# Spectra and Cohomology Theories

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These are notes taken in preparation for a talk on spectra for *Chroma 2022*, a summer seminar in stable/chromatic homotopy theory. Most of the stuff I talk about can be found in [BR20], [Mal14], [Rud98], and [Ada74]; if you want sources which are less likely to have typos, please take a look there.

## 1 Introduction

Last week, we were introduced to the *stable homotopy category* **HoSpectra**, as a convenient place to do stable homotopy theory. This week, I'd like to spend some time describing in a bit more detail what this category is, and some reasons why the masses have decided it to be the desired setting for stable work.

In what follows, the word *pointed space* will mostly refer to a "nice"[1] space with a choice of basepoint; the category of such spaces we will denote simply by $\mathbf{Top}_*$.[2] For such a space $X$ with basepoint $x_0$, $\Sigma X$ will denote the *(reduced) suspension* of $X$:

$$\Sigma X = \frac{[0,1] \times X}{\{(0,x) \sim (0,y), (1,x) \sim (1,y), (s,x_0) \sim (t,x_0)\}} \tag{1.1}$$

for all $x, y$ in $X$ and $s, t$ in $[0,1]$. $\Sigma X$ inherits its basepoint from $X$, given by the class of $(x_0, t)$. More concisely, one can write this as the smash product $S^1 \wedge X$. Given $f \in \mathrm{Hom}_{\mathbf{Top}_*}(X,Y)$, one can define $\Sigma f \in \mathrm{Hom}_{\mathbf{Top}_*}(\Sigma X, \Sigma Y) = \mathrm{Hom}_{\mathbf{Top}_*}(S^1 \wedge X, S^1 \wedge Y)$ as

$$\Sigma f = \mathbf{1}_{S^1} \wedge f,$$

where $\mathbf{1}_{S^1}$ denotes the identity on $S^1$. This makes suspension into a functor $\Sigma : \mathbf{Top}_* \to \mathbf{Top}_*$; as discussed last week, this functor is central to stable homotopy theory.

[1] If this is too vague for you, you can take "nice" to mean something like "weak Hausdorff compactly generated", as per [Rud98]. Or you can just pretend everything is a CW complex.

[2] For technical reasons, we will require our basepoints to be *nondegenerate*; that is, for $(X, x_0) \in \mathbf{Top}_*$, we will require $\{x_0\} \hookrightarrow X$ to be a cofibration.

We will denote the category of pointed CW complexes by **CW**$_*$. With the same definitions as above, suspension is a functor $\Sigma$ : **CW**$_*$ $\to$ **CW**$_*$. In both cases, we let $\Sigma^n$ denote the $n$-fold composition of $\Sigma$.

We will denote by **Top** the category of (unpointed) “nice” topological spaces; the word *space* will be used to refer to such a space. Given a space $X$, we can define a pointed space $X_+$ to be $X \sqcup \{*\}$, with basepoint being the point $*$. This gives us a functor $(-)_+$ : **Top** $\to$ **Top**$_*$. Similarly, this descends to a functor $(-)_+$ : **CW** $\to$ **CW**$_*$.

**Definition 1.2.** A *spectrum* $E$ is a sequence $\{E_n\}_{n\in\mathbb{Z}}$, with $E_n$ being pointed spaces, along with pointed maps $s_n : \Sigma E_n \to E_{n+1}$ called the *structure maps* of the spectrum.

For the pictorially-oriented, here is a diagram illustrating the data contained in a spectrum:

$$\begin{tikzcd}
\cdots & E_{n-1} \arrow[d, dotted, mapsto] & E_n \arrow[d, dotted, mapsto] & E_{n+1} \arrow[d, dotted, mapsto] & \cdots \\
\cdots \arrow[ur, "s_{n-2}"] & \Sigma E_{n-1} \arrow[ur, "s_{n-1}"] & \Sigma E_n \arrow[ur, "s_n"] & \Sigma E_{n+1} \arrow[ur, "s_{n+1}"] & \cdots
\end{tikzcd}$$

Let’s start with some examples.

