Spread Options: From Margrabe to Kirk

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Abstract

Kirk provided an approximate closed-form solution for the price of a spread option [1]. This paper is written in response to ref. [2] published in Applied Mathematics Letters in which the author believes no explicit derivation of Kirk's approximation from Margrabe's exchange option formula is available or has ever been published. Here we provide such an explicit derivation.

Keywords: Spread option, Kirk, Margrabe, Black-Scholes, Energy market

1 Introduction

Derivatives are among the most important financial instruments. They are used not only for speculation purposes but most importantly for hedging the risk. They allow us to remove the uncertainty from the future cash flows of an investment. Depending on the type of risk many different kinds of Derivatives have been developed [3, 4]. In this work, our concern is the *Spread Options* [5]. These kinds of Derivatives have many important applications such as the valuation of power plants [6, 7, 5]. An option as its name says gives its owner the option (but not obligation) to buy the underlying asset at a specific price at a specific time in the future. This is of course the definition for the European call options. There are many other types of options in the market

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[4, 7]. For the case of spread options, the underlying is not a single asset price rather the spread between the price of several assets. For example, the spread between the electric power price and fuel price (rescaled by heat rate which tells us how inefficient the power plant is) which is known as the spark spread. It means that using a series of spark spread options we can hedge the future operational incomes of a power plant or assess its present value [6]. In this work, our focus is on the spread between just two underlying assets.

To hedge the risk we should build a portfolio out of the Derivatives and underlying assets so that the uncertainties offset each other. It is possible to do that because the sources of uncertainty in a derivative and its underlying asset are the same. But the question is which portion of underlying assets with respect to the Derivatives we should use in our portfolio. Using Ito's lemma we can simply see that these portions are given by *Greeks* [4, 3]. But the main challenge of practitioners is that the Greeks are changing from time to time and so to have a risk-free portfolio we need to calculate them fast and accurately to rebalance the portfolio dynamically. Having a closed-form solution for the Derivative prices solves this problem as the Greeks are the derivatives of the Derivative prices with respect to their underlying parameters and so we can have a closed-form solution for the Greeks as well. The other benefit of having a closed-form solution for the Derivative prices is that we can invert them to get the implied quantities such as implied volatility that later can be used to price exotic Derivatives. In this paper, we address an approximate closed-form solution for the price of a (call) spread option given by Kirk [1] (the put option price is reduced to call one through the put-call parity). We will show that Kirk's approximation is indeed an extension of Margrabe's exact solution for exchange options [8].

2 Black and Scholes' paradigm

The main characteristic of a spread option is its payoff at maturity, T:

$$C(T, S_2(T), S_1(T)) = max\{S_2(T) - S_1(T) - K, 0\} \tag{1}$$

where $C_t = C(t, S_2(t), S_1(t))$ is the value of option at time t, K is the strike price and $S_{1,2}$ are the price of underlying assets. Our objective is to find the present value of the option $C_0 = C(0, S_2(0), S_1(0))$. Since the payoff at maturity is a nonlinear function of the price of underlying assets we need to have a model for the underlying asset prices. Here we utilize all the Black-Scholes' assumptions including the Geometric Brownian Motion (GBM) as the model for the price of underlying assets [9]. I.e. :

$$dS_i(t)/S_i(t) = \mu_i dt + \sigma_i d\omega_i(t) \quad i = 1, 2 \tag{2}$$

where $\omega_1(t) = \tilde{\omega}_1(t)$ and $\omega_2(t) = \rho\tilde{\omega}_1(t) + \sqrt{1-\rho^2}\tilde{\omega}_2(t)$ are two correlated Wiener processes and ρ denotes the strength of correlation. $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are two independent Wiener processes. $\omega_{1,2}$ are simply derived through the Cholesky decomposition of the correlation matrix. Constants μ and σ are the expected rate of return and volatility, respectively.

We know from the Feynman-Kac representation theorem that there are two ways to get the C_0 [10, 11]. The first way is to solve a backward parabolic partial differential equation having the terminal value for the price, Eq. (1), which guarantees the uniqueness of the solution [2]. The second way is to use the solution of the underlying stochastic processes at maturity and then take the average over all possible payoffs and finally discount it back to the present time (Equivalent Martingale Measure):

$$C_0 = e^{-rT} E_0^Q \left[max\{S_2(T) - S_1(T) - K, 0\} \right] \tag{3}$$

where Q and 0 indicate that we take the expectation with respect to the risk-neutral probability measure and the stochastic processes are adapted to the natural filtration, respectively. In a risk-neutral world, the expected rate of return for all traded assets is the risk-free rate, $\mu_i = r$, and moreover to get the present value we discount back the future expected payoff using the risk-free rate [3]. We choose the latter way to reach our objective because using Ito's lemma we can have the underlying asset prices at maturity $S_i(T) = S_i(0) \exp\left((\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i\omega_i(T)\right)$. The main challenge that remains is to take the expectation which we will deal with in the coming sections.

