Delta and Vega

Black Scholes model for call option

$$
\begin{aligned}
C(S) &= SN(d_1) - K'N(d_2) \\
K' &= Ke^{-rT} \\
d_1, d_2 &= \frac{\ln(S/K) + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}} \\
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-y^2/2)dy
\end{aligned}
$$

Black Scholes model for put option

$$C(S_T) + K = P(S_T) + S_T \qquad \text{call put parity at maturity}$$

$$C(S_t) + K' = P(S_t) + S_t \qquad \text{call put parity before maturity}$$

$$
\begin{aligned}
P(S) &= C(S) + K' - S && \text{omit index t} \\
&= SN(d_1) - K'N(d_2) + K' - S \\
&= S(N(d_1) - 1) - K'(N(d_2) - 1) \\
&= K'N(-d_2) - SN(-d_1) && \text{see remark [1]}
\end{aligned}
$$

Remark

[1]

$$
\begin{aligned}
N(-x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} \exp(-y^2/2)dy \\
&= 1 - \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} \exp(-y^2/2)dy && \text{since } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-y^2/2)dy = 1 \\
&= 1 + \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-x} \exp(-y^2/2)dy \\
&= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-y^2/2)dy && \text{put y = -y} \\
&= 1 - N(x)
\end{aligned}
$$

[2]

$$
\begin{aligned}
N'(x) &= \lim_{\Delta x \to 0} \frac{1}{\sqrt{2\pi}\Delta x} \left[\int_{-\infty}^{x+\Delta x} \exp(-y^2/2)dy - \int_{-\infty}^{x} \exp(-y^2/2)dy \right] \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{2\pi}\Delta x} \int_{x}^{x+\Delta x} \exp(-y^2/2)dy \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{2\pi}\Delta x} \exp(-x^2/2) \int_{x}^{x+\Delta x} dy \\
&= \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \ = \ G(x) && \text{i.e. Gaussian}
\end{aligned}
$$

[3]

$$
\begin{aligned}
\frac{\partial d_{1,2}}{\partial S} &= \frac{1}{\sigma\sqrt{T}} \frac{\partial \ln(S/K)}{\partial S} \\
&= \frac{1}{\sigma\sqrt{T}} \frac{K}{S} \frac{\partial(S/K)}{\partial S} \\
&= \frac{1}{\sigma\sqrt{T}} \frac{K}{S} \frac{1}{K} \\
&= \frac{1}{S\sigma\sqrt{T}}
\end{aligned}
$$

[4]

$$
\begin{aligned}
\frac{\partial d_{1,2}}{\partial \sigma} &= \frac{\partial}{\partial \sigma} \frac{\ln(S/K) + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}} \\
&= \pm \frac{\sigma T}{\sigma\sqrt{T}} - \frac{\ln(S/K) + (r \pm \sigma^2/2)T}{\sigma^2\sqrt{T}} \\
&= -\frac{\ln(S/K) + (r \mp \sigma^2/2)T}{\sigma^2\sqrt{T}} \quad = \quad -\frac{d_{2,1}}{\sigma}
\end{aligned}
$$

[5] Prove that $SG(d_1) = K'G(d_2)$

$$SG(d_1) = \frac{S}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left[\ln S-\frac{1}{2}\left(\frac{\ln(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{-2\sigma^2T\ln S+[\ln(S/K)+(r+\sigma^2/2)T]^2}{\sigma^2T}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{-2\sigma^2T\ln S+(\ln(S/K))^2+2\ln(S/K)(r+\sigma^2/2)T+((r+\sigma^2/2)T)^2}{\sigma^2T}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{-2\sigma^2T\ln S+(\ln(S/K))^2+2\ln(S/K)(r-\sigma^2/2)T+((r-\sigma^2/2)T)^2+2\ln(S/K)\sigma^2T+2r\sigma^2T^2}{\sigma^2T}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{-2\sigma^2T(\ln K-rT)+(\ln(S/K))^2+2\ln(S/K)(r-\sigma^2/2)T+((r-\sigma^2/2)T)^2}{\sigma^2T}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{-2\sigma^2T(\ln K-rT)+[\ln(S/K)+(r-\sigma^2/2)T]^2}{\sigma^2T}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left[\ln K-rT-\frac{1}{2}\left(\frac{\ln(S/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right)^2\right]$$

