Taylor Series

Taylor series for single variable (real or complex)

Proof 1 : Given function $f : C^1 \to C^1$, which can be expressed as a infinite series sum.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{where } x \in C^1$$

Differentiate both sides recursively, we have :

$$\begin{aligned}
df(x)/dx &= \sum_{n=1}^{\infty} a_n n (x-x_0)^{n-1} \\
d^2 f(x)/dx^2 &= \sum_{n=2}^{\infty} a_n n(n-1)(x-x_0)^{n-2} \\
d^3 f(x)/dx^3 &= \sum_{n=3}^{\infty} a_n n(n-1)(n-2)(x-x_0)^{n-3} \\
d^4 f(x)/dx^4 &= \sum_{n=4}^{\infty} a_n n(n-1)(n-2)(n-3)(x-x_0)^{n-4} \\
\ldots &= \ldots \\
d^m f(x)/dx^m &= \sum_{n=m}^{\infty} a_n \frac{n!}{(n-m)!}(x-x_0)^{n-m}
\end{aligned}$$

Substituting x = x_0 in each of the above, then only $(x\text{-}x_0)^0$ term remains.

$$\begin{aligned}
f(x_0) &= a_0(x_0-x_0)^0 + a_1(x_0-x_0)^1 + +a_2(x_0-x_0)^2 + \ldots &&= a_0 \\
df(x_0)/dx &= \frac{1!}{0!}a_1(x_0-x_0)^0 + \frac{2!}{1!}a_2(x_0-x_0)^1 + \frac{3!}{2!}a_3(x_0-x_0)^2 + \ldots &&= \frac{1!}{0!}a_1 \\
d^2 f(x_0)/dx^2 &= \frac{2!}{0!}a_2(x_0-x_0)^0 + \frac{3!}{1!}a_3(x_0-x_0)^1 + \frac{4!}{2!}!a_4(x_0-x_0)^2 + \ldots &&= \frac{2!}{0!}a_2 \\
d^3 f(x_0)/dx^3 &= \frac{3!}{0!}a_3(x_0-x_0)^0 + \frac{4!}{1!}a_4(x_0-x_0)^1 + \frac{5!}{2!}!a_5(x_0-x_0)^2 + \ldots &&= \frac{3!}{0!}a_3 \\
d^4 f(x_0)/dx^4 &= \frac{4!}{0!}a_4(x_0-x_0)^0 + \frac{5!}{1!}a_5(x_0-x_0)^1 + \frac{6!}{2!}!a_6(x_0-x_0)^2 + \ldots &&= \frac{4!}{0!}a_4 \\
\ldots &= \ldots \\
d^m f(x_0)/dx^m &= \frac{m!}{0!}a_m(x_0-x_0)^0 + \frac{(m+1)!}{1!}a_{m+1}(x_0-x_0)^1 + \frac{(m+2)!}{2!}!a_{m+2}(x_0-x_0)^2 + \ldots &&= \frac{m!}{0!}a_m
\end{aligned}$$

Hence we have :

$$a_n = (d^n f(x_0)/dx^n)/n!$$

$\Rightarrow \quad f(x) = \sum_{n=0}^{\infty}(d^n f(x_0)/dx^n)(x-x_0)^n/n!$ ***Taylor series***

$\Rightarrow \quad df(x) = \sum_{n=1}^{\infty}(d^n f(x)/dx^n)(dx)^n/n!$ ***Taylor series in differential form***

$\Rightarrow \quad f(x) = \sum_{n=0}^{\infty}(d^n f(x_0)/dx^n)x^n/n!$ ***Maclaurin series*** (i.e. Taylor series anchored at zero)

Taylor series is sometimes diverging or converges slowly, thus we use **rational function approximation** (also known as the **Pade approximant**) instead.

