Heston Model – Bakshi Madan approach *from scratch*

Price in terms of *ITM* probabilities

Bakshi Madan approach starts with vanilla call option price *(copied from my "Fundamental theorem of asset pricing.doc")* :

$$
\begin{aligned}
C(S_t) &= e^{-r(T-t)}E_{Q_B}[(S_T-K)^+] && \textit{note } C(S_t) \textit{ is option price, } C(S_T) \textit{ is option payoff}\\
&= e^{-r(T-t)}E_{Q_B}[(S_T-K)1_{S_T>K}]\\
&= e^{-r(T-t)}E_{Q_B}[S_T 1_{S_T>K}]-e^{-r(T-t)}KE_{Q_B}[1_{S_T>K}]\\
&= S_t E_{Q_B}\left[\frac{e^{-r(T-t)}}{S_t/S_T}1_{S_T>K}\right]-e^{-r(T-t)}KE_{Q_B}[1_{S_T>K}]\\
&= S_t E_{Q_S}[1_{S_T>K}]-e^{-r(T-t)}KE_{Q_B}[1_{S_T>K}]\\
&= S_t P_1-e^{-r(T-t)}KP_2 && \textit{(equation A)}
\end{aligned}
$$

both are not physical measure

where $P_1(t,s,v) = \Pr_{Q_S}(S_T>K\,|\,S_t=s, v_t=v)$ *risk neutral measure for stock numeraire*

$P_2(t,s,v) = \Pr_{Q_B}(S_T>K\,|\,S_t=s, v_t=v)$ *risk neutral measure for cash numeraire*

Hence option price is expressed in terms of two *ITM* probabilities (under different measures). Unlike Lewis approach, which works theoretically for all payoffs and all models, Bakshi Madan approach works only for vanilla payoff, though it is applicable to various models by plugging in appropriate *ITM* probabilities. In order to solve the *ITM* probabilities for Heston model, we have to derive a *PDE* of *ITM* probabilities from Heston *SDE*, and prior to that, we have to derive a *PDE* of derivative price from Heston *SDE*, which can be done by setting up a risk free portfolio (like Black Scholes and Lewis approach), or simply by Dr Yan's fast method, which is known as ***backward Kolmogorov equation*** formally. Without step by step proof, we give the result as following, which is the same *equation 4* in *"Heston model1 – Lewis approach.doc"* (don't forget the terminal conditions) :

$$
\begin{aligned}
rV_t &= \partial_t V_t+(r-q)S_t\partial_s V_t+\kappa^*(\theta^*-v_t)\partial_v V_t+\frac{1}{2}v_tS_t^2\partial_{ss}V_t+\rho\sigma S_tv_t\partial_{sv}V_t+\frac{1}{2}\sigma^2v_t\partial_{vv}V_t && \textit{by Dr Yan's fast lane}\\
&= \partial_t V_t+(r-q)S_t\partial_s V_t+(\kappa(\theta-v_t)-\Lambda v_t)\partial_v V_t+\frac{1}{2}v_tS_t^2\partial_{ss}V_t+\rho\sigma S_tv_t\partial_{sv}V_t+\frac{1}{2}\sigma^2v_t\partial_{vv}V_t && \textit{(equation B)}
\end{aligned}
$$

$$
\begin{aligned}
s.t\quad V(T,S_T,v_T) &= (S_T-K)^+ && \textit{payoff} && \textit{where } \kappa^*=\kappa+\Lambda \textit{ and } \vartheta^*=\kappa\vartheta/\kappa^*=\kappa\vartheta/(\kappa+\Lambda)\\
V(t,0,v_t) &= 0 && \textit{asymptotic}\\
V(t,S_t,\infty) &= S_t && \textit{asymptotic}\\
\partial_s V(t,\infty,v_t) &= 1 && \textit{asymptotic}
\end{aligned}
$$