*Example* 1.3. Take $X$ in **CW**$_*$. We can define a spectrum $\Sigma^\infty X$, where

$$(\Sigma^\infty X)_n = \Sigma^n X, \tag{1.4}$$

and with structure maps

$$s_n : \Sigma(\Sigma^\infty X)_n = \Sigma^{n+1} X \xrightarrow{\sim} \Sigma^{n+1} X = (\Sigma^\infty X)_{n+1} \tag{1.5}$$

being the identity. This is called a *suspension spectrum* of the space $X$; you can think of it as an object which consolidates all the information about a space we want for stable work into one object.

Particularly important is the *sphere spectrum*,

$$\mathbb{S} = \Sigma^\infty S^0. \tag{1.6}$$

We have that $\mathbb{S}_n = S^n$.

*Example* 1.7. Take an abelian group $G \in$ **Ab**. We can define a spectrum $HG$, where

$$(HG)_n = K(G, n), \tag{1.8}$$

and with structure maps $s_n : \Sigma(HG)_n = \Sigma K(G, n) \to K(G, n+1) = (HG)_{n+1}$ given to be the adjoints to the isomorphisms

$$K(G, n) \xrightarrow{\sim} \Omega K(G, n+1). \tag{1.9}$$

This spectrum is called an *Eilenberg-Maclane spectrum*.

Next, let’s define what it means to map between spectra.

**Definition 1.10.** A *map* $f$ between a spectrum $E = \{E_n\}$ with structure maps $\{s_n\}$, and a spectrum $F = \{F_n\}$ with structure maps $\{t_n\}$ is a collection

$$\{f_n : E_n \to F_n\} \tag{1.11}$$

which commute with the structure maps. That is, the following squares commute for all $n$:

$$\begin{tikzcd} \Sigma E_n \arrow[r, "s_n"] \arrow[d, "\Sigma f_n"'] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma F_n \arrow[r, "t_n"'] & F_{n+1} \end{tikzcd}$$

**Definition 1.12.** The collection of spectra and their maps form the category of spectra, aptly named **Spectra**.

*Remark* 1.13. Maybe one way to get a feel for the data in a spectrum is to think of it as a topological version of a chain complex, where instead of differentials from one of our abelian groups to the next, we have these structure maps between a suspension of one of our spaces and the next.

Now, our goal is to do homotopical stuff, so let's begin to work towards that.

**Definition 1.14.** Take a pointed space $X$, and a spectrum $E = \{E_n\}$ with structure maps $\{s_n\}$. We can form a spectrum $E \wedge X$, by letting

$$(E \wedge X)_n = E_n \wedge X, \tag{1.15}$$

with structure maps $\{s_n \wedge \mathbf{1}_X\}$.

We define the spectrum $X \wedge E$ symmetrically.

**Definition 1.16.** Let $f, g : E \to F$ be two maps of spectra. A *homotopy* between $f$ and $g$ is a map

$$H : E \wedge [0,1]_+ \to F, \tag{1.17}$$

where the restriction of $H$ to $E \wedge \{0\}_*$ is $f$ and the restriction of $H$ to $E \wedge \{1\}_+$ is $g$.

Naïvely, we might expect to define the stable homotopy category **HoSpectra** as the category whose objects are spectra and morphisms are now homotopy classes of maps of spectra. However, this yields some problems when we want to state 3.13 (see [BR20], p.35, or [Ada74], p.141, for some discussion); it turns out that calling homotopy equivalent spectra isomorphic is too strong in our case.

So towards describing **HoSpectra**, we make the following very important definition.

