On a higher codim'l analogue of Ueda theory and its application.

10:30 - 12:00.

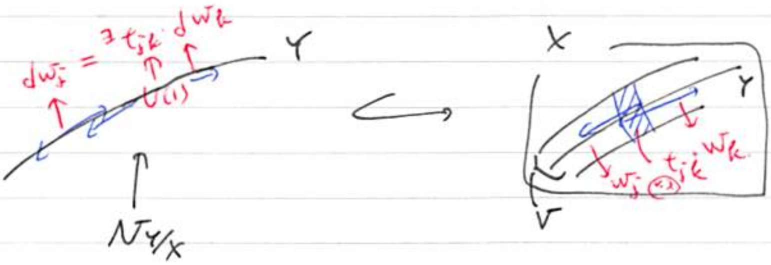
$\cdots$ nbhd. of.

[Ueda '83] Y : cpt sm. curve $\hookrightarrow X$: sm. surf. with $\deg N_{Y/X} = 0$. (c.f. H.Grauert '62, O.Suzuki '75)

Ueda classified the pair (Y, X) into the following two types:

$$\begin{cases} \text{type}(Y,X) < \infty : [Y] \not\cong \widetilde{N_{Y/X}} \text{ formally along } Y. \\ \text{type}(Y,X) = \infty : [Y] \cong \widetilde{N_{Y/X}} \text{ formally along } Y, \end{cases}$$

where $\begin{cases} [Y] : \text{the line bdl} / X \text{ corresp. to. the diviser } Y. \\ \widetilde{N_{Y/X}} : \text{the flat ext. of } N_{Y/X} \in \mathrm{Pic}^0(Y) \underset{\text{fact}}{\cong} H^1(Y, U(1)) \end{cases}$



$\widetilde{N_{Y/X}} \in H^1(\underset{\text{tub. nbhd of } Y}{V}, U(1))$

<u>Ueda's thm</u>

- ◎ type$(Y,X) < \infty \Rightarrow Y$ admits a str. psd concave nbhds system.
- ◎ type$(Y,X) = \infty$ + some condits on $N_{Y/X}$. $\Rightarrow {}^\exists V$: tub. nbhd of Y in X s.t. $[Y] \cong \widetilde{N_{Y/X}}$ on V. //

$\Downarrow$

$$\begin{cases} \text{(i) } {}^\exists \mathcal{F} : \text{sm. hol. foliation on } V \\ \quad \text{s.t. } \begin{cases} Y \in \mathcal{F} \text{ (i.e. } Y : \text{a leaf of } \mathcal{F}) \\ \underbrace{\mathrm{Hol}_{Y,\mathcal{F}}}_{\text{holonomy along } Y.} = \underbrace{\rho_{N_{Y/X}}}_{\text{monodromy of } (N_{Y/X}, \text{flat conn.})} : \pi_1(Y,*) \to U(1) \end{cases} \\ \text{(ii) } [Y]|_V \text{ admits a metric } h \text{ with } \sqrt{-1}\Theta_h \equiv 0 \end{cases}$$ //

also true in the case Y : cpt mfd. with $r := \mathrm{codim}_Y X = 1$

✗ [Ohsawa '07] A rmk on psd conv. dom.s with analytic J. Math. Kyoto U.

X : cpt Kä, $\Rightarrow X \setminus Y$ は 3つ以上の Levi-form の posi. eigen val をもつ exh. をもち得ない

$\boxed{\text{Goal}}$

- ◎ Generalize Ueda's classification
- ◎ Generalize the Ueda's thm for type$(Y,X) = \infty$.
- ◎ Apply it to the semi-positivity criterion for nef line bdls.

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§1. Main result, application to the semi-positivity criterion.
~~§2. definition of Obstr. class and type(Y, X).~~
§3 Outline of the prf.

§1 Main result

Main Thm $Y \subset X$; cpt submfd of $\begin{cases} \text{codim} = r \geq 1. \\ N_{Y/X} \in H^1(Y, U(r)) \end{cases}$.

Assume

- $\text{type}(Y, X) = \infty$
- $N_{Y/X}$ satisfies

"torsion condition" $\# \text{Image}(\rho_{N_{Y/X}} : \pi_1(Y, *) \to U(r)) < \infty$.