Changing variables

In Lewis approach, we convert the *PDE* of $V_t = V(t, S_t, v_t)$ into the *PDE* of $U_t = U(\tau, X'_t, v_t)$, where :

$$
\begin{aligned}
\tau &= T-t\\
X_t &= \ln S_t\\
X'_t &= \ln F = \ln(e^{(r-q)\tau}S_T)\\
U_t &= e^{r\tau}V_t
\end{aligned}
$$

We will explain in next page that :

- *taking log helps to remove S_t term*
- *looking forward for **both** option and stock helps to handle dividend q*

Various approaches may apply different conversions. For Bakshi Madan approach, we convert the *PDE* into $V_t = V(\tau, X_t, v_t)$:

$$
\begin{aligned}
&(i) & \partial_t X_t &= \partial_t \ln S_t = 0\\
&(ii) & \partial_s X_t &= \partial_s \ln S_t = 1/S_t\\
&(iii) & \partial_{ss} X_t &= \partial_{ss} \ln S_t = -1/S_t^2
\end{aligned}
$$

$$
\begin{aligned}
\textit{then}\quad \partial_t V_t &= -\partial_\tau V_t\\
\partial_s V_t &= \partial_x V_t\partial_s X_t = (\partial_x V_t)/S_t && \textit{using remark (ii)}\\
\partial_{ss}V_t &= \partial_s((\partial_x V_t)/S_t)\\
&= (\partial_{sx}V_t)/S_t-(\partial_x V_t)/S_t^2\\
&= (\partial_{xx}V_t)(\partial_s X_t)/S_t-(\partial_x V_t)/S_t^2\\
&= (\partial_{xx}V_t-\partial_x V_t)/S_t^2 && \textit{using remark (ii)}\\
\partial_{sv}V_t &= (\partial_{xv}V_t)/S_t && \textit{(equation set C\#)}
\end{aligned}
$$

whereas theta, vega and vomma remain unchanged, as they do not involve underlying. Plug the above into *equation B*, we have :

$$\begin{aligned}
rV_t &= \partial_t V_t + (r-q)S_t\partial_s V_t + (\kappa(\theta - v_t) - \Lambda v_t)\partial_v V_t + \frac{1}{2}v_t S_t^2 \partial_{ss} V_t + \rho\sigma S_t v_t \partial_{sv} V_t + \frac{1}{2}\sigma^2 v_t \partial_{vv} V_t \\
&= \partial_t V_t + (r-q)\partial_x V_t + (\kappa(\theta - v_t) - \Lambda v_t)\partial_v V_t + \frac{1}{2}v_t(\partial_{xx} V_t - \partial_x V_t) + \rho\sigma v_t \partial_{xv} V_t + \frac{1}{2}\sigma^2 v_t \partial_{vv} V_t \\
&= -\partial_\tau V_t + (r-q-v_t/2)\partial_x V_t + (\kappa(\theta - v_t) - \Lambda v_t)\partial_v V_t + \frac{1}{2}v_t \partial_{xx} V_t + \rho\sigma v_t \partial_{xv} V_t + \frac{1}{2}\sigma^2 v_t \partial_{vv} V_t \qquad \textit{(equation C)}
\end{aligned}$$

We apply log to underlying price in both Lewis and Bakshi, the effect of log is to remove S_t term that goes along with delta, gamma and vanna. Besides, log does also introduce a shift of $-v_t/2$ in the delta term, which can then be explaint by considering *equation C* as the ***backward Kolmogorov equation*** of the following system :

$$\begin{aligned}
dX_t &= (r-q-v_t/2)dt + \sqrt{v_t}\,dz_{1t} \qquad \textit{where } dX_t = d\ln S_t = (1/S_t)dS_t - (1/S_t^2)(dS_t)^2 = \ldots \qquad \textit{by Itos lemma} \\
dv_t &= \kappa(\vartheta - v_t)dt + \sigma\sqrt{v_t}\,dz_{2t} \\
dz_{1t}dz_{2t} &= \rho dt
\end{aligned}$$

