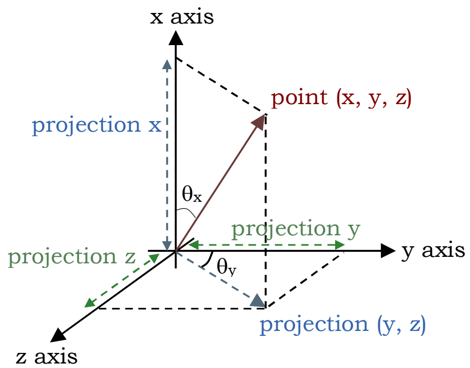
N Dimensional Sphere

This document derives the N dimensional spherical coordinates in two approaches and the volume of N dimensional sphere in terms of gamma function. Finally, the curse of dimensionality is discussed. N dimensional spherical space is composed of 1 radial space (i.e. r) and N-1 angular space (i.e. θ_1, θ_2, θ_3, …θ_{N-2}, θ_{N-1}).

Spherical coordinate (geometric approach)

Lets consider 3D case. Assume the 3 axes are x–axis, y–axis and z–axis, a point on the sphere with radius r be (x, y, z). The point (x, y, z) is projected on the x–axis first, the orthogonal complement is the projection on yz–plane, i.e. (0, y, z). The projection (0, y, z) is then projected on y–axis, the orthogonal complement is the projection on the last axis, i.e. z–axis. Red color indicates original point, blue color indicates the first projections (i.e. on x–axis and on yz–plane), green color indicates the second projections (i.e. on y–axis and on z–axis).



step 1 : projection of (x, y, z)

on x–axis with angle ϑ_x $= r\cos\vartheta_x$ (1)

on yz–plane with angle $\pi/2-\vartheta_x$ $= r\sin\vartheta_x$

step 2 : projection of (0, y, z)

on y-axis with angle ϑ_y $= r\sin\vartheta_x\cos\vartheta_y$ (2)

on z-axis with angle $\pi/2-\vartheta_y$ $= r\sin\vartheta_x\sin\vartheta_y$ (3)

where ϑ_x = angle between (x, y, z) and x–axis

ϑ_y = angle between (0, y, z) and y–axis

Equation (1), (2) and (3) are the transformation that we need :

x	=	$r\cos\vartheta_x$	$r\in[0,\infty]$	called the radial coordinate
y	=	$r\sin\vartheta_x\cos\vartheta_y$	$\vartheta_x\in[0,\pi]$	called the polar angle (or zenith / normal / inclination angle)
z	=	$r\sin\vartheta_x\sin\vartheta_y$	$\vartheta_y\in[-\pi,\pi]$	called the azimuth angle

Lets consider 4D case. Assume the 4 axes are x–axis, y–axis, z–axis and w–axis, a point on the sphere with radius r be (x, y, z, w). The point (x, y, z, w) is projected on the x–axis first, the orthogonal complement is the projection on yzw–space, i.e. (0, y, z, w). The projection (0, y, z, w) is then projected on y–axis, the orthogonal complement is the projection on zw–plane, i.e. (0, 0, z, w). The projection (0, 0, z, w) is further projected on z–axis, the orthogonal complement is the projection on the last axis, i.e. w–axis. Hence we have :

Step 1	projection of (x, y, z, w) on x–axis with angle ϑ_x	= $r\cos\vartheta_x$	(1*)
	projection of (x, y, z, w) on yzw–space with angle $\pi/2-\vartheta_x$	= $r\sin\vartheta_x$	
Step 2	projection of (0, y, z, w) on y-axis with angle ϑ_y	= $r\sin\vartheta_x\cos\vartheta_y$	(2*)
	projection of (0, y, z, w) on zw-plane with angle $\pi/2-\vartheta_y$	= $r\sin\vartheta_x\sin\vartheta_y$	
Step 3	projection of (0, 0, z, w) on z-axis with angle ϑ_z	= $r\sin\vartheta_x\sin\vartheta_y\cos\vartheta_z$	(3*)
	projection of (0, 0, z, w) on w-axis with angle $\pi/2-\vartheta_z$	= $r\sin\vartheta_x\sin\vartheta_y\sin\vartheta_z$	(4*)