$$= \frac{K}{\sqrt{2\pi}}\exp\left[-rT-\frac{1}{2}\left(\frac{\ln(S/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right)^2\right]$$

$$= K'G(d_2)$$

Derive delta for call option

$$\frac{\partial C(S)}{\partial S} = \frac{\partial(SN(d_1)-K'N(d_2))}{\partial S}$$

$$= N(d_1)+SG(d_1)\frac{\partial d_1}{\partial S}-K'G(d_2)\frac{\partial d_2}{\partial S}$$ see remark [2]

$$= N(d_1)+(SG(d_1)-K'G(d_2))\frac{1}{S\sigma\sqrt{T}}$$ see remark [3]

$$= N(d_1)$$ see remark [5]

Derive vega for call option

$$\frac{\partial C(S)}{\partial \sigma} = \frac{\partial(SN(d_1)-K'N(d_2))}{\partial \sigma}$$

$$= SG(d_1)\frac{\partial d_1}{\partial \sigma}-K'G(d_2)\frac{\partial d_2}{\partial \sigma}$$

$$= SG(d_1)\left(\frac{\partial d_1}{\partial \sigma}-\frac{\partial d_2}{\partial \sigma}\right)$$ see remark [5]

$$= SG(d_1)\left(-\frac{\ln(S/K)+(r-\sigma^2/2)T}{\sigma^2\sqrt{T}}+\frac{\ln(S/K)+(r+\sigma^2/2)T}{\sigma^2\sqrt{T}}\right)$$ see remark [4]

$$= SG(d_1)\frac{\sigma^2T}{\sigma^2\sqrt{T}}$$

$$= S\sqrt{T}G(d_1)$$

$$\Rightarrow \frac{\partial C(S)}{\partial \sigma} = S\sqrt{T}G(d_1) = K'\sqrt{T}G(d_2)$$ see remark [5]

Derive delta for put option

$$\begin{aligned}
\frac{\partial P(S)}{\partial S} &= \frac{\partial (K'N(-d_2) - SN(-d_1))}{\partial S} \\
&= -K'G(-d_2)\frac{\partial d_2}{\partial S} + SG(-d_1)\frac{\partial d_1}{\partial S} - N(-d_1) && \text{see remark [2]} \\
&= (SG(d_1) - K'G(d_2))\frac{1}{S\sigma\sqrt{T}} - N(-d_1) && \text{see remark [3]} \\
&= -N(-d_1) && \text{see remark [5]}
\end{aligned}$$

Derive vega for put option

$$\begin{aligned}
\frac{\partial P(S)}{\partial \sigma} &= \frac{\partial (K'N(-d_2) - SN(-d_1))}{\partial \sigma} \\
&= -K'G(-d_2)\frac{\partial d_2}{\partial \sigma} + SG(-d_1)\frac{\partial d_1}{\partial \sigma} \\
&= SG(d_1)\left(-\frac{\partial d_2}{\partial \sigma} + \frac{\partial d_1}{\partial \sigma}\right) && \text{see remark [5]} \\
&= SG(d_1)\left(\frac{\ln(S/K) + (r+\sigma^2/2)T}{\sigma^2\sqrt{T}} - \frac{\ln(S/K) + (r-\sigma^2/2)T}{\sigma^2\sqrt{T}}\right) && \text{see remark [4]} \\
&= SG(d_1)\frac{\sigma^2 T}{\sigma^2\sqrt{T}} \\
&= S\sqrt{T}G(d_1)
\end{aligned}$$

$$\Rightarrow \quad \frac{\partial P(S)}{\partial \sigma} = S\sqrt{T}G(d_1) = K'\sqrt{T}G(d_2) \qquad \text{see remark [5]}$$

Derive vega variance for call option

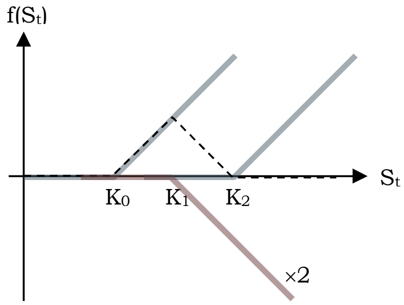
$$\begin{aligned}
\frac{\partial^2 C(S)}{\partial \sigma^2} &= \frac{\partial}{\partial \sigma}\frac{\partial C(S)}{\partial \sigma} \\
&= \frac{\partial}{\partial \sigma} S\sqrt{T}G(d_1) && \text{where } G(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2) \\
&= S\sqrt{T}G'(d_1)\frac{\partial d_1}{\partial \sigma} && \text{where } G'(x) = \frac{-x}{\sqrt{2\pi}}\exp(-x^2/2) = -xG(x) \\
&= S\sqrt{T}G(d_1)(-d_1)\frac{\partial d_1}{\partial \sigma} \\
&= \frac{S\sqrt{T}}{\sigma}G(d_1)d_1 d_2
\end{aligned}$$