In Lewis, we have done further conversions, i.e. applying forward on stock and forward on option, so as to remove rV_t term and r-q term. This is a technique to handle underlying dividend (as forward of option grows at rate of r, while forward of dividend paying stock grows at rate of r-q, looking forward does help to remove the mismatch between r term and r-q term in *LHS&RHS* of equation *B* respectively). As no dividend is considered in Rouah's book, there is no need to look forward when we derive the *PDE*. There is a comparison of *equation 5* for Lewis and *equation C* for Bakshi for stock with zero dividend (we also remove *t* index for simplicity) :

$$\begin{aligned}
0 &= -\partial_\tau U + (\kappa(\theta - v_t) - \Lambda v_t)\partial_v U + \frac{1}{2}v_t(\partial_{x'x'} U - \partial_{x'} U) + \rho\sigma v_t \partial_{x'v} U + \frac{1}{2}\sigma^2 v_t \partial_{vv} U \qquad \textit{equation 5 in Lewis} \\
rV &= -\partial_\tau V + (r - v_t/2)\partial_x V + (\kappa(\theta - v_t) - \Lambda v_t)\partial_v V + \frac{1}{2}v_t \partial_{xx} V + \rho\sigma v_t \partial_{xv} V + \frac{1}{2}\sigma^2 v_t \partial_{vv} V \quad \textit{(put } q = 0\textit{)} \qquad \textit{equation C in Bakshi}
\end{aligned}$$

***PDE* of *ITM* probabilities**
Vanilla option $C(S_t)$ in equation *1* must satisfy *PDE* in equation *II*, hence by plugging $V_t = C(S_t)$:

$$\begin{aligned}
rC &= -\partial_\tau C + (r - v_t/2)\partial_x C + (\kappa(\theta - v_t) - \Lambda v_t)\partial_v C + \frac{1}{2}v_t \partial_{xx} C + \rho\sigma v_t \partial_{xv} C + \frac{1}{2}\sigma^2 v_t \partial_{vv} C \\
C &= e^{X_t}P_1 - e^{-r\tau}KP_2
\end{aligned}$$

$$\begin{aligned}
\textit{where} \quad \partial_\tau C &= e^{X_t}(\partial_\tau P_1) - e^{-r\tau}K(-rP_2 + \partial_\tau P_2) \\
\partial_x C &= e^{X_t}(P_1 + \partial_x P_1) - e^{-r\tau}K(\partial_x P_2) \\
\partial_v C &= e^{X_t}(\partial_v P_1) - e^{-r\tau}K(\partial_v P_2) \\
\partial_{xx} C &= \partial_x(e^{X_t}(P_1 + \partial_x P_1) - e^{-r\tau}K(\partial_x P_2)) = e^{X_t}(P_1 + 2\partial_x P_1 + \partial_{xx} P_1) - e^{-r\tau}K(\partial_{xx} P_2) \\
\partial_{xv} C &= \partial_v(e^{X_t}(P_1 + \partial_x P_1) - e^{-r\tau}K(\partial_x P_2)) = e^{X_t}(\partial_v P_1 + \partial_{xv} P_1) - e^{-r\tau}K(\partial_{xv} P_2) \\
\partial_{vv} C &= e^{X_t}(\partial_{vv} P_1) - e^{-r\tau}K(\partial_{vv} P_2) \qquad \textit{(equation D\#)}
\end{aligned}$$

