Fundamental Theorem of Asset Pricing *(FTAP)*

Introduction
Fundamental theorem of asset pricing states that, in a complete market (market with no transaction costs, perfect information, there exists a price for every asset under every state of the economy) then primary securities are all arbitrage free, iff there exists a unique measure, under which all numeraire-deflatted asset prices are martingale (or driftless), i.e. $\exists Q_N$ such that :

$$S_t / N_t = E_{Q_N}[S_T / N_T \mid I_t]$$

or

$$S_t = E_{Q_N}[\underbrace{N_t / N_T}_{DF_{QN}} \times S_T \mid I_t]$$

Redundant securities, also known as contingent claims, are contracts with future payoffs depending on realization of some random variables. According to the *Law of one price*, contracts with the same future payoffs at maturity *T*, should share the same price today. Therefore all contingent claims can be priced via risk neutral pricing as long as their payoffs can be replicated by primary securities. Finally, according to *Girsanov theorem*, a change in measure becomes a change in drift in stock *SDEs*, hence there is no need to solve for the risk neutral measure explicitly.

Expected log normal	$E[e^{\varepsilon(\mu,\sigma)}] = e^{\mu+\sigma^2/2}$	
Stock price	$S_T = F_t(T)e^{-v/2+\sqrt{v}\varepsilon}$	$E[S_T] = F_t(T)$ and $E[S_T^2] = F_t^2(T)e^v$ and $v = V_Q[\ln(S_T/S_t)]$
Option price	$d_{1,2} = \dfrac{\ln F/K \pm v/2}{\sqrt{v}}$	$d_1(F) = \dfrac{\ln Fe^v/K - v/2}{\sqrt{v}} = d_2(Fe^v)$ *convexity adjustment*
Girsanov theorem	$\dfrac{dQ}{dP} = e^{-v/2-\sqrt{v}\varepsilon}$	

Single period economy

1.1 Fundamental theorem of asset pricing (FTAP) and Law of one price (LOOP)

Given a single period economy having only two time points, current time t and future time T, prices at t are deterministic, whereas payoffs at T are stochastic, they depend on one economic risk factor $\pi \in \Omega$ having M possible states $\Omega = \{\pi_1, \pi_2, \ldots, \pi_M\}$, which will be realized at T. Physical probabilities of the states are not involved in this setting. There are one risk free asset and N-1 risky assets in the economy, price of the assets at t form a N vector and payoff of the assets under various economic states at T form a $N \times M$ matrix. As these assets are limited liability securities, all entries in ***price vector*** S_t and in ***payoff matrix*** S_T are non-negative.

Important, 3 different notations :

a column vector, a row vector and an entry

price $$S_t = \begin{bmatrix} S_{1,t} \\ S_{2,t} \\ S_{3,t} \\ \ldots \\ S_{N,t} \end{bmatrix}$$

payoff vector for state m $S_T(\pi_m)$

payoff structure of asset n, $S_{n,T}$

payoff $$S_T = \begin{bmatrix} S_{1,T}(\pi_1) & S_{1,T}(\pi_2) & S_{1,T}(\pi_3) & \ldots & S_{1,T}(\pi_M) \\ S_{2,T}(\pi_1) & S_{2,T}(\pi_2) & S_{2,T}(\pi_3) & \ldots & S_{2,T}(\pi_M) \\ S_{3,T}(\pi_1) & S_{3,T}(\pi_2) & S_{3,T}(\pi_3) & \ldots & S_{3,T}(\pi_M) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ S_{N,T}(\pi_1) & S_{N,T}(\pi_2) & S_{N,T}(\pi_3) & \ldots & S_{N,T}(\pi_M) \end{bmatrix}$$

asset index *state index*

$$S_{n,T}(\pi_m)$$

time index

$$= \begin{bmatrix} S_{1,T} \\ S_{2,T} \\ S_{3,T} \\ \ldots \\ S_{N,T} \end{bmatrix} \qquad \textit{in terms of row vectors}$$

$$= [S_T(\pi_1) \quad S_T(\pi_2) \quad S_T(\pi_3) \quad \ldots \quad S_T(\pi_M)] \qquad \textit{in terms of column vectors}$$

<u>1.1a Primary securities / payoff structure / payoff space / numeraire deflatted</u>

The n-th row vector in matrix S_T is the ***payoff structure*** of asset n, which denotes how payoff of ***primary securities*** n varies with the single risk factor. Hence there are N payoff vectors (one for each asset) in the $\Re^M$ ***payoff space***. Risk free asset is the one with payoff independent of economic states, the risk free asset is not necessarily $S_{1,T}$.

$$S_{n,T}(\pi_m) = const \qquad \forall m \in [1, M]$$

In order to define time value of money (or equivalently, the exchange rate between two time points) we pick one out of the N assets as an accounting unit, known as the numeraire. The numeraire is not necessarily the risk free asset, it can be a cash account, a bond, an annuity or even a stock. Without loss of generality, we pick S_1 as numeraire and normalize both price and payoff of all assets as :

price $$S^*_{n,t} = S_{n,t} / S_{1,t} \qquad \forall n \in [1, N]$$

payoff $$S^*_{n,T}(\pi_m) = S_{n,T}(\pi_m) / S_{1,T}(\pi_m) \qquad \forall n \in [1, N] \ \textit{and} \ \forall m \in [1, M]$$

$$S^*_{1,T}(\pi_m) = 1 \qquad \textit{for numeraire and} \ \forall m \in [1, M]$$

Normalized ***price vector*** and ***payoff matrix*** become :

price $$S^*_t = \begin{bmatrix} 1 \\ S_{2,t} / S_{1,t} \\ S_{3,t} / S_{1,t} \\ \ldots \\ S_{N,t} / S_{1,t} \end{bmatrix}$$

payoff $$S^*_T = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ S_{2,T}(\pi_1)/S_{1,T}(\pi_1) & S_{2,T}(\pi_2)/S_{1,T}(\pi_2) & S_{2,T}(\pi_3)/S_{1,T}(\pi_3) & \ldots & S_{2,T}(\pi_M)/S_{1,T}(\pi_M) \\ S_{3,T}(\pi_1)/S_{1,T}(\pi_1) & S_{3,T}(\pi_2)/S_{1,T}(\pi_2) & S_{3,T}(\pi_3)/S_{1,T}(\pi_3) & \ldots & S_{3,T}(\pi_M)/S_{1,T}(\pi_M) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ S_{N,T}(\pi_1)/S_{1,T}(\pi_1) & S_{N,T}(\pi_2)/S_{1,T}(\pi_2) & S_{N,T}(\pi_3)/S_{1,T}(\pi_3) & \ldots & S_{N,T}(\pi_M)/S_{1,T}(\pi_M) \end{bmatrix}$$

1.1b Redundant securities / strategy / asset span / complete market

Trading strategy is defined as a portfolio of primary securities with weights w_n, $n \in [1.N]$, which is held from t to T, where w_n can be positive for long position, or negative for short position. Numeraire S_1 is also included in the portolio. Normalized portfolio ***payoff*** structure at T for various economic states can be written as a function of π :

$$\textit{payoff} \qquad V_T^*(\pi_m) = \sum_{n=1}^{N} w_n S_{n,T}^*(\pi_m) \qquad \forall m \in [1, M]$$

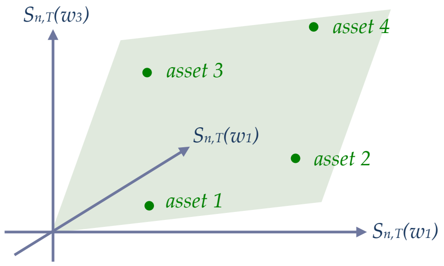
or in matrix form :

$$\begin{aligned}
\textit{payoff} \qquad V_T^* &= [V_T^*(\pi_1) \quad V_T^*(\pi_2) \quad V_T^*(\pi_3) \quad ... \quad V_T^*(\pi_M)] \\
&= \left[\sum_{n=1}^{N} w_n S_{n,T}^*(\pi_1) \quad \sum_{n=1}^{N} w_n S_{n,T}^*(\pi_2) \quad \sum_{n=1}^{N} w_n S_{n,T}^*(\pi_3) \quad ... \quad \sum_{n=1}^{N} w_n S_{n,T}^*(\pi_M)\right] \\
&= W S_T^* \\
\textit{where} \qquad W &= [w_1 \quad w_2 \quad w_3 \quad ... \quad w_N]
\end{aligned}$$

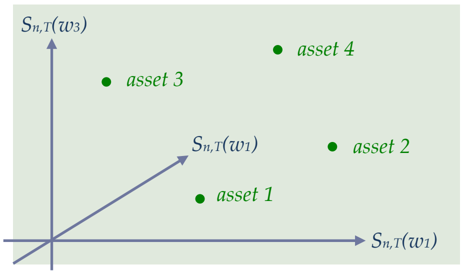
By changing portfolio vector W, we can synthesize various payoff structures.

- Securities with payoff structures replicable by ***primary securities S*** are called ***redundant securities V***.
- The whole set of redundant securities is known as the asset span in $\Re^{rank(S)}$ space, where $rank(S) \leq min(N,M)$.
- In general scenario, ***asset span*** is a subset of ***payoff space***, i.e. $rank(S) \leq M$.
- In complete market, ***asset span*** is identical to ***payoff space***, i.e. $rank(S) = M$, as all claims must have a price for all states.

Suppose $N = 4$ and $M = 3$, here is an illustration of payoff space versus asset span. LHS shows 4 linear dependent assets with rank 2, while RHS shows 4 linear dependent assets with rank 3. Blue axes denote ***payoff space***, while green area denotes ***asset span***.



Incomplete market : not all payoffs have a price



Complete market : all payoffs have a price

1.1c Arbitrage opportunity

Arbitrage opportunity is the existence of strategies having *(1)* zero initial investment *(2)* positive expected future payoff and *(3)* non negative future payoff for all economic states. That is :

$$\begin{aligned}
\textit{price} \qquad V_t^* &= 0 \\
\textit{payoff} \qquad E[V_T^*] &> 0 \\
\textit{payoff} \qquad V_T^*(\pi_m) &\geq 0 \qquad \forall m \in [1, M]
\end{aligned}$$

1.1d Fundamental theorem of asset pricing

A complete market is arbitrage free iff there exists unique measure, under which all numeraire-deflatted asset prices are martingale.

$$\boxed{\begin{array}{lcll} S_{n,t}^* & = & E_Q[S_{n,T}^*(\pi) \,|\, I_t] & \forall n \in [1,N]\text{, where } \pi \text{ is the single risk factor} \\ S_t^* & = & S_T^* Q & \forall n \in [1,N] \end{array}}$$

$$\text{where} \quad Q = \begin{bmatrix} Q(\pi_1) \\ Q(\pi_2) \\ Q(\pi_3) \\ \dots \\ Q(\pi_M) \end{bmatrix} \qquad \text{such that } \sum_{m=1}^{M} Q(\pi_m) = 1 \text{ and } Q(\pi_m) > 0 \text{ , } \forall m \in [1,M]$$

Theoretically we can calibrate Q by matching ***expected normalized payoff*** with ***normalized price prevailing in market*** :

$$\begin{array}{lcll} S_t^* & = & S_T^* Q & \text{the first row ensures that } \sum_{m=1}^{M} Q(\pi_m) = 1 \\ Q & = & (S_T^*)^{-1} S_t^* & \text{(1) which is known as risk neutral measure calibration formula} \end{array}$$

1.1e Law of one price LOOP

However, *FTAP* relates the ***price*** and ***payoff*** for primary securities only, we can extend it to all redundant securities with law of one price. Law of one price states that if two portfolios offer identical ***payoffs*** under all economic states, then they should have the same current ***price***. In other words if we can replicate the ***payoffs*** of a contingent claim with a portfolio W of primary securities S, then its current ***price*** is the same as that of the portfolio.

