Heston Model – Carr Madan approach *from scratch*

CarrMadan vs Lewis

Let's start with a comparison between Carr Madan approach and Lewis approach. Both are based on Fourier transform, the former applies Fourier transform on current option price in log-strike domain, i.e. *logK*, while the latter applies Fourier transform on future payoff in log-underlying-price domain, i.e. *log*S_T, hence, Carr Madan can price multiple options with various strikes simultaneously, whereas Lewis can price only one option each time. We denote Fourier transform as *tilde*. Now suppose :

$$
\begin{aligned}
V_t &= \textit{option price at current time } t \\
f(x) &= \textit{payoff function} \\
X_T &= \ln S_T && \textit{where } x \textit{ is a realization of } X_T \textit{ and we abbreviate } X_T \textit{ as } X \\
k &= \ln K && \textit{where } K \textit{ is vanilla strike}
\end{aligned}
$$

Carr Madan approach

$$
\begin{aligned}
V_t(K) &= FT^{-1}(\tilde{V}_t(z)) \\
&= FT^{-1}(FT(e^{-r(T-t)}E[f(x)])) \\
&= FT^{-1}(FT(e^{-r(T-t)}E[(S_T-K)^+])) \\
&= FT^{-1}(FT(e^{-r(T-t)}E[(e^x-e^k)^+])) \\
&= FT^{-1}(\underbrace{\int_{-\infty}^{\infty} e^{izk}e^{-r(T-t)}E[(e^x-e^k)^+]dk}_{objective})
\end{aligned}
$$

Lewis approach

$$
\begin{aligned}
V_t(S_t) &= e^{-r(T-t)}E[f(x)] \\
&= e^{-r(T-t)}E[FT^{-1}(\tilde{f}(z))] \\
&= \frac{e^{-r(T-t)}}{2\pi}E[\int_{-\infty+iz_i}^{\infty+iz_i}\tilde{f}(z)e^{-izx}dz] \\
&= \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty+iz_i}^{\infty+iz_i}\tilde{f}(z)E[e^{-izx}]dz \\
&= \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty+iz_i}^{\infty+iz_i}\tilde{f}(z)\underbrace{\Phi_X(-z)}_{objective}dz
\end{aligned}
$$

Therefore objective of Carr Madan approach is the derivation of V_t *tilde*, while objective of Lewis approach is the derivation of $\Phi_X(z)$, which includes deriving Heston *PDE*, breaking down into *2 ODEs*, applying Riccati technique, as shown in *Lewis.doc*. Both methods above involve *3* integrations : Fourier transform, inverse Fourier transform and risk neutral expectation. In fact, there are *2* versions for Carr Madan, *f* is vanilla call payoff in version *1*, yet its Fourier inversion is oscillatory for short maturity, thus we have version *2*, in which *f* is out the money payoff, in short it is call for $K > S_t$ and put for $K < S_t$ *(we consider in lnK axis, not in ln*S_T *axis)*.

$$
\begin{aligned}
&\textit{ver 1} \quad & f(X_T=x) &= (e^x-e^k)^+ \\
& & &= (S_T-K)^+ && \textit{consider vanilla call option} \\
&\textit{ver 2} \quad & f(X_T=x) &= (e^x-e^k)^+1_{k>\ln S_t}+(e^k-e^x)^+1_{k<\ln S_t} \\
& & &= (S_T-K)^+1_{K>S_t}+(K-S_T)^+1_{K<S_t} && \textit{consider OTM option (i.e. call and put)}
\end{aligned}
$$

Carr Madan version 1

Let's evaluate the Fourier transform, that is V_t *tilde*. [We use $f(X_T)$ for payoff and $V_t(k)$ for option price, and beware $V_T(k) = f(X_T)$]

$$
\begin{aligned}
V_t(k) &= e^{-r(T-t)}E[f(X_T=x)] \\
\tilde{V}_t(z) &= \int_{-\infty}^{\infty} e^{izk}\underbrace{e^{-r(T-t)}E[(e^x-e^k)^+]}_{V_t(k)}dk
\end{aligned}
$$

Fourier transform exists if $V_t(k)$ is square integrable, or equivalently, vanishes on approaching infinity, however this requirement is not fulfilled by $V_t(k)$ as shown in the following *(Please refer to "Improper integral", i.e. integral with infinite range)*.

