# Bachelier model

As opposed to Black Scholes model which assumes log normal underlying asset price, Bachelier model assumes normal underlying asset price. Here is the risk neutral process for forward price (matured@$T$, as of $t$) under Black Scholes and Bachelier respectively :

$$\begin{aligned}
dF_{t,T} &= \sigma_{black} F_{t,T} dz_t && \textit{for Black Scholes} \\
dF_{t,T} &= \sigma_{bach} dz_t && \textit{for Bachelier} \\
\sigma_{black} &\neq \sigma_{bach} && \textit{log normal volatility vs normal volatility} \\
& && \textit{plesae note that in TDMS, normal vol means Hull White volatility}
\end{aligned}$$

As $F_t = S_t e^{r(T-t)}$ under risk neutral measure, we can derive the $SDEs$ of $S_t$ as :

*for Black Scholes*

$$\begin{aligned}
\sigma_{black} S_t e^{r(T-t)} dz_t &= d(S_t e^{r(T-t)}) \\
\sigma_{black} S_t e^{r(T-t)} dz_t &= e^{r(T-t)} dS_t + 0(dS_t)^2 + S_t d(e^{r(T-t)}) \\
\sigma_{black} S_t e^{r(T-t)} dz_t &= e^{r(T-t)} dS_t + S_t e^{r(T-t)}(-r)dt \\
\sigma_{black} S_t dz_t &= dS_t - rS_t dt \\
dS_t &= rS_t dt + \sigma_{black} S_t dz_t
\end{aligned}$$

*for Bachelier*

$$\begin{aligned}
\sigma_{bach} dz_t &= d(S_t e^{r(T-t)}) \\
\sigma_{bach} dz_t &= e^{r(T-t)} dS_t + 0(dS_t)^2 + S_t d(e^{r(T-t)}) \\
\sigma_{bach} dz_t &= e^{r(T-t)} dS_t + S_t e^{r(T-t)}(-r)dt \\
\sigma_{bach} e^{-r(T-t)} dz_t &= dS_t - rS_t dt \\
dS_t &= rS_t dt + \sigma_{bach} e^{-r(T-t)} dz_t \quad \Leftarrow \textit{pls check}
\end{aligned}$$

Now, let's derive vanilla call on asset following Bachelier model using risk neutral pricing using with cash numeraire (as we do not assume stochastic interest rate). First of all, solve for underlying :

*for Black Scholes*

$$\begin{aligned}
d\ln F_{t,T} &= (1/F_{t,T})dF_{t,T} - (1/2)(1/F_{t,T})^2 (dF_{t,T})^2 \\
&= -(\sigma_{black}^2/2)dt + \sigma_{black} dz_t \\
\ln(F_{T,T}/F_{t,T}) &= -(\sigma_{black}^2/2)(T-t) + \sigma_{black}(z_T - z_t) \\
F_{T,T} &= F_{t,T} e^{-(\sigma_{black}^2/2)(T-t) + \sigma_{black}(z_T - z_t)}
\end{aligned}$$

*for Bachelier*

$$\begin{aligned}
F_{T,T} - F_{t,T} &= \sigma_{bach}(z_T - z_t) \\
F_{T,T} &= F_{t,T} + \sigma_{bach}(z_T - z_t) \\
&\sim norm(F_{t,T}, \sqrt{v_{bach}}) \quad \textit{where} \quad v_{bach} = \sigma_{bach}^2 (T-t)
\end{aligned}$$

Let's continue with the vanilla call for Bachelier model only :

$$\begin{aligned}
call_t &= e^{-r(T-t)} E_Q[(F_{T,T} - K)^+ \mid I_t] \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi v_{bach}}} \int_{-\infty}^{\infty} (x-K)^+ e^{-\frac{(x-F_{t,T})^2}{2v_{bach}}} dx
\end{aligned}$$

Just like what we have done for Black Scholes, we convert integration range in asset price to range in Brownian motion $z_T$ :

$$\begin{aligned}
F_{T,T} &> K \\
F_{t,T} + \sigma_{bach}(z_T - z_t) &> K \\
F_{t,T} + \sigma_{bach}\sqrt{T-t}\, z &> K && \textit{where} \;\; z \sim N(0,1) \\
z &> (K - F_{t,T})/\sqrt{v_{bach}} = -d && \textit{where} \;\; d = (F_{t,T} - K)/\sqrt{v_{bach}}
\end{aligned}$$

