On the minimality of canonically attached s.H.m. on certain nef l.b.

X : sm. proj. var $/\mathbb{C}$

L : (hol) l.b. $/X$.

Def L: nef $\overset{\text{def}}{\iff}$ ${}^\forall C \subset X$: cpt curve, $L.C \geq 0$.

L: str. nef $\overset{\text{def}}{\iff}$ ${}^\forall C \subset X$: ——〃——, $L.C \gneq 0$ //

Q When does a singular Hermitian metric (s.H.m) with minimal singularity of L with s.p. curvature (semi-positive) have singularities when L is (str.) nef?

Def. h : a minimal singular metric of L

$\overset{\text{def}}{\iff}$ ◎ $\sqrt{-1}\,\Theta_h \geq 0$. (i.e. ${}^\forall$local weight of h is psh) $\leftarrow \varphi$, when $h = e^{-\varphi}$ (locally)

◎ ${}^\forall h'$: s.H.m of L s.t. $\sqrt{-1}\,\Theta_{h'}$,

${}^\forall x \in X$, ${}^\exists C \in \mathbb{R}$, $\varphi' \leq \varphi + C$ around x

where φ (resp. φ') is the loc. weight of h (resp. h') //

known

◎ (L: str. nef $\Rightarrow$ L: nef $\Rightarrow$) L: psd. eff $\Rightarrow$ ${}^\exists$ min. sing. metric on L //

◎ L : ample $\overset{\text{Kodaira's emb. thm}}{\iff}$ L : positive. $\Rightarrow$ a min. sig. metric on L has no sing.

◎ L : semi-ample $\Rightarrow$ L : semi-positive $\Rightarrow$ ——〃——
(i.e. L admit a sm metric with semi-positive curvature)

◎ L : semi-positive $\Rightarrow$ nef.

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◎ L : nef $\not\Rightarrow$ L : semi-positive.

e.g. (Demailly–Peternell–Schneider)

C_0 : sm. ellipt. curve.

E : rank 2 - vect. bdl $/C_0$ triv. line bdl $/C_0$

s.t. $0 \to \mathbb{1}_{C_0} \to E \to \mathbb{1}_{C_0} \to 0$; ex, non-splitting.

$X := \mathbb{P}(E) \xrightarrow{\pi} C_0$,

$C :=$ the section of π, $\rightsquigarrow (C^2) = 0$,

$\rightsquigarrow$ ~~$-K_X$~~ $[C]$ is nef,

fact. h : s.H.m on $[C]$ with semi-positive curvature

$\Rightarrow h = {}^{\exists}M \cdot |f_C|^{-2}$ where $M > 0$: const.

$f_C \in H^0(X, [C])$: canonical section.

Main results We restrict ourselves in the ~~case~~ surface case.

Thm 1 X : ~~cpx~~ ~~proj.~~ sm. surf.
$\cup$
C : sm. embedded ~~cpt~~ curve with $\deg N_{C/X} = 0$.

Assume

(C, X) : of finite type in the sense of Ueda. (*)

Then $|f_C|^{-2}$ is a min. sing. metric of L, where $f_C \in H^0(X, [C])$ is the can. section.

Thm 2 There exists a proj. 3-fold ~~X~~ Y and a l.b. $\tilde{L}$ on Y

s.t. $\tilde{L}$ is str. nef, however is not semi-positive.

"Example 5.9" in [Fujino. '13] (A transcendental approach to Kollár's inj. thm II)

Def

— (*) $\underset{\text{def.}}{\Longleftrightarrow}$ $\exists\, n \in \mathbb{N}_{\geq 1}$ s.t.,
for sufficiently small tub. nbhd V of C in X,
$\widetilde{N_{C/V}} \otimes \mathcal{O}_V/\mathcal{O}_V(-nC) \not\cong \mathcal{O}_V(C) \otimes \mathcal{O}_V/\mathcal{O}_V(-nC)$
where $\widetilde{N_{C/V}}$ is the flat ext. of $N_{C/V}$ to V //

§1. some examples and outline of the proof of Thm1.
§2. On Thm 2.

§1

e.g. 1 --- a generalization of D.P.S. example.

C_0 : sm. curve.
E : a rank 2 - vect. bdl / C_0.
s.t. $\exists F$: a flat line bdl / C_0
s.t. $0 \to F \to E \to \mathbb{1}_{C_0} \to 0$: ex. --- (*)
$X := \mathbb{P}(E) \xrightarrow{\pi} C_0$,
$C :=$ the section of π.
$\rightsquigarrow (C^2) = 0$, $[C]$: nef.

Cor 1 $[C]$: semi-positive $\Longleftrightarrow$ (*) splits.
Moreover, if (*) ~~sp~~ does not split,
then $|f_C|^{-2}$ is a minimal sing. metric of $[C]$ //

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e.g.2 ~~Cor 2~~.

C_0 : a sm. curve of genus $=2$.

$C_0 \hookrightarrow Y$: the Jacobian of C_0.

p, q. : conjugate to each other by the hyperelliptic involution

$X :=$ the b-up of Y at $\{p, q\}$

$\check{C} :=$ the str. transf of C_0. $((C^2) = 0)$

~~Then~~ $\boxed{\text{Cor 2}}$ $|f_C|^{-2}$ is a min. sing. metric on $[C]$ //

prf. of Cor 1, 2.

