Heston Model – Lewis approach *from scratch*

Introduction
There are multiple approaches to Heston model, with slightly different closed-form solutions.

year	author	methodology
1999	Carr and Madan	Fourier transform of vanilla price with damping
2000	Bakshi and Madan	Moneyness and change of measure
2001	Lewis	Fourier transform of option payoff (more generic than Carr)

<u>Carr and Madan approach</u>
- Option valuation using the fast Fourier transform, chapter 3
  paper written by Peter Carr and Dilip Madan.

<u>Bakshi and Madan</u>
- The Heston model and its extensions in Matlab and C#, chapter 1
  book written by Fabrice Douglas Rouah, Wiley.

<u>Lewis approach</u>
- Option pricing within the Heston model, chapter 4
  master thesis written by Alma Dögg Helgadóttir, Aarhus University.

- Option Pricing via the FFT and its application to calibration, chapter 5
  master thesis written by Man Wo Ng, Delft University of Technology.

Framework for derivative pricing - Why Fourier transform?
Derivative pricing (from model calibration with market data to pricing of exotic derivatives) consists of the following steps :

(1) define underlying dynamics by stochastic differential equation *SDE*, solve for *PDF* of underlying random variable
(2) derive derivative dynamics by partial differential equation *PDE* (somehow making use of *SDE* in step 1)
(3) plug vanilla option payoff intro *PDE*, solve for analytic vanilla option price
(4) with underlying PDF, calculate price of any exotic derivatives using tree or Monte Carlo

Step 1 defines the underlying model using *SDE*, possible models include Black Scholes, Heston, Bates, Hull White, underlying *PDF* <u>may</u> then be solved. Step 2 is derived by dynamic hedging and the construction of risk free portfolio, it governs the dynamic of any contingent claims. Step 3 gives an analytic solution for vanilla option price in terms of model parameters, it makes the calibration of model parameters to market data possible, as the analytic solution *f(parameter, payoff, exercise)* is fast. Step 4 prices exotic derivatives numerically, we usually pick tree if the exotic derivatives can be early exercised (backward propagation in tree allows observations of expected price in the future), we usually pick simulation if the exotic derivatives is path dependent (forward propagation in path during simulation allows observations of simulated price in the past).

The problem with step1 is that, it is often that analytical distribution *PDF* of underlying random variable S_t does not exist, however analytic characteristic function of the same random variable S_t is usually available, this is why Fourier transform becomes an useful approach for exotic derivative pricing.

<u>Physical meaning of Fourier transform</u>
Fourier transform can be regarded as the breakdown of a time domain function into frequency components (breakdown smoothies into recipes), the strength of each component is calculated as the inner product between the function and a complex sinusoids with corresponding frequency. Inner product between 2 vectors measures the projection of one vector on another, it is maximum when 2 vectors are identical or linearly dependent (recall Cauchy Schwarz inequality $|x \cdot y| < |x||y|$). Similarly, the inner product between 2 time functions measures the projection of one on another (as a function can be sampled to form a vector of infinite length).

$$<u,v> \; = \; \int u(t)v(t)dt \qquad \textit{definition of inner product}$$

$$<u,v>^2 \; \leq \; <u,u>\cdot<v,v> \qquad \textit{Cauchy Schwarz inequality, equality holds if } v = ku, \textit{ i.e. linearly dependent}$$

Complex analysis basic

Here is a brief revision of conjugates. For $x \in \Re$, $z \in C$, whereas f, g are functions : $\Re \to C$, then we have :

$$
\begin{aligned}
\overline{z_1 z_2} &= \overline{(z_{1r}+iz_{1i})(z_{2r}+iz_{2i})} \\
&= \overline{z_{1r}z_{2r}+iz_{1r}z_{2i}+iz_{1i}z_{2r}+z_{1i}z_{2i}} \\
&= z_{1r}z_{2r}-iz_{1r}z_{2i}-iz_{1i}z_{2r}+z_{1i}z_{2i} \\
&= (z_{1r}-iz_{1i})(z_{2r}-iz_{2i}) \\
&= \overline{z_1}\cdot\overline{z_2} \\
\overline{z^n} &= \overline{zz...z} \\
&= \overline{z}\cdot\overline{zz...z} \\
&= \overline{z}\cdot\overline{z}\cdot\overline{z...z} \\
&= \overline{z}^n
\end{aligned}
$$

$$
\begin{aligned}
e^{\overline{z}} &= e^{z_r}e^{-iz_i} \\
&= e^{z_r}(\cos z_i - i\sin z_i) \\
&= e^{z_r}\overline{(\cos z_i + i\sin z_i)} \\
&= \overline{e^{z_r}e^{iz_i}} \\
&= \overline{e^{z}} \\
k^{\overline{z}} &= e^{\ln k^{\overline{z}}} && \forall k \in \Re \\
&= e^{\overline{z}\ln k} \\
&= e^{\overline{z\ln k}} && \textit{since } k \in \Re \\
&= \overline{e^{z\ln k}} && \textit{since } e^{\overline{x}} = \overline{e^{x}} \\
&= \overline{e^{\ln k^{z}}} && = \overline{k^{z}}
\end{aligned}
$$

$$
\begin{aligned}
\overline{f(x)+g(x)} &= \overline{f_r(x)+if_i(x)+g_r(x)+ig_i(x)} \\
&= f_r(x)-if_i(x)+g_r(x)-ig_i(x) \\
&= \overline{f(x)}+\overline{g(x)} \\
&\neq \overline{f(\overline{x})}+\overline{g(\overline{x})} \\
\textit{where } f_r(x) &= \mathrm{Re}(f(x)) \in \Re \\
\textit{and } f_i(x) &= \mathrm{Im}(f(x)) \in \Re \\
\overline{f(x)g(x)} &= \overline{f(x)}\cdot\overline{g(x)} \\
&\neq \overline{f(\overline{x})}\cdot\overline{g(\overline{x})}
\end{aligned}
$$

$$
\begin{aligned}
\overline{\int f(x)dx} &= \lim_{\Delta x \to 0}\overline{\sum f(x)\Delta x} \\
&= \lim_{\Delta x \to 0}\sum \overline{f(x)}\Delta x = \int \overline{f(x)}dx
\end{aligned}
$$

All above proofs have to be rewritten if f,g are functions : $C \to C$.

Fourier transform is defined slightly different from the one in signal and system (the damping will be explaint later) :

Fourier in quant finance	*Fourier in signal and system*	
		different in notation
$\tilde{f}(z) = \int_{-\infty}^{+\infty} f(x)e^{izx}dx$	$F(w) = \int_{-\infty}^{+\infty} f(x)e^{-iwx}dx$	*different in exp's sign*
$f(x) = \frac{1}{2\pi}\int_{-\infty+iz_i}^{+\infty+iz_i}\tilde{f}(z)e^{-izx}dz$	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(w)e^{+iwx}dw$	*different in integration range*

The damped Fourier transform :

$$\tilde{f}(z)e^{z_i x} = \tilde{f}(z)e^{-i(iz_i)x}$$

i.e. $\int_{-\infty}^{\infty}|\tilde{f}(z)e^{z_i x}|dx < \infty$ *is square integrable within strip* $S = \{z = z_r + iz_i \in C : a < z_i < b\}$

<u>Characteristic function</u>

Characteristic function of random variable X is defined as :

$$
\begin{aligned}
\Phi_X(z) &\equiv E[e^{izX}] \\
&= \int_{-\infty}^{+\infty} p_X(x)e^{izx}dx && \textit{where } p_X(x) \textit{ is pdf of } X \\
&= \tilde{p}_X(z) && \textit{hence characteristic function of X is the Fourier transform of X's pdf}
\end{aligned}
$$

Characteristic function of linearly transformed random variable $Y = aX+b$:

$$
\begin{aligned}
\Phi_Y(z) &= \int_{-\infty}^{+\infty} p_Y(y)e^{izy}dy \\
&= \int_{-\infty}^{+\infty}\underbrace{(p_X(x)/a)}_{remark}e^{iz(ax+b)}d(ax) \\
&= \int_{-\infty}^{+\infty} p_X(x)e^{izax}e^{izb}dx \\
&= e^{izb}\int_{-\infty}^{+\infty} p_X(x)e^{i(za)x}dx \\
&= e^{izb}\Phi_X(az)
\end{aligned}
$$

Remark :

$$
\begin{aligned}
\Pr(Y<y) &= \Pr(X<x) \\
\partial_y \Pr(Y<y) &= \partial_y \Pr(X<x) \\
\Rightarrow \quad p_Y(y) &= \partial_x \Pr(X<x)\partial_x y \\
&= p_X(x)/a
\end{aligned}
$$

If X is log price at maturity T, Y is log normalized price at maturity T, then we have :

$$
\begin{aligned}
X &= \ln S_T && \textit{log underlying price} \\
Y &= \ln(S_T/\underbrace{S_0 e^{(r-q)T}}_{forward}) = \ln(S_T/F) = \underbrace{1}_{a}\cdot X \underbrace{-\ln F}_{b} && \textit{log underlying price, normalized with forward F} \\
\Phi_Y(z) &= e^{-iz\ln F}\Phi_X(z) \\
\textit{or} \quad \Phi_X(z) &= e^{+iz\ln F}\Phi_Y(z)
\end{aligned}
$$

Hermitian

Hermitian function is a complex function such that :

$$\overline{f(z)} = f(-z)$$

Please don't confuse with the following :

$$\overline{f(z)} \neq f(\overline{z})$$

To avoid confusion, we may write : $\bar{f}(z)$ vs $f(\bar{z})$

For example, given :

$$\begin{aligned} & f(z) &=& e^{izx} && \text{where } x \in C \\ \Rightarrow\ & \overline{f(z)} &=& e^{-iz\cdot\bar{x}} && \overline{f(z)} \text{ means applying conjugate to inside of } f(z)\text{, but not to } z \\ \Rightarrow\ & \overline{f(\bar{z})} &=& e^{-i\bar{z}\cdot\bar{x}} \end{aligned}$$

Show that characteristic function is Hermitian :

$$\begin{aligned} \overline{\Phi_X(z)} &= \overline{E[e^{izx}]} \\ &= \overline{\int_{-\infty}^{+\infty} p(x)e^{izx}dx} \\ &= \int_{-\infty}^{+\infty} p(x)\overline{e^{izx}}dx && \text{Probability } p(x) \text{ must be real.} \\ &= \int_{-\infty}^{+\infty} p(x)e^{-izx}dx && \text{We suppose underlying random variable X is real.} \\ &= E[e^{-izx}] \\ &= \Phi_X(-z) \end{aligned}$$

Show that the following function is also Hermitian (this result will be used in Lewis's proof) :

$$\begin{aligned} f(z) &= \Phi_X(-z)\cdot\overbrace{\frac{k^{1+iz}}{z^2-iz}}^{A} && \text{As we will see later, A is the Fourier transform of covered call payoff.} \\ \overline{f(z)} &= \overline{\Phi_X(-z)}\cdot\overline{\left(\frac{k^{1+iz}}{z^2-iz}\right)} \\ &= \Phi_X(z)\cdot\frac{k^{1-iz}}{z^2+iz} && \text{since characteristic function is Hermitian} \\ &= f(-z) && \text{hence } f(z) \text{ is Hermitian} \end{aligned}$$

What is the implication of a Hermitian function?

