Halley's method

Both Newton's method and Halley's method are univariate solvers based on Taylor series expansion. Newton's method gives quadratic convergence, while Halley's method gives cubic convergence. There are two types of Halley's method, i.e. (1) Halley's irrational method, which involves a square root (thus there are chances of getting complex number) and (2) Halley's rational method, which is a combination of Halley's irrational method with Newton's method, does not involve square root (we will provide two proofs for Halley's rational method). Lets review the Newton's method first.

Newton's method

$$f(x) = f(x_t) + f'(x_t)(x - x_t) + \ldots$$ $f(x) = 0$ is the function that we want to solve.

$$g(x) = f(x_t) + f'(x_t)(x - x_t)$$ $g(x) = 0$ is the function approximation nearby $x = x_t$.

$$x_{t+1} = x_t - f(x_t)/f'(x_t)$$

$$= x_t - u_t$$ suppose $u_t = f(x_t)/f'(x_t)$ and $v_t = f'(x_t)/f''(x_t)$

Halley's irrational method

$$f(x) = f(x_t) + f'(x_t)(x - x_t) + (1/2)f''(x_t)(x - x_t)^2 + \ldots$$ $f(x) = 0$ is the function that we want to solve.

$$g(x) = f(x_t) + f'(x_t)(x - x_t) + (1/2)f''(x_t)(x - x_t)^2$$ $g(x) = 0$ is the function approximation nearby $x = x_t$.

$$x_{t+1} = x_t + \frac{-f'(x_t) \pm \sqrt{f'(x_t)^2 - 2f(x_t)f''(x_t)}}{f''(x_t)}$$

$$= x_t + \frac{-1 \pm \sqrt{1 - 2f(x_t)f''(x_t)/(f'(x_t))^2}}{f''(x_t)/f'(x_t)}$$

$$= x_t + \frac{-1 + \sqrt{1 - 2f(x_t)f''(x_t)/(f'(x_t))^2}}{f''(x_t)/f'(x_t)}$$ suppose $f(x_t) \sim 0$, we pick a solution closer to zero

$$= x_t - (1 - \sqrt{1 - 2u_t/v_t})v_t$$ suppose $u_t = f(x_t)/f'(x_t)$ and $v_t = f'(x_t)/f''(x_t)$

$$= x_t - (v_t - \sqrt{v_t^2 - 2u_t v_t})$$

Halley's rational method – proof 1 from Peter John Acklam

$$f(x) = f(x_t) + f'(x_t)(x - x_t) + (1/2)f''(x_t)(x - x_t)^2 + \ldots$$ $f(x) = 0$ is the function that we want to solve.

$$g(x) = f(x_t) + f'(x_t)(x - x_t) + (1/2)f''(x_t)(x - x_t)^2$$ $g(x) = 0$ is the function approximation nearby $x = x_t$.

$$0 = f(x_t) + f'(x_t)(x - x_t) + (1/2)f''(x_t)(x - x_t)^2$$

$$= f(x_t) + [f'(x_t) + (1/2)f''(x_t)(x - x_t)](x - x_t)$$

$$x_{t+1} = x_t - f(x_t)/[f'(x_t) + (1/2)f''(x_t)(x - x_t)]$$ Note : There is still variable x in the RHS.

$$= x_t - f(x_t)/[f'(x_t) + (1/2)f''(x_t)(-f(x_t)/f'(x_t))]$$ Note : c.f. Newton's method, $x_{t+1} - x_t = -f(x_t)/f'(x_t)$.

$$= x_t - 1/[u_t^{-1} - (1/2)v_t^{-1}]$$

$$= x_t - 2u_t/(2 - u_t/v_t)$$

Halley's rational method – proof 2 from wikipedia

Consider $g(x) = f(x)/|f'(x)|^{1/2}$, since $f(x)$ becomes zero, $g(x)$ becomes zero, hence roots of $f(x)$ are also roots of $g(x)$. Besides, absolute value is used inside the square root in order to ensure no complex number resulted.

$$g(x) = \frac{f(x)}{|f'(x)|^{1/2}}$$

$$g'(x) = \frac{f'(x)}{|f'(x)|^{1/2}} - \frac{1}{2}\frac{f(x)}{|f'(x)|^{3/2}}\frac{d|f'(x)|}{dx}$$

For $f(x) \neq 0$ and $f(x) \sim 0$, then

if $f'(x) \geq 0$ $\Rightarrow$ $|f'(x)| = +f'(x)$ $\Rightarrow$ $d|f'(x)|/dx = +f''(x)$ $\Rightarrow$ $\dfrac{d|f'(x)|/dx}{|f'(x)|} = +\dfrac{f''(x)}{f'(x)}$

if $f'(x) < 0$ $\Rightarrow$ $|f'(x)| = -f'(x)$ $\Rightarrow$ $d|f'(x)|/dx = -f''(x)$ $\Rightarrow$ $\dfrac{d|f'(x)|/dx}{|f'(x)|} = +\dfrac{f''(x)}{f'(x)}$ (same result)

Thus, we can simplify $g'(x)$ as :

$$
\begin{aligned}
g'(x) &= \frac{f'(x)}{|f'(x)|^{1/2}} - \frac{1}{2}\frac{f(x)}{|f'(x)|^{1/2}}\frac{f''(x)}{f'(x)} \\
&= \frac{2(f'(x))^2 - f(x)f''(x)}{2f'(x)|f'(x)|^{1/2}}
\end{aligned}
$$

Since roots of $f(x)$ are also roots of $g(x)$. We can then apply Newton's method on $g(x)$, we have :

$$
\begin{aligned}
x_{t+1} &= x_t - g(x_t)/g'(x_t) \\
&= x_t - \frac{f(x_t)}{|f'(x_t)|^{1/2}}\frac{2f'(x_t)|f'(x_t)|^{1/2}}{2(f'(x_t))^2 - f(x_t)f''(x_t)} \\
&= x_t - \frac{2f(x_t)f'(x_t)}{2(f'(x_t))^2 - f(x_t)f''(x_t)} \\
&= x_t - \frac{2f(x_t)/f'(x_t)}{2 - f(x_t)f''(x_t)/(f'(x_t))^2} \\
&= x_t - \frac{2u_t}{2 - u_t/v_t} \qquad \text{(same as proof 1 by Peter John Acklam)}
\end{aligned}
$$

Summary

$x_{t+1} =$	$x_t - u_t$	Newton's method	(quadratic convergence)
$x_{t+1} =$	$x_t - (v_t - \sqrt{v_t^2 - 2u_t v_t})$	Halley's irrational method	(cubic convergence)
$x_{t+1} =$	$x_t - u_t/[1-(1/2)(u_t/v_t)]$	Halley's rational method	(combine Newton and Halley)
$u_t =$	$f(x_t)/f'(x_t)$		
$u_t =$	$f'(x_t)/f''(x_t)$		

What is rate of converge?

Suppose $\lim_{t\to\infty} x_t = L$, then we can define linear convergence, quadratic convergence and cubic convergence as :

$$\lim_{t\to\infty}\frac{|x_{t+1}-L|}{|x_t-L|} = const > 0 \qquad \text{linear convergence}$$

$$\lim_{t\to\infty}\frac{|x_{t+1}-L|}{|x_t-L|^2} = const > 0 \qquad \text{quadratic convergence}$$

$$\lim_{t\to\infty}\frac{|x_{t+1}-L|}{|x_t-L|^3} = const > 0 \qquad \text{cubic convergence}$$

Reference : A small paper on Halley's method, Peter John Acklam, 2002.