X : (proj.) cpx mfd.

$\uparrow$

$L = \mathcal{O}_X(S)$; effective net line bdl $\left(\text{i.e. } \forall C \subset X : \text{curve}, \; S.C = \int_C c_1(\mathcal{O}_X(S)) \geq 0 \right)$

Q When is L semi-positive ?
(i.e. when does L admit a C^∞ Herm. metric with s.p. curv. ?)

Known
- L : s.p. $\Rightarrow$ L : net
- $\not\Rightarrow$

eg. (Serre, Demailly-Peternell-Schneider)
C_0 : sm. ellipe. curve.
$\uparrow$
E : non-triv. ext. of $\mathbb{1}_{C_0}$ (triv. line bdl) by $\mathbb{1}_{C_0}$.
$X := \mathbb{P}(E) \xrightarrow{\pi} C_0$
$\cup$
s : the section $\rightsquigarrow$ $L = \mathcal{O}_X(S)$: eff. net, not s.p.

11.

Our main interest is in...

eg. (P. Ueda, Brunella)
$\mathbb{P}^2 \supset C_0$: sm. ellipe. curve.
$\Downarrow$
$P_1, P_2, \cdots, P_8$: fix. ($P_i \neq P_\ell$ if $i \neq \ell$)
$\xi \in C_0(-[P_1, \cdots, P_8])$
$\rightsquigarrow$ $X_\xi := Bl_{\{P_1, \cdots, P_8, \xi\}} \mathbb{P}^2, \xrightarrow{\pi} \mathbb{P}^2$
$\cup$
$C_\xi := (\pi^{-1})_* C_0$.
$L_\xi := \mathcal{O}_{X_\xi}(C_\xi)$: eff. net, $(= K_{X_\xi}^{-1})$
$\rightsquigarrow N_{C_\xi / X_\xi} \in \mathcal{E}_0(C_\xi) \cup \mathcal{E}_1(C_\xi) \Rightarrow L_\xi$: S.P,
where $\mathcal{E}_0(C_\xi) := \{ F \in \mathrm{Pic}^0(C_\xi) \mid \exists n \geq 1, F^n = \mathbb{1}_{C_\xi} \}$
$\mathcal{E}_1(C_\xi) := \{ \text{---''---} \mid \log d(\mathbb{1}_{C_0}, F^n) = O(\log n) \}$

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Rmk

⊙ $C_0 \cong \mathrm{Pic}^0(C_q)$

$q \longmapsto \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}(-p_1-p_2-\cdots-p_8-q) \cong N_{C_q/X_q}$.

⊙ $\mathrm{Pic}^0(C_0) \supset \mathcal{E}_1(C_0)$: full-measure,

$= \bigcup_{\nu=1}^{\infty} {}^{\exists}K_\nu$ nowhere dense cl.-sub of $\mathrm{Pic}^0(C_0)$

⊙ $N_{C_q/X_q} \in \mathcal{E}_0(C_q) \iff X_q$ admits an ellipt. fibr. str.

In this case, L_q : semi-ample.

⊙ $N_{C_q/X_q} \notin \mathcal{E}_0(C_q) \Rightarrow \dim H^0(X_q, L_q^m) \equiv 1$ for ${}^{\forall}m$ //

Question (Demailly, ...) ∃? $q \in C_0$ s.t. L_q : not s.p.?

Main results

↙ ArXiv 1507.00109.

Thm A (singular analogue of U-B-e.g.)

$C_0 \subset \mathbb{P}^2$; curve of deg $= 3$, with only nodes (for simplicity)

$p_1, \cdots p_8, q \in C_0 \setminus (C_0)_{\mathrm{sing}}$.

X_q, C_q, L_q : as above.

Then (i) $N_{C_q/X_q} \in \mathcal{E}_0(C_q) \cup \mathcal{E}_1(C_q) \Rightarrow L_q$: s.p.

