Ueda theory for compact curves with nodes

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X を滑らかな複素曲面, $C \subset X$ をコンパクトな部分曲線とする. 本講演では, 法線則 $N_{C/X} := j^*\mathcal{O}_X(C)$ が位相的に自明である場合に, C の近傍について議論する. ここで包含射 $C \hookrightarrow X$ を, 記号 j で表している. また以下では, C 上の位相的に自明な直線束全体の集合を $\mathcal{P}(C)$ で, また C 上の Hermitian flat な (つまり変換関数が $U(1)$ 値局所定数関数としてとれるような) 直線束全体の集合を $\mathcal{P}_0(C)$ で表す.

C が滑らかである場合には, 上田による結果がある [U83]. この場合には $\mathcal{P}(C) = \mathcal{P}_0(C)$ である. このとき上田は, X 中での C の近傍と $N_{C/X}$ 中での 0 切断の近傍とを ν-jet で比較することで, 障害類 $u_\nu(C,X) \in H^1(C, \mathcal{O}_C(N_{C/X}^{-\nu}))$ を定義した.

ここでは

[U91]) for a curve C with only nodes included in a non-singular surface X.

Prof. Tetsuo Ueda investigated complex analytic properties of a neighborhood of C when $N_{C/X}$ is topologically trivial in the case where C is non-singular ([U83]) and the case where C is a rational curve with a node ([U91]). When C is a curve with nodes, we define the "type" of the pair (C,X) as the supremum of the set of all integers n such that $u_\nu(C,X) = 0$ holds for all integer $\nu < n$, where $u_\nu(C,X) \in H^1(C, \mathcal{O}_C(N_{C/X}^{-\nu}))$ is the class we will define in §3 as an analogue of Ueda's obstruction class posed in [U83]. Before describing our main results, we first explain our notations. We denote by $\mathcal{E}_0(C)$ the set of all torsion elements of $\mathcal{P}_0(C)$, and by $\mathcal{E}_1(C)$ the set of all elements L of $\mathcal{P}_0(C)$ which satisfies the condition $\log d(\mathcal{O}_C, L^n) = O(\log n)$ as $n \to \infty$, where d is an invariant distance of $\mathcal{P}_0(C)$ ($\mathcal{E}_1(C)$ does not depend on the choice of d, see [U83, §4.1]). For $L \in \mathcal{P}(C)$, we denote by $\mathbb{C}(L)$ the sheaf of constant sections of L. Note that the notion of the constant section is well-defined for $L \in \mathcal{P}(C)$, since L admits a flat connection (even when $L \notin \mathcal{P}_0(C)$, or equivalently, even when L admits no Hermitian flat metric, see Lemma **??**). Note also that, the sheaf $\mathbb{C}(L)$ is independent of the choice of the flat connection up to sheaf isomorphism.

The main results of this paper are the follows. The first one is an analogue of [U83, Theorem 3]:

定理 1. *Let X be a non-singular complex surface, C be a 1-dimensional reduced compact subvariety of X with only nodes such that $N_{C/X} \in \mathcal{E}_0(C) \cup \mathcal{E}_1(C)$. Assume that $i^*N_{C/X} \in \mathcal{E}_0(\widetilde{C})$, where $i\colon \widetilde{C} \to C$ is the normalization of C. Assume also that $H^1(C, \mathbb{C}(N_{C/X}^{-n})) = 0$ holds for each $n \in \mathbb{Z}_{>0}$. Then, if the pair (C,X) is of infinite type, then there exists a neighborhood V of C in X such that $\mathcal{O}_V(C)$ is Hermitian flat.*

Next one is an analogue of [U83, Theorem 1, 2]:

定理 2. *Let X be a non-singular complex surface, C be a 1-dimensional reduced compact subvariety of X with only nodes such that $G(C)$ is a tree and $N_{C/X} = \mathcal{O}_C$, where $G(C)$ is the dual graph of C; i.e. $G(C)$ is the graph such that the vertex set of $G(C)$ is the set of all irreducible components of C and the edge set of $G(C)$ is the set*

