2018/12/10 (月).

13:30 − 14:30 (1h)

On the nbhd of a torus leaf and dynamics of hol. foliations.

j.w./ N. Ogawa.

(C [K-Ogawa, ArXiv : 1808.10219])

Set-up X : cpx surf, $\cup$ Y : cpt curve (smooth) with $c_1(N_{Y/X}) = 0$.

[T. Ueda '83] : cpx analytical classification of a nbhd of Y in X.

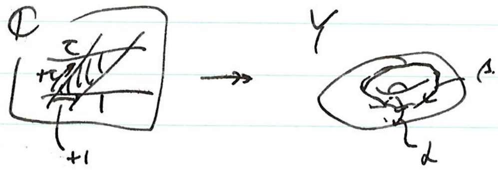
Today, we additionally assume the existence of a hol. (sm.) foliation $\mathcal{F}$ by curves on a nbhd V of Y in X.

s.t. Y is a leaf of $\mathcal{F}$.

⤳ **Q** holonomy of $\mathcal{F}$ alg Y $(\mathrm{Hol}_{\mathcal{F},Y})$ versus v.s. Ueda's classification?

(cpx dynamical classification of) $\pi_1(Y,*) \longrightarrow \mathcal{O}^*_{\mathbb{C},0}$.

We will investigate this Question when $Y = \mathbb{C}/\langle 1, \tau\rangle$: ellipt. curve $(\tau \in \mathbb{H})$



$$\left.\begin{aligned} f &:= \mathrm{Hol}_{\mathcal{F},Y}(\alpha) \\ g &:= \mathrm{Hol}_{\mathcal{F},Y}(\beta) \end{aligned}\right\} \in \mathcal{O}^*_{\mathbb{C},0}.$$

$$f(w) = \lambda \cdot w + O(w^2), \quad g(w) = \mu \cdot w + O(w^2),$$

$f \circ g = g \circ f$ on a nbhd of $\{w=0\} \subset \mathbb{C}$.

One conclusion of our main result is, for e.g when $g = \mathrm{id}$;

Thm 1 Assume that $g(w) = w$. and $|\lambda| = 1$.
Then the followings are equivalent:

(i) f : linearizable around $\{w=0\} \subset \mathbb{C}$.

(ii) $\exists V$: a nbhd of Y, $\exists \Phi : V \to \mathbb{R} \cup \{-\infty\}$: psh s.t. $dd^c\Phi = \int_Y$ ".

(iii) $L := \mathcal{O}_X(Y)$: admits a C^∞ Herm. metric h with $\sqrt{-1}\,\Theta_h \geq 0$. //

§1. Background (Ueda's ~~theory~~ classification).
§2. Outline of the prf.
~~§3~~.

$U(1) := \{ t \in \mathbb{C} \mid |t| = 1 \}$.

§1

X : cpx non-sing surf., $\supset Y$: cpt curve, non-sing, $c_1(N_{Y/X}) = 0$.

V : tub. nbhd of Y. $\rightsquigarrow$ $H^1(V, U(1)) \cong H^1(Y, U(1))$

$\tilde{N} \longleftarrow N_{Y/X}$ (Fact)

(flat ext. of $N_{Y/X}$)

"Ueda type".

type$(Y, X) := \max_{\text{(~~sup~~)}} \{ n \in \mathbb{Z}_{\geq 1} \cup \{\infty\} \mid \forall \nu \in \{0, 1, \cdots, n-1\},\ \mathcal{O}_X(Y) \cong \tilde{N}$ formally along Y in ν-jet $\}$.

i.e. $\mathcal{O}_V(Y) \otimes \mathcal{O}_V / I_Y^{\nu+1} \cong \tilde{N} \otimes \mathcal{O}_V / I_Y^{\nu+1}$

Def (Y, X) : of type (α) $\overset{\text{def}}{\iff}$ type$(Y, X) < \infty$.

(Y, X) : of type (β) $\overset{\text{def}}{\iff}$ $\exists V$: tub. nbhd of Y s.t. $\tilde{N} \cong \mathcal{O}_V(Y)$. ($\Rightarrow$ type $= \infty$)

(γ) $\overset{\text{def}}{\iff}$ type $= \infty$, however is not of type (β)

e.g. ① C : ellipt. curve $\quad 0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0$: ex, non-splitting. (E : rank 2 vect. bdl $/C$)

$X := \mathbb{P}(E)$
$\cup$
$Y := \mathbb{P}(\mathcal{O}_C)$ (Serre's example) $\rightsquigarrow$ type$(Y, X) = 1 < \infty$.

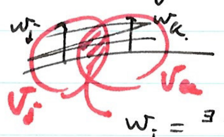
② $L \longrightarrow C$: flat line bdl. (C : cpt curve) $\rightsquigarrow$ $X := L \supset Y :=$ the zero sect'n.
$\rightsquigarrow$ (Y, X) : of type (β).

Thm (Ueda '83). $\exists$ example of (Y, X) : type (γ)

this example admits Z as in §0. with $\begin{cases} \partial = \mathrm{id}. \\ |\lambda| = 1. \end{cases}$

~~Motivation~~ : §2. Outline of the prf

@ (i) $\Longrightarrow$ (Y, X) : of type (β) (by considering the foliation chart)



$w_j = {}^{\exists} t_{jk} \cdot w_k$ on V_{jk}.
$t_{jk} \in U(1)$.