--- Neeman showed that (C, X) is of finite type (if (*) does not split)

//.

Outline of the prf of Thm 1

--- a simple application of Ueda's thm.

Let h be a s.H.m. of $[C]$ with s.p. curvature.

$\Phi := -\log |f_C|_h^2$

$\rightsquigarrow \Phi \underset{\text{local}}{=\!=} -\log |\underbrace{f_C}_{\text{local defining fct of } C}|^2 + \underbrace{(\text{local weight of } h)}_{\text{psh}}$

Thus ~~where~~ Φ is psh on $X \setminus C$,

with ~~singularity along~~ ~~the~~ growth $\Phi(p) = o(\text{dist}(p, C)^{-\frac{1}{2}})$ as $p \to C$

It follows from Ueda's thm on the growth of psh functions defined around C when (C, X) : of fin. type.

that $\Phi \equiv \exists M$: constant around C //.

Therefore $|t_C|_h^2 \equiv {}^\exists M'$ around C,

$\Updownarrow$

$h \equiv M' \cdot |t_C|^{-2}$. //

§2 .... strict nefness v.s. semi-positivity.

Q1 Does the strict nefness imply the ~~semi-pos~~ semi-ampleness ?

No ... e.g. (Mumford)

$\tilde{C}$: a sm. cpt curve of genus $= g > 1$.

fact. $\left({}^\exists F : \text{rank } 2\text{-vect. bdl} \ /\tilde{C} \right.$ s.t. $\deg(F) = 0$, $S^m F$: stable for ${}^\forall m \geq 1$.

$\rightsquigarrow Y := \mathbb{P}(F)$, $L_Y := \mathcal{O}_{\mathbb{P}(F)}(1)$

then L_Y is str. nef, but not semi-ample //

~~Q2~~

Rmk. L_Y is semi-positive.

prf. Consider the fiberwise Fubini-Study metric h_{L_Y} on L_Y constructed by using ~~the~~ a flat metric on F.

$\subset$ [Narasimhan - Seshadri] //

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e.g (= Example. 5.7. in [Fujino '13])

$\tilde{C}$; Y, L_Y : as above.

Fix a rank 2-vect. bdl E on $\tilde{C}$ s.t.

$0 \to \mathbb{1}_{\tilde{C}} \to E \to \mathbb{1}_{\tilde{C}} \to 0$: ex non-splitting.

$\tilde{X} := \mathbb{P}(E) \xrightarrow{\pi} \tilde{C}$.

$\cup$

$\tilde{D} :=$ the section of π.

$$\begin{tikzcd} \tilde{Y} := \tilde{X} \times_{\tilde{C}} Y \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\ulcorner"] & \tilde{X} \arrow[d] \\ Y \arrow[r] & \tilde{C}. \end{tikzcd}$$

$\tilde{L} := [\tilde{D} \times_{\tilde{C}} Y] \otimes p_2^* L_Y$.

$\rightsquigarrow$ $\tilde{L}$ is strictly nef, but not semi-ample.

Q (Fujino) Is $\tilde{L}$ semi-positive?

NO

Thm 2 Let $f_{\tilde{D}}$ be a canonical section of $H^0(X, [\tilde{D}])$

Then $(p_1^* |f_{\tilde{D}}|^{-2}) \otimes (p_2^* h_{L_Y})$ is a min. sing. metric on $\tilde{L}$ //

sm. Herm. metric on L_Y with s.p. curvature

Especially, $\tilde{L}$ is not semi-positive //.

<u>Outline of the prf of Thm 2</u>.

Let $h_{\tilde{L}}$ be a s.H.m. of $\tilde{L}$ with s.p. curvature.

Fix a sm. metric h_∞ on $[\tilde{D}]$ ~~or~~

a l.b. on $\tilde{X}$.

$\rightsquigarrow$ $h_{\tilde{L}} = (p_1^* h_\infty) \otimes (p_2^* h_{L_Y}) \cdot e^{-\exists \chi}$

<u>Consider</u> ~~$\tilde{\chi}(p) := \max$~~

$\tilde{\chi} : \tilde{X} \longrightarrow \mathbb{R} \cup \{-\infty\}$.

$\phantom{\tilde{\chi} : } x \longmapsto \max_{\tilde{x} \in p_1^{-1}(x)} \chi(x)$.

$\rightsquigarrow$ We can show that

$h_\infty \cdot e^{-\tilde{\chi}}$ is a s.H.m. of $[\tilde{D}]$ with s.p. curvature.

$\rightsquigarrow$ <u>Cor. 1</u> $\rightsquigarrow$ $\exists \varepsilon > 0$,

$$\varepsilon \cdot |f_{\tilde{D}}|^{-2} \leq h_\infty \cdot e^{-\tilde{\chi}}$$

$$\rightsquigarrow \varepsilon \cdot ((p_1^* |f_{\tilde{D}}|^{-2}) \otimes (p_2^* h_{L_Y}))$$

$$\leq (p_1^* h_\infty) \otimes (p_2 h_{L_Y}) \cdot e^{-\chi}$$

$$= h_{\tilde{L}}$$

//.