$$
\begin{aligned}
\lim_{k\to\infty} V_t(k) &= \lim_{K\to\infty} V_t(K) &&= 0 && V_t(k) \textit{ is bounded in positive } k \textit{ axis} \\
\lim_{k\to-\infty} V_t(k) &= \lim_{K\to 0} V_t(K) &&= S_t && V_t(k) \textit{ is unbounded in negative } k \textit{ axis} \\
\int_{-\infty}^{\infty}|V_t(k)|^2\,dk &= \lim_{k\to-\infty}\int_k^{\infty}|V_t(k')|^2\,dk' &&= \infty && \textit{its area is thus unbounded in negative } k \textit{ axis}
\end{aligned}
$$

Therefore we intentionally introduce a damping factor with positive α :

$$
U_t(k) = e^{\alpha k}V_t(k)
$$

$$= \quad e^{\alpha k} e^{-r(T-t)} E[(e^x - e^k)^+]$$

so that $U_t(k)$ is bounded in negative k axis :

$$\begin{aligned}
\lim_{k\to-\infty} U_t(k) &= \lim_{k\to-\infty} e^{\alpha k} e^{-r(T-t)} E[(e^x - e^k)^+] && \textit{where } \alpha > 0 \\
&= 0 \times S_t \\
&= 0 && \textit{damped price vanishes at +ve inf log strike}
\end{aligned}$$

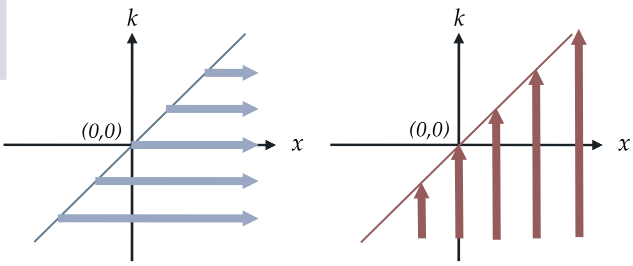
however $U_t(k)$ may become unbounded in positive k axis *(this is what "aggravate" means in Carl Madan's paper)* :

$$\begin{aligned}
\lim_{k\to\infty} U_t(k) &= \lim_{k\to\infty} e^{\alpha k} e^{-r(T-t)} E[(e^x - e^k)^+] && \textit{where } \alpha > 0 \\
&= \lim_{k\to\infty} e^{\alpha k} e^{-r(T-t)} \int_k^\infty (e^x - e^k) p_X(x)dx && \textit{this is } \infty \times 0 \\
&= e^{-r(T-t)} \lim_{k\to\infty} \frac{\int_k^\infty (e^x - e^k) p_X(x)dx}{e^{-\alpha k}} && \textit{apply L Hospital rule} \\
&= e^{-r(T-t)} \lim_{k\to\infty} \frac{\int_k^\infty \partial_k (e^x - e^k) p_X(x)dx - (e^k - e^k) p(x)}{-\alpha e^{-\alpha k}} && \textit{apply Leibniz rule} \\
&= e^{-r(T-t)} \lim_{k\to\infty} \frac{\int_k^\infty -e^k p_X(x)dx}{-\alpha e^{-\alpha k}} \\
&= e^{-r(T-t)} \lim_{k\to\infty} \frac{\int_k^\infty p_X(x)dx}{\alpha e^{-(1+\alpha)k}} \\
&= e^{-r(T-t)} \frac{0}{0} && (\#) \\
&= \ldots && \textit{damped price doesn't vanish at -ve inf log strike}
\end{aligned}$$

We cannot found an appropriate value of α such that $U_t(k)$ is bounded in positive k axis in the above, thus we will try again using a different method in later sections.