(ii) N_{C_q/X_q} : not Hermitian flat

$\Rightarrow L_q$: not s.p. //

Cor $\exists \{p_1, \cdots, p_8, q\} \subset \mathbb{P}^2$

s.t. $K_{X_q}^{-1}$: not s.p //

(F : Herm. flat $\overset{\text{def}}{\iff}$ F admits a C^∞ Herm. metric with curv. $\equiv 0$

$\iff F \in H^1(C_q, U(1)) \subset H^1(C_q, \mathcal{O}^*_{C_q})$

e.g. $C_q =$ rat'l curve with a node. $\rightsquigarrow$ $U(1) \subset \mathbb{C}^*$).

c.f. $F \to C$ (cpt Kä sm) $\rightsquigarrow$ F : Herm. flat $\iff$ F : top. triv. //

↙ "Toward a higher codim'l. Ueda theory", Math. Z. 281, 3 2015, 967–991.

Thm B (higher dim'l analogue of U-B-e.g.)

$\{p_j\}_{j=1}^{8} \subset \mathbb{P}^3$; general

⇝ Take $Q_0, Q_1 \subset \mathbb{P}^3$: quadric surfaces s.t. $C_0 := Q_0 \pitchfork Q_1$ passes through $\{p_j\}_{j=1}^{8}$.

$X := Bl_{\{p_j\}_{j=1}^{8}} \mathbb{P}^3 \xrightarrow{\pi} \mathbb{P}^3$
$\cup$
$S := (\pi^{-1})_* Q_0$
$\cup$
$C_{(1)} := (\pi^{-1})_* C_0$ ⇝ $N := N_{C/S} = N_{S/X}|_C$
$(\cong \mathcal{O}_{\mathbb{P}^3}(2)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - \cdots - p_8))$

⇝ $N \in \mathcal{E}_0(C) \cup \mathcal{E}_1(C) \Rightarrow K_X^{-1}$: s.p.
$\quad \| $
$\mathcal{O}_X(2S)$ //

c.f. Q (Totaro's question)

$\exists ? (X, L)$ s.t. $\# \{C' \subset X \mid L.C' = 0\} = \# \mathbb{Z}$? (✡)
($\dim X \geq 3$, L nef, C' cpt curve)

↑

Thm (Lesieutre–Otten '14)
$\{p_j\}^8 \subset \mathbb{P}^3$: very general $\Rightarrow (X, K_X^{-1})$ satisfies (✡) //

Cor $\exists (X, L)$ s.t. (✡) holds and L : s.p. //

Outline of the prf. ... run an analogue of "Ueda theory"!

(∗) ~~Outline of the prf of Thm A (1)~~

Ueda theory ; $S^n \subset X^{n+1}$ s.t. $c_1(N_{S/X}) = 0$.
(S: sm. cpt Kä. (or rat'l curve with a node))

⇝ $\text{type}(S, X) := \max \left\{ n \in \mathbb{Z}_{\geq 1} \;\middle|\; \begin{array}{l} 0 \leq {}^\forall \nu < n, \\ \mathcal{O}_V(C) \otimes \mathcal{O}_V / I_C^{\nu+1} \cong \mathcal{O}_V(\widetilde{N_{C/X}}) \otimes \mathcal{O}_V / I_C^{\nu+1} \end{array} \right.$

V: suff. small tub. nbhd of C. I_C: def. ideal of C. $\widetilde{N_{C/X}}$: Herm. flat l.b. / V s.t. $\widetilde{N_{C/X}}|_C = N_{C/X}$.

[Ueda '83] ... a suff. cond. for "$\mathcal{O}_V(C) \cong \widetilde{N_{C/X}}$" holds.

when type$(C,X) = \infty$ ($\Leftarrow$ $\forall n \geq 1$, $H^1(C, N_{C/X}^{-n}) = 0$)

[Ueda '83, '91] ... a suff. cond for the bdd. ness of

$$\Psi : X \setminus C \longrightarrow \mathbb{R}_{\geq 0} \quad : \text{psh.}$$

when type$(C,X) \neq \infty$, C : sm.

or $N_{C/X}$: not Herm flat, C : "rat'l curve with a node"

($\Psi(p) = o(\mathrm{dist}(p,C)^{-a})$ for $\exists a$, $0 < a <$ type(C,X) $\Rightarrow \Psi$: bdd from above) ($\Psi(p) = o((-\log \mathrm{dist}(p,C))^{2a})$ for $\exists a$, $0 < a < 1$ $\Rightarrow$ —"—).