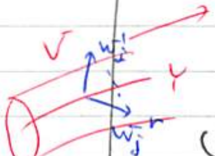
or

"Diophantine condition"

$\exists N_1, N_2, \cdots, N_r \in H^1(Y, U(1))$
$\exists A > 0$ s.t.
$N_{Y/X} \cong N_1 \oplus N_2 \oplus \cdots \oplus N_r$
$\forall \alpha = (\alpha_1, \cdots, \alpha_r) \in \mathbb{Z}^r$,
$d(\mathbb{1}_Y, \bigotimes_{\lambda=1}^{r} N_\lambda^{\alpha_\lambda}) \geq \frac{1}{(2|\alpha|)^A}$
$(|\alpha| := \alpha_1 + \cdots + \alpha_r \geq 1)$.

Then

(i) $\exists V$: a nbhd of $Y \subset X$.
$\exists \mathcal{F}$; sm. hol. foliation on V of codim $= r$. s.t. $\begin{cases} Y \in \mathcal{F} \\ \text{Hol}_{\mathcal{F}, Y} = \rho_{N_{Y/X}}. \end{cases}$



(ii) $\forall S$: sm. hyp. surf $\subset V$ with $\begin{cases} Y \subset S \\ N_{Y/S} \in H^1(Y, U(r-1)), \end{cases}$

$\exists V_S \subset V$: Y-nbhd.
$\exists \mathcal{G}_S$: sm. hol. foliation of codim $= 1$ on V_S.
s.t. $\begin{cases} \cdot\ S \cap V_S \in \mathcal{G}_S \\ \cdot\ \text{Hol}_{\mathcal{G}_S, S \cap V_S} : U(1)\text{-linear} \\ \cdot\ \text{each leaf of } \mathcal{G}_S \text{ is the union of leaves of } \mathcal{F}. // \end{cases}$

$\left(\Longrightarrow [S \cap V_S] \in H^1(V_S, U(1)) \right)$
"flat"

$\begin{pmatrix} w_j^1 \\ \vdots \\ w_j^r \end{pmatrix} = \exists T_{jk} \begin{pmatrix} w_k^1 \\ \vdots \\ w_k^r \end{pmatrix}$, $T_{jk} \in U(r)$

$\boxed{\mathcal{F} =_{\text{loc.}} \{(w_j = \text{const})\}}$

$S =_{\text{locally}} \{w_j^1 = 0\}$

$T_{jk} = \begin{pmatrix} t_{jk} & 0 \\ 0 & U(r-1) \end{pmatrix}$

$\boxed{\mathcal{G}_S =_{\text{l.c}} \{(w_j^1 = \text{const})\}}$

~~Schedule §1. Some examples~~
~~§2. definition of Obser. classes and type(Y, X).~~
§3,

<u>Rmk</u> ◎ $\mathrm{type}(Y, X) = \infty$ if $H^1(Y, N_{Y/X} \otimes S^{n+1} N^*_{Y/X}) = 0$ for $\forall n \geq 1$.

$\Uparrow$

((Y: sm. ellipt. curve, + $N_{Y/X}$ satisfies "Dioph. cond.".

◎ Arnol'd '76; Main thm(i) for this case. ($r = 1$?)

◎ Ueda '83; Main thm(i) for $r=1$.

◎ (~~Kotke~~ '15) K—, N. Ogawa '16 ... $r=2$

//

<u>e.g.</u> $\pi : X \twoheadrightarrow B$ (dom $\subset \mathbb{C}^r$) : surj. hol. submersion. $\pi^{-1}(x)$: proj. for $\forall x \in B$, $Y := \pi^{-1}(0)$: sm. fiber.

$\underset{\text{free}}{\Longrightarrow}$ $N_{Y/X}$: hol. triv, $\mathrm{type}(Y, X) = \infty$

<u>In this case</u>: ◎ Main thm (i) ... triv. ($\mathcal{F} = \{\pi^{-1}(x)\}_{x \in B}$)

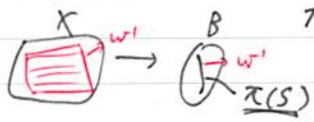
◎ Simple prf of Mainthm (ii) in this case:

$S \subset X$: hyp. surf as in (ii).

$\rightsquigarrow$ $[S]|_{\pi^{-1}(x)}$: top. triv. for $\forall x$.