Recall Leibniz rule (both sides on the following are functions of k)

$$\frac{d}{dk} \int_{a(k)}^{b(k)} f(x,k)dx = \int_{a(k)}^{b(k)} \partial_k f(x,k)dx + f(b(k),k)\frac{db(k)}{dk} - f(a(k),k)\frac{da(k)}{dk}$$



Let's continue to evaluate the new Fourier transform, that is U_t *tilde*.

$$\begin{aligned}
\tilde{U}_t(z) &= \int_{-\infty}^\infty e^{izk} e^{\alpha k} e^{-r(T-t)} E[(e^x - e^k)^+]dk \\
&= \int_{-\infty}^\infty e^{(\alpha+iz)k} e^{-r(T-t)} \left(\int_k^\infty (e^x - e^k) p_X(x)dx\right)dk \\
&= e^{-r(T-t)} \int_{-\infty}^\infty p_X(x)\left(\int_{-\infty}^x e^{(\alpha+iz)k}(e^x - e^k)dk\right)dx && \textit{reverse integration order as diagram above} \\
&= e^{-r(T-t)} \int_{-\infty}^\infty p_X(x)\left(e^x \int_{-\infty}^x e^{(\alpha+iz)k}dk - \int_{-\infty}^x e^{(1+\alpha+iz)k}dk\right)dx \\
&= e^{-r(T-t)} \int_{-\infty}^\infty p_X(x)\left(e^x [e^{(\alpha+iz)k}/(\alpha+iz)]_{k=-\infty}^{k=x} - [e^{(1+\alpha+iz)k}/(1+\alpha+iz)]_{k=-\infty}^{k=x}\right)dx \\
&= e^{-r(T-t)} \int_{-\infty}^\infty p_X(x)\left(e^x e^{(\alpha+iz)x}/(\alpha+iz) - e^{(1+\alpha+iz)x}/(1+\alpha+iz)\right)dx && \textit{where } e^{(\alpha+iz)(-\infty)} = 0 \textit{ and } e^{(1+\alpha+iz)(-\infty)} = 0 \\
&= e^{-r(T-t)} \frac{\int_{-\infty}^\infty p_X(x)e^{(1+\alpha+iz)x}dx}{(\alpha+iz)(1+\alpha+iz)} \\
&= e^{-r(T-t)} \frac{\int_{-\infty}^\infty p_X(x)e^{i(z-i(1+\alpha))x}dx}{(\alpha+iz)(1+\alpha+iz)} \\
&= e^{-r(T-t)} \frac{\Phi_X(z-i(1+\alpha))}{(\alpha+iz)(1+\alpha+iz)} && \textit{where } \Phi_X(z) = \int_{-\infty}^\infty p_X(x)e^{izx}dx \\
&= e^{-r(T-t)} \frac{\Phi_X(z-i(1+\alpha))}{\alpha+\alpha^2+iz\alpha+iz+iz\alpha-z^2} \\
&= e^{-r(T-t)} \frac{\Phi_X(z-i(1+\alpha))}{\alpha^2+\alpha-z^2+iz(1+2\alpha)} && \textit{equation 1}
\end{aligned}$$

Price for vanilla call of various strike $K = e^k$ can be calculated by inverse Fourier transform *(implemented as FFT)* :

$$V_t(k) = e^{-\alpha k} U_t(k)$$

$$= e^{-\alpha k} FT^{-1}(\tilde{U}_t(z))$$

$$= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-izk}\tilde{U}_t(z)dz$$
as $V_t(k)$ is real, $U_t(z)$ tilde has even real odd imag

$$= \frac{e^{-\alpha k}}{\pi}\int_{0}^{\infty} e^{-izk}\tilde{U}_t(z)dz$$
this is equation 5 in Carl Madan's paper

$$= \frac{e^{-\alpha k}}{\pi}\int_{0}^{\infty} e^{-izk} e^{-r(T-t)} \frac{\Phi_X(z-i(1+\alpha))}{\alpha^2+\alpha-z^2+iz(1+2\alpha)}dz$$
this is equation 6 in Carl Madan's paper

$$= \frac{e^{-\alpha k}e^{-r(T-t)}}{\pi}\int_{0}^{\infty} \frac{e^{-izk}\Phi_X(z-i(1+\alpha))}{\alpha^2+\alpha-z^2+iz(1+2\alpha)}dz$$
equation 2

