

Why ain't American call exercised?

American call on stock without dividend

Suppose current time is 0, let's consider *two* portfolios, the former contains only a vanilla call option with maturity T while the latter contains the underlying stock, together with a short position in riskless cash that amounts to K at maturity T , therefore PV of cash is $K \times DF_{0,T}$. There are two possible outcomes (economic states) at T , which are S_T lying above and below K .

time	portfolio 1	portfolio 2	comparison
0	C_t	$S_t - K \times DF_{0,T}$	
T with state $S_T > K$	$C_T = S_T - K$	$S_T - K$	portfolio1 = portfolio 2
T with state $S_T < K$	$C_T = 0$	$S_T - K$	portfolio1 > portfolio 2

Since the future value of portfolio 1 is greater than or the same as that of portfolio 2 under both states, the present value of portfolio 1 is thus greater than that of portfolio 2. PV of portfolio 1 is simply current call price, while PV of portfolio 2 is intrinsic value, hence their difference is the time value, which is non-negative.

$$\begin{aligned}
 C_t &> S_t - K \times DF_{0,T} && \text{equation 1} \\
 \text{hence } C_t &= \underbrace{S_t - K \times DF_{0,T}}_{\text{part1}} + \underbrace{V_t}_{\text{part2}} \\
 \text{intrinsic value} &= S_t - K \times DF_{0,T} && \text{can be positive or negative depending moneyness} \\
 \text{time value} &= V_t > 0 && \text{must be positive, according to equation 1}
 \end{aligned}$$

Decision of early exercise at time 0 depends on its profit and loss :

$$\begin{aligned}
 \text{profit of early exercise} &= (S_t - K) - C_t \\
 \text{loss of early exercise} &= C_t - (S_t - K) \\
 &= \underbrace{C_t - (S_t - K \times DF_{0,T})}_{V_t} + \underbrace{K - K \times DF_{0,T}}_{\text{strike-loss}} > 0 && \text{since } V_t > 0 \text{ and } K - K \times DF_{0,T} > 0
 \end{aligned}$$

The loss of early exercise can be decomposed into two parts : abandoned time value and strike loss. Strike loss incurred is the result of a higher strike than it supposed to be, which is PV of strike ($K \times DF_{0,T}$), when exercise at time 0. As both time value and strike loss are positive, there is always a positive loss on early exercise of American call for underlying without dividend.

American put on stock without dividend

Lets repeat the same procedure for put option.

time	portfolio 1	portfolio 2	comparison
0	P_t	$K \times DF_{0,T} - S_t$	
T with state $S_T < K$	$P_T = K - S_T$	$K - S_T$	portfolio1 = portfolio 2
T with state $S_T > K$	$P_T = 0$	$K - S_T$	portfolio1 > portfolio 2

Since the future value of portfolio 1 is greater than or the same as that of portfolio 2 under both states, so are the PVs.

$$\begin{aligned}
 P_t &> K \times DF_{0,T} - S_t && \text{equation 2} \\
 \text{hence } P_t &= \underbrace{K \times DF_{0,T} - S_t}_{\text{part1}} + \underbrace{V_t}_{\text{part2}} \\
 \text{intrinsic value} &= K \times DF_{0,T} - S_t && \text{can be positive or negative depending moneyness} \\
 \text{time value} &= V_t > 0 && \text{must be positive, according to equation 2}
 \end{aligned}$$

Decision of early exercise at time 0 depends on its profit and loss :

$$\begin{aligned}
 \text{profit of early exercise} &= (K - S_t) - P_t \\
 \text{loss of early exercise} &= P_t - (K - S_t) \\
 &= \underbrace{P_t - (K \times DF_{0,T} - S_t)}_{V_t} - \underbrace{(K - K \times DF_{0,T})}_{\text{strike-gain}}
 \end{aligned}$$

Unlike American call, the loss of early exercise of American put equals to the abandoned time value minus a strike gain. Strike gain is the result of a higher strike received than it supposed to be, which is PV of strike ($K \times DF_{0,T}$), when exercise at time 0. Though time value is positive, it can be compensated by strike gain, there exists possibility of early exercise for American put.

American call on stock with dividend

Lets repeat the same procedure for call option on stock with dividend D at time t such that $0 < t < T$. This time portfolio 1 is a combo of call option, together with a long position in riskless cash that amounts to D at maturity t .

<i>time</i>	<i>portfolio 1</i>	<i>portfolio 2</i>
0	$C_t + D \times DF_{0,t}$	$S_t - K \times DF_{0,T}$
T with state $S_T > K$	$C_T + D / DF_{t,T} = S_T - K + D / DF_{t,T}$	$S_T + D / DF_{t,T} - K$
T with state $S_T < K$	$C_T + D / DF_{t,T} = D / DF_{t,T}$	$S_T + D / DF_{t,T} - K$

Since the future value of portfolio 1 is greater than or the same as that of portfolio 2 under both states, so are the PVs .

$$\begin{aligned}
 C_t + D \times DF_{0,t} &> S_t - K \times DF_{0,T} && \text{equation 3} \\
 \text{hence } C_t &= \underbrace{S_t - K \times DF_{0,T} - D \times DF_{0,t}}_{\text{part1}} + \underbrace{V_t}_{\text{part2}} \\
 \text{intrinsic value} &= S_t - K \times DF_{0,T} - D \times DF_{0,t} && \text{can be postive or negative depending moneyness} \\
 \text{time value} &= V_t > 0 && \text{must be positive, according to equation 3}
 \end{aligned}$$

Decision of early exercise at time 0 depends on its profit and loss :

$$\begin{aligned}
 \text{profit of early exercise} &= (S_t - K) - C_t \\
 \text{loss of early exercise} &= C_t - (S_t - K) \\
 &= \underbrace{C_t - (S_t - K \times DF_{0,T} - D \times DF_{0,t})}_{V_t} + \underbrace{K - K \times DF_{0,T}}_{\text{strike-loss}} - \underbrace{D \times DF_{0,t}}_{\text{dividend-gain}}
 \end{aligned}$$

The loss can either be positive or negative, depending on whether the dividend gain can compensate time value loss and strike loss. Therefore, there exists possibility of early exercise for American call on stock with dividend.