Cubic equation

This document shows the way to find explicit solutions for cubic equation, which involves three transformations. The **first step** is to transform the cubic equation into depressed cubic by substitution : x = y + k. The objective is the find the value of k such that y quadratic term vanishes, and find the depressed cubic equation in terms of y.

$$
\begin{aligned}
0 &= ax^3+bx^2+cx+d \\
&= a(y+k)^3+b(y+k)^2+c(y+k)+d \\
&= a(y^3+3y^2k+3yk^2+k^3)+b(y^2+2yk+k^2)+c(y+k)+d \\
&= ay^3+(3ak+b)y^2+(3ak^2+2bk+c)y+(ak^3+bk^2+ck+d) \\
&= ay^3+(3ak^2+2bk+c)y+(ak^3+bk^2+ck+d) && \text{if we choose : } 3ak+b=0\text{, i.e. } k=-b/(3a) \text{ or } x=y-b/(3a) \\
0 &= y^3+By+C && \text{if we scale by } 1/a
\end{aligned}
$$

where

$$
\begin{aligned}
B &= 3k^2+2(b/a)k+(c/a) &&= (b/a)^2/3-2(b/a)^2/3+(c/a) &&= -(b/a)^2/3+(c/a) \\
C &= k^3+(b/a)k^2+(c/a)k+(d/a) &&= -(b/a)^3/27+(b/a)^3/9-(c/a)(b/a)/3+(d/a) &&= 2(b/a)^3/27-(bc/a^2)/3+(d/a)
\end{aligned}
$$

The **second step** is to transform the depressed cubic equation into quadratic equation using Vieta's substitution.

$$
\begin{aligned}
& y = z+s/z \\
\Rightarrow \quad 0 &= y^3+By+C \\
&= (z+s/z)^3+B(z+s/z)+C \\
&= z^3+3sz+3s^2/z+s^3/z^3+Bz+Bs/z+C \\
&= z^3+(3s+B)z+(3s+B)s/z+s^3/z^3+C \\
&= z^3+s^3/z^3+C && \text{if we choose : } 3s+B=0\text{, i.e. } s=-B/3 \text{ or } y=z-B/(3z) \\
0 &= z^6+Cz^3-B^3/27
\end{aligned}
$$

The **third step** is to transform the above equation into quadratic by : $w=z^3$. We then have :

$$
0 = w^2+Cw-B^3/27
$$

We can then solve the equations step by step : w → z → y → x. The quadratic equation gives two solutions : $w = w_1$ or w_2, which may be real or complex. For each w_i, we can solve for three solutions : $z = z_{m,1}$ or $z_{m,2}$ or $z_{m,3}$. Hence there are six solutions of z in total, and for each $z_{m,n}$, we can solve for one $y_{m,n}$ and finally one $x_{m,n}$. Since three values of $y_{m,n}$ will be identical to the other three values of $y_{m,n}$ (can we prove it?), we will have exactly three solutions for x finally. These three solutions can be real or complex. Suppose we have :

$$
\begin{aligned}
w_1 &= (-C+\sqrt{C^2+4B/27})/2 &&= r_1e^{i\vartheta_1} \\
w_2 &= (-C-\sqrt{C^2+4B/27})/2 &&= r_2e^{i\vartheta_2}
\end{aligned}
$$

then the solutions for z are found by **De Moivre's formula** (see complex analysis) :

$$
\begin{aligned}
z_{m,1} &= r_m^{1/3}e^{i(\vartheta_m/3)} &&= r_m^{1/3}e^{i\phi_{m,1}} && \text{for } m=1,2 \\
z_{m,2} &= r_m^{1/3}e^{i(\vartheta_m/3+2\pi/3)} &&= r_m^{1/3}e^{i\phi_{m,2}} && \text{for } m=1,2 \\
z_{m,3} &= r_m^{1/3}e^{i(\vartheta_m/3+4\pi/3)} &&= r_m^{1/3}e^{i\phi_{m,3}} && \text{for } m=1,2 \\
z_{m,n} &= r_m^{1/3}e^{i\phi_{m,n}} && && \text{for } m=1,2 \text{ and } n=1,2,3\text{, hence } |z_{m,1}|=|z_{m,2}|=|z_{m,3}|=r_m^{1/3}=|w_m|^{1/3}
\end{aligned}
$$

In the **Argand diagram**, $z_{m,1}$, $z_{m,2}$ and $z_{m,3}$ form the corners of an equilateral triangle centred at origin. Now both $y_{m,n}$ and $x_{m,n}$ can be found :

$$
\begin{aligned}
y_{m,n} &= z_{m,n}-B/(3z_{m,n}) \\
x_{m,n} &= z_{m,n}-B/(3z_{m,n})-b/(3a) && \text{for } m=1,2 \text{ and } n=1,2,3
\end{aligned}
$$

Physical interpretation when all three roots are real

When all three roots are real, we can derive a physical interpretation for the cubic solutions, and visualise them in a diagram. The **first transformation**, i.e. shifting x axis by $-b/3a$ to form depressed cubic, is in fact shifting the centre of mass (the average of all three roots) so as to match with the origin. Here is the proof. Suppose the three roots in x domain are x_1, x_2 and x_3 :

$$ax^3 + bx^2 + cx + d \equiv a(x - x_1)(x - x_2)(x - x_3)$$

Consider the quadratic term on both sides, we have :

$$\begin{aligned} LHS &= bx^2 \\ RHS &= -a(x_1 + x_2 + x_3)x^2 \qquad \rightarrow \qquad -b/(3a) = (x_1 + x_2 + x_3)/3 \end{aligned}$$