where ϑ_x = angle between (x, y, z, w) and x–axis

ϑ_y = angle between (0, y, z, w) and y–axis

ϑ_z = angle between (0, 0, z, w) and z–axis

Equation (1*), (2*), (3*) and (4*) are the transformation that we need :

x	=	$r\cos\vartheta_x$	$r\in[0,\infty]$	called the radial coordinate
y	=	$r\sin\vartheta_x\cos\vartheta_y$	$\vartheta_x\in[0,\pi]$	called the polar angle
z	=	$r\sin\vartheta_x\sin\vartheta_y\cos\vartheta_z$	$\vartheta_y\in[0,\pi]$	called the polar angle
w	=	$r\sin\vartheta_x\sin\vartheta_y\sin\vartheta_z$	$\vartheta_z\in[-\pi,\pi]$	called the azimuth angle

Finally, lets consider the ND case, assume N axes are : x_1, x_2, x_3, … x_{N-1}, x_N. Following the same arguments as 3D case and 4D case, we have the following results. In the spherical coordinates, there is one **radial** coordinate, and N-1 angular coordinates, and within the N-1 angular coordinates, N-2 are called **polar angles** (ranges from 0 to $+\pi$), and the last one is called **azimuth angle** (ranges from $-\pi$ to $+\pi$).

Step 1	proj of ($x_1, x_2, x_3, \ldots x_{N-1}, x_N$) on x_1–axis with angle ϑ_1	=	$r\cos\vartheta_1$
	proj of ($x_1, x_2, x_3, \ldots x_{N-1}, x_N$) on complement space	=	$r\sin\vartheta_1$
Step 2	proj of ($0, x_2, x_3, \ldots x_{N-1}, x_N$) on x_2–axis with angle ϑ_2	=	$r\sin\vartheta_1\cos\vartheta_2$
	proj of ($0, x_2, x_3, \ldots x_{N-1}, x_N$) on complement space	=	$r\sin\vartheta_1\sin\vartheta_2$
Step 3	proj of ($0, 0, x_3, \ldots x_{N-1}, x_N$) on x_3–axis with angle ϑ_3	=	$r\sin\vartheta_1\sin\vartheta_2\cos\vartheta_3$
	proj of ($0, 0, x_3, \ldots x_{N-1}, x_N$) on complement space	=	$r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3$
…			
Step N-1	proj of ($0, \ldots 0, x_{N-1}, x_N$) on x_{N-1}–axis with angle ϑ_{N-1}	=	$r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3 \ldots \sin\vartheta_{N-2}\cos\vartheta_{N-1}$
	proj of ($0, \ldots 0, x_{N-1}, x_N$) on x_N–axis with angle $\pi/2-\vartheta_{N-1}$	=	$r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3 \ldots \sin\vartheta_{N-2}\sin\vartheta_{N-1}$

where	ϑ_n	=	angle between (n-1)th projection and x_n–axis		for $n \in [1, N-1]$
	ϑ_n	$\in$	$[0,\pi]$	is known as polar angle	for $n \in [1, N-2]$
	ϑ_{N-1}	$\in$	$[-\pi,+\pi]$	is known as azimuth angle	for $n = N-1$

$$
\begin{aligned}
x_1 &= r\cos\vartheta_1 \\
x_2 &= r\sin\vartheta_1\cos\vartheta_2 \\
x_3 &= r\sin\vartheta_1\sin\vartheta_2\cos\vartheta_3 \\
&\ldots \\
x_{N-2} &= r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\sin\vartheta_4 \ldots \sin\vartheta_{N-3}\cos\vartheta_{N-2} \\
x_{N-1} &= r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\sin\vartheta_4 \ldots \sin\vartheta_{N-3}\sin\vartheta_{N-2}\cos\vartheta_{N-1} \\
x_N &= r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\sin\vartheta_4 \ldots \sin\vartheta_{N-3}\sin\vartheta_{N-2}\sin\vartheta_{N-1}
\end{aligned}
$$

