CHROMATIC HOMOTOPY THEORY AND p-DIVISIBLE GROUPS

KEITA ALLEN
BABYTOP FALL 2023 - TOPOLOGICAL AUTOMORPHIC FORMS
TALK 2

CONTENTS

These are notes for a talk provided for Babytop, a student-led algebraic topology learning seminar, during fall 2023. In my talk, I'll be trying to motivate some of the big ideas in chromatic homotopy theory and introduce p-divisible groups, a class of objects which we will see down the line in the seminar. As usual, if you would like more detailed/reliable sources for what I'm talking about here, take a look at the sources cited.

1. STABLE HOMOTOPY THEORY AND FORMAL GROUPS

This will be a lightning-fast introduction to some of the big ideas in stable and chromatic homotopy theory. I will only give a couple of proper definitions and definitely not be proving anything, but if any of this piques your interest you can take a look at [Mal14] (for a short introduction to the ideas of stable homotopy theory) and [Lur10] (for an introduction to chromatic homotopy theory).

1.1. **Spectra and the stable homotopy category.**

Description 1.1. Spectra are a "variation on the idea of spaces"; we can talk about them like we talk about spaces, but they also conveniently package a lot of the information that we talk about in homotopy theory. For our purposes, the following facts will be important.

(1) For a spectrum X, we can talk about its *homotopy groups*, denoted $\pi_*(X)$.
(2) To a spectrum E, we have associated its *E-cohomology* $E^*(-)$, a cohomology theory in the sense of satisfying the Eilenberg-Steenrod axioms (minus the dimension axiom). In fact, this association is an equivalence, so often the words "spectra" and "cohomology theory" are used interchangeably.

Example 1.2. For A an abelian group, there is a spectrum $\mathrm{H}A$, known as the *Eilenberg-Maclane spectrum* of A. This spectrum represents singular (ordinary) cohomology with coefficients in A.

Example 1.3. For R a (commutative) ring, there is a spectrum $K(R)$, known as the *algebraic K-theory spectrum* of R. The homotopy groups $\pi_n(K(R))$ are the algebraic K-theory groups of the ring R, commonly denoted $K_n(R)$.

Spectra form an (∞-)category, denoted Sp, which is also known as the stable homotopy category. For now, stable homotopy theory can be described as the study of the stable homotopy category.[1]

1.2. **Formal groups and the chromatic approach.** We now have an interlude to the world of ordinary algebra.

Definition 1.4. Fix a positive integer n, and let $\underline{x}$ denote the collection of polynomial generators $(x_1, \ldots, x_n)$. A *n-dimensional (commutative) formal group law* (FGL) over a commutative ring R is a collection of n power series

$$f(\underline{x}, \underline{y}) = \left(f_1(\underline{x}, \underline{y}), \ldots, f_n(\underline{x}, \underline{y})\right),$$

where $f_i(\underline{x}, \underline{y}) \in R[\![\underline{x}, \underline{y}]\!]$, the power series ring in $2n$ variables, such that the following identities hold.

(1) $f_i(\underline{x}, 0) = f_i(0, \underline{x}) = x_i$ for all i.
(2) $f(\underline{x}, \underline{y}) = f(\underline{y}, \underline{x})$.
(3) $f(f(\underline{x}, \underline{y}), \underline{z}) = f(\underline{x}, f(\underline{y}, \underline{z}))$ (in $R[\![\underline{x}, \underline{y}, \underline{z}]\!]$).

Remark 1.5. When $n = 1$, the above definition just becomes one power series $f \in R[\![x, y]\!]$, satisfying the identities (1)-(3).

Remark 1.6. If the abundance of coordinates in the definition above made your skin crawl, do not despair; there is a nice, coordinate-free generalization of formal group laws, called *formal groups*. I won't define these here, but they may come up in later talks.

It turns out that formal group laws play a big part in the modern study of stable homotopy theory; this starts with the notion of a complex orientation on a spectrum.

Definition 1.7. Let E be a spectrum with a multiplication $E \otimes E \to E$. E is called *complex-orientable* if the map $E^2(\mathbb{CP}^\infty) \to E^2(S^2)$ is surjective.

We can actually retrieve a formal group law from such a spectrum.