3 Margrabe's exchange option

When $K = 0$ in fact we have the option to change the underlying assets at maturity. In such a case, Margrabe showed that there is a closed-form solution for Eq. (3) [8]. To get this solution first we factor $S_1(T)$ and then move to an equivalent probability measure, Q', using Radon–Nikodym density $dQ'/dQ = \exp\left(-\frac{1}{2}\sigma_1^2 T + \sigma_1 \tilde{\omega}_1(T)\right)$, so that we can bring S_1 out of expectation:

$$\begin{aligned} C_0 &= e^{-rT} E_0^{Q'}\left[max\{(S_2(T)/S_1(T) - 1)\, S_1(T), 0\} \times \frac{dQ}{dQ'}\right] \\ &= S_1(0) E_0^{Q'}\left[max\{S_2(T)/S_1(T) - 1, 0\}\right]. \end{aligned} \tag{4}$$

Using Ito's lemma and Eqs. (2), we can find the dynamic of S_2/S_1:

$$\begin{aligned} &d\left(\frac{S_2}{S_1}\right) / \left(\frac{S_2}{S_1}\right) = (\rho\sigma_2 - \sigma_1)(-\sigma_1 dt + d\tilde{\omega}_1) + \sqrt{1-\rho^2}\sigma_2 d\tilde{\omega}_2 \\ &= (\rho\sigma_2 - \sigma_1)\, d\tilde{\omega}_1^{Q'} + \sqrt{1-\rho^2}\sigma_2 d\tilde{\omega}_2 = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\, d\omega = \sigma d\omega \end{aligned} \tag{5}$$

where $-\sigma_1 dt + d\tilde{\omega}_1$ becomes drift-less ($d\tilde{\omega}_1^{Q'}$) under probability measure Q' as we expect from Girsanov's theorem [10]. $\tilde{\omega}_1^{Q'}$ and $\tilde{\omega}_2$ are still independent Wiener processes. As can be seen $S_2(t)/S_1(t)$ is a GBM. It means we can use the Black-Scholes formula to get C_0 in Eq. (4), as the expectation is indeed equivalent to the European call option price when strike price and risk-free rate are 1 and 0, respectively:

$$C_0 = S_2(0)\Phi(d_1) - S_1(0)\Phi(d_2), \quad d_{1,2} = \frac{\ln\left(S_2(0)/S_1(0)\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \tag{6}$$

where Φ is the cumulative distribution function of standard normal distribution.

4 Kirk's approximation

In the previous section, we dealt with the case $K = 0$. Now let us see how we can deal with the case $K \neq 0$. Kirk provided an approximate solution for such a case [1]. Here we will show that Kirk's approximation is indeed an immediate extension of Margrabe's exact solution. We write Eq. (1) as following :

$$C_T = max\{S_2(T) - S_1'(T), 0\} \tag{7}$$

where $S_1'(T) = S_1(T) + K$. Now it is very tempting to use Margrabe's formula, but the problem is that $S_1'(t) = S_1(t) + Ke^{-r(T-t)}$ is not necessarily a GBM. Using Ito's lemma, we have

$$\begin{aligned} d \ln S_1'(t) &= \frac{dS_1(t) + rKe^{-r(T-t)}}{S_1(t) + Ke^{-r(T-t)}} - \frac{1}{2} \frac{\left(dS_1(t) + rKe^{-r(T-t)}\right)^2}{\left(S_1(t) + Ke^{-r(T-t)}\right)^2} \\ &= \left(r - \frac{1}{2}{\sigma_1'}^2\right) dt + \sigma_1' d\omega_1(t) \end{aligned} \tag{8}$$

where

$$\sigma_1'(t) = \frac{S_1(t)}{S_1(t) + Ke^{-r(T-t)}} \sigma_1. \tag{9}$$

Eq. (8) reminds us of a GMB, but it is not yet necessarily unless σ_1' be constant. That is exactly the place Kirk's approximation comes into play: *If σ_1' be constant (or at least a slowly-varying function of t and $S_1(t)$), then we can use Margrabe's formula, Eqs. (6) as a (approximate) solution for Eq. (3):*

$$C_0 = S_2(0)\Phi(d_1) - (S_1(0) + Ke^{-rT})\Phi(d_2), \quad d_{1,2} = \frac{\ln\left(\frac{S_2(0)}{S_1(0)+Ke^{-rT}}\right) \pm \frac{1}{2}\sigma'^2 T}{\sigma' \sqrt{T}} \tag{10}$$

where $\sigma' = \sqrt{[\sigma_1'(0)]^2 + \sigma_2^2 - 2\rho[\sigma_1'(0)]\sigma_2}$. We are done. Note that when $K = 0$ the Kirk's approximation is reduced to Margrabe's exact solution.

5 Summary

In this paper, we addressed the (approximate) closes-form solutions for a spread option with two underlying assets (Eq. (1)). We kept all the assumption of Black and Scholes and first derived the Margrabe's exact solution (Eqs. (6)) in which the strike price is zero. Then we went to non-zero strike prices and showed that Kirk's approximation (Eqs. (10)) is indeed the immediate extension of Margrabe's exact solution assuming σ_1', Eq. (9), is a slowly-varying function of t and $S_1(t)$.

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