Spherical coordinate (analytic approach)

A point ($x_1, x_2, x_3, \ldots x_{N-1}, x_N$) lying on ND hypershpere with radius r can be decomposed into a projection on x_1–axis and a projection on its orthogonal complement, by noticing that the latter projection ($0, x_2, x_3, \ldots x_{N-1}, x_N$) is a point lying on a 'N-1'D hypersphere (composed of x_2–axis, x_3–axis, … x_N–axis) with radius $r\sin\theta_1$, we can derive the spherical coordinates in a recursion manner. Since $x_1 \in [-r,+r]$, it is reasonable to let :

$$
\begin{aligned}
x_1 &= r\cos\vartheta_1 \\
r^2 &= x_1^2 + x_2^2 + x_3^2 + \ldots x_{N-1}^2 + x_N^2 \\
\Rightarrow \quad r^2 - x_1^2 &= x_2^2 + x_3^2 + \ldots x_{N-1}^2 + x_N^2 \\
\Rightarrow \quad r^2 - (r\cos\theta_1)^2 &= x_2^2 + x_3^2 + \ldots x_{N-1}^2 + x_N^2 \\
\Rightarrow \quad (r\sin\theta_1)^2 &= x_2^2 + x_3^2 + \ldots x_{N-1}^2 + x_N^2
\end{aligned}
$$

The point ($x_2, x_3, \ldots x_{N-1}, x_N$) lying on 'N-1'D hypershpere with radius $r\sin\theta_1$ can be decomposed into a projection on x_2–axis and a projection on its orthogonal complement. Since $x_2 \in [-r\sin\theta_1, +r\sin\theta_1]$, it is reasonable to let :

$$
\begin{aligned}
x_2 &= r\sin\vartheta_1\cos\vartheta_2 \\
(r\sin\theta_1)^2 &= x_2^2 + x_3^2 + x_4^2 + \ldots x_{N-1}^2 + x_N^2 \\
\Rightarrow \quad (r\sin\theta_1)^2 - x_2^2 &= x_3^2 + x_4^2 + \ldots x_{N-1}^2 + x_N^2 \\
\Rightarrow \quad (r\sin\theta_1)^2 - (r\sin\vartheta_1\cos\vartheta_2)^2 &= x_3^2 + x_4^2 + \ldots x_{N-1}^2 + x_N^2 \\
\Rightarrow \quad (r\sin\theta_1\sin\theta_2)^2 &= x_3^2 + x_4^2 + \ldots x_{N-1}^2 + x_N^2
\end{aligned}
$$

The point ($x_3, x_4, \ldots x_{N-1}, x_N$) lying on 'N-2'D hypershpere with radius $r\sin\theta_1\sin\theta_2$ can be decomposed into a projection on x_3–axis and a projection on its orthogonal complement. Since $x_3 \in [-r\sin\theta_1\sin\theta_2, +r\sin\theta_1\sin\theta_2]$, it is reasonable to let :

$x_3 = r\sin\vartheta_1\sin\vartheta_2\cos\vartheta_3$ by following the same argument, we have :

