Spectra and Cohomology Theories

Chroma 2022 Week 2

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Overview

Last week: crash course on the stable homotopy category.

Claim (Woodruff, 2022)

The stable homotopy category **HoSpectra** is a nice place to do stable homotopy theory.

This week

Explore this in more detail; what is this category, and what makes it so nice?

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Preliminaries

Some categories we'll be seeing:

Top: the category of "nice" topological spaces and continuous maps.

CW: the category of CW complexes and cellular maps.

$\mathbf{Top}_*$: the category of *pointed* "nice" topological spaces and basepoint-preserving continuous maps.

$\mathbf{CW}_*$: the category of *pointed* CW complexes and basepoint-preserving cellular maps.

Suspension

For X in $\mathbf{Top}_*/\mathbf{CW}_*$,

$$\Sigma X = S^1 \wedge X.$$

Defines a functor $\Sigma : \mathbf{Top}_*/\mathbf{CW}_* \to \mathbf{Top}_*/\mathbf{CW}_*$.

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Spectra

Definition

A *spectrum* E is a sequence $\{E_n\}_{n\in\mathbb{Z}}$, with $E_n \in \mathbf{Top}_*$, along with pointed maps $s_n : \Sigma E_n \to E_{n+1}$ called the *structure maps* of the spectrum.

The data in a spectrum:

$$
\begin{tikzcd}
\cdots & E_{n-1} \arrow[d, dotted, mapsto] & E_n \arrow[d, dotted, mapsto] & E_{n+1} \arrow[d, dotted, mapsto] & \cdots \\
\cdots \arrow[ur, "s_{n-2}"] & \Sigma E_{n-1} \arrow[ur, "s_{n-1}"] & \Sigma E_n \arrow[ur, "s_n"] & \Sigma E_{n+1} \arrow[ur, "s_{n+1}"] & \cdots
\end{tikzcd}
$$

Remark

Might be helpful to think of spectra as “topological chain complexes.”

Examples

Example (Suspension spectra)

Take X in $\mathbf{CW}_*$. We can define a spectrum $\Sigma^\infty X$, where

$$(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X & n \geq 0 \\ \{*\} & n < 0 \end{cases}$$

and with structure maps

$$s_n : \Sigma(\Sigma^\infty X)_n = \Sigma^{n+1} X \stackrel{\mathrm{id}}{\to} \Sigma^{n+1} X = (\Sigma^\infty X)_{n+1}.$$

This is called a *suspension spectrum* of X.

Example (Sphere spectrum)

We have the *sphere spectrum*

$$\mathbb{S} := \Sigma^\infty S^0.$$

Examples

Example (Eilenberg-Maclane spectrum)

Take an abelian group $G \in \mathbf{Ab}$. We can define a spectrum HG, where

$$(HG)_n = K(G, n),$$

and with structure maps
$s_n : \Sigma(HG)_n = \Sigma K(G, n) \to K(G, n+1) = (HG)_{n+1}$ given to be the *adjoints* to the isomorphisms

$$K(G, n) \xrightarrow{\sim} \Omega K(G, n+1).$$

This spectrum is called an *Eilenberg-Maclane spectrum.*

The category **Spectra**

Definition (Maps of spectra)

A *map* f between a spectrum $E = \{E_n\}$ with structure maps $\{s_n\}$, and a spectrum $F = \{F_n\}$ with structure maps $\{t_n\}$ is a collection

$$\{f_n : E_n \to F_n\}$$

such that the following squares commute for all n:

$$\begin{tikzcd} \Sigma E_n \arrow[r, "s_n"] \arrow[d, "\Sigma f_n"'] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma F_n \arrow[r, "t_n"'] & F_{n+1} \end{tikzcd}$$

Definition (**Spectra**)

Spectra and their maps form the *category of spectra*, **Spectra**.

Homotopy of spectra

Definition (Smash product of space and spectrum)

Take $X \in \mathbf{Top}_*$, and a spectrum $E = \{E_n\}$ with structure maps $\{s_n\}$. We can form a spectrum $E \wedge X$, by letting

$$(E \wedge X)_n = E_n \wedge X,$$

with structure maps $\{s_n \wedge 1_X\}$.

