Siegel Paradox and Quanto

Question 1 - Correlated trio

Suppose X, Y and Z are correlated random variables with <u>unknown distributions</u>, find correlation $\rho_{X,Z}$ given $\rho_{X,Y}$ and $\rho_{Y,Z}$. Firstly, if sample sets for X, Y and Z are given as $\{x_n, y_n, z_n \mid n \in [1,N]\}$, then the empirical formula for variance and covariance are :

$$
\begin{aligned}
V(X) &= \frac{1}{N}\sum_n (x_n - \frac{1}{N}\sum_m x_m)^2 \\
Cov(X,Y) &= \frac{1}{N}\sum_n (x_n - \frac{1}{N}\sum_m x_m)(y_n - \frac{1}{N}\sum_m y_m)
\end{aligned}
$$

Secondly, we remove bias by subtracting mean from the dataset :

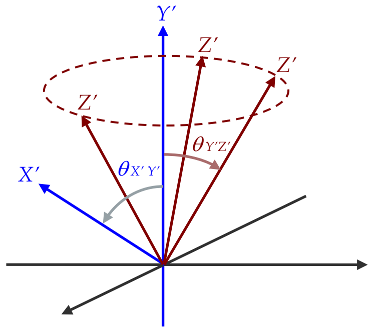
$$
\begin{aligned}
V(X') &= \frac{1}{N}\sum_n (x'_n - \frac{1}{N}\sum_m x'_m)^2 && \textit{where } x'_n = x_n - \frac{1}{N}\sum_n x_n \textit{ hence } \frac{1}{N}\sum_n x'_n = 0 \\
&= \frac{1}{N}\sum_n (x'_n)^2 \\
&= \frac{1}{N}|x'|^2 && \textit{which is normalized L2 norm} \\
Cov(X',Y') &= \frac{1}{N}\sum_n (x'_n - \frac{1}{N}\sum_m x'_m)(y'_n - \frac{1}{N}\sum_m y'_m) && \textit{where } y'_n = y_n - \frac{1}{N}\sum_n y_n \textit{ hence } \frac{1}{N}\sum_n y'_n = 0 \\
&= \frac{1}{N}\sum_n x'_n y'_n \\
&= \frac{1}{N}(x' \cdot y') && \textit{which is normalized dot product}
\end{aligned}
$$

Consider in regressor space, that is treating $\{x_1', x_2', .., x_n'\}$, $\{y_1', y_2', .., y_n'\}$ and $\{z_1', z_2', .., z_n'\}$ as 3 vectors in $\Re^N$ space, variance can be regarded as $L2$ norm, whereas covariance can be regarded as dot product, both normalized by dimension of regression space. Since correlation is the ratio between covariance and $L2$ norm, so it is considered as cosine of included angle in this geometry analogy.

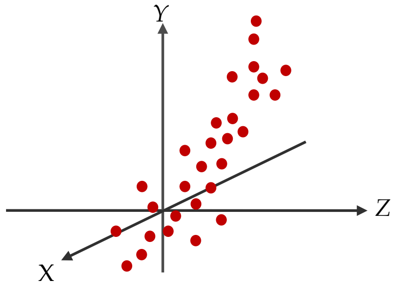
$$
\begin{aligned}
\rho_{X'Y'} &= Cov(X',Y')/\sqrt{V(X')V(Y')} \\
&= (x' \cdot y')/(|x'||y'|) \\
&= \cos\theta_{x',y'}
\end{aligned}
$$

By plotting vector X', Y' and Z' in regression space, with Y' pointing upwards, while included angles $\theta_{X'Y'}$ and $\theta_{Y'Z'}$ are fixed, vector Z' is free to rotate along the Y' axis, forming the red cone. We can find feasible range of $\rho_{X'Z'}$ based on given values of $\rho_{X'Y'}$ and $\rho_{Y'Z'}$.

$\Re^N$ regressor space



$\Re^3$ data space should look like this :

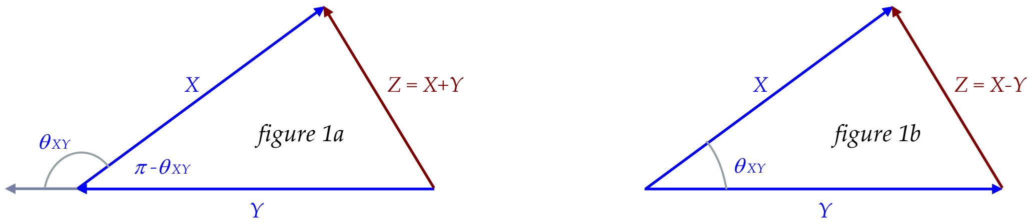


$$
\begin{aligned}
\rho_{X'Z'} &= \cos\theta_{X'Z'} \\
&\in \cos(\theta_{X'Y'} \pm \theta_{Y'Z'}) && \textit{when } X'\ Y'\ Z' \textit{ are coplanar} \\
&= \cos\theta_{X'Y'}\cos\theta_{Y'Z'} \mp \sin\theta_{X'Y'}\sin\theta_{Y'Z'} \\
&= \rho_{X'Y'}\rho_{Y'Z'} \mp \sqrt{1-\rho^2_{X'Y'}}\sqrt{1-\rho^2_{Y'Z'}} \\
\max(\rho_{X'Z'}) &= \rho_{X'Y'}\rho_{Y'Z'} + \sqrt{1-\rho^2_{X'Y'}}\sqrt{1-\rho^2_{Y'Z'}} && \textit{when } Z' \textit{ is on the nearest side} \\
\min(\rho_{X'Z'}) &= \rho_{X'Y'}\rho_{Y'Z'} - \sqrt{1-\rho^2_{X'Y'}}\sqrt{1-\rho^2_{Y'Z'}} && \textit{when } Z' \textit{ is on the furthest side}
\end{aligned}
$$

Correlation is cosine of included angle between two arms both pointing outwards.
Reference - Using Correlation in studies of studies, by Alan Safer and Saleem Watson.