$$\begin{aligned} \text{If}\quad & \overline{f(z)} &=& f(-z) \\ \Rightarrow\ & \overline{f_r(z)+if_i(z)} &=& f_r(-z)+if_i(-z) \\ \Rightarrow\ & f_r(z)-if_i(z) &=& f_r(-z)+if_i(-z) && \text{where } f_r() \text{ and } f_i() \text{ are both real-valued functions} \\ \Rightarrow\ & \begin{bmatrix} f_r(z) \\ f_i(z) \end{bmatrix} &=& \begin{matrix} f_r(-z) \\ -f_i(-z) \end{matrix} && \text{for all } z\text{, therefore } f \text{ has even real part and odd imaginary part} \end{aligned}$$

Lets consider the integration of a Hermitian function. Suppose we have a Hermitan function $f:\Re\to C$

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x)dx &= \left[\int_{-\infty}^{0} f_r(x)dx+\int_0^{\infty} f_r(x)dx\right]+i\left[\int_{-\infty}^{0} f_i(x)dx+\int_0^{\infty} f_i(x)dx\right] \\ &= \left[-\int_{\infty}^{0} f_r(-y)dy+\int_0^{\infty} f_r(x)dx\right]+i\left[-\int_{\infty}^{0} f_i(-y)dy+\int_0^{\infty} f_i(x)dx\right] && \text{putting } x=-y \\ &= \left[-\int_{\infty}^{0} f_r(y)dy+\int_0^{\infty} f_r(x)dx\right]+i\left[\int_{\infty}^{0} f_i(y)dy+\int_0^{\infty} f_i(x)dx\right] && \text{for Hermitian, } f_r \text{ is even while } f_i \text{ is odd} \\ &= \left[\int_0^{\infty} f_r(y)dy+\int_0^{\infty} f_r(x)dx\right]+i\left[-\int_0^{\infty} f_i(y)dy+\int_0^{\infty} f_i(x)dx\right] \\ &= 2\int_0^{\infty} f_r(x)dx \\ &= 2\int_{-\infty}^{0} f_r(x)dx && \text{via similar argument} \end{aligned}$$

Two theorems for Fourier Transform

Inverse theorem and Parseval theorem are involved in option pricing using Fourier transform, inverse theorem is applied in Bakshi Madan approach, while Parseval theorem is applied in Carr Madan approach. Inverse theorem connects characteristic function of a distribution with its *cdf*, Parseval theorem relates dot product of two functions in time domain to dot product of the same functions in frequency domain. Please note that these two theorems are not involved in Lewis approach.

<u>Inverse theorem – for Bakshi Madan</u>

Inverse theorem recovers cdf from characteristic function via inverse Fourier transform :

$$P_X(x_1) - P_X(x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-izx_0} - e^{-izx_1}}{iz}\Phi_X(z)dz$$

identical to "Note on inverse theorem", by GilPelaez

As a result, *pdf*, *cdf* and characteristic function of a distribution are related by the following triangle.

$$\begin{tikzcd}
P_X(x) \arrow[rr, "\text{differentiation wrt } x"] & & p_X(x) \arrow[dl, "\int_{-\infty}^{+\infty} p_X(x)e^{izx}dx"] \\
& \Phi_X(z) \arrow[ul, "\text{inverse theorem}"] &
\end{tikzcd}$$

Proof of inverse theorem starts with considering $x_0 < x_1$:

Can we repeat this proof without x_1?

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-izx_0} - e^{-izx_1}}{iz}\Phi_X(z)dz &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-izx_0} - e^{-izx_1}}{iz}E_X[e^{izx}]dz && \textit{then reverse order of 2 integrations} \\
&= \frac{1}{2\pi}E_X\left[\int_{-\infty}^{\infty} \frac{e^{-izx_0} - e^{-izx_1}}{iz}e^{izx}dz\right] \\
&= \frac{1}{2\pi}E_X\left[\int_{-\infty}^{\infty} \frac{e^{iz(x-x_0)} - e^{iz(x-x_1)}}{iz}dz\right] \\
\int_{-\infty}^{\infty} \frac{e^{iz(x-x_0)} - e^{iz(x-x_1)}}{iz}dz &= \int_{-\infty}^{0} \frac{e^{iz(x-x_0)} - e^{iz(x-x_1)}}{iz}dz + \int_{0}^{\infty} \frac{e^{iz(x-x_0)} - e^{iz(x-x_1)}}{iz}dz \\
&= -\int_{0}^{\infty} \frac{e^{-iz'(x-x_0)} - e^{-iz'(x-x_1)}}{iz'}dz' + \int_{0}^{\infty} \frac{e^{iz(x-x_0)} - e^{iz(x-x_1)}}{iz}dz && \textit{put } z'=-z \\
&= \int_{0}^{\infty} \frac{2i\sin(z(x-x_0)) - 2i\sin(z(x-x_1))}{iz}dz && \textit{as } e^{i\theta} - e^{-i\theta} = 2i\sin\theta \\
&= 2\left[\int_{0}^{\infty} \frac{\sin(z(x-x_0))}{z}dz - \int_{0}^{\infty} \frac{\sin(z(x-x_0))}{z}dz\right] && \textit{as } \mathrm{sgn}(x) = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin(zx)}{z}dz \\
&= \pi(\mathrm{sgn}(x-x_0) - \mathrm{sgn}(x-x_1)) && \textit{where } \mathrm{sgn}(x) = \begin{bmatrix} -1 & if & x<0 \\ 0 & if & x=0 \\ +1 & if & x>0 \end{bmatrix} \\
\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-izx_0} - e^{-izx_1}}{iz}\Phi_X(z)dz &= E_X[(\mathrm{sgn}(x-x_0) - \mathrm{sgn}(x-x_1))]/2 \\
&= E_X[2\times 1_{x\in(x_0,x_1)}]/2 \\
&= \Pr_X(x\in(x_0,x_1)) \\
&= P_X(x_1) - P_X(x_0) && \textit{hence proved}
\end{aligned}$$

The above inverse theorem is identical to that in most statistic references, including "Note on inverse theorem" by Gil Pelaez. Yet in quant finance, most references state inverse theorem in a slightly different version. The major difference is that Gil Pelaez's version involves two bounds, which are x_0 and x_1, while Bakshi Madan's version does not involve x_1.

$$\begin{aligned}
\Pr_X(x > x_0) &= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-izx_0}}{iz}\Phi_X(z)dz && \textit{recall that } P_X(x_0) = \Pr_X(x < x_0)\textit{, beware > and <} \\
&= \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} \mathrm{Re}[\frac{e^{-izx_0}}{iz}\Phi_X(z)]dz && \textit{since characteristic function must be Hermitian}
\end{aligned}$$

A formal proof from Gil Pelaez version to quant finance version involve contour integration in complex plane, please refer to thesis "Four generations of asset pricing models and volatility dynamics", by Sascha Desmettre, in page 63-66 for the proof. As it involves complex analysis, we will not go through it here, let's take it in face value for now. It is wrong to remove x_1 like the following!!!

$$\begin{aligned}
\Pr_X(x > x_0) &= P_X(\infty) - P_X(x_0) \\
&= \cancel{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-izx_0} - e^{-iz\infty}}{iz}\Phi_X(z)dz} \\
&= \cancel{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-izx_0}}{iz}\Phi_X(z)dz} && \textit{wrong, contour integral is needed}
\end{aligned}$$

Therefore *ITM* probability can be found using characteristic function in two approaches : (1) by integrating *cdf*, which is the inverse Fourier transform of characteristic function and (2) application of inverse theorem. *Why most Heston materials prefer the latter?*

$$\begin{aligned}
\Pr_X(x > x_0) &= \int_{x_0}^{\infty}[\overbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-izx}\Phi_X(z)dz}^{p_X(x)=FT^{-1}[\Phi_X(z)]}]dx && \textit{approach 1 : integrating cdf} \\
\Pr_X(x > x_0) &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \text{Re}[\frac{e^{-izx_0}}{iz}\Phi_X(z)]dz && \textit{approach 2 : inverse theorem}
\end{aligned}$$

Parseval theorem – for Carr Madan

Parseval theorem is about inner product in price domain and in frequency domain :

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(z)\overline{\tilde{g}(z)}dz$$

The proof involves inverse Fourier transform of delta function $\delta(x-x_0) = (1/2\pi)\int_{-\infty}^{\infty} e^{iz(x-x_0)}dz$ (please read *Dirac Delta.doc*) :

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx &= \int_{-\infty}^{\infty}(\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(z)e^{-izx}dz)\overline{(\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(z)e^{-izx}dz)}dx && \textit{plug in inverse Fourier transform of f and g} \\
&= \int_{-\infty}^{\infty}(\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(z)e^{-izx}dz)(\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\tilde{g}(z')}e^{+iz'x}dz')dx \\
&= \frac{1}{2\pi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(z)\overline{\tilde{g}(z)}(\int_{-\infty}^{\infty}e^{i(z'-z)x}dx)dzdz' && \textit{we have delta function } \delta(x-x_0) = (1/2\pi)\int_{-\infty}^{\infty} e^{iz(x-x_0)}dz \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(z)\overline{\tilde{g}(z)}\delta(z'-z)dzdz' && \textit{reverse x/z, we have } \delta(z'-z) = (1/2\pi)\int_{-\infty}^{\infty} e^{i(z'-z)x}dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(z)(\int_{-\infty}^{\infty}\overline{\tilde{g}(z)}\delta(z'-z)dz')dz \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(z)\overline{\tilde{g}(z)}dz && \textit{by definition of delta } \int_{-\infty}^{\infty} h(x)\delta(x-x_0)dx = h(x_0)
\end{aligned}$$

Derivative pricing

In both Black Scholes and Heston, derivatives are functions of current time t,underlying S_t and volatility v_t (for Heston only), where S_t and v_t are filtrations at time t, in other words, ***derivative prices*** depend only on current state (not on any historical states), so they are Markovian. Besides, according to *FTAP*, ***numeraire deflatted derivative prices*** are martingale under risk neutral measure.

Derivative *PDE* from underlying *SDE*

In stochastic calculus, we never take derivative *wrt* Brownian motion z_t, we take derivative *wrt* values of current state only, i.e.

$$df(t, S_t, v_t) = \partial_t f dt + \partial_s f dS_t + \partial_v f dv_t + \frac{1}{2}(\partial_{ss} f(dS_t)^2 + 2\partial_{sv} f(dS_t)(dv_t) + \partial_{vv} f(dv_t)^2) + ...$$

where S_t and v_t are random processes defined on Brownian motion z_t. We never treat it the following way :

$$\cancel{df(t, S(t, z_{1t}), v(t, z_{2t})) = \partial_t f dt + \partial_{z1} f dz_{1t} + \partial_{z2} f dz_{2t} + \frac{1}{2}(\partial_{z1z1} f(dz_{1t})^2 + 2\partial_{z1z2} f(dz_{1t})(dz_{2t}) + \partial_{z2z2} f(dz_{2t})^2) + ...}$$

Formally, we should define function as : $f(t, S_t = s, v_t = v)$, it has 3 inputs, the first one is time, the other two are random processes.

Black Scholes vs Heston

Black Scholes starts with deriving *PDE* modelling the dynamics of contingent claim making use of *SDE* of Black Scholes model, this *PDE* can be solved analytically if we plug in the payoff function of a vanilla option (or a barrier option, a lookback option, an Asian option) as boundary conditions. However in Heston, it is not easy to solve the *PDE*, instead we have to go for the Fourier transform approach, it does not mean the *PDE* becomes redundant, it is still critical to solving the characteristic function of underlying (or log underlying) on maturity.