<u>*Issue 1 : Determine α value*</u>

There are two unsolved issues, *(1)* determination of α value, *(2)* truncation of integration range in equation *2*. We have attempted to address *issue 1* in *(#)*, however it failed, now we are going to tackle it again in a different way. Making damped price $U_t(k)$ vanished on positive infinite log strike is equivalent to making the *DC* component of U_t *tilde* infinite, that is :

$$\infty > \tilde{U}_t(z=0)$$

$$= \int_{-\infty}^{\infty} e^{i0k}U_t(k)dk$$

$$= \int_{-\infty}^{\infty} U_t(k)dk$$
as $U_t(k)$ is bounded on –ve log strike axis

$$= A + \int_{0}^{\infty} U_t(k)dk$$
there exists $A<\infty$ such that $\int_{-\infty}^{0} U_t(k)dk = A$

$$\infty > \int_{0}^{\infty} U_t(k)dk$$

$$\infty > \int_{0}^{\infty} |U_t(k)|\, dk$$
since $U_t(k)$ is real and positive for all k

$$\lim_{k\to\infty} U_t(k) = 0$$
Is there any formal reason for this step?

From equation *1*, a finite *DC* component of U_t *tilde* is equivalent to :

$$\infty > \tilde{U}_t(z=0)$$

$$\infty > e^{-r(T-t)}\frac{\Phi_X(-i(1+\alpha))}{\alpha^2+\alpha-z^2+iz(1+2\alpha)}$$

$$\infty > \Phi_X(-i(1+\alpha))$$

$$= E[e^{izx}]_{z=-i(1+\alpha)}$$

$$= E[e^{-ii(1+\alpha)x}]$$

$$= E[(e^x)^{1+\alpha}]$$

$$= E[(S_T)^{1+\alpha}]$$
this is equation 7 in Carl Madan's paper

For implementation, we may have :

$$E[(S_T)^{1+\alpha}] = E[(S_t e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\varepsilon})^{1+\alpha}]$$

$$= (S_t)^{1+\alpha}E[e^{(r-\sigma^2/2)(T-t)(1+\alpha)+\sigma\sqrt{T-t}(1+\alpha)\varepsilon}]$$

$$= (S_t)^{1+\alpha}e^{(r-\sigma^2/2)(T-t)(1+\alpha)+(\sigma^2/2)(T-t)(1+\alpha)^2}$$
since $E[e^{\mu+\sigma\varepsilon}] = e^{\mu+\sigma^2/2}$

$$10^8 = (S_t)^{1+\alpha^*}e^{(r-\sigma^2/2)(T-t)(1+\alpha^*)+(\sigma^2/2)(T-t)(1+\alpha^*)^2}$$
pick a large number, which is 10^8 here

Then solve the above equation for α^*, and Carl Madan suggests to use $\alpha = \alpha^*/4$.

<u>*Issue 2 : Truncation of integration*</u>

For implementation, when we calculate the improper integral in equation *2*, the upper limit is truncated to a large number *L* so that the truncation error is acceptable. That is :

$$V_t(k) = \frac{e^{-\alpha k}}{\pi}\int_{0}^{L} e^{-izk}\tilde{U}_t(z)dz$$

$$err(L) = \frac{e^{-\alpha k}}{\pi}\int_L^\infty e^{-izk}\tilde{U}_t(z)dz \qquad \textit{equation 3}$$

Now, let's estimate the truncation error :

$$\begin{aligned}
|\tilde{U}_t(z)|^2 &= e^{-r(T-t)}\left|\frac{\Phi_X(z-i(1+\alpha))}{\alpha^2+\alpha-z^2+iz(1+2\alpha)}\right|^2 \\
&\le \frac{|\Phi_X(z-i(1+\alpha))|^2}{|\alpha^2+\alpha-z^2+iz(1+2\alpha)|^2} \\
&= \frac{|\Phi_X(z-i(1+\alpha))|^2}{(\alpha^2+\alpha-z^2)^2+z^2(1+2\alpha)^2} \\
&= \frac{|\Phi_X(z-i(1+\alpha))|^2}{(\alpha^2+\alpha)^2-2(\alpha^2+\alpha)z^2+z^4+z^2(1+2\alpha)^2} \\
&= \frac{|\Phi_X(z-i(1+\alpha))|^2}{(\alpha^2+\alpha)^2+(1-2\alpha+2\alpha^2+4\alpha^4)z^2+z^4} \\
&= \frac{|\Phi_X(z-i(1+\alpha))|^2}{(\alpha^2+\alpha)^2+((1-\alpha)^2+\alpha^2+4\alpha^4)z^2+z^4} \\
&\le \frac{|\Phi_X(z-i(1+\alpha))|^2}{z^4} \\
&\le \frac{E[(S_T)^{1+\alpha}]}{z^4} = \frac{10^8}{z^4} \qquad \textit{why? Carl said that } \Phi \textit{ is bounded by } E[(S_T)^{1-\alpha}] \\
|\tilde{U}_t(z)| &\le \frac{10^4}{z^2}
\end{aligned}$$

