Linear Programming Optimization in Finance

Khrystyna Trowbridge
MS Data Science, CIS Dept, Fordham University, Lincoln Center
113 W 60th Street, New York, NY 10023, USA
ktrowbridge@fordham.edu

***Abstract*—This project aims to study the use of linear programming (LP) in finance and explore in which way linear programming can be brought up and used for Cost of Capital with the emphasis on the Investment Portfolio Management for Capital Asset Pricing Model. The tasks of this project are 1) review and propose linear programming approach for corporate financial planning; 2) sensitivity analysis in linear programming review and evaluation; 3) review of practical difficulties of the models' implementation.**

Keywords—linear programming, optimization function, linear equality and inequality constrains, feasible region, half spaces, capital, cost of capital, rate of return, debt, equity, weighted-average, financial markets, sensitivity analysis, tax implications, tax rate, hedging, risk estimate, market risk premium.

I. Introduction

Financial decisions in both corporate and investment worlds always required thorough analysis and detailed and accurate mathematical calculations, along with understanding of the importance of possible economic, social, and political interventions into a technically perfect forecasting. That is the reason why hedging grew in its popularity during the last few decades – option to leverage your risks and losses related to them. However, if the risks are too high, it is harder to forecast possible returns and the need to find the most efficient and profitable source of capital.

A linear programming model is widely used to determine the most economical arrangement of finance suitable for company, private equity, or personal investments. Many large and medium businesses use linear programming and similar algorithms and either dedicate the whole department of their staff or hire third party consultants to perform the analyses for capital cost planning and scheduling processes. The analysis requires information about debt/equity proportion, as well as the cost of each. These are incorporated into a linear programming which determines the highest profit, with maximized revenue and minimized costs of borrowings and investments. Model results can be used by the management to decide on the best combination of capital sources, to arrange the best times to start and finish projects and to select projects and investment strategies, which would be the most suitable for company's or personal financial objectives.

When budgeting, businesses of all kinds typically focus on three types of capital: working capital, equity capital, and debt capital. Working capital is the difference between current assets and current liabilities. In simple terms, it is what the company already had and/or owed. This capital can be optimized by strategic management and operational planning. Equity capital is typically viewed in the form of shares of stock of the company. Debt capital most often referred to borrowing from banks and other financial institutions or issuing bond. When the need to expand or finance a new operational unit shows up on horizon, management needs to decide how operationally and financially functionally this need is. Each particular idea of implementation the need is called a project. Whenever the decision to start a new project comes up, the question of how it will be sponsored appears, meaning what capital will be used and where to obtain it from. Also, usually there is never only one option for business projects for developments and growth, and therefore it makes even more difficult for the decision makers to choose which option(s) is/are the best. The solution should be the one, that will provide the highest return on investment (ROI) either by choosing one project or a combination of them. Here is where linear programming can be of great use.

In this paper I will concentrate the attention on corporate projects and review the most optimized ways to implement them in terms of profit. Each project is evaluated based on its rates of return and opportunity cost. We will review how the linear programming can help us to formulate a decision proposal for rationing the undertaken risk. I will describe how sensitivity analysis techniques can determine how the output can be influenced by changing the input variables and changing the constrains of the model, how the difference can influence the decision of capital allocation and why is that so important for linear programming tasks. I will also review the states of possible projects being dependent and independent from each other, and what practical difficulties can be seen in real life.

II. ANALYSIS OF THE PROBLEM

A. Linear Programming

Linear Programming (LP) models are mathematical programming models dealing with the optimization of a linear objective function under linear equality and linear inequality constraints.

Linear programming is viewed as an important algorithmic technique that is used to find the optimum resource usage. The "linear" part reflects the relationship between multiple variables with degree one. The "programming" part reflects the process of selecting the best solution from the variety of alternatives depending on constrains and optimization function itself. The characteristics of the linear programming are the following: objective function (linear function that defines the problem), constrains (the limitations set by the problem to define the resource in mathematical terms), linearity (meaning that the relationship between the variables is linear), finiteness (the function has to have finite factors, otherwise the optimal solution is not feasible), decision variables (they determine the output; for any problem this is the first step to identify the decision variables).