Proof 2 : Given function $f : C^1 \to C^1$

$$\begin{aligned}
f(x+\Delta x) &= f(x) + \int_x^{x+\Delta x} f'(y)dy && \text{(eq 1)} \\
&= f(x) + [f'(y)y]_x^{x+\Delta x} - \int_x^{x+\Delta x} y df'(y) && \text{(integration by parts)} \\
&= f(x) + [f'(y)y]_x^{x+\Delta x} - \int_x^{x+\Delta x} y f''(y)dy \\
&= f(x) - f'(x)(x) + f'(x+\Delta x)(x+\Delta x) - \int_x^{x+\Delta x} y f''(y)dy \\
&= f(x) - f'(x)(x) + f'(x)(x+\Delta x) - f'(x)(x+\Delta x) + f'(x+\Delta x)(x+\Delta x) - \int_x^{x+\Delta x} y f''(y)dy \\
&= f(x) + f'(x)\Delta x + (x+\Delta x)\int_x^{x+\Delta x} f''(y)dy - \int_x^{x+\Delta x} y f''(y)dy \\
&= f(x) + f'(x)\Delta x + \int_x^{x+\Delta x}(x+\Delta x - y)f''(y)dy && \text{(eq 2)} \\
&= f(x) + f'(x)\Delta x + 1/2 f''(x)(\Delta x)^2 + 1/2\int_x^{x+\Delta x}(x+\Delta x-y)^2 f'''(y)dy && \text{(applied eq 1} \Rightarrow \text{eq 2 again)} \\
&= f(x) + f'(x)\Delta x + 1/2 f''(x)(\Delta x)^2 + 1/3! f'''(x)(\Delta x)^3 + \ldots && \text{(applied eq 1} \Rightarrow \text{eq 2 again)} \\
&\quad \ldots + 1/m! f^{(m)}(x)(\Delta x)^m + 1/m!\int_x^{x+\Delta x}(x+\Delta x-y)^m f^{(m+1)}(y)dy && \text{(residue term is obtained)}
\end{aligned}$$

Taylor series for multiple variables (real or complex)

Taylor series for two variables in **differential form** :

$$
\begin{aligned}
df(x,y) &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{C_m^n}{n!}\frac{\partial f^n(x,y)}{\partial x^m \partial y^{n-m}}(dx)^m(dy)^{n-m} \\
&\quad \frac{\partial f(x,y)}{\partial x}dx+\frac{\partial f(x,y)}{\partial y}dy+ \\
&= \frac{1}{2}\frac{\partial^2 f(x,y)}{\partial x^2}(dx)^2+\frac{\partial^2 f(x,y)}{\partial x\partial y}dxdy+\frac{1}{2}\frac{\partial^2 f(x,y)}{\partial y^2}(dy)^2+ \\
&\quad \frac{1}{6}\frac{\partial^3 f(x,y)}{\partial x^3}(dx)^3+\frac{1}{2}\frac{\partial^3 f(x,y)}{\partial x^2\partial y}(dx)^2dy+\frac{1}{2}\frac{\partial^3 f(x,y)}{\partial x\partial y^2}dx(dy)^2+\frac{1}{6}\frac{\partial^3 f(x,y)}{\partial y^3}(dy)^3+\ldots
\end{aligned}
$$

Taylor series for three variables in **differential form** :

$$
\begin{aligned}
df(x,y) &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{n}\frac{C_m^n C_l^n}{n!}\frac{\partial f^n(x,y,z)}{\partial x^m\partial y^l\partial z^{n-m-l}}(dx)^m(dy)^l(dz)^{n-m-l} \qquad (f \text{ is used instead of } f(x,y,z) \text{ for simplicity}) \\
&\quad \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz+ \\
&= \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(dx)^2+\frac{\partial^2 f}{\partial y^2}(dy)^2+\frac{\partial^2}{\partial z^2}(dz)^2\right]+\left[\frac{\partial^2 f}{\partial x\partial y}dxdy+\frac{\partial^2 f}{\partial y\partial z}dydz+\frac{\partial^2 f}{\partial z\partial x}dzdx\right]+ \\
&\quad \frac{1}{6}\left[\frac{\partial^3 f}{\partial x^3}(dx)^3+\frac{\partial^3 f}{\partial y^3}(dy)^3+\frac{\partial^3 f}{\partial z^3}(dz)^3\right]+\frac{1}{2}\left[\begin{matrix}\frac{\partial^3 f}{\partial x^2\partial y}(dx)^2dy+\frac{\partial^3 f}{\partial x^2\partial z}(dx)^2dz+\frac{\partial^3 f}{\partial y^2\partial z}(dy)^2dz+ \\ \frac{\partial^3 f}{\partial y^2\partial x}(dy)^2dx+\frac{\partial^3 f}{\partial z^2\partial x}(dz)^2dx+\frac{\partial^3 f}{\partial z^2\partial y}(dz)^2dy\end{matrix}\right]+\ldots
\end{aligned}
$$

