Arnol'd's type thms on a nbhd of a curve and gluing construction of K3 surfaces

東北大 セミナー 15:00〜16:30

Goal of this talk

§1 ⊙ Give a "rigidity" result for the cpx str. (uniformization) of a small nbhd of a curve C surf.

(§2) ⊙ Apply it to the gluing construction of K3 surfaces (in progress, with T. Uehara)

§1. 設定.

$\underset{\text{cpx cpx curve, reduced}}{C} \underset{\text{hol'ly}}{\subset} \underset{\text{non-sing cpx surf.}}{S}$ s.t. $N_{C/S} \cong [C]|_C$: top. triv.

C.f. Thm1 (Grauert '62)

- $(C^2) := \deg N_{C/S} < 0 \Rightarrow C$ can be "contracted."
- $(C^2) < \min\{0, 4-4g(C)\}$ (C: sm) $\Rightarrow C$ admits a hol. tub. nbhd."

i.e. $\exists V$: a nbhd of C in S, $\exists V'$: a nbhd of $\tilde{C} :=$ 0-section $\subset N_{C/S}$.

s.t.
$$\begin{array}{ccc} V & \cong & V' \\ \cup & & \cup \\ C & \xrightarrow{\cong} & \tilde{C} \end{array}$$ "

Rmk $\nexists$ hol. tub. nbhd in general. (e.g. Serre's e.g., Ueda's e.g. ⋮)

ellipt. fibr. ← C: ellipt, $N_{C/S} = \mathbb{1}$

C: ellipt, $N_{C/S} \neq \mathbb{1} \in \mathrm{Pic}^0(C)$, non-tor, "with str. Cremona cond"

Thm2 (Arnol'd '76)

C: sm. ellipt. s.t. $N_{C/S} \in \mathrm{Pic}^0(C)$: Dioph. (i.e. $\exists A > 0, \exists d > 0$ s.t. $\mathrm{dist}_{\mathrm{Euc}}(\mathbb{1}_C, N_{C/S}^n) \geq A \cdot n^{-d}$ for $\forall n \geq 1$.)

$\Rightarrow C$ admits a hol. tub. nbhd //

Ueda generalized Arnold's thm as: $\{t\in\mathbb{C} \mid |t|=1\}$ $= U(1)$

Thm 3 (Ueda '83) [cpt ~~Kä~~ with $N_{C/S}$: $U(1)$-flat ($\Leftarrow$ cpt Kä with $(C^2)=0$ dy $N_{C/S}$)]

C: cpt R.S. $\subset S$, $(C^2)=0$.

Assume $N_{C/S}$: torsion or Dioph.

$\forall n\geq 1$, $u_n(C,S)=0 \in H^1(C, N_{C/S}^{-n})$

[difference between $\tilde{C}\subset N_{C/S}$ and $C\subset S$ in n-jet]

Then $\exists \{V_j\}$: open cov. of a nbhd V of C in S,

$\exists w_j : V_j \to \mathbb{C}$: loc. def. func of C

s.t. $w_j = {}^{\exists}t_{jk}\cdot w_k$ on V_{jk}.

$t_{jk} \in U(1)$

($\Downarrow$ ∂V: psd flat i.e. ∂V: Levi-flat) //

c.f. C: ellipt. $N_{C/S}$: non-tor $\Rightarrow H^1(C,N^{-n})\equiv 0$

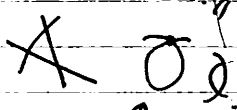
c.f. (C,S): Serre's ex.

$\Rightarrow$ C admits a str. psd conv nbhd ($S\setminus C \cong \mathbb{C}^*\times\mathbb{C}^*$) //

Thm 4 (K-'17) Indiana.

