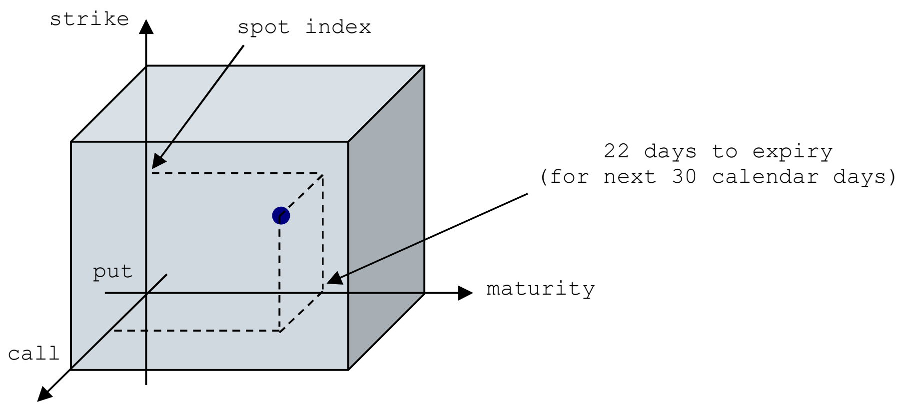
Volatility Index VIX

Old VIX is introduced by Chicago Board Options Exchange (CBOE) in 1993. It is defined as minute by minute snapshot of expected index volatility over the next 30 calendar days (22 trading days). It is simply calculated as a tri-linear interpolation in a three dimensional space, using exactly 8 nearest neighbouring index options in the space. The three dimensions are :

- call and put option (binary space)
- different maturity (real space)
- different strike price (real space)

The 8 nearest neighbouring options are :

- next expiring, nearest in the money, call & put option
- next expiring, nearest out the money, call & put option
- second next expiring, nearest in the money, call & put option
- second next expiring, nearest out the money, call & put option



The VIX is then defined as the weighted sum of the 8 implied volatilities.

$$VIX = \begin{array}{l} w_{N_0,K_0,call}\times\sigma_{N_0,K_0,call}+w_{N'_0,K'_0,put}\times\sigma_{N'_0,K'_0,put}+ \\ w_{N_0,K_1,call}\times\sigma_{N_0,K_1,call}+w_{N'_0,K'_1,put}\times\sigma_{N'_0,K'_1,put}+ \\ w_{N_1,K_0,call}\times\sigma_{N_1,K_0,call}+w_{N'_1,K'_0,put}\times\sigma_{N'_1,K'_0,put}+ \\ w_{N_1,K_1,call}\times\sigma_{N_1,K_1,call}+w_{N'_1,K'_1,put}\times\sigma_{N'_1,K'_1,put} \end{array}$$

where N_0 and N_1 are the number of days to the next and second next expiring index option respectively, such that $N_0 \le 22 \le N_1$, while K_0 and K_1 are the nearest out the money and nearest in the money strike price (for call option). Primed N_0, N_1, K_0, K_1 are for put option.

$$w_{N_0,K_0,call} = \frac{1}{2}\frac{K_1-S}{K_1-K_0}\frac{N_1-22}{N_1-N_0} \qquad w_{N'_0,K'_0,put} = \frac{1}{2}\frac{K'_1-S}{K'_1-K'_0}\frac{N'_1-22}{N'_1-N'_0}$$

$$w_{N_0,K_1,call} = \frac{1}{2}\frac{S-K_0}{K_1-K_0}\frac{N_1-22}{N_1-N_0} \qquad w_{N'_0,K'_1,put} = \frac{1}{2}\frac{S-K'_0}{K'_1-K'_0}\frac{N'_1-22}{N'_1-N'_0}$$

$$w_{N_1,K_0,call} = \frac{1}{2}\frac{K_1-S}{K_1-K_0}\frac{22-N_0}{N_1-N_0} \qquad w_{N'_1,K'_0,put} = \frac{1}{2}\frac{K'_1-S}{K'_1-K'_0}\frac{22-N'_0}{N'_1-N'_0}$$

$$w_{N_1,K_1,call} = \frac{1}{2}\frac{S-K_0}{K_1-K_0}\frac{22-N_0}{N_1-N_0} \qquad w_{N'_1,K'_1,put} = \frac{1}{2}\frac{S-K'_0}{K'_1-K'_0}\frac{22-N'_0}{N'_1-N'_0}$$