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of all nodal points of C. Assume that the type of the pair (C, X) is a finite number $n \in \mathbb{Z}_{>0}$. Assume also that $u_n(C, X)|_{C_\nu} \neq 0 \in H^1(C_\nu, \mathcal{O}_{C_\nu})$ holds for all irreducible component C_ν of C. Then the following holds:
(i) For each real number $\lambda > 1$, There exists a neighborhood V of C and a strongly psh function $\Phi_\lambda \colon V \setminus C \to \mathbb{R}$ such that $\Phi_\lambda(p) \to \infty$ and $\Phi_\lambda(p) = O(d(p, C)^{-\lambda n})$ hold as $p \to C$, where $d(p, C)$ is the distance from p to C calculated by using a local Euclidean metric on a neighborhood of a point of C in V.
(ii) Let V be a neighborhood of C in X, Ψ be a psh function defined on $V \setminus C$. If there exists a real number $0 < \lambda < 1$ such that $\Psi(p) = O(d(p, C)^{-\lambda n})$ as $p \to C$, then there exists a neighborhood V_0 of C in V such that $\Psi|_{V_0 \setminus C}$ is a constant function.

The third one is an generalization of [U91, Theorem 1, 2]:

定理 3. *Let X be a non-singular complex surface, C be a 1-dimensional reduced compact subvariety of X with only nodes such that the dual graph $G(C)$ is a cycle graph ($G(C)$ may be the graph with one vertex and one edge) and $N_{C/X} \in \mathcal{P}(C) \setminus \mathcal{P}_0(C)$. Assume that there exists a flat connection of $N_{C/X}$ such that the type of the pair (C, X) is larger than or equal to 4 with respect to it. Then the following holds:*
(i) For each real number $\lambda > 1$, There exists a neighborhood V of C and a strongly psh function $\Phi_\lambda \colon V \setminus C \to \mathbb{R}$ such that $\Phi_\lambda(p) \to \infty$ and $\Phi(p) = O((-\log d(p, C))^{2\lambda})$ hold as $p \to C$, where $d(p, C)$ is the distance from p to C calculated by using a local Euclidean metric on a neighborhood of a point of C in V.
(ii) Let V be a neighborhood of C in X, Ψ be a psh function defined on $V \setminus C$. If there exists a real number $0 < \lambda < 1$ such that $\Psi(p) = O((-\log d(p, C))^{2\lambda})$ as $p \to C$, then there exists a neighborhood V_0 of C in V such that $\Psi|_{V_0 \setminus C}$ is a constant function.

[U83, Theorem 3] is shown by using L^∞-norm estimates for 0-cochains whose coboundary define the obstruction class $u_\nu(C, X)$ for each ν. Refining this technique by considering the exterior derivatives of such 0-cochains, we prove Theorem 1. However the conditions "$i^* N_{C/X} \in \mathcal{E}_0(\widetilde{C})$" and "$H^1(C, \mathbb{C}(N_{C/X}^{-n})) = 0$" actually play important roles under this strategy of the proof, these conditions are not needed in the case where C is non-singular ([U83, Theorem 3]). In this sense, they seem to be technical conditions and thus we are interested in the following question: Does [U83, Theorem 3] hold also when C is a compact curve with only nodes?

[U83, Theorem 1, 2] and [U91, Theorem 1, 2] are shown by constructing a suitable function Φ_λ on a neighborhood of the curve for each $\lambda > 0$. We prove Theorem 2 and Theorem 3 by generalizing this construction. However the condition "$N_{C/X} = \mathcal{O}_C$" in Theorem 2 is needed to use Proposition **??**, this condition is not needed in the case where C is non-singular ([U83, Theorem 1, 2]). In this sense, this seem to be a technical condition and thus we are interested in the following question also in this case: Does [U83, Theorem 1, 2] hold also when C is a compact curve with only nodes?

参考文献

[1] M. Brunella, On Kähler surfaces with semipositive Ricci curvature, Riv. Mat. Univ. Parma, **1**, 441–450 (2010).

[2] T. Ueda, On the neighborhood of a compact complex curve with topologically trivial normal bundle, Math. Kyoto Univ., **22** (1983), 583–607.

[U83] T. Ueda, On the neighborhood of a compact complex curve with topologically trivial normal bundle, J. Math. Kyoto Univ., **22** (1983), 583–607.

[U91] T. Ueda, Neighborhood of a rational curve with a node, Publ. RIMS, Kyoto Univ., **27** (1991), 681–693.