We then group terms common to P_1 and terms common P_2 separately. As the *PDE* is valid for all X_t K and r, we can substitute $X_t = 0$ and $K = 0$ to get a *PDE* with P_1 terms only, we can also substitute $S_t = 0$, $K = 1$ and $r = 0$ to get a *PDE* with P_2 terms only. In conclusion, we generate *2 PDEs*, each governs the *ITM* probability under different measures :

$$\begin{aligned}
\textit{(1)} \quad \cancel{rP_1} &= -(\partial_\tau P_1) + (r - v_t/2)(\cancel{P_1} + \partial_x P_1) + (\kappa(\theta - v_t) - \Lambda v_t)(\partial_v P_1) + \frac{1}{2}v_t(\cancel{P_1} + 2\partial_x P_1 + \partial_{xx} P_1) + \rho\sigma v_t(\partial_v P_1 + \partial_{xv} P_1) + \frac{1}{2}\sigma^2 v_t(\partial_{vv} P_1) \\
0 &= -(\partial_\tau P_1) + (r + v_t/2)(\partial_x P_1) + (\kappa(\theta - v_t) - \Lambda v_t + \rho\sigma v_t)(\partial_v P_1) + \frac{1}{2}v_t(\partial_{xx} P_1) + \rho\sigma v_t(\partial_{xv} P_1) + \frac{1}{2}\sigma^2 v_t(\partial_{vv} P_1) \\
\textit{(2)} \quad rP_2 &= -(-rP_2 + \partial_\tau P_2) + (r - v_t/2)(\partial_x P_2) + (\kappa(\theta - v_t) - \Lambda v_t)(\partial_v P_2) + \frac{1}{2}v_t(\partial_{xx} P_2) + \rho\sigma v_t(\partial_{xv} P_2) + \frac{1}{2}\sigma^2 v_t(\partial_{vv} P_2) \\
0 &= -(\partial_\tau P_2) + (r - v_t/2)(\partial_x P_2) + (\kappa(\theta - v_t) - \Lambda v_t)(\partial_v P_2) + \frac{1}{2}v_t(\partial_{xx} P_2) + \rho\sigma v_t(\partial_{xv} P_2) + \frac{1}{2}\sigma^2 v_t(\partial_{vv} P_2)
\end{aligned}$$

These two *PDEs* differ in the delta term and vega term only, they can be combined together for clarity :

$$-\partial_\tau P_n + (r \pm \frac{1}{2}v_t)(\partial_x P_n) + (\kappa\theta - b_n v_t)(\partial_v P_n) + \frac{1}{2}v_t(\partial_{xx}P_n) + \rho\sigma v_t(\partial_{xv}P_n) + \frac{1}{2}\sigma^2 v_t(\partial_{vv}P_n) = 0 \quad \forall n \in [1,2]$$

where $b_1 = \kappa + \Lambda - \rho\sigma$

$b_2 = \kappa + \Lambda$ *(equation D)*

When n = 2 and setting r = 0, equation III is equivalent to equation II in Lewis.doc.

When we solve a *PDE*, we need two more components which are terminal condition and an ansatz. Terminal condition is necessary, different terminal conditions will result in different solutions. Ansatz is a reasonable guess so that it matches the terminal condition on reaching boundary (i.e. maturity in option pricing). For Heston model, here is the terminal condition and ansatz are :

$$P_n(T, X_T = x, v_T = v) = 1_{x > \ln K} \quad \textit{terminal condition} \quad \forall n \in [1,2]$$

$$P_n(t, X_t = x, v_t = v) = \textit{function}(t,x,v)??? \quad \textit{ansatz that approaches to terminal conditions}$$

An ansatz must approach terminal condition as time moves towards maturity, while the above terminal condition is discontinuous along the stock price axis, hence it is difficult to make a reasonable ansatz for *ITM* probabilities. That's why most Heston references do not solve the *PDE* directly for ITM probabilities, instead we solve for characteristic functions under the two measures, Q_B and Q_S. The latter approach is feasible because of three reasons, i.e. *(1)* inversion theorem which gives *ITM* probabilities using characteristic functions, (2) backward Kolmogorov equation which implies that *ITM* probabilities and characteristic functions are solutions to the same *SDE* system with respect to different terminal conditions, and *(3)* continuity of characteristic functions in $X_T = x$ domain at T.

