Contraction Mapping Theorem

In mathematics, pair and tuple are defined the same as `std::pair` and `std::tuple` in C++, an ordered pair is a pair in which the order of the items cannot be changed. A metric space is an ordered pair (X, d), where X is a set, and d is a distance function, $d : M \times M \to \Re$, such that the following properties are satisfied $\forall x, y, z \in X$:

- $d(x,y) = 0 \quad$ *iff* $x = y$
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$
- $d(x,y) \geq 0$

The last property can be derived from the others as :

$$\begin{aligned} d(x,z) &\leq d(x,y)+d(y,z) \\ d(x,x) &\leq d(x,y)+d(y,x) && \textit{putting } z = x \\ 0 &\leq 2d(x,y) \\ d(x,y) &\geq 0 \end{aligned}$$

A mapping $f : X \to X$ is a contraction mapping if the distance between any two points is shortened after applying the mapping (also called *"taking image under f "*), or mathematically, there exists $\lambda \in [0,1)$, such that :

$$d(f(x), f(y)) \leq \lambda d(x,y) \qquad \forall x,y \in X$$

A fixing point of a mapping is the point which is unchanged before and after mapping (analogous to the reference point for image transformation), that is for $x^* \in X$, then it is the point where the curve $y = f(x)$ intersects the line $y = x$:

$$f(x^*) = x^*$$

Contraction mapping theorem (also called Banach fixed point theorem) states that, given a metric space together with a contraction mapping $f : X \to X$, (1) there exists one unique fixed point x^*, and (2) it can be obtained by repeatedly applying the mapping starting from arbitary point $x_0 \in X$, i.e. $x^* = f(f(\ldots f(x_0)))$. Here is an intuitive proof (not official proof). Let $x_n = f(x_{n-1})$, then :

$$\begin{aligned} d(x_n, x_{n-1}) &= d(f(x_{n-1}), f(x_{n-2})) \\ &\leq \lambda d(x_{n-1}, x_{n-2}) \\ &\leq \lambda^2 d(x_{n-2}, x_{n-3}) \\ &\leq \ldots \\ &\leq \lambda^{n-1} d(x_1, x_0) \\ \lim_{n\to\infty} d(x_n, x_{n-1}) &\leq \lim_{n\to\infty} \lambda^{n-1} d(x_1, x_0) = 0 \end{aligned}$$

For example, if X is the real space $\Re$, then any concave function, i.e. a function with $f''(x) < 0$, can be a contraction mapping, repeated application of the function with any starting point will end up with the intersection between $y = f(x)$ and $y = x$.