Construction 1.8. Given a complex-orientable spectrum E, we have isomorphisms

$$E^*(\mathbb{CP}^\infty) \simeq \pi_*(E)[\![t]\!], \quad E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq \pi_*(E)[\![x, y]\!].$$

Using a topological group structure on $\mathbb{CP}^\infty$, this gives a map

$$\pi_*(E)[\![t]\!] \simeq E^*(\mathbb{CP}^\infty) \longrightarrow E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq \pi_*(E)[\![x, y]\!].$$

The image of t gives us a 1-dimensional formal group law.

[1]This description has the advantage of being short and sweet, but has the disadvantage of not really being descriptive. A word of caution; using too many descriptions like this may cause your peers to think you are obnoxious.

Example 1.9. Singular cohomology $H\mathbb{Z}$ is complex-orientable, and its associated formal group law is the *additive formal group law*, $x + y$.

Example 1.10. There is a spectrum MU, representing *complex bordism*, which is complex-orientable. In fact, MU is *universal* amongst such theories; a spectrum X is complex-orientable if and only if there is a map of spectra $MU \to X$. A choice of such a map is known as a *complex orientation* of X.

Chromatic homotopy theory can be summarized as the study of how the structure of formal groups can be exploited to understand stable homotopy theory. We'll begin to describe some of this structure in the next section.

1.3. **Heights and Morava K-theory.** An invariant of formal groups which turns out to be of great use is height.

Definition 1.11. Let $f \in R[\![x, y]\!]$ be a 1-dimensional formal group law. For every nonnegative integer m, we define a power series $[m](t) \in R[\![t]\!]$, known as the *m-series* of f, inductively as follows.

(1) $[0](t) = 0$.
(2) $[m](t) = f([m-1](t), t)$ for $m > 0$.

Definition 1.12. Let f be a 1-dimensional formal group law over R, and fix a prime p. We denote by v_n the coefficient of t^{p^n} in $[p](t)$. We say f has

(1) *height* $\geq n$ if $v_i = 0$ for $i < n$.
(2) *height exactly* n if $v_i = 0$ for $i < n$ and v_n is a unit in R.

Example 1.13. If $f = x + y$ is the additive formal group law over a ring with $p = 0$, then $[p](t) = pt = 0$, and so f has height $\geq n$ for every n. In this case, we say f has infinite height.

The above example shows us that the formal group associated to ordinary cohomology $H\mathbb{Z}/p$ is of infinite height for p. One might then wonder if we can produce spectra which yield finite-height formal group laws. Luckily, we can.

Description 1.14. For each prime p and each positive integer n, there is a spectrum $K(n)$ which is of height exactly n (for p). These are collectively referred to as *Morava K-theory* spectra.

These spectra turn out to be central to chromatic homotopy theory. Through a theory of localization of spectra known as *Bousfield localization*, given a spectrum X we can study its $K(n)$-*localization* $L_{K(n)}X$, which roughly corresponds to studying the "height n portion" of X. This lets us break up the study of Sp by height, which turns out to be very fruitful. We won't have time to discuss this anymore today, but perhaps it will be explored further in the coming talks.

2. p-DIVISIBLE GROUPS AND FRIENDS

We now make a pivot to discussing p-divisible groups, a likely unfamiliar family of objects, and their relationships with some objects which may be a bit more familiar. My main source for this section is [Tat67], but a healthy amount of Googling also helped a lot.

2.1. **p-divisible groups and algebraic groups.** In what follows, let k be a field. For our purposes, an *algebraic group* will be a finite-type k-scheme with a group structure (i.e. we will not be assuming reducedness or separatedness, although you can if you would like to). We will also assume all groups are abelian.

Definition 2.1. Let A be an algebraic group, and fix a positive integer m. We say A is *finite of order* m (over k) if A is finite over k (as a scheme) of constant rank m.

Definition 2.2. Fix an nonnegative integer h. A *p-divisible group of height* h over k is a system of algebraic groups over k

$$G = \{G_i, f_i\}_i = \{G_0 \overset{f_0}{\to} G_1 \overset{f_1}{\to} \cdots\}$$

such that

(1) G_i is finite of order p^{ih} over k.
(2) For each i,

$$0 \to G_i \to G_{i+1} \overset{\times p^i}{\to} G_{i+1}$$

is exact.

Remark 2.3. Let's think about what such a system would look like if we were working with ordinary finite abelian groups. The axioms tell us that G_i is an abelian group of order p^{ih}, so G_i has to be the direct sum of $\mathbb{Z}/p^n$s. Further, it is also identified with the p^i-torsion of G_{i+1}. So for any G_i, given that its p^{i-1}-torsion, G_{i-1}, is of order $p^{(i-1)h}$, a little bit of thought tells us that we must have $G_i = \mathbb{Z}/p^i$, with $G_i \to G_{i+1}$ given by multiplication by p.