Thm 2 also holds for the case where C is a cycle of rat'l curves,

if $N_{C/S}$: $U(1)$-flat



rat'l curve with a node

In this case,

c.f. ① $H^1(C, L) = \begin{cases} \mathbb{C} & \text{if } L=\mathbb{1}_C \\ 0 & \text{if } L\neq\mathbb{1}_C \end{cases}$

$\rightarrow$ $N_{C/S}$: non-tor $\Rightarrow H^1(C,N^{-n})\equiv 0$,

② $\mathrm{Pic}^0(C) \cong H^1(C,\mathbb{C}^*) = \mathbb{C}^*$, $L \mapsto t(L)$

$\cup$ $\qquad\qquad\qquad\qquad \cup$

$\{U(1)\text{-flat}\} = U(1)$

Rmk [Ueda '91] [K-'17] C: cycle of $\mathbb{P}^1$'s, $|t(N_{C/S})|\neq 1$ $\Rightarrow$ C admits str psd conv. nbhd sys.

Main result

Improvement of Thm4(K-'17),
an analogue of Thm2(Arnol'd)

Thm5(K-)

(C, S) : as above, C: a cycle of $\mathbb{P}^1$'s.
(C', S') —— 〃

s.t. $\begin{cases} C = C', \\ N_{C/S} \cong N_{C'/S'}, \\ t(N_{C/S}) = t(N_{C'/S'}) = e^{2\pi\sqrt{-1}\theta} \end{cases}$
for $\exists \theta \in \mathbb{R} \setminus \mathbb{Q}$: Dioph.

Then $\exists V$: nbhd of C in S,
$\exists V'$: —〃— C' in S',
s.t. $\begin{matrix} V \cong V' \\ \cup \quad \cup \\ C = C' \end{matrix}$ //.

In the prf, we use some techniques from Cpx Dynamics.
origined from...

Thm6(Siegel '42)

Ω: a nbhd of 0 in $\mathbb{C}$.
$f: \Omega \to \mathbb{C}$: $\begin{cases} f(0) = 0 \\ t := f'(0) : \text{Dioph.} \ (\text{as above}) \\ \qquad \in U(1) \end{cases}$

$\Rightarrow$ f: linble

i.e. $\exists\, 0 \in \Omega' \Subset \Omega$, $\exists\, \varphi: \Omega' \to \mathbb{C}$, $0 \mapsto 0$, $\varphi(w) = w + O(w^2)$.
s.t. $\varphi^{-1} \circ f \circ \varphi(w) = t \cdot w$ //

c.f. $\varphi(w) = w + a_2 \cdot w^2 + \cdots$ $\quad f(z) = tz + b_2 \cdot z^2 + \cdots$

$f \circ \varphi(w) = \varphi(t \cdot w)$

$t(w + a_2 \cdot w^2 + \cdots) + b_2 \cdot (w^2 + \cdots)$ $\quad \| \quad$ $tw + a_2 \cdot t^2 w^2 + \cdots$

$\longrightarrow$ $t \cdot a_2 + b_2 = a_2 \cdot t^2$

$a_2 = \dfrac{b_2}{t^2 - t}$.

"small denominator"

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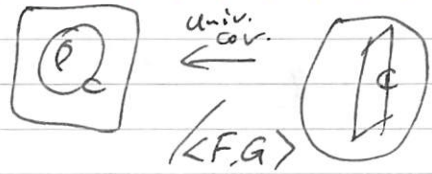
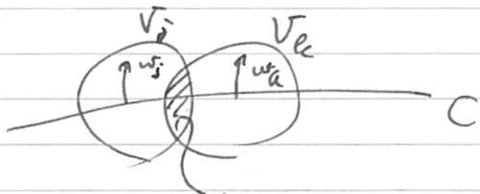
c.f.

- ~~Thm 2~~ (Arnold's thm) $\Longleftarrow$



$\rightsquigarrow$ linearize F and G !

- Thm.3 (Ueda's thm) $\Leftarrow$



$w_j = \underbrace{t_{jk} \cdot w_k + O(w_k^2)}_{\text{linearize!}}$

Problem : what for linearize for cycle case?

