Euler’s Number

Euler’s number

Euler’s number is defined as :

(1) $e \quad = \quad \lim_{n\to\infty}(1+1/n)^n$

(2) $e \quad = \quad \sum_{n=0}^{\infty} 1/n!$

Lets derive from definition (1) to get definition (2).

$$
\begin{aligned}
e &= \lim_{n\to\infty}(1+1/n)^n \qquad \text{(expand using binomial theorem)}\\
&= \lim_{n\to\infty} 1+\frac{n!}{(n-1)!1!}(\frac{1}{n})^1+\frac{n!}{(n-2)!2!}(\frac{1}{n})^2+\frac{n!}{(n-3)!3!}(\frac{1}{n})^3+\frac{n!}{(n-4)!4!}(\frac{1}{n})^4+\ldots+\frac{n!}{(n-k)!k!}(\frac{1}{n})^k+\ldots+\frac{n!}{1!(n-1)!}(\frac{1}{n})^{n-1}+\frac{n!}{0!n!}(\frac{1}{n})^n\\
&= \lim_{n\to\infty} 1+\frac{1}{1!}(\frac{1}{n})^0+\frac{n-1}{2!}(\frac{1}{n})^1+\frac{(n-1)(n-2)}{3!}(\frac{1}{n})^2+\frac{(n-1)(n-2)(n-3)}{4!}(\frac{1}{n})^3+\ldots+\frac{(n-1)(n-2)\ldots(n-k+1)}{k!}(\frac{1}{n})^{k-1}+\ldots\\
&= \lim_{n\to\infty} 1+\frac{1}{1!}+\frac{1}{2!}(1-\frac{1}{n})+\frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\frac{1}{4!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})+\ldots+\frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\ldots(1-\frac{k-1}{n})+\ldots\\
&= 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\\
&= \sum_{n=0}^{\infty} 1/n!
\end{aligned}
$$

Exponential function

Exponential function is defined as : e^x , lets find it in terms of (1) limit and (2) summation.

(1)
$$
\begin{aligned}
e^x &= \lim_{n\to\infty}(1+1/n)^{nx}\\
&= \lim_{m\to\infty}(1+x/m)^m \qquad \text{(substitute } m=nx\text{)}
\end{aligned}
$$

(2)
$$
\begin{aligned}
e^x &= \lim_{n\to\infty}(1+x/n)^n \qquad \text{(expand using binomial theorem)}\\
&= \lim_{n\to\infty} 1+\frac{n!}{(n-1)!1!}(\frac{x}{n})^1+\frac{n!}{(n-2)!2!}(\frac{x}{n})^2+\frac{n!}{(n-3)!3!}(\frac{x}{n})^3+\frac{n!}{(n-4)!4!}(\frac{x}{n})^4+\ldots+\frac{n!}{(n-k)!k!}(\frac{x}{n})^k+\ldots+\frac{n!}{1!(n-1)!}(\frac{x}{n})^{n-1}+\frac{n!}{0!n!}(\frac{x}{n})^n\\
&= \lim_{n\to\infty} 1+\frac{1}{1!}\frac{x^1}{n^0}+\frac{n-1}{2!}\frac{x^2}{n^1}+\frac{(n-1)(n-2)}{3!}\frac{x^3}{n^2}+\frac{(n-1)(n-2)(n-3)}{4!}\frac{x^4}{n^3}+\ldots+\frac{(n-1)(n-2)\ldots(n-k+1)}{k!}\frac{x^k}{n^{k-1}}+\ldots\\
&= \lim_{n\to\infty} 1+\frac{x^1}{1!}+\frac{x^2}{2!}(1-\frac{1}{n})+\frac{x^3}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\frac{x^4}{4!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})+\ldots+\frac{x^k}{k!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\ldots(1-\frac{k-1}{n})+\ldots\\
&= \sum_{n=0}^{\infty} x^n/n!
\end{aligned}
$$

Please note that we derive exponential without using Taylor series (we simply use binomial theorem), now we know the derivative of exponential, lets verify if Taylor series can give a consistent result.

$$e^x = e^0+\frac{1}{1!}e^0x+\frac{1}{2!}e^0x^2+\frac{1}{3!}e^0x^3+\ldots \qquad \text{(same as above)}$$

Beauty of Euler's number

(1) for continuous compounding
(2) for differentiation

$$\begin{aligned}
\frac{de^x}{dx} &= \frac{d}{dx}\sum_{n=0}^{\infty} x^n / n! \\
&= \sum_{n=1}^{\infty} nx^{n-1} / n! && \text{(The first term in } \textstyle\sum_{n=0}^{\infty} x^n / n! \text{ is a constant.)} \\
&= \sum_{n=1}^{\infty} x^{n-1} / (n-1)! \\
&= \sum_{m=0}^{\infty} x^m / m! && \text{(substitute } m = n-1\text{)} \\
&= e^x && \text{This is useful for taking gradient of } \textbf{\textit{composition of functions}}.
\end{aligned}$$

(3) Gaussian distribution, which is important for :
- central limit theorem
- product of Gaussian functions is a Gaussian function
- convolution of Gaussian functions is a Gaussian function } the same thing
- sum of Gaussian random variables is Gaussian random variable
- Fourier transform of Gaussian is also a Gaussian

(4) Euler formula, which describes circular (or period) motion. Fourier transform is defined using Euler's formula.

Other properties

Exponential is useful for finding $\frac{d}{dx} f(x)^{g(x)}$.

$$\begin{aligned}
\frac{d}{dx} f(x)^{g(x)} &= \frac{d}{dx} e^{g(x)\ln f(x)} \\
&= e^{g(x)\ln f(x)}(g'(x)\ln f(x) + g(x)f'(x)/f(x)) \\
&= f(x)^{g(x)}(g'(x)\ln f(x) + g(x)f'(x)/f(x))
\end{aligned}$$