Quantitative Finance

Atlantic Lock-In Synthetic Zero

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Introduction

In this assignment, we will consider an investment product that can be bought from BNP Paribas, namely the "Atlantic Lock-In Synthetic Zero". This product can be bought for the price of £100. Whether this amount will be returned and whether it will accrue a return depends on the performance of two indices, namely the FTSE100 and the SP500. We will consider the pay-off in-depth in Part B.

Data

In our analysis we will use daily prices of FTSE 100 index and S&P 500 index downloaded from finance.yahoo.com. Since, we have to analise a contract starting on 04 May 2011, we therefore chose a sample period of 10 years starting from 01 May 2006 till 04 May 2011.

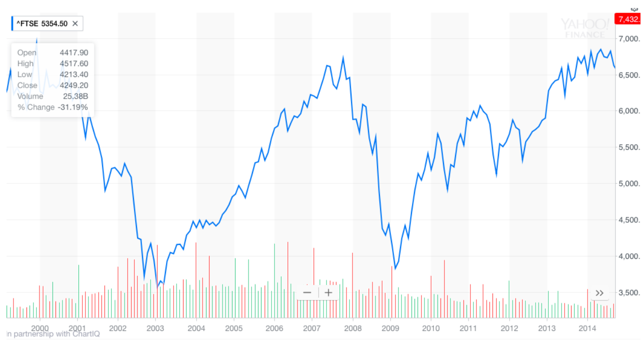

Figure 1: FTSE 100 daily prices

Figure 2: S&P 500 daily prices

Using this daily prices we obtained daily returns with the formula:

$$\frac{R_t - R_{t-1}}{R_{t-1}} \tag{1}$$

Moreover, assuming that there are 252 trading days in one year we transform daily returns to yearly returns.

Part A: Formulate a model in terms of SDE's for the underlying assets and use the "standard" model for the money-market account

First, we formulate a model for the model-market account and the SDE's of the underlying assets, i.e. the FTSE100 and SP500. In order to formulate such a model we will use the standard multivariate Black-Scholes model with 2 risky assets, being the **FTSE 100 index** and the **S&P 500 index**. The risk-free asset will be the money-market account.
In order to formulate the model in terms of SDE's, we introduce the following notation:

$$\begin{aligned} \mathbf{B_t} &: \textit{the value of one unit of the bond at time } t \\ \mathbf{S_{1,t}} &: \textit{the value of the FTSE 100 index at time } t \\ \mathbf{S_{2,t}} &: \textit{the value of the S\&P 500 index at time } t \end{aligned} \tag{2}$$

For the bond price, we will assume the exponential model with a fixed rate r (the lending rate equals the borrow rate). For the stock prices, we assume the model of geometric Brownian Motion. We then obtain 3 SDE's, corresponding to the bond, the FTSE100 index and the S&P500 index:

$$\begin{aligned} dB_t &= rB_t dt \\ dS_{1,t} &= \mu_1 S_{1,t} dt + \sigma_1 S_{1,t} dW_{1,t} \\ dS_{2,t} &= \mu_2 S_{2,t} dt + \sigma_2 S_{2,t} dW_{2,t} \end{aligned} \tag{3}$$

Where $W_{1,t}$ and $W_{2,t}$ standard Brownian Motions, μ_1 and μ_2 the mean returns (drift parameters) of the FTSE 100 index and the S&P 500 index, and σ_1 and σ_2 the volatility parameters of the respective indices.
We assume that $B_0 = 1$ and we make assumption that the trade is frictionless, meaning that there are no transaction costs and trading can be done continuously. We will also assume there are no restrictions on short sales: investors are free to take long and short positions in both assets and we finally assume that it is only possible to invest in these 3 assets.

Part B: Formulating the payoffs of the contract in terms of the underlying assets.

The specification of the Atlantic Lock-In Synthetic Zero product [1] clearly specifies the pay-off structure of the bond. As mentioned, one can buy the product for the price of £100. The pay-off depends on the yearly performance of both indices relative to the starting level up to and including the final valuation date (i.e. $t = 1, ...6$, see also Table).

t	Moment	date
0	Start Date	04-May-11
1	Observation Date 1	04-May-12
2	Observation Date 2	07-May-13
3	Observation Date 3	06-May-14
4	Observation Date 4	05-May-15
5	Observation Date 5	04-May-16
6	Final Valuation Date	04-May-17

Table 1: Moments and dates corresponding to the time indices.

Buyers of the product face two types of uncertainty when it comes to their pay-off at maturity:
(i) Uncertainty about how much gross return is accrued over the duration of the contract (We denote this by P^A).
(ii) Uncertainty about how much of the issue price is returned at maturity by the bank (we denote this by P^I).
With respect to (i), the gross return can be at most 50.70%. This maximum of 50.70% of gross return can be occurred in 2 cases.
1: Both indices during all 6 years close above 50%(on every Observation Date and on the Final Valuation Date)
2:Both indices simultaneously close above 110% of their Starting Index Levels (FTSE 100 Index: 6,582.48, S&P 500 Index: 1,482.05) on any of 5 Observation Dates or on the Final Valuation Date: Lock-In.
If no lock-in occurs but t=1,2,3,4,5,6, both indices close above 50% of their initial levels in a total of z periods, this lead to $z * 8.45$ % of gross return.