E.g 7 (analogue of Ueda's e.g)

Easily we can construct an e.g. of $S \supset \underset{\sim}{C}$ a rat'l curve with a node (cycle)

s.t. $\exists \mathcal{F}$: hol. foliatn. on S

s.t. $\begin{cases} \mathrm{Sing}\,\mathcal{F} = C_{sing} \\ \mathcal{F} \ni C \text{ leaf} \\ \forall \mathcal{L} \in \mathcal{F} \setminus \{C\} : \text{ellipr. curve or } \mathbb{C}^* \end{cases}$

with "holonomy" $= \varphi$ for $\forall \varphi(w) = t \cdot w + O(w^2)$ $|t| = 1$.

t : "str. Cremer cond"

$\Rightarrow$ φ has "small cycles" around 0.

$\Rightarrow$ $\forall V$: nbhd of C,

$\exists E \subset V \setminus C$: sm. ellipt. curve

$\Rightarrow$ $\nexists V$: psd flat $\Rightarrow$ $\nexists \{w_j\}$ as in Thm 2.4.

$\Downarrow$

~~$\nexists$ hol. tub. nbhd~~

$\nexists$ nbhd as "Standard eg" in the following E.g. 9

Pull

(40)

E.g. 8

$C_0 \subset \mathbb{P}^2$: curve of deg = 3, sm. ellipc or ... or ... or ... .

$\cup$

$Z := \{p_1, \cdots, p_9\}$: nine pts. $\subset (C_0)_{reg}$.

$S := Bl_Z \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$: b-up.

$\cup$

$C := (\pi^{-1})_* C_0$. : str. transf.

$\rightsquigarrow$ $N_{C/S}$: top. triv, ~~moves~~ attains $\forall$ p.t. $\in Pic^0(C_0)$ by changing Z. //

E.g. 9. (the standard model of a nbhd of C with $t(N_{C/S}) = \underbrace{t \in \mathbb{C}^*}_{\text{given}}$)

for C: a rat'l curve with a node.

$\pi : \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \mathbb{P}^1$.

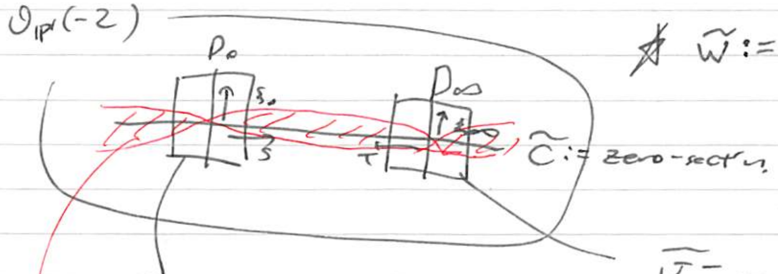
non-homog. coord $S = T^{-1}$.

fiber coord ξ_0 around $\{s=0\} =: D_0$

ξ_∞ around $\{s=\infty\} =: D_\infty$.

$\not\exists\ \widetilde{W} := S\cdot\xi_0 = T\cdot\xi_\infty$

$(\Leftarrow \xi_\infty = S^2\xi_0)$



$\widetilde{V_0^+} := \{|s|<\varepsilon, |\xi_0|<\varepsilon\}$

$\widetilde{V_0^-} := \{|T|<\varepsilon, |\xi_\infty|<\varepsilon\}$

$\widetilde{V_1} :=$ a small nbhd of $\widetilde{C} \setminus \{s=0,\infty\}$.

$\widetilde{V} := \widetilde{V_0^+} \cup \widetilde{V_1} \cup \widetilde{V_0^-}$

$V := \widetilde{V}/\sim$ $\left(\widetilde{V_0^+} \ni (s, \xi_0) \sim F(s,\xi_0) := (\underset{T}{\underset{\shortparallel}{t\xi_0}}, \underset{\xi_\infty}{\underset{\shortparallel}{s}}) \in \widetilde{V_0^-}\right)$

$i : \widetilde{V} \to V$: natural

$C := i(\widetilde{C}) \subset V$; $t(N_{C/S}) = t$. //

(T_0) (~60)

(E.g.9)

<u>Obs</u> $i : \tilde{V} \xrightarrow[/\!\!\sim]{} V \quad : ℚ$

$V_0 := i(\tilde{V}_0^+) = i(\tilde{V}_0^-)$ ··· a nbhd of C_{sing}.