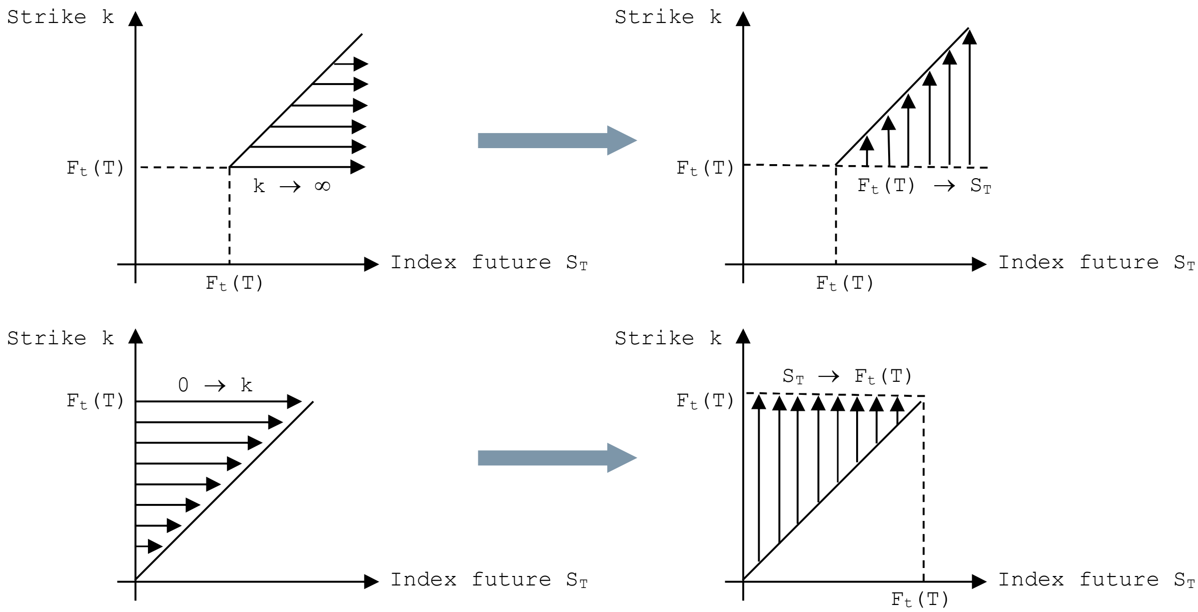
As underlying spot index changes, the 8 nearest neighbouring index option change. Since 2003, CBOE introduced the new VIX, which includes two major changes : (1) estimation is done with a wider range of index option (not just 8 neighbourhood), (2) estimation is done without involving solving implied volatility using Newton Raphson method. To calculate $VIX_t(T)$, where t is current time, T is future time (in years), lets consider a portfolio consisting of all "expected" out the money index call option and "expected" out the money index put option (all with maturity T), it is simply a linear combination, with weighting inverse proportional to square of strike. S_t and S_T denote spot index and future index , $F_t(T)$ is current expected future index, while $C_t(k,T)$ and $P_t(k,T)$ denote continuous spectrum of call and put option with strike price k and maturity T.

$$G_t(T) = \int_{F_t(T)}^{\infty} C_t(k,T)/k^2 dk + \int_0^{F_t(T)} P_t(k,T)/k^2 dk$$

$$C_t(k,T) \;=\; e^{-r(T-t)}\int_0^{\infty}(S_T-k)^+\,pdf(S_T)dS_T \;=\; e^{-r(T-t)}\int_k^{\infty}(S_T-k)\,pdf(S_T)dS_T$$
$$P_t(k,T) \;=\; e^{-r(T-t)}\int_0^{\infty}(k-S_T)^+\,pdf(S_T)dS_T \;=\; e^{-r(T-t)}\int_0^{k}(k-S_T)\,pdf(S_T)dS_T$$

Putting into the portfolio and change the order of double integration :

$$\begin{aligned}G_t(T) &= e^{-r(T-t)}\left[\left(\int_{F_t(T)}^{\infty}\tfrac{1}{k^2}\int_k^{\infty}(S_T-k)\,pdf(S_T)dS_T dk\right)+\left(\int_0^{F_t(T)}\tfrac{1}{k^2}\int_0^{k}(k-S_T)\,pdf(S_T)dS_T dk\right)\right]\\ &= e^{-r(T-t)}\left[\left(\int_{F_t(T)}^{\infty}pdf(S_T)\int_{F_t(T)}^{S_T}(S_T-k)/k^2\,dk dS_T\right)+\left(\int_0^{F_t(T)}pdf(S_T)\int_{S_T}^{F_t(T)}(k-S_T)/k^2\,dk dS_T\right)\right]\end{aligned}$$