$$\begin{aligned}
\textit{Hence} \quad call_t &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (F_{t,T} + \sqrt{v_{bach}}\, z - K)^+ e^{-z^2/2} dz \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-d}^{\infty} (F_{t,T} + \sqrt{v_{bach}}\, z - K) e^{-z^2/2} dz \\
&= e^{-r(T-t)} \left[ \frac{F_{t,T} - K}{\sqrt{2\pi}} \int_{-d}^{\infty} e^{-z^2/2} dz + \frac{\sqrt{v_{bach}}}{\sqrt{2\pi}} \int_{-d}^{\infty} z e^{-z^2/2} dz \right] \\
&= e^{-r(T-t)} \left[ (F_{t,T} - K)(1 - N(-d)) + \frac{\sqrt{v_{bach}}}{\sqrt{2\pi}} \int_{-d}^{\infty} z e^{-z^2/2} dz \right] && \textit{where} \;\; N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-z^2/2} dz \\
&= e^{-r(T-t)} \left[ (F_{t,T} - K)N(d) + \frac{\sqrt{v_{bach}}}{\sqrt{2\pi}} \int_{-d}^{\infty} z e^{-z^2/2} dz \right] && \textit{where} \;\; N(-d) = 1 - N(d)
\end{aligned}$$

$$= \quad e^{-r(T-t)}\left[(F_{t,T}-K)N(d)+\frac{\sqrt{v_{bach}}}{\sqrt{2\pi}}\int_{z=-d}^{z=\infty}e^{-x}dx\right] \qquad where\;\; x=z^2/2\;\; and\;\; dx=2dz/2=dz$$

$$= \quad e^{-r(T-t)}\left[(F_{t,T}-K)N(d)-\sqrt{v_{bach}}\frac{1}{\sqrt{2\pi}}e^{-x}\Big|_{z=-d}^{z=\infty}\right]$$

$$= \quad e^{-r(T-t)}\left[(F_{t,T}-K)N(d)-\sqrt{v_{bach}}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\Big|_{-d}^{\infty}\right]$$

$$= \quad e^{-r(T-t)}\left[(F_{t,T}-K)N(d)-\sqrt{v_{bach}}(\underbrace{n(\infty)}_{0}-n(-d))\right] \qquad where\;\; n(d)=\frac{1}{\sqrt{2\pi}}e^{-d^2/2}$$

$$= \quad e^{-r(T-t)}[(F_{t,T}-K)N(d)+\sqrt{v_{bach}}\,n(d)] \qquad where\;\; n(-d)=n(d)$$

Relationship between Black Scholes volatility and Bachelier volatility for the same vanilla call price can thus be solved by :

$$F_{t,T}N(d_1)-KN(d_2) \quad = \quad (F_{t,T}-K)N(d)+\sqrt{v_{bach}}\,n(d)$$

where

$$d_{1,2} \quad = \quad \frac{\ln(F_{t,T}/K)\pm v_{black}/2}{\sqrt{v_{black}}} \qquad and \qquad v_{black}=\sigma_{black}^2(T-t)$$

$$d \quad = \quad \frac{F_{t,T}-K}{\sqrt{v_{bach}}} \qquad and \qquad v_{bach}=\sigma_{bach}^2(T-t)$$

which can only be solved numerically. Finally lets find forward delta for vanilla call under Bachelier model.

$$\begin{aligned}
delta &= \partial_F e^{-r(T-t)}[(F_{t,T}-K)N(d)+\sqrt{v_{bach}}\,n(d)] \\
&= e^{-r(T-t)}[\partial_F((F_{t,T}-K)N(d))+\partial_F(\sqrt{v_{bach}}\,n(d))] \\
&= e^{-r(T-t)}[N(d)+\underbrace{(F_{t,T}-K)\partial_F N(d)+\sqrt{v_{bach}}\,\partial_F n(d)}_{0}] \\
&= e^{-r(T-t)}N(d)
\end{aligned}$$

*where*

$$\begin{aligned}
\partial_F N(d) &= n(d)\partial_F d \\
\partial_F n(d) &= (-d)n(d)\partial_F d \\
\partial_F d &= \partial_F\frac{F_{t,T}-K}{\sqrt{v_{bach}}} = \frac{1}{\sqrt{v_{bach}}}
\end{aligned}$$

*hence*

$$\begin{aligned}
& (F_{t,T}-K)\partial_F N(d)+\sqrt{v_{bach}}\,\partial_F n(d) \\
= \;& (F_{t,T}-K)n(d)\partial_F d+\sqrt{v_{bach}}(-d)n(d)\partial_F d \\
= \;& (F_{t,T}-K)n(d)\partial_F d-(F_{t,T}-K)n(d)\partial_F d \\
= \;& 0
\end{aligned}$$

Recall that inside the interest rate world of *TDMS*, lognorm volatility to normal volatility conversion does not refer to Black Scholes volatility to Bachelier volatility conversion, instead it refers to Black Scholes volatility to Hull White volatility conversion.