**Definition 1.18.** The *homotopy groups* of a spectrum $E$, denoted $\pi_n(E)$ is $\varinjlim_a \pi_{n+a}(E_a)$, where the direct limit is taken over the sequence below:

$$\cdots \longrightarrow [S^{n+a}, E_a] \xrightarrow{\Sigma} [S^{n+a+1}, \Sigma E_a] \xrightarrow{(s_a)_*} [S^{n+a+1}, E_{a+1}] \longrightarrow \cdots$$

*Example* 1.19. Let $X$ be a pointed space, and let's take a look at the homotopy groups of the suspension spectrum $\Sigma^\infty X$. Then we get a sequence that looks like the following:

$$\cdots \longrightarrow [S^{n+a}, \Sigma^a X] \xrightarrow{\Sigma} [S^{n+a+1}, \Sigma^{a+1} X] = [S^{n+a+1}, \Sigma^{a+1} X] \longrightarrow \cdots$$

Freudenthal's suspension theorem tells us that for $a$ large enough, $[S^{n+a}, \Sigma^a X] = [S^{n+a+1}, \Sigma^{a+1} X]$; this is what we call the *$n$th stable homotopy group* of $X$, $\pi_n^{\mathrm{st}}(X)$. So when we take the direct limit, we retrieve

$$\pi_n(\Sigma^\infty X) = \pi_n^{\mathrm{st}}(X) \tag{1.20}$$

Again, particularly important is the case of $\mathbb{S} = \Sigma^\infty S^0$. Here, we retrieve

$$\pi_n(\mathbb{S}) = \pi_n^{\mathrm{st}}(S^0).$$

This is what we call the *$n$th stable homotopy group of spheres.*

Note that $\pi_n$ is a functor **Spectra** $\to$ **Ab**, in a way similar to how "unstable" homotopy groups $\pi_n$ : **Top**$_*$ $\to$ **Ab** are. In particular, it makes sense to ask about the induced map $\pi_n(f)$, where $f$ is a map of spectra.

**Definition 1.21.** Let $E$ and $F$ be spectra, and $f : E \to F$ a map. We say $f$ is a *weak homotopy equivalence* if the induced map $\pi_n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

We use this to finally make our definition of the stable homotopy category.

**Proposition 1.22.** *There is a category* **HoSpectra**, *whose objects are spectra and whose morphisms are maps of spectra, but modified so that every weak homotopy equivalence of spectra becomes an isomorphism. That is, for every weak homotopy equivalence $f : E \to F$, we formally add an inverse $f^{-1} : F \to E$. This category what we will call* the stable homotopy category.

A priori, it is not at all clear how we would go about constructing such a category. The modern approach taken seems to be via model categories; exposition can be found in chapter 2 of [BR20].

For practical purposes, it can be helpful to have a more explicit understanding of these equivalences. It turns out that restricting our attention to "nicer" classes of spectra makes this a little easier.

**Definition 1.23.** A *CW*-spectrum is a spectrum where all spaces are pointed CW complexes, and where all structure maps are pointed cellular maps.

**Definition 1.24.** An $\Omega$-spectrum is a spectrum $E$ where for all structure maps $s_n : \Sigma E_n \to E_{n+1}$, their adjoint maps

$$\widetilde{s_n} : E_n \to \Omega E_{n+1} \tag{1.25}$$

are weak homotopy equivalences.

**Proposition 1.26** ([Ada74], p.141). *Let $E$ be a CW-spectrum, and $F$ an $\Omega$-spectrum. Then $f : E \to F$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

If this is not satisfying, we refer the reader to the development of the homotopy category of CW spectra, as given in [Ada74] or [Rud98]; this gives a more explicit version of **HoSpectra** which has all of the properties wanted from it.

## 2 Extra structures on HoSpectra

So far, all we have done is describe a category somehow "derived"[3] from topological spaces, and see that there is a (somewhat) natural way to recover stable homotopy groups from this category. However, last week we claimed[4] that we make the move from $\mathbf{HoTop}_*$ to **HoSpectra** because this category not only captures the information we want for stable homotopy theory, but because it is "nicer" as a category in some ways. The punchline is the following:

**Proposition 2.1.** **HoSpectra** *is a*

- *symmetric monoidal* (has a "tensor product" that is commutative),
- *closed* (has an "internal Hom"),
- *triangulated* (has "distinguished triangles" which lead to long exact sequences)

*category.*

Some of the explicit constructions giving these structures get kind of nasty, so we will not spend time describing those. Instead, we'll review what precisely it means for a category to be as above, and the pieces from **(Ho)Spectra** which satisfy the necessary axioms. I will cite [Mal14] as my source as my main source for this section; if you want another good starting point, I honestly think the Wikipedia articles are pretty useful, in addition to [nLa22].