This definition might seem a bit mysterious, but hopefully the following construction helps to illuminate why we might consider such objects.

Construction 2.4. Let A be an (abelian) algebraic group over k, and let $A_n = \ker(A \overset{\times n}{\to} A)$. Then

$$A(p) := \{A_1 = 0 \to A_p \to A_{p^2} \to \cdots\}$$

is a p-divisible group (where the maps are inclusions). If A is of dimension d, then $A(p)$ is of height $2d$.

Here, we can think of $\mathrm{coker}(A_{p^{i-1}} \to A_{p^i})$ as the p-torsion of A which is killed by p^i but not p^{i-1}. In this way, this p-divisible group naturally encodes the p-torsion information of A.

2.2. **p-divisible groups and formal groups.** It turns out that p-divisible groups are also closely related to our favorites, formal group laws. For what follows, k will be a field of characteristic $p > 0$.

Definition 2.5. Let $A = k[\![x_1, \ldots, x_n]\!]$, $I = (x_1, \ldots, x_n) \subset A$, and take an n-dimensional formal group law $f = (f_1(\underline{x}, \underline{y}), \ldots, f_n(\underline{x}, \underline{y}))$. Define a homomorphism $\psi : A \to A$ via $x_i \mapsto f_i(\underline{x}, \underline{x})$. We say that the formal group law f is *(p-)divisible* if ψ^p is finite free.

Construction 2.6. We continue using the notation from the definition above. If we let

$$A_i = A/\psi^{p^i}(I)A,$$

then $\operatorname{Spec} A_i$ inherits an abelian group structure from f. This gives us a p-divisible group

$$f_p = \{\operatorname{Spec} A_0 \to \operatorname{Spec} A_1 \to \cdots\}.$$

This p-divisible group is of height h, where p^h is the degree of ψ.

This construction seems a bit mysterious, at least to me. The next result we will mention, however, will show us that it is a natural one. First, a definition.

Definition 2.7. A p-divisible group $G = \{G_i\}$ is *connected* if the G_i are connected.

Proposition 2.8. *There is an* equivalence *between the categories of divisible FGLs over k and connected p-divisible groups over k, given via $f \mapsto f_p$*

The proof is apparently long and technical, so I'll omit it here. But if we accept this result, we now have a familiar way to think about connected p-divisible groups. Given this, one may wonder if we have a convenient way to encode the *failure* of a p-divisible group to be connected. It turns out we do; we'll start working towards describing this now.

Definition 2.9. A p-divisible group $G = \{G_i\}$ is *étale* if each G_i are étale over k.

Proposition 2.10. *For an algebraic group A of finite order over k, there is a canonical exact sequence*

$$0 \to A^0 \to A \to A^{\text{ét}} \to 0,$$

where A^0 is connected and $A^{\text{ét}}$ is étale (Concretely, A^0 is the connected component of the identity of A, and $A^{\text{ét}}$ is obtained by considering the maximal étale subalgebras of an affine open cover of A).

If G is a p-divisible group, this extends to an exact sequence

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0,$$

where G^0 is connected and $G^{\text{ét}}$ is étale.

This proposition tells us that we can think of such an algebraic group/p-divisible group as decomposable into a connected part and an étale part, with the étale part measuring the failure to be connected (and vice versa).

2.3. **Duality.** To close, we will briefly discuss duality and how it interacts with what we discussed above.

Definition 2.11. Let $\hat{A}$ denote the Cartier dual of an algebraic group A. Given a p-divisible group $G = \{G_i\}_i$, we can define the *dual p-divisible group*

$$\hat{G} = \{\hat{G}_i, f_i\},$$

where f_i are the duals of the maps $G_{i+1} \to G_i$ induced by multiplication by p.

References

[Lur10] Jacob Lurie. Chromatic homotopy theory. *Lecture series*, 2010.

[Mal14] Cary Malkiewich. The stable homotopy category. *Online Lecture Notes. Binghamton University*, 2014.

[Tat67] John T Tate. p-divisible groups. In *Proceedings of a Conference on Local Fields: NUFFIC Summer School held at Driebergen (The Netherlands) in 1966*, pages 158–183. Springer, 1967.