Taylor series for K variables in **differential form** :

$$
df(x,y) = \sum_{n=1}^{\infty}\sum_{N_1=0}^{n}\sum_{N_2=0}^{n}\cdots\sum_{N_{K-1}=0}^{n}\frac{C_{N_1}^n C_{N_2}^n\ldots C_{N_{K-1}}^n}{n!}\frac{\partial f^n(x_1,x_2,\ldots,x_K)}{\partial x_1^{N_1}\partial x_2^{N_2}\ldots\partial x_{K-1}^{N_{K-1}}\partial x_K^{N_K}}(dx_1)^{N_1}(dx_2)^{N_2}\ldots(dx_{K-1})^{N_{K-1}}(dx_K)^{N_K}
$$

where $\quad n = N_1+N_2+N_3+\ldots+N_{K-1}+N_K$

Taylor series for K variables in **vector form** (or matrix form) :

x	$=$	$[x_1,x_2,x_3,\ldots,x_K]^T$	(column matrix)
Δx	$=$	$[\Delta x_1,\Delta x_2,\Delta x_3,\ldots,\Delta x_K]^T$	(column matrix)
∇	$=$	$[\partial_{x_1},\partial_{x_2},\partial_{x_3},\ldots,\partial_{x_K}]^T$	(column matrix : Gradient operator)
$\nabla f(x)$	$=$	$[\partial_{x_1}f(x),\partial_{x_2}f(x),\partial_{x_3}f(x),\ldots,\partial_{x_K}f(x)]^T$	(column matrix : Jacobian)
$\nabla^2 f(x)$	$=$	$\nabla(\nabla f(x))^T \quad = \quad [\partial_{x_i}\partial_{x_j}f(x)]_{i,j\in[1,K]}$	($K\times K$ matrix : Hessian)
$\nabla\cdot\nabla f(x)$	$=$	$\nabla^T(\nabla f(x)) \quad = \quad \sum_{i=1}^{K}(\partial_{x_i}\partial_{x_i}f(x))$	(scalar : Laplacian operator)
$\nabla\cdot\nabla f(x)$	$\neq$	$\nabla^2 f(x)$	

Taylor series :

$$
\begin{aligned}
f(x+\Delta x) &= f(x)+J^T\Delta x+1/2\,(\Delta x)^T H\Delta x+\ldots \\
&= f(x)+(\nabla f(x))^T\Delta x+1/2\,(\Delta x)^T(\nabla^2 f(x))\Delta x+\ldots
\end{aligned}
$$

Taylor series for linear motion with constant acceleration

Lets consider linear motion with constant acceleration, i.e. $a_t = a$, with initial condition x_0 and $dv/dt\big|_{t=0} = v_0$, we have :

$$
\begin{aligned}
x_t &= x_0 + \frac{dx}{dt}t + \frac{1}{2}\frac{d^2x}{dt^2}t^2 &&= x_0 + v_0 t + \frac{1}{2}at^2 && (1) \\
v_t &= v_0 + \frac{dv}{dt}t &&= v_0 + at && (2) \\
v_t^2 &= v_0^2 + 2v_0\frac{dv}{dt}t + \left(\frac{dv}{dt}t\right)^2 &&= v_0^2 + \left(2v_0 + \frac{dv}{dt}t\right)\frac{dv}{dt}t && \text{from (2)} \\
& &&= v_0^2 + (2v_0 t + at^2)a && \text{from (1)} \\
& &&= v_0^2 + 2(x_t - x_0)a && (3)
\end{aligned}
$$

the three fundalmental formula in linear motion easily derived from Taylor series.