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On a nbhd of a torus leaf of a certain class of hol. foliations on cpx surface.

§0. Introd.

興味 … $\underset{\text{(non-sing.) cpt cpx submfd.}}{Y} \subset \underset{\text{cpx mfd}}{X}$ の近傍の複素解析幾何的性質.

以下, $\dim_{\mathbb{C}} Y = 1$, $\dim_{\mathbb{C}} X = 2$, $c_1(N_{Y/X}) = 0$ とする.
~~以下~~

例1 $Y_1 := \mathbb{C}_{z_1}/\langle 1, \tau\rangle$ $(\mathrm{Im}\,\tau > 0)$

$\cap$

$X_1 := \underset{(z_1,\, w_1)}{Y_1 \times \mathbb{P}^1}$ $(Y_1 = \{w_1 = 0\})$

例2 (Serre の例) $Y_1 := \mathbb{C}_{z_2}/\langle 1, \tau\rangle$

$\uparrow$

$\underset{Y_1 \text{上 } \mathbb{P}^1\text{-bdl.}}{X_2} := \overline{\mathbb{C}_{z_2} \times \mathbb{C}_{w_2}} / \left\langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}\tau\\\tau\end{pmatrix} \right\rangle$ ← "無限遠"~~直線~~による cpt 化 ← Y_1 上 affine bdl.

Y_1 と"無限遠"を同一視して, $Y_1 \subset X_2$ とみる.

例1 v.s. 例2 … ともに $N_{Y/X} = \mathbb{I}_Y$

	例1	例2
$N_{Y/X}$	$\mathbb{I}_Y$	$\mathbb{I}_Y$ ← hol. triv. l.b.
$X \setminus Y$	$Y_z \times \mathbb{C}_w$	$\mathbb{C}^*_z \times \mathbb{C}^*_w$
plurisubharm (p.s.h.) exhaustion on $X \setminus Y$ ($i\partial\bar\partial \geq 0$; sublevel set が rel. cpt.)	$\log\lvert w\rvert^2$ (harmonic) "Yは psd flat nbhd をもつ"	$\lvert w - z\rvert^2$ (strictly psh.) i.e. $i\partial\bar\partial \gneq 0$ "Yは str. psd concave nbhd をもつ"
$X \setminus Y$ の cpt cpx sub.	$Y_z \times \{*\}$ たち.	なし.

C.f.
H. Grauert '62 … $N_{Y/X}$: negative $\Rightarrow$ Y : str. psd convex nbhd をもつ
O. Suzuki '75 … $N_{Y/X}$: positive $\Rightarrow$ Y : str. psd concave nbhds をもつ
T. Ueda '83 … $N_{Y/X}$: flat. case.

§1 Ueda's classification. (type α, β, γ)

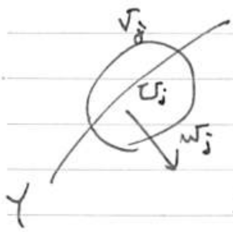
§2. 応用

§3. type γ の pair について.

§1. $Y^1 \subset X^2$, $c_1(N_{Y/X}) = 0$

$\overset{\longrightarrow}{\text{Hodge theory}}$ $N_{Y/X}$ は $U(1)$-flat.

i.e. $^\exists\{U_j\}$: open cov. of Y,
$^\exists\, t_{jk} \in \mathbb{C}$, $|t_{jk}| = 1$,
$N_{Y/X} = [\{U_{jk}, t_{jk}\}] \in \check{H}^1(\{U_j\}, U(1))$.



V_j : U_j-nbhd $\subset X$.
w_j : local def. hol. func of Y in V_j $\quad (U_j = \{w_j = 0\}_{\text{in } V_j})$

$\rightsquigarrow t_{jk} \cdot w_k = w_j + {}^\exists f_{kj,2}(z_j) \cdot w_j^2 + {}^\exists f_{kj,3}(z_j) \cdot w_j^3 + \cdots$

Ueda-type

$\text{type}(Y, X) := \max\left\{ n \in \mathbb{Z}_{\geq 2} \cup \{\infty\} \;\middle|\; \begin{array}{l} ^\forall 2 \leq m < n,\ ^\exists\{(V_j, w_j)\} \text{ を } (\text{: as above}), \\ ^\forall j,k \text{ で} \\ f_{kj,m} \equiv 0 \text{ とできる} \end{array} \right\} - 1$

(c.f. Ueda class $u_n \in \check{H}^1(\{U_j\}, \mathcal{O}_Y(N_{Y/X}^{-n}))$)

Cut?