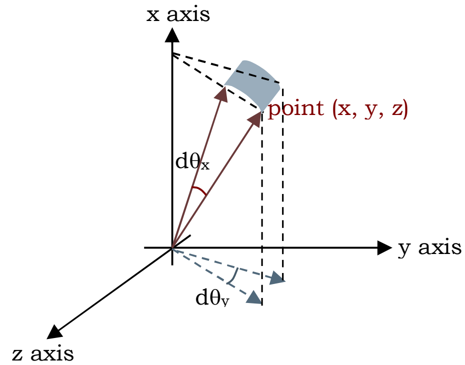
$$
\begin{aligned}
x_4 &= r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\cos\vartheta_4 \\
x_5 &= r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\sin\vartheta_4\cos\vartheta_5 \\
&\ldots \\
x_{N-2} &= r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\sin\vartheta_4 \ldots \sin\vartheta_{N-3}\cos\vartheta_{N-2}
\end{aligned}
$$

The point (x_{N-1}, x_N) lying on 2D circle with radius $r\sin\theta_1\sin\theta_2 \ldots \sin\theta_{N-2}$ can be decomposed into a projection on x_{N-1}–axis and a projection on x_N–axis. Finally, we have :

$$
\begin{aligned}
x_{N-1} &= r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\sin\vartheta_4 \ldots \sin\vartheta_{N-3}\sin\vartheta_{N-2}\cos\vartheta_{N-1} \\
x_N &= r\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\sin\vartheta_4 \ldots \sin\vartheta_{N-3}\sin\vartheta_{N-2}\sin\vartheta_{N-1}
\end{aligned}
$$

Differential surface, differential solid angle and differential volume
Differential surface is the surface area of a hypersphere spanned by differential change in each angular component, while keeping radius component constant at r. Since there are N-1 angular dimensions, when there is delta change in any one angular component (while keeping other angular components unchanged), an arc is spanned, with length equals to radius multipled by delta angle. Differential surface is thus the product of all N-1 arcs spanned by delta change in all angular components.

For 3D case



$$\begin{aligned}
dl_x &= \text{arc length spanned by delta change in } \theta_x \\
&= \text{radius of original sphere} \times d\theta_x \\
&= r d\theta_x \\
dl_y &= \text{arc length spanned by delta change in } \theta_y \\
&= \text{radius of 1}^{\text{st}} \text{ projected sphere} \times d\theta_y \\
&= r\sin\theta_x d\theta_y \\
dS &= dl_x dl_y \\
&= r^2 \sin\theta_x d\theta_x d\theta_y
\end{aligned}$$

For ND case

$$\begin{aligned}
dl_n &= \text{arc length spanned by delta change in } \theta_n && \text{(for all n} \in [1, \text{N-1]}) \\
&= \text{radius of n-1}^{\text{th}} \text{ projected sphere} \times d\theta_n \\
&= r\sin\vartheta_1 \sin\vartheta_2 ... \sin\vartheta_{n-1} d\vartheta_n \\
dS &= dl_1 dl_2 ... dl_{N-1} && \text{(remark : product of N-1 arc length)} \\
&= (rd\vartheta_1)\times(r\sin\vartheta_1 d\vartheta_2)\times(r\sin\vartheta_1\sin\vartheta_2 d\vartheta_3)\times...\times(r\sin\vartheta_1\sin\vartheta_2...\sin\vartheta_{N-2}d\vartheta_{N-1}) \\
&= r^{N-1}(\sin\vartheta_1)^{N-2}(\sin\vartheta_2)^{N-3}(\sin\vartheta_3)^{N-4}...(\sin\vartheta_{N-2})^1 d\vartheta_1 d\vartheta_2 d\vartheta_3 ... d\vartheta_{N-2} d\vartheta_{N-1} \\
&= r^{N-1} d\Omega \\
d\Omega &= (\sin\vartheta_1)^{N-2}(\sin\vartheta_2)^{N-3}(\sin\vartheta_3)^{N-4}...(\sin\vartheta_{N-2})^1 d\vartheta_1 d\vartheta_2 d\vartheta_3 ... d\vartheta_{N-2} d\vartheta_{N-1}
\end{aligned}$$

where Ω is known as the differential solid angle, which is defined as differential surface with unit radius (i.e. r = 1). Note that : **surface area S is dependent on radius** (i.e. it should be S(r) formally), while solid angle Ω is independent on radius, **Ω is a constant for a specific dimension N**. Differential volume is then defined as the volume spanned by the differential change in surface area multipled by the differential change in radius.