Question 2 – Sum/difference of correlated variables

Similar scenairo as question *1*, however this time instead of given $\rho_{Y,Z}$ we have to solve for variance of Z provided that $Z=X\pm Y$, omit apostrophe for simplicity). By plotting the vector addition as a triangle in $\Re^N$ space, we have :



Recall that, empirical variance is analogous to *L2* norm of vectors, empirical covariance is analogous to dot product and correlation is analogous to cosine of included angle whereas included angle is the angle in between vectors pointing outwards, then covariance of sum or difference can be derived as :

$$
\begin{aligned}
V(Z) &= V(X\pm Y) && \textit{for case } Z = X+Y \textit{ and case } Z = X\text{-}Y \textit{ respectively} \\
&= V(X)+V(Y)\pm 2Cov(X,Y) \\
&= V(X)+V(Y)\pm 2\rho_{X,Y}\sqrt{V(X)V(Y)} \\
\frac{1}{N}|z|^2 &= c\frac{1}{N}|x|^2+\frac{1}{N}|y|^2 \pm 2\cos\theta_{X,Y}\sqrt{\frac{1}{N}|x|^2\frac{1}{N}|y|^2} && \textit{substitute the analogies} \\
|z|^2 &= |x|^2+|y|^2 \pm 2\cos\theta_{X,Y}|xy| && \textit{recall that cosine rule takes negative sign}
\end{aligned}
$$

It gives a perfect analogy to cosine rule for case $Z = X\text{-}Y$, however for the case $Z = X+Y$, we need a slight adjustment :

$$
\begin{aligned}
|z|^2 &= |x|^2+|y|^2 + 2\cos\theta_{X,Y}|xy| && \textit{note : } \rho_{X,Y} = \cos\theta_{X,Y} \textit{ for both cases} \\
&= |x|^2+|y|^2 - 2\cos(\pi-\theta_{X,Y})|xy| && \textit{note : included angle is } (\pi\text{-}\theta_{XY}) \textit{ for fig1a and } (\theta_{XY}) \textit{ for fig1b}
\end{aligned}
$$

It is an analogy to a triangle, once ***3 out of 6*** pieces of informations (*3* sides plus *3* angles) are given, the others are known :

$$
\frac{1}{N}\begin{bmatrix} |x|^2 & \cos\theta_{x,y}|x||y| & \cos\theta_{x,z}|x||z| \\ & |y|^2 & \cos\theta_{y,z}|y||z| \\ & & |z|^2 \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \rho_{x,y}\sigma_x\sigma_y & \rho_{x,z}\sigma_x\sigma_z \\ & \sigma_y^2 & \rho_{y,z}\sigma_y\sigma_z \\ & & \sigma_z^2 \end{bmatrix}
$$

However if you find the cosine rule difficult to understand, simply treat it as variance of sum, $V(X\pm Y) = V(X) + V(Y) \pm 2Cov(X,Y)$. By comparing question *1* and *2*, correlated trio is analogous to <u>cosine of sum or difference</u>, while correlated sum is analogous to <u>cosine rule</u>, the former does not assume any relation between X, Y and Z, whereas the latter assumes that $Z = X \pm Y$. This technique doesnt deal with $Z = X \pm Y$ only, we can extend it to multiplication and division by taking log, for example :

$$
\begin{aligned}
Z &= XY \\
\ln Z &= \ln X + \ln Y \\
V(\ln Z) &= V(\ln X)+V(\ln Y)+2\rho_{\ln X,\ln Y}\sqrt{V(\ln X)V(\ln Y)} \\
&= \dots
\end{aligned}
$$

Sum (or multiplication) of correlated log-normal processes

Given two correlated Brownian motions *(both are under risk neutral measure of the same numeraire)* :

$$
\begin{array}{lllllll}
dX_t & = & \mu_x X_t dt + \sigma_x X_t dz_{xt} & \Rightarrow & d\ln X_t & = & (\mu_x - \frac{1}{2}\sigma_x^2)dt + \sigma_x dz_{xt} \\
dY_t & = & \mu_y Y_t dt + \sigma_y Y_t dz_{yt} & \Rightarrow & d\ln Y_t & = & (\mu_y - \frac{1}{2}\sigma_y^2)dt + \sigma_y dz_{yt} \\
dz_{xt} & = & \varepsilon_x \sqrt{dt} & & & & \\
dz_{yt} & = & \varepsilon_y \sqrt{dt} & \textit{and} & dz_{xt} dz_{yt} & = & \rho_{x,y} dt
\end{array}
$$

then product of the two processes can be derived using *[method 1] multi-dimensional Ito's lemma* :

$$
\begin{array}{llll}
Z_t & = & X_t Y_t & \textit{ensure X\&Y under same RN measure by applying Girsanov} \\
dZ_t & = & d(X_t Y_t) & \\
 & = & Y_t dX_t + X_t dY_t + dX_t dY_t & \\
 & = & X_t Y_t(\mu_x dt + \sigma_x dz_{xt}) + X_t Y_t(\mu_y dt + \sigma_y dz_{yt}) + X_t Y_t(\sigma_x \sigma_y dz_{xt} dz_{yt}) & \\
 & = & Z_t \underbrace{(\mu_x + \mu_y + \rho_{x,y}\sigma_x\sigma_y)}_{\mu_z} dt + Z_t \underbrace{(\sigma_x dz_{xt} + \sigma_y dz_{yt})}_{\sigma_z dz_{zt}} & \\
 & = & \mu_z Z_t dt + \sigma_z Z_t dz_{zt} & \textit{still under RN measure of the same numeraire}
\end{array}
$$

$$
\begin{array}{lllll}
\textit{hence} \quad \mu_z & = & \mu_x + \mu_y + \rho_{x,y}\sigma_x\sigma_y & \textit{drift is not analogous to anything} & (2a) \\
\textit{and} \quad dz_{zt} & = & (\sigma_x dz_{xt} + \sigma_y dz_{yt}) / \sigma_z & \textit{diffusion is analogous to cosine rule, we will see ...} & (2^*)
\end{array}
$$

As weighted sum of correlated Gaussian is Gaussian, thus dz_{zt} is also Gaussian with mean and variance :