**Definition 2.2.** A category $\mathcal{C}$ is *monoidal* if there exists an operation $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ which is unital and associative; that is, we have some unit object, say $\mathbf{1}$, and natural isomorphisms

$$\begin{aligned} l_A :& \quad \mathbf{1} \otimes A \xrightarrow{\sim} A \\ r_A :& \quad A \otimes \mathbf{1} \xrightarrow{\sim} A \\ a_{A,B,C} :& \quad (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C) \end{aligned} \tag{2.3}$$

satisfying some coherence conditions. $\mathcal{C}$ is called *symmetric monoidal* if in addition this operation is commutative; that is, we have a natural isomorphism

$$s_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A \tag{2.4}$$

[3]The word "derived" here does not have any precise meaning, or at least I do not mean it to.
[4]Or maybe we didn't; sorry if I'm putting words in your mouth, Dora.

*Example* 2.5. The category of abelian groups is a symmetric monoidal category, with the tensor product $\otimes_{\mathbb{Z}}$. More generally, for any (commutative) ring $R$, the category of $R$-modules is a monoidal category, with the tensor product $\otimes_R$.

*Example* 2.6. Closer to our current focus, the category $\mathbf{Top}_*$ is a symmetric monoidal category, with the operation being the smash product $\wedge$.

**Proposition 2.7** ([Rud98] Thm. 2.1)**.** *There exists a smash product $\wedge$ on* **HoSpectra**, *making it into a symmetric monoidal category, with unit object being the sphere spectrum $\mathbb{S}$.*

A lot of familar categories are symmetric monoidal, so it is nice that our new one is too. Let's see how far we can go.

**Definition 2.8.** A symmetric monoidal category $(\mathcal{C}, \otimes)$ is called *closed* if for any objects $B, C$, there is an associated object $F(B, C)$, with natural bijections

$$\mathrm{Hom}(A \otimes B, C) \simeq \mathrm{Hom}(A, F(B, C)) \tag{2.9}$$

This object $F(B, C)$ is often called an *internal Hom*.

*Example* 2.10. The category of abelian groups is closed; for any groups $A, B$, one can make $\mathrm{Hom}(A, B)$ an abelian group via pointwise addition of maps. Similarly, the category of $R$-modules is also closed.

*Example* 2.11. $\mathbf{Top}_*$ is also closed, by giving $\mathrm{Hom}_{\mathbf{Top}_*}(X, Y)$ a suitable topology.

**Proposition 2.12.** *For any two spectra $X, Y$, there exists a* function spectrum $F(X, Y)$ *making* **HoSpectra** *into a closed symmetric monoidal category.*

This is already pretty cool, but you may still be a little underwhelmed; we have yet to describe a structure on **HoSpectra** that we can't find on spaces. If so, hopefully what follows will entertain you a bit more.

**Definition 2.13.** A *shift functor* on a category $\mathcal{C}$ is an autoequivalence

$$T : \mathcal{C} \to \mathcal{C} \tag{2.14}$$

*Example* 2.15. On any category $\mathcal{C}$, the identity functor is a shift functor.

*Example* 2.16. Consider the category **ChAb** of chain complexes of abelian groups. If $C_* = \{C_n, d_n\}$ is a chain complex,

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

we can define the *right shift* of $C_*$ to be the complex $C[1]_*$,

$$\cdots \longrightarrow C[1]_{n+1} \xrightarrow{d[1]_{n+1}} C[1]_n \xrightarrow{d[1]_n} C[1]_{n-1} \longrightarrow \cdots$$

with $C[1]_n = C_{n+1}$, $d[1]_n = d_{n+1}$. Similarly, we can define the *left shift* of $C_*$ to be the complex $C[-1]_*$,

$$\cdots \longrightarrow C[-1]_{n+1} \xrightarrow{d[-1]_{n+1}} C[-1]_n \xrightarrow{d[-1]_n} C[-1]_{n-1} \longrightarrow \cdots$$

with $C[-1]_n = C_{n-1}$, $d[-1]_n = d_{n-1}$. These two operations let us define functors $(-)[1], (-)[-1]$ : **ChAb** $\to$ **ChAb** which are inverses of each other; these are more interesting examples of shift functors.