◎ Take $x \in B$ with $\pi^{-1}(x) \not\subset S$ $\rightsquigarrow$ $S|_{\pi^{-1}(x)}$: eff. div. on $\pi^{-1}(x)$ $\rightarrow$ $\pi^{-1}(x) \cap S = \phi$.

$\rightsquigarrow$ $S = \pi^{-1}(\pi(S))$



<u>Application to the semi-positivity criterion</u>

X: cpx mfd of dim $= n$.

$\uparrow$

L: hol. line bdl, $D_1, D_2, \cdots, D_{n-1} \in |L|$ $C := \bigcap_{\lambda=1}^{n-1} D_\lambda$

$\rightsquigarrow$ ◎ $\exists$ sing. Herm metric h_{sing} on L with analytic sing and $\{h_{sing} = \infty\} = C$. ($\Leftarrow$ "Bergman type constr.")

◎ If $\exists V$: C-nbhd s.t. $L|_V$: flat.,
Then L: semi-positive (i.e. L admits a C^∞ Herm metric with s.p. curv.)
($\Leftarrow$ "regularized max. constr.")

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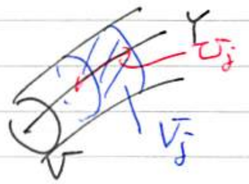
Cor Assume that $D_1, \cdots, D_{n-1}$ intersect transversally along C,
$L|_C \in H^1(C, U(1))$, satisfies Dioph. cond.
Then L: semi-positive. //

e.g. (V, F) : sm. del Pezzo mfd of deg $= 1$ $\left(\begin{array}{l} F : \text{ample} / V^n, \\ F^{n-1} \cong K_V^{-1} \end{array}\right)$
$p \in V$: "general".
$\pi : X := Bl_p V \longrightarrow V$, $E := \pi^{-1}(p)$,
$L := \pi^* F \otimes [-E]$. $\underset{\text{cor}}{\Longrightarrow}$ $\underset{(K_X^{-1})}{L}$: semi-positive //

§2. Definition of obstr. class and type (Y, X).



V : tub. nbhd of Y, (suff. small)
$\{V_j\}$: open cov. of V,
$U_j := V_j \cap Y$.
$w_j = \begin{pmatrix} w_j^1 \\ \vdots \\ w_j^r \end{pmatrix}$: def. func. of U_j on V_j.

$(N_{Y/X} \in H^1(Y, U(r)) \Rightarrow)$ One can take $\{(V_j, w_j)\}$
s.t. $dw_j|_{U_j} = {}^\exists T_{jk} \cdot dw_k|_{U_k}$, $T_{jk} \in U(r)$.

z_j : coord of U_j

$$\rightsquigarrow \underset{\left(\sum_{\mu=1}^r (T_{jk})^\lambda_\mu \cdot w_k^\mu\right)_\lambda}{T_{jk} \cdot w_k} = \underset{\begin{pmatrix} w_j^1 \\ \vdots \\ w_j^r \end{pmatrix}}{w_j} + \sum_{\substack{\alpha \in (\mathbb{Z}_{\geq 0})^r \\ |\alpha| \geq 2}} {}^\exists \underset{\begin{pmatrix} f^1_{kj,\alpha} \\ \vdots \\ f^r_{kj,\alpha} \end{pmatrix}}{f_{kj,\alpha}(z_j)} \cdot w_j^\alpha \quad \text{on } V_{jk} = V_j \cap V_k.$$

$$\rightsquigarrow \left[\left\{ \sum_{\lambda, |\alpha| = 2} f^\lambda_{kj,\alpha} \cdot \frac{\partial}{\partial w_j^\lambda}\Big|_{U_{jk}} \otimes (dw_j|_{U_{jk}})^\alpha \right\}\right] \in H^1(Y, N_{Y/X} \otimes S^2 N^*_{Y/X})$$

$= u_1(Y, X)$. (fact)

Fact $u_1(Y, X) = 0 \Rightarrow$ One can take $\{(V_j, w_j)\}$ s.t.
$$T_{jk} \cdot w_k = w_j + \sum_{|\alpha| \geq 3} f_{kj,\alpha}(z_j) \cdot w_j^\alpha$$

$\rightsquigarrow$ $u_2(Y,X) := [\{ \sum_{\lambda, |\alpha|=3} \text{---} \}] \in H^1(Y, N_{Y/X} \otimes S^3 N_{Y/X}^*)$

$\vdots$

<u>Fact</u> ⊙ "$\underset{H^1(Y, N_{Y/X}\otimes S^{n+1}N_{Y/X}^*)}{u_n(Y,X)} \overset{?}{=} 0$" : ~~well-def.~~ does not depend on $\{w_j\}$.