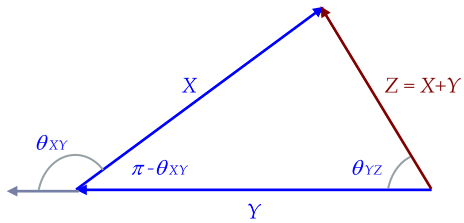
$$
\begin{array}{lllll}
E[dz_{zt}] & = & (\sigma_x E[dz_{xt}] + \sigma_y E[dz_{yt}]) / \sigma_z \quad = \quad 0 & & \\
V[dz_{zt}] & = & (\sigma_x^2 V[dz_{xt}] + \sigma_y^2 V[dz_{yt}] + 2\sigma_x\sigma_y Cov(dz_{xt}, dz_{yt})) / \sigma_z^2 & & \\
 & = & (\sigma_x^2 dt + \sigma_y^2 dt + 2\rho_{x,y}\sigma_x\sigma_y dt) / \sigma_z^2 & \textit{now force LHS to dt, then we have :} & \\
\Rightarrow \quad \sigma_z^2 & = & \sigma_x^2 + \sigma_y^2 + 2\rho_{x,y}\sigma_x\sigma_y & \textit{we derived +ve version cosine rule for Z = X+Y} & (2b)
\end{array}
$$

Alternatively we can do the same thing with *[method 2] sum of correlated variables, i.e. apply +ve version cosine rule in figure 1a* :

$$
\begin{array}{llll}
d\ln Z_t & = & d\ln X_t + d\ln Y_t & \textit{multiplication of X\&Y becomes sum of dlnX and dlnY} \\
 & = & (\mu_x - \frac{1}{2}\sigma_x^2)dt + dz_{xt} + (\mu_y - \frac{1}{2}\sigma_y^2)dt + dz_{yt} & \\
 & = & (\mu_x + \mu_y - \frac{1}{2}\sigma_x^2 - \frac{1}{2}\sigma_y^2)dt + \sigma_x dz_{xt} + \sigma_y dz_{yt} & \textit{now apply +ve version cosine rule for Z = X+Y} \\
 & = & \underbrace{(\mu_x + \mu_y - \frac{1}{2}\sigma_x^2 - \frac{1}{2}\sigma_y^2)}_{\mu_z - (1/2)\sigma_z^2} dt + \sigma_z dz_{zt} & \textit{where } \sigma_z^2 = \sigma_x^2 + \sigma_y^2 + 2\rho_{x,y}\sigma_x\sigma_y \textit{, which is identical to (2b)} \\
\textit{hence} \quad \mu_z & = & \mu_x + \mu_y - \frac{1}{2}(\sigma_x^2 + \sigma_y^2 - \sigma_z^2) & \\
 & = & \mu_x + \mu_y - \frac{1}{2}(-2\rho_{x,y}\sigma_x\sigma_y) & \\
 & = & \mu_x + \mu_y + \rho_{x,y}\sigma_x\sigma_y & \textit{which is identical to (2a)}
\end{array}
$$

Finally, let's visit the *3 out of 6 property*. By referring to *figure 1a* :

$$
\begin{array}{lllll}
dz_{xt} dz_{zt} & = & dz_{xt}(\sigma_x dz_{xt} + \sigma_y dz_{yt}) / \sigma_z & \textit{apply (2*)} \\
 & = & (\sigma_x dz_{xt} dz_{xt} + \sigma_y dz_{xt} dz_{yt}) / \sigma_z & \\
\rho_{x,z} dt & = & ((\sigma_x + \rho_{x,y}\sigma_y) / \sigma_z) dt & \\
\textit{hence} \quad \rho_{x,z} & = & (\sigma_x + \rho_{x,y}\sigma_y) / \sigma_z & (2c) \\
 & & & \\
dz_{yt} dz_{zt} & = & dz_{yt}(\sigma_x dz_{xt} + \sigma_y dz_{yt}) / \sigma_z & \textit{apply (2*)} \\
 & = & (\sigma_x dz_{xt} dz_{yt} + \sigma_y dz_{yt} dz_{yt}) / \sigma_z & \\
\rho_{y,z} dt & = & ((\rho_{x,y}\sigma_x + \sigma_y) / \sigma_z) dt & \\
\textit{hence} \quad \rho_{y,z} & = & (\rho_{x,y}\sigma_x + \sigma_y) / \sigma_z & (2d)
\end{array}
$$



Drift, diffusion and correlation are given by (2a-d) respectively.

Difference (or division) between correlated log-normal processes

Given the original correlated Brownian motions *(again both are under risk neutral measure of the same numeraire)*, division between the processes can be derived using *[method 1] multi-dimensional Ito's lemma* :

$$
\begin{aligned}
Z_t &= X_t / Y_t && \textit{ensure X\&Y under same RN measure by applying Girsanov} \\
dZ_t &= d(X_t / Y_t) \\
&= (1/Y_t)dX_t - (X_t/Y_t^2)dY_t - (1/Y_t^2)(dX_t)(dY_t) + (X_t/Y_t^3)(dY_t)^2 \\
&= \left[\begin{array}{l} (1/Y_t)(\mu_x X_t dt + \sigma_x X_t dz_{xt}) - (X_t/Y_t^2)(\mu_y Y_t dt + \sigma_y Y_t dz_{yt}) \\ -(1/Y_t^2)(\sigma_x \sigma_y X_t Y_t \rho_{x,y} dt) + (X_t/Y_t^3)(\sigma_y^2 Y_t^2 dt) \end{array}\right. \\
&= \left[\begin{array}{l} (X_t/Y_t)(\mu_x dt + \sigma_x dz_{xt}) - (X_t/Y_t)(\mu_y dt + \sigma_y dz_{yt}) \\ -(X_t/Y_t)(\rho_{x,y}\sigma_x\sigma_y dt) + (X_t/Y_t)(\sigma_y^2 dt) \end{array}\right. \\
&= Z_t \underbrace{(\mu_x - \mu_y + \sigma_y^2 - \rho_{x,y}\sigma_x\sigma_y)}_{\mu_z} dt + Z_t \underbrace{(\sigma_x dz_{xt} - \sigma_y dz_{yt})}_{\sigma_z dz_{zt}} \\
&= \mu_z Z_t dt + \sigma_z Z_t dz_{zt} && \textit{still under RN measure of the same numeraire} \\
\text{hence} \quad \mu_z &= \mu_x - \mu_y + \sigma_y^2 - \rho_{x,y}\sigma_x\sigma_y && \textit{drift is not analogous to anything} \quad (3a) \\
\text{and} \quad dz_{zt} &= (\sigma_x dz_{xt} - \sigma_y dz_{yt})/\sigma_z && \textit{diffusion is analogous to cosine rule, we will see …} \quad (3^*)
\end{aligned}
$$