Now, before we mentioned that we might be able to get a feel for spectra by thinking of them as "topological chain complexes." This suggests that we might be able to define a shift in a similar fashion, and indeed we can.

**Definition 2.17** (Suspension of spectra)**.** Given a spectrum $E = \{E_n, s_n\}$, we can define the *suspension* of $E$, denoted by $\Sigma E = \{(\Sigma E)_n, (\Sigma s)_n\}$, to be the spectrum where

$$(\Sigma E)_n = E_{n+1}, \quad (\Sigma s)_n = s_{n+1}$$

This lets us define an shift functor $\Sigma$ : **HoSpectra** $\to$ **HoSpectra**.

*Remark* 2.18. As we can see, in **(Ho)Spectra**, suspension is now an invertible operation!

**Definition 2.19.** Given a (additive) shift functor $T$ on an (additive) category $\mathcal{C}$, a *triangle* in $\mathcal{C}$ is a sequence of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

A *triangulated category* is an additive category $\mathcal{C}$ with a translation functor $T$ and a class of *distinguished triangles*; triangles which satisfy a couple of important axioms which I will not list here.

We have the following fact about triangulated categories:

**Proposition 2.20.** *Triangulated categories have special functors (known as* (co)homological functors*) which take triangles to long exact sequences.*

**Proposition 2.21. HoSpectra***, along with suspension $\Sigma$, is a triangulated category. The distinguished triangles are called* fiber/cofiber sequences[5]*; in fact, for spectra fiber and cofiber sequences are the same. These distinguished triangles are what give rise to long exact sequences in homotopy groups.*

For a more detailed discussion of this, please see 1.4 in [Mal14].

[5] Note that these haven't been defined here; just know that they can be defined topologically in a way analogous to spaces

*Remark* 2.22. This makes the analogy between chain complexes a little more convincing; given a ring $R$, the *derived category* **D(R-mod)**, whose objects are chain complexes of $R$-modules, is also a closed symmetric monoidal triangulated category. This offers a pedagogical unification between long exact sequences in homological algebra and homotopy theory.

If you want, you can maybe think of the move from $\mathbf{Top}_*$ to **HoSpectra** ($X \mapsto \Sigma^\infty X$) as analogous to the move from **R-mod** to **D(R-mod)** ($M \mapsto$ some resolution of $M$). I do not want to accept any responsibility for wrong intuition caused by these analogies, however, so please use everything with a couple of grains of salt.

## 3 Connections to cohomology theories

So now we know a bit about the structure of the category of spectra. It is worth taking a look back now and seeing if we can find a stronger connection between spectra and spaces; the place we will look is our favorite collection of invariants, (co)homology theories.

**Definition 3.1.** An *(extraordinary) homology theory* $E_*$ is a collection of covariant functors $\{E_n\}_{n\in\mathbb{Z}}$ from pairs of spaces $\mathbf{Top}^2$ to abelian groups $\mathbf{Ab}$, satisfying some axioms.

- (homotopy invariance) If $f, g : (X, A) \to (Y, B)$ are homotopic, then the induced homomorphisms $E_n(f), E_n(g)$ are the same for all $n$.
- (exactness) For every pair $(X, A)$, there is a natural long exact sequence

  $$\cdots \longrightarrow E_n(A, \varnothing) \xrightarrow{i_*} E_n(X, \varnothing) \xrightarrow{j_*} E_n(X, A) \xrightarrow{\partial} E_{n-1}(A) \longrightarrow \cdots$$

  where $i$ and $j$ are the maps $(A, \varnothing) \hookrightarrow (X, \varnothing)$ and $(X, \varnothing) \to (X, A)$ respectively.
- (excision) For every pair $(X, A)$, the collapse $(X, A) \to (X/A, \{*\})$ induces an isomorphism $E_n(X, A) \to E_n(X/A, \{*\})$.