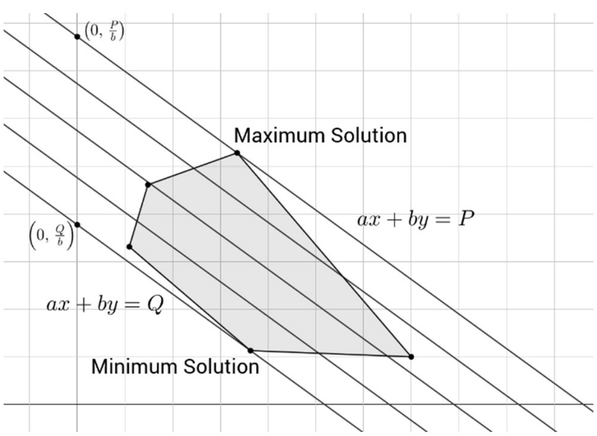

Fig. 1 Illustration of the Linear Programming Graphed System Constraints

The linear programming problems use different methods to find the solution, such as the simplex method, graphical method, and/or by using tools such as R, open solver, Python etc.

B. Project Capital

First, let's understand what capital is and how it can be derived by the company.

Capital is anything that brings value or benefit to its owner, such as a plant and equipment, intellectual property like patents and licenses, or the financial assets of a business or an individual. Business capital may be obtained from the operations of the business or be raised from equity and/or debt financing. Equity financing means the company goes public (IPO) and sells its shares of stock to the investors. The earned revenue then is considered a part of capital. Debt financing means the company issues bonds - a documented company's promise to pay their principal and interest on a loan to a lender. And the earned revenue then also is considered a part of capital. We will concentrate on the latter two to see how we can optimize our cost of capital function and what constrains have to be analyzed and reviewed.

Most common way to calculate the cost of capital is by using a Weighted Average Cost of Capital formula, which represents weighted cost of capital from all the resources the company used.

Weighted Average Cost of Capital:

$$\text{WACC} = w_d \times r_d + w_e \times r_e$$

where:
w_d - weight of debt
w_e - weight of equity
r_d - cost of debt
r_e - cost of equity

Now, let's look at the two parts of the WACC equation and see how each of the calculated:

$$\text{Cost of Debt} = \text{I} \times (1\text{-T})$$

where:
I – interest on the company's current debt
T – the company's marginal tax rate

$$\text{Cost of Equity} = R_f + \beta(R_m - R_f)$$

where:
R_f - risk-free rate of return
R_m - market rate of return
β - estimated risk

The cost of capital needs to be minimized, as the company's goal is to obtain the maximum return from the capital usage. The cost of debt and the cost of equity are constants that would be calculated and based on the time of market conditions as of the date of project implementation. The sum of the equity and debt weight should equal to 1, and they should be non-negative. However, particularly the weights will be the variables the task would be to find.

For example, if the calculated r_e = 7% and the calculated r_d = 5%, the linear programming task would have the following look:

$$w_d \times 0.05 + w_e \times 0.07 \text{ – to minimize}$$

The constraints would be:

$$w_d + w_e = 1,$$

$$w_d \geq 0$$

$$w_e \geq 0$$

$$w_e \geq k \times w_d$$

where:
k – coefficient determined by the company based on tax laws, company's objectives, present debt-to-equity ratio

As we can see, by analyzing what the capital consists of and what parts of the formula are constants, we can find an optimized solution for the lowest possible cost, counting those costs of both debt and equity were calculated based on the market projected estimated risk, risk-free rate of return, market rate of return, corporate tax rate and present interest expense.

C. Linear Programming Implementation For Investment Decisions

Now, let's see how Linear Programming can be implemented in the project's capital cost on a straightforward example, but for this task we will assume that the company is actually seeking a return for its current investment options, so it can be utilized for the project implementation, rather than finding the solution for minimized capital cost.

Let's assume there was a project approved with a budget of $12,000. The possible additional expenses could come, and the management wanted to see what the best options are to invest the budget amount in order to get the maximum return, which could cover the initial project cost plus additional expenses throughout the project's duration. Let B be investment in Bonds, C – investment in Certificate of Deposits (CD), and H – investment in high-yield savings account. The calculated return on B = 7%, return on C = 8%, return on H = 12%.