From equation *3*, the truncation error is :

$$\begin{aligned}
err(L) &= \frac{e^{-\alpha k}}{\pi}\int_L^\infty e^{-izk}\tilde{U}_t(z)dz \\
&\le \frac{e^{-\alpha k}}{\pi}\int_L^\infty |\tilde{U}_t(z)|\,dz \qquad \textit{why?} \\
&\le \frac{e^{-\alpha k}}{\pi}\int_L^\infty \frac{10^4}{z^2}dz \\
&= \frac{e^{-\alpha k}}{\pi}\left[-\frac{10^4}{z}\right]_L^\infty \\
&= \frac{e^{-\alpha k}}{\pi}\frac{10^4}{L} \\
L &= \frac{e^{-\alpha k}}{\pi}\frac{10^4}{error-bound}
\end{aligned}$$

Hyperbolic cosine and Hyperbolic sine

Before we talk about another approach to Carr Madan, let's take a look at *hyperbolic cosine* and *sine*, abbreviated as *cosh* and *sinh*.

$$\cosh(x) = \frac{e^x+e^{-x}}{2} \qquad \cosh(0) = 1 \qquad \lim_{x\to\pm\infty}\cosh(x) = \infty$$

$$\sinh(x) = \frac{e^x-e^{-x}}{2} \qquad \sinh(0) = 0 \qquad \lim_{x\to\pm\infty}\sinh(x) = \pm\infty$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x-e^{-x}}{e^x+e^{-x}} \qquad \tanh(0) = 0 \qquad \lim_{x\to\pm\infty}\tanh(x) = \pm\infty$$

Beware that *sinh*, *sinc* and *sgn* are different mathematical functions. Don't confuse :

$$\sin c(x) = \frac{\sin(x)}{x} \qquad \textit{for mathematics} \qquad \int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx = \pi$$

$$\begin{aligned}
\sin c(x) &= \frac{\sin(\pi x)}{\pi x} \qquad \textit{for engineering} \qquad \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\pi x} dx = 1 \\
\text{sgn}(x) &= 1_{x>0}
\end{aligned}$$

It is called *hyperbolic sine* because it has some *sine-liked* properties :

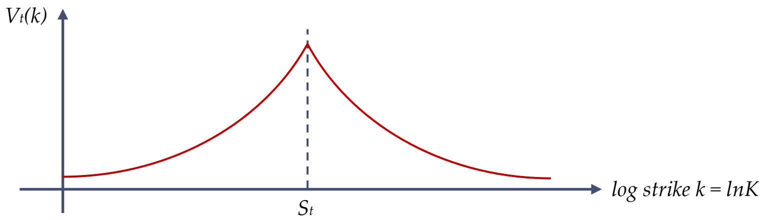
$$\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= ((e^x + e^{-x})^2 - (e^x - e^{-x})^2)/4 && \textit{note : it is a minus, not a plus} \\
&= (e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x}))/4 \\
&= 1 \\
\sinh(x+y) &= (e^{(x+y)} - e^{-(x+y)})/2 \\
&= ((e^{x+y} - e^{-x-y}) + (e^{x+y} - e^{-x-y}))/4 \\
&= ((e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}))/4 \\
&= ((e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y}))/4 \\
&= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \\
\cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) && \textit{note : it is a plus, not a minus} \\
\sinh(2x) &= 2\sinh(x)\cosh(x) \\
\cosh(2x) &= \cosh^2(x) + \sinh^2(x) && \textit{note : it is a plus, not a minus}
\end{aligned}$$