$V_1 := i(\tilde{V}_1)$ ···· (C_{reg})-nbhd.

$W_0 :=$ "$S\cdot\xi_0$" on V_0

$W_1 :=$ "$S\cdot\xi_0$" on V_1

$\rightsquigarrow \quad W_1 = \begin{cases} 1\cdot W_0 & \text{on } V^+ := i(\ \ ) \\ t\cdot W_0 & \text{on } V^- := i(\ \ ) \end{cases}$ //

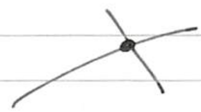
<u>Q</u> Classification of $(C, \underset{\sim}{S})$ ~~with~~ opt s.t. C admits a "standard" nbhd?

<u>Outline of the prf of Thm 5</u> = Main result

<u>Strategy</u> Starting from (general) (C, S) as in Thm 5, we show the existence of "standard" nbhd.

<u>Assume</u> C : a rat'l curve with a node (for simplicity)

① U_0 : a nbhd of the nodal pt. $\subset C$
$\quad = U_0^+ \cup U_0^-$



$U_1 := C_{reg}$.

$U^{\pm} := U_0^{\pm} \cap U_1$.

② V_j : a nbhd of U_j s.t. $V_j \cap C = U_j$.

$V_0 \cap V_1 = \underset{\cup\, U^+}{V^+} \sqcup \underset{\cup\, U^-}{V^-}$

③ $\tilde{V}_0^{\pm}$: 2-copies of V_0,

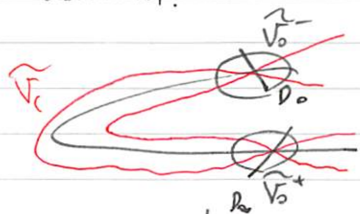
$\tilde{V}_1$: a copy of V_1,

$\widetilde{V} := \widetilde{V}_0^+ \sqcup \widetilde{V}_1 \sqcup \widetilde{V}_0^- / \sim$ ← gen'd by

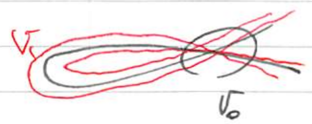
$V^+ \rightrightarrows \widetilde{V}_0^+, \widetilde{V}_1$

$V^- \rightrightarrows \widetilde{V}_1, \widetilde{V}_0^-$

$i : \widetilde{V} \to V$: natural. (univ. cov)



$\downarrow$



$\rightsquigarrow \quad i^*C = \underset{\text{cpt.}}{\widetilde{C}} + \underset{\widetilde{V}_0^+}{D_0} + \underset{\widetilde{V}_0^-}{D_\infty}$

⊙ $\deg(i^* N_{C/S})|_{\widetilde{C}} = 0$

$\overset{\shortparallel}{(\widetilde{C}^2) + (\widetilde{C}.\underset{1}{D_0}) + (\widetilde{C}.\underset{1}{D_\infty})} \rightsquigarrow (\widetilde{C}^2) = -2$

$< \min\{0, 4-4g\}$

$\overset{\text{Th-1 (Camacho)}}{\Longrightarrow}$ $\widetilde{V}$ can be embedded in $\mathcal{O}_{\mathbb{P}^1}(-2)$

$\widetilde{V} \cup \widetilde{C}$ ═══ $\cup$ 0-section.

⊙ $s = \tau^{-1}$: non-hom. coord of $\widetilde{C}$ s.t. $D_0 \cap \widetilde{C} = \{s=0\}$, $D_\infty \cap \widetilde{C} = \{s=\infty\}$.