The lock-in occurs if in any of the periods both indices close above 110% of their initial levels. Thus, we have:

$$P^A = \begin{cases} 50.7 & \text{if a lock-in occurs, i.e. if there is at least one } k \text{ s.t. } \frac{S_{1,k}}{S_{1,0}}, \frac{S_{2,k}}{S_{2,0}} \geq 1.1, \\ 8.45 \times z & \text{if no lock-in occurs, with } z = |\{k \in 1,2,3,4,5,6| \frac{S_{1,k}}{S_{1,0}}, \frac{S_{2,k}}{S_{2,0}} \geq 0.5\}|, \end{cases}$$

[1] http://www.londonstockexchange.com/prices-and-markets/structured-products/documents/rbs-gb00b6hz3n37-rs68.pdf

We will use indicator functions to formulate the payoff function.

$$
\begin{aligned}
P^A &= 50.7 * \mathbf{1}\{\text{"lock-in"}\} + 8.45 * \mathbf{1}\{\text{"no lock-in"}\} * z \\
&= (50.7 - 8.45 * z) * \mathbf{1}\{\text{"lock-in"}\} + 8.45 * z \\
&= (50.7 - 8.45 * z) * \mathbf{1}\Big\{\Big(\sum_{k=1}^{6} \mathbf{1}\{\frac{S_{1,k}}{S_{1,0}} \geq 1.1\}\mathbf{1}\{\frac{S_{2,k}}{S_{2,0}} \geq 1.1\}\Big) \geq 1\Big\} + 8.45 * z, \qquad (4)
\end{aligned}
$$

where $z = \sum_{k=1}^{6} \mathbf{1}\{\frac{S_{1,k}}{S_{1,0}} \geq 0.5\}\mathbf{1}\{\frac{S_{2,k}}{S_{2,0}} \geq 0.5\}$

So,in case lock-in occurs, gross return equals to £50,70 and in case no lock-in occurs then the gross return equals to z*£8.75.

With respect to (ii), the full issue price is returned in all cases except when either one of the indices closes below 50% at the Final Valuation Date $t = 6$. In that case, the returned amount of the issue price will be reduced by the same percentage that by which the closing level of worst performing Index at t = 6 is below its Starting Index Level:it returns the percentage equal to the relative level of the lowest-performing index.
Thus, in terms of indicator functions the payoff function can be formulated as follows:

$$
P^I = 100 * [\mathbf{1}\{\frac{S_{1,6}}{S_{1,0}} \geq 0.5\}\mathbf{1}\{\frac{S_{2,6}}{S_{2,0}} \geq 0.5\}] + 100 * \mathrm{Min}(\frac{S_{1,6}}{S_{1,0}}, \frac{S_{2,6}}{S_{2,0}}) * \Big(1 - \mathbf{1}\{\frac{S_{1,6}}{S_{1,0}} \geq 0.5\}\mathbf{1}\{\frac{S_{2,6}}{S_{2,0}} \geq 0.5\}\Big) \qquad (5)
$$

So, if both indices closes above 50% then P^I = 100*1+0 = 100 meaning that the full issue price of £100 will be returned (per certificate). In case at least one of them closes below 50% then the amount to be returned per certificate will be equal to £100 times the minimum of two ratio levels.

The total net payoff is the sum of the two separate payoff functions (3) and (4) minus the issue price, i.e. $P_T net$=P^A+P^I - 100 and the total payoff is $\mathrm{P}_T = \mathrm{P}^A + \mathrm{P}^I$

Part C: Estimation of the SDE parameters

To estimate the volatility and drift parameters of our SDE's, we would need to perform Ito calculus on our set of SDE's (see 3). We will take into account the correlation of two geometric Brownian Motions of our SDE's(W_1 & W_2). Therefore, we will assume that the multivariate price process of S follows a multivariate geometric Brownian motion, i.e. that the Wiener processes are joint Wiener process with a correlation coefficient ρ, i.e. $\mathbf{E}[dW_1 dW_2] = \rho_{1,2} dt$.

Before we start with the main estimation part of our model we will analyze 3 formulated models in part A:(2).Note that well known property of the asset price process is that $S_{1,t}$ and $S_{2,t}$ follows log-normal distributions. We will determine the d(log($S_{1,t}$)) and d(log($S_{2,t}$)). Therefore we will use It$\hat{o}$ rule.

$$\begin{aligned} d(F(S_{1,t})) &= F'(S_{1,t})dS_{1,t} + \frac{1}{2}F"(S_{1,t})d[S_1, S_1]_t \\ d(F(S_{2,t})) &= F'(S_{2,t})dS_{2,t} + \frac{1}{2}F"(S_{2,t})d[S_2, S_2]_t \end{aligned} \tag{6}$$