Carr Madan version 2 – OTM options

For short maturity option, call price approaches its intrinsic value, which is non-analytic (analytic function in complex plane means function that is infinitely differentiable, i.e. locally given by Taylor series), as a result, Fourier inversion by equation 2 becomes very oscillatory, thus we need an alternative approach that focuses on time value only, as opposed to intrinsic value. It is done by setting up an *OTM* option, i.e. a call when $k > \ln S_t$ and a put when $k < \ln S_t$.

$$f(k) = (e^x - e^k)^+ 1_{k>\ln S_t} + (e^k - e^x)^+ 1_{k<\ln S_t} \qquad \textit{where } X_T = x \textit{ and } X_T = \ln S_T$$

If the probability density function of $X_T = \ln S_T$, i.e. $p_X(x)$, is unimodal (i.e. single-peak), then option price $V_t(k)$ should peak at $k = \ln S_t$, and declines as k moves to either positive or negative infinity.



$$\begin{aligned}
V_t(k) &= e^{-r(T-t)} E[f(k)] \\
&= e^{-r(T-t)} E[(e^x - e^k)^+ 1_{k>\ln S_t} + (e^k - e^x)^+ 1_{k<\ln S_t}] \\
\tilde{V}_t(z) &= \int_{-\infty}^{\infty} e^{izk} e^{-r(T-t)} E[(e^x - e^k)^+ 1_{k>\ln S_t} + (e^k - e^x)^+ 1_{k<\ln S_t}] dk \\
&= e^{-r(T-t)} \times \left[\int_{-\infty}^{\infty} e^{izk} [\int_{-\infty}^{\infty} ((e^x - e^k) 1_{x>k,k>\ln S_t} + (e^k - e^x) 1_{x<k,k<\ln S_t}) p_X(x) dx] dk \right] \\
&= e^{-r(T-t)} \times \left[\int_{\ln S_t}^{\infty} e^{izk} \int_k^{\infty} (e^x - e^k) p_X(x) dx dk + \int_{-\infty}^{\ln S_t} e^{izk} \int_{-\infty}^{k} (e^k - e^x) p_X(x) dx dk \right] \\
&= e^{-r(T-t)} \times \left[\int_{\ln S_t}^{\infty} p_X(x) \int_{\ln S_t}^{x} e^{izk} (e^x - e^k) dk dx + \int_{-\infty}^{\ln S_t} p_X(x) \int_x^{\ln S_t} e^{izk} (e^k - e^x) dk dx \right] \\
&= e^{-r(T-t)} \times \left[\int_{\ln S_t}^{\infty} p_X(x) \int_{\ln S_t}^{x} (e^x e^{izk} - e^{(1+iz)k}) dk dx + \int_{-\infty}^{\ln S_t} p_X(x) \int_x^{\ln S_t} (e^{(1+iz)k} - e^x e^{izk}) dk dx \right] \\
&= e^{-r(T-t)} \times \left[\begin{array}{l} \int_{\ln S_t}^{\infty} p_X(x) ((\frac{e^x e^{izx}}{iz} - \frac{e^{(1+iz)x}}{1+iz}) - (\frac{e^x e^{iz\ln S_t}}{iz} - \frac{e^{(1+iz)\ln S_t}}{1+iz})) dx + \\ \int_{-\infty}^{\ln S_t} p_X(x) ((\frac{e^{(1+iz)\ln S_t}}{1+iz} - \frac{e^x e^{iz\ln S_t}}{iz}) - (\frac{e^{(1+iz)x}}{1+iz} - \frac{e^x e^{izx}}{iz})) dx \end{array} \right] \\
&= e^{-r(T-t)} \times \left[\int_{-\infty}^{\infty} p_X(x) ((\frac{e^x e^{izx}}{iz} - \frac{e^{(1+iz)x}}{1+iz}) - (\frac{e^x e^{iz\ln S_t}}{iz} - \frac{e^{(1+iz)\ln S_t}}{1+iz})) dx \right] \\
&= e^{-r(T-t)} \times \left[\int_{-\infty}^{\infty} p_X(x) (e^{(1+iz)x} (\frac{1}{iz} - \frac{1}{1+iz}) - e^{iz\ln S_t} (\frac{e^x}{iz} - \frac{e^{\ln S_t}}{1+iz})) dx \right]
\end{aligned}$$

