From volatility term structure to local and stochastic volatility
by Dr Ken Yan

Dr Yan's derivative pricing framework

It is composed of : (1) deal term, (2) market data, (3) model and pricing engine. Suppose x_t is a stochastic underlying variable which is also known as the reference variable, then a deal's price is :

$$
\begin{aligned}
price &= DF \times \hat{E}[f(x_t) \mid I_t] && I_t \textit{ is market data, from which DF curve is constructed and prob model is calibrated} \\
&= DF \times \int \overbrace{f(x_t)}^{dealterm} dprob(x_t \mid I_t) \\
&= DF \times \underbrace{\int f(x) pdf(x \mid para_{opt}) dx}_{pricing-engine} && para_{opt} = \overbrace{\arg\min_{para}}^{calibrate}(\textstyle\sum_n error(vanillaoption_n(para), marketquote_n))
\end{aligned}
$$

Black Scholes revision

Black Scholes equation is derived in two steps : define stock model and risk neutral pricing of derivatives. The first step involves (1) defining stochastic process for stock price, (2) solving the differential equation, and as a result obtaining the stock price distribution. This distribution is critical, as it will be used in the calculation of risk neutral expectation of an instrument's payoff, different stock models (example, constant volatility model, local volatility model and stochastic volatility model) will result in different stock price distributions, and hence different instrument prices. Let's start with the simplest stock model, with the drift term containing minus dividend rate (minus is used, as part of the stock value goes to the dividend payment, i.e. value of a share = stock price + dividend).

$$
\begin{aligned}
dS_t &= (r-g)S_t dt + \sigma S_t dZ_t \\
r &= interest\ rate \\
g &= dividend\ rate
\end{aligned}
$$

Solve the differential equation for stock price, it will be a random variable (don't forget Ito's lemma) :

$$
\begin{aligned}
d\ln S_t &= (1/S_t)dS_t - 1/2(1/S_t)^2(dS_t)^2 \\
&= (1/S_t)((r-g)S_t dt + \sigma S_t dZ_t) - 1/2(1/S_t)^2((r-g)S_t dt + \sigma S_t dZ_t)^2 \\
&= (1/S_t)((r-g)S_t dt + \sigma S_t dZ_t) - 1/2(1/S_t)^2 \sigma^2 S_t^2 dt && since\ \ ((r-g)S_t dt + \sigma S_t dZ_t)^2 = \sigma^2 S_t^2 dt \\
&= (r-g-\sigma^2/2)dt + \sigma dZ_t \\
[\ln S_t]_0^t &= (r-g-\sigma^2/2)\int_0^t dt + \sigma \int_0^t dZ_t \\
&= (r-g-\sigma^2/2)[t]_0^t + \sigma [Z_t]_0^t \\
S_t &= S_0 \exp((r-g-\sigma^2/2)t + \sigma Z_t)
\end{aligned}
$$

we consider time $0 \rightarrow t$ for stock

we consider time $t \rightarrow T$ for derivatives

Therefore, stock price is a log-normal random variable :

$$
\begin{aligned}
\ln(S_t/S_0) &\sim \varepsilon((r-g-\sigma^2/2)t, \sigma\sqrt{t}) \\
d\ln S_t &\sim \varepsilon((r-g-\sigma^2/2)dt, \sigma\sqrt{dt})
\end{aligned}
$$

Now lets go to step two : risk neutral pricing, the price of an equity option given current price of underlying can be derived as :

$$
\begin{aligned}
\hat{E}[(S_T-K)^+ \mid I_t] &= \hat{E}[(S_t \exp((r-g-\sigma^2/2)(T-t) + \sigma Z_{T-t}) - K)^+] \\
&= \hat{E}[(S_t \exp((r-g-\sigma^2/2)(T-t) + \sigma\sqrt{T-t}\,x) - K)^+] && where\ \ x \sim \varepsilon(0,1) \\
&= \frac{1}{\sqrt{2\pi}} \int_d^\infty (S_t \exp((r-g-\sigma^2/2)(T-t) + \sigma\sqrt{T-t}\,x) - K) e^{-x^2/2} dx \ = \ \ldots && perform\ completing\ square
\end{aligned}
$$

where one of the critical part is to identify d when payoff equals to zero :

$$
\begin{aligned}
K &= S_t \exp((r-g-\sigma^2/2)(T-t) + \sigma\sqrt{T-t}\,d) \\
d &= \frac{\ln(K/S_t) - (r-g-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} && both\ d_{1,2}\ look\ like\ but\ not\ equal\ to\ d
\end{aligned}
$$

Volatility term structure

Here comes a question, why are there rate and volatility term structure? This is because a single constant rate and a single constant volatility cannot explain the prices of forward contracts and option contracts prevailing in the market. Implied rates and volatilities must be ***tenor-dependent*** so as to be consistent with market data. Therefore, when we price forwards and options having tenors not lying on term structures' nodes, interpolation must kick in. Once the required rate and volatility are interpolated, they then become constant inputs for Black Scholes equation, or constant inputs for numerical methods like tree or Monte Carlo. These are known as term-rate (zero rate) and term volatility, as opposed to what we are going to introduced : instantaneous rate and instantaneous vol.