An *(extraordinary) cohomology theory* $E^*$ is a collection of contravariant functors $\{E^n\}_{n\in\mathbb{Z}}$ from pairs of spaces $\mathbf{Top}^2$ to abelian groups $\mathbf{Ab}$, satisfying some dual axioms:

- (homotopy invariance) If $f, g : (X, A) \to (Y, B)$ are homotopic, then the induced homomorphisms $E^n(f), E^n(g)$ are the same for all $n$.
- (exactness) For every pair $(X, A)$, there is a natural long exact sequence

  $$\cdots \longleftarrow E_n(A) \xleftarrow{i^*} E_n(X) \xleftarrow{j^*} E_n(A, X) \xleftarrow{\partial} E_{n-1}(A) \longleftarrow \cdots$$

  where $i$ and $j$ are the maps $(A, \varnothing) \hookrightarrow (X, \varnothing)$ and $(X, \varnothing) \to (X, A)$ respectively.

- (excision) For every pair $(X, A)$, the collapse $(X, A) \to (X/A, \{*\})$ induces an isomorphism $E^n(X/A, \{*\}) \to E^n(X, A)$

Let's recall some examples.

*Example* 3.2. Of course, we have our favorite; singular (co)homology, $H_n(-; A)$ and $H^n(-; A)$, with coefficients in some abelian group $A$.

*Example* 3.3. More surprisingly, it turns out that stable homotopy is a homology theory; the functors $\pi_n^{\mathrm{st}}(-)$ satisfy the axioms above.

*Example* 3.4. An important class of (co)homology theories are given by (co)bordism theories, which are defined by considering certain equivalence classes of manifolds and maps between them.

We continue with a related definition.

**Definition 3.5.** A *reduced homology theory* $\widetilde{E}_n$ is a sequence of covariant functors from pointed spaces $\mathbf{Top}_*$ to abelian groups $\mathbf{Ab}$ satisfying the following properties:

- (homotopy invariance) If $f, g : (X, x_0) \to (Y, y_0)$ are homotopic, then the induced homomorphisms $\widetilde{E}_n(f), \widetilde{E}_n(g)$ are the same for all $n$.
- (exactness) For a triple $(X, A, x_0)$, we have that the sequence

$$\widetilde{E}_n(A) \xrightarrow{i_*} \widetilde{E}_n(X) \xrightarrow{j_*} \widetilde{E}_n(C_i)$$

  is exact for all $n$, where $C_i$ is the mapping cone of $i$, and $j$ is the inclusion $X \hookrightarrow C_i$.
- (suspension isomorphism) There is a natural isomorphism

$$s_n : \widetilde{E}_n(X) \to \widetilde{E}_{n+1}(\Sigma X) \tag{3.6}$$

  for all $n$.

Dually, a *reduced cohomology theory* $\widetilde{E}^n$ is a sequence of contravariant functors from pointed spaces $\mathbf{Top}_*$ to abelian groups $\mathbf{Ab}$ satisfying the following properties:

- (homotopy invariance) If $f, g : (X, x_0) \to (Y, y_0)$ are homotopic, then the induced homomorphisms $\widetilde{E}^n(f), \widetilde{E}^n(g)$ are the same for all $n$.
- (exactness) For a triple $(X, A, x_0)$, we have that the sequence

$$\widetilde{E}^n(A) \xleftarrow{i^*} \widetilde{E}^n(X) \xleftarrow{j^*} \widetilde{E}^n(C_i)$$

  is exact for all $n$, where $C_i$ is the mapping cone of $i$, and $j$ is the inclusion $X \hookrightarrow C_i$.