例

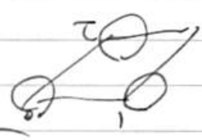
例1の
(Y_1, X_1) では…… $w := \frac{1}{w_1}$ $\rightsquigarrow$ $t_{jk} \cdot w_j = w_k$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{1} \quad \overset{\shortparallel}{w|_{V_j}} \quad \overset{\shortparallel}{w|_{V_k}}$.

$\rightsquigarrow$ type $= \infty$.

例2の
(Y_2, X_2) では…… $w = \frac{1}{w_2}$

$\frac{w_j}{\tau w_j + 1}$

$\rightsquigarrow z \mapsto z+1$ で $\frac{1}{w_k} = \frac{1}{w_j} + 1 \iff w_k = w_j - w_j^2 + \cdots$
$z \mapsto z + \tau$ で $\frac{1}{w_k} = \frac{1}{w_j} + \tau \iff w_k = w_j - \tau w_j^2 + \cdots$



$\underset{\text{thus}}{\rightsquigarrow}$ type $= 1$.

Def (Ueda's classification)

- ⓐ (Y, X) : type(α) $\underset{\text{def}}{\iff}$ type $< \infty$.
- ⓑ : type(β') $\underset{\text{def}}{\iff}$ type $= \infty$ かつ $N_{Y/X}$: torsion $\in \mathrm{Pic}^0(Y)$.
- ⓒ : type(β'') $\underset{\text{def}}{\iff}$ type $= \infty$ かつ $N_{Y/X}$: non-tor. かつ Y: psd flat nbhds sys をもつ
- ⓓ : type(γ) $\underset{\text{def}}{\iff}$ ——〃—— かつ Y: 〃 をもたない.

Thm (Ueda).

(1) type (α) $\Longrightarrow$ Y: str. psd concave nbhds system をもつ

(2) type (β') $\Longrightarrow$ $\exists (Y\text{-nbhd}) \xrightarrow{f} \underset{\text{disc} \subset \mathbb{C}}{\Delta}$: ~~[illegible]~~. cpt sub mfd の変形族

とくに Y: psd-flat nbhds sys. をもつ.

(3) type=∞ かつ $N_{Y/X}$: Diophantine $\Longrightarrow$ type (β'').

$\left(-\log d(\mathbb{I}_Y, N_{Y/X}^n) = \underset{n\to\infty}{O(\log n)}\right)$

(4) ∃(Y, X) : type (γ) //

cut?

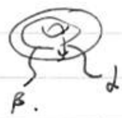
§2. 応用.

- Barrett – Inaba (D.E., T.), K~~[illegible]~~-Ogawa ('90, '92) ; Levi-flat mfd の実現可能性 (モジュライ)
  (Thm の (2) ⇝ ∃hol. func f ⇝ ...)
- Thm (2), (3) ⇝ $\exists !$ hol. foliation $\mathcal{F}$ on a nbhd of Y s.t. $Y \in \mathcal{F}$. (non-sing.)
- M. Brunella '10 (K– '14...) : type (β'), (β'') のときの Y が定める line bdl に ∃C^∞ Herm. metric with semi-positive curvature
- K— '14 : type (α) のとき, ∄ —— // —— .
- H. Tsuji '84, K— '17 : ~~X: $\mathbb{P}^2$ の 9点 blowup~~, $Y \subset X$: type (β'') (Y: ellipt. curve)
  $Y' \subset X'$ 〃,
  $N_{Y/X} \cong N_{Y'/X'}^{-1}$
  ⇝ $X \setminus (Y\text{-nbhd})$ と $X' \setminus (Y'\text{-nbhd})$ とはり合わせて 新しい cpx mfd (Tsuji ... $\approx S^3 \times S^3$, K— ... K3.)
  cf. [Arnol'd's thm]

⋮

§3. type γ について

Y : ellipt. curve,



f : (ℂ の 0-nbhd) → ℂ : f(0) = 0, f'(0) ≠ 0 なるもの

を決めるごとに, ...