As weighted difference of correlated Gaussian is Gaussian, thus dz_{zt} is also Gaussian with mean and variance :

$$
\begin{aligned}
E[dz_{zt}] &= (\sigma_x E[dz_{xt}] - \sigma_y E[dz_{yt}])/\sigma_z \quad = \quad 0 \\
V[dz_{zt}] &= (\sigma_x^2 V[dz_{xt}] + \sigma_y^2 V[dz_{yt}] - 2\sigma_x\sigma_y Cov(dz_{xt}, dz_{yt}))/\sigma_z^2 \\
&= (\sigma_x^2 dt + \sigma_y^2 dt - 2\rho_{x,y}\sigma_x\sigma_y dt)/\sigma_z^2 && \textit{now force LHS to dt, then we have :} \\
\Rightarrow \quad \sigma_z^2 &= \sigma_x^2 + \sigma_y^2 - 2\rho_{x,y}\sigma_x\sigma_y && \textit{we derived -ve version cosine rule for } Z = X - Y \quad (3b)
\end{aligned}
$$

Alternatively we can do the same thing with *[method 2] diff of correlated variables, i.e. apply –ve version cosine rule in figure 1b.* This part is similar to that in previous section, and is thus omitted for clarity. Finally, let's visit the *3 out of 6 property*. By referring to *figure 1b* :

$$
\begin{aligned}
dz_{xt}dz_{zt} &= dz_{xt}(\sigma_x dz_{xt} - \sigma_y dz_{yt})/\sigma_z && \textit{apply } (3^*) \\
&= (\sigma_x dz_{xt}dz_{xt} - \sigma_y dz_{xt}dz_{yt})/\sigma_z \\
\rho_{x,z}dt &= ((\sigma_x - \rho_{x,y}\sigma_y)/\sigma_z)dt \\
\text{hence} \quad \rho_{x,z} &= (\sigma_x - \rho_{x,y}\sigma_y)/\sigma_z && (3c)
\end{aligned}
$$

$$
\begin{aligned}
dz_{yt}dz_{zt} &= dz_{yt}(\sigma_x dz_{xt} - \sigma_y dz_{yt})/\sigma_z && \textit{apply } (3^*) \\
&= (\sigma_x dz_{xt}dz_{yt} - \sigma_y dz_{yt}dz_{yt})/\sigma_z \\
\rho_{y,z}dt &= ((\rho_{x,y}\sigma_x - \sigma_y)/\sigma_z)dt \\
\text{hence} \quad \rho_{y,z} &= (\rho_{x,y}\sigma_x - \sigma_y)/\sigma_z && (3d) \textit{ which is identical to eq(9) in Quanto option, by Wystup}
\end{aligned}
$$

<u>*Consistency between (2) and (3)*</u>

Drift, diffusion and correlation given by *(3a-d)* must all be consistent with (2*a-d*), they can be proved by firstly renaming *X*, *Y* and *Z* in *(3a-d)* as *X'*, *Y'* and *Z'* respectively, then followed by putting *X'* = *Z*, *Y'* = *Y* and *Z'* = *X*. Let's try *(2a, 3a)* :

$$
\begin{aligned}
\mu_{z'} &= \mu_{x'} - \mu_{y'} + \sigma_{y'}^2 - \rho_{x',y'}\sigma_{x'}\sigma_{y'} && \textit{rewrite of (3a)} \\
\mu_x &= \mu_z - \mu_y + \sigma_y^2 - \rho_{y,z}\sigma_y\sigma_z && \textit{substituting } X' = Z,\ Y' = Y \text{ and } Z' = X \\
\mu_z &= \mu_x + \mu_y - \sigma_y^2 + \rho_{y,z}\sigma_y\sigma_z \\
&= \mu_x + \mu_y - \sigma_y^2 + ((\rho_{x,y}\sigma_x + \sigma_y)/\sigma_z)\sigma_y\sigma_z && \textit{substituting (2c)} \\
&= \mu_x + \mu_y + \rho_{x,y}\sigma_x\sigma_y && \textit{thus equivalent to (2a)}
\end{aligned}
$$

Others are left as an exercise.

Summary
Lets recall *(2a-c)* and *(3a-c)* here for convenience. Given the *SDE* for X_t and Y_t, then :

(1) for multiplication (or addition in log)

$$\begin{aligned} dZ_t &= d(X_tY_t) \\ &= \mu_z Z_t dt + \sigma_z Z_t dz_{zt} \end{aligned}$$

where $dz_{xt}dz_{zt} = \rho_{x,z}dt$ *and* $dz_{yt}dz_{zt} = \rho_{y,z}dt$ *(triangle analogy)*

we have

$$\begin{aligned} \rho_{x,z} &= (\sigma_x + \rho_{x,y}\sigma_y)/\sigma_z && \text{(2c) or equivalently } \rho_{x,y}\sigma_x\sigma_y = \rho_{x,z}\sigma_x\sigma_z - \sigma_x^2 \\ \rho_{y,z} &= (\rho_{x,y}\sigma_x + \sigma_y)/\sigma_z && \text{(2d) or equivalently } \rho_{x,y}\sigma_x\sigma_y = \rho_{y,z}\sigma_y\sigma_z - \sigma_y^2 \\ \sigma_z^2 &= \sigma_x^2 + \sigma_y^2 + 2\rho_{x,y}\sigma_x\sigma_y && \text{(2b) which is not useful in FX, as we have market data for } \sigma_z \\ \mu_z &= \mu_x + \mu_y + \rho_{x,y}\sigma_x\sigma_y && \text{(2a) however we want to have Z in correlation, so we convert as :} \\ &= \mu_x + \mu_y + \rho_{x,z}\sigma_x\sigma_z - \sigma_x^2 && \text{(2a') remove } \sigma_y \text{ by putting (2c) in (2a)} \\ &= \mu_x + \mu_y + \rho_{y,z}\sigma_y\sigma_z - \sigma_y^2 && \text{(2a'') remove } \sigma_x \text{ by putting (2d) in (2a)} \end{aligned}$$