- (suspension isomorphism) There is a natural isomorphism

$$s^n : \widetilde{E}^{n+1}(\Sigma X) \to \widetilde{E}^n(X) \tag{3.7}$$

for all $n$.

*Example* 3.8. Reduced singular (co)homology is probably the most familiar example of a reduced (co)homology theory.

*Example* 3.9. Any unreduced (co)homology theory defines a reduced one; see [Rud98], page 56, for example. In fact, *all* reduced (co)homology theories are obtained in this way; there is a bijection between unreduced and reduced cohomology theories (up to suitable equivalence).

The next proposition is what begins to tie together the study of spectra with (co)homology theories.

**Proposition 3.10.** *Spectra define reduced homology and cohomology theories. Given a spectrum $E$, we can define a reduced homology theory via*

$$\widetilde{E}_n(X) = [\mathbb{S}, (\Sigma^\infty X) \wedge E] \simeq \pi_n((\Sigma^\infty X) \wedge E) \tag{3.11}$$

*and a reduced cohomology theory via*

$$\widetilde{E}^n(X) = [\Sigma^\infty X, E]_{-n} \simeq \pi_{-n}(F(\Sigma^\infty X, E)) \tag{3.12}$$

In fact, we have something stronger:

**Proposition 3.13** (Brown representability)**.** *If $E$ is a cohomology theory on* **CW**$_*$*, then it is represented by a spectrum which is unique up to equivalence in* **HoSpectra**.

For this reason, oftentimes people will blur the line between a reduced cohomology theory $E$ and the spectrum representing it. This is another marvelous property of **HoSpectra**; you can think of its objects as exactly cohomology theories!

*Example* 3.14. Singular (co)homology with coefficients in an abelian group $G$ is represented by the Eilenberg-Maclane spectrum $HG$, just like how we learned that Eilenberg-Maclane spaces represent singular cohomology; the functors $H^n(-; G)$ and $[-, K(G, n)]$ are naturally isomorphic.

*Example* 3.15. Stable homotopy is represented by $\mathbb{S}$.

*Example* 3.16. Bordism theories are represented by *Thom spectra* (to be introduced in next talk!)

Now so far, we have only been considering (co)homology theories as an assignment of graded abelian groups. But if we look back at singular cohomology, one of the useful structures was the cup product, giving us a ring structure on the cohomology of any space. It is worth asking if there is some structure on spectra which will correspond to this multiplicative structure, and indeed there is!

**Definition 3.17.** A *ring spectrum* is a spectrum $E$, along with a map $m : E \wedge E \to E$ (called the *multiplication map*) and a map $\iota : \mathbb{S} \to E$ (called the *unit map*), with the following properties.

- (associativity) The following diagram commutes in **HoSpectra**

$$\begin{tikzcd}
(E\wedge E)\wedge E \arrow[rr, "a_{E,E,E}"] \arrow[d, "m\wedge 1_E"'] & & E\wedge(E\wedge E) \arrow[d, "1_E\wedge m"] \\
E\wedge E \arrow[dr, "m"'] & & E\wedge E \arrow[dl, "m"] \\
& E &
\end{tikzcd}$$

- (unitarity) The following diagram commutes in **HoSpectra**:

$$\begin{tikzcd}
S\wedge E \arrow[r, "\iota\wedge 1"] \arrow[dr, "l_E"'] & E\wedge E \arrow[d, "m"] & E\wedge S \arrow[l, "1\wedge\iota"'] \arrow[dl, "r_E"] \\
& E &
\end{tikzcd}$$

$E$ is *commutative* if, in addition, we have the following property:

- (commutativity) The following diagram commutes in **HoSpectra**:

$$\begin{tikzcd}
E\wedge E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E\wedge E \arrow[dl, "\mu"] \\
& E &
\end{tikzcd}$$

  where $\tau$ is the *twist map*, interchanging the two factors.

**Proposition 3.18.** *Ring spectra correspond to multiplicative cohomology theories under Brown representability.*

*Example* 3.19. All of the representing spectra given above are ring spectra.

*Example* 3.20. $HR$ is a ring spectra for $R$ a ring.

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