Then we consider both inner integrations :

$$\int_{F_t(T)}^{S_T}(S_T-k)/k^2\,dk \;=\; \int_{F_t(T)}^{S_T}(S_T/k^2-1/k)\,dk \;=\; [-S_T/k-\ln k]_{F_t(T)}^{S_T}$$
$$\int_{S_T}^{F_t(T)}(k-S_T)/k^2\,dk \;=\; \int_{S_T}^{F_t(T)}(1/k-S_T/k^2)\,dk \;=\; [\ln k+S_T/k]_{S_T}^{F_t(T)}$$

and they are the same. Hence, we have :

$$\begin{aligned}G_t(T) &= e^{-r(T-t)}\left[\left(\int_{F_t(T)}^{\infty}pdf(S_T)[\ln k+S_T/k]_{S_T}^{F_t(T)}dS_T\right)+\left(\int_0^{F_t(T)}pdf(S_T)[\ln k+S_T/k]_{S_T}^{F_t(T)}dS_T\right)\right]\\ &= e^{-r(T-t)}\left[\int_0^{\infty}pdf(S_T)(\ln F_t(T)-\ln S_T+S_T/F_t(T)-1)dS_T\right]\\ &= e^{-r(T-t)}\left[\ln F_t(T)-1+\left(\int_0^{\infty}pdf(S_T)S_T dS_T\right)/F_t(T)-\left(\int_0^{\infty}pdf(S_T)\ln S_T dS_T\right)\right]\\ &= e^{-r(T-t)}\left[\ln F_t(T)-1+E(S_T)/F_t(T)-E(\ln S_T)\right]\end{aligned}$$

Expected index future and expected log index future are :

$$\begin{aligned}S_T &= S_t\exp((r-\sigma^2/2)(T-t)+\sigma Z_{T-t}) \;=\; \exp(\varepsilon(\ln S_t+(r-\sigma^2/2)(T-t),\sigma\sqrt{T-t})) \;=\; \exp(\varepsilon(M,S))\\ E(S_T) &= \exp(M+S^2/2) \;=\; \exp(\ln S_t+(r-\sigma^2/2)(T-t)+\sigma^2(T-t)/2) \;=\; S_t\exp(r(T-t))\\ E(\ln S_T) &= M \;=\; \ln S_t+(r-\sigma^2/2)(T-t)\end{aligned}$$

and thus we have :

$$
\begin{aligned}
G_t(T) &= e^{-r(T-t)}[\ln F_t(T) - 1 + S_t \exp(r(T-t))/F_t(T) - \ln S_t - (r-\sigma^2/2)(T-t)] \\
&= e^{-r(T-t)}[\ln F_t(T)/S_t - 1 + S_t \exp(r(T-t))/F_t(T) - (r-\sigma^2/2)(T-t)] \\
&= e^{-r(T-t)}[r(T-t) - 1 + 1 - (r-\sigma^2/2)(T-t)] \\
&= e^{-r(T-t)}[\sigma^2(T-t)/2]
\end{aligned}
$$

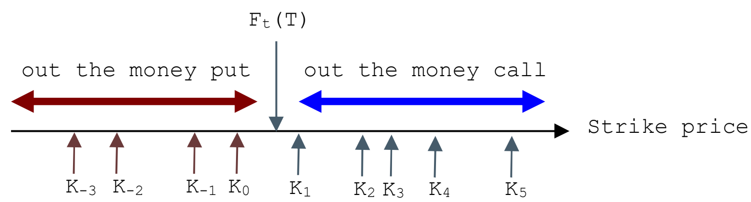
Theoretically, market index future is the same as theoretical index future, however they may be different slightly in reality. For VIX, in practice, we choose $F_t(T)$ to be theoretical index future (rather than direct using market index future), i.e. $F_t(T) = S_t(T)\exp(r(T-t))$.

$$
\begin{aligned}
\sigma^2 &= \frac{2e^{r(T-t)}}{T-t} G_t(T) \\
&= \frac{2e^{r(T-t)}}{T-t}\left[\int_{F_t(T)}^{\infty} C_t(k,T)/k^2 dk + \int_0^{F_t(T)} P_t(k,T)/k^2 dk\right] \\
&= \frac{2e^{r(T-t)}}{T-t}\sum_i Q_t(k_i,T)\Delta k_i / k_i^2 \qquad \text{where } Q_t(k_i,T) = \begin{bmatrix} C_t(k_i,T) & for & F_t(T) < k_i \\ P_t(k_i,T) & for & F_t(T) > k_i \end{bmatrix}
\end{aligned}
$$

For preserving continuity of VIX, a second term is added, please see remark [2].

$$
\sigma^2 = \frac{2e^{r(T-t)}}{T-t}\sum_i Q_t(k_i,T)\Delta k_i / k_i^2 - \frac{(F_t(T)/K_0 - 1)^2}{T-t} \qquad \text{where } \Delta k_i = \frac{k_{i+1} - k_{i-1}}{2}
$$

where K_0 is first strike below index future $F_t(T)$. Please be noted that the spectrum of strikes are not necessarily uniformly distributed. Besides, we defined $C_t(k,T)$ and $P_t(k,T)$ as the mid point between best bid and best ask in the corresponding option order book, rather than the most recently traded price.