Definition (Homotopy of spectra)

Let $f, g : E \to F$ be two maps of spectra. A *homotopy* between f and g is a map

$$H : E \wedge [0,1]_+ \to F, \tag{10}$$

where the restriction of H to $E \wedge \{0\}_+$ is f and the restriction of H to $E \wedge \{1\}_+$ is g.

Definition of **HoSpectra**?

Definition?

Spectra and homotopy classes of maps between them form the *stable homotopy category*, **HoSpectra**.

$\Rightarrow$ Not quite... it turns out that we run into some problems if we choose this to be our category. Roughly, we "don't have enough maps," and so some spectra we want to be (homotopy) equivalent are not.

Homotopy groups of spectra

Definition (Homotopy groups of spectra)

The *homotopy groups* of a spectrum E, denoted $\pi_n(E)$, are $\varinjlim_a \pi_{n+a}(E_a)$, where the direct limit is taken over the sequence below:

$$\cdots \longrightarrow [S^{n+a}, E_a] \xrightarrow{\Sigma} [S^{n+a+1}, \Sigma E_a] \xrightarrow{(s_a)_*} [S^{n+a+1}, E_{a+1}] \longrightarrow \cdots$$

Example (Stable homotopy groups)

The homotopy groups of a suspension spectrum $\Sigma^\infty X$ are the *stable homotopy groups* of the space X;

$$\pi_n(\Sigma^\infty X) = \pi_n^{\mathrm{st}}(X) = \pi_{n+a}(\Sigma^a X) \quad \text{(for } a \text{ large enough)}$$

In particular, $\pi_n(\mathbb{S}) = \pi_{n+a}(S^a)$ (for a large) is the *nth stable homotopy group of spheres.*

The category **HoSpectra** (for real this time)

Note: π_n is a functor $\textbf{Spectra} \to \textbf{Ab} \Rightarrow$ It makes sense to ask about the induced map $\pi_n(f)$, where f is a map of spectra.

Definition (Weak homotopy equivalence of spectra)

Take $E, F \in \textbf{Spectra}$, and $f : E \to F$. We say f is a *weak homotopy equivalence* if the induced map $\pi_n(f)$ is an isomorphism for *all* $n \in \mathbb{Z}$.

Proposition (The category **HoSpectra**)

There is a category **HoSpectra**, *whose objects are spectra and whose morphisms are maps of spectra, where every weak homotopy equivalence of spectra is an isomorphism. That is, for every weak homotopy equivalence $f : E \to F$, we formally add an inverse $f^{-1} : F \to E$. This category what we will call the stable homotopy category.*

Some remarks

Remark

It is not obvious how we can build such a category; the modern approach seems to be via the theory of *model categories*.
$\Rightarrow$ Barnes-Roitzheim *Foundations of Stable Homotopy Theory*, Ch.2

Remark

If we restrict to spectra which are "nice" enough, then homotopy equivalence $\Leftrightarrow$ weak homotopy equivalence.

Remark

The *homotopy category of CW spectra* offers a more explicit description of **HoSpectra**.
$\Rightarrow$ Rudyak *On Thom Spectra, Orientability and Cobordism* (Ch. 2) or Adams *Stable Homotopy and Generalized Homology* (Part 3)

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How is **HoSpectra** nice?

Proposition (Structure of HoSpectra)

HoSpectra *is a*

- *symmetric monoidal (has a "tensor product" that is commutative)*
- *closed (has an "internal Hom"),*
- *triangulated (has "distinguished triangles" which lead to long exact sequences)*

category.

Remark

A lot of familiar categories are closed symmetric monoidal (e.g. $\mathbf{Top}_*$ with $\wedge$, $\mathbf{Ab}$ with $\oplus$).

Smash product on **HoSpectra**

Proposition (Existence of smash product of spectra)

There exists a smash product

$$\wedge : \mathbf{HoSpectra} \times \mathbf{HoSpectra} \to \mathbf{HoSpectra},$$

making it into a symmetric monoidal category, with unit object being the sphere spectrum $\mathbb{S}$. *That is, we have natural isomorphisms*

$$\begin{aligned} l_X : &\quad \mathbb{S} \wedge A \xrightarrow{\sim} A \\ r_X : &\quad A \wedge \mathbb{S} \xrightarrow{\sim} A \\ a_{X,Y,Z} : &\quad (X \wedge Y) \wedge Z \xrightarrow{\sim} X \wedge (Y \wedge Z) \\ s_{X,Y} : &\quad X \wedge Y \xrightarrow{\sim} Y \wedge X \end{aligned}$$