Black Scholes model

$$dS_t = (r-q)S_t dt + \sigma S_t dz_t$$

Heston model

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{v_t}\, S_t dz_{1t} \\ dv_t &= \kappa(\vartheta - v_t)dt + \sigma\sqrt{v_t}\, dz_{2t} \\ dz_{1t}dz_{2t} &= \rho dt \end{aligned}$$

(with init instantaneous volatility v_0)

Black Scholes PDE

$$rV_t = \partial_t V_t + (r-q)S_t(\partial_S V_t) + \frac{1}{2}\sigma^2 S_t^2(\partial_S \partial_S V_t)$$

i.e. $$rV_t = theta + (r-q)S_t(delta) + \frac{1}{2}\sigma^2 S_t^2(gamma)$$

1 factor	dz_t	*2 factors*	dz_{1t}, dz_{2t}
1 parameter	σ	*5 parameters*	$v_0, \kappa, \vartheta, \sigma, \rho$
2 PDE variables	t, S_t	*3 PDE variables*	t, S_t, v_t
3 PDE Greeks	*1st order : theta, delta* *2nd order : gamma*	*6 PDE Greeks*	*1st order : theta, delta, vega* *2nd order : gamma, vanna, volga*
market data	$S_0, r, q, vol-matrix$	*market data*	S_0, r, q *(vol-matrix is absorbed by the 5 parameters)*

Counting factors, parameters and variables

Black Scholes is a 1-factor model, whereas Heston is a 2-factor model. Here factor means risk factor or the source of randomness, i.e. dz_t in Black Scholes, dz_{1t} and dz_{2t} in Heston. In terms of model calibration, Black Scholes is a single parameter model (i.e. volatility σ, strictly speaking, there is no model calibration but vol-surface interpolation in Black Scholes), while Heston is a 5 parameters model (i.e. initial volatility v_0, mean reverting rate κ, long run volatility ϑ, vol-of-vol σ and correlation between the two randomness ρ, we need to solve for all 5 parameters through calibration to market data). Finally, in *PDE* perspective, Black Scholes has 2 independent variables (i.e. time t and underlying S_t, as Black Scholes dynamics is governed by time derivatives theta and underlying derivatives delta & gamma), whereas Heston has 3 independent variables (i.e. time t, log underlying x_t, instantaneous volatility v_t, since Heston dynamics is governed by time derivatives theta, log underlying derivatives delta & gamma and instantaneous volatility derivatives vega, vanna & volga). Can we call these independent variables (t, x_t, v_t) endogenous, while other variables (payoff K, exercise T and term structure r&g) exogenous? Both *BSPDE* and Heston *PDE* are equations that link all relevant Greeks together. There is no vega, vanna and volga in *BSPDE*, because by assumption, Black Scholes model does not capture volatility-related risk factor.

BSPDE for dividend paying underlying

Please read "Deriving the Black Scholes PDE for a dividend paying underlying using a hedging portfolio", by Ophir Gottlieb. It is a generic version that adapts to dividend paying underlying, the result is useful in deriving Heston *PDE* later. Suppose S_t, Q_t & V_t are underlying, accumulated dividend and option respectively, the riskless portfolio is :

$$\Pi_t = -V_t + \Delta_t S_t + Q_t$$ *where Q_t is accumulated dividend from time 0*

where $$Q_t = \int_0^t \Delta_\tau S_\tau q_\tau d\tau$$ *where q_t is dividend rate per share of underlying*

hence $$dQ_t = \Delta_t S_t q_t dt$$ *where $q_t dt$ is dividend per share of underlying*

Recall $$dS_t = (r_t - q_t)S_t dt + \sigma_t S_t dz_t$$ *by Black Scholes model*

and $$dV_t = \partial_t V_t dt + \partial_S V_t dS_t + \frac{1}{2}\partial_{SS}V_t(dS_t)^2$$ *by Ito's lemma*

Therefore delta change in portfolio is :

$$\begin{aligned} d\Pi_t &= -dV_t + \Delta_t dS_t + dQ_t \\ &= -\partial_t V_t dt - \partial_S V_t dS_t - \frac{1}{2}\partial_{SS}V_t(dS_t)^2 + \Delta_t dS_t + dQ_t \\ &= -\partial_t V_t dt - \partial_S V_t((r_t - q_t)S_t dt + \sigma_t S_t dz_t) - \frac{1}{2}\partial_{SS}V_t(\sigma_t^2 S_t^2 dt) + \Delta_t dS_t + \Delta_t S_t q_t dt \end{aligned}$$

$=$ *group dt term and dS_t term, then remove the latter by picking suitable Δ_t, our aim is to derive BSPDE*

Heston basic 1 – Feller's square root condition

Stochastic volatility in Heston model is proved (read *"A short remark on Feller's square root condition"* by Ilya Gikhman) to be positive for all time according to Feller's condition, which states, given :

$$dv_t = \kappa(\vartheta - v_t)dt + \sigma\sqrt{v_t}\,dz_t$$

where v_0, κ, θ are positive, then $v_t > 0$, for all $t \in [0,\infty)$ if :

$$\sigma^2 < 2\kappa\vartheta$$

Heston basic 2 – Change of measure from P to Q_B

Heston model in physical measure should be *(please note that Heston model contains a variance process, not a volatility process)* :

$$\begin{aligned} dS_t &= (\mu - q)S_t dt + \sqrt{v_t}\,S_t dz_{1t} && \textit{price process} \\ dv_t &= \kappa(\vartheta - v_t)dt + \sigma\sqrt{v_t}\,dz_{2t} && \textit{variance process} \\ dz_{1t}dz_{2t} &= \rho dt \end{aligned}$$

Yet we need Heston model under risk neutral measure with cash as numeraire. We create a new stochastic process z_{1t} *tilde*, which is drifted version of z_{1t}, according to Girsanov theorem there exists measure Q_B under which the new process is Brownian, while cash-deflatted price process becomes martingale under measure Q_B (i.e. price process has the same drift as cash numeraire, which is r_t-q_t). The theorem does also tell us the corresponding Radon Nikodym derivative, yet it is irrelevant here.

$$\begin{aligned} & d\tilde{z}_{1t} = \lambda_{1t}dt + dz_{1t} \\ \textit{and} \quad & dS_t = (\mu - q)S_t dt + \sqrt{v_t}\,S_t dz_{1t} = (r-q)S_t dt + \sqrt{v_t}\,S_t d\tilde{z}_{1t} \\ & (\mu - q)S_t dt + \sqrt{v_t}\,S_t dz_{1t} = (r-q)S_t dt + \sqrt{v_t}\,S_t(\lambda_{1t}dt + dz_{1t}) \\ & \lambda_{1t} = (\mu - r)/\sqrt{v_t} \qquad \textit{which is called market price of risk} \end{aligned}$$

Similarly for variance process, we create a new stochastic process z_{2t} *tilde*, also according to Girsanov theorem, there exists measure Q_B under which the new process is Brownian, whereas drift of the variance process is shifted by Λ_t which is known as volatility risk premium (volatility risk means risk-of-risk). Please note that, unlike the price process, we do not simply replace the drift by r_t-q_t.

$$\begin{aligned} & d\tilde{z}_{2t} = \lambda_{2t}dt + dz_{2t} \\ \textit{and} \quad & dv_t = \kappa(\vartheta - v_t)dt + \sigma\sqrt{v_t}\,dz_{2t} = (\kappa(\vartheta - v_t) - \Lambda_t)dt + \sigma\sqrt{v_t}\,d\tilde{z}_{2t} \\ & \kappa(\vartheta - v_t)dt + \sigma\sqrt{v_t}\,dz_{2t} = (\kappa(\vartheta - v_t) - \Lambda_t)dt + \sigma\sqrt{v_t}\,(\lambda_{2t}dt + dz_{2t}) \\ & \lambda_{2t} = \frac{\Lambda_t dt}{\sigma\sqrt{v_t}} \end{aligned}$$

According to the references I read there is no clue whether λ_{1t} and λ_{2t} are the same or different, please note that λ_{1t} is market price of risk, where Λ_t (not λ_{2t}) is volatility risk premium. The consumption model in "An intertemporal asset pricing model with stochastic consumption and investment opportunities", by Breeden in 1979, postulates that $\Lambda_t = \Lambda \times v_t$, we can therefore define κ^* θ^* to ***absorb*** volatility risk premium. Note that volatility risk premium is just another name for market price of volatility risk.

$$\begin{aligned} dv_t &= (\kappa(\vartheta - v_t) - \Lambda_t))dt + \sigma\sqrt{v_t}\,d\tilde{z}_{2t} \equiv \kappa^*(\vartheta^* - v_t)dt + \sigma\sqrt{v_t}\,d\tilde{z}_{2t} && \textit{for all } v_t \\ & (\kappa(\vartheta - v_t) - \Lambda v_t))dt + \sigma\sqrt{v_t}\,d\tilde{z}_{2t} \equiv \kappa^*(\vartheta^* - v_t)dt + \sigma\sqrt{v_t}\,d\tilde{z}_{2t} && \textit{assuming that } \Lambda_t = \Lambda \times v_t \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & (\kappa + \Lambda_t)v_t = \kappa^* v_t \quad \textit{and} \quad \kappa\vartheta = \kappa^*\vartheta^* \\ \Rightarrow \quad & \kappa^* = \kappa + \Lambda_t \quad \textit{and} \quad \vartheta^* = \kappa\vartheta/\kappa^* = \kappa\vartheta/(\kappa + \Lambda_t) \end{aligned}$$

Finally we have Heston model under risk neutral measure with cash as numeraire :

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{v_t}\,S_t d\tilde{z}_{1t} \\ dv_t &= \kappa^*(\vartheta^* - v_t)dt + \sigma\sqrt{v_t}\,d\tilde{z}_{2t} \\ d\tilde{z}_{1t}d\tilde{z}_{2t} &= (\lambda_{1t}dt + dz_{1t})(\lambda_{2t}dt + dz_{2t}) = dz_{1t}dz_{2t} = \rho dt \end{aligned}$$

We may will omit the *tilde* in the following discussions.

Lewis approach

The idea of Lewis approach can be summarised as follows. First of all, price of any contingent claim can be proved to be equivalent to the inverse Fourier transform of the product between Fourier transform of payoff in log price domain and characteristic function of log price at maturity. Secondly, we go for the Fourier transform of payoff, yet there are two stuffs that make maths easier : (1) the payoff function is written in ***log price*** domain and (2) the payoff of a covered call is used instead of a vanilla call. Covered call is an option strategy which shorts a vanilla put and long cash that amounts to the discounted strike. After that, we don't immediately go for the characteristic function, instead we substitute the Fourier transform of covered call payoff to result of step 1, and simplify the call option equation, which is frequency-domain integration of the characteristic function with damping factor and frequency shift. With this equation, we can price vanilla option for any model, as long as the corresponding characteristic function is available. The third step is derivation of the characteristic function of ***log price at maturity***, yet it is a more complicated step which can be divided into substeps, (*3a*) derivation of Heston *PDE* via construction of a riskless portfolio with dynamic hedging, like what we have done for *BSPDE*, then transform it into forward *PDE*, (*3b*) make an ansatz for the forward *PDE*, then breakdown the three variables (τ, x_t, v_t) *PDE* into two *ODE*s, each of which contains the time derivative only, thus making it easy to solve, (*3c*,*d*) solve the two *ODE*s, the former applies Riccati equation technique. Finally, we can put the pieces together. Equations after main steps are labelled.

