Estimation of Pi

There are many ways to estimate pi. Here we introduce the simplest one : Leibniz's formula for pi. In order to derive Leibniz's formula, we need to start by reviewing "Chain Rule.doc" that :

Given $\quad y = f(x)$

then $\quad f^{-1}{}'(f) = 1/f'(x)$

i.e. $\quad \dfrac{df^{-1}(y)}{dy} = \left(\dfrac{df(x)}{dx}\right)^{-1}_{x=f^{-1}(y)}$

Lets consider arctan(x).

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x} - \frac{\sin x}{\cos^2 x}(-\sin x) = 1+\tan^2 x$$

$$\frac{d}{dy}\tan^{-1} y = \frac{1}{1+\tan^2 x} = \frac{1}{1+(\tan(\tan^{-1} y))^2} = \frac{1}{1+y^2}$$

thus

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\tan^{-1} x = \int_0^x \frac{1}{1+s^2}ds$$

$$= \int_0^x \sum_{n=0}^{\infty}(-s^2)^n ds \qquad \textit{geometric series} \quad 1+(-s^2)+(-s^2)^2+(-s^2)^3+\ldots = 1/(1+s^2)$$

$$= \sum_{n=0}^{\infty}(-1)^n \int_0^x s^{2n} ds$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \qquad \textit{this is called Leibniz seies}$$

Lets put x=1, we have :

$$\pi/4 = \tan^{-1}(1)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$