⊙ <u>Either</u> $\exists n \geq 1$ s.t. $u_n(Y,X) \neq 0$. $\Big] \leftarrow$ type$(Y,X) := n$

<u>or</u> $\forall n \geq 1$, $u_n(Y,X) = 0$. $\longleftarrow$ type$(Y,X) := \infty$ //.

<u>Rmk</u> I_Y : def. ideal of Y $(\subset \mathcal{O}_X)$

$\rightsquigarrow$ $0 \to I_Y/I_Y^2 \to \mathcal{O}_Y/I_Y^2 \to \mathcal{O}_Y/I_Y \to 0$

$\rightsquigarrow$ $\otimes \widetilde{N_{Y/X}}$ $\quad \underbrace{H^0(\mathcal{O}_Y/I_Y^2)}_{\otimes N_{Y/X}} \to \underbrace{H^0(\mathcal{O}_Y/I_Y)}_{\otimes N_{Y/X}} \to \underbrace{H^1(I_Y/I_Y^2)}_{\otimes N_{Y/X}}$

"$\{w_j\}$" $\longmapsto$ $u_1 \in H^1(Y, N_{Y/X} \otimes S^2 N_{Y/X}^*)$

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§3. Outline of the prf of Main thm.

$\{w_j\}$: as in §2. $\left(T_{jk} \cdot w_k = w_j + \sum_{|\alpha| \geq 2} f_{kj,\alpha} \cdot w_j^\alpha . \right)$

<u>prf of Main thm(i)</u> .... (c.f. prf of Ueda's thm).

Solve a "Schröder type functional equation"

$$w_j = u_j + \sum_{|\alpha| \geq 2} \underset{\left(\begin{smallmatrix} F_j^1 \\ \vdots \\ F_j^r \end{smallmatrix}\right)}{F_{j,\alpha}(z_j)} \cdot u_j^\alpha \quad \text{-----} (*)$$

to construct $\{u_j\}$

with $\begin{cases} u_j : \text{def. func. system of } U_j \text{ in } V_j, \\ u_j = T_{jk} \cdot u_k \quad \text{on } V_{jk} \end{cases}$

⊙ Existence of "nice" $\{F_{j,\alpha}\}_{|\alpha|=n+1} \Longleftarrow u_n = 0$.

⊙ Conv. of $X \mapsto X + \sum_{|\alpha| \geq 2} \|F_{j,\alpha}\|_{L^\infty} \cdot X^\alpha \Longleftarrow$ "torsion cond." or "Dioph. cond."

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prf of Mainthm (ii) ;

$N_{Y/S} \in H^1(Y, U(r-1))$

$\Rightarrow$ One can take an initial system $\{w_j\}$ so that.

$$\begin{cases} \{w_j' = 0\} = S \cap V_j \text{ in } V_j. \\ T_{jk} = \left(\begin{array}{c|c} U(1) & 0 \\ \hline 0 & U(r-1) \end{array}\right). \end{cases}$$

$\downarrow$ Solve func. eq. (*)

Fact $\underbrace{\{u_j\}}_{\text{the solution}}$ satisfies $\{u_j' = 0\} = S \cap V_j$ in V_j.

($+$ $T_{jk} u_k = u_j$ on V_{jk}.)

Q $\{u_j\}$ in the prf of Mainthm (i) v.s. $\{u_j\}$ in $\cancel{H}$ (ii) ?

$\{u_j\} \rightsquigarrow \mathcal{F}$ $\quad$ $\{\hat{u}_j\} \rightsquigarrow \hat{\mathcal{F}}$

Fact $\hat{u}_j = u_j + \sum_{|\alpha| \geq 2} \underbrace{a_{j\alpha}(z_j)}_{\text{loc. const.}} \cdot u_j^\alpha$

$\Rightarrow \mathcal{F} = \hat{\mathcal{F}}$

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