Since we have to decide SDE of logarithmic $S_{1,t}$ we define:
F($S_{1,t}$)= log($S_{1,t}$) and F($S_{2,t}$) = log($S_{2,t}$) Moreover, notice that F'(x) and F"(x) are first and second derivatives of the function F(x).
We then obtain:

$$\begin{aligned} dlog(S_{1,t}) &= \frac{1}{S_{1,t}} * dS_{1,t} - 0.5 * \frac{1}{S_{1,t}^2} d[S_1, S_1]_t \\ &= \frac{1}{S_{1,t}} * dS_{1,t} - 0.5 * \frac{1}{S_{1,t}^2} * \sigma_1^2 S_{1,t}^2 dt \\ &= \frac{1}{S_{1,t}} * (\mu_1 S_{1,t} dt + \sigma_1 S_{1,t} dW_{1,t}) - 0.5\sigma_1^2 dt \\ &= \mu_1 dt + \sigma_1 dW_{1,t} - 0.5\sigma_1^2 dt \\ &= (\mu_1 - 0.5\sigma_1^2)dt + \sigma_1 dW_{1,t} \end{aligned} \tag{7}$$

$$\begin{aligned} dlog(S_{2,t}) &= \frac{1}{S_{2,t}} * dS_{2,t} - 0.5 * \frac{1}{S_{2,t}^2} d[S_2, S_2]_t \\ &= (\mu_2 - 0.5\sigma_2^2)dt + \sigma_1 dW_{2,t} \end{aligned}$$

Notice that all steps in (7) for $S_{1,t}$ hold for $S_{2,t}$ as well. We then obtain:

$$\begin{aligned} dlog(S_{1,t}) &= (\mu_1 - 0.5\sigma_1^2)dt + \sigma_1 dW_{1,t} \\ dlog(S_{2,t}) &= (\mu_2 - 0.5\sigma_2^2)dt + \sigma_2 dW_{2,t} \end{aligned} \tag{8}$$

Notice that in (7) and (8) we used the results about quadratic variations presented in *Theorems 2.4* and *Theorems 2.5* of the lecture notes. Thus for the stochastic processes $d[S_1,S_1]_t = \sigma_1^2 S_{1,t}^2 dt$ and $d[S_2,S_2]_t = \sigma_2^2 S_{2,t}^2 dt$.

Then we take an integral from both sides in order to analyse the change over time interval [t,t+1]:

$$\begin{aligned} \int_t^{t+1} dlog(S_{1,u}) &= \int_t^{t+1} ((\mu_1 - 0.5\sigma_1^2)du + \sigma_1 dW_{1,u}) \\ \int_t^{t+1} dlog(S_{2,u}) &= \int_t^{t+1} ((\mu_2 - 0.5\sigma_2^2)du + \sigma_2 dW_{2,u}) \end{aligned} \tag{9}$$

Applying Telescope rule we get in the left hand-side of our equations:

$$\begin{aligned} \int_t^{t+1} dlog(S_{1,u}) &= log(S_{1,t+1}) - log(S_{1,t}) \\ \int_t^{t+1} dlog(S_{2,u}) &= log(S_{2,t+1}) - log(S_{2,t}) \end{aligned} \tag{10}$$

Note that μ_1, μ_2, σ_1 and σ_2 are just constants.Then in the right hand-side of our equations we get:

$$\begin{aligned} \int_t^{t+1} ((\mu_1 - 0.5\sigma_1^2)du + \sigma_1 dW_{1,u}) &= (\mu_1 - 0.5\sigma_1^2)1 + \sigma_1(W_{1,t+1} - W_{1,t}) \\ \int_t^{t+1} ((\mu_2 - 0.5\sigma_2^2)du + \sigma_2 dW_{2,u}) &= (\mu_2 - 0.5\sigma_2^2)1 + \sigma_2(W_{1,t+1} - W_{2,t}) \end{aligned} \tag{11}$$

Combining above results we obtain:

$$\begin{aligned} log(S_{1,t+1}) - log(S_{1,t}) = log(\frac{S_{1,t+1}}{S_{1,t}}) &= (\mu_1 - 0.5\sigma_1^2) + \sigma_1(W_{1,t+1} - W_{1,t}) \\ log(S_{2,t+1}) - log(S_{2,t}) = log(\frac{S_{2,t+1}}{S_{2,t}}) &= (\mu_2 - 0.5\sigma_2^2) + \sigma_2(W_{2,t+1} - W_{2,t}) \end{aligned} \tag{12}$$

As we have already mentioned $W_{1,t}$ and $W_{2,t}$ are standard Brownian Motions. Using the property of standard BM that increment of the process has a standard normal distribution, we get:$W_{1,t+1}$-$W_{1,t}$ $\sim$ N(0,1) and $W_{2,t+1}$-$W_{2,t}$ $\sim$ N(0,1).
We will apply the bivariate case of the Ito formula, i.e.

$$dF = \frac{\delta F}{dt} + \frac{\delta F}{\delta S_{1,t}}\delta S_{1,t} + \frac{\delta F}{\delta S_{2,t}}\delta S_{2,t} + 2 * \frac{1}{2} * \frac{\delta^2 F}{\delta S_{1,t}\delta S_{2,t}}\delta S_{1,t}\delta S_{2,t}.$$

Let us apply the transformation $Y_t = (\log(S_{1,t}), \log(S_{2,t}))$. Since the transformation happens per entry, the partial derivatives w.r.t. x and y become 0. Hence, the final term in the multivariate formula above reduces to 0 (quadratic covariation is bounded since each are Brownian). As a result, we obtain the same result as in the univariate case for dY_t, namely:

$$dY_t = \begin{bmatrix} (\mu_1 - \frac{1}{2}\sigma_1^2)\delta t + \sigma_1 dW_{1,t} \\ (\mu_2 - \frac{1}{2}\sigma_2^2)\delta t + \sigma_2 dW_{2,t} \end{bmatrix} = \begin{bmatrix} \nu_1\delta t + \sigma_1 dW_{1,t} \\ \nu_2\delta t + \sigma_2 dW_{2,t} \end{bmatrix} \tag{13}$$

Where we denote for simplicity (μ_1 - $\frac{1}{2}$ σ_1^2) and (μ_2 - $\frac{1}{2}$ σ_2^2) with ν_1 and ν_2 respectively.