Summary of Lewis approach

step 1 *Fourier pricing formula for all payoffs and all models*

step 2 *Fourier pricing formula for vanilla payoff and all models*

- *derive Fourier transform of covered call*
- *derive Fourier pricing formula by plugging in Fourier transform of covered call payoff*

step 3 *characteristic function of log price at maturity*

- *derive Heston PDE, transform into PDE of forward option in terms of log forward underlying*
- *breakdown Heston PDE into two ODEs by plugging in an ansatz*
- *solve ODE2 (it is a Riccati equation)*
- *solve ODE1*

step 4 *putting the pieces together, derive an analytic solution for Heston*
hence by replacing characteristic function, we obtain solutions for other models

Step 1 – Fourier pricing formula for all payoffs and all models

In quant finance we use tilde ~ instead of capital letter to represent Fourier transform. Price domain is analogous to time domain in signal and system. Now given a contingent claim with payoff $f(S_T = s)$ traded at time 0, which will be matured at time T, its value at current time is t can be found with risk neutral pricing as follows (i.e. $0 < t < T$ are 3 timepoints).

$$
\begin{aligned}
V_t &= e^{-r(T-t)}\hat{E}[f(x)] \\
&= \frac{e^{-r(T-t)}}{2\pi}\hat{E}\left[\int_{-\infty+iz_i}^{+\infty+iz_i}\tilde{f}(z)e^{-izx}dz\right] && \textit{replace payoff by } FT^{-1} \textit{ of its Fourier transform} \\
&= \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty+iz_i}^{+\infty+iz_i}\tilde{f}(z)\hat{E}[e^{-izx}]dz \\
&= \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty+iz_i}^{+\infty+iz_i}\tilde{f}(z)\Phi_X(-z)dz && \textit{characteristic function of } X \textit{ depends on underlying model SDE (equation 1)} \\
&= \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty+iz_i}^{+\infty+iz_i}\tilde{f}(z)\overline{\Phi_X(z)}dz && \textit{characteristic function of } X \textit{ is Hermitian}
\end{aligned}
$$

where

$$
\begin{aligned}
X_t &= \ln S_t && X_t \textit{ is deterministic, } X_T \textit{ is random (abbreviated as X, its value is x)} \\
X'_t &= \ln e^{(r-q)(T-t)}S_t && X'_t \textit{ is deterministic, } X'_T \textit{ is random (abbreviated as X', its value is x')} \\
\Phi_X(z) &= E[e^{izX}] = \tilde{p}_X(z) && \textit{characteristic function of } X_T \\
V_t &= f(S_t) + \textit{timevalue} && V_t \textit{ stands for option value, } f \textit{ stands for payoff, they are different, until on } T \\
V_T &= f(S_T) && \textit{on boundary}
\end{aligned}
$$

Result in step 1 shows that the value of a contingent claim (for any payoff and any model) can be calculated as a frequency integral of product between the Fourier transform of payoff and the Fourier transform of log underlying's pdf, while the latter is equivalent to characteristic function of log underlying on maturity $X_T = x$.

Step 2 – Fourier pricing formula for vanilla payoff and all models

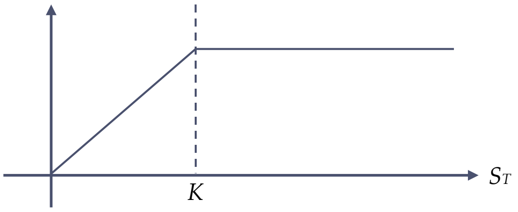
Instead of finding the Fourier transform of a vanilla call, we look for the Fourier transform of a covered call. What is a covered call? Lets take a look at the call put parity.

$$
\begin{aligned}
C_t + Ke^{-r(T-t)} &= P_t + S_t e^{-q(T-t)} \\
Ke^{-r(T-t)} - P_t &= S_t e^{-q(T-t)} - C_t = \textit{definition of covered call}
\end{aligned}
$$

On reaching maturity T, covered call becomes :

$$
\begin{array}{llllll}
 & K-P_T & = & K-(K-S_T)^+ & = & \min(S_T,K) \\
\text{or} & S_T-C_T & = & S_T-(S_T-K)^+ & = & \min(S_T,K)
\end{array}
$$

Payoff function of a covered call on maturity :



Now we declare payoff *f(x)* as the following, which is a covered call :

$$
\begin{array}{lll}
f(x) & = & K-(K-e^x)^+ \\
\tilde{f}(x) & = & \int_{-\infty}^{\infty}(K-(K-e^x)^+)e^{izx}dx \qquad \textit{beware, this is a complex integral} \\
 & = & \int_{-\infty}^{\infty}\min(e^x,K)e^{izx}dx \\
 & = & \int_{-\infty}^{\ln K}\min(e^x,K)e^{izx}dx+\int_{\ln K}^{\infty}\min(e^x,K)e^{izx}dx \\
 & = & \int_{-\infty}^{\ln K}e^{(1+iz)x}dx+K\int_{\ln K}^{\infty}e^{izx}dx \\
 & = & \left.\dfrac{e^{(1+iz)x}}{1+iz}\right|_{x=-\infty}^{x=\ln K}+K\left.\dfrac{e^{izx}}{iz}\right|_{x=\ln K}^{x=\infty} \qquad \textit{since } \int e^{kx}dx=(1/k)\int e^{kx}dkx=e^{kx}/k \\
 & = & \dfrac{e^{(1+iz)\ln K}}{1+iz}-\underbrace{\lim_{x\to-\infty}\dfrac{e^{(1+iz)x}}{1+iz}}_{0,remark1}+K\underbrace{\lim_{x\to\infty}\dfrac{e^{izx}}{iz}}_{0,remark1}-K\dfrac{e^{(iz)\ln K}}{iz} \\
 & = & \dfrac{e^{(1+iz)\ln K}}{1+iz}-K\dfrac{e^{(iz)\ln K}}{iz} \qquad \textit{given that } 0<z_i<1\textit{, please refer to remark 1} \\
 & = & \dfrac{(e^{\ln K})^{1+iz}}{1+iz}-K\dfrac{(e^{\ln K})^{iz}}{iz} \\
 & = & \dfrac{K^{1+iz}}{1+iz}-K\dfrac{K^{iz}}{iz} \\
 & = & K^{1+iz}\left(\dfrac{1}{1+iz}-\dfrac{1}{iz}\right) \\
 & = & \dfrac{K^{1+iz}}{z^2-iz} \qquad \textit{(equation 2)}
\end{array}
$$

[Remark 1] We need to find a region in complex plane, such that $\lim_{x\to\infty}\dfrac{e^{izx}}{iz}=0$ and $\lim_{x\to-\infty}\dfrac{e^{(1+iz)x}}{1+iz}=0$, we then have :

$$
\begin{array}{lll}
\lim_{x\to\infty}\dfrac{e^{izx}}{iz} & = & \lim_{x\to\infty}\dfrac{e^{i(z_r+iz_i)x}}{i(z_r+iz_i)} \qquad \textit{where } z_r,z_i\in\Re \\
 & = & \lim_{x\to\infty}\dfrac{e^{-z_ix}e^{iz_rx}}{i(z_r+iz_i)} \\
 & = & \lim_{x\to\infty}\dfrac{e^{-z_ix}\overbrace{(\cos(z_rx)+i\sin(z_rx))}^{cycle-and-bounded}}{i(z_r+iz_i)}=0 \qquad \textit{if damping factor } \lim_{x\to\infty}e^{-z_ix}=0\textit{, i.e. when } -z_i<0 \\
\lim_{x\to-\infty}\dfrac{e^{(1+iz)x}}{1+iz} & = & \lim_{x\to-\infty}\dfrac{e^{(1+i(z_r+iz_i))x}}{1+i(z_r+iz_i)} \\
 & = & \lim_{x\to-\infty}\dfrac{e^{(1-z_i)x}e^{iz_rx}}{1+i(z_r+iz_i)} \\
 & = & \lim_{x\to-\infty}\dfrac{e^{(1-z_i)x}\overbrace{(\cos(z_rx)+i\sin(z_rx))}^{cycle-and-bounded}}{1+i(z_r+iz_i)}=0 \qquad \textit{if damping factor } \lim_{x\to-\infty}e^{(1-z_i)x}=0\textit{, i.e. when } 1-z_i>0
\end{array}
$$

Strip of regularity for covered call is $0<z_i<1$ (while that for vanilla call is $z_i>1$). The former is prefered as it is easier to work with.

if we repeat the above procedure for vanilla call

Lets put Fourier transform of payoff into contingent claim price equation, i.e. plugging equation *2* into equation *1*.

$$
\begin{aligned}
V_t &= \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty+iz_i}^{+\infty+iz_i}\Phi_X(-z)\tilde{f}(z)dz \\
&= \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty+iz_i}^{+\infty+iz_i}\Phi_X(-z)\frac{K^{1+iz}}{z^2-iz}dz
\end{aligned}
$$

On plugging payoff, V_t becomes covered call price, then we have $V_t = Ke^{-r(T-t)} - P_t$, or alternatively $V_t = S_t e^{-q(T-t)} - C_t$:

$$
\begin{aligned}
C_t &= S_t e^{-q(T-t)} - \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty+i/2}^{+\infty+i/2}\Phi_X(-z)\cdot\frac{K^{1+iz}}{z^2-iz}dz && \textit{we pick } z_i = 1/2 \textit{ as it is symmetrically located in strip } 0 < z_i < 1 \\
&= S_t e^{-q(T-t)} - \frac{e^{-r(T-t)}}{\pi}\int_{-\infty+i/2}^{0+i/2}\mathrm{Re}\left(\Phi_X(-z)\cdot\frac{K^{1+iz}}{z^2-iz}\right)dz && \textit{we have proved that } \Phi_X(-z)\cdot\frac{k^{1+iz}}{z^2-iz} \textit{ is Hermitian}
\end{aligned}
$$

We can shift the integation path to the real axis by substituting $z' = i/2-z$, i.e. $z = -z'+i/2$, hence z' lies on real axis :

$$
\begin{aligned}
C_t &= S_t e^{-q(T-t)} - \frac{e^{-r(T-t)}}{\pi}\int_{\infty}^{0}\mathrm{Re}\left(\Phi_X(-(-z'+i/2))\cdot\frac{K^{1+i(-z'+i/2)}}{(-z'+i/2)^2-i(-z'+i/2)}\right)(-1)dz' \\
&= S_t e^{-q(T-t)} - \frac{e^{-r(T-t)}}{\pi}\int_{0}^{\infty}\mathrm{Re}\left(\Phi_X(z'-i/2)\cdot\frac{K^{1-iz'-1/2}}{z'^2-iz'-1/4+iz'+1/2}\right)dz' \\
&= S_t e^{-q(T-t)} - \frac{e^{-r(T-t)}}{\pi}\int_{0}^{\infty}\mathrm{Re}\left(\Phi_X(z'-i/2)\cdot\frac{K^{1/2-iz'}}{z'^2+1/4}\right)dz' \\
&= S_t e^{-q(T-t)} - \frac{e^{-r(T-t)}\sqrt{K}}{\pi}\int_{0}^{\infty}\frac{\mathrm{Re}[\Phi_X(z'-i/2)K^{-iz'}]}{z'^2+1/4}dz' && \textit{since K and } z' \textit{ are real} \\
&= e^{-r(T-t)}\left[S_t e^{(r-q)(T-t)} - \frac{\sqrt{K}}{\pi}\int_{0}^{\infty}\frac{\mathrm{Re}[\Phi_X(z'-i/2)K^{-iz'}]}{z'^2+1/4}dz'\right] \\
&= \underbrace{e^{-r(T-t)}}_{DF}\left[F - \frac{\sqrt{K}}{\pi}\int_{0}^{\infty}\frac{\mathrm{Re}[\Phi_X(z'-i/2)K^{-iz'}]}{z'^2+1/4}dz'\right] && \textit{where DF is discount factor, i.e. } e^{-r(T-t)} \textit{ (equation 3)}
\end{aligned}
$$

Equation *3* gives vanilla call price for any underlying model, as long as the characteristic function of <u>log price at maturity</u> is known. Unlike the result in *P29* of *Alma Dogg*, which expresses it in terms of characteristic function of <u>log normalised price at maturity</u>.

