On a nbhd of a torus leaf of a certain class of hol. foliations on cpx surfaces

X : cpx mfd. (surf)
$\cup$
Y : cpt cpx submfd s.t. $N_{Y/X}$: "flat". (curve) (i.e. $\deg N_{Y/X} = 0$) $\Leftrightarrow$ top. triv.

Q What kind of <u>cpx analytic property</u> does a tub. nbhd of Y have?
(or $\{$p.s.h. functions on a tub. nbhd of $Y\}$?)

Q1. $\exists$? hol. foliation $\mathcal{F}$ on a tub. nbhd of Y s.t. $Y \in \mathcal{F}$?

today's topic → **Q** <u>when</u> $\exists$? $\mathcal{F}$ —//— s.t. $\begin{cases} Y \in \mathcal{F} \\ \text{and } \cancel{\mathrm{Hol}_{\mathcal{F},Y} : \pi_1(Y,*) \to U(1)} \\ \mathcal{F} \text{ has (lin.) holonomy?} \end{cases}$

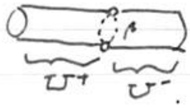
Q <u>when</u> $\exists$? a tub. nbhd of Y which is bihol. to a tub. nbhd of the 0-sect'n in $N_{Y/X}$?

Example 1

C: sm. ellipt. curve.

$f \in \mathcal{O}_{\mathbb{C},0}$: a germ of hol. func s.t. $\begin{cases} f(0) = 0 \\ |f'(0)| = 1. \end{cases}$

$U_1 := C \setminus$ "β."
$U_2 :=$ "β-nbhd" $\rightsquigarrow$ $C = U_1 \cup U_2$.

$U_1 \cap U_2 =$ 

disc $\subset \mathbb{C}$

$$X = X_{(C,f)} := \underset{(z,w)}{U_1 \times \Delta} \cup \underset{(z,w)}{U_2 \times \Delta} \Big/ \sim .$$

$$\left(\begin{array}{l} (z,w) \sim (z,w) \text{ on } U^+ \\ (z,w) \sim (z, f(w)) \text{ on } U^- \end{array} \right).$$

$\rightsquigarrow$ ◎ $U_j \times \Delta \xrightarrow{Pr_1} U_j$ glue up to define $\pi : X \twoheadrightarrow C$.

◎ $\mathcal{F} :=$ "$\{w = \text{const}\}$" ; hol. fol. of X

◎ $Y :=$ "$\{w = 0\}$" $\subset X$: sub mfd, <u>$Y \cong C$</u>,

$N_{Y/X} \leftrightarrow \rho_{N_{Y/X}} : \pi_1(\underset{Y}{\overset{\shortparallel}{C}}, *) \to U(1)$
$\alpha \longmapsto f'(0)$
$\beta \longmapsto 1$.

◎ $\mathrm{Hol}_{\mathcal{F},Y} : \pi_1(Y,*) \to \mathcal{O}^*_{\mathbb{C},0}$
$\alpha \longmapsto f$
$\beta \longmapsto (\mathrm{id})$

//

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Obs f : lin'ble (i.e. $\exists g \in \mathcal{O}_{\mathbb{C},0}$ s.t. $g^{-1} \circ f \circ g(\xi) = \lambda \cdot \xi$ $(\lambda := f'(0))$)

$\Rightarrow (Y, X)$: "type β" ($\underset{\text{def}}{\Leftrightarrow}$ $\exists \mathcal{F}$: hol. foliation on X with $Y \in \mathcal{F}$ s.t. $\pi_1(Y, *) \xrightarrow[\mathrm{Hol}_{\mathcal{F}, Y}]{} U(1)$: linear)

$\Updownarrow$

(Y has a sys. of a psd-flat nbhds) the hol. l.b. $[Y]$: flat on a nbhd of Y

$\Leftarrow$?

(*)

Main result :

Thm 2 $(C, (\alpha, \beta))$, f, $X = X_{(C,f)}$, Y : as in Example 1, $\lambda := f'(0)$; $\boxed{\text{non-torsion } \in U(1)}$

Then (Y, X) ; type β $\Leftrightarrow$ f : lin'ble //

[§1. Motivation (from Ueda theory)
§2. Outline of prf of Thm 2.

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§1

X : cpx surf.
$\cup$
Y : cpt cpx curve, $N_{Y/X}$: flat.