(2) for division (or subtraction in log)

$$\begin{aligned} dZ_t &= d(X_t/Y_t) \\ &= \mu_z Z_t dt + \sigma_z Z_t dz_{zt} \end{aligned}$$

where $dz_{xt}dz_{zt} = \rho_{x,z}dt$ *and* $dz_{yt}dz_{zt} = \rho_{y,z}dt$ *(triangle analogy)*

we have

$$\begin{aligned} \rho_{x,z} &= (\sigma_x - \rho_{x,y}\sigma_y)/\sigma_z && \text{(3c) or equivalently } \sigma_x^2 - \rho_{x,y}\sigma_x\sigma_y = \rho_{x,z}\sigma_x\sigma_z \\ \rho_{y,z} &= (\rho_{x,y}\sigma_x - \sigma_y)/\sigma_z && \text{(3d) or equivalently } \rho_{x,y}\sigma_x\sigma_y - \sigma_y^2 = \rho_{y,z}\sigma_y\sigma_z \\ \sigma_z^2 &= \sigma_x^2 + \sigma_y^2 - 2\rho_{x,y}\sigma_x\sigma_y && \text{(3b) which is not useful in FX, as we have market data for } \sigma_z \\ \mu_z &= \mu_x - \mu_y - \rho_{x,y}\sigma_x\sigma_y + \sigma_y^2 && \text{(3a) however we want to have Z in correlation, so we convert as :} \\ &= \mu_x - \mu_y + \rho_{x,y}\sigma_x\sigma_y - \sigma_x^2 + \sigma_z^2 && \text{putting (3b) in (3a)} \\ &= \mu_x - \mu_y - \rho_{x,z}\sigma_x\sigma_z + \sigma_z^2 && \text{(3a') remove } \sigma_y \text{ by putting (3c) in the above line} \\ &= \mu_x - \mu_y - \rho_{y,z}\sigma_y\sigma_z && \text{(3a'') remove } \sigma_x \text{ by putting (3d) in (3a)} \end{aligned}$$

Remark : In *FX*, *Z* is a standard currency pair, while either one of the *{X, 1/X, Y, 1/Y}* is also a standard currency pair. We must pick one formula out of *(2a')*, *(2a'')*, *(3a')* or *(3a'')* depending on standard pairs we are dealing with.

Recap of multi-dimensional Ito's lemma
Multi-dimensional Itos lemma kicks in when we need to calculate differential change of function containing two or more Brownian motions, it can be derived from Taylor series in just a few lines. Let's consider both the product and division of Brownian motions :

$$\begin{aligned} d(X_tY_t) &= \left[\begin{array}{l} \partial_X(X_tY_t)dX_t + \partial_Y(X_tY_t)dY_t + \\ \frac{1}{2}\partial_{XX}(X_tY_t)(dX_t)^2 + \frac{1}{2}2\partial_{XY}(X_tY_t)(dX_t)(dY_t) + \\ \frac{1}{2}\partial_{YY}(X_tY_t)(dY_t)^2 + higher-order-terms... \end{array}\right. \\ &= Y_t dX_t + X_t dY_t + \frac{1}{2}0(dX_t)^2 + 1(dX_t)(dY_t) + \frac{1}{2}0(dY_t)^2 \\ &= Y_t dX_t + X_t dY_t + dX_t dY_t \qquad (*) \\ d\left(\frac{X_t}{Y_t}\right) &= \left[\begin{array}{l} \partial_X(X_t/Y_t)dX_t + \partial_Y(X_t/Y_t)dY_t + \\ \frac{1}{2}\partial_{XX}(X_t/Y_t)(dX_t)^2 + \frac{1}{2}2\partial_{XY}(X_t/Y_t)(dX_t)(dY_t) + \\ \frac{1}{2}\partial_{YY}(X_t/Y_t)(dY_t)^2 + higher-order-terms... \end{array}\right. \\ &= (1/Y_t)dX_t - (X_t/Y_t^2)dY_t + \frac{1}{2}0(dX_t)^2 - (1/Y_t^2)(dX_t)(dY_t) + \frac{1}{2}2(X_t/Y_t^3)(dY_t)^2 \\ &= (1/Y_t)dX_t - (X_t/Y_t^2)dY_t - (1/Y_t^2)(dX_t)(dY_t) + (X_t/Y_t^3)(dY_t)^2 \qquad (**) \end{aligned}$$

***FX SDE* under quanto currency numeraire**

Fixed income contracts are denominated in a currency, such as *USD* for *USD-LIBOR-3M* swap, swaption, range accrual and *JPY* for *USDJPY* option, however the financial institution participating these trades may have a different cost of fund, such as *EURIBOR* for European banks, or for banks taking *EUR* collaterals, in the above examples *USD* and *JPY* are called deal currency, whereas *EUR* is called quanto currency. When quanto currency is different from deal currency, then we need to do two things :

- risk neutral pricing under *RN* measure of quanto currency, and
- risk free discounting using risk free rate of quanto currency.

Luckily, there is no need to rewrite the Black Scholes for quanto option, what we need to do is simply a quanto adjustment. First of all, lets derive *SDE* for contract denominated in deal currency under risk neutral measure using quanto currency as numeraire. Lets consider *FX* pair *A/B* quantoed in currency *C*, according to real world market convention *A/B* must be a standard pair, that is either *A* or *B* must be *USD* (*B* is *USD* for *EURUSD*, *GBPUSD*, *AUDUSD* or *NZDUSD*, otherwise A is *USD*), moreover either *A* or *B* should link to *C* as another standard pair, hence there are *4* cases, having different quanto adjustment factors :

- *case 1 : when B is USD and B/C is standard pair* — *method 1 : use (3a")*
- *case 2 : when B is USD and C/B is standard pair* — *method 1 : use (3a") and method 2 : use (2a")*
- *case 3 : when A is USD and A/C is standard pair* — *method 1 : use (3a')*
- *case 4 : when A is USD and C/A is standard pair* — *method 1 : use (3a') and method 2 : use (2a?)*

Original *SDE* for *A/B* under risk neutral measure in numeraire *B* is given as :

$$dX_{AB,t} = (r_B - r_A)X_{AB,t}dt + \sigma_{AB}X_{AB,t}dz_{AB,t}$$

The *SDE* for *A/B* under risk neutral measure in numeraire *C* can be derived by multiplication or division between the two standard pairs, which are two of the possibilities *(A/C, C/A, B/C or C/B)* using equations *(2a', 2a", 3a' or 3a")*. Here are some remarks when we do the multiplication or division *(note that Z is A/B, whereas X is either A/C or C/A, and Y is either B/C or C/B)* :

- *SDEs* for both standard pairs should be under the same *RN* measure, which is *RN* measure for numeraire *C*,
- result *SDE* after multiplication or division is also in the same *RN* measure, which is *RN* measure for numeraire *C*,
- failure to guarantee the above will result in incorrect result,
- multiplication (or division) between standard pairs is equivalent to addition (or subtraction) in *log* domain.