$$\begin{aligned}
dV &= drdS \\
&= r^{N-1} dr d\Omega && \text{(equation A)} \\
&= r^{N-1}(\sin\vartheta_1)^{N-2}(\sin\vartheta_2)^{N-3}(\sin\vartheta_3)^{N-4}...(\sin\vartheta_{N-2})^1 dr d\vartheta_1 d\vartheta_2 d\vartheta_3 ... d\vartheta_{N-2} d\vartheta_{N-1}
\end{aligned}$$

Relation between those differentials
Why do we define differential surface, differential solid angle and differential volume? This is because we want to find out the relationship between differential change in rectangular space and differential change in spherical space. Consider a function f : $R^N \to R^1$, the integration of the function in the whole ND space should be unchanged, no matter whether we perform integration in the rectangular space or in the spherical space. Hence we have :

$$\begin{aligned}
\int f(x_1, x_2, ... x_N) dV_{rect} &= \int f(r, \theta_1, \theta_2, ... \theta_{N-1}) dV_{spher} && \text{(remark : } dV_{rect} \neq dV_{spher}) \\
\int_{-\infty}^{+\infty} f(x_1, x_2, ... x_N) dx_1 dx_2 ... dx_N &= \int f(r, \theta_1, \theta_2, ... \theta_{N-1}) drdS && \text{(remark : } drdS \neq dx_1 dx_2 ... dx_N) \\
&= \int f(r, \theta_1, \theta_2, ... \theta_{N-1}) r^{N-1} dr d\Omega && \text{(remark : } drdS = r^{N-1} dr d\Omega) \quad \text{(equation B)} \\
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty} f(x_1, x_2, ... x_N) dx_1 dx_2 ... dx_N &= \int_{-\pi}^{+\pi}\int_0^{\pi}...\int_0^{\pi}\int_0^{\infty} f(r, \vartheta_1, \vartheta_1, ..., \vartheta_{N-1}) r^{N-1}(\sin\vartheta_1)^{N-2}(\sin\vartheta_2)^{N-3}...(\sin\vartheta_{N-2})^1 dr d\vartheta_1 d\vartheta_2 ... d\vartheta_{N-2} d\vartheta_{N-1}
\end{aligned}$$

$$\text{note the integration range} = \left[\begin{array}{ll} r \in [0, \infty] & \\ \theta_n \in [0, \pi] & \forall n \in [1, N-2] \\ \theta_{N-1} \in [-\pi, +\pi] & \end{array}\right.$$

Please notice differences in integration range between rectangular coordinates and spherical coordinates, and the differences in integration range between the polar angles (0, π) and the azimuth angle (-π, +π).

Remark (1) For 2D space

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1,x_2)dx_1dx_2 = \int\int_0^{\infty} f(r,\theta)drdl = \int_{-\pi}^{+\pi}\int_0^{\infty} f(r,\theta)rdrd\theta$$ (remark : the integration range is important)

$$drdl = rdrd\theta$$ (remark : for 2D, θ is azimuth angle)

$$dx_1dx_2 \neq rdrd\theta$$ (remark : for 2D, there is no polar angle)

Remark (2) For volume of hypersphere

When function f(**x**) returns 1 if the vector **x** lies within a distance of R from the origin, and returns zero otherwise, then both sides of the integration below give the volume of the ND hypersphere.