$$\text{if} \quad V_T^* = W S_T^* \quad \text{then} \quad V_t^* = W S_t^*$$

1.1f Risk neutral pricing

Combining *FTAP* and *LOOP* we can derive risk neutral pricing. Given V_T as ***payoff structure*** of redundant claim, we have :

$$\begin{array}{lcllll} V_t^* & = & W S_t^* & & & \text{by LOOP (assume that W replicates claim's payoff)} \\ & = & W S_T^* Q & = & W E_Q[S_T^*(\pi) \,|\, I_t] & \text{by FTAP (where } \pi \text{ is the only risk factor)} \\ & = & V_T^* Q & = & E_Q[V_T^*(\pi) \,|\, I_t] & \text{assumption of LOOP} \end{array}$$

$$V_t^* = \underbrace{W \overbrace{S_T^* Q}^{\text{FTAP}}}_{\text{LOOP}}$$

Thus we have risk neutral pricing formula as following (from now on, we denote numeraire S_{1t} as N_t) :

$$\boxed{\begin{array}{lcl} \dfrac{V_t}{N_t} & = & E_Q\left[\dfrac{V_T(\pi)}{N_T(\pi)} \,\Big|\, I_t\right] \\ V_t & = & E_Q\left[\dfrac{N_t}{N_T(\pi)} \times V_T(\pi) \,\Big|\, I_t\right] \end{array}} \quad \begin{array}{l} \text{(2) where } N_t \text{ is the numeraire and } \pi \text{ is the only risk factor} \\ \\ \text{(3) which is numeraire-discounted expectation of payoff} \end{array}$$

Any asset can be used as the numeraire. Given the same market data set, different risk neutral measures can be implied when using different numeraires. Although a change in numeraires invokes a change in measures, price of a contingent claim is invariant to the choice of numeraires. Therefore, we tend to pick a numeraire to make calculation of expectation easier.

$$\boxed{E_{Q_{N1}}\left[\frac{N_{1,t}}{N_{1,T}(\pi)} \times V_T(\pi) \,\Big|\, I_t\right] V_t = E_{Q_{N2}}\left[\frac{N_{2,t}}{N_{2,T}(\pi)} \times V_T(\pi) \,\Big|\, I_t\right]} \quad \text{(4) which is the generic form}$$

Risk neutral pricing - Intuition
As asset price can be find simply by taking expectation without considering risk premium, as if being neutral to risk, thus :
- [1] the above pricing is called ***RN pricing***
- the above measure Q is called ***RN measure***
- ***RN measure*** is different for different numeraires, but asset price in invariant to numeraires

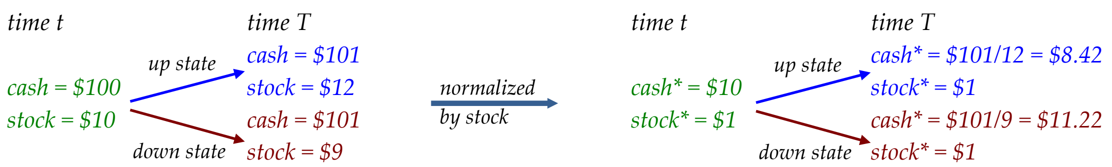
Risk neutral pricing is effectively an extrapolation of contingent claim (redundant securities) price, using prevailing market price of the primary securities set in a complete market, through 2 steps :
- [2] calibration step : ***RN pricing*** of primary securities using *(1)* to match with market price for implication of Q
- pricing step : ***RN pricing*** of redundant securities using *(2)* with the implied Q
- ***RN price*** is not a predicted price nor a speculated price, it is a no-arbitrage price
- [3] ***RN measure*** is an imaginary measure that facilitates the extrapolation (we like to work with expectation)
- [4] ***RN measure*** and ***RN price*** are two sides of a coin, they are *1-1* corresponding, we can transform between the two domains

P, Q and W ***do not*** appear in Black Scholes, why?
- Physical measure P of the economy does not exist at all, hence ***RN pricing*** is not related to physical measure.
- *FTAP* generates Q in ***RN pricing formula***, yet we don't solve it in practice, instead we convert it into drift of *SDE* by Girsanov.
- *LOOP* generates W in ***RN pricing proof***, yet it doesn't exist in ***RN pricing formula***, hence there is no need to solve it.

Why explicit solving of Q is not needed?
In discrete time economy, Q can be solved explicitly by matrix inverse in equation *(1)*.

In continous time economy, stock price is modeled by *SDEs* in physical measure, such as geometric Brownian motion with a drift μ. According to Girsanov theorem, a change from physical measure P to risk neutral measure Q with cash as numeraire, is equivalent to adding a shift to the drift, so that the new drift under Q equals to the drift r of cash numeraire. In short, we can model stock price ***directly under** Q* with a geometric Brownian motion having cash deposit rate (such as LIBOR or FedFund OIS depending on cost of fund of the bank) as the drift, that is $dS_t = rS_t dt + \sigma S_t dz_t$ which is the starting point of most *MSFE* materials. The above reasoning is a construction of martingale stochastic process, it makes risk neutral pricing possible. Main role of a quant is to ***construct martingale***.

Risk neutral pricing - Example
In an economy with two states *(M = 2)*, up and down, we need two assets *(N = 2)* to replicate the whole payoff space, one of the two assets is risk free cash, while another is risky stock. Let's use stock as the numeraire.



We can also plot the tree under cash numeraire, which is omited here. By solving equation *(1)*, we have risk neutral measure :

$$Q_{cash} = \begin{bmatrix} 101/101 & 101/101 \\ 12/101 & 9/101 \end{bmatrix}^{-1} \begin{bmatrix} 100/100 \\ 100/10 \end{bmatrix}$$

$$Q_{stock} = \begin{bmatrix} 101/12 & 101/9 \\ 12/12 & 9/9 \end{bmatrix}^{-1} \begin{bmatrix} 100/10 \\ 10/10 \end{bmatrix}$$

1.2 Proof of FTAP

We promote payoff space in $\Re^M$ to cashflow space in $\Re^{M+1}$ by putting ***initial price*** and ***future payoff*** together in a row vector. Define ***cashflow set*** $A \subset \Re^{M+1}$ such that for all $a \in A$, the first entry denotes initial cashflow for portfolio W at time t, a negative sign is added because this is an outward cashflow :

$$
\begin{aligned}
a &= [-WS_t^* \mid WS_T^*(\pi_1) \quad WS_T^*(\pi_2) \quad WS_T^*(\pi_3) \quad \dots \quad WS_T^*(\pi_M)] && \text{where } W = [w_1 \quad w_2 \quad w_3 \quad \dots \quad w_N] \\
&= W[-S_t^* \mid S_T^*(\pi_1) \quad S_T^*(\pi_2) \quad S_T^*(\pi_3) \quad \dots \quad S_T^*(\pi_M)] \\
&= \underbrace{W}_{1\times N}\underbrace{[-S_t^* \mid S_T^*]}_{N\times(M+1)} && \text{which is weighted average of N row vectors}
\end{aligned}
$$

If $a \in A$, then $-a \in A$, because weight inside W can be negative. Next we define ***arbitrage opportunity set*** $B \subset \Re^{M+1}$ such that :

$$
\begin{aligned}
B &= \Re_+^{M+1} \setminus \{0\} \\
&= \textit{positive orthant (quadrant in M+1 space) excluding origin} \\
b &= \textit{an arbitrage opportunity} && \forall b \in B
\end{aligned}
$$

Set B is also a convex set. The absence of arbitrage opportunities implies that ***cashflow set A*** and ***arbitrage opportunity set B*** do not intersect. If they do, then $x \in A \cap B$ would be a strategy having cash inflows at both t and T for all possible economic states, implying an arbitrage opportunity. *Separating hyperplane theorem* states that two disjoint convex sets can always be separated by a hyperplane. As A and B are two disjoint convex sets in $\Re^{M+1}$, then there exists $f = [f_0, f_1, f_2, \dots, f_M] \in \Re^{M+1}$ such that :

$$
\begin{aligned}
f \cdot a &< f \cdot b && && \forall a \in A \text{ and } \forall b \in B && (1) \\
f \cdot (-a) &< f \cdot b && (\textit{since } -a \in A \textit{ if } a \in A) && \forall a \in A \text{ and } \forall b \in B && (2)
\end{aligned}
$$

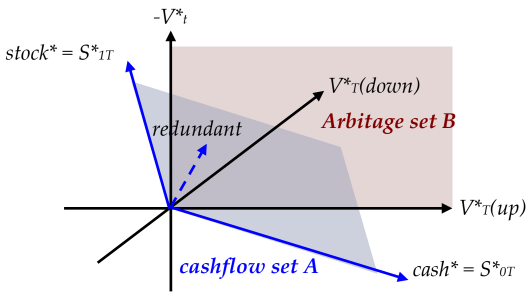
Both *(1)* and *(2)* can be true simultaneously only if :

$$
\begin{aligned}
f \cdot a &= 0 && \forall a \in A && (3) \\
f \cdot b &> 0 && \forall b \in B = \Re_+^{M+1} \setminus \{0\}
\end{aligned}
$$

As all dimensions of vector b are positive, so does f :

$$
f_m > 0 \qquad \forall m \in [0, M]
$$

For example, in a bistate economy with *1* risk free asset (cash) and *1* risky asset (stock), we can plot set A and set B as :



In this example, we have M = N = 2. If there is one more risky asset (N becomes 3) such as a derivative on the stock, then its payoff structure must lie on the same asset span A, that is the cash, stock* and derivative* should be coplanar and the plane must cut through the origin, otherwise there is arbitrage.*

Rearrange cashflows in *(3)*, move initial cashflow to *LHS* while keeping final cashflow on *RHS*, we have :

$$
\begin{aligned}
f_0(WS_t^*) &= \sum_{m=1}^{M}[f_m \times WS_T^*(\pi_m)] \\
WS_t^* &= \sum_{m=1}^{M}[\frac{f_m}{f_0} \times WS_T^*(\pi_m)] \\
&= \sum_{m=1}^{M}[Q(\pi_m) \times WS_T^*(\pi_m)] \qquad \text{where } Q(\pi_m) = \frac{f_m}{f_0} > 0, \forall m \in [1, M] \longleftarrow (4) \\
WS_t^* &= \underbrace{W}_{1\times N} \underbrace{S_T^*}_{N\times M} \underbrace{Q}_{M\times 1} \longleftarrow (5) \qquad \text{where } Q = \begin{bmatrix} Q(\pi_1) \\ Q(\pi_2) \\ \ldots \\ Q(\pi_M) \end{bmatrix}
\end{aligned}
$$

- The 1st matrix multiplication in *(5)* generates a portfolio.
- The 2nd matrix multiplication in *(5)* generates an expectation.
- For complete market, $rank(S^*_T) = M$.
- For complete market with N independent primary securities, $rank(S^*_T) = N = M$, i.e. S^*_T is a square matrix, thus invertible.
- We cannot simply remove W from both sides of *(5)* as W is non-invertible, instead we have to go through following steps …
- Equation *(5)* is valid for all W, we consider two special cases of W :

(case 1) Numeraire

$$
\begin{aligned}
\text{when} \quad W &= [1 \;\; 0 \;\; 0 \;\; \ldots \;\; 0] \\
S_{1,t}^* &= S_{1,T}^* Q \\
\frac{S_{1,t}}{S_{1,t}} &= \sum_{m=1}^{M} \frac{S_{1,T}(\pi_m)}{S_{1,T}(\pi_m)} Q(\pi_m) \\
1 &= \sum_{m=1}^{M} Q(\pi_m) \qquad (6)
\end{aligned}
$$

(case 2) Primary securities n

$$
\begin{aligned}
\text{when} \quad W &= \begin{array}{c} \\ \end{array}[0 \;\; \ldots \;\; \overset{\overset{n}{\downarrow}}{1} \;\; \ldots \;\; 0] \qquad \forall n \in [2, N] \\
S_{n,t}^* &= S_{n,T}^* Q \qquad (7a) \\
\frac{S_{n,t}}{S_{1,t}} &= \sum_{m=1}^{M} \frac{S_{n,T}(\pi_m)}{S_{1,T}(\pi_m)} Q(\pi_m) \\
S_{n,t} &= S_{1,t} \times E_Q\left[\frac{S_{n,T}(\pi)}{S_{1,T}(\pi)} | I_t\right] \qquad (7b)
\end{aligned}
$$

- Equation *(4)* and *(6)* together imply that Q is a measure, thus *(7a)* can be written as an expectation under Q as in *(7b)*.
- Equation *(7a)* and *(7b)* can be generalized for all $n \in [1,N]$ to get a formal formulation for *FTAP* :

$$
\begin{aligned}
S_t^* &= S_T^* Q && \textit{(8a) the matrix form} \\
S_{n,t} &= S_{1,t} \times E_Q\left[\frac{S_{n,T}(\pi)}{S_{1,T}(\pi)} | I_t\right] \quad \forall n \in [1, N] && \textit{(8b) the expectation form}
\end{aligned}
$$

1.3 Remarks

<u>What is risk free rate?</u>
This is the drift of asset having no risk, such as FedFund OIS. It is the cost of fund for hedging or replication of redundant securities. It is the opportunity cost of the issuer, the best return if it invests in an alternative project with no risk. Risk free rate is different for different financial institutions, those with lower cost of fund are more competitive.