However, it is usually impossible to find option with maturity exactly equal to T, hence we need to do interpolation in maturity time domain. Suppose T is next 30 calendar days, the next expiring option is called near term, the one after that is called next term, we then do an interpolation between near term volatility and next term volatility. Read remark 1 for the minor difference between T and N.

$$VIX_{30} = 100\sigma_{30}$$

$$
\begin{aligned}
\sigma_{30}^2 T_{30} &= w_{near}(\sigma_{near}^2 T_{near}) + w_{next}(\sigma_{next}^2 T_{next}) \\
w_{near} &= (N_{next} - N_{30})/(N_{next} - N_{near}) \\
w_{next} &= (N_{30} - N_{near})/(N_{next} - N_{near})
\end{aligned}
$$

T_{near} and N_{near} denotes time in year and number of days to 1st next expiring option
T_{next} and N_{next} denotes time in year and number of days to 2nd next expiring option
T_{30} and N_{30} denotes 30/365 in year and 30 days

Questions
[1] Why consider "expected" out the money option only?
[2] Why add weighting inversely proportional to strike?

Reference

[1] The Old VIX vs New VIX, Mark Loffe, Egar Tech http://www.egartech.com/docs/the_old_vix_vs_New.pdf

Remark

[1] For this document, T denotes time in years, N denotes number of days.

[2] When $F_t(T) > K_0$, all out the money call and out the money put are counted.

$$\sigma^2 = \frac{2e^{r(T-t)}}{T-t}\sum_i Q_t(k_i,T)\Delta k_i / k_i^2$$

$$= \frac{2e^{r(T-t)}}{T-t}\begin{bmatrix} & +P_t(k_0,T)(k_{+1}-k_{-1})/2k_0^2 \\ +C_t(k_1,T)(k_2-k_0)/2k_1^2 & +P_t(k_{-1},T)(k_0-k_{-2})/2k_{-1}^2 \\ +C_t(k_2,T)(k_3-k_1)/2k_2^2 & +P_t(k_{-2},T)(k_{-1}-k_{-3})/2k_{-2}^2 \\ +C_t(k_3,T)(k_4-k_2)/2k_3^2 & +P_t(k_{-3},T)(k_{-2}-k_{-4})/2k_{-3}^2 \\ \ldots & \ldots \end{bmatrix}$$

When $F_t(T) = K_0$, all at/out the money call and out the money put are counted.

$$\sigma^2 = \frac{2e^{r(T-t)}}{T-t}\sum_i Q_t(k_i,T)\Delta k_i / k_i^2$$

$$= \frac{2e^{r(T-t)}}{T-t}\begin{bmatrix} +C_t(k_0,T)(k_1-k_0)/2k_0^2 & +P_t(k_0,T)(k_{+1}-k_{-1})/2k_0^2 \\ +C_t(k_1,T)(k_2-k_0)/2k_1^2 & +P_t(k_{-1},T)(k_0-k_{-2})/2k_{-1}^2 \\ +C_t(k_2,T)(k_3-k_1)/2k_2^2 & +P_t(k_{-2},T)(k_{-1}-k_{-3})/2k_{-2}^2 \\ +C_t(k_3,T)(k_4-k_2)/2k_3^2 & +P_t(k_{-3},T)(k_{-2}-k_{-4})/2k_{-3}^2 \\ \ldots & \ldots \end{bmatrix}$$

Hence, we add a term to make sure VIX is continuous as $F_t(T) > K_0$ approaches to K_0.

$$\sigma^2(F_t(T)) = \frac{2e^{r(T-t)}}{T-t}\sum_i Q_t(k_i,T)\Delta k_i / k_i^2 + term \qquad \text{s.t.} \qquad \lim_{F_t(T)\to K_0} \sigma^2(F_t(T)) = \sigma^2(F_t(T) = K_0)$$

Now, we have to find the term. If term = 0 when $F_t(T) = K_0$, then

$$term = \frac{2e^{r(T-t)}}{T-t}\frac{C_t(k_0,T)(k_1-k_0)}{2k_0^2}$$

$$= \frac{2e^{r(T-t)}}{T-t}\frac{e^{-r(T-t)}E[(S_T-k_0)^+](k_1-k_0)}{2k_0^2}$$

$$\sim \frac{1}{T-t}\frac{(E(S_T)-k_0)(k_1-k_0)}{k_0^2} \qquad \text{(why?)}$$

$$\sim \frac{1}{T-t}\frac{(F_t(T)-k_0)^2}{k_0^2} \qquad \text{(why?)}$$

= missing a minus sign (you need to review practical calculation of VIX)