Ueda's classification

(Y, X) : of type (α) $\underset{\text{def}}{\Leftrightarrow}$ $\exists n \geq 1$, $[Y] \not\cong \widetilde{N_{Y/X}}$ in n-jet along Y.
$\left(\begin{matrix} H^1(Y, U(1)) \cong H^1(\text{tub. nbhd}, U(1)) \\ N_{Y/X} \longmapsto \widetilde{N}. \end{matrix} \right)$

of type (β) $\underset{\text{def}}{\Leftrightarrow}$ (*) holds. "$[Y] \cong_{\text{formally}} \widetilde{N}$"

of type (γ) $\underset{\text{def}}{\Leftrightarrow}$ $\forall n$, $[Y] \cong \widetilde{N_{Y/X}}$ in n-jet along Y, however (*) does not hold //

Thm (Ueda '83)

(◎ (Y, X) : of type (α) $\Rightarrow$ Y admits a sys. of str. psdconvex nbhds.)

◎ $[Y] \cong_{\text{formally}} \widetilde{N}$ and $N_{Y/X}$: torsion in $H^1(Y, U(1))$ $\Rightarrow$ (Y, X) : type (β).

◎ $[Y] \cong_{\text{formally}} \widetilde{N}$ and $N_{Y/X}$: "Diophantine" $\Rightarrow$ — " — //
$(-\log d(\mathbb{1}, N_{Y/X}^n) = O(\log n))$

	$N_{Y/X}$: torsion	non-torsion
$[Y] \not\cong_{\text{formally}} \widetilde{N}$	(α)	
$[Y] \cong_{\text{formally}} \widetilde{N}$	(β)	(β) or (γ)

Obs $X = X_{(C,f)} \supset Y$: as in Example 1.

$\rightsquigarrow$ ① $N_{Y/X}$: torsion $\iff \lambda := f'(0)$: torsion $\in U(1)$.

① $N_{Y/X}$: Dioph $\iff \lambda$: Dioph. $\in U(1)$

$\Downarrow$ Siegel's linearization thm

f : lin'ble ($\Rightarrow$ type (β))

◎ $\lambda \in U(1)$: non-torsion $\Rightarrow f$: "formally lin'ble"

$\Rightarrow [Y] \cong_{\text{formally}} \widetilde{N}$ ($\Rightarrow$ type (β) or (γ))

Q $\exists$? Criteria for "(Y,X) : type (β) / (γ)" ?

Rmk Ueda's example of type (γ)-pair. "強 Cremer cond".

... Take ◎ $\lambda \in U(1)$, s.t. $\exists A > 1$, $\exists d \geq 2$, $\liminf_{\ell \to \infty} A^{\ell} \cdot |1 - \lambda^{\ell}|^{\frac{1}{d^{\ell - 1}}} = 0$.

◎ $f \in \mathcal{O}_{\mathbb{C},0}$: poly of deg $= d$, $f(0) = 0$, $f'(0) = \lambda$.

$\xrightarrow{\text{Cremer-type argument.}}$ $\exists \{m_\nu\} \subset \mathbb{N}$, $m_\nu \nearrow \infty$,

$\exists \{c_{\nu,1}, \cdots, c_{\nu,m_\nu}\} \subset \mathbb{C}$: periodic cycle of f, order $= m_\nu$.

s.t. $\max_{1 \leq k \leq m_\nu} |c_{\nu,k}| \to 0$. as $\nu \to \infty$. ("small cycle")

$\rightsquigarrow \Gamma_\nu :=$ "$\{w \in \{c_{\nu,1}, \cdots, c_{\nu,m_\nu}\}\}$" $\subset X = X_{(C,f)}$

; cpt leaf of $\mathcal{F}$. , $\Gamma_\nu \sim_{\text{homologically}} m_\nu Y$

$\to (C, X)$: of type (γ) (∵ type (α) $\Rightarrow_{\text{Ueda's thm.}}$ $\nexists$ such cpt leaf around C; type (β) $\Rightarrow$ SMR_f : const.)

Thm (K-' ArXiv/1510.02287)

略?

for this Ueda's e.g. of type (γ)-pair (Y, X),

$\forall W$: Y-nbhd $\subset X$, $\forall \varphi : W \to (-\infty, \infty]$: conti,

$\varphi|_{W \setminus C}$: psh (u.s.c + "$\partial\bar{\partial} \geq 0$") $\Rightarrow \varphi$: bdd from above "

$\lambda := f'(0) \in U(1)$: non-torsion //

§2. Outline of prf ← prf of "f : non-lin'ble $\Rightarrow$ type (P)" (Y, X)

Obs ... type$(P) \Rightarrow \exists \psi : X \setminus Y \to \mathbb{R}$: p.h (harmonic)

s.t. $\psi(p) \sim -\log \mathrm{dist}(p, Y)$ as $p \to Y$

⇝ Enough to show;

$\forall W$: Y-nbhd, $\forall \psi : W \to (-\infty, \infty]$: conti.