<u>How to explain risk neutral pricing to a layman?</u>
Consider a simple economy with coin toss as the only risk factor, having two states. Primary assets of the economy are interest free cash and a security that pays *$1* for head and *$0* for tail. The coin is biased with physical measure being unknown, yet market price of the security is available to all investors. Suppose banks issue a derivative that pays owner *$200* for head and punishes owner *$80* for tail, what is the arbitrage free price of that derivative, given that price of security is *s*?

There are 3 assets in the bistate economy forming a complete market :
- cash
- stock
- derivatives, which is linear-dependent on cash and stock.

(method 1) hedging (this is used in Chap1 Introduction.doc part 3.1 : Discrete Pricing)
We hedge by forming portfolio :

$$-f_t + \Delta S_t \quad \text{such that} \quad -f_T(up) + \Delta S_T(up) = -f_T(down) + \Delta S_T(down)$$

$$\Delta = \frac{f_T(up) - f_T(down)}{S_T(up) - S_T(down)} = \frac{200-(-80)}{1-0} = 280$$

then by no arbitrage, we have :

$$\begin{aligned} f_t &= \Delta S_t + f_T(down) - \Delta S_T(down) \\ &= 280s + (-80) - 280 \times 0 \\ &= 280s - 80 \end{aligned}$$

method 1 forms *portfolio of derivative and stock*
method 2 forms *portfolio of cash and stock*
method 3 is derived from method 1

(method 2) replication
We replicate by forming portfolio :

$$as + b\$1 \quad \text{such that} \quad \begin{aligned} aS_T(up) + b\$1 &= f_T(up) \\ aS_T(down) + b\$1 &= f_T(down) \end{aligned}$$

With two equations and two unknowns, we have : $a = 280$ and $b = -80$. Thus replication cost is $280s - 80$.

(method 3) risk neutral pricing
Risk neutral pricing is derived from hedging in method 1. Risk neutral probability p is :

$$p = \frac{s - S_T(down)}{S_T(up) - S_T(down)} = \frac{s-0}{1-0} = s$$

By risk neutral pricing, derivative price is $p200 + (1-p)(-80)$, discounted by 1 (no interest rate), giving is $280s - 80$.

How to explain intuitively that, why drifts are not in Black Scholes formula?
(Answer 1) Mathematically
In the proof of *BSPDE*, we cancel the drift when we put $\Delta = \partial_s f$ to remove Brownian term.

(Answer 2) FTAP and Girsanov
In the proof of *FTAP*, ***physical measure*** is not involved, only ***risk neutral measure*** kicks in when we calculate arbitrage-free price of redundant securities. By Girsanov, a change in measure invokes a change in drift of SDE, as a result, μ becomes r.

(Answer 3) Intuition
Options are insurance. Once options are sold, issuers hedge by a sequence of buy-high sell-low transactions, which incurs costs. No arbitrage price of options is thus equivalent to hedging cost, which depends on volatility of the underlying only (not on the drift), a volatile underlying requires more frequent hedging and involves higher cost.

Given stocks having same volatility but different drifts, does the one with greater drift have a higher call price?
- According to physical measure, yes. This is ***physically predicted price***.
- FTAP does not consider physical measure, as it is for solving ***no-arbitrage price***.
- Why do stocks have same risk but different expected return in physical measures? Is there problems in this assumption?

Radon Nikodym and Girsanov

There are two definitions for martingale : expectation form and differential form. The former will lead us to Radon Nikodym, while the latter will lead us to Girsanov.

$$
\begin{array}{lcll}
X_t & = & E[X_T \mid I_t] & \textit{in expectation form} \\
dX_t & = & \sigma_t dz_t & \textit{in differential form}
\end{array}
$$

2.1 Radon Nikodym for random variable

Given X is a random variable defined in probability space (Ω,F,P) and (Ω,F,Q) where $P:F\to\Re$ and $Q:F\to\Re$ are equivalent measures. Function expectation of X under measure Q can be calculated using measure P through reweighting with likelihood ratio.

$$
\begin{array}{lcl}
E_Q[f(X)] & = & \int f(x)q_X(x)dx \\
 & = & \int f(x)L(x)p_X(x)dx \\
 & = & E_P[f(X)L(X)]
\end{array}
$$

$$
\begin{array}{llcll}
\textit{where} & L(x) & = & q_X(x)/p_X(x) & \textit{likelihood ratio in terms of probability density} \\
 & & = & q_X(x)dx/p_X(x)dx & \\
 & & = & dQ_X(x)/dP_X(x) & \textit{likelihood ratio in terms of cumulative density}
\end{array}
$$

- ***reweighting*** is also known as ***importance sampling***
- ***likelihood ratio*** is also known as ***Radon Nikodym derivative***
- $L(x)$ is ***deterministic*** while $L(X)$ is ***random***, its expectation under original measure P is 1.

$$
\begin{array}{lcl}
E_P[L(X)] & = & E_P[q(x)/p(x)] \\
 & = & \int (q(x)/p(x))p(x)dx \\
 & = & \int q(x)dx \\
 & = & 1
\end{array}
$$

Change in measure is not change in variable

Change in measure of a random variable is different from change in variable under the same measure. From *Chap1.doc* we have :

$$
\begin{array}{lcl}
y & = & g(x) \\
p_Y(y) & = & p_X(x)/g'(x)
\end{array}
$$

Can we find a change in variable from x to y so that it has the same expectation to a change in measure from P to Q?

$$
\begin{array}{lcll}
E_P[f(Y)] & = & E_Q[f(X)] & \textit{change in variable in the same measure } P \\
 & = & E_P[f(X)L(X)] & \textit{change in measure for the same variable } X \\
\int f(y)p_Y(y)dy & = & \int f(x)L(x)p_X(x)dx & \\
\int f(y)p_X(x)dx & = & \int f(x)L(x)p_X(x)dx & \textit{since } \; p_Y(y)dy = p_X(x)dx
\end{array}
$$

Since the above is true for all f, we have :

$$
\begin{array}{lcl}
f(y) & = & f(x)L(x) \\
g(x) & = & f^{-1}(f(x)L(x))
\end{array}
$$

The Gaussian example

Consider a change between two Gaussians measures for a random variable (not a process) :

$$
\begin{aligned}
p(x) &= \frac{1}{\sqrt{2\pi\sigma_P^2}} e^{-(x-\mu_P)^2/2\sigma_P^2} \\
q(x) &= \frac{1}{\sqrt{2\pi\sigma_Q^2}} e^{-(x-\mu_Q)^2/2\sigma_Q^2} \\
L(x) = \frac{q(x)}{p(x)} &= \frac{\sigma_P}{\sigma_Q} e^{-((x-\mu_Q)^2/\sigma_Q^2-(x-\mu_P)^2/\sigma_P^2)/2}
\end{aligned}
$$

Expectation reweighted by likelihood ratio becomes :

$$
\begin{aligned}
E_Q[f(X)] &= E_P[f(X)L(X)] \\
&= E_P[f(X)e^{-((X-\mu_Q)^2/\sigma_Q^2-(X-\mu_P)^2/\sigma_P^2)/2}]\times\frac{\sigma_P}{\sigma_Q} \\
&= E_P[f(X)e^{-(-2X\mu_Q+\mu_Q^2+2X\mu_P-\mu_P^2)/2\sigma^2}] && \text{when } \sigma_P = \sigma_Q = \sigma \\
&= E_P[f(X)e^{(\mu_Q-\mu_P)X/\sigma^2}]\times e^{-(\mu_Q^2-\mu_P^2)/2\sigma^2} && \text{when } \sigma_P = \sigma_Q = \sigma \\
&= E_P[f(X)e^{\mu X}]\times e^{-\mu^2/2} && \text{when } \sigma_P = \sigma_Q = 1,\ \mu_P = 0 \text{ and } \mu_Q = \mu, \text{ i.e. } p \text{ is unit normal}
\end{aligned}
$$

The mean of standard normal via can be shifted to the right by reweighting with $e^{\mu X}$, suppose μ is positive :

- it weighs positive X more
- it weighs negative X less

Radon Nikodym in Monte Carlo

Importance sampling is a useful technique in Monte Carlo simulation for estimating probability of a rare event. Samples X_n $n\in[1,N]$ are drawn from measure Q, a sample is considered as a hit if $X_n \in A$, estimation of hitting probability is done by :

$$
\tilde{Q}(A) = \frac{1}{N}\sum_{n=1}^{N} f(X_n) = \frac{1}{N}\sum_{n=1}^{N} 1_A(X_n)
$$

If A is a rare event and $Q(A)$ is small, this estimator is inaccurate when N is not large enough say $Q(A) = 10^{-6}$ and $N = 10^6$. It is solved by simulation with another measure P such that $P(A) >> Q(A)$ and reweighting by likelihood ratio $L(x) = dQ(x)/dP(x)$.

Radon Nikodym in FTAP

According to *FTAP*, we have equation *(4)* in section 2.1 :

$$
\begin{aligned}
E_{Q_{N1}}\left[\frac{N_{1,t}}{N_{1,T}(\pi)}\times V_T(\pi)\,|\,I_t\right] &= E_{Q_{N2}}\left[\frac{N_{2,t}}{N_{2,T}(\pi)}\times V_T(\pi)\,|\,I_t\right] \\
&= E_{Q_{N1}}\left[\frac{N_{2,t}}{N_{2,T}(\pi)}\times V_T(\pi)\times\frac{Q_{N_2}(\pi)}{Q_{N_1}(\pi)}\,|\,I_t\right]
\end{aligned}
$$

As the above is true for all contingent claims, or by substituting $V_T(\pi) = \delta(\pi)$ for all π, then we have :

$$
\begin{aligned}
\frac{N_{1,t}}{N_{1,T}(\pi)}\times V_T(\pi) &= \frac{N_{2,t}}{N_{2,T}(\pi)}\times V_T(\pi)\times\frac{Q_{N_2}(\pi)}{Q_{N_1}(\pi)} && \forall\pi\in\Omega \\
\frac{Q_{N_2}(\pi)}{Q_{N_1}(\pi)} &= \frac{N_{1,t}/N_{1,T}(\pi)}{N_{2,t}/N_{2,T}(\pi)} \\
&= \frac{DF_1(t,T)}{DF_2(t,T)}
\end{aligned}
$$

Radon Nikodym derivative for change in numeraire is the *ratio between total return* or *inverse ratio between discount factors*.

2.2 Girsanov theorem for random process

Formal statement of Girsanov theorem

Given Brownian motion z_t under measure P, we introduce a new process :

$$d\tilde{z}_t = \lambda_t dt + dz_t \qquad (8)$$

which is a drifted Brownian motion in measure P, if there exists a measure Q under which, the new process is martingale, then :

$$dQ = \underbrace{\exp\left(-\frac{1}{2}\int_0^T \lambda_t^2 dt - \int_0^T \lambda_t dz_t\right)dP}_{Radon-Nikodym} \qquad (9)$$

$$\frac{dQ}{dP} = e^{-v/2-\sqrt{v}\varepsilon}$$

Girsanov theorem tells us about the relation between P and Q in terms of a Radon Nikodym derivative.

Physical meaning

According to *FTAP*, given an asset and a numeraire, we have to solve for risk neutral measure of that numeraire. Instead of solving it explicitly, we can simply find the new *SDE* of that asset under Q without solving for Q, making use of Girsanov theorem.

