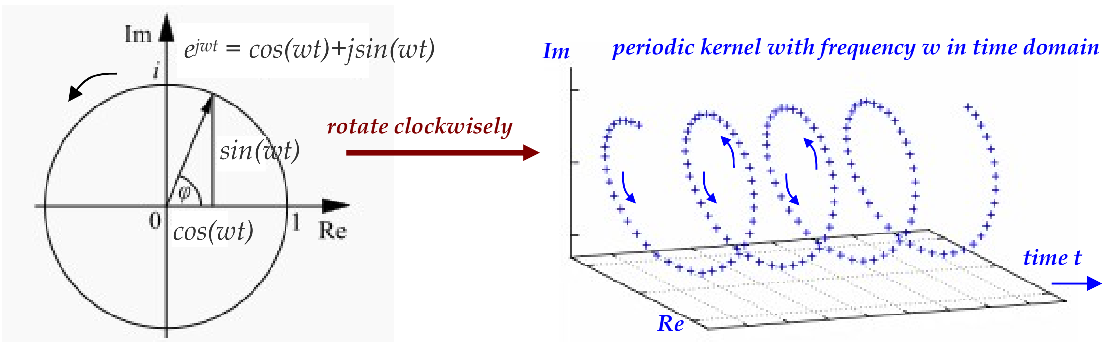
Comparison of CTFS (FS), CTFT, DTFS (DFS), DTFT, DFT, FFT, LT and ZT

All these transformations aim at mapping a function in time domain t or spatial domain x-y to frequency domain f, w or Z, by performing inner product between the time function (or spatial function) and a set of periodic kernels (also in time domain) with changing frequency w. With the help of Euler's formula, periodic kernels with changing frequency in time domain can be represented as rotation of a complex vector with changing angular speed w in Argand diagram. Lets take a look at continuous time Fourier transform (CTFT) as the general case :

$F(w)$	=	$\int_{-\infty}^{+\infty} f(t)e^{-jwt}dt$	$w \in \Re$	$F(w) \in C$	Fourier transform is complex integral.
$f(t)$	=	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(w)e^{+jwt}dw$	$t \in \Re$	$f(t) \in \Re$ (usually)	Is Fourier transform a functional?

Please note that t and w are real. In general *f(t)* can be complex (yet it is usually real practically), while *F(w)* is complex. Since *F(w)* is complex, it has a magnitude and a phase, intuitively *f(t)* can be interpreted as the weighted sum of phase shifted periodic kernels, with $|F(w)|$ as weight and *arg(F(w))* as phase shift. Here is a complex vector *F(w)*, it generates a periodic kernel (in time domain) when it rotates in anticlockwise direction with angular speed w radian per second.



Here is a question : in case *f(t)* is real (which is mostly the case), how do we ensure that the inverse transform formula returns a real result? Lets prove it. *Remark 1 : we can move Re or Im into the integrand only when dt and dw are real.*

	$f(t)$	=	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(w)e^{+jwt}dw$	
	$\mathrm{Im}(f(t))$	=	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Im}(F(w)e^{+jwt})dw$	*remark 1*
		=	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Re}(F(w))\sin(wt)dw + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Im}(F(w))\cos(wt)dw$	*equation 1*
as	$\mathrm{Re}(F(w))$	=	$\int_{-\infty}^{+\infty} \mathrm{Re}(f(t)e^{-jwt})dt$	
		=	$\int_{-\infty}^{+\infty} f(t)\cos(wt)dt$	*given f(t) is real*
and	$\mathrm{Im}(F(w))$	=	$\int_{-\infty}^{+\infty} \mathrm{Im}(f(t)e^{-jwt})dt$	
		=	$-\int_{-\infty}^{+\infty} f(t)\sin(wt)dt$	*given f(t) is real*