Investment in high yield savings should be no more of $2,000. Investment in bonds should be at least three times as in CDs, as the occurring interest expense would offset the total profit on which the corporate tax would be applied at the end of the tax quarter/year.

After the consideration of corporate tax expense and proportion of weights of equity and debt, the optimization function constructed was the following:

$$0.07B + 0.08C + 0.12H \text{ - to maximize}$$

The constrains:

$$B + C + H \leq 12{,}000$$

$$H \leq 2{,}000,$$

$$B \geq 3C,$$

$$B \geq 0,$$

$$C \geq 0,$$

$$H \geq 0$$

After building and running a Python program, we received the following results:

Bonds = $7,500

CDs = $2,500

High yield savings = $2,000

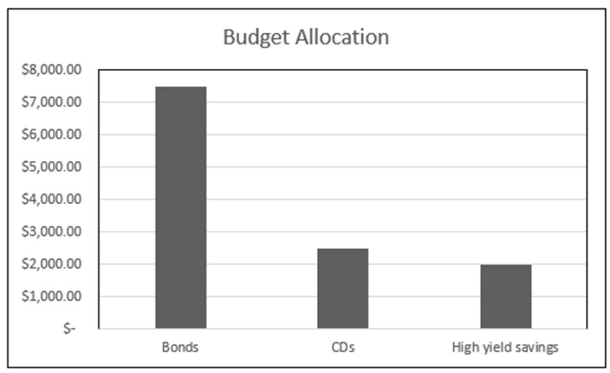

Fig. 2 Budget allocation visualization

By implementing a linear programming, we were able to solve the problem and get the most optimized solution for our project capital source.

After we saw how LP can be implemented on already pre-set budget and corporate calculations, we want to build another more generalized optimized function and review the possible constraints that need to be considered while initializing the problem to source the project.

Let's look at our problem under a different angle. Let's combine two objectives in one – choosing the project(s) that would bring in the biggest cash flow and be of the highest possible value. This would be a completely different task comparing to the two described above, as this one will not concentrate on cost of the capital or on options to invest and maximize the returns on the financial assets. This problem asks states another problem – what project(s) should we give a green light to maximize their values (and therefore the company's value) and how much potential cash flow would this/these project(s) bring. By putting the problem in the abovementioned boundaries, we would need to obtain an objective function that would need to be maximized and the constraints that would define the requirement.

We will initialize the company's objective function.

To maximize:

$$\Delta = \sum_{k=1}^{n} X\,k * Ak + \sum_{k=1}^{m} Y\,k * Bk$$

where:
Δ - change in stock price
Xk = 0 if the investment was denied, 1 – if accepted
Ak – k^{th} project's present value (PV)

Yk – financing obtained by investing in the chosen k^{th} project
Bk – change in stock price of financing obtained from project k

The objective function is subject to:

$$Y_k \leq Z_k$$

$$X_k \leq 1$$

where:
Z_k – accepted projects' debt capacity

It is important to notice that for setting our optimization function the following way, we assumed that decision to invest in the project k is independent of other investment decisions made by the company, and therefore there is no correlation of obtaining any particular stock price change.

III. Sensitivity Analysis

A. *The Sensitivity Analysis*

The sensitivity analysis is an analysis based on the variables that affect valuation, which a financial model can depict using the variables' price and Earning Per Share (EPS). The sensitivity analysis isolates these variables and then records the range of possible outcomes.

The sensitivity analysis helps identify the best data that needs to be collected for analyses to evaluate a project's return on investment (ROI). The sensitivity analysis helps engineers create more reliable, robust mathematical designs by assessing points of uncertainty in the design's structure.

B. *What Role The Sensitivity Analysis Plays*

The sensitivity analysis main purpose is to help with the uncertainty inherent in mathematical models where the values for the inputs used in the model can vary. It goes hand-to-hand with initialization of linear programming optimization function and the constraints to it. All of the models composed, and studies executed, to draw conclusions or inferences for policy decisions, are based on assumptions regarding the validity of the inputs used in calculations.

