Computing the homology of the $\mathbb{C}$-motivic lambda algebra

Keita Allen[1], mentored by Mark Behrens[2] & Peter May[3]

[1]Leadership Alliance @ The University of Chicago
[2]University of Notre Dame
[3]University of Chicago

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Example

The *hairy ball theorem* states that if you have a sphere with hair on it, then there is no way to comb the hair flat.

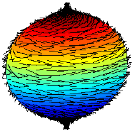

Figure: A (failed) attempt to comb the sphere. PC: Wikimedia Commons

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Further, there is a procedure for computing this homology, called the *Curtis algorithm*.

The input to the Adams spectral sequence

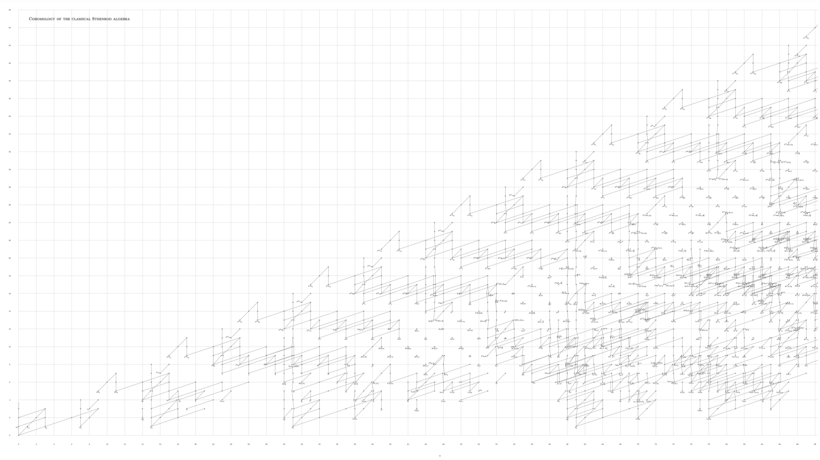

Figure: The E_2 page of the Adams spectral sequence. PC: Isaksen-Wang-Xu

The motivic story

Algebraic geometry

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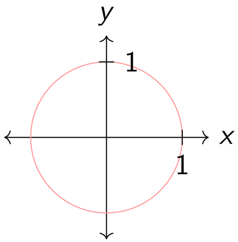
Consider the equation $x^2 + y^2 = 1$. The solutions to this equation form the unit circle in the plane $\mathbb{R}^2$.



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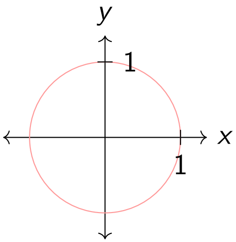

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The motivic innovations

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$\Rightarrow$ Here, we have motivic analogues of much of the things discussed before; there are *motivic (stable) homotopy groups*, *motivic Adams spectral sequences*, and *motivic lambda algebras,* which vary in form depending on our choice of *base field* ("coefficients") k.

The input to the $\mathbb{C}$-motivic Adams spectral sequence

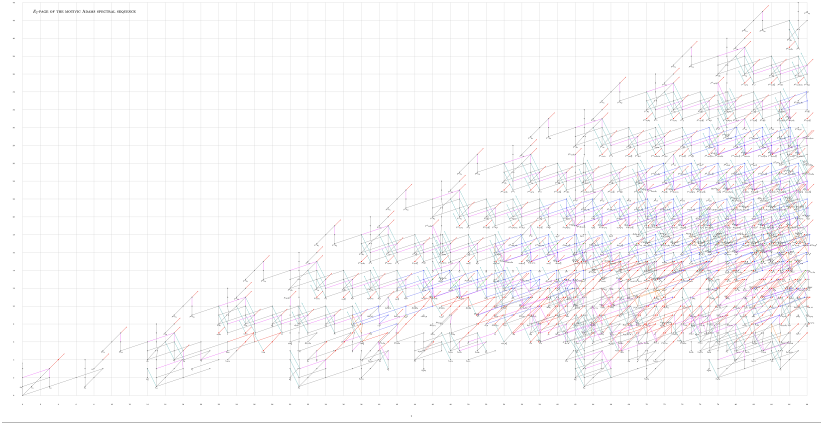

Figure: The E_2 page of the $\mathbb{C}$-motivic Adams spectral sequence. PC: Isaksen-Wang-Xu

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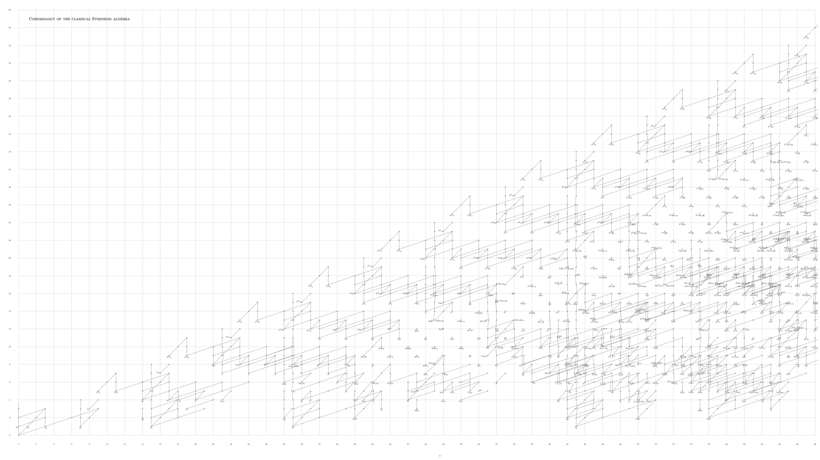

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Conjecture

In this case, we *do* have a Curtis algorithm. In fact, one can effectively use the *classical* Curtis algorithm to read off the input to the $\mathbb{C}$-motivic Adams spectral sequence.

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