$$f(x_1,x_2,...x_N) = \begin{bmatrix} 1 & if & |x|^2 < R \\ 0 & if & otherwise \end{bmatrix}$$

$$\begin{aligned}
\text{volume of sphere} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty} f(x_1,x_2,...x_N)dx_1dx_2...dx_N \\
&= \int_{-\pi}^{+\pi}\int_0^{\pi}...\int_0^{\pi}\int_0^{\infty} f(r,\vartheta_1,\vartheta_1,...,\vartheta_{N-1})r^{N-1}(\sin\vartheta_1)^{N-2}(\sin\vartheta_2)^{N-3}...(\sin\vartheta_{N-2})^1 drd\vartheta_1 d\vartheta_2...d\vartheta_{N-2}d\vartheta_{N-1} \\
&= \int_{-\pi}^{+\pi}\int_0^{\pi}...\int_0^{\pi}\int_0^{R} r^{N-1}(\sin\vartheta_1)^{N-2}(\sin\vartheta_2)^{N-3}...(\sin\vartheta_{N-2})^1 drd\vartheta_1 d\vartheta_2...d\vartheta_{N-2}d\vartheta_{N-1} \\
&= \int_0^R\left[\int_{-\pi}^{+\pi}\int_0^{\pi}...\int_0^{\pi} r^{N-1}(\sin\vartheta_1)^{N-2}(\sin\vartheta_2)^{N-3}...(\sin\vartheta_{N-2})^1 d\vartheta_1 d\vartheta_2...d\vartheta_{N-2}d\vartheta_{N-1}\right]dr \\
&= \int_0^R\int_{Surf} dS(r)dr \\
&= \int_0^R S(r)dr
\end{aligned}$$

where

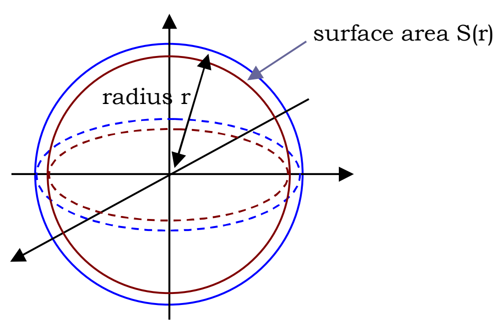
$$dS = r^{N-1}(\sin\vartheta_1)^{N-2}(\sin\vartheta_2)^{N-3}(\sin\vartheta_3)^{N-4}...(\sin\vartheta_{N-2})^1 d\vartheta_1 d\vartheta_2 d\vartheta_3...d\vartheta_{N-2}d\vartheta_{N-1}$$

$$dS = r^{N-1}d\Omega$$

$$S = r^{N-1}\Omega$$

Recall :

- S grows with r^{N-1}.
- Ω is a constant in $\Re^N$.



Volume of sphere

Volume of sphere can be found by considering the integral of the following Gaussian.

$$\text{Area of gaussian} = \int_{-\infty}^{+\infty} e^{-x^2}dx$$

Direct calculation of the above integration is not easy, it can be made easier by considering the square of the integral, followed by a transformation from rectangular coordinates into spherical coordinates.

$$\begin{aligned}
(\int_{-\infty}^{+\infty} e^{-x^2}dx)^2 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x_1^2+x_2^2)}dx_1dx_2 \\
&= \int_{-\pi}^{+\pi}\int_0^{\infty} e^{-r^2}rdrd\theta \qquad \text{(remark : } r^2 = x_1^2 + x_2^2 \text{ and } \theta \text{ is azimuth angle in 2D)} \\
&= \int_{-\pi}^{+\pi}d\theta\int_0^{\infty} e^{-r^2}rdr \\
&= 2\pi\int_0^{\infty} e^{-r^2}dr^2 \\
&= \pi[e^{-r^2}]_0^{\infty} = \pi
\end{aligned}$$

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi} \qquad \text{(equation C)}$$

Lets consider the power of N.