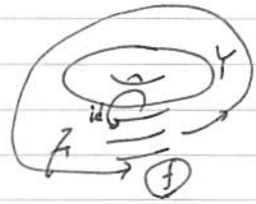
No.

Date

$X_{(Y,f)} \approx_{C^\infty} Y \times \mathrm{disc}_{(\subset \mathbb{C})}$

Known $\exists$ X : open cpx mfd. $\exists \mathcal{F}$; X 上 hol. foliation, (non-sing.)

s.t. $Y \in \mathcal{F}$, $\mathrm{Hol}_{\mathcal{F},Y}$: $\alpha \mapsto \mathrm{id}$
$\beta \mapsto f$.



Ueda's proof of Thm(4)

(cont??)

f について、多項式で、かつ $|f'(0)| = 1$, "str. Cremer cond" をみたすもの とする.

$\rightsquigarrow$ $\exists \Omega_n$: 0-nbhds $\subset \mathbb{C}$ $\Omega_1 \supset \Omega_2 \supset \cdots$

$\exists \{p_1^{(n)}, p_2^{(n)}, \cdots p_{m_n}^{(n)}\} \subset \Omega_n$: periodic cycle of f, period $= m_n$

s.t. $m_n \nearrow \infty$.

$\rightsquigarrow$ $\exists V_n$: Y-nbhd $\subset X = X_{(Y,f)}$, $V_1 \supset V_2 \supset \cdots$

$\exists \Gamma_n \subset V_n$: ~~sm~~ ellipt. curve $\subset V_n \setminus Y$

s.t. $[\Gamma_n] \sim_{\text{homol.}} [m_n \cdot Y]$, $m_n \nearrow \infty$.

もし (Y, X) が… type α, β'' $\Rightarrow$ $V_n \setminus Y$ は cpt curve なし

type β' $\Rightarrow$ m_n は定数.

(Rmk) $|f'(0)| \neq 1 \Rightarrow$ type β' or β'',
$f'(0) \in U(1)$: torsion $\Rightarrow$ type β' or type α.
$f'(0) \in U(1)$: non-tor $\Rightarrow$ type$(Y,X) = \infty$. (β'' or γ) //

Thm (K— '16)

Ueda's Thm(4) の一般化.

(Y, X) : as above

: type γ $\Leftrightarrow$ ~~$|f'(0)| = 1$~~
~~$|f'(0)| = 1$~~, $f'(0) \in U(1)$; non-tor
(*) f は 0 を Cremer fixed pt. として持つ
(i.e. 0-nbhd で f は non-lin'ble) //

idea of the prf ;

$\forall \Omega$: (small) disc $\subset \mathbb{C}$, center $= 0$,
$\exists K = K(\Omega)$: "Hedgehog"
s.t. K : cpt conn, $\mathbb{C} \setminus K$: conn. $0 \in K$, $K = f(K) = f^{-1}(K)$, $f \cap K \cap \partial\Omega \neq \phi$, μ-a.e. $x \in K$, $O(x)$ は dense in K

-- Use "Hedgehog" by [Pérez-Marco '97]

(*) $\Rightarrow$ $\forall \Omega$: 0-nbhd $\subset \mathbb{C}$, $\exists p_1, p_2, \cdots \in \Omega$ ~~$\subset \mathbb{C}$~~ : $p_{\nu+1} = f(p_\nu)$, $\overline{\{p_\nu\}} \ni 0$.

$\rightsquigarrow$ 対応する leaf 上で最大値原理を考え, ~~A~~ $X \setminus Y$ 上 psh. は Y で有界と分かる $\rightsquigarrow$ $\nexists$ psh'ble nbhds ...