The conclusions obtained from studies of linear programming calculations can be significantly altered, depending on such things as how a certain variable is defined, or the parameters initialized for a study. When the results of a study or computation do not significantly change due to variations in underlying assumptions, such as optimized function initialization and constraints building, they are considered to be robust. If variations in foundational inputs or assumptions significantly change outcomes of the model, sensitivity analysis can be employed to determine how changes in inputs, definitions, or modeling can improve the accuracy or robustness of any outcoming results.

C. *How The Sensitivity Analysis is Used*

The sensitivity analysis is highly helpful in engineering techniques to use computer models to test the design of structures before they are built. Sensitivity analysis helps mathematicians and engineers create more trustworthy, robust designs by assessing points of uncertainty or wide variations in possible inputs and their corresponding effects on the viability of the model. Refinement of computer models such as linear programming models can tremendously impact the accuracy of evaluations of such.

IV. Practical Difficulties of The Model's Implementation

In the first section we reviewed the formula of Weighted Average Cost of Capital, which says that the capital cost consists of debt and equity with some proportion of each other. That proportion is critical to find. We also noticed that two variables need to be independent of each other. However, in practice it is an issue that needs to have a closer consideration.

First, it is important to notice that the size of possible obtained debt is proportional to a lot of variables characterizing the company – its size, credit history, balance sheet liability numbers, reputation, etc. But mostly, it is dependent on the company's equity base. If the company is highly leveraged (meaning that company already has a higher debt than the average of the industry it belongs to, and/or its present capital debt base is high comparing to equity – standard ration debt to equity usually has to be no more of 2.0) than it will be more difficult for it to obtain any additional debt to serve the purpose of the sponsoring the project. Also, the interest rate most probably would be higher, which would influence our optimization function constrains. In return, the equity will be affected too, since higher coupon/interest rates would imply less cash (and/or other liquid assets) left to invest back in the company, which inevitably would affect the company's stocks and similar financial assets' (like derivatives – if pertaining to the company's segment) prices.

Moreover, if the company decides to do a stock split or issue more stocks (depending on company's financial prerogatives and future strategies), but it has a lot of debt, it would be seen by potential investors as either a possible candidate for bankruptcy or a high-risk investment option. The first view will make additional stock issue an impossible task, the second view will make the prices of additionally issued stocks very low, which in turn can become a snowball effect and make the company's market value go tremendously low.

Second important issue is bond convexity. Bond convexity is a risk-measurement tool, which is used to measure and manage the exposure of the investment portfolio risk. It is a measure of the curvature in the relationship between bond prices and bond yields, which demonstrates how the duration of a bond changes as the interest rate changes. If market rates rise by 1%, a one-year maturity bond price should decline by an equal 1%. However, for bonds with long-dated maturities, the reaction increases. As a general rule of thumb, if rates rise by 1%, bond prices fall by 1% for each year of maturity. For example, if rates rise by 1%, the two-year bond price would fall 2%, the three-year bond price by 3%, and the 10-year price by 10%.

If the company issued bonds, it means it will have a liability which needs to be fulfilled within some timeframe and due to maturity date (for bonds). However, if the company's management decided to source the project's capital in installments, meaning the total cost was calculated, designed, and considered to be fulfilled and to be divided within the timeframe instead of being received as a lump sum at the initial start of the project implementation, the proposed calculations, and estimations could be at risk due to convexity. If the market interest rates rise the risk to a fixed-income portfolio means that the existing fixed-rate instruments are not as attractive. Therefore, it would be more difficult to obtain debt base capital and especially with the same calculated revenue and associated cost. As convexity decreases, the exposure to market interest rates decreases and the bond portfolio can be considered hedged. As a conclusion, all previously made calculations based on linear programming model would not work as the function and its constrains were influenced and therefore changed.

Third issue is corporate tax estimation. Whenever the company forecasting its budget and revenue estimation, the tax implications play an important role in those calculations. Depending on what accounting principal the company implements, the profit on which the tax will be imposed would differ. Meaning, whenever the proposed project's ROI calculated, its optimization function and constrains might not represent the full picture as the tax variable could be miscalculated due to inaccurate projected revenue and expenses.

It is important to notice that as of today, thankfully to the Tax Cuts and Jobs Act of 2017 and its established fixed tax rate of 21% this issue is covered. However, counting that prior to 2017 there were taxable income brackets and that the current act can be reversed, we cannot fully disregard such a possible difficulty.