Now let us consider the covariance between $Y_{1,t}$ and $Y_{2,t}$.
Since both are martingales, their expectations are 0.Hence, we find $\text{Cov}(Y_{1,t}, Y_{2,t}) = \mathbf{E}(Y_{1,t}Y_{2,t}) - \mathbf{E}(Y_{1,t})\mathbf{E}(Y_{2,t}) = \mathbf{E}(Y_{1,t}Y_{2,t}) - 0 = \mathbf{E}(Y_{1,t}Y_{2,t})$.
Let us work this term further out:

$$\begin{aligned}
&\mathbf{E}(Y_{1,t}Y_{2,t}) = \mathbf{E}\big[(\nu_1\delta t + \sigma_1 dW_{1,t})(\nu_2\delta t + \sigma_2 dW_{2,t})\big] \\
&= \mathbf{E}[\nu_1\nu_2(\delta t)^2] + \mathbf{E}[\sigma_2\nu_1 dW_{2,t}] + \mathbf{E}[\sigma_1\nu_2 dW_{1,t}] + \mathbf{E}[\sigma_1\sigma_2 dW_{1,t}dW_{2,t}] \\
&= 0 + 0 + 0 + \sigma_1\sigma_2\mathbf{E}[dW_{1,t}dW_{2,t}] = \sigma_1\sigma_2\rho_{1,2}dt
\end{aligned}$$

So we get $\mathbf{E}(\text{Y}_{1,t}\text{Y}_{2,t}) = \sigma_1\sigma_2\rho_{1,2}\text{dt}$.
We observe $\text{d}[\log\text{S}_1,\log\text{S}_2]_t = \sigma_1\sigma_2\rho_{1,2}\text{dt}$ and recognize that log-returns yield i.i.d. draws from a bivariate normal distribution namely, Y_t follows a bivariate normal distribution $N_2(\nu dt, \Sigma dt)$ with

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{1,2} \\ \sigma_1\sigma_2\rho_{1,2} & \sigma_2^2 \end{bmatrix} \text{ and } \nu = \begin{bmatrix} \mu_1 - \frac{1}{2}\sigma_1^2 \\ \mu_2 - \frac{1}{2}\sigma_2^2 \end{bmatrix}$$

Since the RHS of equation 13 does not depend on Y_t itself, we can simply solve it by direct integration. Then an estimation procedure for σ_1, σ_2, μ_1, μ_2 follows naturally: we can obtain historical data for $S_{1,t}$ and $S_{2,t}$, log-transform the data, calculate the differences between subsequent observations of Y_t and exploit that these should follow the discussed bivariate normal distribution. In particular, we can use the MLE estimator for multivariate normal distributions to obtain estimates $\hat{v}dt$ and $\hat{\Sigma}dt$, divide by the fixed time interval dt to obtain MLE estimates for Σ and ν and use equation 29 to derive the MLE estimates for the parameters we want to know.

Estimation

We observe that $\hat{\nu}_1 = \hat{\mu}_1$-$\frac{1}{2}\hat{\sigma}_1^2$ and $\hat{\nu}_2 = \hat{\mu}_2$-$\frac{1}{2}\hat{\sigma}_2^2$

For the estimation procedure above, a critical choice would be the data sample used as well as the time interval (dt). We chose to use a relatively short time interval such that we would have a large number of observations. We want to estimate σ and μ. More information about data and which sources is used can be found in the Introduction section. We chose our sampling period to span 5 years. In table 2 are presented MLE estimates for the

	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\nu}$
1: FTSE 100	.0223	.2129	-.0004
2: S&P 500	.0327	.2622	.00715

volatility parameters and drift parameters for each of the two indices.The MLE estimate of $\rho_{1,2}$ is .87, indicating the two indices are highly correlated in the period 2006-2011.

Part D: Determining the no-arbitrage price for the contract

In order to determine the no-arbitrage price of the contract we will consider the contract at 4 May 2011. In order to do this we will use our model formulated in part A and the model parameters. We will use the risk-neutral pricing in order to determine the no-arbitrage price of the contract.
First we pick a numeraire N_t, we will consider the money market account as the numeraire: $\mathrm{N}_t = \mathrm{B}_t$ where the model corresponding to money market account is formulated in Part A. $\mathrm{dB}_t = \mathrm{rB}_t\mathrm{dt}$ $\mathrm{N}_t = \mathrm{B}_t$ Applying First Fundamental Theorem of asset pricing (FFT) which states that the market is free of arbitrage opportunities if and only if following properties are satisfied:
- There exists probability measure Q equivalent to P: Q $\sim$ P.
- For $\mathrm{S}_{1,t}$ and $\mathrm{S}_{2,t}$ the discounted prices:$\mathrm{S}_{1,t}/\mathrm{N}_t$ and $\mathrm{S}_{2,t}/\mathrm{N}_t$ are martingales under probability measure Q.