Further, this smash product plays nicely with Σ, Σ^∞

Function spectra

Proposition

For any two spectra X, Y, there exists a function spectrum $F(X, Y)$ making **HoSpectra** *into a closed symmetric monoidal category. That is, we have an adjunction*

$$\mathrm{Hom}(X \wedge Y, Z) \simeq \mathrm{Hom}(X, F(Y, Z))$$

Triangles

Definition (Shift functor)

A *shift functor* on a category $\mathcal{C}$ is an autoequivalence $T : \mathcal{C} \to \mathcal{C}$.

Example

Consider the category **ChAb** of chain complexes of abelian groups. If $C_* = \{C_n, d_n\}$ is a chain complex, we can define the *right shift* of C_* to be the complex $C[1]_*$ with $C[1]_n = C_{n+1}$, $d[1]_n = d_{n+1}$. This lets us define a shift functor $[1]$: **ChAb** $\to$ **ChAb**.

Triangulated categories

Definition (Triangle)

Given a (additive) shift functor T on an (additive) category $\mathcal{C}$, a *triangle* in $\mathcal{C}$ is a sequence of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

Definition (Triangulated category)

A *triangulated category* is an additive category $\mathcal{C}$ with a translation functor T and a class of *distinguished triangles*; triangles which satisfy a couple of important axioms which I will not list here.

Fact

Triangulated categories have special functors (called (co)homological functors) which take triangles to long exact sequences.

HoSpectra is triangulated

Definition (Suspension of spectra)

Given a spectrum $E = \{E_n, s_n\}$, we can define the *suspension* of E, denoted by $\Sigma E = \{(\Sigma E)_n, (\Sigma s)_n\}$, to be the spectrum where

$$(\Sigma E)_n = E_{n+1}, \quad (\Sigma s)_n = s_{n+1}$$

This lets us define an shift functor Σ : **HoSpectra** $\to$ **HoSpectra**.

Proposition

HoSpectra, *along with suspension* Σ, *is a triangulated category. The distinguished triangles are called the fiber/cofiber sequences; for spectra, these agree. These distinguished triangles are what give rise to long exact sequences in homotopy groups.*

Remark

This makes the analogy with chain complexes a little more convincing; given a ring R, the *derived category* **D(R-mod)**, whose objects are chain complexes of R-modules, is also a symmetric monoidal, closed, triangulated category.

This offers a pedagogical unification between long exact sequences in homological algebra and homotopy theory.

You can maybe think of the move from $\mathbf{Top}_*$ to **HoSpectra** ($X \mapsto \Sigma^\infty X$) as analogous to the move from **R-mod** to **D(R-mod)** ($M \mapsto$ some resolution of M).[a]

[a] I do not want to accept any responsibility for wrong intuition caused by this analogy, so please use with a couple of grains of salt.

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(Co)homology theories

Definition

An *(extraordinary)* homology theory E_* is a collection of covariant functors $\{E_n\}_{n\in\mathbb{Z}}$ from pairs of spaces $\mathbf{Top}^2 \to \mathbf{Ab}$, satisfying:

(homotopy invariance) Homotopic maps induce the same map.

(exactness) For every pair (X, A), there is a natural long exact sequence

$$\cdots \longrightarrow E_n(A, \varnothing) \xrightarrow{i_*} E_n(X, \varnothing) \xrightarrow{j_*} E_n(X, A) \xrightarrow{\partial} E_{n-1}(A) \longrightarrow \cdots$$

where $i : (A, \varnothing) \hookrightarrow (X, \varnothing)$ and $j : (X, \varnothing) \to (X, A)$.

(excision) For every pair (X, A), the collapse $(X, A) \to (X/A, \{*\})$ induces an isomorphism $E_n(X, A) \to E_n(X/A, \{*\})$.

An *(extraordinary) cohomology theory E^** is a collection of contravariant functors $\{E^n\}_{n\in\mathbb{Z}}$ $\mathbf{Top}^2 \to \mathbf{Ab}$, satisfying axioms dual to above.