V. Conclusion

In this study we reviewed what linear programming is and how useful linear programming for financial optimization for corporate and wealth management decisions. We looked at a few different ways linear programming can be implemented, and different optimization functions can be constructed, depending on the company's objective and particular requirements and/or interests. We emphasized the importance of decision variables (activities) - in order to arrive at the ideal value of the objective function, we must analyze several alternatives (courses of action). Obviously, we wouldn't need linear programming if we didn't have any other options. The nature of the objective function and the availability of resources guide the evaluation of potential options.

We stated the importance of sensitivity analysis and its role in initializing the objective functions, building constraints, and deciding on what would be the constants and variables. Sensitivity analysis also aids in understanding the decision model's uncertainties, constraints, and scope. After all, we make the majority of our judgments in the face of uncertainty, therefore it's useful to know how much distinct independent variables can influence the dependent variable. The thorough research is always needed when the tasks given are not that straightforward and cannot be easily described. And for financial purposes, it is required to get all information from the outer world based on the present conditions – simply because the previous mathematical forecasts cannot guarantee the accuracy of such.

We reviewed technical and real-life difficulties that the companies of different size can meet, as the real business cycle cannot be perfectly, and hundred percent accurately described by mathematical algorithms due to stochastic constrains. Sometimes it is more practical to assume independence of variables, or state some variables as a constant (i.e., estimated risk based on the present market characteristics). But it is needed to be mentioned, that we need to be careful with those assumption as sometimes they are based on other estimations as well.

As a summary, it is evidently that linear programming is highly practical for the goal to identify the capital allocation that maximizes total expected return while minimizing risk within specific constraints.

REFERENCES

[1] “Introduction to practical linear programming,” *Computers & Mathematics with Applications*, vol. 33, no. 4, p. 133, Feb. 1997, doi: 10.1016/s0898-1221(97)90047-x.

[2] A. Charnes, W. W. Cooper, and M. H. Miller, “Application of Linear Programming to Financial Budgeting and the Costing of Funds,” *The Journal of Business*, vol. 32, no. 1, p. 20, Jan. 1959, doi: 10.1086/294232.

[3] C. S. Warren, J. M. Reeve, and J. E. Duchac, *Managerial accounting*. Boston] Cengage Learning, 2016.

[4] W. T. Morris, “‘Application of Linear Programming To Financial Budgeting and The Costing Of Funds,’” *The Engineering Economist*, vol. 5, no. 3, pp. 55–56, Jan. 1960, doi: 10.1080/001379x6008546907.

[5] C. W. Young, “LINEAR PROGRAMMING AND SHORT-TERM FINANCIAL PLANNING*,” *The Journal of Finance*, vol. 24, no. 4, pp. 738–739, Sep. 1969, doi: 10.1111/j.1540-6261.1969.tb00406.x.

[6] A. Geyer, M. Hanke, and A. Weissensteiner, “Life-cycle asset allocation and consumption using stochastic linear programming,” *The Journal of Computational Finance*, vol. 12, no. 4, pp. 29–50, Jun. 2009, doi: 10.21314/jcf.2009.203.

[7] G. Salkin and J. Kornbluth, *Linear programming in financial planning*. London Haymarket, 1973.

[8] G. Cornuejols, Javier Francisco Peña, and Reha Tütüncü, *Optimization methods in finance*. Cambridge, United Kingdom ; New York, Ny: Cambridge University Press, 2018.

[9] Veikko Jääskeläinen, *Linear programming and budgeting*. Lund: Studentlitteratur, 1977.

[10] Ales Cerny, *Mathematical techniques in finance : tools for incomplete markets*. Princeton, N.J. ; Oxford: Princeton University Press, 2009.

[11] N. N. Wijeratneand and F. C. Harris, “Development of linear programming for capital budgeting of construction projects,” *Building Research & Information*, vol. 21, no. 6, pp. 339–345, Nov. 1993, doi: 10.1080/09613219308727332.

[12] J. W. Hayes, “DUAL VARIABLES IN PURE CAPITAL RATIONING LINEAR PROGRAMMING FORMULATIONS,” *The Engineering Economist*, vol. 34, no. 3, pp. 255–260, Jan. 1989, doi: 10.1080/00137918908902990.