Girsanov theorem states that given a shifted Brownian, there exists a Q under which the shifted Brownian becomes non-shifted, the shift and the Q is *(1)* one-one correspondent and *(2)* related by a Radon Nikodym derivative. Therefore instead of :

- solving for a Q under which numeraire-deflatted asset is martingale, alternatively, we can …
- solving for a λ under which numeraire-deflatted asset has a driftless *SDE*

Solving λ, we have SDE of numeraire-deflatted asset under Q, which is solved by integration and plugged into risk neutral pricing. The relationship between dz_t and $d\tilde{z}_t$ is ***fixed*** and ***independent*** on the measure

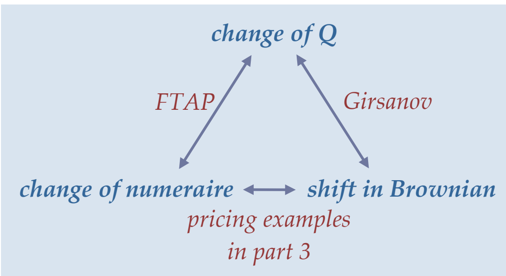
Likelihood ratios for these 4 arrows are different!

	original Brownian		*shifted Brownian*
under measure P	$dz_t = \varepsilon\sqrt{dt}$ *(driftless)*	$\xrightarrow{\text{by definition in (8)}}$	$d\tilde{z}_t = \lambda_t dt + \varepsilon\sqrt{dt}$ *(with drift)*
	$\downarrow$ *see the proof below*		$\downarrow$ *find Q under which **shifted** Brownian becomes driftless*
under measure Q	$dz_t = -\lambda_t dt + \varepsilon\sqrt{dt}$ *(with drift)*	$\xleftarrow{\text{by definition in (8)}}$	$d\tilde{z}_t = \varepsilon\sqrt{dt}$ *(driftless)*
SDE of derivative	$dX_t = \mu dt + \sigma dz_t$		$dX_t = k d\tilde{z}_t$ *(when is* $d\tilde{z}_t$ *driftless,* dX_t *becomes martingale)*

Proof of Girsanov

Now we are going to find the likelihood ratio L ***(see green dotted line above)***, so that :

$$\begin{aligned} E_Q[f(z_{(0:T]})] &= \int f(z_{(0,T]})q(z_{(0:T]})dz_{(0:T]} \\ &= \int f(z_{(0,T]})L(z_{(0,T]})p(z_{(0:T]})dz_{(0:T]} \\ &= E_P[f(z_{(0:T]})L(z_{(0,T]})] \end{aligned}$$



Discretizing the Markov process into N segments, such that $T = N\Delta t$:

$$\begin{aligned} \Delta z_k &= \sqrt{\Delta t}\varepsilon && \textit{under measure P} \\ \Delta z_k &= -\lambda_k \Delta t + \sqrt{\Delta t}\varepsilon && \textit{under measure Q} \end{aligned}$$

$$
\begin{aligned}
L(z_{(0,T]}) &= \lim_{\Delta t\to 0}\frac{q(z_{(0:T]})}{p(z_{(0:T]})} \\
&= \lim_{\Delta t\to 0}\frac{q(z_1\mid z_0)q(z_2\mid z_1)q(z_3\mid z_2)...q(z_N\mid z_{N-1})}{p(z_1\mid z_0)p(z_2\mid z_1)p(z_3\mid z_2)...p(z_N\mid z_{N-1})} && \textit{where } T=N\Delta t \\
&= \lim_{\Delta t\to 0}\exp\left(-\sum_{k=0}^{N-1}\frac{1}{2\Delta t}[(z_{k+1}-(z_k-\lambda_k\Delta t))^2-(z_{k+1}-z_k)^2]\right) && \textit{where } \left[\begin{array}{ll} z_{k+1}=\varepsilon(z_k,\sqrt{\Delta t}) & \textit{under } P \\ z_{k+1}=\varepsilon(z_k-\lambda_k\Delta t,\sqrt{\Delta t}) & \textit{under } Q \end{array}\right. \\
&= \lim_{\Delta t\to 0}\exp\left(-\sum_{k=0}^{N-1}\frac{1}{2\Delta t}[(\Delta z_k+\lambda_k\Delta t)^2-\Delta z_k{}^2]\right) \\
&= \lim_{\Delta t\to 0}\exp\left(-\sum_{k=0}^{N-1}\frac{1}{2\Delta t}[2\lambda_k\Delta t\Delta z_k+(\lambda_k\Delta t)^2]\right) \\
&= \lim_{\Delta t\to 0}\exp\left(-\frac{1}{2}\sum_{k=0}^{N-1}\lambda_k{}^2\Delta t-\sum_{k=0}^{N-1}\lambda_k\Delta z_k\right) \\
&= \exp\left(-\frac{1}{2}\int_0^T\lambda_t^2dt-\int_0^T\lambda_t dz_t\right) && \textit{hence (9) is proved}
\end{aligned}
$$

Expectation of Radon Nikodym under P

$$
\begin{aligned}
E_P[L(z_{(0,T]})] &= E_P\left[\exp\left(-\frac{1}{2}\int_0^T\lambda_t^2dt-\int_0^T\lambda_t dz_t\right)\right] && \textit{recall that } E_P\left[\int_0^T\lambda_t dz_t\right]=0 \\
&= E_P\left[\exp\left(\varepsilon(-\frac{1}{2}\int_0^T\lambda_t^2dt,\sqrt{\int_0^T\lambda_t^2dt})\right)\right] && \textit{since } dz_t \textit{ is Brownian under P, but not Q} \\
&= \exp\left(-\frac{1}{2}\int_0^T\lambda_t^2dt+\frac{1}{2}\int_0^T\lambda_t^2dt\right) && \textit{recall that } E[\exp(\varepsilon(\mu,\sigma))]=\exp(\mu+\sigma^2/2) \\
&= 1 && \textit{consistent with section 3.1}
\end{aligned}
$$

Reversing Girsanov theorem

Girsanov theorem states, given driftless Brownian dz_t in measure P, then a shifted Brownian :

$$
d\tilde{z}_t = \lambda_t dt+dz_t \qquad \textit{is Brownian under Q, if } L_0=\frac{dQ}{dP}=\exp\left(-\frac{1}{2}\int_0^T\lambda_t^2dt-\int_0^T\lambda_t dz_t\right)
$$

Reversely, given driftless Brownian $d\tilde{z}_t$ in measure Q, then a shifted Brownian :

$$
dz_t = -\lambda_t dt+d\tilde{z}_t \qquad \textit{is Brownian under P, if } L_1=\frac{dP}{dQ}=\exp\left(-\frac{1}{2}\int_0^T(-\lambda_t)^2dt-\int_0^T(-\lambda_t)d\tilde{z}_t\right)
$$

The product of these two Radon Nikodym derivatives is :

$$
\begin{aligned}
L_0\times L_1 &= \exp\left(-\frac{1}{2}\int_0^T\lambda_t^2dt-\int_0^T\lambda_t dz_t\right)\exp\left(-\frac{1}{2}\int_0^T(-\lambda_t)^2dt-\int_0^T(-\lambda_t)d\tilde{z}_t\right) \\
&= \exp\left(-\int_0^T\lambda_t^2dt-\int_0^T\lambda_t dz_t+\int_0^T\lambda_t d\tilde{z}_t\right) \\
&= \exp\left(-\int_0^T\lambda_t^2dt-\int_0^T\lambda_t dz_t+\int_0^T\lambda_t(\lambda_t dt+dz_t)\right) \\
&= 1
\end{aligned}
$$

Intuition of Girsanov theorem

Given a fair coin as the only risk factor under original physical measure P, we define random process X_k, which is incremented by *1* when we get a head and *-1* otherwise, this process is martingale as $E_P[X_k | X_0] = X_0$. We then define drifted random process Y_k, which is incremented by $1+\lambda$ when we get a head and $-1+\lambda$ otherwise, the new process is non-martingale under measure P, yet there exists a measure Q, so that the new process becomes martingale. Now, let's find Q, that is $Q(Head)$ and $Q(Tail)$.

$$
\begin{aligned}
Y_k &= E_Q[Y_{k+1} | I_k] \\
Y_k &= Q(Head)(Y_k + 1 + \lambda) + (1 - Q(Head))(Y_k - 1 + \lambda) \\
0 &= Q(Head)(1+\lambda) + (1 - Q(Head))(-1+\lambda) \\
0 &= \lambda - 1 + 2Q(Head) \\
Q(Head) &= (1-\lambda)/2 && \textit{which is less than } P(Head) \textit{ for } \lambda > 0 \\
Q(Tail) &= (1+\lambda)/2 && \textit{which is greater than } P(Tail) \textit{ for } \lambda > 0
\end{aligned}
$$

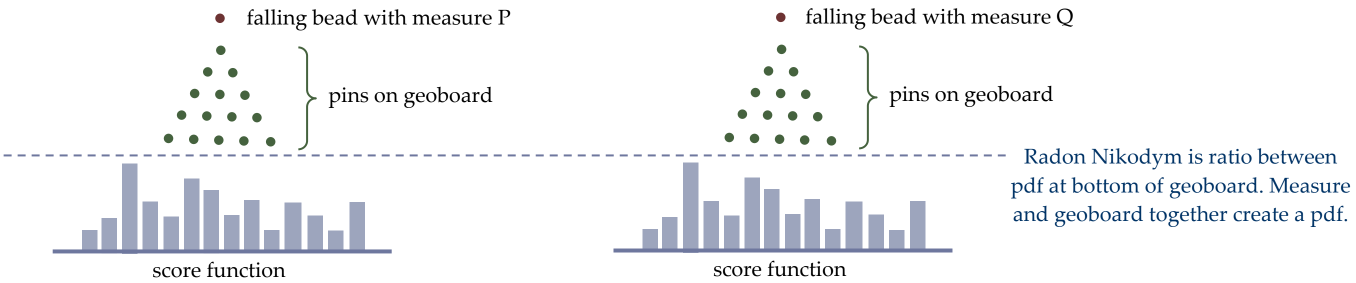
The geoboard analogy

Imagine rolling a set of beads down a geoboard having a bucket-array at the bottom. The score marked on a bucket can be obtained if a bead falls into it. Find the average score we get with the set of beads. Assumption :

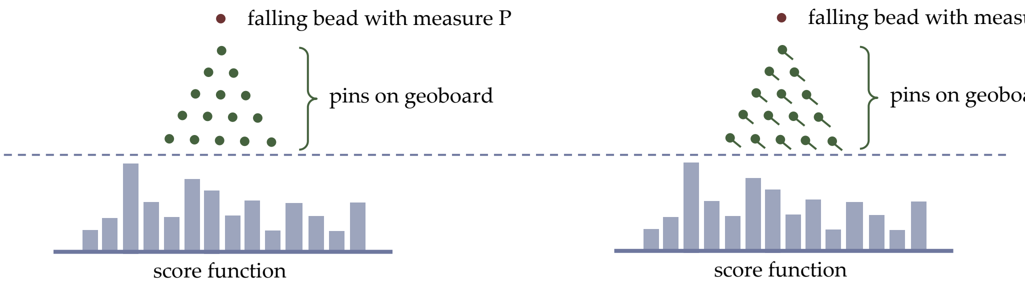
- the beads can be biased or unbiased
- the geoboard can be empty
- the geoboard pattern can be shifted
- either a ***change in bias*** or a ***change in geoboard*** results in ***change in average score***
- Radon Nikodym is the ratio between *pdf* at the bottom of geoboard ***after*** the change and ***before*** the change

objects	*analogy*
ordinary beads	measure P, bead has probability $P(LHS)=0.5$ to go LHS on hitting a pin, $P(RHS)=0.5$ otherwise
biased beads	measure Q, bead has probability $Q(LHS)\neq 0.5$ to go LHS on hitting a pin, $Q(RHS)\neq 0.5$ otherwise
ordinary geoboard	$dX_t = \mu_t dt + \sigma_t dz_t$ or dz_t
shifted geoboard	$d\tilde{X}_t = \tilde{\mu}_t dt + \sigma_t dz_t$ or $d\tilde{z}_t$
score array	contingent claim $f(X_{(0:T]})$ or $f(\tilde{X}_{(0:T]})$
average score	$E_P[f(X_{(0:T]}) | X_0]$ or $E_Q[f(X_{(0:T]}) | X_0]$

Radon Nikodym for random variable is about change-in-bias with the same (binomial) geoboard :



Radon Nikodym for random process is about change-in-geoboard with unbiased beads :



Girsanov theorem is about cancelling the effect of change-in-geoboard with biased beads (or a change-in-bias). The theorem targets at finding a set of biased beads, when rolled down a shifted geoboard, gives the same average score as the original setting (rolling a set of unbiased beads down an ordinary geoboard). Relation between bias of beads and shift in geoboard is governed by Girsanov.