Inversion theorem

Inverse theorem allows us to find *ITM* probability given characteristic function (please read *Lewis.doc* for proof) :

$$\Pr_X(x > x_0) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\frac{e^{-izx_0}}{iz}\Phi_X(z)]dz \quad \textit{approach 1 : inverse theorem}$$

Besides, we can also find *ITM* probability given characteristic function using its definition :

$$\Pr_X(x > x_0) = \int_{x_0}^\infty [\overbrace{\frac{1}{2\pi}\int_{-\infty}^\infty e^{-izx}\Phi_X(z)dz}^{p_X(x) = FT^{-1}[\Phi_X(z)]}]dx \quad \textit{approach 2 : integrating cdf}$$

Backward Kolmogorov equations

Equation D is in fact the backward Kolmogorov equations of the following two *SDE* systems :

$$dX_t = (r \pm v_t/2)dt + \sqrt{v_t}\,dz_{1t}$$

$$dv_t = (\kappa\theta - b_n)dt + \sigma\sqrt{v_t}\,dz_{2t} \quad \forall n \in [1,2]$$

$$dz_{1t}dz_{2t} = \rho dt$$

Therefore by solving *PDE equation D* with respect to different terminal conditions, we have different solutions :

$$\textit{wrt condition : } P_n(T, X_T = x, v_T = v) = 1_{x > \ln K} \quad \textit{gives solution : } P_n(t, X_t = x, v_t = v) = E_{Q_n}[1_{X_T > \ln K} \mid I_t] = \overbrace{\Pr_{Q_n}(X_T > \ln K)}^{ITM}$$

$$\textit{wrt condition : } P_n(T, X_T = x, v_T = v) = e^{izx} \quad \textit{gives solution : } P_n(t, X_t = x, v_t = v) = E_{Q_n}[e^{izX_T} \mid I_t] = \Phi_{Q_n}(z)$$

Therefore, we can either solve for *ITM* probabilities using the former boundary condition, or solve for characteristic functions using the latter boundary condition. Most Heston references prefer the latter, as we cannot make a good ansatz for the former.

Continuity of characteristic function

Ansatz for characteristic function is possible because it is continuous. Let's check.

$$\lim_{\Delta \to 0} e^{iz(x+\Delta)} = e^{izx}$$

Thus we postulate the following ansatz :

$$\Phi_{Q_n}(t, X_t = x, v_t = v) = e^{A_n(z,\tau) + B_n(z,\tau)v + izx} \quad \textit{please standardise the convention in Lewis.doc and Bakshi.doc}$$

This ansatz must satisfy the terminal condition that X'_T becomes deterministic when $\tau = 0$:

$$\Phi_{Q_n}(T, X_T = x, v_T = v) = e^{A_n(z,0)+B_n(z,0)v+izx}$$ *this ansatz is identical to that in Lewis*

$$= e^{izx}$$ *for all x and v only if A(z,0) =0 and B(z,0) =0 at maturity*

PDE of characteristic functions

Next, we are going to solve *equation D* for characteristic functions under the two measures using the above ansatz *wrt* new terminal conditions *A(z,0) =0 and B(z,0) =0* at maturity. Since this ansatz is identical to that in *Lewis.doc*, its derivatives are also the same, thus *equation set 6#* of *Lewis.doc* is also applicable here. By plugging *equation set 6#* of *Lewis.doc* in *equation D* of *Bakshi.doc*, we have :