Which means that:

$$\frac{S_{1,0}}{N_0} = E_Q[\frac{S_{1,T}}{N_T}]$$
$$\frac{S_{2,0}}{N_0} = E_Q[\frac{S_{2,T}}{N_T}]$$

We will then derive a SDE's for discounted price processes: $\mathrm{d}[\mathrm{S}_{1,t}/\mathrm{N}_t]$ and $\mathrm{d}[\mathrm{S}_{2,t}/\mathrm{N}_t]$ using the model formulated in part A (2) , where we take $\mathrm{N}_t = \mathrm{B}_t$ as mentioned earlier.
Applying It$\hat{o}$'s lemma one more time and $\mathrm{dB}_t = \mathrm{rB}_t\mathrm{dt}$ we obtain:

$$d(\frac{1}{B_t}) = -\frac{1}{B_t^2}d(B_t) + \frac{2}{B_t^3}d[B,B]_t = -\frac{1}{B_t^2}d(B_t) = -\frac{1}{B_t^2} * (rB_t)dt = -\frac{r}{B_t}dt$$

Then,using the stochastic product rule we obtain:

$$\begin{aligned}
&d(\frac{S_{1,t}}{B_t}) = \frac{1}{B_t}d(S_{1,t}) + S_{1,t}d(\frac{1}{B_t}) + [S_1, B]_t \\
&= \frac{1}{B_t}d(S_{1,t}) + S_{1,t}d(\frac{1}{B_t}) \\
&= \frac{1}{B_t}(\mu_1 S_{1,t} + \sigma_1 S_{1,t}dW_{1,t}) + S_{1,t}d(\frac{1}{B_t}) \\
&= \frac{1}{B_t}(\mu_1 S_{1,t} + \sigma_1 S_{1,t}dW_{1,t}) - \frac{r * S_{1,t}}{B_t}dt \\
&= \frac{S_{1,t}}{B_t}((\mu_1 - r)dt + \sigma_1 dW_{1,t})
\end{aligned} \tag{14}$$

So we get for two models:

$$d(\frac{S_{1,t}}{B_t}) = \frac{S_{1,t}}{B_t}((\mu_1 - r)dt + \sigma_1 dW_{1,t})$$
$$d(\frac{S_{2,t}}{B_t}) = \frac{S_{2,t}}{B_t}((\mu_2 - r)dt + \sigma_2 dW_{2,t}) \tag{15}$$

Finally, applying Girsanov theorem, we define new processes $W^Q_{1,t}$ and $W^Q_{2,t}$ such that:

$$W^Q_{1,0} = 0 \quad dW^Q_{1,t} = \gamma_{1,t}dt + dW_{1,t}$$
$$W^Q_{2,0} = 0 \quad dW^Q_{2,t} = \gamma_{2,t}dt + dW_{2,t} \tag{16}$$

The goal is to choose the values of $\gamma_{1,t}$ and $\gamma_{2,t}$ such that we make the drift parameters of $d(\frac{S_{2,t}}{B_t})$ equivalent to zero. Then these two processes $W^Q_{1,t}$ and $W^Q_{2,t}$ will be standard Brownian Motions under the probability measure Q, since BM's have no drift terms and their initial values are zero. For this purpose, we replace $dW_{1,t}$ and $dW_{2,t}$ in the expressions of $d[S_{1,t}/B_t]$ and $d[S_{2,t}/B_t]$ in (9) with $dW_{1,t} = dW^Q_{1,t} - \gamma_{1,t}dt$ and $dW_{2,t} = dW^Q_{2,t} - \gamma_{2,t}dt$. We then obtain:

$$d(\frac{S_{1,t}}{B_t}) = \frac{S_{1,t}}{B_t}((\mu_1 - r)dt + \sigma_1(dW^Q_{1,t} - \gamma_{1,t}dt))$$
$$= ((\mu_1 - r) - \sigma_1\gamma_{1,t})\frac{S_{1,t}}{B_t}dt + \sigma_1\frac{S_{1,t}}{B_t}dW^Q_{1,t}$$
$$d(\frac{S_{2,t}}{B_t}) = \frac{S_{2,t}}{B_t}((\mu_2 - r)dt + \sigma_2(dW^Q_{2,t} - \gamma_{2,t}dt))$$
$$= ((\mu_2 - r) - \sigma_2\gamma_{2,t})\frac{S_{2,t}}{B_t}dt + \sigma_2\frac{S_{2,t}}{B_t}dW^Q_{2,t} \tag{17}$$

We then have to choose $\gamma_{1,t} = \gamma_{2,t}$ such that the drift parameters: $((\mu_1\text{-r}) - \sigma_1\gamma_{1,t})$ and $((\mu_2\text{-r}) - \sigma_2\gamma_{2,t})$ becomes zero. Therefore, we choose :

$$\gamma_{1,t} = \frac{\mu_1 - r}{\sigma_1} \quad \gamma_{2,t} = \frac{\mu_2 - r}{\sigma_2} \tag{18}$$

