Ornstein Uhlenbeck Process

Ornstein process is a mean reverting process, it is used to model volatility square in Heston model.

$$dx_t = \kappa(\theta - x_t)dt + \sigma dz_t$$

Lets solve for x_t. First of all, put :

$$\begin{aligned}
y_t &= x_t - \theta && \textit{i.e. } y_t \textit{ is zero-reverting} \\
dy_t &= dx_t && \textit{by Ito's lemma} \\
&= \kappa(\theta - x_t)dt + \sigma dz_t \\
&= -\kappa y_t dt + \sigma dz_t
\end{aligned}$$

Then we remove the drift by putting :

$$\begin{aligned}
z_t &= e^{\kappa t} y_t \\
dz_t &= \partial_t(e^{\kappa t} y_t)dt + \partial_y(e^{\kappa t} y_t)dy_t + \frac{1}{2}\partial_{yy}(e^{\kappa t} y_t)(dy_t)^2 && \textit{by Ito's lemma} \\
&= \kappa e^{\kappa t} y_t dt + e^{\kappa t} dy_t + \frac{1}{2}0(dy_t)^2 \\
&= \kappa e^{\kappa t} y_t dt + e^{\kappa t}(-\kappa y_t dt + \sigma dz_t) \\
&= \sigma e^{\kappa t} dz_t
\end{aligned}$$

Finally, integrate both sides from s to t. RHS is an Ito's integral, it is a Gaussian with zero mean and variance as a time integral :

$$\begin{aligned}
z_t - z_s &= \int_s^t \sigma e^{\kappa\tau} dz_\tau \\
&= \varepsilon\left(mean = 0, \text{var} = \int_s^t \sigma e^{2\kappa\tau} d\tau \right) \\
&= \varepsilon\left(mean = 0, \text{var} = \frac{1}{2\kappa}\int_s^t \sigma e^{2\kappa\tau} d2\kappa\tau \right) \\
&= \varepsilon\left(mean = 0, \text{var} = \frac{\sigma}{2\kappa}(e^{2\kappa t} - e^{2\kappa s}) \right) \\
&= \varepsilon\sqrt{\frac{\sigma}{2\kappa} e^{2\kappa}(e^t - e^s)}
\end{aligned}$$

Reversing the change of variables, we have :

$$\begin{aligned}
y_t &= e^{-\kappa t} z_t \\
&= e^{-\kappa t}\left(z_s + \sigma\int_s^t e^{\kappa\tau} dz_\tau \right) \\
&= e^{-\kappa t} z_s + \sigma\int_s^t e^{-\kappa(t-\tau)} dz_\tau \\
&= e^{-\kappa(t-s)} e^{-\kappa s} z_s + \sigma\int_s^t e^{-\kappa(t-\tau)} dz_\tau \\
&= e^{-\kappa(t-s)} y_s + \sigma\int_s^t e^{-\kappa(t-\tau)} dz_\tau \\
x_t &= \theta + y_t \\
&= \theta + e^{-\kappa(t-s)} y_s + \sigma\int_s^t e^{-\kappa(t-\tau)} dz_\tau \\
&= \theta + e^{-\kappa(t-s)}(x_s - \theta) + \sigma\int_s^t e^{-\kappa(t-\tau)} dz_\tau \\
&= \theta + e^{-\kappa(t-s)}(x_s - \theta) + \sigma\left(\int_s^t e^{-2\kappa(t-\tau)} d\tau \right)\varepsilon
\end{aligned}$$