Risk neutral pricing in practice

3.1 Risk neutral pricing

Risk neutral pricing of redundant security is done by :

$$f(S_t) \quad = \quad E_Q\left[\frac{N_t}{N_T}\times f(S_T)\,|\,I_t\right]$$

First of all we need to pick a numeraire so as to facilitate the calculation of the above expectation. After that Q is implied by :

$$S_t \quad = \quad E_Q\left[\frac{N_t}{N_T}\times S_T\,|\,I_t\right]$$

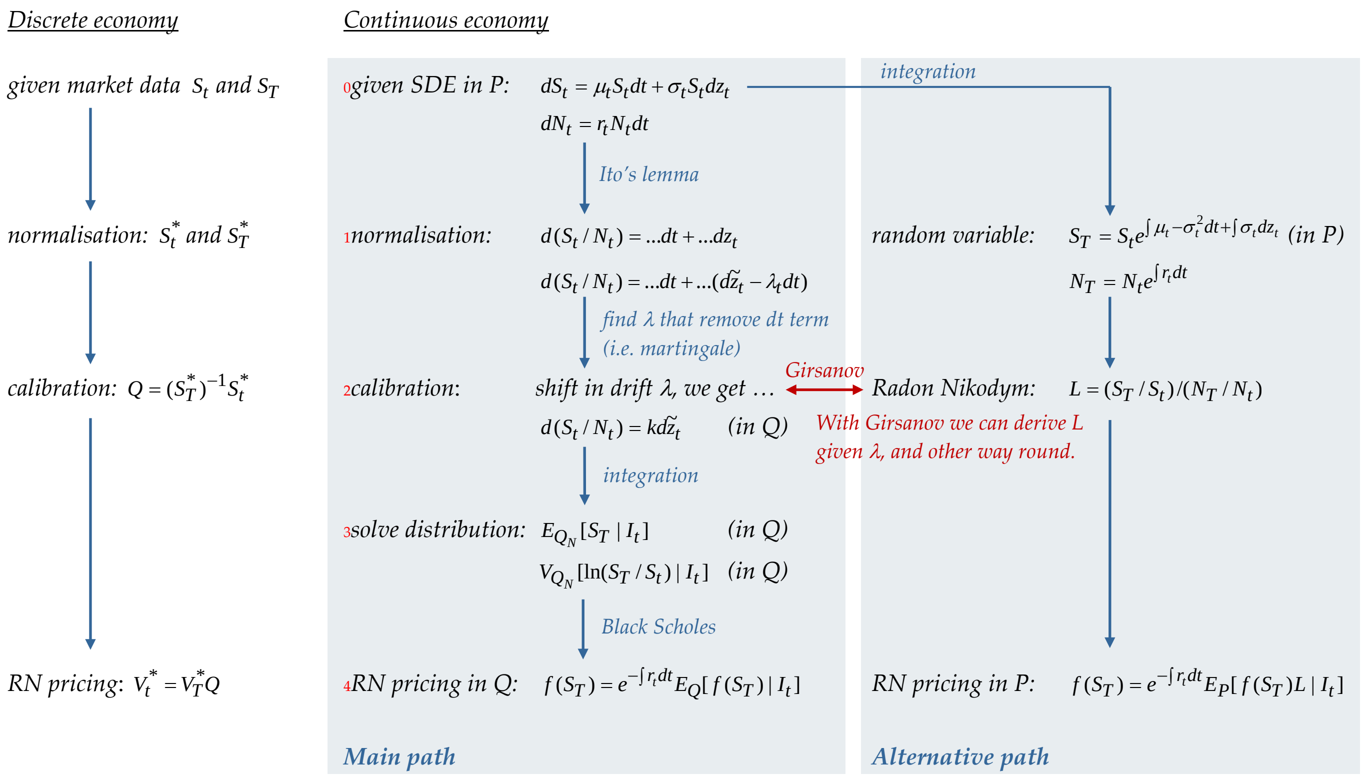
However in practice, instead of solving Q explicitly, we solve for a shift in drift so that the *SDE* of $dX_t = d(S_t / N_t)$ is Brownian :

$$\begin{aligned} dX_t &= d(S_t / N_t) && \textit{normalization by Ito's lemma (corresponds to step}_1\textit{ in DAG below)} \\ &= A_t X_t dt + B_t X_t dz_t \\ &= A_t X_t dt + B_t X_t (d\tilde{z}_t - \lambda dt) \\ &= B_t X_t d\tilde{z}_t && \textit{calibration by making it driftless (corresponds to step}_2\textit{ in DAG below)} \end{aligned}$$

If the risk neutral expectation is simplified as something depending on S_T, then solve distribution of S_T. However, if the expectation is simplified as something depending on X_T, then solve distribution of X_T. Normally, both of them are log normal (S_T is not driftless while X_T is driftless), thus we can apply Black Scholes solution by substituting *(corresponds to step*$_3$ *and step*$_4$ *in DAG below)* :

- $E_{Q_N}[S_T \,|\, I_t]$ *and* $v = V_{Q_N}[\ln(S_T / S_t) \,|\, I_t]$ for expectation depending on non-driftless S_T under Q_N measure
- $E_{Q_N}[X_T \,|\, I_t]$ *and* $v = V_{Q_N}[\ln(X_T / X_t) \,|\, I_t]$ for expectation depending on driftless X_T under Q_N measure

Example of S_T	Example of X_T
3.2 cash numeraire for vanilla call option	3.3 stock numeraire for exchange option
3.4 stock numeraire for 2nd order option	3.5 forward numeraire for bond option



"An Elementary Introduction to Changes of Numeraire", by Simon Ellersgaard Nielsen

3.2 Cash numeraire for vanilla call option

Find vanilla call option price given under P :

$$
\begin{aligned}
dS_t &= \mu_t S_t dt + \sigma_t S_t dz_t \\
dB_t &= r_t B_t dt
\end{aligned}
$$

Firstly we aim at finding distribution of S_T under Q. The *SDE* of cash-deflatted stock $X_t = S_t / B_t$ under either P or Q is :

$$
\begin{aligned}
dX_t &= (1/B_t)dS_t - (S_t/B_t^2)dB_t + \frac{1}{2}0(dS_t)^2 - \frac{1}{2}(-2)(S_t/B_t^3)\overbrace{(dB_t)^2}^{0} + \ldots \\
&= (1/B_t)dS_t - (S_t/B_t^2)dB_t \\
&= (\mu_t - r_t)X_t dt + \sigma_t X_t dz_t && \textit{by substituting SDEs under P} \\
&= (\mu_t - r_t)X_t dt + \sigma_t X_t (d\tilde{z}_t - \lambda_t dt) && \textit{introduce shifted Brownian } d\tilde{z}_t = \lambda_t dt + dz_t \\
&= (\mu_t - r_t - \lambda_t \sigma_t)X_t dt + \sigma_t X_t d\tilde{z}_t \\
&= \sigma_t X_t d\tilde{z}_t
\end{aligned}
$$

Secondly in order to make cash-deflatted stock to be martingale under Q, we set :

$$
\begin{aligned}
0 &= \mu_t - r_t - \lambda_t \sigma_t \\
\lambda_t &= \frac{\mu_t - r_t}{\sigma_t}
\end{aligned}
$$

which is ***market price of risk*** *or* ***Sharpe ratio***

The *SDE* of stock S_t under measure Q is :

$$
\begin{aligned}
dS_t &= \mu_t S_t dt + \sigma_t S_t dz_t && dz_t \textit{ is not Brownian under Q} \\
&= \mu_t S_t dt + \sigma_t S_t (d\tilde{z}_t - \lambda_t dt) && d\tilde{z}_t \textit{ is Brownian under Q} \\
&= \mu_t S_t dt + \sigma_t S_t \left(d\tilde{z}_t - \frac{\mu_t - r_t}{\sigma_t} dt\right) \\
&= r_t S_t dt + \sigma_t S_t d\tilde{z}_t && \textit{which has the same drift as cash-numeraire}
\end{aligned}
$$

Thirdly we solve for random variable S_T :

$$
S_T = S_t e^{\int_t^T (r_s - \sigma_s^2/2)ds + \int_t^T \sigma_s d\tilde{z}_s} \qquad \textit{which is lognormal under Q}
$$

We calculate its expectation under Q to facilitate risk neutral pricing :

$$
\begin{aligned}
E_Q[S_T \mid I_t] &= S_t e^{\int_t^T (r_s - \sigma_s^2/2)ds + (\int_t^T \sigma_s^2 ds)/2} \\
&= S_t e^{\int_t^T r_s ds} && \textit{woo… it grows in the same rate as cash under Q}
\end{aligned}
$$

Finally calculate expected payoff under Q :

$$
\begin{aligned}
f(S_t) &= B_t E_Q\left[\frac{(S_T - K)^+}{B_T} \mid I_t\right] \\
&= e^{-\int_t^T r_s ds} E_Q[(S_T - K)^+ \mid I_t] && \textit{interest rate is deterministic in Black Scholes} \\
&= e^{-\int_t^T r_s ds} (E_Q[S_T \mid I_t] N(d_1) - K N(d_2)) \\
\textit{where} \quad d_{1,2} &= \frac{\ln(S_t e^{\int_t^T r_s ds}/K) \pm (\int_t^T \sigma_s^2 ds)/2}{\sqrt{\int_t^T \sigma_s^2 ds}}
\end{aligned}
$$

3.3 Stock numeraire for exchange option

[Here P refers to physical measure, Q refers to risk neutral measure with stock1 as numeraire.]

Exchange option gives holder the right to exchange one share of stock*1* with one share of stock*2* on maturity *T*. Given under *P* :

$$dS_{i,t} = \mu_i S_{i,t}dt + \sigma_i S_{i,t}dz_{i,t} \qquad \forall i = (1,2) \qquad \textit{where } dz_{1,t}dz_{2,t} = \rho dt \ \ \textit{(they are correlated)}$$

Firstly we aim at finding distribution under *Q*. The *SDE* of stock*1*-deflatted stock*2* $X_t = S_{2t}/S_{1t}$ under either *P* or *Q* is :

$$\begin{aligned}
dX_t &= (1/S_{1,t})dS_{2,t} - (S_{2,t}/S_{1,t}^2)dS_{1,t} + \frac{1}{2}0(dS_{2,t})^2 - (1/S_{1,t}^2)(dS_{2,t})(dS_{1,t}) - \frac{1}{2}(-2)(S_{2,t}/S_{1,t}^3)(dS_{1,t})^2 + \ldots \\
&= \begin{bmatrix} (1/S_{1,t})(\mu_2 S_{2,t}dt + \sigma_2 S_{2,t}dz_{2,t}) - (S_{2,t}/S_{1,t}^2)(\mu_1 S_{1,t}dt + \sigma_1 S_{1,t}dz_{1,t}) \\ -(1/S_{1,t}^2)(\mu_2 S_{2,t}dt + \sigma_2 S_{2,t}dz_{2,t})(\mu_1 S_{1,t}dt + \sigma_1 S_{1,t}dz_{1,t}) + (S_{2,t}/S_{1,t}^3)(\mu_1 S_{1,t}dt + \sigma_1 S_{1,t}dz_{1,t})^2 \end{bmatrix} \\
&= \begin{bmatrix} (\mu_2 X_t dt + \sigma_2 X_t dz_{2,t}) - (\mu_1 X_t dt + \sigma_1 X_t dz_{1,t}) \\ -(1/S_{1,t}^2)(\sigma_2 S_{2,t}dz_{2,t})(\sigma_1 S_{1,t}dz_{1,t}) + (S_{2,t}/S_{1,t}^3)\sigma_1^2 S_{1,t}^2 dt \end{bmatrix} \\
&= \begin{bmatrix} (\mu_2 X_t dt + \sigma_2 X_t dz_{2,t}) - (\mu_1 X_t dt + \sigma_1 X_t dz_{1,t}) \\ -\rho\sigma_1\sigma_2 X_t dt + \sigma_1^2 X_t dt \end{bmatrix} \\
&= (\mu_2 - \mu_1 - \rho\sigma_1\sigma_2 + \sigma_1^2)X_t dt - \sigma_1 X_t dz_{1,t} + \sigma_2 X_t dz_{2,t} \\
&= \mu X_t dt + \sigma X_t dz_t && \textit{see remark} \\
&= \mu X_t dt + \sigma X_t(d\tilde{z}_t - \lambda dt) && \textit{introduce shifted Brownian } d\tilde{z}_t = \lambda dt + dz_t \\
&= (\mu - \lambda\sigma)X_t dt + \sigma X_t d\tilde{z}_t \\
&= \sigma X_t d\tilde{z}_t
\end{aligned}$$

