_0$ on $V^- := i($  $)$

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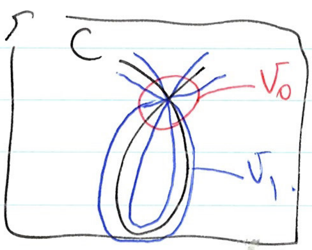
§3. Outline of the pf of Thm 5.

(C, S) : as before, $C =$  for simplicity. $t(N_{C/S}) = e^{2\pi\sqrt{-1}\theta}$ (θ: Dioph.)

$\rightsquigarrow$ construct a nbhd as in E.g. 7 !.



$U_0 := V_0 \cap C$, $U_1 := V_1 \cap C$,

$U_0 \cap U_1 = U^+ \cup U^-$,

$V_0 \cap V_1 = \hat{V}^+ \cup \hat{V}^-$.

$\widetilde{V_0^\pm}$: 2-copies of V_0, $\widetilde{V_1} := V_1$.

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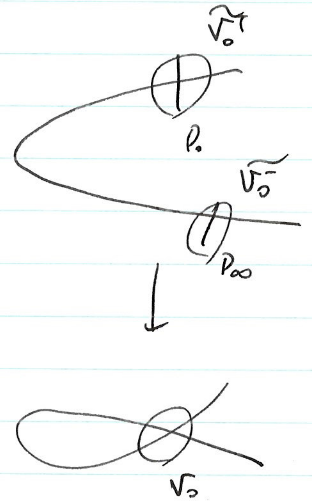
$\tilde{V} := \tilde{V}_0^+ \sqcup \tilde{V}_1 \sqcup \tilde{V}_0^- / \sim$ ← gen'd by

$V^+ \hookrightarrow \tilde{V}_0^+$, $V^+ \hookrightarrow \tilde{V}_1$, $V^- \hookrightarrow \tilde{V}_1$, $V^- \hookrightarrow \tilde{V}_0^-$.

$i : \tilde{V} \longrightarrow V$: natural.
$\cap$
(univ. cov. of V).



$\rightsquigarrow$ $i^* C$ (as div.) $= \underbrace{\tilde{C}}_{\mathbb{P}^1} + \underset{\tilde{V}_0^+}{\underset{\cap}{D_0}} + \underset{\tilde{V}_0^-}{\underset{\cap}{D_\infty}}$.

<u>Simple calculation</u> $\rightsquigarrow$ $(\tilde{C}^2) = -2 < \min\{0, 4-4g(\tilde{C})\}$.

$\xRightarrow{[\text{Grauert '62}]}$ $\tilde{V}$ can be embedded in $\mathcal{O}_{\mathbb{P}^1}(-2)$.

$\tilde{V} \cup \tilde{C} \xrightarrow{\cong}$ zero-sect'n $\subset \mathcal{O}_{\mathbb{P}^1}(-2)$

① $S = T^{-1}$: non-homog. coord of $\tilde{C}$

s.t. $\begin{cases} D_0 \cap \tilde{C} = \{S = 0\} \\ D_\infty \cap \tilde{C} = \{S = \infty\} \end{cases}$

② [Ohsawa '84] $\rightsquigarrow$ One can extend $S = T^{-1}$ to $\tilde{V}$ as mero. funcs (by shrinking if ness.)

s.t. $D_0 = \{S = 0\}$, $D_\infty = \{T = 0\}$

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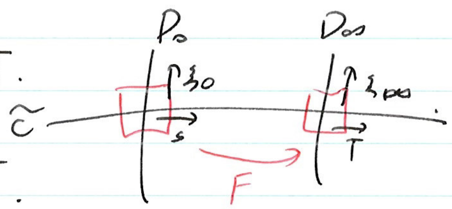
Thm 4 $\Longrightarrow$ $\left.\begin{array}{l} \exists\, w_0 : V_0 \to \mathbb{C} \\ \exists\, w_1 : V_1 \to \mathbb{C} \end{array}\right)$: local def. functions on $V_\bullet$'s, of C.

s.t. $w_1 = \begin{cases} {}^{\exists}t_+ \cdot w_0 & \text{on } V^+ \\ {}^{\exists}t_- \cdot w_1 & \text{on } V^- \end{cases}$ ($t_\pm \in U(1)$)

$$\tilde{w} := \begin{cases} t_+ w_0 & \text{on } \tilde{V}_0^+ \\ w_1 & \tilde{V}_1 \\ t_- \cdot w_0 & \tilde{V}_0^- \end{cases} \rightsquigarrow \tilde{w} : \tilde{V} \to \mathbb{C} : \text{glob. def. func. of } i^*C = \tilde{C} + D_0 + D_\infty.$$

$\xi_0 := \tilde{w}/s$, $\xi_\infty := \tilde{w}/T$.



Then (by a simple obs,) "deck transf." F of $i : \tilde{V} \to V$ can be written as

$$F(s, \xi_0) = \left(\frac{t \cdot \xi_0}{g(s,\xi_0)},\ g(s,\xi_0)\cdot s \right)$$

for ${}^{\exists} g : \tilde{V}_0^+ \to \mathbb{C}^*$ with $g(0,0) = 1$. ($\tilde{w} \rightsquigarrow \tilde{w}/g(0,0)$)

Obs $H : \tilde{V} \to \mathbb{C}^*$.

$$\left[\begin{array}{l} s \rightsquigarrow \hat{s} = H^{-1}\cdot s \\ T \rightsquigarrow \hat{T} = H \cdot T \end{array}\right] \rightsquigarrow \text{“}\hat{g}\text{''} = g \cdot \frac{H}{F^* H}.$$

$\to$ enough to show:

~~Thm~~ Prop 8 $\varinjlim_{V^* : \text{nbhd of } C \text{ in } V} H^1(V^*, \mathcal{O}_{V^*}) \xrightarrow[\text{restr.}]{} H^1(C, \mathcal{O}_C)$: inj.

or ~~equivalently~~, $\varinjlim H^1(V^*, \mathcal{O}_{V^*}(-C)) = 0$ //

Shown by Siegel's linearization technique "