<u>Step 3a – Derive Heston PDE</u>

In order to find the characteristic function of log price at maturity, we have to started with the derivation of Heston *PDE*. Similar to what we have done for Black Scholes, we need to construct a riskless portfolio via dynamic hedging. As Heston is a two risk factors model, we need to dynamically hedge with two independent instruments, i.e. long *1* option contract being priced, hedge by selling some units of the underlying and selling some units of another option that depends on volatility. Let S_t be the underlying, V_t and U_t be the option being priced and the option for hedging respectively. The portfolio is (*read <u>BSPDE for dividend paying underlying</u>*) :

$$
\begin{aligned}
&& \Pi_t &= V_t - \Delta S_t - Q_t - \Lambda U_t && \textit{where } Q_t \textit{ is accumulated dividend from time 0} \\
&\textit{where} & Q_t &= \int_0^t \Delta_\tau S_\tau q_\tau d\tau && \textit{where } q_t \textit{ is dividend rate per share of underlying} \\
&\textit{hence} & dQ_t &= \Delta_t S_t q_t dt && \textit{where } q_t dt \textit{ is dividend per share of underlying}
\end{aligned}
$$

$$
\begin{aligned}
&\textit{Recall} & dS_t &= (r-q)S_t dt + \sqrt{v_t}S_t dz_{1t} && \textit{by Heston model} \\
&& dv_t &= \kappa(\vartheta - v_t)dt + \sigma\sqrt{v_t}dz_{2t} \textit{ and } dz_{1t}dz_{2t} = \rho dt && \textit{don't confuse volatility } v_t \textit{ with option price } V_t \\
&& dV_t &= \partial_t V_t dt + \partial_S V_t dS_t + \partial_v V_t dv_t + \frac{1}{2}\partial_{SS}V_t(dS_t)^2 + \partial_{Sv}V_t(dS_t dv_t) + \frac{1}{2}\partial_{vv}V_t(dv_t)^2 && \textit{by 2 dimensional Ito's lemma} \\
&& &= \partial_t V_t dt + \partial_S V_t dS_t + \partial_v V_t dv_t + \frac{1}{2}v_t S_t^2\partial_{SS}V_t dt + \rho\sigma S_t v_t \partial_{Sv}V_t dt + \frac{1}{2}\sigma^2 v_t \partial_{vv}V_t dt \\
&& &= \left(\partial_t V_t + \frac{1}{2}v_t S_t^2\partial_{SS}V_t + \rho\sigma S_t v_t\partial_{Sv}V_t + \frac{1}{2}\sigma^2 v_t\partial_{vv}V_t\right)dt + \partial_S V_t dS_t + \partial_v V_t dv_t \\
&\textit{Similarly} & dU_t &= \left(\partial_t U_t + \frac{1}{2}v_t S_t^2\partial_{SS}U_t + \rho\sigma S_t v_t\partial_{Sv}U_t + \frac{1}{2}\sigma^2 v_t\partial_{vv}U_t\right)dt + \partial_S U_t dS_t + \partial_v U_t dv_t
\end{aligned}
$$

Therefore delta change in portfolio is :

$$
\begin{aligned}
d\Pi_t &= dV_t - \Delta dS_t - dQ_t - \Lambda dU_t \\
&= dV_t - \Delta dS_t - qS_t\Delta dt - \Lambda dU_t \\
&= \left[\left(\partial_t V_t + \frac{1}{2}v_t S_t^2 \partial_{ss}V_t + \rho\sigma S_t v_t \partial_{sv}V_t + \frac{1}{2}\sigma^2 v_t \partial_{vv}V_t\right)dt + \partial_s V_t dS_t + \partial_v V_t dv_t\right] - \Delta dS_t - \Delta S_t q dt \\
&\quad - \Lambda\left[\left(\partial_t U_t + \frac{1}{2}v_t S_t^2 \partial_{ss}U_t + \rho\sigma S_t v_t \partial_{sv}U_t + \frac{1}{2}\sigma^2 v_t \partial_{vv}U_t\right)dt + \partial_s U_t dS_t + \partial_v U_t dv_t\right] \\
&= \left(\partial_t V_t + \frac{1}{2}v_t S_t^2 \partial_{ss}V_t + \rho\sigma S_t v_t \partial_{sv}V_t + \frac{1}{2}\sigma^2 v_t \partial_{vv}V_t - \Delta S_t q\right)dt \quad \longleftarrow \textit{dividend} \\
&\quad - \Lambda\left(\partial_t U_t + \frac{1}{2}v_t S_t^2 \partial_{ss}U_t + \rho\sigma S_t v_t \partial_{sv}U_t + \frac{1}{2}\sigma^2 v_t \partial_{vv}U_t\right)dt + \underbrace{(\partial_s V_t - \Lambda\partial_s U_t - \Delta)}_{hedge}dS_t + \underbrace{(\partial_v V_t - \Lambda\partial_v U_t)}_{hedge}dv_t
\end{aligned}
$$

With perfect hedging, we can eliminate both risk factors :

$$
\begin{array}{lllllll}
0 = \partial_v V_t - \Lambda\partial_v U_t & \Rightarrow & \Lambda & = & \dfrac{\partial_v V_t}{\partial_v U_t} & = & \textit{ratio of two vegas} \\
0 = \partial_s V_t - \Lambda\partial_s U_t - \Delta & \Rightarrow & \Delta & = & \partial_s V_t - \Lambda\partial_s U_t & = & \textit{weighted sum of two deltas}
\end{array}
$$

We are going to introduce differential operators D and D' which make the maths neater.

$$
d\Pi_t = \begin{array}{l}
\overbrace{(\partial_t V_t + \frac{1}{2}v_t S_t^2 \partial_{ss}V_t + \rho\sigma S_t v_t \partial_{sv}V_t + \frac{1}{2}\sigma^2 v_t \partial_{vv}V_t}^{DV_t} - \Delta S_t q)dt \\
-\Lambda\underbrace{(\partial_t U_t + \frac{1}{2}v_t S_t^2 \partial_{ss}U_t + \rho\sigma S_t v_t \partial_{sv}U_t + \frac{1}{2}\sigma^2 v_t \partial_{vv}U_t)dt}_{DU_t}
\end{array} = r(V_t - \Delta S_t - \Lambda U_t)dt \qquad \textit{why exclude dividend?}
$$

$$
\begin{aligned}
DV_t - \Delta S_t q - \Lambda DU_t &= r(V_t - \Delta S_t - \Lambda U_t) && \textit{substitute } \Delta \\
DV_t - qS_t(\partial_s V_t - \Lambda\partial_s U_t) - \Lambda DU_t &= rV_t - rS_t(\partial_s V_t - \Lambda\partial_s U_t) - \Lambda r U_t \\
DV_t - qS_t\partial_s V_t + \Lambda q S_t \partial_s U_t - \Lambda DU_t &= rV_t - rS_t\partial_s V_t + \Lambda r S_t \partial_s U_t - \Lambda r U_t \\
DV_t + (r-q)S_t\partial_s V_t - rV_t &= \Lambda(DU_t + (r-q)S_t\partial_s U_t - rU_t) && \textit{move V/U to LHS/RHS} \\
\frac{DV_t + (r-q)S_t\partial_s V_t - rV_t}{\partial_v V_t} &= \frac{DU_t + (r-q)S_t\partial_s U_t - rU_t}{\partial_v U_t} && \textit{substitute } \Lambda \\
\frac{D'V_t - rV_t}{\partial_v V_t} &= \frac{D'U_t - rU_t}{\partial_v U_t}
\end{aligned}
$$

$$
\begin{aligned}
\textit{where} \quad D &= \partial_t + \frac{1}{2}v_t S_t^2 \partial_{ss} + \rho\sigma S_t v_t \partial_{sv} + \frac{1}{2}\sigma^2 v_t \partial_{vv} \\
D' &= \partial_t + (r-q)S_t\partial_s + \frac{1}{2}v_t S_t^2 \partial_{ss} + \rho\sigma S_t v_t \partial_{sv} + \frac{1}{2}\sigma^2 v_t \partial_{vv} \\
D'V_t &= V_{theta} + (r-q)S_t V_{delta} + \frac{1}{2}v_t S_t^2 V_{gamma} + \rho\sigma S_t v_t \underbrace{V_{vanna}}_{\partial_{sv}V_t} + \frac{1}{2}\sigma^2 v_t \underbrace{V_{vomma}}_{\partial_{vv}V_t}
\end{aligned}
$$

Since V and U can be any contingent claim, therefore in general the Greeks of a contingent claim can be written as a 3 variables *(t, S_t and v_t) PDE* like *A'* or *B'* above, in other words, we have :

$$
\begin{aligned}
\frac{V_{theta} + (r-q)S_t V_{delta} + \frac{1}{2}v_t S_t^2 V_{gamma} + \rho\sigma S_t v_t V_{vanna} + \frac{1}{2}\sigma^2 v_t V_{vomma} - rV_t}{V_{vega}} &= f(t, S_t, v_t) \\
&= -\kappa(\theta - v_t) + \underbrace{\Lambda(t, S_t, v_t)}_{=\lambda\Lambda_t}
\end{aligned}
$$

RHS function is selected according to Heston's original paper in *1993*, "A closed-form solution for options with stochastic volatility with applications to bond and currency option". Besides, according to consumption model by Breeden in 1979, we have $\Lambda(t,S_t,v_t) = \Lambda v_t$ (we can omit Λ if κ^* and θ^* are used instead of κ and θ). Finally we have the Heston *PDE* :

$$
rV_t = \underbrace{V_{theta}}_{time-decay} + (r-q)S_t V_{delta} + (\kappa(\theta - v_t) - \Lambda v_t)V_{vega} + \frac{1}{2}v_t S_t^2 V_{gamma} + \rho\sigma S_t v_t V_{vanna} + \frac{1}{2}\sigma^2 v_t V_{vomma} \qquad \textit{(equation 4)}
$$

This is a deterministic *PDE*, since S_t and v_t are known as of current time *t*.