differences between Lewis and Bakshi

$$\begin{aligned}
0 &= -\partial_\tau P_n + \boxed{(r \pm \tfrac{1}{2}v_t)}(\partial_x P_n) + \boxed{(\kappa\theta - b_n v_t)}(\partial_v P_n) + \tfrac{1}{2}v_t(\partial_{xx}P_n) + \rho\sigma v_t(\partial_{xv}P_n) + \tfrac{1}{2}\sigma^2 v_t(\partial_{vv}P_n) \\
&= -\Phi(\partial_\tau A_n + \partial_\tau B_n v_t) + (r \pm \tfrac{1}{2}v_t)(\Phi iz) + (\kappa\theta - b_n v_t)(\Phi B_n) - \tfrac{1}{2}v_t(\Phi z^2) + \rho\sigma v_t(\Phi izB_n) + \tfrac{1}{2}\sigma^2 v_t(\Phi B_n^2) \\
&= -(\partial_\tau A_n + \partial_\tau B_n v_t) + (r \pm \tfrac{1}{2}v_t)iz + (\kappa\theta - b_n v_t)B_n - \tfrac{1}{2}v_t z^2 + \rho\sigma v_t(izB_n) + \tfrac{1}{2}\sigma^2 v_t B_n^2 && \textit{cancel } \Phi \textit{ on both sides} \\
&= -\partial_\tau A_n + riz + \kappa\theta B_n + \left[-\partial_\tau B_n \pm \tfrac{1}{2}iz - b_n B_n - \tfrac{1}{2}z^2 + \rho\sigma(izB_n) + \tfrac{1}{2}\sigma^2 B_n^2\right]v_t && \textit{same as eq1.46 in Rouah book}
\end{aligned}$$

Following the same logic as *Lewis*, we can breakdown the *PDE* into two *ODEs*, they are similar to *equation 6a-b* in *Lewis.doc*.

$$ODE1 \quad \partial_\tau A_n = riz + \kappa\theta B_n \qquad \textit{such that } A(z,0) = 0 \qquad (\textit{equation Ea})$$

$$ODE2 \quad \partial_\tau B_n = \underbrace{-\tfrac{1}{2}(z^2 \mp iz)}_{P_n} + \underbrace{(\rho\sigma iz - b_n)}_{Q_n}B_n + \underbrace{\tfrac{1}{2}\sigma^2}_{R_n}B_n^2 \qquad \textit{such that } B(z,0) = 0 \qquad (\textit{equation Eb})$$

Using Riccati technique, which gives analytic solution to the following ODE :

$$\begin{aligned}
\partial_x f &= P + Qf + Rf^2 \\
\Rightarrow \quad f(x) &= \frac{1-e^{Dx}}{1-Ge^{Dx}}r_+
\end{aligned}$$

where D,G and r_+ are determinant, root ratio (i.e. positive root : negative root) and the positive root respectively.

Therefore we have an analytic solution for B_n, which is the same as equation *1.58* in *Rouah book (identical to B in Lewis when n = 2)* :

$$B_n(z,\tau) = \frac{1-e^{D_n\tau}}{1-G_n e^{D_n\tau}}r_{n+} \qquad \forall n \in [1,2] \qquad \textit{apply equation 7 in Lewis.doc}$$

$$\begin{aligned}
\textit{where} \quad D_n &= \sqrt{Q_n^2 - 4P_nR_n} &&= \sqrt{(\rho\sigma iz - b_n)^2 + (z^2 \mp iz)\sigma^2} \\
G_n &= \frac{r_{n+}}{r_{n-}} = \frac{-Q_n + D_n}{-Q_n - D_n} &&= \frac{b_n - \rho\sigma iz + D_n}{b_n - \rho\sigma iz - D_n} \\
r_{n+} &= \frac{-Q_n + D_n}{2R_n} &&= \frac{b_n - \rho\sigma iz + D_n}{\sigma^2} \\
\textit{recall} \quad b_1 &= \kappa + \Lambda - \rho\sigma \\
b_2 &= \kappa + \Lambda
\end{aligned}$$

