Dirac Delta Function

This document discusses the two definitions of dirac delta function, the first one is an intuitive definition, while the second one is a functional definition. Delta function plays an important role in (1) defining Fourier transform and (2) defining the Green function for linear differential operators.

Intuitive definition
The first definition of dirac dellta function is quite intuitive. It is defined as the limit of a sequence of functions $f_n(x)$ where $n \in [1, 2, ..., \infty)$. There are many possible choices of function $f_n(x)$, however it must fulfill two properties : (1) integral of $f_n(x)$ equals to one for all n, (2) as n tends to infinity, $f_n(x)$ tends to infinity if $x = x_0$, and tends to zero if $x \neq x_0$.

if $\int_{-\infty}^{+\infty} f_n(x)dx = 1 \quad \forall n \in [1,\infty)$ and

if $\lim_{n\to\infty} f_n(x) = \begin{cases} \infty & x = x_0 \\ 0 & x \neq x_0 \end{cases} \quad \forall n \in [1,\infty)$

then delta function is defined as :

$$\delta(x - x_0) = \lim_{n\to\infty} f_n(x)$$

The above defines delta function in 1D, which can be easily generalized in N dimensional space in the same way. Possible choices of $f_n(x)$ include (1) rectangular function, (2) triangular function, (3) Gaussian function and (4) sinc function etc. The integral of all these function sequences are 1, and they tend to be an impulse as n tends to infinity.

(1) $f_n(x) = \begin{cases} n & |x - x_0| \leq 1/(2n) \\ 0 & |x - x_0| > 1/(2n) \end{cases}$ rectangular function with peak n and width 1/n

(2) $f_n(x) = \begin{cases} -n^2 |x - x_0| + n & |x - x_0| \leq 1/n \\ 0 & |x - x_0| > 1/n \end{cases}$ triangular function with peak n and width 2/n

(3) $f_n(x) = \frac{n}{\sqrt{2\pi}} e^{-(x-x_0)^2 n^2/2}$ Gaussian function with $\sigma = 1/n$

(4) $f_n(x) = \frac{n}{\pi}\frac{\sin(nx)}{nx} = \frac{\sin(nx)}{\pi x}$ sinc function

Lets verify the area for Gaussian function.

$$\begin{aligned}
\int_{-\infty}^{+\infty} f_n(x)dx &= \int_{-\infty}^{+\infty} \frac{n}{\sqrt{2\pi}} e^{-(x-x_0)^2 n^2/2} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy && \text{remark : } y = n(x - x_0) \\
\left(\int_{-\infty}^{+\infty} f_n(x)dx\right)^2 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-(x^2+y^2)/2} dxdy \\
&= \int_{-\pi}^{+\pi}\int_{0}^{\infty} \frac{1}{2\pi} e^{-r^2/2} rdrd\theta \\
&= \frac{1}{2\pi}\int_{0}^{\infty} e^{-r^2/2} rdr \times \int_{-\pi}^{+\pi} d\theta \\
&= \frac{1}{2\pi}[e^{-r^2/2}]_0^{\infty} \times 2\pi \\
&= 1
\end{aligned}$$

Lets verify the area for sinc function.

$$\begin{aligned}
\int_{-\infty}^{+\infty} f_n(x)dx &= \int_{-\infty}^{+\infty} \frac{n}{\pi}\frac{\sin(nx)}{nx} dx \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\sin(y)}{y} dy && \text{remark : } y = nx \\
&= 1 && \text{remark : } \int_{-\infty}^{+\infty} (\sin x)/x dx = \pi \text{ (please read doc about sinc function)}
\end{aligned}$$

This is why delta is known as the approximate identity.

Functional definition
Functional definition is a mapping from a function to a value. Given any function g(x), delta function is defined by the following functional, which returns the value $g(x_0)$.

$$\begin{aligned} F[g(x)] &= \langle \delta(x-x_0), g(x) \rangle \\ &= \int_{-\infty}^{+\infty} \delta(x-x_0) g(x) dx \quad = \quad g(x_0) \end{aligned}$$

We now derive the second definition from the first definition. We take rectangular function as $f_n(x)$. You can repeat the proof with other choices of $f_n(x)$.

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta(x-x_0) g(x) dx &= \int_{-\infty}^{+\infty} \lim_{n\to\infty} f_n(x) g(x) dx \\ &= \lim_{n\to\infty} \int_{-\infty}^{+\infty} f_n(x) g(x) dx \\ &= \lim_{n\to\infty} n \int_{x_0-1/2n}^{x_0+1/2n} g(x) dx \\ &= \lim_{n\to\infty} n \int_{-1/2n}^{+1/2n} g(y+x_0) dy \\ &= \lim_{n\to\infty} n \int_{-1/2n}^{+1/2n} (g(x_0) + g'(x_0)y + \tfrac{1}{2} g''(x_0) y^2 + \tfrac{1}{6} g'''(x_0) y^3 + \tfrac{1}{24} g''''(x_0) y^4 + \ldots) dy \\ &= \lim_{n\to\infty} n \left(g(x_0) y + \tfrac{1}{2} g'(x_0) y^2 + \tfrac{1}{6} g''(x_0) y^3 + \tfrac{1}{24} g'''(x_0) y^4 + \tfrac{1}{120} g''''(x_0) y^5 + \ldots \right)_{-1/2n}^{+1/2n} \\ &= \lim_{n\to\infty} n \left(g(x_0) \tfrac{1}{n} + \tfrac{1}{2} g'(x_0) 0 + \tfrac{1}{6} g''(x_0) \tfrac{1}{4n^3} + \tfrac{1}{24} g'''(x_0) 0 + \tfrac{1}{120} g''''(x_0) \tfrac{1}{16n^5} + \ldots \right) \\ &= \lim_{n\to\infty} \left(g(x_0) + \tfrac{1}{24n^2} g''(x_0) + \tfrac{1}{1920n^4} g''''(x_0) + \ldots \right) \\ &= g(x_0) \end{aligned}$$