Remark

The combined drift is :

$$\mu = \mu_2 - \mu_1 - \rho\sigma_1\sigma_2 + \sigma_1^2$$

The combined diffusion is sum of two correlated Brownian motions :

$$\begin{aligned}
\sigma X_t dz_t &= -\sigma_1 X_t dz_{1,t} + \sigma_2 X_t dz_{2,t} \\
\textit{where} \quad \sigma &= \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}
\end{aligned}$$

Proof of combined diffusion :

$$\begin{aligned}
E[RHS] &= E[-\sigma_1 X_t dz_{1,t} + \sigma_2 X_t dz_{2,t}] \\
&= -\sigma_1 X_t E[dz_{1,t}] + \sigma_2 X_t E[dz_{2,t}] \\
&= 0 \\
&= E[LHS]
\end{aligned}$$

$$\begin{aligned}
V[RHS] &= V[-\sigma_1 X_t dz_{1,t} + \sigma_2 X_t dz_{2,t}] \\
&= (\sigma_1 X_t)^2 V[dz_{1,t}] + (\sigma_2 X_t)^2 V[dz_{2,t}] + 2(-\sigma_1 X_t)(\sigma_2 X_t)Cov[dz_{1,t}, dz_{2,t}] \\
&= (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)X_t^2 dt \\
&= V[LHS]
\end{aligned}$$

Secondly in order to make cash-deflatted stock to be martingale under Q, we set :

$$\begin{aligned} \mu - \lambda\sigma_t &= 0 \\ \lambda &= \frac{\mu}{\sigma} \end{aligned}$$

The *SDE* of stock*1*-deflatted stock*2* X_t under measure Q is :

$$dX_t = \sigma X_t d\tilde{z}_t \qquad d\tilde{z}_t \text{ is Brownian under } Q$$

Thirdly we solve for random variable X_T :

$$X_T = X_t e^{-\sigma^2(T-t)/2+\sigma(z_T-z_t)} \qquad \text{which is lognormal under } Q$$

We calculate its expectation under Q to facilitate risk neutral pricing :

$$\begin{aligned} E_Q[X_T \mid I_t] &= X_t e^{-\sigma^2(T-t)/2+\sigma^2(T-t)/2} \\ &= X_t \qquad \text{woo… it is martingale under } Q \end{aligned}$$

Finally calculate expected payoff under Q :

$$\begin{aligned} f(S_{1,t}, S_{2,t}) &= S_{1,t} E_Q\left[\frac{(S_{2,T}-S_{1,T})^+}{S_{1,T}} \mid I_t\right] \\ &= S_{1,t} E_Q[(X_T - 1)^+ \mid I_t] \\ &= S_{1,t}(E_Q[X_T \mid I_t] N(d_1) - 1 \times N(d_2)) \qquad \text{plug into Black Schole equation} \\ &= S_{1,t}(X_t N(d_1) - N(d_2)) \\ &= S_{2,t} N(d_1) - S_{1,t} N(d_2) \qquad \text{since } X_t = S_{2,t}/S_{1,t} \end{aligned}$$

$$\text{where} \quad d_{1,2} = \frac{\ln(X_t/1) \pm \sigma^2(T-t)/2}{\sigma\sqrt{T-t}} \qquad \text{where } \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\text{ , } \mu \text{ isnt involved}$$

3.4 Stock numeraire for 2nd-order option

[Here P refers to physical measure, Q_B refers to risk neutral measure with cash as numeraire, Q_S refers to risk neutral measure with stock as numeraire.]

Let's try another path in the *DAG (starting with Q_B)* with a 2nd-order option. The payoff of this 2nd-order option is defined as :

$$payoff = S_T(S_T - K)^+$$

Given under Q_B (instead of physical measure P) :

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dz_t \\ dB_t &= rB_t dt \end{aligned}$$

Firstly we aim at finding Radon Nikodym for a change from Q_B *to* Q_S. Lets solve the above *SDEs* under Q_B :

$$\begin{aligned} S_T &= S_t e^{(r-\sigma^2/2)(T-t)+\sigma(z_T - z_t)} \\ B_T &= B_t e^{r(T-t)} \end{aligned}$$

Radon Nikodym derivative for change from cash-numeraire to stock-numeraire is :

$$\begin{aligned} \frac{dQ_S}{dQ_B} &= \frac{DF_B(t,T)}{DF_S(t,T)} \\ &= e^{-r(T-t)}(S_T / S_t) \\ &= e^{(-\sigma^2/2)(T-t)+\sigma(z_T - z_t)} \end{aligned}$$

Secondly we solve λ by comparing Girsanov theorem, i.e. given $d\tilde{z}_t = \lambda_t dt + dz_t$, we have Radon Nikodym $\exp\left(-\frac{1}{2}\int_0^T \lambda_t^2 dt - \int_0^T \lambda_t dz_t\right)$:

$$\begin{aligned} \Rightarrow \quad \lambda_t &= -\sigma \\ i.e. \quad d\tilde{z}_t &= -\sigma dt + dz_t \end{aligned}$$

The *SDE* of stock S_t under measure Q_S is :

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dz_t && dz_t \textit{ is not Brownian under } Q_S \\ &= rS_t dt + \sigma S_t (d\tilde{z}_t + \sigma dt) && d\tilde{z}_t \textit{ is Brownian under } Q_S \\ &= (r+\sigma^2) S_t dt + \sigma S_t d\tilde{z}_t \end{aligned}$$

Thirdly we solve for random variable S_T :

$$\begin{aligned} S_T &= S_t e^{(r+\sigma^2-\sigma^2/2)(T-t)+\sigma(\tilde{z}_T - \tilde{z}_t)} \\ &= S_t e^{(r+\sigma^2/2)(T-t)+\sigma(\tilde{z}_T - \tilde{z}_t)} && \textit{which is lognormal under } Q_S \end{aligned}$$

We calculate its expectation under Q to facilitate risk neutral pricing :

$$\begin{aligned} E_{Q_S}[S_T \mid I_t] &= S_t e^{(r+\sigma^2/2)(T-t)+\sigma^2(T-t)/2} \\ &= S_t e^{(r+\sigma^2)(T-t)} \\ &= E_{Q_B}[S_T \mid I_t] e^{\sigma^2(T-t)} \end{aligned}$$

Finally calculate expected payoff under Q_S :

$$\begin{aligned} f(S_t) &= S_t E_{Q_S}\left[\frac{S_T(S_T-K)^+}{S_T} \mid I_t\right] \\ &= S_t E_{Q_S}[(S_T - K)^+ \mid I_t] \\ &= S_t (E_{Q_S}[S_T \mid I_t] N(d_1) - KN(d_2)) \\ \textit{where} \quad d_{1,2} &= \frac{\ln E_{Q_S}[S_T \mid I_t]/K \pm \sigma^2 (T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

3.5 Forward numeraire for bond option

When interest rate is stochastic, we cannot use cash as numeraire, as we cannot move integral of interest rate out of the expectation.

$$\begin{aligned} f(LIBOR_t) &= B_t E_{Q_B}\left[\frac{f(LIBOR_T)}{B_T} \mid I_t\right] \\ &= \$1 E_{Q_B}\left[\frac{f(LIBOR_T)}{e^{\int_t^T r_s ds}} \mid I_t\right] \\ &\neq e^{-\int_t^T r_s ds} E_{Q_B}[f(LIBOR_T) \mid I_t] \qquad \textit{since } r_s \textit{ is stochastic} \end{aligned}$$

Instead, we use zero coupon bond with same maturity as option as numeraire, known as forward numeraire.

Given Merton's short rate model, the price of zero coupon bond which matures at T, as of current time t is :

$$dr_t = a dt + b dz_t$$

$$\begin{aligned} \Rightarrow \quad P_t(t,T) &= e^{-A(t,T)-B(t,T)r_t} \qquad \textit{where } r_t \textit{ is known at current time } t \\ A(t,T) &= \frac{1}{2}a(T-t)^2 - \frac{1}{6}b^2(T-t)^3 \\ B(t,T) &= T-t \end{aligned}$$

There are two approaches to prove the above (please refer to *Chap5 - Interest rate model.doc*) :

1. Risk neutral expectation method (*RNE*)
2. Partial differential equation method (*PDE*)

In general, we have forward bond defined as (please read *Chap4 - Interest rate derivatives.doc*) :

$$\begin{aligned} P_t(T,\Gamma) &= \textit{forward bond price (deterministic) with start-date } T \textit{ to end-date } \Gamma\textit{, given market data at time } t \\ P_t(t,\Gamma) &= \textit{spot bond price (deterministic) at current time } t \\ P_T(T,\Gamma) &= \textit{spot bond price (stochastic) at future time } T \end{aligned}$$

Bond option pricing

Bond option gives holder the right, not the obligation, to enter a bond, which starts at T and ends at Γ, at a price of K on maturity T, hence the underlying of bond option is a ***spot bond in the future*** (not a forward bond, as forward bond is deterministic). By picking spot bond that matures at T as numeraire, risk neutral price of bond option is made easy, as numeraire becomes *1* at T.

$$\begin{aligned} f(r_t) &= \underbrace{P_t(t,T)}_{\text{numeraire}} E_{Q_T}\left[\frac{(\overbrace{P_T(T,\Gamma)}^{\text{underlying variable}} - K)^+}{P_T(T,T)} \mid I_t\right] \\ &= P_t(t,T) E_{Q_T}[(P_T(T,\Gamma)-K)^+ \mid I_t] \qquad \textit{since } P_T(T,\Gamma) = 1 \end{aligned}$$

Correspondence to stock option :

	at time t	*at time T*
underlying	$S_t = P_t(t,\Gamma)$	$S_T = P_T(T,\Gamma)$
numeraire	$N_t = P_t(t,T)$	$N_T = P_T(T,T) = 1$
deflatted underlying	$X_t = P_t(t,\Gamma)/P_t(t,T) = P_t(T,\Gamma)$	$X_T = P_T(T,\Gamma)/P_T(T,T) = P_T(T,\Gamma)$

By Girsanov, we have to look for a shift so that deflatted asset is driftless (martingale), i.e. $dX_t = dP_t(T,\Gamma) = k d\tilde{z}_t$.