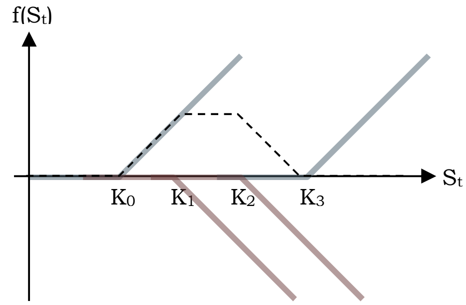
$$\begin{aligned}
d_1 d_2 &= \frac{\ln(S/K) + (r\pm\sigma^2/2)T}{\sigma\sqrt{T}}\frac{\ln(S/K) + (r\mp\sigma^2/2)T}{\sigma\sqrt{T}} \\
&= \frac{(\ln(S/K) + rT)^2 - (\sigma^2 T/2)^2}{\sigma^2 T} \\
&= \frac{(\ln(S/K) + \ln\exp(rT))^2 - (\sigma^2 T/2)^2}{\sigma^2 T} \\
&= \frac{(\ln(S\exp(rT)/K))^2 - (\sigma^2 T/2)^2}{\sigma^2 T} \\
&= -\frac{(\sigma^2 T/2)^2}{\sigma^2 T} = -\frac{\sigma^2 T}{4} && \text{for at the money option } S\exp(rT) = K
\end{aligned}$$

Summary	price	delta	vega
Call option	$SN(+d_1) - K'N(+d_2)$	$+N(d_1)$	$S\sqrt{T}G(d_1)$
Put option	$K'N(-d_2) - SN(-d_1)$	$-N(-d_1)$	$S\sqrt{T}G(d_1)$

Butterfly Strategy

The butterfly strategy involves different positions in a set of call options (or put options) with the same underlying and expiry date, but different strike price. Here are two portfolios of butterfly strategy : one with long positions at K_0 and K_2, together with a double size short position at K_1, where $K_0 < K_1 < K_2$, providing a triangular shape payoff, while the other with long poition at K_0 and K_3, together with short position at K_1 and K_2, where $K_0 < K_1 < K_2 < K_3$, providing a trapezoidal shape payoff. The black curve is final payoff, which is the sum of individual options' payoff.





The premium (risk neutral price) of the butterfly portfolio is $B(S_t)$:

$$\begin{aligned} B(S_t) &= e^{-r(T-t)}\hat{E}[f(S_t) \mid I_t] \\ &= e^{-r(T-t)}\int_0^\infty f(S_t)p(S_t)dS_t \\ &= e^{-r(T-t)} \times \text{area under combined payoff curve weighted by } p(S_t) \end{aligned}$$

where $f(S_t)$ is the payoff of the butterfly portfolio (not individual option). Now we can find the arbitrage opportunity by checking if there exists non positive premium at time t, i.e. for the first butterfly portfolio, if $2K_1 > K_0 + K_2$, then there will be positive cashflow when you buy the portfolio, and positive cashflow when you offset the portfolio later. However, normally, $2K_1 < K_0 + K_2$, since option price vs strike price exhibits convexity (true for both call and put option), i.e.

$$\begin{aligned} C(S_t, rK_0 + (1-r)K_1) &< rC(S_t, K_0) + (1-r)C(S_t, K_1) \\ P(S_t, rK_0 + (1-r)K_1) &< rP(S_t, K_0) + (1-r)P(S_t, K_1) \end{aligned}$$

where $C(S_t, K)$ and $P(S_t,K)$ are the call and put price with underlying S_t and strike K respectively. This can be proved formally by taking derivative of Black Scholes formula with respect to K. Please note that : call price vs strike price is strictly decreasing and convex, while put price vs strike price is strictly increasing and convex.