Important examples

Example

Our favorite; *singular (co)homology*, $H_n(-; A)$ and $H^n(-; A)$, with coefficients in some abelian group A.

Example

It turns out that *stable homotopy* is a homology theory; the functors $\pi_n^{\mathrm{st}}(-)$ satisfy the axioms above.

Example

(Co)bordism theories, which are defined by considering certain equivalence classes of manifolds and maps between them.

Reduced (co)homology

Definition

A *reduced homology theory* $\widetilde{E}_n$ is a sequence of covariant functors $\mathbf{Top}_* \to \mathbf{Ab}$ satisfying:

(homotopy invariance) Homotopic maps induce the same map.

(exactness) For a triple (X, A, x_0), we have that the sequence

$$\widetilde{E}_n(A) \xrightarrow{i_*} \widetilde{E}_n(X) \xrightarrow{j_*} \widetilde{E}_n(C_i)$$

is exact for all n, where C_i is the mapping cone of i.

(suspension isomorphism) There are natural isomorphisms

$$s_n : \widetilde{E}_n(X) \to \widetilde{E}_{n+1}(\Sigma X)$$

A *reduced cohomology theory* $\widetilde{E}^n$ is a sequence of contravariant functors $\mathbf{Top}_* \to \mathbf{Ab}$ satisfying axioms dual to above.

Examples

Example

Reduced singular (co)homology $\widetilde{H}_n$ and $\widetilde{H}^n$.

Example

Any unreduced (co)homology theory defines a reduced one.
$\Rightarrow$ In fact, this gives a bijection between unreduced and reduced cohomology theories (up to suitable equivalence).

Spectra $\Leftrightarrow$ (co)homology theories

Proposition (Spectra define (co)homology theories)

Given a spectrum E, we can define a reduced homology theory via

$$\widetilde{E}_n(X) = [\mathbb{S}, (\Sigma^\infty X) \wedge E] \simeq \pi_n((\Sigma^\infty X) \wedge E)$$

and a reduced cohomology theory via

$$\widetilde{E}^n(X) = [\Sigma^\infty X, E]_{-n} \simeq \pi_{-n}(F(\Sigma^\infty X, E))$$

In fact, we have something stronger:

Proposition (Brown representability)

If $\widetilde{E}^*$ is a reduced cohomology theory on $\mathbf{CW}_*$*, then it is represented by a spectrum which is unique up to equivalence in* **HoSpectra**.

$\Rightarrow$ *Spectra and cohomology theories are the same thing!*

For our previous examples...

Example

Singular (co)homology with coefficients in an abelian group G is represented by the *Eilenberg-Maclane spectrum HG*.
$\Rightarrow$ like how Eilenberg-Maclane *spaces* represent singular cohomology; the functors $H^n(-; G)$ and $[-, K(G, n)]$ are naturally isomorphic.

Example

Stable homotopy is represented by $\mathbb{S}$.

Example

(Co)bordism theories are represented by *Thom spectra* (to be introduced in next talk!)

Rings

Definition (Ring spectrum)

A *ring spectrum* is a spectrum E, along with a map $m : E \wedge E \to E$ (called the *multiplication map*) and a map $\iota : \mathbb{S} \to E$ (called the *unit map*) if the *left and middle* diagrams commute in **HoSpectra**. If additionally the diagram on the right commutes, E is a *commutative ring spectrum*.

(associativity)

$$\begin{tikzcd} (E \wedge E) \wedge E \arrow[rr, "a_{E,E,E}"] \arrow[d, "m \wedge 1_E"'] & & E \wedge (E \wedge E) \arrow[d, "1_E \wedge m"] \\ E \wedge E \arrow[dr, "m"'] & & E \wedge E \arrow[dl, "m"] \\ & E & \end{tikzcd}$$

(unitarity)

$$\begin{tikzcd} S \wedge E \arrow[r, "\iota \wedge 1"] \arrow[dr, "l_E"'] & E \wedge E \arrow[d, "m"] & E \wedge S \arrow[l, "1 \wedge \iota"'] \arrow[dl, "r_E"] \\ & E & \end{tikzcd}$$

(commutativity)

$$\begin{tikzcd} E \wedge E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \wedge E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

Proposition

E is a ring spectrum $\Leftrightarrow$ E represents a multiplicative cohomology theory (a cohomology theory with a cup product.)