With these chosen γ's the expressions of $d[S_{1,t}/B_t]$ and $d[S_{2,t}/B_t]$ become:

$$d(\frac{S_{1,t}}{B_t}) = \sigma_1\frac{S_{1,t}}{B_t}dW^Q_{1,t}$$
$$d(\frac{S_{2,t}}{B_t}) = \sigma_2\frac{S_{2,t}}{B_t}dW^Q_{2,t} \tag{19}$$

From this we can conclude that all $S_{1,t}/B_t$ and $S_{2,t}/B_t$ are martingales for t = 0,1,...,6. We then repeat the same steps as in part C for deriving $d\log(S_{1,t})$ and $d\log(S_{2,t})$ only this time we use the probability measure Q by replacing $dW_{1,t}$ and $dW_{2,t}$ by $dW^Q_{1,t} - \gamma_{1,t}dt$ and $dW^Q_{2,t} - \gamma_{2,t}dt$.

We obtain:

$$
\begin{aligned}
&dlog(S_{1,t}) = (\mu_1 - 0.5\sigma_1^2)dt + \sigma_1 dW_{1,t} \\
&= (\mu_1 - 0.5\sigma_1^2)dt + \sigma_1(dW_{1,t}^Q - \gamma_{1,t}dt) \\
&= (\mu_1 - 0.5\sigma_1^2)dt + \sigma_1(dW_{1,t}^Q - \frac{\mu_1 - r}{\sigma_1}dt) \\
&= (r - 0.5\sigma_1^2)dt + \sigma_1 dW_{1,t}^Q \\
\\
&dlog(S_{2,t}) = (\mu_2 - 0.5\sigma_2^2)dt + \sigma_2 dW_{2,t} \\
&= (\mu_2 - 0.5\sigma_2^2)dt + \sigma_2(dW_{2,t}^Q - \gamma_{2,t}dt) \\
&= (\mu_2 - 0.5\sigma_2^2)dt + \sigma_2(dW_{2,t}^Q - \frac{\mu_2 - r}{\sigma_2}dt) \\
&= (r - 0.5\sigma_2^2)dt + \sigma_2 dW_{2,t}^Q
\end{aligned}
\tag{20}
$$

Notice that μ_1 and μ_2 disappear. Then replicating the steps of (6), only this time we take the integrals over time-interval [0,t].By definition in (10) that $W_{1,0}^Q = 0$ and $W_{2,0}^Q = 0$ we obtain:

$$
\begin{aligned}
&log(S_{1,t}) - log(S_{1,0}) = \int_0^t dlog(S_{1,u}) \\
&= \int_0^t ((r - 0.5\sigma_1^2)du + \sigma_1 dW_{1,u}^Q) \\
&= (r - 0.5\sigma_1^2)dt + \sigma_1(W_{1,t}^Q - W_{1,0}^Q) \\
&= (r - 0.5\sigma_1^2)dt + \sigma_1 W_{1,t}^Q \\
&log(S_{2,t}) - log(S_{2,0}) = \int_0^t dlog(S_{2,u}) \\
&= \int_0^t ((r - 0.5\sigma_2^2)du + \sigma_2 dW_{2,u}^Q) \\
&= (r - 0.5\sigma_2^2)dt + \sigma_2(W_{2,t}^Q - W_{2,0}^Q) \\
&= (r - 0.5\sigma_2^2)dt + \sigma_2 W_{2,t}^Q
\end{aligned}
\tag{21}
$$

We conclude:

$$
\begin{aligned}
log(S_{1,t}) = log(S_{1,0}) + (r - 0.5\sigma_1^2)dt + \sigma_1 W_{1,t}^Q \\
log(S_{2,t}) = log(S_{2,0}) + (r - 0.5\sigma_2^2)dt + \sigma_2 W_{2,t}^Q
\end{aligned}
\tag{22}
$$

Taking exponential power from both sides and using the assumtions that $S_{1,0} = S_{1,0}$ & $S_{2,0} = S_{2,0}$ we get:

$$
\begin{aligned}
S_{1,t} = S_{1,0} * exp((r - 0.5\sigma_1^2)dt + \sigma_1 W_{1,t}^Q) \\
S_{2,t} = S_{2,0} * exp((r - 0.5\sigma_2^2)dt + \sigma_2 W_{2,t}^Q)
\end{aligned}
\tag{23}
$$

As mentioned earlier:

$$
\begin{aligned}
S_{1,0} &= B_0 * E_Q[\frac{S_{1,t}}{B_t}] \\
S_{2,0} &= B_0 * E_Q[\frac{S_{2,t}}{B_t}]
\end{aligned} \tag{24}
$$