Firstly we aim at finding distribution of $P_T(T,\Gamma)$ under Q_T using *SDE* of *T-bond*-deflatted *Γ-bond* $\frac{P_t(t,\Gamma)}{P_t(t,T)}$ is the *SDE* of $P_t(T,\Gamma)$:

$$\begin{aligned}
P_t(T,\Gamma) &= \frac{P_t(t,\Gamma)}{P_t(t,T)} \\
&= e^{-(A(t,\Gamma)-A(t,T))-(B(t,\Gamma)-B(t,T))r_t} \\
\partial_t P_t(T,\Gamma) &= P_t(T,\Gamma)\times[-(\partial_t A(t,\Gamma)-\partial_t A(t,T))-(\overbrace{\partial_t B(t,\Gamma)}^{-1}-\overbrace{\partial_t B(t,T)}^{-1})r_t] && \textit{forward bond } P_t(T,\Gamma) \textit{ is stochastic as of time 0} \\
&= P_t(T,\Gamma)\times[-\partial_t A(t,\Gamma)+\partial_t A(t,T)] \\
\partial_r P_t(T,\Gamma) &= P_t(T,\Gamma)\times[-(B(t,\Gamma)-B(t,T))] \\
&= P_t(T,\Gamma)\times[-(\Gamma-T)] \\
\partial_{rr} P_t(T,\Gamma) &= P_t(T,\Gamma)\times[-(\Gamma-T)]^2 \\
\Rightarrow\quad dP_t(T,\Gamma) &= \partial_t P_t(T,\Gamma)dt+\partial_r P_t(T,\Gamma)dr_t+\frac{1}{2}\partial_{rr}P_t(T,\Gamma)\overbrace{(dr_t)^2}^{b^2dt} \\
&= \mu dt+\partial_r P_t(T,\Gamma)bdz_t && \textit{where } \mu=\partial_t P_t(T,\Gamma)+\partial_r P_t(T,\Gamma)a+\frac{1}{2}\partial_{rr}P_t(T,\Gamma)b^2 \\
&= \mu dt-b(\Gamma-T)P_t(T,\Gamma)dz_t \\
&= \mu dt-b(\Gamma-T)P_t(T,\Gamma)(d\tilde{z}_t-\lambda_t dt) && \textit{introduce shifted Brownian } d\tilde{z}_t=\lambda_t dt+dz_t \\
&= (\mu+\lambda_t b(\Gamma-T)P_t(T,\Gamma))dt-b(\Gamma-T)P_t(T,\Gamma)d\tilde{z}_t \\
&= -b(\Gamma-T)P_t(T,\Gamma)d\tilde{z}_t
\end{aligned}$$

Secondly in order to make *T-bond*-deflatted *Γ-bond* to be martingale under Q_T, we set :

$$\begin{aligned}
0 &= \mu+\lambda_t b(\Gamma-T)P_t(T,\Gamma) \\
\lambda_t &= -\frac{\mu}{b(\Gamma-T)P_t(T,\Gamma)}
\end{aligned}$$

The *SDE* of stock S_t under measure Q_T is :

$$dP_t(T,\Gamma) = -b(\Gamma-T)P_t(T,\Gamma)d\tilde{z}_t \qquad d\tilde{z}_t \textit{ is Brownian under } Q_T$$

Thirdly we solve for random variable $P_T(T,\Gamma)$:

It doesn't matter.

$$P_T(T,\Gamma) = P_t(T,\Gamma)e^{-(b(\Gamma-T))^2(T-t)/2\pm b(\Gamma-T)(z_T-z_t)} \qquad \textit{which is lognormal under } Q_T$$

We calculate its expectation under Q_T to facilitate risk neutral pricing :

$$\begin{aligned}
E_{Q_T}[P_T(T,\Gamma)\,|\,I_t] &= P_t(T,\Gamma)e^{-(b(\Gamma-T))^2(T-t)/2+(b(\Gamma-T))^2(T-t)/2} \\
&= P_t(T,\Gamma) && \textit{woo… it is martingale under } Q_T
\end{aligned}$$

Finally calculate expected payoff under Q :

beware : viewing time is T

$$\begin{aligned}
f(r_t) &= P_t(t,T)E_{Q_T}[(P_T(T,\Gamma)-K)^+\,|\,I_t] \\
&= P_t(t,T)(E_{Q_T}[P_T(T,\Gamma)\,|\,I_t]N(d_1)-K\times N(d_2)) \\
&= P_t(t,T)(P_t(T,\Gamma)N(d_1)-K\times N(d_2)) \\
&= \underbrace{P_t(t,\Gamma)}_{DF(t,\Gamma)}N(d_1)-\underbrace{P_t(t,T)}_{DF(t,T)}KN(d_2) && \textit{since } P_t(t,T)P_t(T,\Gamma)=P_t(t,\Gamma) \\
\textit{where}\quad d_{1,2} &= \frac{\ln(P_t(T,\Gamma)/K)\pm(b(\Gamma-T))^2(T-t)/2}{b(\Gamma-T)\sqrt{T-t}}
\end{aligned}$$

optionality starts at t and ends at T

3.6 Forward numeraire for vanilla option with stochastic interest rate

Cash numeraire should be changed to forward numeraire even for pricing vanilla stock option if interest rate is stochastic.

$$
\begin{aligned}
f(S_t) &= B_t E_{Q_B}\left[\frac{f(S_T)}{B_T}\right] && \textit{B_T is stochastic, better use other numeraire} \\
&= P_t(t,T) E_{Q_T}\left[\frac{f(S_T)}{P_T(T,T)}\right] \\
&= P_t(t,T) E_{Q_T}[f(S_T)] && \textit{$P_T(T,T)$ is always 1}
\end{aligned}
$$

3.7 Stock numeraire for vanilla option with Heston model

Stock numeraire should be used in the proof of Heston model for deterministic interest rate.

$$
\begin{aligned}
f(S_t) &= B_t E_{Q_B}\left[\frac{f(S_T)}{B_T}\right] \\
&= e^{-r(T-t)} E_{Q_B}[(S_T - K)^+] && \textit{since interest rate is deterministic} \\
&= e^{-r(T-t)} E_{Q_B}[(S_T - K) 1_{S_T > K}] \\
&= e^{-r(T-t)} E_{Q_B}[S_T 1_{S_T > K}] - e^{-r(T-t)} K E_{Q_B}[1_{S_T > K}] \\
&= e^{-r(T-t)} E_{Q_S}\left[\frac{dQ_B}{dQ_S} S_T 1_{S_T > K}\right] - e^{-r(T-t)} K E_{Q_B}[1_{S_T > K}] \\
&= e^{-r(T-t)} E_{Q_S}\left[\frac{S_t / S_T}{e^{-r(T-t)}} S_T 1_{S_T > K}\right] - e^{-r(T-t)} K E_{Q_B}[1_{S_T > K}] \\
&= S_t E_{Q_S}[1_{S_T > K}] - e^{-r(T-t)} K E_{Q_B}[1_{S_T > K}] \\
&= S_t \Pr_{Q_S}(ITM) - e^{-r(T-t)} K \Pr_{Q_B}(ITM) && \textit{then calculate ITM prob, refer to Heston.doc}
\end{aligned}
$$

3.8 Stock numeraire for inverse FX process

[Reference] Siegel paradox about exchange rates, Presh Talwalkar.

Siegel paradox

Consider FX pair *USDJPY* with current rate is *100*. Suppose the market expects that there is a *60%* chance *USD* will be *10%* stronger and *40%* chance *USD* will be *10%* weaker in one year time, what is the expected rate in a year? In *JPY* perspective, it should be $0.6 \times 110 + 0.4 \times 90 = 102$, whereas in *USD* perspective it should be $0.6/110 + 0.4/90 \approx 1/101.02$, which is inconsistent with the former, why's that? This is called Siegel paradox.

Explanation 1

The root cause of paradox is that :

$$E[S_T \mid I_t] \neq 1/E[1/S_T \mid I_t] \qquad \text{where } S_T \text{ is USDJPY in a year}$$

$$\text{or} \qquad 1/E[S_T \mid I_t] \neq E[1/S_T \mid I_t]$$

Explanation 2

Whenever we price using ***probability-weighted sum*** we are doing ***risk neutral pricing***. In that case we need to ensure expectation is calculated in appropriate measures. Suppose probability *(60%,40%)* is implied from market data of *JPY* yield curve, then *(60%,40%)* is considered as risk neutral measure of *JPY* numeraire. To solve the paradox, we introduce *USD* measure so that equality holds :

$$1/E_{Q_{JPY}}[S_T \mid I_t] = E_{Q_{USD}}[1/S_T \mid I_t]$$

Risk neutral measure of *USD* numeraire is probably different from *(60%, 40%)*, it is implied from market data of *USDJPY* forwards and *JPY* yield curve. Yet in practice, as *USD* is dominating currency, we must calibrate to *USD* yield curve and *FX* market instead.

Inverse process

Given risk neutral process S_t of pair *A/B* under numeraire *B*, lets derive risk neutral process S'_t of pair *B/A* under numeraire *A*.

$$dS_t = (r_B - r_A)S_t dt + \sigma S_t dz_t \qquad \text{under } Q_B$$

By Ito's lemma, we have :

The point is to find a transformation of Brownian motion, so that the new one is martingale in desired RN measure.

$$\begin{aligned}
dS'_t &= d(1/S_t) \\
&= -(S_t)^{-2}(dS_t) - \frac{1}{2}(-2)(S_t)^{-3}(dS_t)^2 \\
&= -(S_t)^{-2}((r_B - r_A)S_t dt + \sigma S_t dz_t) + (S_t)^{-3}((r_B - r_A)S_t dt + \sigma S_t dz_t)^2 \\
&= (r_A - r_B)S'_t dt - \sigma S'_t dz_t + \sigma^2 S'_t dt \\
&= (r_A - r_B + \sigma^2)S'_t dt - \sigma S'_t dz_t \\
&= (r_A - r_B + \sigma^2)S'_t dt - \sigma S'_t(-d\tilde{z}_t - \lambda_t dt) && \text{introduce shifted Brownian } d\tilde{z}_t = -(\lambda_t dt + dz_t) \\
&= (r_A - r_B + \sigma^2 + \sigma\lambda_t)S'_t dt + \sigma S'_t d\tilde{z}_t && \text{Why } -ve \text{ sign? Because it leads to } +\sigma \text{ instead of } -\sigma.
\end{aligned}$$

In order to make *B-deflatted pair B/A* martingale under *RN* measure of numeraire *A*, i.e. S'_t should have a drift of r_A-r_B. We set :

$$\begin{aligned}
& \sigma^2 + \sigma\lambda_t = 0 \\
\text{or} \quad & \lambda_t = -\sigma \\
\text{or} \quad & d\tilde{z}_t = -(-\sigma dt + dz_t) = \sigma dt - dz_t
\end{aligned}$$

Thus risk neutral process S'_t of under numeraire *A* is :

$$dS'_t = (r_A - r_B)S'_t dt + \sigma S'_t d\tilde{z}_t \qquad \text{under } Q_A\text{, under which } d\tilde{z}_t \text{ is Brownian}$$

Intuition of $N(d_1)$ and $N(d_2)$

4.1 Two digital options and vanilla option

Let's consider two digital call options :

- digital call option $d_B(S_t)$ that pays *\$1*, if S_T finishes *ITM* at maturity *T*
- digital call option $d_S(S_t)$ that pays *1* share of stock, if S_T finishes *ITM* at maturity *T*

Approach 1

By risk neutral pricing under Q measure, we have :

$$
\begin{aligned}
d_B(S_t) &= e^{-r(T-t)} E_{Q_B}[1_{S_T>K} \mid I_t] \\
&= e^{-r(T-t)} E_{Q_B}[1_{F_{QB}\exp(-v/2+\sqrt{v}\varepsilon)>K} \mid I_t] \\
&= e^{-r(T-t)} E_{Q_B}[1_{\varepsilon>(\ln(K/F_{QB})+v/2)/\sqrt{v}} \mid I_t] \\
&= e^{-r(T-t)} E_{Q_B}[1_{\varepsilon>-d_2} \mid I_t] \\
&= e^{-r(T-t)} E_{Q_B}[1_{\varepsilon<+d_2} \mid I_t] \\
&= e^{-r(T-t)} N(d_2) & (1)
\end{aligned}
$$

These steps are the same.

$$
\begin{aligned}
d_S(S_t) &= e^{-r(T-t)} E_{Q_B}[S_T 1_{S_T>K} \mid I_t] \\
&= e^{-r(T-t)} E_{Q_B}[S_T 1_{F_{QB}\exp(-v/2+\sqrt{v}\varepsilon)>K} \mid I_t] \\
&= e^{-r(T-t)} E_{Q_B}[S_T 1_{\varepsilon>(\ln(K/F_{QB})+v/2)/\sqrt{v}} \mid I_t] \\
&= e^{-r(T-t)} E_{Q_B}[S_T 1_{\varepsilon>-d_2} \mid I_t] \\
&= e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} F_{Q_B} e^{-v/2+\sqrt{v}x} e^{-x^2/2} dx \\
&= S_t \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-(x^2-2\sqrt{v}x+v)/2} dx & \text{since } e^{-r(T-t)} F_{Q_B} = S_t \\
&= S_t \frac{1}{\sqrt{2\pi}} \int_{-d_1}^{\infty} e^{-y^2/2} dy & \text{let } y = x-\sqrt{v}\text{, hence } y=-d_2-\sqrt{v} \text{ when } x=-d_2 \\
&= S_t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-y^2/2} dy \\
&= S_t N(d_1) & (2)
\end{aligned}
$$

Approach 2

By writing expectation as probabilities by $E_Q[1_{x>K}] = \int_K^{\infty} dQ(x) = \Pr_Q(X > K)$, we have :

$$
\begin{aligned}
d_B(S_t) &= e^{-r(T-t)} E_{Q_B}[1_{S_T>K} \mid I_t] \\
&= e^{-r(T-t)} \Pr_{Q_B}(ITM \mid I_t) & (3)
\end{aligned}
$$

$$
\begin{aligned}
d_S(S_t) &= e^{-r(T-t)} E_{Q_B}[S_T 1_{S_T>K} \mid I_t] \\
&= e^{-r(T-t)} E_{Q_S}\left[\frac{B_T/B_t}{S_T/S_t} S_T 1_{S_T>K} \mid I_t\right] \\
&= S_t E_{Q_S}[1_{S_T>K} \mid I_t] & \text{since } e^{-r(T-t)} = B_t/B_T \\
&= S_t \Pr_{Q_S}(ITM \mid I_t) & (4)
\end{aligned}
$$

These steps are the same.