$$= e^{-r(T-t)} \times \left[\int_{-\infty}^{\infty} p_X(x)(e^{(1+iz)x}(\frac{1}{iz} - \frac{1}{1+iz}) - S_t^{iz}(\frac{e^x}{iz} - \frac{S_t}{1+iz}))dx \right]$$

$$= e^{-r(T-t)} \times \left[\frac{1}{iz(1+iz)} \int_{-\infty}^{\infty} p_X(x)e^{(1+iz)x}dx - S_t^{iz} \int_{-\infty}^{\infty} p_X(x)(\frac{e^x}{iz} - \frac{S_t}{1+iz})dx \right]$$

$$= e^{-r(T-t)} \times \left[\frac{1}{iz(1+iz)} \int_{-\infty}^{\infty} p_X(x)e^{(1+iz)x}dx - \frac{S_t^{iz}}{iz} \int_{-\infty}^{\infty} p_X(x)e^x dx + \frac{S_t^{1+iz}}{1+iz} \int_{-\infty}^{\infty} p_X(x)dx \right]$$

$$= e^{-r(T-t)} \times \left[\frac{1}{iz(1+iz)} \int_{-\infty}^{\infty} p_X(x)e^{i(z-i)x}dx - \frac{S_t^{iz}}{iz} \int_{-\infty}^{\infty} p_X(x)e^{i(-i)x} dx + \frac{S_t^{1+iz}}{1+iz} \right]$$

$$= e^{-r(T-t)} \times \left[\frac{1}{iz(1+iz)} \Phi_X(z-i) - \frac{S_t^{iz}}{iz} \Phi_X(-i) + \frac{S_t^{1+iz}}{1+iz} \right]$$ *generic version of equation 14 in Carl Madan*

Unfortunately, for nearly *ATM* option approaching maturity, $V_t(k)$ is approximately a dirac delta function, hence again, the Fourier inverse becomes oscillatory again, hence it is better to add a damping using *sinh* function, it is picked because it vanishes at $k = lnSt$, suppressing the dirac delta, besides it is easy to calculate, as it becomes a frequency shift in Fourier domain.

$$\begin{aligned}
U_t(k) &= \sinh(\alpha k)V_t(k) \\
\tilde{U}_t(z) &= \int_{-\infty}^{\infty} e^{izk} \sinh(\alpha k)V_t(k)dk \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{izk}(e^{\alpha k} - e^{-\alpha k})V_t(k)dk \\
&= \frac{1}{2}\int_{-\infty}^{\infty} (e^{i(z-i\alpha)k} - e^{i(z+i\alpha)k})V_t(k)dk \\
&= \frac{1}{2}(\tilde{V}_t(z-i\alpha) - \tilde{V}_t(z+i\alpha)) \\
V_t(k) &= \frac{1}{\sinh(\alpha k)}U_t(k) \\
&= \frac{1}{\sinh(\alpha k)} FT^{-1}(\tilde{U}_t(z)) \\
&= \frac{1}{\sinh(\alpha k)} FT^{-1}(\frac{1}{2}(\tilde{V}_t(z-i\alpha) - \tilde{V}_t(z+i\alpha))) \\
&= \frac{1}{\sinh(\alpha k)} \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izk}(\tilde{V}_t(z-i\alpha) - \tilde{V}_t(z+i\alpha))dz
\end{aligned}$$

Summary

Solution to vanilla call for various strikes and various models can be found by plugging appropriate characteristic function :

<u>*Approach 1*</u>

$$V_t(k) = \frac{e^{-\alpha k} e^{-r(T-t)}}{\pi} \int_0^{\infty} \frac{e^{-izk}\Phi_X(z - i(1+\alpha))}{\alpha^2 + \alpha - z^2 + iz(1+2\alpha)} dz$$

<u>*Approach 2*</u>

$$\begin{aligned}
V_t(k) &= \frac{1}{\sinh(\alpha k)} \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izk}(\tilde{V}_t(z-i\alpha) - \tilde{V}_t(z+i\alpha))dz \\
\tilde{V}_t(z) &= e^{-r(T-t)} \times \left[\frac{1}{iz(1+iz)} \Phi_X(z-i) - \frac{S_t^{iz}}{iz} \Phi_X(-i) + \frac{S_t^{1+iz}}{1+iz} \right]
\end{aligned}$$