[Remark 1] We can come up with the same Heston *PDE* in a much faster way using Dr Yan's approach. Recall Heston model :

$$
\begin{aligned}
dS_t &= \overbrace{(r-q)S_t}^{\alpha_1}dt + \overbrace{\sqrt{v_t}\,S_t}^{\beta_1}dz_{1t} \\
dv_t &= \underbrace{\kappa(\vartheta - v_t)}_{\alpha_2}dt + \underbrace{\sigma\sqrt{v_t}}_{\beta_2}dz_{2t}
\end{aligned}
$$

α_1 = *drift of underlying*
α_2 = *drift of volatility*
β_1 = *diffusion of underlying*
β_2 = *diffusion of volatility, i.e. volatility of volatility*

Then the Heston *PDE* can be written down directly as the following. *LHS* is the riskfree rate of return of the option sales (premium) in sell-side's perspective, while *RHS* decomposes the change in option value in buy-side's perspective, intuitively, it is just a Taylor series expansion. Both sides equal at the time the option is traded. This approach does also explain why there is no vega, vanna nor volga in Black Scholes (as both α_2 and β_2 are diminished in Black Scholes).

$$
\begin{aligned}
rV_t &= V_{theta} + \alpha_1 V_{delta} + \alpha_2 V_{vega} + \frac{1}{2}\beta_1^2 V_{gamma} + \rho\beta_1\beta_2 V_{vanna} + \frac{1}{2}\beta_2^2 V_{vomma} \\
&= V_{theta} + (r-q)S_t V_{delta} + \kappa(\theta - v_t)V_{vega} + \frac{1}{2}v_t S_t^2 V_{gamma} + \rho\sigma S_t v_t V_{vanna} + \frac{1}{2}\sigma^2 v_t V_{vomma}
\end{aligned}
$$

here V_{vega} *means* $\partial_v V$, *not* $\partial_\sigma V$

here Λv_t *is missing in vega*

[Remark 2] Notation V_t is not perfect, it is better denoted as a 3 variable function $V(t,S_t,v_t)$. Boundary conditions for Heston *PDE* :

(a)	$V(T,S_T,v_T)$	=	$(S_T - K)^+$	*option payoff*
(b)	$V(t,0,v_t)$	=	0	*when underlying price is zero, there is no diffusion (A_2=0), hence option remains OTM*
(c)	$V(t,S_t,\infty)$	=	S_t	*We can prove it, but what makes it a boundary condition? What is its physical meaning?*
(d)	$\partial_S V(t,\infty,v_t)$	=	1	*We can prove it, but what makes it a boundary condition? What is its physical meaning?*

We have not considered the last 3 boundary conditions in Black Scholes, now let's check if they are satified by the *BS equation*.

(b) $\lim_{S\to 0} N(d_{1,2}) = \lim_{F\to 0} N(\overbrace{(\overbrace{\ln F/K}^{-\infty} \pm v/2)/\sqrt{v}}^{0}) = 0$ *where* $v = \sigma^2(T-t)$

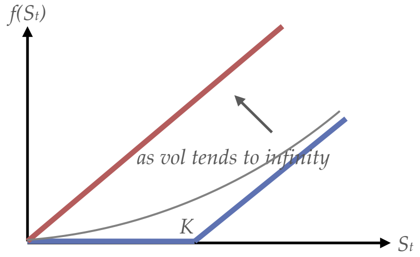
$V(t,0,v_t) = \lim_{F\to 0}(FN(d_1) - KN(d_2))\times DF = (0\times 0 - K\times 0)\times DF = 0$

(c) $\lim_{\sigma\to\infty} N(d_{1,2}) = \lim_{v\to\infty} N((\ln F/K \pm v/2)/\sqrt{v}) = \lim_{v\to\infty} N(\overbrace{(\ln F/K)/\sqrt{v}}^{0} \pm \sqrt{v}/2) = N(\pm\infty) = 1,0$

$\boxed{\Pr_S(ITM) = 1, \Pr_Q(ITM) = 0}$

$V(t,S_t,\infty) = \lim_{v\to 0}(FN(d_1) - KN(d_2))\times DF = (F\times 1 - K\times 0)\times DF = F\times DF = S_t$

(d) $\partial_S V(t,\infty,v_t) = \lim_{F\to\infty} N(\underbrace{(\ln F/K + v/2)/\sqrt{v}}_{d_1}) = N(\infty) = 1$



[Remark 3] We transform Heston *PDE* into a forward *PDE*, which include the following modifications. With these changes, we can remove *r-q* term from the *PDE*, which makes things easier. The forward *PDE* helps to find characteristic function of log moneyness, which is then converted into characteristic function of log price. Please note that X_t is not a random variable as of time *t*.

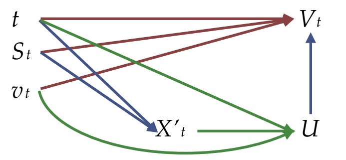
- *Replace timepoints t and T by time interval τ = T-t, hence as t increases, τ decreases, and dt = -dτ*
- *Replace spot stock price S_t by log forward stock price X'_t = logF = log($e^{(r-q)\tau}S_t$) (NOT log spot stock price X_t = logS_t)*
- *Replace spot option price V_t by forward option price U = $e^{r\tau}V_t$*

Here is the *DAG* linking all the variables. Distinguish between total derivative or full derivative and partial derivative, they are the same only when there are only 2 layers in DAG, in that case, I use ∂ to indicate both full derivative and partial derivative. There are 2 layers in V_t tree, but 3 layers in *U* tree. Our objective is to convert the *PDE* for the ***red edges*** to a *PDE* for the ***green edges***.

$$\text{(i)}\quad \partial_t X'_t = \partial_t \ln(e^{(r-q)\tau}S_t) = (e^{(r-q)\tau}S_t)^{-1}(-(r-q)e^{(r-q)\tau}S_t) = -(r-q)$$ *since* $\partial_t\tau = \partial_t(T-t) = -1$

$$\text{(ii)}\quad \partial_s X'_t = \partial_s \ln(e^{(r-q)\tau}S_t) = (e^{(r-q)\tau}S_t)^{-1}e^{(r-q)\tau} = 1/S_t$$

$$\text{(iii)}\quad \partial_{ss} X'_t = \partial_{ss} \ln(e^{(r-q)\tau}S_t) = \partial_s(1/S_t) = -1/S_t^2$$ *remark (i-iii) are blue edges*



$$\text{(iv)}\quad dU/dt = \partial_{x'}U\partial_t X'_t + \partial_t U = -(r-q)\partial_{x'}U + \partial_t U$$ *using remark (i)*

$$\text{(v)}\quad dU/dS_t = \partial_{x'}U\partial_s X'_t = (1/S_t)\partial_{x'}U$$ *using remark (ii)*

$$\text{(vi)}\quad dU/dv_t = \partial_v U$$

then

$$\partial_t V_t = \partial_t e^{-r\tau}U = e^{-r\tau}(dU/dt) - -re^{-r\tau}U$$ *recall* $\partial_t\tau = \partial_t(T-t) = -1$

$$= e^{-r\tau}(-(r-q)\partial_{x'}U + \partial_t U) + re^{-r\tau}U$$ *using remark (iv)*

$$= e^{-r\tau}(\boxed{-(r-q)\partial_{x'}U} + \partial_t U + rU)$$ *used to remove r-q term in PDE*

$$\partial_s V_t = \partial_s e^{-r\tau}U = e^{-r\tau}(1/S_t)\partial_{x'}U$$ *using remark (v)*

$$\partial_v V_t = \partial_v e^{-r\tau}U = e^{-r\tau}\partial_v U$$ *using remark (vi)*

$$\begin{aligned}
\partial_{ss}V_t = \partial_{ss}e^{-r\tau}U &= \partial_s(e^{-r\tau}(1/S_t)\partial_{x'}U) \\
&= -e^{-r\tau}(1/S_t^2)\partial_{x'}U + e^{-r\tau}(1/S_t)\partial_{x'x'}U\partial_s X'_t \\
&= -e^{-r\tau}(1/S_t^2)\partial_{x'}U + e^{-r\tau}(1/S_t^2)\partial_{x'x'}U \qquad \textit{using remark (ii)} \\
&= e^{-r\tau}(1/S_t^2)(\partial_{x'x'}U - \partial_{x'}U)
\end{aligned}$$

$$\begin{aligned}
\partial_{sv}V_t = \partial_{sv}e^{-r\tau}U &= \partial_v(e^{-r\tau}(1/S_t)\partial_{x'}U) \\
&= e^{-r\tau}(1/S_t)\partial_{x'v}U
\end{aligned}$$

$$\partial_{vv}V_t = \partial_{vv}e^{-r\tau}U = e^{-r\tau}\partial_{vv}U$$ *(equation set 5#)*

Substituting all the above into the Heston *PDE*, then substitute *t* by τ, we then have the forward Heston *PDE*.

$$rV_t = V_{theta} + (r-q)S_tV_{delta} + (\kappa(\theta-v_t)-\Lambda v_t)V_{vega} + \frac{1}{2}v_tS_t^2V_{gamma} + \rho\sigma S_tv_tV_{vanna} + \frac{1}{2}\sigma^2v_tV_{vomma}$$

$$e^{-r\tau}rU = \left[\begin{array}{l} e^{-r\tau}(-(r-q)\partial_{x'}U + \partial_tU + rU) + e^{-r\tau}(r-q)\partial_{x'}U + e^{-r\tau}(\kappa(\theta-v_t)-\Lambda v_t)\partial_vU \\ +\frac{1}{2}e^{-r\tau}v_t(\partial_{x'x'}U - \partial_{x'}U) + e^{-r\tau}\rho\sigma v_t\partial_{x'v}U + \frac{1}{2}e^{-r\tau}\sigma^2v_t\partial_{vv}U \end{array}\right.$$

$$\cancel{rU} = \left[\begin{array}{l} (\cancel{-(r-q)\partial_{x'}U} + \partial_tU + \cancel{rU}) + \cancel{(r-q)\partial_{x'}U} + (\kappa(\theta-v_t)-\Lambda v_t)\partial_vU \\ +\frac{1}{2}v_t(\partial_{x'x'}U - \partial_{x'}U) + \rho\sigma v_t\partial_{x'v}U + \frac{1}{2}\sigma^2v_t\partial_{vv}U \end{array}\right.$$

$$0 = \partial_tU + (\kappa(\theta-v_t)-\Lambda v_t)\partial_vU + \frac{1}{2}v_t(\partial_{x'x'}U - \partial_{x'}U) + \rho\sigma v_t\partial_{x'v}U + \frac{1}{2}\sigma^2v_t\partial_{vv}U$$ *all* S_t *and r-q terms removed*

$$0 = -\partial_\tau U + (\kappa(\theta-v_t)-\Lambda v_t)\partial_vU + \frac{1}{2}v_t(\partial_{x'x'}U - \partial_{x'}U) + \rho\sigma v_t\partial_{x'v}U + \frac{1}{2}\sigma^2v_t\partial_{vv}U$$ *where* $\partial_\tau U = -\partial_t U$

(equation 5)

Step 3b – Breakdown Heston PDE into two ODEs by plugging in an ansatz

We don't need to solve the Heston *PDE*, instead we go for characteristic function of ***log forward price*** X'_T making use of the Heston *PDE*, as it is relatively easier. Please note that as of current time *t*, X'_t is deterministic, whereas X'_T is random.

$$\Phi_{X'}(z,\tau) = E[e^{izX'_T} \mid X'_t = x', v_t = v]$$ *price of contingent claim with complex payoff* $e^{izX'_T}$