<u>Case 1 : when B is USD and B/C is standard pair</u>

For example, *EURUSD* option quantoed in *JPY* (and modified *Wystup* example as shown in appendix). We then divide the *SDE* for *A/C* by the *SDE* for *B/C* with *(3a")*, we have the *SDE* for *A/B* in quanto numeraire :

$$\begin{aligned}
dX_{AC,t} &= (r_C - r_A)X_{AC,t}dt + \sigma_{AC}X_{AC,t}dz_{AC,t} && \textit{under RN measure of numeraire C} \\
dX_{BC,t} &= (r_C - r_B)X_{BC,t}dt + \sigma_{BC}X_{BC,t}dz_{BC,t} && \textit{under RN measure of numeraire C} \\
\Rightarrow \quad dX_{AB,t} &= d(X_{AC,t}/X_{BC,t}) \\
&= (\mu_x - \mu_y - \rho_{y,z}\sigma_y\sigma_z)Z_t dt + \sigma_z Z_t dz_{zt} && \textit{applying (3a")} \\
&= ((r_C - r_A) - (r_C - r_B) - \rho_{AB,BC}\sigma_{AB}\sigma_{BC})X_{AB,t}dt + \sigma_{AB}X_{AB,t}dz_{AB,t} && \textit{with Z = A/B, X = A/C and Y = B/C} \\
&= (r_B - r_A - \rho_{AB,BC}\sigma_{AB}\sigma_{BC})X_{AB,t}dt + \sigma_{AB}X_{AB,t}dz_{AB,t} && \textit{under RN measure of numeraire C}
\end{aligned}$$

This case is equivalent to the modified example in *Quanto options* by *Wystup* (please refer to the appendix). Besides this is consistent with Girsanov theorem that a change in measure corresponds to a shift in drift only, the volatility part is unchanged, value of σ_{AB} is available in market data (as *A/B* is a standard pair).

<u>Case 2 : when B is USD and C/B is standard pair</u>

For example, *EURUSD* option quantoed in *AUD* (and *John Hull* example as shown in *P665*). We can solve this case using *2* methods. ***Method 1*** simply reuses the result in case *1*, followed by a reverse in sign :

$$\begin{aligned}
dX_{AB,t} &= (r_B - r_A - \rho_{AB,BC}\sigma_{AB}\sigma_{BC})X_{AB,t}dt + \sigma_{AB}X_{AB,t}dz_{AB,t} && \textit{under RN measure of numeraire C} \\
&= (r_B - r_A + \rho_{AB,CB}\sigma_{AB}\sigma_{CB})X_{AB,t}dt + \sigma_{AB}X_{AB,t}dz_{AB,t} && \textit{woo done !!!}
\end{aligned}$$

The reason is :

$$\begin{aligned}
\rho_{AB,BC}\sigma_{AB}\sigma_{BC} &= +Cov(d\ln X_{AB,t}, d\ln X_{BC,t}) \\
&= -Cov(d\ln X_{AB,t}, d\ln X_{CB,t}) \\
&= -\rho_{AB,CB}\sigma_{AB}\sigma_{CB}
\end{aligned}$$

Method 2 is done via multiplication between the *SDE* for *A/C* and the *SDE* for *C/B* with *(2a')*, remember to change the numeraire of *C/B* from *B* to *C* (like what we did for stock numeraire in *FTAP.doc*) before performing the multiplication.

$$\begin{aligned}
& dX_{AC,t} &&= (r_C - r_A) X_{AC,t} dt + \sigma_{AC} X_{AC,t} dz_{AC,t} && \textit{under RN measure of numeraire C} \\
& dX_{CB,t} &&= (r_B - r_C) X_{CB,t} dt + \sigma_{CB} X_{CB,t} dz_{CB,t} && \textit{under RN measure of numeraire B} \\
& &&= (r_B - r_C + \sigma_{CB}^2) X_{CB,t} dt + \sigma_{CB} X_{CB,t} dz_{CB,t} \qquad \textit{(important)} && \textit{under RN measure of numeraire C} \\
\Rightarrow \quad & dX_{AB,t} &&= d(X_{AC,t} X_{CB,t}) && \\
& &&= \mu_x + \mu_y + \rho_{y,z}\sigma_y\sigma_z - \sigma_y^2 && \textit{applying (2a″)} \\
& &&= ((r_C - r_A) + (r_B - r_C + \sigma_{CB}^2) + \rho_{AB,CB}\sigma_{AB}\sigma_{CB} - \sigma_{CB}^2) X_{AB,t} dt + \sigma_{AB} X_{AB,t} dz_{AB,t} && \textit{with Z = A/B, X = A/C and Y = C/B} \\
& &&= (r_B - r_A + \rho_{AB,CB}\sigma_{AB}\sigma_{CB}) X_{AB,t} dt + \sigma_{AB} X_{AB,t} dz_{AB,t} && \textit{under RN measure of numeraire C}
\end{aligned}$$

which is the same as the result from method *1*. Besides example in *John Hull P665* belongs to this case, here is the comparison :

	this doc	*John Hull P665*		
asset	*A*	*V*	=	*Nikkei 225*
deal ccy	*B*	*Y*	=	*JPY*
quanto ccy	*C*	*X*	=	*USD*
original SDE	*A/B*	*V/Y*		
2nd standard pair	*C/B*	*X/Y*	=	*W*

Case 3 : when A is USD and A/C is standard pair

For example, *USDJPY* option quantoed in *HKD*. Unliked case *1*, we divide the *SDE* for *A/C* by the *SDE* for *B/C* using *(3a')* instead of using *(3a″)*, as we want to obtain an expression in terms of σ_x and σ_z (rather than σ_y).