Monte Carlo Simulation

We need to calculate $E_Q[P_T]$ in order to find P_0. P_0 can be approximated by simulations. Using time change of 1 unit such that dt=1 and (17) for times t = 0,1,...,6 we obtain:

$$
\begin{aligned}
S_{1,0} &= S_{1,0} \\
S_{1,1} &= S_{1,0} * exp((r - 0.5\sigma_1^2) + \sigma_1 W_{1,1}^Q) \\
S_{1,2} &= S_{1,1} * exp((r - 0.5\sigma_1^2) + \sigma_1 W_{1,2}^Q) \\
S_{1,3} &= S_{1,2} * exp((r - 0.5\sigma_1^2) + \sigma_1 W_{1,3}^Q) \\
S_{1,4} &= S_{1,3} * exp((r - 0.5\sigma_1^2) + \sigma_1 W_{1,4}^Q) \\
S_{1,5} &= S_{1,4} * exp((r - 0.5\sigma_1^2) + \sigma_1 W_{1,5}^Q) \\
S_{1,6} &= S_{1,5} * exp((r - 0.5\sigma_1^2) + \sigma_1 W_{1,6}^Q)
\end{aligned} \tag{25}
$$

and

$$
\begin{aligned}
S_{2,0} &= S_{2,0} \\
S_{2,1} &= S_{2,0} * exp((r - 0.5\sigma_2^2) + \sigma_2 W_{2,1}^Q) \\
S_{2,2} &= S_{2,1} * exp((r - 0.5\sigma_2^2) + \sigma_2 W_{2,2}^Q) \\
S_{2,3} &= S_{2,2} * exp((r - 0.5\sigma_2^2) + \sigma_2 W_{2,3}^Q) \\
S_{2,4} &= S_{2,3} * exp((r - 0.5\sigma_2^2) + \sigma_2 W_{2,4}^Q) \\
S_{2,5} &= S_{2,4} * exp((r - 0.5\sigma_2^2) + \sigma_2 W_{2,5}^Q) \\
S_{2,6} &= S_{2,5} * exp((r - 0.5\sigma_2^2) + \sigma_2 W_{2,6}^Q)
\end{aligned} \tag{26}
$$

Notice that here $W_{1,t}$ is here random variable following standard normal distribution for t=1,...,6 where 1 refares to first index and $W_{2,t}$ for second index for t=1,...,6. For the estimation we use:

$$
S_T = \begin{bmatrix} S_{1,T} \\ S_{2,T} \end{bmatrix} \quad S_0 = \begin{bmatrix} S_{1,0} \\ S_{2,0} \end{bmatrix} \tag{27}
$$

$$
\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{1,2} \\ \sigma_1\sigma_2\rho_{1,2} & \sigma_2^2 \end{bmatrix} \tag{28}
$$

We assume that the interest rate is r = 0.02 and T = 6.
So,in order to determine P_0 We use exact simulation and we randomly draw 10.000 times from bivariate normal distribution (we simulate 10.000 times values of W_T under Q) and

we then compute the corresponding S_T's for each of this 10.000 cases. We then get 10000 times different matrices with dimension 7 by 2 for i = 1,....,10000:

$$Matrix\ S_T^i = \begin{bmatrix} S_{1,0}^i & S_{2,0}^i \\ S_{1,1}^i & S_{2,1}^i \\ S_{1,2}^i & S_{2,2}^i \\ S_{1,3}^i & S_{2,3}^i \\ S_{1,4}^i & S_{2,4}^i \\ S_{1,5}^i & S_{2,5}^i \\ S_{1,6}^i & S_{2,6}^i \end{bmatrix} \tag{29}$$

Then using this matrix S_T^i we compute for each scenario i corresponding payoff:
$P_{Ti} = F(S_T^i)$ using the payoff function of part B: $P_T^i = P^{Ai} + P^{Ii}$

$$\begin{aligned} P^{Ai} &= \Big(50.7 - 8.45 * \sum_{k=1}^{6} \mathbf{1}\{\frac{S_{1,k}^i}{S_{1,0}^i} \geq 0.5\}\mathbf{1}\{\frac{S_{2,k}^i}{S_{2,0}^i} \geq 0.5\}\Big) * \mathbf{1}\Big\{\Big(\sum_{k=1}^{6} \mathbf{1}\{\frac{S_{1,k}^i}{S_{1,0}^i} \geq 1.1\}\mathbf{1}\{\frac{S_{2,k}^i}{S_{2,0}^i} \geq 1.1\}\Big) \geq 1\Big\}, \\ P^{Ii} &= 100 * \Big[\mathbf{1}\{\frac{S_{1,6}^i}{S_{1,0}^i} \geq 0.5\}\mathbf{1}\{\frac{S_{2,6}^i}{S_{2,0}^i} \geq 0.5\} + \mathrm{Min}\big(\frac{S_{1,6}^i}{S_{1,0}^i}, \frac{S_{2,6}^i}{S_{2,0}^i}\big) * \Big(1 - \mathbf{1}\{\frac{S_{1,6}^i}{S_{1,0}^i} \geq 0.5\}\mathbf{1}\{\frac{S_{2,6}^i}{S_{2,0}^i} \geq 0.5\}\Big)\Big] \end{aligned} \tag{30}$$

Applying FFT,while using B_T as numeraire we observe that:

$$P_0 = B_0 E_Q\Big[\frac{P_T}{B_T}\Big] = exp(-rT)E_Q\big[P_T\big] \tag{31}$$

For simplification we created payoff function in *Matlab* called *PayoffFunction* in order to calculate the payoffs of the contract.The expectation $E_Q[P_T]$ we estimate as follows: we create a payoff vetctor called *Payoff-Vector* with dimension 1 by 10000 where we store calculated payoffs for all 10000 scenarios of (27).We then calculate the mean of Payoff-Vector, which will be then the estimated $E_Q[P_T]$.