Since $\phi_{X'}(z)$ can be regarded as the ***forward price*** of contingent claim having complex payoff $e^{izX'T}$, $\phi_{X'}(z)$ must then fulfill the ***forward Heston PDE***. Now, we postulate an ansatz for $\phi_{X'}(z)$, and substitute it into the forward Heston *PDE*. The ansatz is :

$$\Phi_{X'}(z,\tau) = e^{A(z,\tau)+B(z,\tau)v+C(z,\tau)x'}$$ *payoff when* $X'_T = x'$ *at maturity T (i.e. when* $\tau = 0$*)*

which separates *underlying-terms volatility-terms* and *other-terms* into $e^{C(z,\tau)x'}\,e^{B(z,\tau)v}$ and $e^{A(z,\tau)}$ respectively, furthermore this ansatz must satisfy the boundary condition that X'_T becomes deterministic when $\tau = 0$:

$$\begin{aligned}
\Phi_{X'}(z,0) &= E[e^{izX'_T} \mid X'_T = x', v_T = v] \\
&= e^{izx'}
\end{aligned}$$

$$\Rightarrow \quad e^{izx'} = e^{A(z,0)+B(z,0)v+C(z,0)x'}$$ *for all* $X'_T = x'$

$$
\begin{aligned}
\Rightarrow \quad & A(z,0) &=& \; 0 \\
& B(z,0) &=& \; 0 \\
& C(z,0) &=& \; iz \\
\Rightarrow \quad & \Phi_{X'}(z,\tau) &=& \; e^{A(z,\tau)+B(z,\tau)v+izx'}
\end{aligned}
$$

we update the ansatz, which must fulfill the forward Heston PDE

Hence by putting $U = \phi_{X'}(z,\tau)$, then calculate derivatives of U :

$$
\begin{aligned}
\partial_\tau \Phi &= \Phi \cdot \partial_\tau (A(z,\tau) + B(z,\tau)v + izx') &&= \Phi \cdot (\partial_\tau A + \partial_\tau Bv) \\
\partial_{x'} \Phi &= \Phi \cdot \partial_{x'} (A(z,\tau) + B(z,\tau)v + izx') &&= \Phi \cdot iz \\
\partial_v \Phi &= \Phi \cdot \partial_v (A(z,\tau) + B(z,\tau)v_t + izx') &&= \Phi \cdot B \\
\partial_{x'x'} \Phi &= \partial_{x'} (\Phi \cdot iz) = \Phi \cdot (iz)^2 &&= -\Phi \cdot z^2 \\
\partial_{x'v} \Phi &= \partial_v (\Phi \cdot iz) = \Phi \cdot iz \cdot B \\
\partial_{vv} \Phi &= \partial_v (\Phi \cdot B) = \Phi \cdot B^2
\end{aligned}
$$

(equation set 6#)

Plugging them into the forward Heston *PDE*, all *x′* terms are removed (recall that v and v_t are the same thing):

$$
\begin{aligned}
0 &= -\partial_\tau U + (\kappa(\theta - v_t) - \Lambda v_t)\partial_v U + \frac{1}{2} v_t (\partial_{x'x'} U - \partial_{x'} U) + \rho \sigma v_t \partial_{x'v} U + \frac{1}{2}\sigma^2 v_t \partial_{vv} U \\
&= -\Phi(\partial_\tau A + \partial_\tau B v_t) + (\kappa(\theta - v_t) - \Lambda v_t)(\Phi B) + \frac{1}{2} v_t (-\Phi z^2 - \Phi iz) + \rho \sigma v_t (\Phi iz B) + \frac{1}{2}\sigma^2 v_t (\Phi B^2) \\
&= -(\partial_\tau A + \partial_\tau B v_t) + (\kappa(\theta - v_t) - \Lambda v_t)B - \frac{1}{2} v_t (z^2 + iz) + \rho \sigma v_t (izB) + \frac{1}{2}\sigma^2 v_t B^2 && \textit{cancel } \Phi \textit{ on both sides} \\
&= -\partial_\tau A + \kappa\theta B + \left[-\partial_\tau B - (\kappa + \Lambda) B - \frac{1}{2}(z^2 + iz) + \rho\sigma(izB) + \frac{1}{2}\sigma^2 B^2 \right] v_t && \textit{group terms by } v_t
\end{aligned}
$$

which is true for all given information v_t as of time *t*, hence we can breakdown the *PDE* into two *ODEs*. Solutions to the ODEs must be function *A* and *B* in terms of time to maturity τ, frequency *z* and Heston parameters v_0, κ, θ, σ and ρ, there should be no S_t nor v_t in the solution of *A* and *B*. As *ODE1* depends on another, we work on *ODE2* first. RHS of *ODE2* is quadratic in *B*, called *Riccati Eq*.

ODE1	$\partial_\tau A$	=	$\kappa\theta B$	*such that* $A(z,0) = 0$	*(equation 6a)*
ODE2	$\partial_\tau B$	=	$-\frac{1}{2}(z^2 + iz) + (\rho\sigma iz - (\kappa + \Lambda))B + \frac{1}{2}\sigma^2 B^2$	*such that* $B(z,0) = 0$	*(equation 6b)*

IF the forward *PDE* is expressed in terms of X_t rather than X_t', then the *ODE2* is the same, while *ODE1* becomes :

ODE1	$\partial_\tau A$	=	$\kappa\theta B + iz(r-q)$	*repeat above steps for proof, or see Rouah P11*
ODE2	$\partial_\tau B$	=	$-\frac{1}{2}(z^2 + iz) + (\rho\sigma iz - (\kappa + \Lambda))B + \frac{1}{2}\sigma^2 B^2$	*more complicated, this is why we prefer* X_t'

<u>Brief summary</u>

What we have done so far is to derive a *PDE* in terms of :

Function	$V = V(t, S_t, v_t)$	
Greeks	$[\partial_t V, \partial_S V, \partial_v V, \partial_{SS} V, \partial_{Sv} V, \partial_{vv} V]$	
Parameters	$[v_0, \kappa, \theta, \sigma, \rho]$	

which is then converted into a *PDE* in terms of :

Function	$U = U(\tau, X'_t, v_t)$	*where* $X'_t = \log F = \log(e^{(r-q)\tau} S_t)$
Greeks	$[\partial_\tau U, \partial_{x'} U, \partial_v U, \partial_{x'x'} U, \partial_{x'v} U, \partial_{vv} U]$	
Parameters	$[v_0, \kappa, \theta, \sigma, \rho]$	

We then plug in a complex payoff and convert into two *ODEs* in terms of time derivative only :

Function	$A(z,\tau)$, $B(z,\tau)$ *and* $B^2(z,\tau)$	*where filtration* $I_t = (x', v)$ *is removed, freq z is introduced*
Greeks	$[\partial_\tau A, \partial_\tau B]$	*beauty of complex payoff is that most Greeks are removed*
Parameters	$[v_0, \kappa, \theta, \sigma, \rho]$	*hence solutions of ODEs are functions of* τ *only*

Step 3c – Solution to *ODE2*

We will solve *ODE2* using Riccati equation technique.

$$\partial_\tau B = \overbrace{-\frac{1}{2}(z^2+iz)}^{P}+\overbrace{(\rho\sigma iz-(\kappa+\Lambda))}^{Q}B+\overbrace{\frac{1}{2}\sigma^2}^{R}B^2 \qquad \textit{such that } B(z,0)=0$$

$$\partial_\tau B = P(\tau)+Q(\tau)B+R(\tau)B^2 \qquad \textit{This is called Riccati equation.}$$

$$= R(B-r_+)(B-r_-) \qquad \textit{analytic solution to quadratic equation}$$

$$\textit{where} \quad r_+,r_- = \frac{-Q\pm D}{2R} \quad \textit{and } D = \sqrt{Q^2-4PR} \qquad \textit{P, Q, R, D, } r\pm\textit{, G depend on } (v_0,\kappa,\theta,\sigma,\rho)\textit{, } \Lambda \textit{ and } (i,z) \textit{ only}$$

Separating variables, we have :

$$d\tau = \frac{1}{R(B-r_+)(B-r_-)}dB$$

$$= \frac{1}{R(r_+-r_-)}\times\left(\frac{1}{B-r_+}-\frac{1}{B-r_-}\right)dB \qquad \textit{can be integrated directly once becomes 1/B}$$

$$\tau+const = \frac{1}{R(r_+-r_-)}\times\left(\int(B-r_+)^{-1}dB-\int(B-r_-)^{-1}dB\right)$$

$$= \frac{1}{R(r_+-r_-)}\times(\ln(B-r_+)-\ln(B-r_-))$$

$$= \frac{1}{D}\ln\frac{B-r_+}{B-r_-} \qquad \textit{since } R(r_+-r_-)=R\left[\frac{-Q+D}{2R}-\frac{-Q-D}{2R}\right]=D$$

When $\tau=0$, $B(z,0)=0$, we can solve for the *const* :

$$const = \frac{1}{D}\ln\frac{0-r_+}{0-r_-}$$

$$= \frac{1}{D}\ln G \qquad \textit{where } G=r_+/r_-\textit{, i.e. ratio of the two roots}$$

$$\Rightarrow \quad \tau = \frac{1}{D}\ln\frac{B-r_+}{B-r_-}-\frac{1}{D}\ln G$$

$$= \frac{1}{D}\ln\left(\frac{B-r_+}{B-r_-}\frac{1}{G}\right)$$

$$(B-r_-)Ge^{D\tau} = (B-r_+)$$

$$B = \frac{-r_++r_-Ge^{D\tau}}{-1+Ge^{D\tau}}$$

$$= \frac{-1+e^{D\tau}}{-1+Ge^{D\tau}}r_+$$

$$= \frac{1-e^{D\tau}}{1-Ge^{D\tau}}r_+ \qquad \textit{(equation 7)}$$

The above is a generic *Riccati* technique that works for all *P*, *Q*, *R*. Now lets focus on our case, plugging in Heston parameters :

$$\Rightarrow \quad B(z,\tau) = \frac{1-e^{D\tau}}{1-Ge^{D\tau}}\frac{\kappa+\Lambda-\rho\sigma iz+D}{\sigma^2} \qquad \textit{which is the same as equation 1.58 in Rouah book}$$

$$\textit{where} \quad D = \sqrt{Q^2-4PR} = \sqrt{(\rho\sigma iz-(\kappa+\Lambda))^2+(z^2+iz)\sigma^2}$$

$$G = \frac{r_+}{r_-}=\frac{-Q+D}{-Q-D} = \frac{\kappa+\Lambda-\rho\sigma iz+D}{\kappa+\Lambda-\rho\sigma iz-D}$$

$$r_+ = \frac{-Q+D}{2R} = \frac{\kappa+\Lambda-\rho\sigma iz+D}{\sigma^2}$$

Step 3d – Solution to *ODE1*

We substitute solution of *B* above into *ODE1*, it can be solved by in the same way as *Rouah's* book, chapter *1*, page *14*.

$$
\begin{array}{rcll}
\partial_\tau A &=& \kappa\theta B & P,\ Q,\ R,\ D,\ r\pm,\ G \text{ do not depend on } \tau \\
dA &=& \kappa\theta\cdot r_+ \dfrac{1-e^{D\tau}}{1-Ge^{D\tau}}d\tau & \\
\Rightarrow\ A(z,\tau)-\underbrace{A(z,0)}_{0} &=& \displaystyle\int_0^\tau \kappa\theta\cdot r_+ \frac{1-e^{Ds}}{1-Ge^{Ds}}ds & \\
&=& \dfrac{\kappa\theta\cdot r_+}{D}\displaystyle\int_{s=0}^{s=\tau}\frac{1-x}{1-Gx}\frac{1}{x}dx & \text{using } x=e^{Ds} \text{ and } dx=De^{Ds}ds=Dxds \\
\Rightarrow\ \displaystyle\int_{s=0}^{s=\tau}\frac{1-x}{1-Gx}\frac{1}{x}dx &=& \displaystyle\int_{s=0}^{s=\tau}\left(\frac{1}{x}-\frac{1-G}{1-Gx}\right)dx & \text{using partial fraction} \\
&=& \displaystyle\int_{s=0}^{s=\tau}\frac{1}{x}dx+\frac{1-G}{G}\int_{s=0}^{s=\tau}\frac{1}{1-Gx}d(1-Gx) & \\
&=& (\ln e^{D\tau}-\ln e^{D0})+\dfrac{1-G}{G}(\ln(1-Ge^{D\tau})-\ln(1-Ge^{D0})) & \text{recall that } x=e^{Ds} \\
&=& D\tau+\dfrac{1-G}{G}\ln\dfrac{1-Ge^{D\tau}}{1-G} & \\
\Rightarrow\ A(z,\tau) &=& \dfrac{\kappa\theta\cdot r_+}{D}\left[D\tau+\dfrac{1-G}{G}\ln\dfrac{1-Ge^{D\tau}}{1-G}\right] & \\
&=& \kappa\theta\cdot\left[r_+\tau+\dfrac{r_+}{D}\dfrac{1-G}{G}\ln\dfrac{1-Ge^{D\tau}}{1-G}\right] & \text{(equation 8)} \\
&=& \kappa\theta\cdot\left[\dfrac{\kappa+\Lambda-\rho\sigma iz+D}{\sigma^2}\tau-\dfrac{2}{\sigma^2}\ln\dfrac{1-Ge^{D\tau}}{1-G}\right] & \text{where } \dfrac{r_+}{D}\dfrac{1-G}{G}=-\dfrac{1}{R}=-\dfrac{2}{\sigma^2}\text{, see remark} \\
&=& \dfrac{\kappa\theta}{\sigma^2}\cdot\left[\tau(\kappa+\Lambda-\rho\sigma iz+D)-2\ln\dfrac{1-Ge^{D\tau}}{1-G}\right] & \text{which is the same as equation 1.62 in Rouah book}
\end{array}
$$