We now derive the first definition (i.e. area of delta function equals to 1) from the second definition, by taking g(x) to be constant function such that g(x) = 1 for all x.

$$\begin{aligned} & \int_{-\infty}^{+\infty} \delta(x-x_0) g(x) dx = g(x_0) \\ \Rightarrow \quad & \int_{-\infty}^{+\infty} \delta(x-x_0) dx = 1 \end{aligned}$$

Inner product vs correlation vs convolution
Inner product (also known as the dot product) of 2 functions is an integral that returns a value, while correlation and convolution of 2 functions are integrals that returns a function. Suppose the 2 functions are f and g, for inner product, the variable of integration is x, while for correlation and convolution, the variable of integration is s (i.e. the dummy variable), besides they return a function in variable x.

inner product	$\langle f(x), g(x) \rangle$	$= \int_{-\infty}^{+\infty} f(x) g^*(x) dx$	(i.e. output is a scalar.)
correlation	$(f \bullet g)(x)$	$= \int_{-\infty}^{+\infty} f(s)^* g(x+s) ds$	(i.e. output is a function.)
convolution	$(f * g)(x)$	$= \int_{-\infty}^{+\infty} f(s) g(x-s) ds$	(i.e. output is a function.)

Inner product, correlation and convolution involving delta function become (we consider real functions only) :

inner product	$\langle \delta(x-x_0), g(x) \rangle$	$= \int_{-\infty}^{+\infty} \delta(x-x_0) g(x) dx$	$= g(x_0)$
correlation	$(\delta(s-x_0) \bullet g(s))(x)$	$= \int_{-\infty}^{+\infty} \delta(s-x_0) g(x+s) ds$	$= g(x+x_0)$
convolution	$(\delta(s-x_0) * g(s))(x)$	$= \int_{-\infty}^{+\infty} \delta(s-x_0) g(x-s) ds$	$= g(x-x_0)$

Hence we have the following useful equations, where $f_n(x)$ is any function sequence defining delta function $\delta(x-x_0)$.

inner product	$\lim_{n\to\infty} \langle f_n(x), g(x) \rangle$	$= \int_{-\infty}^{+\infty} \delta(x-x_0) g(x) dx$	$= g(x_0)$
correlation	$\lim_{n\to\infty} (f_n \bullet g)(x)$	$= \int_{-\infty}^{+\infty} \delta(s-x_0) g(x+s) ds$	$= g(x+x_0)$
convolution	$\lim_{n\to\infty} (f_n * g)(x)$	$= \int_{-\infty}^{+\infty} \delta(s-x_0) g(x-s) ds$	$= g(x-x_0)$

The star indicates complex conjugate. These 3 integrals have different conjugates, and thus different commutativity, convolution is pure commutative, while inner product and correlation are not purely commutative.

Fourier transform of delta function
Fourier transform of delta function gives the its integral representation. Lets consider the oridinary frequency.

$$FT_{of}[f(x)] = \int_{-\infty}^{+\infty} f(x)e^{-j2\pi ux}dx$$

$$FT_{of}[\delta(x-x_0)] = \int_{-\infty}^{+\infty} \delta(x-x_0)e^{-j2\pi ux}dx$$

$$= e^{-j2\pi ux_0}$$

$$\delta(x-x_0) = FT_{of}^{-1}e^{-j2\pi ux_0}$$

$$= \int_{-\infty}^{+\infty} e^{-j2\pi ux_0}e^{j2\pi ux}du$$

$$= \int_{-\infty}^{+\infty} e^{j2\pi u(x-x_0)}du$$

$\Rightarrow$ $\delta(x) = \int_{-\infty}^{+\infty} e^{j2\pi ux}du$ (in ordinary frequency)

or $\delta(x) = \int_{-\infty}^{+\infty} e^{jwx}dw\frac{1}{2\pi}$ (in angular frequency) by substituting $w = 2\pi u$

Summary
Together with the integral representation, there are three different representations of delta function.

(1) $\delta(x-x_0) = \int_{-\infty}^{+\infty} e^{j2\pi u(x-x_0)}du$

$= \int_{-\infty}^{+\infty} e^{jwx}dw\frac{1}{2\pi}$

(2) $\delta(x-x_0) = \lim_{n\to\infty} f_n(x)$ for all n, where $\int_{-\infty}^{+\infty} f_n(x)dx = 1$ and $\lim_{n\to\infty} f_n(x) = \begin{cases} \infty & x = x_0 \\ 0 & x \neq x_0 \end{cases}$

(3) $\int_{-\infty}^{+\infty} \delta(x-x_0)g(x)dx = g(x_0)$ for all function g

Delta function in discrete domain can be denoted as

$$\delta_{x,k} = \delta(x-k) = \begin{cases} 1 & when \quad x = k \\ 0 & when \quad x \neq k \end{cases}$$

Delta function is very useful in modelling point source in N dimensional space, such as heat source, point charge etc.