Instantaneous volatility

However, it is not reasonable for options with different tenors, but sharing the same underlying, to have different volatilities, as vol is a parameter modeling the underlying process regardless of the options. In order to justify the tenor-dependent volatility, we then introduce the concept of instantaneous volatility, which is deterministic but non-observable. Use of instantaneous volatility implies introducing infinite number of parameters, so instantaneous volatility must be modelled with limited number of parameters. There are two types of model (1) parametric model like Dupire, which is a local volatility model, or Heston, which is a stochastic volatility model and (2) non parametric model, interpolation of values of nodes in the term structure, in fact, interpolation can be regarded as parameterization with values of the node set. The idea is to model the non-observable instantaneous volatility with limited number of parameters or node values, which are tuned so as to construct some observable quantities (known as implied quotes in QuantLib, for example, implied swap rate, implied swap point, implied vol etc) being consistent with market data. If consistent tuning can be done, the rate and volatility term structures are said to be explaint by the model parameters or the interpolation node sets. Tuning process for parametric model is known as calibration, while tuning process for interpolation node value is known as bootstrapping. Lets look at bootstrapping first, the discussion of model calibration is delayed until Dupire and Heston sections. Here is the revised stock model with instantaneous rate and volatility :

$$\begin{aligned}
dS_t &= (r_{d,t}-r_{f,t})S_t dt+\sigma_t S_t dZ_t && \textit{note the time index t instead of index T, now lets solve it}\\
d\ln S_t &= (1/S_t)dS_t - 1/2(1/S_t)^2(dS_t)^2 \\
&= \dots && \textit{follow exactly the same steps as constant rate and vol model}\\
&= (r_{d,t}-r_{f,t}-\sigma_t^2/2)dt+\sigma_t dZ_t \\
[\ln S_\tau]_0^t &= \int_0^t (r_{d,\tau}-r_{f,\tau})d\tau - 1/2\int_0^t \sigma_\tau^2 d\tau + \underbrace{\int_0^t \sigma_t dZ_t}_{Ito's} && \textit{Ito's integral : } \int_0^t \sigma_\tau dZ_\tau \sim \varepsilon\left(0, stdev=\sqrt{\int_0^t \sigma_\tau^2 d\tau}\right)
\end{aligned}$$

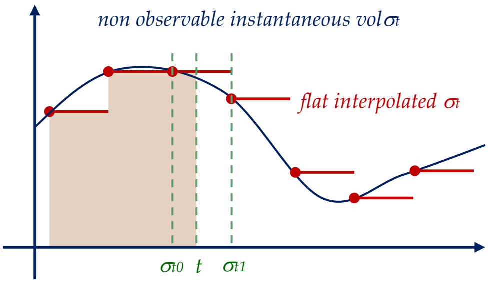
The Ito's integral is a linear combination of Gaussian with deterministic weights, the sum is a zero mean Gaussian. Therefore, stock model with instantaneous rate and vol does still follow log-normal distribution. Lets put it side by side with the constant vol model. We have to use different notations to denote zero rate and term vol (so as to differentiate from the instantaneous counterparts).

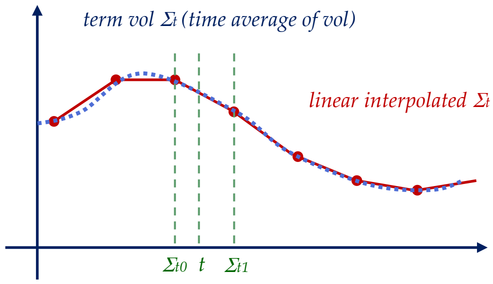
$$\begin{aligned}
S_t &= S_0 \exp((R_{d,t}-R_{f,t})t - \Sigma_t^2 t/2 + \Sigma_t\sqrt{t}\times\varepsilon) && \textit{constant rate and vol model}\\
S_t &= S_0 \exp\left(\int_0^t (r_{d,\tau}-r_{f,\tau})d\tau - 1/2\int_0^t \sigma_\tau^2 d\tau + \sqrt{\int_0^t \sigma_\tau^2 d\tau}\times\varepsilon\right) && \textit{instantaneous rate and vol model, where } \varepsilon \textit{ is unit normal}\\
\textit{where}\quad R_{d,t}, R_{f,t} &= \frac{1}{t}\int_0^t r_{d,\tau}d\tau \ \text{ and } \ \frac{1}{t}\int_0^t r_{f,\tau}d\tau && \textit{known as term rate or zero rate}\\
\Sigma_t &= \sqrt{\frac{1}{t}\int_0^t \sigma_\tau^2 d\tau} && \textit{known as term volatility}
\end{aligned}$$