$$
\begin{aligned}
(\int_{-\infty}^{+\infty} e^{-x^2} dx)^N &= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} e^{-(x_1^2+x_2^2+...+x_N^2)} dx_1 dx_2 ... dx_N \\
\pi^{N/2} &= \int_\Omega \int_0^\infty e^{-r^2} r^{N-1} dr d\Omega && \text{(from equation B)} \\
\pi^{N/2} &= \int_\Omega \int_0^\infty e^{-r^2} r^{N-1} dr d\Omega && \text{(from equation C)} \\
\pi^{N/2} &= \int_\Omega d\Omega \times \int_0^\infty e^{-r^2} r^{N-1} dr \\
\pi^{N/2} &= \Omega \int_0^\infty e^{-r^2} r^{N-1} dr \\
\pi^{N/2} &= \Omega \Gamma(N/2)/2 && \text{(see remark)} \\
\Omega &= 2\pi^{N/2} / \Gamma(N/2) && \text{(equation D)}
\end{aligned}
$$

Remark : Definition of gamma function

$$
\begin{aligned}
\int_0^\infty e^{-r^2} r^{N-1} dr &= \int_0^\infty e^{-r^2} r^{N-2} r dr && \text{(substitute } x = r^2 \text{ and } dx = 2rdr \text{)} \\
&= \frac{1}{2} \int_0^\infty e^{-x} x^{(N-2)/2} dx \\
&= \frac{1}{2} \int_0^\infty e^{-x} x^{(N/2)-1} dx \\
&= \Gamma(N/2)/2 && \text{(since } \Gamma(z) \text{ is defined as } \int_0^\infty e^{-r} r^{z-1} dr \text{)}
\end{aligned}
$$

The final step is to find the volume by integrating the solid angle.

$$
\begin{aligned}
S(R) &= R^{N-1}\Omega \\
&= \frac{2R^{N-1}\pi^{N/2}}{\Gamma(N/2)} && \text{(from equation D)}
\end{aligned}
$$

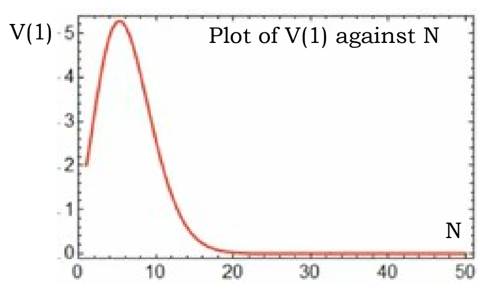
$$
\begin{aligned}
V(R) &= \int_0^R S(r) dr \\
&= \int_0^R r^{N-1} \Omega dr \\
&= \Omega \int_0^R r^{N-1} dr && \text{(since solid angle is independent on radius)} \\
&= \Omega \times [r^N / N]_0^R \\
&= \frac{2R^N \pi^{N/2}}{N\Gamma(N/2)} && \text{(from equation D)} \\
&= \frac{R^N \pi^{N/2}}{\Gamma(N/2+1)} && \text{(since } \Gamma(N/2)N/2 = \Gamma(N/2+1) \text{, in general } \Gamma(z)z = \Gamma(z+1) \text{)}
\end{aligned}
$$

To summarise :

$$
\begin{aligned}
S(R) &= \frac{2R^{N-1}\pi^{N/2}}{\Gamma(N/2)} \\
V(R) &= \frac{R^N \pi^{N/2}}{\Gamma(N/2+1)}
\end{aligned}
$$

Consider $R = 1$, we have :

$$
V(1) = \frac{\pi^{N/2}}{\Gamma(N/2+1)}
$$



Curse of dimensionality
In machine learning, we usually form classes or clusters making use of Euclidean distance from sample points. Now, with the formula of N dimensional sphere, we can see that sparsity that can be represented by the hypersphere formed by a sample data point. The sparsity is more serious as N increases. This is called the curse of dimensionality. We can find the ratio between a N dimensional sphere with its enclosing cube (recall : volume of cube is $(2R)^N$) :

$$
\frac{V_{sphere}(R=1)}{V_{cubic}(R=1)} = \frac{\pi^{N/2}}{\Gamma(N/2+1)} \frac{1}{2^N} \quad \text{tends to zero as N increases.}
$$