Put the above into equation 1, we have :

$$\mathrm{Im}(f(t)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(t)\cos(wt)dt\right)\sin(wt)dw - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(t)\sin(wt)dt\right)\cos(wt))dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(t)\cos(wt)\sin(wt) - f(t)\sin(wt)\cos(wt)dt\right)dw \;=\; 0$$

Hence, theoretically, given real *f(t)*, we will reconstruct real result from *CTFT*$^{-1}$*(CTFT(f(t)))*. Now we specialize the idea by discretizing either time domain or frequency domain. We see that periodicity in one domain implies discreteness in the counter domain, while aperiodicity in one domain implies continuity in the counter domain. Both time and frequency domain are infinite, i.e. $t,w \in (-\infty, +\infty)$ and $n,m \in (-\infty, +\infty)$ for CTFS, CTFT, DFTS and DTFT.

	time domain			frequency domain		
CTFS	continuous t	periodic T	infinite	aperiodic	discrete $m \times 2\pi/T$	infinite
CTFT	continuous t	aperiodic	infinite	aperiodic	continuous w	infinite
DTFS	discrete n	periodic N	infinite	periodic 2π	discrete $m \times 2\pi/N$	infinite
DTFT	discrete n	aperiodic	infinite	periodic 2π	continuous w	infinite

Here are the formulae for forward and inverse transform :

duality

CTFS	$f(t) = \sum_{-\infty}^{+\infty} a_m e^{+j(m\Delta w)t}$	$a_m = \frac{1}{T}\int_T f(t) e^{-j(m\Delta w)t} dt$	$\Delta w = 2\pi / T$
CTFT	$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(w) e^{+jwt} dw$	$F(w) = \int_{-\infty}^{+\infty} f(t) e^{-jwt} dt$	
DTFS	$f[n] = \sum_{0}^{N-1} a_m e^{+j(m\Delta w)n}$	$a_m = \frac{1}{N}\sum_{0}^{N-1} f[n] e^{-j(m\Delta w)n}$	$\Delta w = 2\pi / N$
DTFT	$f[n] = \frac{1}{2\pi}\int_{2\pi} F(w) e^{+jwn} dw$	$F(w) = \sum_{-\infty}^{+\infty} f[n] e^{-jwn}$	

Please note that :
(1) CTFS and DTFT form a duality.
(2) CTFS is also called Fourier series (FS).
(3) DTFS is also called discrete Fourier series (DFS).
(4) DFT assumes aperiodic finite time and aperiodic finite frequency.
(5) DFT can be regarded as sampling the spectrum of DTFT.
(6) DFT is mathematically identical to DFS.
(7) FFT is a fast implementation of DFT, which assumes N to be a power of 2.

<u>Question 1 : Why cant we apply CTFT for periodic continuous signal?</u>
Answer : since CTFT is defined for finite energy signal only, otherwise *F(w)* will not converge to a finite value.

$$E_f = \int_{-\infty}^{+\infty} f(t)^2 dt < \infty \qquad \text{for } f : \Re \to \Re$$

$$E_f = \int_{-\infty}^{+\infty} |f(t)|^2 dt < \infty \qquad \text{for } f : \Re \to C$$

Periodic signal does have infinite energy, for periodic *f(t)* with period *T* :

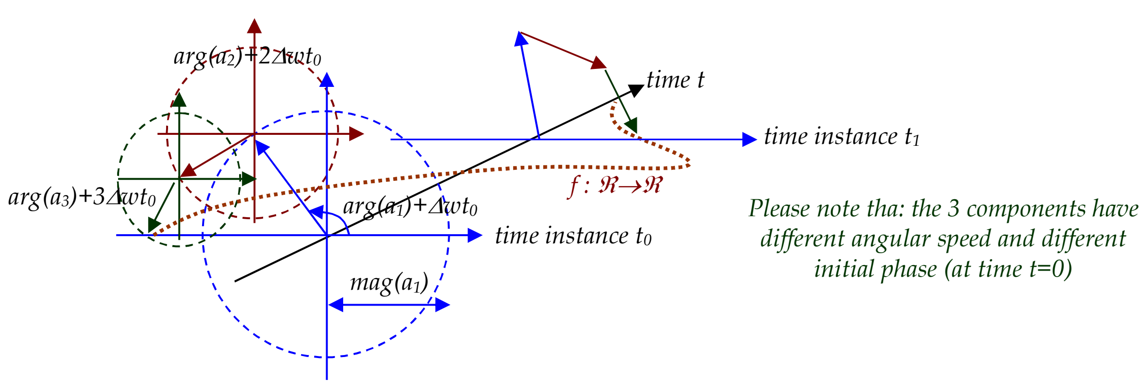
$$f(t) = f(t - nT) \qquad \forall n \in \aleph$$