$\Longrightarrow$ (ii) $\left(\Phi := \frac{\sqrt{-1}}{2\pi} \log |w_j|^2 \right)$.

@ (Y, X) : of type (β) $\Longrightarrow$ (iii) (cf. [Brunella '10], by [K-'15])

($h :=$ (reg. of) $\min \{$ flat metric of $\mathcal{O}_X(Y)|_V$, ψ $\}$. Bergman type singular Herm metric of $\mathcal{O}_X(Y)$)

⇝ Enough to show ① (ii) $\Rightarrow$ (i)
② (iii) $\Rightarrow$ (i)

① … Assume that f : not linearizable $(\neg$(i)$)$ $\left[f = \lambda \cdot w + O(w^2) \right]$
and that (ii) ${}^{\exists} V$: nbhd of Y. ${}^{\exists} \Phi : V \to \mathbb{R} \cup \{-\infty\}$: psh s.t. $dd^c \Phi =$ "δ_Y".
(prf. by contradiction). we will ~~use~~ apply …

[$\lambda \in U(1)$: tor $\underset{\text{simply}}{\Longrightarrow}$ (Y, X) : of type (β) $\underset{\text{[Ueda '83]}}{\Longrightarrow}$ $\nexists$ such Φ.]

Thm (Pérez-Marco '97, …)

$f(w) = \lambda \cdot w + O(w^2)$, $\lambda \in U(1)$: non-torsion, f : not lin'ble.
$U \subset \mathbb{C}$: open nbhd of 0, with $\begin{cases} \partial U : \text{Jordan curve} \\ f : \text{univalent on } \overline{U}. \end{cases}$

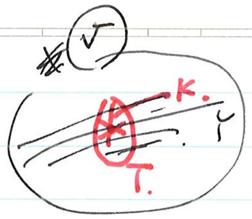
$\Longrightarrow$
@ ${}^{\exists} K \subset \overline{U}$: cpt. conn. sub. "Hedgehog" s.t. $\begin{cases} \mathbb{C} \setminus K : \text{conn}, \quad 0 \in K, \\ K \neq \{0\}, \quad K \cap \partial U \neq \emptyset, \quad f(K) = f^{-1}(K) = K. \end{cases}$

@ for almost every $w \in K$ in the harmonic measure,
$\{ f^n(w) \mid n \in \mathbb{Z} \} \subset K$: dense.

Take a (suff. small) transversal T of $\mathcal{F}$

$\cong$ disc $\subset \mathbb{C}$.

~~regard~~ identify it ~~as~~ with "U" and use apply Pérez-Marco's thm.



$\to \exists K \subset \overline{T}$: Hedgehog of f.

Take $w_0 \in K (\subset T \subset X)$. s.t. $\overline{\{f^n(w_0) \mid n\in\mathbb{Z}\}} = K \subset \overline{T}$.

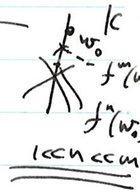
$\rightsquigarrow \exists! \mathcal{L}$: a leaf of $\mathcal{F}$ s.t. $\{f^n(w_0)\mid n\in\mathbb{Z}\} = \mathcal{L}\cap T$.

$\mathcal{L} \neq Y$

$i_{\mathcal{L}} : \mathbb{C}^* \cong \mathcal{L} \hookrightarrow V$, $\underbrace{(-i_{\mathcal{L}}^*\Phi)}_{\Phi_{\mathcal{L}}} : \mathbb{C}^* \longrightarrow \mathbb{R}$: (sub) pluriharmonic.

Then it is easily observed that

$\exists \gamma_1, \gamma_2, \gamma_3 \subset \mathbb{C}^*$: curves s.t.

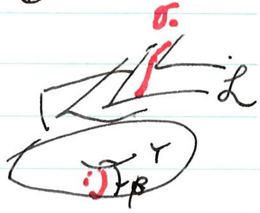


s.t. ~~max $\Phi_{\mathcal{L}}$ is attained by a point "around" γ_2.~~

$\max_{\gamma_1 \cup \gamma_3} \Phi_{\mathcal{L}} < \Phi_{\mathcal{L}}(p)$

p: close to γ_2.

$\to$ contradicts to the Maximal principle.



② ...((iii) $\Rightarrow$ (i))

Assume $\neg$(i). $\rightsquigarrow$ type$(Y, X) = \begin{cases} \exists n < \infty \ (\alpha) & \text{if } \lambda \in U(1): \text{torsion.} \\ \infty & \text{if } \lambda \in U(1): \text{non-torsion} \end{cases}$

Fact([K-'14]) : (Y, X): of type(α) $\Rightarrow$ $\mathcal{O}_X(Y)$: not semi-positive (i.e. $\neg$((iii))

If type$(Y, X) = \infty$, λ: non-torsion. $\rightsquigarrow \exists \mathcal{L}$ as above.

"$\neg$(iii)" is shown by considering $(-\log|f_Y|_h^2)|_{\mathcal{L}}$: $\left(\begin{array}{l} f_Y \in H^0(X, \mathcal{O}_X(Y)) \text{ : can. section} \\ h\text{: as in (iii)} \end{array} \right)$