Besides, we scale the depressed cubic equation by coefficient 'a', making the cubic coefficient (in y domain) equals to one. Suppose the three roots in y domain are y_1, y_2 and y_3 :

$$y^3 + By + C \equiv (y - y_1)(y - y_2)(y - y_3) \qquad \rightarrow \qquad 0 = y_1 + y_2 + y_3 \quad \text{(zero centre of mass)} \qquad \text{(equation 1)}$$

Consider the linear term on both sides, we have :

$$\begin{aligned} LHS &= By \\ RHS &= (y_1y_2 + y_2y_3 + y_3y_1)y \qquad \rightarrow \qquad B = y_1y_2 + y_2y_3 + y_3y_1 \qquad \text{(equation 2)} \end{aligned}$$

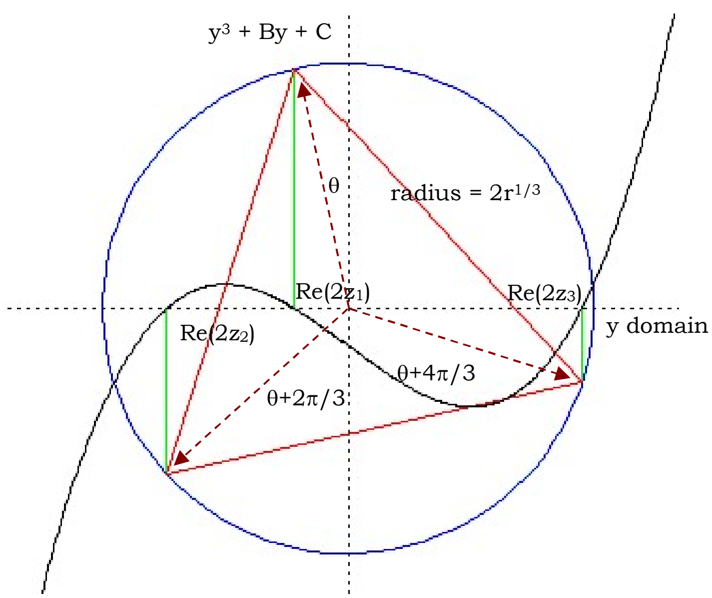
Consider the constant term on both sides, we have :

$$\begin{aligned} LHS &= C \\ RHS &= y_1y_2y_3 \qquad \rightarrow \qquad C = y_1y_2y_3 \qquad \text{(equation 3)} \end{aligned}$$

The **second transformation** gives the relation between y and z. Now consider the case when all x are real, i.e. when all y are real. Since z can be complex, lets assume

$$\begin{aligned} y &= z - B/(3z) \\ &= z - B\bar{z}/(3z\bar{z}) \\ &= \text{Re}(z) + i\,\text{Im}(z) - B\,\text{Re}(z)/(3z\bar{z}) + iB\,\text{Im}(z)/(3z\bar{z}) \\ &= \text{Re}(z) - B\,\text{Re}(\bar{z})/(3z\bar{z}) \qquad \text{(we consider cases when all three roots are real)} \\ \rightarrow \quad 0 &= \text{Im}(z) + B\,\text{Im}(\bar{z})/(3z\bar{z}) \\ 0 &= 1 + B/(3z\bar{z}) \\ B &= -3z\bar{z} \\ \rightarrow \quad y &= \text{Re}(z) - B\,\text{Re}(\bar{z})/(3z\bar{z}) \\ &= 2\,\text{Re}(z) \end{aligned}$$

Hence for cases when all three roots are real, the three roots in y domain are the real part of double z, which form the three corners of an equilateral triangle, centred at the origin, with radius equals to $2r^{1/3}$, we can visualise it, as shown in the following figure, which overlays the depressed cubic equation together with the Argand diagram.



Deriving the condition when all three roots are real
From previous page, we have shown that the only condition for all real roots is :

$$
\begin{aligned}
B &= -3z\bar{z} \\
&= -3|z|^2 \\
&= -3|w|^{2/3} \\
&= -3\times\left|(-C\pm\sqrt{C^2+4B/27})/2\right|^{2/3}
\end{aligned}
$$

$$-B^3/27 = \left((-C\pm\sqrt{C^2+4B/27})/2\right)^2 \qquad \text{when } C^2+4B/27\ge 0$$

$$-B^3/27 = \left((-C\pm\sqrt{C^2+4B/27})/2\right)\times\left((-C\mp\sqrt{C^2+4B/27})/2\right) \qquad \text{when } C^2+4B/27<0$$

Hence we have :

$$-4B^3/27 = C^2\pm 2C\sqrt{C^2+4B/27})+C^2+4B/27 \qquad \text{when } C^2+4B/27\ge 0$$

$$-4B^3/27 = C^2-C^2-4B/27 \qquad \text{when } C^2+4B/27<0$$

and finally :

$$-4B^3/27-4B/27 = 2C^2\pm 2C\sqrt{C^2+4B/27}) \qquad \text{when } C^2+4B/27\ge 0$$

$$-4B^3/27+4B/27 = 0 \qquad \text{when } C^2+4B/27<0$$