$$E_f = \int_{-\infty}^{+\infty} f(t)^2 dt = \infty \times \int_T f(t)^2 dt \qquad \text{unless } \int_T f(t)^2 dt = 0$$

<u>Question 2 : Why does discreteness in one domain imply periodicity in another domain?</u>
Answer : Consider discreteness in frequency domain, in which $w = m\Delta w$, then in time domain, *f(t)* is a summation of periodic kernels with period $2\pi/(1\Delta w)$, $2\pi/(2\Delta w)$, $2\pi/(3\Delta w)$ … in time domain (recall : period = 2π/angular frequency), summation of a set of periodic kernels is also a periodic function with period given by the least common multiple of the set, which is $2\pi/(1\Delta w)$, thus *f(t)* is period in time domain with $T = 2\pi/\Delta w$, in other words, $\Delta w = 2\pi/T$.

<u>Question 3 : How are the complex vectors added together?</u>
Answer : Here is an illustration using CTFS. Suppose there are 3 components : blue a_1, red a_2, green a_3, where $a_n \in C$.

$$\begin{aligned} f(t) &= a_1 e^{+j\Delta wt} + a_2 e^{+j2\Delta wt} + a_3 e^{+j3\Delta wt} \\ &= mag(a_1) e^{+j(\arg(a_1)+\Delta wt)} + mag(a_2) e^{+j(\arg(a_2)+2\Delta wt)} + mag(a_3) e^{+j(\arg(a_2)+3\Delta wt)} \end{aligned}$$



<u>Question 4 : For real *f(t)*, why do we express it as a summation of real periodic kernels (instead of complex)?</u>
Answer : Yes, please see discrete cosine transform.

<u>Question 5 : How to derive CTFS, DTFS and DTFT?</u>
Answer : Please check

<u>Question 6 : Why does period of DTFS (and DTFT) equal to 2π?</u>
Answer : Please check

Laplace transform and Z transform

Feedback is common in linear time invariant systems, i.e. delayed outputs are fed back into some of the inputs. The frequency response of the system may therefore contain zeros and poles. Poles are locations where output becomes infinite, like pointing a speaker towards a microphone, or financial market consisted of a lot of trend following algo traders, which place orders by following the trend observed in market, and hence reinforcing the trend. In this case, Fourier transform does not converge, we need to generalize it to Laplace transform, by replacing the unit-magnitude complex exponential with any complex number.

CTFT $\quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(w)e^{+jwt}dw \qquad F(w) = \int_{-\infty}^{+\infty} f(t)e^{-jwt}dt$

LT $\quad f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{+st}ds \qquad F(s) = \int_{-\infty}^{+\infty} f(t)e^{-st}dt \qquad$ where $s = \sigma + jw$

where σ is a damping factor. Laplace transform can be regarded as Fourier transform of damped signal. :

$$f(t)e^{-\sigma t}$$

Fourier transform is a contour integral along contour C in complex plane. In case there exists a pole along the contour C, we define the Cauchy principal value of the integral as :

$$pv\int_C f(s)dz = \lim_{\varepsilon\to 0^+}\int_{C(\varepsilon)} f(s)dz$$

where contour $C(\varepsilon)$ is the same contour as C, while excluding the disk with radius ε around the pole.