Outline of proof of Thm A (i)

Singular analogue of Ueda theory

$\rightsquigarrow$ $N_{C_\varepsilon/X_\varepsilon} \in \mathcal{E}_0(C_\varepsilon) \cup \mathcal{E}_1(C_\varepsilon)$

$\Rightarrow$ $\exists V_\varepsilon$: C_ε - nbhd

s.t. $\mathcal{O}_{V_\varepsilon}(C_\varepsilon)$ admits a Herm flat metric h_ε.

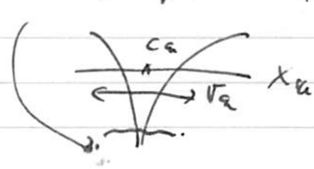
Consider

$$(\text{reg.}) \min \{ M \cdot h_\varepsilon, \; |f_{C_\varepsilon}|^{-2} \}$$

$(M \gg 1)$, f : can. section $\in H^0(X_\varepsilon, \mathcal{O}(C_\varepsilon))$

$\uparrow$

local weight $= (\text{reg}) \max \{ (\text{const}), \log |f_{C_\varepsilon}|^2 \}$.



//

Outline of prf of Thm A (ii)

Assume $\exists h_a$: C^∞ Hem metric on $\mathcal{O}_{X_a}(C_a)$ with semi-positive curvature.

Consider $\psi(p) := -\log |t_{C_a}|^2_{h_a}$

$\overset{\text{locally}}{=}$ (local weight of h_a : psh.) $-$ (harmonic on $X_a \setminus C_a$) $\log |t_{C_a}|^2$.

$\rightsquigarrow \psi : X_a \setminus C_a \to \mathbb{R}$: psh., $= 0\ (\leftarrow (\log \Delta(p, C_a))^{\frac{1}{2}})$

$\rightsquigarrow$ Singular analogue of Ueda's thm (gen. of [U. §1]) ψ : bdd from above. //

Outline of prf of Thm B

$X \supset S \supset C$.

⊙ Difficulty : $\mathcal{O}_{N_{S/X}} c_1(N_{S/X}) \neq 0$.

$\rightsquigarrow$ we can not use Ueda thm in this setting.

However $c_1(N_{S/X}|_V) = 0$ for a nbhd $C \subset V \subset S$.

$\rightarrow$ we posed "~~higher~~ codim 2-analogue of Ueda thm" and showed that

$\exists W \subset X$: a nbhd of C s.t. $\mathcal{O}_X(S)|_W$: flat

Thm

X : cpx mfd.
$\cup$
S : sm. hyp. surf.
$\cup$
C : sm. hyp. surf. cpt käh.

; s.t. $N_{S/X}|_V$: flat. for $\exists V \subset S$: C-nbhd and $\exists Y$: sm. hyp. surf of X s.t. $C = Y \cap S$

Assume
(i) $N_{C/S}, N_{S/X}|_C \in \mathcal{E}_0$. + type $(C,S) = \infty$
or (ii) $N_{C/S} \cong N_{S/X}|_C \in \mathcal{E}_1$ + ″ ,
or (iii) $N_{S/X}|_C \in \mathcal{E}_0$, $N_{S/X}^{-1}$: ample, C admits a fund. system of Stein nbhds

Then type $(C,S,X) = \infty \Longrightarrow \exists W \subset X$: C-nbhd s.t. $\mathcal{O}_X(S)|_W$: flat //

($\forall n \geq 1$, $\exists m \geq 1$: $H^1(C, N_{C/S}^{-n} \otimes N_{S/X}^{-m}) = 0$!!)

No.

Date