Besides we have an analytic solution for A_n, which is the same as equation *1.62* in *Rouah book (identical to A in Lewis when n = 2)* :

beware that r is risk free rate

r_{n+} r_{n-} are roots, don't confuse

$$\begin{aligned}
\partial_\tau A_n &= riz + \kappa\theta B_n \\
dA_n &= rizd\tau + \kappa\theta \cdot r_{n+}\frac{1-e^{D_n\tau}}{1-G_ne^{D_n\tau}}d\tau \\
A_n(z,\tau) &= \int_0^\tau rizds + \overbrace{\int_0^\tau \kappa\theta \cdot r_{n+}\frac{1-e^{D_n\tau}}{1-G_ne^{D_n\tau}}ds}^{follow-Lewis-method} \\
&= riz\tau + \kappa\theta \cdot \left[r_+\tau + \frac{r_{n+}}{D_n}\overbrace{\frac{1-G_n}{G_n}}^{-2/\sigma^2}\ln\frac{1-G_ne^{D_n\tau}}{1-G_n}\right] && \textit{apply equation 8 in Lewis.doc}
\end{aligned}$$

$$= \quad riz\tau + \frac{\kappa\theta}{\sigma^2} \cdot \left[\tau(b_n - \rho\sigma iz + D_n) - 2\ln\frac{1 - G_n e^{D_n \tau}}{1 - G_n} \right]$$

Putting these pieces together, we have the solution of Bakshi Madan approach.

$$C(S_t) = S_t P_1 - e^{-r(T-t)} K P_2$$

$$\Pr_X(x > \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\frac{e^{-iz\ln K}}{iz}\Phi_X(z)]dz \qquad \textit{where } X = \ln S_T$$

or

$$\Pr_X(x > \ln K) = \int_{x_0}^\infty [\frac{1}{2\pi}\int_{-\infty}^\infty e^{-iz\ln K}\Phi_X(z)dz]dx \qquad \textit{no reference mention this approach, why?}$$

where

$$\Phi_{Q_n}(t, X_t = x, v_t = v) = e^{A_n(z,\tau)+B_n(z,\tau)v+izx} \qquad \forall n \in [1,2] \qquad \textit{where } \tau = T - t$$

$$A_n(z,\tau) = riz\tau + \frac{\kappa\theta}{\sigma^2}\cdot\left[\tau(b_n - \rho\sigma iz + D_n) - 2\ln\frac{1-G_n e^{D_n\tau}}{1-G_n}\right]$$

$$B_n(z,\tau) = \frac{1-e^{D_n\tau}}{1-G_n e^{D_n\tau}} r_{n+}$$

and

$$D_n = \sqrt{(\rho\sigma iz - b_n)^2 + (z^2 \mp iz)\sigma^2} \qquad \textit{where } b_1 = \kappa + \Lambda - \rho\sigma$$

$$G_n = \frac{b_n - \rho\sigma iz + D_n}{b_n - \rho\sigma iz - D_n} \qquad \textit{where } b_2 = \kappa + \Lambda$$

$$r_{n+} = \frac{b_n - \rho\sigma iz + D_n}{\sigma^2}$$

Comparison between *Lewis* and *Bakshi Madan*
Here are the steps involved in both approaches :

Lewis	*Bakshi*
value for all options and models	
value for vanilla and all models	*value for vanilla and all models*
derive PDE in V and S	*derive PDE in V and S*
derive PDE in U and X'	*derive PDE in V and X*
	derive PDE in Pr(ITM) by plugging vanilla payoff
derive ODE in A and B by plugging ansatz	*derive ODE in A and B by plugging ansatz*
	inverse theorem for characteristic function

Numerical index and alphabetical index are used for labelling equations in *Lewis* and *Bakshi* respectively. Correspondence is :

Lewis	*Bakshi*
4	*B*
5	*C*
6a	*Ea*
6b	*Eb*

Reference
The proof of Heston model by Bakshi Madan approach in this document is based on :

- *chapter 1 in "The Heston Model and Its Extensions in Matlab and C#", by Fabice Douglas Rouah (no dividend is considered)*
- *chapter 3.3 and 3.4 in "Four Generations of Asset Pricing Models and Volatility Dynamics", by Sascha Desmettre*

The proof of inverse theorem for characteristic function using contour integral in complex plane can be found in Sascha Desmettre's thesis.