$$E_Q[P_T] = \frac{1}{10000}\sum_{i=1}^{10000} P_T^{(i)} \tag{32}$$

Finally, using this estimation (29)of the expectation of the payoff function under the probability measure Q, we are ready to compute the arbitrage-free price of the contract combining (28) and (29) such that:

$$P_0 = \frac{exp(-rT)}{10000}\sum_{i=1}^{10000} P_T^{(i)} = £111.1211 \tag{33}$$

Part E: The probability that investing in the "Atlantic Lock-In Synthetic Zero" yields a return that exceeds the return of investing all the money in the money-market-account

To calculate probability that investing in the "Atlantic Lock-In Synthetic Zero" yields a return that exceeds the return of investing all the money in the money-market-account we first need to look at the simple option,when this all the money is invested in money-market account with interest rate of 2%.Using continuous compounding we compute the return on the money-market account:

$$R^B = e^{0.02*6} * £1000 = £1127.497 \tag{34}$$

This gives us return of 12.75%.
As we have computed in part D, the no-arbitrage price of the contract is £ 111.1211. Therefore, this investor can buy approximately 8 certificates of "Atlantic Lock-In Synthetic Zero" each with the price of £ 111.121.(£1000/£ 111.121 ≈ 8,99919 ≈ 8). This means that the investor with £1000 invests £889.68 in "Atlantic Lock-In Synthetic Zero" and £110.32 on money-market account.
In this final part of our analysis we will use Brownian Motions under probability measure P and not under Q like in part (D).
Using the same exact steps of equations in part C only over the interval of [0,t] we get:

$$\begin{aligned} log(S_{1,t}) &= log(S_{1,0}) + (\mu_1 - 0.5\sigma_1^2) + \sigma_1 W_{1,t} \\ log(S_{2,t}) &= log(S_{2,0}) + (\mu_2 - 0.5\sigma_2^2) + \sigma_2 W_{2,t} \end{aligned} \tag{35}$$

$$\begin{aligned} \int_0^t dlog(S_{1,t}) &= \int_0^t ((\mu_1 - 0.5\sigma_1^2)du + \sigma_1 dW_{1,u}) \\ \int_0^t dlog(S_{2,t}) &= \int_0^t ((\mu_2 - 0.5\sigma_2^2)du + \sigma_2 dW_{2,u}) \end{aligned} \tag{36}$$

Applying Telescope rule we get in the left hand-side of our equations:

$$\begin{aligned} &\int_0^t dlog(S_{1,t}) = log(S_{1,0}) - log(S_{1,t}) \\ &\int_0^t dlog(S_{2,t}) = log(S_{2,0}) - log(S_{2,t}) \\ &\int_0^t ((\mu_1 - 0.5\sigma_1^2)du + \sigma_1 dW_{1,u}) = (\mu_1 - 0.5\sigma_1^2)(t-0) + \sigma_1(W_{1,t} - W_{1,0}) \\ &\int_0^t ((\mu_2 - 0.5\sigma_2^2)du + \sigma_2 dW_{2,u}) = (\mu_2 - 0.5\sigma_2^2)(t-0) + \sigma_2(W_{2,t} - W_{2,0}) \end{aligned} \tag{37}$$

Then we take the exponential power from both sides and we obtain:

$$\begin{aligned} S_{1,t} &= S_{1,0} * exp\big((\mu_1 - 0.5\sigma_1^2) + \sigma_1 W_{1,t}\big) \\ S_{2,t} &= S_{2,0} * exp\big((\mu_2 - 0.5\sigma_2^2) + \sigma_2 W_{2,t}\big) \end{aligned} \tag{38}$$

In order to find the payoffs we go through exact same steps as in Monte Carlo simulation of part D with only difference that instead of r we use μ_1 and μ_2 and in order to take into account the correlation coefficient of two indices we use slightly different SDE's in the *Matlab code*, namely:

$$\begin{aligned} dS_{1,t} &= \mu_1 S_{1,t} dt + \sigma_1 S_{1,t} dW_{1,t} \\ dS_{2,t} &= \mu_2 S_{2,t} dt + \sigma_2 S_{2,t} \sqrt{\rho^2} dW_{1,t} \sigma_2 S_{2,t} \sqrt{1-\rho^2} dW_{2,t} \end{aligned} \tag{39}$$

Where ρ is the correlation coefficient between FTSE 100 Index and S&P 500 index. We then again get a payoff vector called *Payoff-Vector2* with dimension 1 by 10000 using the results of (26)-(30). Notice that this time the payoffs are calculated with the same function *PayoffFunction* in *Matlab* only taking into account that only 8 certificates are bought and the rest of the investment money has been put on the Bank account.
Moreover, for each scenario i of this 10000 cases we compare each time the corresponding payoff of the contract P'$^{(i)}_T$ with the return on money-market account: R$_B$-1000 (34).
Then each time when the second investment payoff is larger than the value of 127.497 (return on money-market account) the variable *CountHigherPayoff* increases by 1. At the and we devide this *CountHigherPayoff* by 10000 and we get that the probability that investing in the "Atlantic Lock-In Synthetic Zero" lead to higher return than investing solely in money-market account is 91.22% (0.9122).