[13] Y. Yang and Fei-Yue Wang, *Budget constraints and optimization in sponsored search auctions*. San Diego, Ca: Academic Press, 2014.

[14] B. D. Craven and Sardar M N Islam, *Optimization in economics and finance : some advances in non-linear, dynamic, multi-criteria and stochastic models*. Dordrecht: Springer, 2005.

[15] H. M. Salkin and Jahar Saha, *Studies in linear programming*. Amsterdam: North-Holland Pub. Co. ; New York, 1975.

[16] Marvin Walter Kottke, *Obtaining identical solutions using linear programming and partial budgeting*. Storrs, Conn.: Storrs Agricultural Experiment Station, 1960.

[17] M.-I. Boloș, I.-A. Bradea, and C. Delcea, “Linear Programming and Fuzzy Optimization to Substantiate Investment Decisions in Tangible Assets,” *Entropy*, vol. 22, no. 1, p. 121, Jan. 2020, doi: 10.3390/e22010121.

[18] K. J. Cohen and Richard Michael Cyert, *Theory of the firm : resource allocation in a market economy*. Englewood Cliffs, N.J.: Prentice-Hall, 1975.

[19] G. A. Fleischer, *Capital allocation theory*. New York: Appleton-Centery-Crofts, 1969.

[20] S. C. Myers, *Linear programming for financial planning under uncertainty*. Cambridge, M.I.T, 1969.

[21] E. Nikbakht and A. A. Groppelli, *Finance*. Hauppauge, New York: Barron’s Educational Series, Inc, 2018.

[22] Robert Willard Johnson, *Capital budgeting*. Dubuque, Iowa: Kendall/Hunt Pub. Co, 1977.

[23] H. J. Johnson, *Strategic capital budgeting : developing and implementing the corporate capital allocation program*. Chicago, Ill.: Probus Pub. Co, 1994.

[24] J. E. Graver, “On the foundations of linear and integer linear programming I,” *Mathematical Programming*, vol. 9, no. 1, pp. 207–226, Dec. 1975, doi: 10.1007/bf01681344.

[25] D. M. Valladão, Á. Veiga, and G. Veiga, “A multistage linear stochastic programming model for optimal corporate debt management,” *European Journal of Operational Research*, vol. 237, no. 1, pp. 303–311, Aug. 2014, doi: 10.1016/j.ejor.2014.01.028.

[26] Sven Danø, *Linear programming in industry: theory and applications; an introduction.* Wien, Springer, 1963.

[27] H. M. Weingartner, “Linear Programming and Optimal Bank Asset Management Decisions: Discussion,” *The Journal of Finance*, vol. 22, no. 2, p. 166, May 1967, doi: 10.2307/2325552.

[28] W. Candler and S. Dano, “Linear Programming in Industry. Theory and Applications. An Introduction,” *Econometrica*, vol. 30, no. 2, p. 394, Apr. 1962, doi: 10.2307/1910236.

[29] Sven Danø, *Linear programming in industry*. Wien: Springer-Verlag, 1974.

[30] R. G. Bland, “The Allocation of Resources by Linear Programming,” *Scientific American*, vol. 244, no. 6, pp. 126–144, Jun. 1981, doi: 10.1038/scientificamerican0681-126.

[31] T. L. Saaty, L. G. Vargas, and K. Dellmann, “The allocation of intangible resources: the analytic hierarchy process and linear programming,” *Socio-Economic Planning Sciences*, vol. 37, no. 3, pp. 169–184, Sep. 2003, doi: 10.1016/s0038-0121(02)00039-3.

[32] M. A. H. Dempster and J. P. Hutton, “Pricing American Stock Options by Linear Programming,” *Mathematical Finance*, vol. 9, no. 3, pp. 229–254, Jul. 1999, doi: 10.1111/1467-9965.00069.

[33] M. Leippold and J. M. Syz, “Trend Derivatives: Pricing, Hedging, and Application to Executive Stock Options,” *SSRN Electronic Journal*, 2006, doi: 10.2139/ssrn.782786.

[34] S. A. Ross, R. Westerfield, and J. F. Jaffe, *Corporate finance*. New York, Ny: Mcgraw-Hill/Irwin, 2013.