Derivative and Differential

Derivative
This is common in microeconomic and financial engineering. Whenever we have bivariate function *f(u,v)* :

(case 1) u,v are independent input to function f

$$\frac{\partial f(u,v)}{\partial v} = \lim_{\Delta v \to 0} \frac{f(u,v+\Delta v) - f(u,v)}{\Delta v}$$

$$\frac{df(u,v)}{dv} = \lim_{\Delta v \to 0} \frac{f(u,v+\Delta v) - f(u,v)}{\Delta v}$$

in this case : $\frac{df(u,v)}{du} = \frac{\partial f(u,v)}{\partial u}$

(case 2) however when we set $u = u(v)$, u,v then become dependent input to function f

$$\frac{\partial f(u(v),v)}{\partial v} = \lim_{\Delta v \to 0} \frac{f(u(v),v+\Delta v) - f(u(v),v)}{\Delta v}$$

$$\frac{df(u(v),v)}{dv} = \lim_{\Delta v \to 0} \frac{f(u(v+\Delta v),v+\Delta v) - f(u(v),v)}{\Delta v}$$

$$\frac{df(u(x),v(x))}{dx} = \lim_{\Delta x \to 0} \frac{f(u(x+\Delta x),v(x+\Delta x)) - f(u(x),v(x))}{\Delta x}$$

in this case : $\frac{df(u(v),v)}{dv} \neq \frac{\partial f(u(v),v)}{\partial v}$

In microeconomics, we have :

$$\frac{df}{dv} = \frac{\partial f}{\partial u}\frac{du}{dv} + \frac{\partial f}{\partial v}$$

where $u = u(v)$

In Ito's lemma, we have :

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial s}ds + \frac{1}{2}\frac{\partial f}{\partial s}(ds)^2$$

where $ds = g(dt, dz_t)$

Differential
This is a relation between differential change in certain variable with the differential change in other variables, it is useful especially when we cannot express derivative explicity in terms of other derivatives.

(case 1) u,v are independent

$$df(u,v) = \frac{\partial f(u,v)}{\partial u}du + \frac{\partial f(u,v)}{\partial v}dv = \frac{\partial f(u,v)}{\partial u}du + \frac{\partial f(u,v)}{\partial v}dv$$

(case 2) u,v are dependent

$$df(u(v),v) = \frac{\partial f(u,v)}{\partial u}du + \frac{\partial f(u,v)}{\partial v}dv = \frac{\partial f(u,v)}{\partial u}\frac{\partial u(v)}{\partial v}dv + \frac{\partial f(u,v)}{\partial v}dv$$

$$df(u(x),v(x)) = \frac{\partial f(u,v)}{\partial u}du + \frac{\partial f(u,v)}{\partial v}dv = \frac{\partial f(u,v)}{\partial u}\frac{\partial u(x)}{\partial x}dx + \frac{\partial f(u,v)}{\partial v}\frac{\partial v(x)}{\partial x}dx$$

Chain rule (please also read the chain rule document)

$$\begin{aligned}
\frac{df(g(x))}{dx} &= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta g \to 0} \frac{f(g(x)+\Delta g(x)) - f(g(x))}{\Delta g} \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= \frac{df(g)}{dg}\frac{dg(x)}{dx}
\end{aligned}$$

$$
\begin{aligned}
\frac{d^2 f(g(x))}{dx^2} &= \frac{d}{dx}\left[\frac{df(g)}{dg}\frac{dg(x)}{dx}\right] \\
&= \frac{d^2 f(g)}{dg^2}\left(\frac{dg(x)}{dx}\right)^2 + \frac{df(g)}{dg}\frac{d^2 g(x)}{dx^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^3 f(g(x))}{dx^3} &= \frac{d}{dx}\left[\frac{d^2 f(g)}{dg^2}\left(\frac{dg(x)}{dx}\right)^2 + \frac{df(g)}{dg}\frac{d^2 g(x)}{dx^2}\right] \\
&= \frac{d^3 f(g)}{dg^3}\left(\frac{dg(x)}{dx}\right)^3 + 2\frac{d^2 f(g)}{dg^2}\frac{d^2 g(x)}{dx^2}\frac{dg(x)}{dx} + \frac{d^2 f(g)}{dg^2}\frac{d^2 g(x)}{dx^2}\frac{dg(x)}{dx} + \frac{df(g)}{dg}\frac{d^3 g(x)}{dx^3} \\
&= \frac{d^3 f(g)}{dg^3}\left(\frac{dg(x)}{dx}\right)^3 + 3\frac{d^2 f(g)}{dg^2}\frac{d^2 g(x)}{dx^2}\frac{dg(x)}{dx} + \frac{df(g)}{dg}\frac{d^3 g(x)}{dx^3}
\end{aligned}
$$

$$
\frac{\partial f_k}{\partial x_n} = \frac{\partial f_k}{\partial g_1}\frac{\partial g_1}{\partial x_n} + \frac{\partial f_k}{\partial g_2}\frac{\partial g_2}{\partial x_n} + \ldots + \frac{\partial f_k}{\partial g_M}\frac{\partial g_M}{\partial x_n} \qquad (\, f:\Re^M \to \Re^K \,,\; g:\Re^N \to \Re^M \,,\; x \in \Re^N \,)
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