<u>Claim</u> we can extend $s = \tau^{-1}$ to $\widetilde{V}$ s.t. $D_0 = \{s=0\}$, $D_\infty = \{\tau = 0\}$. by shrinking $\widetilde{V}$ if ness. //

⊙ (∵) $\exists \mathcal{V}$: str. psd conv nbhd of $\widetilde{C}$ s.t. $\mathcal{V} \supset \widetilde{V}$.

Ohsawa's vanishing thm $\underset{K_\mathcal{V} = \mathcal{O}_\mathcal{V}}{\Longrightarrow}$ $H^1(\mathcal{V}, \mathcal{O}_\mathcal{V}) = 0$.

$\underset{\text{exp-ex. seq.}}{\rightsquigarrow}$ $\forall$ top. triv. l.b. $/\mathcal{V}$: hol. triv ↳ Thus for $\widetilde{V}$.

$\rightsquigarrow$ $\mathcal{O}_\mathcal{V}(D_0 - D_\infty)$ has glob. section //

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def. function of τ_{ξ} in V_0

Thm 4 $\Longrightarrow$ $\exists \{W_0, W_1\}$ s.t. $w_{\Sigma} = t_{\pm} W_0$ on $V^{\pm}$
for $\exists t_{\pm} \in U(1)$.

$\widetilde{w} : \widetilde{V} \to \mathbb{C}$: defined by $\widehat{w} = \begin{cases} t_+ w_0 & \text{on } \widehat{V_0^+} \\ w_1 & \text{on } \widehat{V_1} \\ t_- w_0 & \text{on } \widehat{V_0^-} \end{cases}$

$\left(t(N_{\varphi_S}) = t_+ / t_- \right)$.
$\overset{\shortparallel}{t}$

- $\xi_0 := \widehat{w}/S$, $\xi_\infty := \widehat{w}/T$.
- $F : \widehat{V_0^+} \xrightarrow{\cong} \widehat{V_0^-}$ "deck transf." of i
$\left(i^{-1}(i(p)) = \{p, F(p)\} \right)$

$\rightsquigarrow F(S, \xi_0) = \left(\frac{t\xi_0}{G}, G \cdot S \right)$
for $\exists \, \underbrace{G : \widehat{V_0^+} \to \mathbb{C}^*}$.

- by replacing $\widetilde{w}$ with $\widetilde{w}/G(0,0)$, w.m.a. $\underline{G(0,0) = 1}$.

<u>Enough to show</u> : $G \equiv 1$ by changing S, T.

<u>Obs</u>. for $H : \widehat{V} \to \mathbb{C}^*$,

$\hat{S} := H^{-1} \cdot S$, $\hat{T} = H \cdot T$
$\rightsquigarrow$ "$\hat{\xi}_0$" $:= H \cdot \xi_0$ "$\hat{\xi}_\infty$" $:= H^{-1} \cdot \xi_\infty$ $(\widetilde{w} : \text{fixed})$

$\rightsquigarrow F^*\hat{T} = (F^*T) \cdot (F^*H) = \frac{t \cdot \xi_0}{G} (F^*H)$
$\overset{\shortparallel}{\hat{T} \circ F}$
$= \frac{F^*H}{G \cdot H} \cdot t \hat{\xi}_0$.

$\rightsquigarrow$ "$\hat{G}$" $= G \cdot \frac{H}{F^*H}$.

$\rightsquigarrow$ <u>Enough to show</u> :

<u>Prop 10</u>. $\varinjlim_{V^+ : \text{nbhd of } C} H^1(V^+, \mathcal{O}_{V^+}) \xrightarrow[\text{restr}]{} H^1(C, \mathcal{O}_C)$: inj

$\varinjlim H^1(V^+, \mathcal{O}_{V^+}(-C)) = 0$

for $F_\pm : V^\pm \to \mathbb{C}$, we construct a primitive by using a technique from Cpx Dyn.