$$\begin{aligned}
& dX_{AC,t} &&= (r_C - r_A) X_{AC,t} dt + \sigma_{AC} X_{AC,t} dz_{AC,t} && \textit{under RN measure of numeraire C} \\
& dX_{BC,t} &&= (r_C - r_B) X_{BC,t} dt + \sigma_{BC} X_{BC,t} dz_{BC,t} && \textit{under RN measure of numeraire C} \\
\Rightarrow \quad & dX_{AB,t} &&= d(X_{AC,t} / X_{BC,t}) && \\
& &&= \mu_x - \mu_y - \rho_{x,z}\sigma_x\sigma_z + \sigma_z^2 && \textit{applying (3a')} \\
& &&= ((r_C - r_A) - (r_C - r_B) - \rho_{AB,AC}\sigma_{AB}\sigma_{AC} + \sigma_{AB}^2) X_{AB,t} dt + \sigma_{AB} X_{AB,t} dz_{AB,t} && \textit{with Z = A/B, X = A/C and Y = B/C} \\
& &&= (r_B - r_A - \rho_{AB,AC}\sigma_{AB}\sigma_{AC} + \sigma_{AB}^2) X_{AB,t} dt + \sigma_{AB} X_{AB,t} dz_{AB,t} && \textit{under RN measure of numeraire C}
\end{aligned}$$

Case 4 : when A is USD and C/A is standard pair

For example, *USDJPY* option quantoed in *EUR*. Like case *2* method *1*, we simply derive it based on case *3* plus a sign reverse.

$$\begin{aligned}
dX_{AB,t} &= (r_B - r_A - \rho_{AB,AC}\sigma_{AB}\sigma_{AC} + \sigma_{AB}^2) X_{AB,t} dt + \sigma_{AB} X_{AB,t} dz_{AB,t} && \textit{under RN measure of numeraire C} \\
&= (r_B - r_A + \rho_{AB,CA}\sigma_{AB}\sigma_{CA} + \sigma_{AB}^2) X_{AB,t} dt + \sigma_{AB} X_{AB,t} dz_{AB,t} && \textit{woo done !!!}
\end{aligned}$$

The reason is :

$$\begin{aligned}
\rho_{AB,AC}\sigma_{AB}\sigma_{AC} &= +Cov(d\ln X_{AB,t}, d\ln X_{AC,t}) \\
&= -Cov(d\ln X_{AB,t}, d\ln X_{CA,t}) \\
&= -\rho_{AB,CA}\sigma_{AB}\sigma_{CA}
\end{aligned}$$

Quanto adjustment factors

Let's see how Black Scholes is adapted to quanto option. For simplicity, consider a *FX* quanto option for case *1* :

$$
\begin{aligned}
E_C[(X_{AB,T}-K)^+] &= (E_C[X_{AB,T}]N(d_1)-KN(d_2))\times DF_C \\
d_{1,2} &= \frac{\ln(E_C[X_{AB,T}]/K)\pm\sigma_{AB}^2T}{\sqrt{\sigma_{AB}^2T}}
\end{aligned}
$$

and
$$
\begin{aligned}
E_C[X_{AB,T}] &= X_{AB,0}e^{(r_B-r_A-\rho_{AB,BC}\sigma_{AB}\sigma_{BC})T} \qquad \textit{where } (r_B-r_A-\rho_{AB,BC}\sigma_{AB}\sigma_{BC}) \textit{ is the drift of A/B numeraire C} \\
&= E_B[X_{AB,T}]e^{-\rho_{AB,BC}\sigma_{AB}\sigma_{BC}T}
\end{aligned}
$$

Therefore what we need to do is simply an adjustment on forward value only. Using similar reasoning, we have :

case 1 $\quad E_C[X_{AB,T}] = E_B[X_{AB,T}]\times e^{-\rho_{AB,BC}\sigma_{AB}\sigma_{BC}T}$ $\quad$ *if B/C is a standard pair, such as EURUSD quantoed JPY*

case 2 $\quad E_C[X_{AB,T}] = E_B[X_{AB,T}]\times e^{+\rho_{AB,CB}\sigma_{AB}\sigma_{CB}T}$ $\quad$ *if C/B is a standard pair, such as EURUSD quantoed AUD*

case 3 $\quad E_C[X_{AB,T}] = E_B[X_{AB,T}]\times e^{(-\rho_{AB,AC}\sigma_{AB}\sigma_{AC}+\sigma_{AB}^2)T}$ $\quad$ *if A/C is a standard pair, such as USDJPY quantoed HKD*

case 4 $\quad E_C[X_{AB,T}] = E_B[X_{AB,T}]\times e^{(+\rho_{AB,CA}\sigma_{AB}\sigma_{CA}+\sigma_{AB}^2)T}$ $\quad$ *if C/A is a standard pair, such as USDJPY quantoed EUR*

There are two more special cases, when *A* = *C* for *case 2* and when *A* = *C* for *case 3*, these are called *case 5* and *6* respectively :

case 5 $\quad$ *when B = USD and A = C = EUR*
$$
\begin{aligned}
E_C[X_{AB,T}] &= E_B[X_{AB,T}]\times e^{+\rho_{AB,CB}\sigma_{AB}\sigma_{CB}T} \\
E_A[X_{AB,T}] &= E_B[X_{AB,T}]\times e^{+\rho_{AB,AB}\sigma_{AB}\sigma_{AB}T} \\
&= E_B[X_{AB,T}]\times e^{+\sigma_{AB}^2T}
\end{aligned}
$$

case 6 $\quad$ *when A = C = USD and B = JPY*
$$
\begin{aligned}
E_C[X_{AB,T}] &= E_B[X_{AB,T}]\times e^{(-\rho_{AB,AC}\sigma_{AB}\sigma_{AC}+\sigma_{AB}^2)T} \\
E_A[X_{AB,T}] &= E_B[X_{AB,T}]\times e^{(-\rho_{AB,AA}\sigma_{AB}\sigma_{AA}+\sigma_{AB}^2)T} \qquad \textit{where } \sigma_{AA}=0 \\
&= E_B[X_{AB,T}]\times e^{+\sigma_{AB}^2T}
\end{aligned}
$$

Both *case 5* and *6* share the same quanto adjustment, which is equivalent to a change to stock numeraire like the one in *FTAP.doc*.

Implementation details

The above adjustment is not precise enough when we consider the following issues :

- *T*+*2* settlement and
- volatility surface, i.e. there are smiles (depending on strike) and term structure (depending on optionality)

Suppose *T* is option maturity, while *T+2* is settlement, then all forward should be found by *T+2*, whereas all optionalities should be found by *T*. Consider case *1*, a precise version of adjustment factor should become the following. Beware that, we must use $E[X_{AB,T+2}]$ and *T* as the *2* dimensional indices in volatility surface.

$$quanto = e^{-\rho_{AB,BC}\times\sigma_{AB}(E_B[X_{AB,T+2}],T)\times\sigma_{BC}(E_C[X_{BC,T+2}],T)\times T}$$

Reference

- *Quanto options, by Wystup, MathFinance, Waldems, Germany.*
- *Foreign exchange, ADR and quanto securities, by Martin Haugh (part 2 is very similar to the content in this doc).*

Recap of our notations

At the beginning, we derive sum and difference of random variables as *Z* = *X*± *Y*, then we put *X*, *Y* and *Z* as currency pairs, like *A*/*B*, *A*/*C* and *B*/*C* etc, and finally, we make some examples by substituting some real currencies into *A*, *B* and *C* such as *USD*, *EUR* etc.