Relation between digital and vanilla
Payoff of vanilla call option can be broken down as the payoff of one digital-stock $d_S(S_t)$ minus K digital-cash $d_B(S_t)$:

$$\begin{aligned}(S_T - K)^+ &= (S_T - K)1_{S_T > K} \\ &= S_T \times 1_{S_T > K} - K \times 1_{S_T > K}\end{aligned}$$

By law of one price, vanilla call price (similar for vanilla put price) :

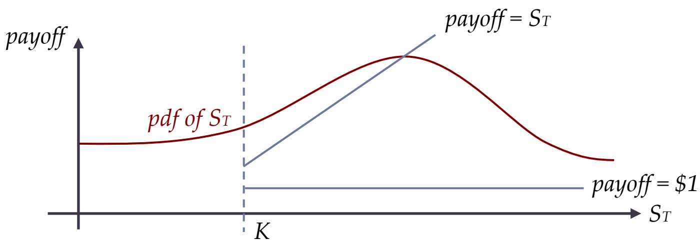
$$\begin{aligned}f_{call}(S_t) &= d_{S,call}(S_t) - Kd_{B,call}(S_t) \\ f_{put}(S_t) &= Kd_{B,put}(S_t) - d_{S,put}(S_t)\end{aligned}$$

Relation between digital and ITM prob
Comparing the result between *(1)* and *(3)*, between *(2)* and *(4)*, we have ITM probabilities in different measures :

$$\begin{aligned}N(d_2) &= \Pr_{Q_B}(ITM \mid I_t) \\ N(d_1) &= \Pr_{Q_S}(ITM \mid I_t) \\ f_{call}(S_t) &= d_{S,call}(S_t) - Kd_{B,call}(S_t) \\ &= S_t \Pr_{Q_S}(ITM \mid I_t) - Ke^{-r(T-t)} \Pr_{Q_B}(ITM \mid I_t) \\ &= \sum_n PV_n \times unit_n \times \Pr_{Q_n}(ITM \mid I_t)\end{aligned}$$

Thus we have a generic form of Black Scholes :
PV_n is present value of 1 unit payment-asset
$unit_n$ is number of units of payment-asset
Q_n is RN measure with payment-asset as numeraire

Difference between the two digital options
The two digital options look similar, yet they are very different in payoff, the digital-cash has constant payoff as S_T goes more deep *ITM*, while the digital-stock has increasing payoff as S_T goes more deep *ITM*, so they have different integration areas as shown :



Convexity adjustment
For geometric Brownian stock model, a change in numeraire can be considered as a convexity adjustment of forward.

$$\begin{aligned}\Pr_{Q_S}(ITM \mid I_t) &= N(d_1(F_{Q_B})) && \text{where } d_1(x) = \frac{\ln(x/K)+v/2}{\sqrt{v}} \\ &= N\left(\frac{\ln(F_{Q_B}/K)+v/2}{\sqrt{v}}\right) \\ &= N\left(\frac{\ln(F_{Q_B}e^v/K)-v/2}{\sqrt{v}}\right) \\ &= N(d_2(F_{Q_B}e^v)) && \text{where } d_2(x) = \frac{\ln(x/K)-v/2}{\sqrt{v}} \\ &= \Pr_{Q_B}(ITM \mid I_t)\Big|_{convexity-adjusted-forward}\end{aligned}$$

Effectively, it is equivalent to calculation in cash-numeraire, but convexity-adjust the forward stock price by the factor e^v. Therefore, we have different perspectives of change in numeraire :

change in numeraire	↔	*Radon Nikodym*	*(for any pdf in general)*
	↔	*Girsanov*	*(for Brownian motion z_t)*
	↔	*convexity adjustment*	*(for stock model $dS_t = rS_t dt + \sigma S_t dz_t$)*

Now we are going to extend the ***two*** digital options in this section to ***four*** digital options for FX scenario.

4.2 FX Option

Let’s consider two *FX* digital options :

- *USDJPY* digital call option that pays *1 yen* if *USDJPY* exceeds *K* at maturity *T*, suppose its current price is *n yen*
- *JPYUSD* digital put option that pays *1 dollar* if *JPYUSD* goes below *1/K* at maturity *T*, suppose its current price is *m dollar*

⇒ as both options are settled in ccy2, is $n = m$?

Solution

The answer is NO, which is counter-intuitive. Suppose S_t denotes *USDJPY* and S'_t denotes *JPYUSD*, they have different drifts while having the same volatility. As $S_t = 1/S'_t$ for all time, there must be relationship between dz_t and dz'_t.

$$
\begin{aligned}
S'_t &= 1/S_t && \textit{for all time } t \\
dS_t &= (r_{JPY} - r_{USD})S_t dt + \sigma S_t dz_t && \textit{under } Q_{JPY}\textit{, besides } dz_t \textit{ is Brownian under } Q_{JPY} \\
dS'_t &= (r_{USD} - r_{JPY})S'_t dt + \sigma S'_t dz'_t && \textit{under } Q_{USD}\textit{, besides } dz'_t \textit{ is Brownian under } Q_{USD}
\end{aligned}
$$

Forward price under corresponding measures are :

$$
\begin{aligned}
E_{Q_{JPY}}[S_T \mid I_t] &= S_t e^{(r_{JPY} - r_{USD})(T-t)} && (1) \\
E_{Q_{USD}}[S'_T \mid I_t] &= S'_t e^{(r_{USD} - r_{JPY})(T-t)} && (2) \\
&= (1/S_t) e^{(r_{USD} - r_{JPY})(T-t)} \\
&= 1/E_{Q_{JPY}}[S_T \mid I_t] && (3)
\end{aligned}
$$

Digital option under corresponding measures are :

$$
\begin{aligned}
n &= e^{-r_{JPY}(T-t)} \times \Pr_{JPY}(USDJPY_T > K \mid I_t) \\
m &= e^{-r_{USD}(T-t)} \times \Pr_{USD}(JPYUSD_T < 1/K \mid I_t) \\
&= e^{-r_{USD}(T-t)} \times \Pr_{USD}(USDJPY_T > K \mid I_t) \\
&\neq n
\end{aligned}
$$

This is not conclusive (as both *ITM* probabilities and discount factors are different), let’s try another way :

$$
\begin{aligned}
n &= e^{-r_{JPY}(T-t)} \times N(d_2) \\
m &= e^{-r_{USD}(T-t)} \times N(d'_2) && \textit{different} \\
&= e^{-r_{USD}(T-t)} \times N(-d_1) && \textit{applying equation (4) below} \\
&\neq n
\end{aligned}
$$

$$
\begin{aligned}
\textit{where} \quad d_{1,2} &= \frac{\ln(E_{Q_{JPY}}[S_T \mid I_t]/K) \pm v/2}{\sqrt{v}} \\
d'_{1,2} &= \frac{\ln(E_{Q_{USD}}[S'_T \mid I_t]/(1/K)) \pm v/2}{\sqrt{v}} \\
&= \frac{\ln((1/E_{Q_{JPY}}[S_T \mid I_t])/(1/K)) \pm v/2}{\sqrt{v}} && \textit{applying equation (3)} \\
&= -\frac{\ln(E_{Q_{JPY}}[S_T \mid I_t]/K) \mp v/2}{\sqrt{v}} \\
&= -d_{2,1} && (4)
\end{aligned}
$$

In fact, there are 4 different possible digital options :

option	*payoff at T*	*currency*		*price at t (applying result from section 5.1)*	
USDJPY call	$(1_{S_T > K}) \times 1_{JPY}$	*pay JPY, settle JPY*		$e^{-r_{JPY}(T-t)} N(d_2)_{JPY}$	
USDJPY call	$(1_{S_T > K}) \times 1_{USD}$	*pay USD, settle JPY*		$e^{-r_{JPY}(T-t)} E_{Q_{JPY}}[S_T \mid I_t] N(d_1)_{JPY}$	
			=	$e^{-r_{USD}(T-t)} S_t N(d_1)_{JPY}$	*applying (1)*
			=	$e^{-r_{USD}(T-t)} N(d_1)_{USD}$	*change ccy at spot*
JPYUSD put	$(1_{S'_T < 1/K}) \times 1_{USD}$	*pay USD, settle USD*		$e^{-r_{USD}(T-t)} N(d'_2)_{USD}$	
			=	$e^{-r_{USD}(T-t)} N(-d_1)_{USD}$	
JPYUSD put	$(1_{S'_T < 1/K}) \times 1_{JPY}$	*pay JPY, settle USD*		$e^{-r_{USD}(T-t)} E_{Q_{USD}}[S'_T \mid I_t] N(d'_1)_{USD}$	
			=	$e^{-r_{JPY}(T-t)} (1/S_t) N(d'_1)_{USD}$	*applying (2)*
			=	$e^{-r_{JPY}(T-t)} N(d'_1)_{JPY}$	*change ccy at spot*
			=	$e^{-r_{JPY}(T-t)} N(-d_2)_{JPY}$	*applying (4)*

Putting them together, they all have different prices (i.e. all *4* combinations of $\pm d_{1,2}$ as *ITM* prob) :

option	*payoff at T*	*price at t*
USDJPY call	$(1_{S_T > K}) \times 1_{JPY}$	$e^{-r_{JPY}(T-t)} N(d_2)_{JPY}$
USDJPY call	$(1_{S_T > K}) \times 1_{USD}$	$e^{-r_{USD}(T-t)} N(d_1)_{USD}$
JPYUSD put	$(1_{S'_T < 1/K}) \times 1_{USD}$	$e^{-r_{USD}(T-t)} N(-d_1)_{USD}$
JPYUSD put	$(1_{S'_T < 1/K}) \times 1_{JPY}$	$e^{-r_{JPY}(T-t)} N(-d_2)_{JPY}$

(1) risk neutral measure *(2) numeraire*

Conclusion

According to FTAP, risk neutral pricing is nothing related to physical measure, risk neutral measure is simply a tools that facilitates extrapolation from 'market price' of vanilla products to 'fair price' of contingent claim. FTAP has to work with a numeraire, which is simply a normalization of present values and future payoffs. When we change the numeraire market price of all securities do not change, however its stochastic properties change, such as :

- pdf for random variable
- SDE for random process
- expected value of payoff

(3) Radon Nikodym and Girsanov

The key is to find the pdf of underlying random variable, or SDE of underlying random process under the risk neutral measure.

- For random variable, we can do it by Radon Nikodym.
- For random process, we can do it by Girsanov theorem.