$$
\begin{array}{rcll}
\text{Remark}\quad \dfrac{r_+}{D}\dfrac{1-G}{G} &=& \dfrac{(-Q+D)/2R}{D}\dfrac{1-(-Q+D)/(-Q-D)}{(-Q+D)/(-Q-D)} & \text{express } r_+ \text{ and } G \text{ in terms of } Q \text{ and } D \text{ only} \\
&=& \dfrac{(-Q+D)/2R}{D}\dfrac{(-Q-D)-(-Q+D)}{-Q+D} & \\
&=& \dfrac{1/2R}{D}\dfrac{-2D}{1} & \\
&=& -1/R & \text{This is true for all } P,\ Q,\ R.
\end{array}
$$

Although in general, the characteristic function of X'_t is ***different*** from the characteristic function of X_t, the characteristic function of X'_T is ***equivalent*** to the characteristic function of X_T (hence they coinside at maturity *T*) :

$$
\begin{array}{rcl}
\Phi_{X'}(z,\tau) &=& E[e^{izX'_T}\mid X'_t=x',v_t=v] \\
&=& E[e^{iz\ln(e^{(r-q)(T-t)}S_t)|_{t=T}}\mid X'_t=x',v_t=v] \\
&=& E[e^{iz\ln S_T}\mid I_t] \\
&=& E[e^{izX_T}\mid I_t] \\
&=& \Phi_X(z,\tau)
\end{array}
$$

Finally, we need to calculate the characteristic function of log spot underlying at maturity for Lewis approach.

$$
\begin{array}{rcl}
\Phi_X(z,\tau) &=& \Phi_{X'}(z,\tau) \\
&=& e^{A(z,\tau)+B(z,\tau)v_t+izX'_t} \\
&=& \exp(izX'_t)\times\exp(B(z,\tau)v_t)\times\exp(A(z,\tau)) \\
&=& \exp(iz\ln(e^{(r-q)(T-t)}S_t))\times\exp(B(z,\tau)v_t)\times\exp(A(z,\tau)) \\
&=& \left[\begin{array}{l}\exp(iz(\underbrace{\ln S_t+(r-q)(T-t)}_{\ln F}))\times\exp\left(\dfrac{1-e^{D(T-t)}}{1-Ge^{D(T-t)}}\dfrac{\kappa+\Lambda-\rho\sigma iz+D}{\sigma^2}v_t\right) \\ \times\exp\left(\dfrac{\kappa\theta}{\sigma^2}\cdot\left[(T-t)(\kappa+\Lambda-\rho\sigma iz+D)-2\ln\dfrac{1-Ge^{D(T-t)}}{1-G}\right]\right)\end{array}\right]
\end{array}
$$

Step 4 – Analytic solution for Heston

By merging results from *step 2* and *step 3*, we have an analytic solution for Heston. Given Heston parameter and trade details :

$$C(S_t) = DF \times \left[F - \frac{\sqrt{K}}{\pi} \int_0^\infty \frac{\mathrm{Re}[\Phi(z'-i/2)K^{-iz'}]}{z'^2+1/4} dz' \right] \qquad \textit{term structure r and q are absorbed into forward F}$$

$$\Phi(z) = \exp(iz\ln F) \times \exp\left(\frac{1-e^{D(T-t)}}{1-Ge^{D(T-t)}} \frac{\kappa+\Lambda-\rho\sigma iz+D}{\sigma^2} v_t \right) \times \exp\left(\frac{\kappa\theta}{\sigma^2} \cdot \left[(T-t)(\kappa+\Lambda-\rho\sigma iz+D) - 2\ln\frac{1-Ge^{D(T-t)}}{1-G} \right] \right)$$

$$= \exp(iz\ln F) \times \exp\left(\frac{1-e^{D(T-t)}}{1-Ge^{D(T-t)}} \frac{\kappa^*-\rho\sigma iz+D}{\sigma^2} v_t \right) \times \exp\left(\frac{\kappa^*\theta^*}{\sigma^2} \cdot \left[(T-t)(\kappa^*-\rho\sigma iz+D) - 2\ln\frac{1-Ge^{D(T-t)}}{1-G} \right] \right)$$

where

$$D = \sqrt{(\rho\sigma iz-(\kappa+\Lambda))^2+(z^2+iz)\sigma^2}$$

$$G = \frac{\kappa+\Lambda-\rho\sigma iz+D}{\kappa+\Lambda-\rho\sigma iz-D}$$

$$\kappa^* = \kappa+\Lambda$$

$$\vartheta^* = \kappa\vartheta/(\kappa+\Lambda)$$

Convention of variables

The Lewis proof above is a crossover between *Alma Dogg* and *ManWoNg*, a few modifications have been made, here is the list.

Comparison	*r and q*	*PDE of option*	*characteristic function variable*
Alma Dogg [bug!!!]	*assume r = q = 0*	*spot option = $V_t(t$, log price $X_t, v_t)$*	$\Phi_{\text{log normalized price}}(z)$
ManWoNg	*support r only*	*forward option = $U_t(t$, log moneyness $M_t, v_t)$*	$\Phi_{\text{log price}}(z)$
this document	*support r & g*	*forward option = $U_t(t$, log forward $X'_t, v_t)$*	$\Phi_{\text{log price}}(z)$

where

log price	$X_t = \ln S_t$		
log normalized price	$Y_t = \ln(S_t / F)$	*not used in this document*	
log moneyness	$M_t = \ln(e^{(r-q)(T-t)}S_t / K)$	*not used in this document*	
log forward	$X'_t = \ln(e^{(r-q)(T-t)}S_t)$		

Variable naming correspondence

	this proof	*Alma Dogg*	*ManWoNg*
Heston parameters	$v_0, \kappa, \theta, \sigma, \rho$	$v_0, \kappa, \theta, \sigma, \rho$	$v_0, \lambda, v, \eta, \rho$
ODE1 & 2	A, B	(α, β) or (C, D)	A, B
ODE2 coeff	P, Q, R	a, b, c	$\alpha, -\beta, \gamma$
ODE2 determinant	D	d	D
ODE2 roots and root ratio	$r_\pm, G$	$r_\pm, g$	$r_\pm, G$

Here a directed acyclic graph (DAG) of the variables.

variables	t, S_t, v_t		*ODE2 coeff*		*determinant, roots*		*decoupled tasks*
parameters	$v_0, \kappa, \theta, \sigma, \rho$	$\longrightarrow$	P,Q,R	$\longrightarrow$	$D, r_\pm, G$ = *root ratio*	$\longrightarrow$	*forward, DF, Φ*
exogenous	r, q, K, T, Λ						

Recall that Λ is related to volatility risk premium, please do not confuse it with λ, which is the market price of risk. Moreover, most references simply omit dividend q and volatility risk premium Λ.

Lewis approach for Black Scholes
Lets review some basic stuffs about Black Scholes :

$$
\begin{aligned}
S_T &= S_t e^{(r-q)(T-t)} e^{\varepsilon(-v/2,\sqrt{v})} && \textit{where variance is } v = \sigma^2(T-t) = \int_t^T \sigma_s^2 ds \\
F &= E[S_T \mid I_t] && \textit{recall } S_T \textit{ is random variable, while F is not, F is the expected value of } S_T \\
&= S_t e^{(r-q)(T-t)} E[e^{\varepsilon(-v/2,\sqrt{v})}] \\
&= S_t e^{(r-q)(T-t)} e^{-v/2+\sqrt{v}^2/2} && \textit{recall expectation of log normal } E[e^{\varepsilon(\mu,\sigma)}] = e^{\mu+\sigma^2/2} \textit{ or } E[e^{k\varepsilon(\mu,\sigma)}] = e^{k\mu+(k\sigma)^2/2} \\
&= S_t e^{(r-q)(T-t)}
\end{aligned}
$$

$$
\begin{aligned}
\textit{Hence} \quad S_T &= F e^{\varepsilon(-v/2,\sqrt{v})} \\
X_T &= \ln S_T \\
&= \ln S_t + (r-q)(T-t) + \varepsilon(-v/2,\sqrt{v}) \\
&= \varepsilon(\ln S_t + (r-q)(T-t) - v/2, \sqrt{v})
\end{aligned}
$$

The k^{th} moment of log normal is somehow related to characteristic function of normal random variable.

Characteristic function of normally distributed random variable is :

$$
\begin{aligned}
\Phi(z) &= E[e^{izx}] && \textit{where } x = \varepsilon(\mu,\sigma) \\
&= e^{i\mu z-(1/2)\sigma^2 z^2} && \textit{putting } k = iz \textit{ into } E[e^{k\varepsilon(\mu,\sigma)}] = e^{k\mu+(k\sigma)^2/2}
\end{aligned}
$$

Characteristic function of X_T for Black Scholes is thus :

$$
\begin{aligned}
\Phi_X(z) &= E[e^{izx}] && \textit{where } x = \varepsilon(\ln S_t + (r-q)(T-t) - v/2, \sqrt{v}) \\
&= e^{i(\ln S_t + (r-q)(T-t) - v/2)z - (1/2)vz^2} \\
&= e^{\ln S_t + (r-q)(T-t)} e^{-(1/2)vz(i+z)} \\
&= F e^{-(1/2)vz(i+z)}
\end{aligned}
$$

Plug the characteristic function into *equation 3*.

$$
\begin{aligned}
C(S_t) &= DF \times \left[F - \frac{\sqrt{K}}{\pi} \int_0^\infty \frac{\mathrm{Re}[\Phi(z-i/2)K^{-iz}]}{z^2+1/4} dz \right] \\
&= DF \times \left[F - \frac{\sqrt{K}}{\pi} \int_0^\infty \frac{\mathrm{Re}[F e^{-(1/2)v(z-i/2)(i+z-i/2)} K^{-i(z-i/2)}]}{(z-i/2)^2+1/4} dz \right] \\
&= DF \times \left[F - \frac{\sqrt{K}}{\pi} \frac{F}{\sqrt{K}} \int_0^\infty \frac{\mathrm{Re}[e^{-(1/2)v(z^2+1/4)} K^{-iz}]}{z^2-iz} dz \right] \\
&= DF \times \left[F - \frac{F}{\pi} \int_0^\infty \frac{\mathrm{Re}[e^{-(1/2)v(z^2+1/4)} e^{-iz\ln K}]}{z^2-iz} dz \right] \\
&= \ldots \\
&= DF \times [FN(d_1) - KN(d_2)]
\end{aligned}
$$