Appendix – Modifications on Quanto option, by Wystup

Modifications on *Wystup* paper so as to make convention being consistent with this document. Firstly replace *EUR* with *JPY*, all the red letters are my modifications. Secondly, $Z = S_t^{(1)} = XAUUSD$, $X = S_t^{(3)} = XAUJPY$, and $Y = S_t^{(2)} = USDJPY$.

1.1 FX Quanto Drift Adjustment

We take the example of a Gold contract with underlying XAU/USD in XAU-USD quotation that is quantoed into *JPY* Since the payoff is in *JPY*, we let *JPY* be the numeraire or domestic or base currency and consider a Black-Scholes model

$$
\begin{aligned}
\text{XAU-}JPY\; dS_t^{(3)} &= (r_{JPY} - r_{XAU}) S_t^{(3)}\, dt + \sigma_3 S_t^{(3)}\, dW_t^{(3)}, && (1)\\
\text{USD-}JPY\; dS_t^{(2)} &= (r_{JPY} - r_{USD}) S_t^{(2)}\, dt + \sigma_2 S_t^{(2)}\, dW_t^{(2)}, && (2)\\
dW_t^{(3)}\, dW_t^{(2)} &= +\rho_{23}\, dt, && (3)
\end{aligned}
$$

where ~~we use a minus sign in front of the correlation~~, because both $S^{(3)}$ and $S^{(2)}$ have the same base currency (DOM), which is *JPY* in this case. The scenario is displayed in Figure 1. The actual underlying is then

$$
\text{XAU-USD: } S_t^{(1)} = \frac{S_t^{(3)}}{S_t^{(2)}}. \qquad (4)
$$

Using Itô's formula, we first obtain

$$
\begin{aligned}
dS_t^{(1)} &= \frac{1}{S_t^{(2)}}\, dS_t^{(3)} + S_t^{(3)}\, d\frac{1}{S_t^{(2)}} + dS_t^{(3)}\, d\frac{1}{S_t^{(2)}}\\
&= \frac{S_t^{(3)}}{S_t^{(2)}}(r_{JPY} - r_{XAU})\, dt + \frac{S_t^{(3)}}{S_t^{(2)}}\sigma_3\, dW_t^{(3)}\\
&\quad + \frac{S_t^{(3)}}{S_t^{(2)}}(r_{USD} - r_{JPY} + \sigma_2^2)\, dt - \frac{S_t^{(3)}}{S_t^{(2)}}\sigma_2\, dW_t^{(2)} + \frac{S_t^{(3)}}{S_t^{(2)}}\rho_{23}\sigma_2\sigma_3\, dt\\
&= (r_{USD} - r_{XAU} + \sigma_2^2 - \rho_{23}\sigma_2\sigma_3) S_t^{(1)}\, dt + S_t^{(1)}(\sigma_3\, dW_t^{(3)} - \sigma_2\, dW_t^{(2)}).
\end{aligned}
$$

Since $S_t^{(1)}$ is a geometric Brownian motion with volatility σ_1, we introduce a new Brownian motion $W_t^{(1)}$ and find

$$
dS_t^{(1)} = (r_{USD} - r_{XAU} + \sigma_2^2 - \rho_{23}\sigma_2\sigma_3) S_t^{(1)}\, dt + \sigma_1 S_t^{(1)}\, dW_t^{(1)}. \qquad (6)
$$

Now Figure 1 and the *law of cosine* imply

since $d\ln S^{(3)} = d\ln S^{(1)} + d\ln S^{(2)}$
hence $V(d\ln S^{(3)}) = V(d\ln S^{(1)}) + V(d\ln S^{(2)}) + 2\rho_{12} \ldots$

$$
\begin{aligned}
\textit{cosine rule +ve version}\quad \sigma_3^2 &= \sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2, && (7)\\
\textit{cosine rule -ve version}\quad \sigma_1^2 &= \sigma_2^2 + \sigma_3^2 - 2\rho_{23}\sigma_2\sigma_3, && (8)
\end{aligned}
$$

since $d\ln S^{(1)} = d\ln S^{(3)} - d\ln S^{(2)}$
hence $V(d\ln S^{(1)}) = V(d\ln S^{(3)}) + V(d\ln S^{(2)}) - 2\rho_{23} \ldots$

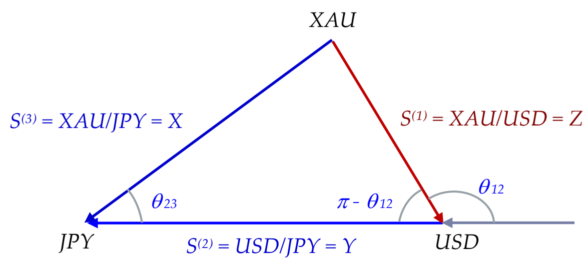
which yields

$$
\sigma_2^2 - \rho_{23}\sigma_2\sigma_3 = -\rho_{12}\sigma_1\sigma_2. \qquad (9)
$$

As explained in the *currency triangle* in Figure 1, ρ_{12} is the correlation between XAU-USD and USD-*JPY* ~~whence $\rho \stackrel{\Delta}{=} -\rho_{12}$ is the correlation between XAU-USD and EUR-USD~~. Inserting this into Equation (6), we obtain the usual formula for the drift adjustment

$$
dS_t^{(1)} = (r_{USD} - r_{XAU} - \rho_{12}\sigma_1\sigma_2) S_t^{(1)}\, dt + \sigma_1 S_t^{(1)}\, dW_t^{(1)}. \qquad (10)
$$

This is the Risk Neutral Pricing process that can be used for the valuation of any derivative depending on $S_t^{(1)}$ which is quantoed into *JPY*



Recall : correlation is cosine of included angle between two arms both pointing outwards.