For model calibration, the random process of rate and volatility are usually considered together, in a single unified model, whereas for interpolation, bootstrapping of rate and bootstrapping of volatility can be done separately. Bootstrapping of rate can be done by interpolation either in discount factor, zero rate or instantaneous rate domain, while bootstrapping of volatility can be done in term volatility or instantaneous volatility. Term rate (zero rate) and term volatility are simply time average of instantaneous counterpart. Interpolation in term rate R and vol Σ is different from interpolation in instantaneous rate r and vol σ. Note that flat interpolation of instantaneous variance is equivalent to linear interpolation of term variance (note variance is defined as volatility square multiplied by time). Let's prove, by consider the case when σ or σ^2 as a step function (i.e. flat interpolation of instantaneous variance) :

$$\begin{aligned}
w_0(\Sigma_{t_0}^2 t_0)+w_1(\Sigma_{t_1}^2 t_1) &= w_0\left(\frac{1}{t_0}\int_0^{t_0}\sigma_s^2 ds\times t_0\right)+w_1\left(\frac{1}{t_1}\int_0^{t_1}\sigma_s^2 ds\times t_1\right) && \textit{where } w_0=(t_1-t)/(t_1-t_0) \text{ and } w_1=(t-t_0)/(t_1-t_0)\\
&= \int_0^{t_0} w_0\sigma_s^2 ds + \int_0^{t_1} w_1\sigma_s^2 ds \\
&= \int_0^{t_0}(w_0+w_1)\sigma_s^2 ds + \int_{t_0}^{t_1} w_1\sigma_s^2 ds \\
&= \int_0^{t_0}\sigma_s^2 ds + \int_{t_0}^{t_1} w_1\sigma_s^2 ds && \textit{where } w_0+w_1=1\\
&= \int_0^{t_0}\sigma_s^2 ds + w_1(t_1-t_0)\sigma_{t_0}^2
\end{aligned}$$

$$= \int_0^{t_0} \sigma_s^2 ds + (t - t_0)\sigma_{t_0}^2 \qquad \text{since } w_1 = (t - t_0)/(t_1 - t_0)$$

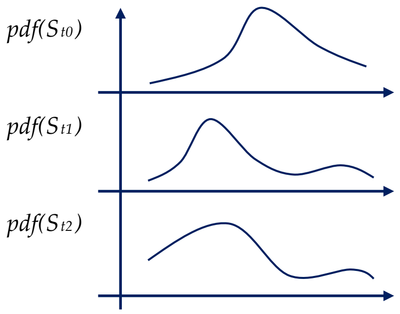




Various interpolations (flat or linear, in volatility domain or variance domain, term or instantaneous) are all feasible in mathematics perspective, yet the optimal choice is a trading decision. In conclusion, we have introduced the abstract idea of instantaneous rate& volatility, from which observable quantities can be constructed, as a result, underlying model or term structure nodes can be tuned to match with prevailing quotes observable in market. The tuning is called calibration (for model) or bootstrapping (for nodes).

Stochastic volatility

How about the strike dimension in option volatility matrix? Market price of options with the same tenor but different strikes imply different volatilities, there is a skew in volatility on strike axis (skew on K), therefore instantaneous volatility cannot explain market behaviour completely, that's why we introduce a model, such that the distribution of S_t is time dependent, i.e. a term structure of S_t distribution, such as the stochastic volatility in Heston model.



Model is an abstraction of prevailing market data (rate&vol).

Lets go back to stock model, suppose we are using Heston model, now σ is stochastic :

$$S_t = S_0 \exp\left(\int_0^t (r_{d,\tau} - r_{f,\tau}) d\tau - 1/2 \int_0^t \sigma_\tau^2 d\tau + \underbrace{\int_0^t \sigma_\tau dZ_\tau}_{Ito's}\right)$$

This time, the integrand inside Ito's integral is stochastic, the Ito's integral is no longer a Gaussian, yet we can still find out its mean, which is zero, and its variance. As Ito's integral is no longer normal, S_t itself is then no longer log-normal. Yes, this is what we need, we are looking for a skew on K axis. Volatility smile tries to explain the skew as the result of option moneyness, however this is not convincing as underlying's volatility has nothing to do with derivatives. In short, conversion from market volatility surface to local volatility can be summarised as :

$$\sigma(T, K) \to \sigma(S_t, t)$$

the former is contract term dependent, whereas the latter is simply underlying dependent.