$\psi|_{W \setminus Y}$: p.(s).h. $\Rightarrow$ ψ : bdd from above //

$\boxed{\text{Main idea}}$ ... Apply Pérez-Marco's theory on "Hedgehog".

Thm 3 (Pérez-Marco '97) (λ : non-tor, f : non-lin'ble)

→ for any suff. small disc or $\Omega \subset \mathbb{C}$, centre $= 0$,

$\exists K = K(\Omega)$: "Hedgehog"

s.t. $\begin{cases} K : \text{cpt conn.} \\ \mathbb{C} \setminus K : \text{conn.} \\ \{0\} \subsetneq K, \\ f(K) = f^{-1}(K) = K, \\ K \cap \partial\Omega \neq \emptyset \end{cases}$

//

Known

⊙ $K(\Omega)$ = the conn. comp. containing 0 of $\{z \in \overline{\Omega} \mid f^n(z) \in \overline{\Omega} \text{ for } \forall n \in \mathbb{Z}\}$.

⊙ $\exists \mu_K$: "harmonic measure" on K. --- supp. $\mu_K = \partial K$, $\mu_K(K) = 1$.

⊙ for μ_K-a.e. $z \in K$,

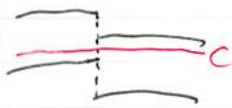
$\partial K \cap \underbrace{\{f^n(z) \mid z \in \mathbb{Z}\}}_{\mathcal{O}(z)}$: dense in ∂K //

Simple argument. $\exists z \in K$, $\exists a \in K \setminus \{0\}$,

s.t. $\{0, a\} \subset$ (the set of all accumulation pts of $\mathcal{O}^+(z) := \{f^n(z) \mid n \geq 1\}$.)

$\boxed{\text{w.m.a}}$

$\Gamma :=$ "$\{w \in \mathcal{O}(z)\}$" $\in \mathcal{A}$. $\Gamma \Subset W$

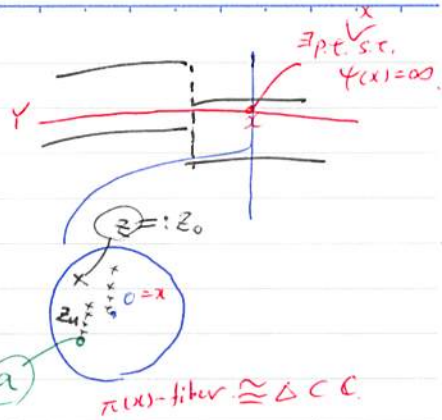


$\cup$

$\Xi :=$ "$\{w \in \mathcal{O}^+(z)\}$"

⇝ $\exists F : \mathbb{C}^* \longrightarrow X$: hol, s.t. $\begin{cases} I_Y(F) = \Gamma, \\ F(\Delta^*) = \Xi. \end{cases}$

$(\Delta^* := \{\zeta \mid 0 < |\zeta| < 1\})$



Assume $\exists\, \psi : W \to (-\infty, \infty]$; $\overset{\text{contl.}}{\text{unbdd}}$,
s.t. $\psi|_{W\setminus Y}$: psh.

Take $x \in Y$ s.t. $\psi(x) = \infty$.

$Z_0 := \{w = z\} \cap \pi^{-1}(\pi(x))$ ← 1点.

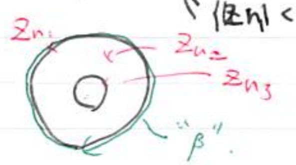
$Z_n := \{w = f^n(z)\} \cap \pi^{-1}(\pi(x))$

⇝ Take appropriate $1 \le n_1 \le n_2 \le n_3$

s.t. $\begin{cases} Z_{n_1}, Z_{n_3} : \text{suff. close to } a \ (= \{w = a\} \cap \pi^{-1}(\pi(x))) \\ Z_{n_2} : \quad 〃 \quad\quad x \ (= Y \cap \pi^{-1}(\pi(x))) \end{cases}$

⇝ $F^*\psi|_{A_{n_1,n_3}} : A_{n_1,n_3} \longrightarrow \mathbb{R} \cup \{+\infty\}$.

$A_{n_1,n_3} = \{|Z_{n_1}| < |\xi| < |Z_{n_3}|\}$



$\begin{cases} F^*\psi|_{\partial A} \sim \psi(a) < \infty. \\ F^*\psi(\underbrace{F^{-1}(z_{n_2})}_{\in\, \mathrm{Int}\, A}) = \psi(z_{n_2}) \sim \psi(x) = \infty. \end{cases}$

$\xrightarrow[\text{maximal principle}]{}$ Contradiction! //

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