FX Spot Delta and Forward Delta

Buy USD sell JPY

Given S_0 as the USDJPY spot rate, current time is $t = 0$, Black Scholes for *USDJPY option@T* with *JPY* as term currency is :

$$c = (Se^{-r_{USD}T})N(d_1) - (Ke^{-r_{JPY}T})N(d_2)_{\;yen}$$

$$\text{and} \quad p = (Ke^{-r_{JPY}T})N(-d_2) - (Se^{-r_{USD}T})N(-d_1)_{\;yen}$$

$$\text{where} \quad d_{1,2} = \frac{\ln(S/K) + (r_{JPY} - r_{USD} \pm \sigma^2/2)T}{\sigma\sqrt{T}} \qquad \text{hence } d_2 = d_1 - \sigma\sqrt{T}$$

Derive call spot delta and put spot delta :

$$\frac{\partial d_{1,2}}{\partial S} = \frac{1}{\sigma\sqrt{T}}\frac{1}{S}$$

$$\begin{aligned}
\frac{\partial c}{\partial S} &= N(d_1)e^{-r_{USD}T} + (Se^{-r_{USD}T})\frac{\partial N(d_1)}{\partial S} - (Ke^{-r_{JPY}T})\frac{\partial N(d_2)}{\partial S} \\
&= N(d_1)e^{-r_{USD}T} + (Se^{-r_{USD}T})\frac{n(d_1)}{\sigma\sqrt{T}S} - (Ke^{-r_{JPY}T})\frac{n(d_2)}{\sigma\sqrt{T}S} && \text{where } n(d) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \text{ and } \frac{\partial d_{1,2}}{\partial S} = \frac{1}{\sigma\sqrt{T}S} \\
&= N(d_1)e^{-r_{USD}T}{}_{\;yen} && \text{where } (Se^{-r_{USD}T})n(d_1) = (Ke^{-r_{JPY}T})n(d_2)\text{, see remark} \\
\frac{\partial p}{\partial S} &= -N(-d_1)e^{-r_{USD}T}{}_{\;yen} && \text{following similar argument as call option}
\end{aligned}$$

Therefore, the spot vector-delta representation in trader's perspective is :

$$\text{spot delta vector for call} = \left(-\frac{N(d_1)e^{-r_{USD}T}}{S}_{\$}, N(d_1)e^{-r_{USD}T}{}_{\;yen} \right)$$

For calculating forward delta, it is prefered to slightly reformulate Black Scholes as Black76 :

$$c = (FN(d_1) - KN(d_2)) \times e^{-r_{JPY}T}{}_{\;yen} \qquad \text{since } F = Se^{(r_{JPY} - r_{USD})T}$$

$$\text{and} \quad p = (KN(-d_2) - FN(-d_1))e^{-r_{JPY}T}{}_{\;yen}$$

$$\text{where} \quad d_{1,2} = \frac{\ln(F/K) \pm v/2}{\sqrt{v}} \qquad \text{where } v = \sigma^2 T \text{ and again } d_2 = d_1 - \sigma\sqrt{T}$$

$$\frac{\partial d_{1,2}}{\partial S} = \frac{1}{\sqrt{v}}\frac{1}{F}$$

$$\begin{aligned}
\frac{\partial c}{\partial F} &= \left(N(d_1) + F\frac{\partial N(d_1)}{\partial S} - K\frac{\partial N(d_2)}{\partial S} \right) \times e^{-r_{JPY}T} \\
&= \left(N(d_1) + F\frac{n(d_1)}{\sqrt{v}F} - K\frac{n(d_2)}{\sqrt{v}F} \right) \times e^{-r_{JPY}T} && \text{where } Fn(d_1) = Kn(d_2) \\
&= N(d_1)e^{-r_{JPY}T}{}_{\;yen} \\
\frac{\partial p}{\partial F} &= -N(-d_1)e^{-r_{JPY}T}{}_{\;yen}
\end{aligned}$$

Remark :

$$\begin{aligned}
(Ke^{-r_{JPY}T})n(d_2) &= (Ke^{-r_{JPY}T})n(d_1 - \sigma\sqrt{T}) \\
&= (Ke^{-r_{JPY}T})e^{-(d_1 - \sigma\sqrt{T})^2/2} / \sqrt{2\pi} \\
&= (Ke^{-r_{JPY}T})n(d_1) \times e^{(2d_1\sigma\sqrt{T} - \sigma^2 T)/2} \\
&= (Ke^{-r_{JPY}T})n(d_1) \times e^{(2\ln(S/K) + 2(r_{JPY} - r_{USD})T)/2} \\
&= Kn(d_1) \times e^{\ln(S/K)}e^{-r_{USD}T} \\
&= (Se^{-r_{USD}T})n(d_1)
\end{aligned}$$

$$\text{besides} \quad Kn(d_2) = Fn(d_1)$$

Buy JPY sell USD

Given $S'=1/S$ as the JPYUSD spot rate, current time is $t = 0$, Black Scholes for *JPYUSD option@T* with *USD* as term currency is :

$$c' = (S'e^{-r_{JPY}T})N(d'_1) - (K'e^{-r_{USD}T})N(d'_2)_{\$}$$

and $$p' = (K'e^{-r_{USD}T})N(-d'_2) - (S'e^{-r_{JPY}T})N(-d'_1)_{\$}$$

where $$d'_{1,2} = \frac{\ln(S'/K') + (r_{USD} - r_{JPY} \pm \sigma'^2/2)T}{\sigma'\sqrt{T}}$$ *hence* $d'_2 = d'_1 - \sigma'\sqrt{T}$

$$S' = 1/S$$
$$K' = 1/K$$
$$\sigma' = \sigma$$ *why? derive inverse process by Girsanov theorem?*

hence $$d'_{1,2} = \frac{-\ln(S/K) - (r_{JPY} - r_{USD} \mp \sigma^2/2)T}{\sigma\sqrt{T}} = -d_{2,1}$$

Derive call spot delta and put spot delta :

$$\frac{\partial d'_{1,2}}{\partial S'} = \frac{1}{\sigma'\sqrt{T}}\frac{1}{S'}$$

$$\frac{\partial c'}{\partial S'} = N(d'_1)e^{-r_{JPY}T}{}_{\$}$$ *following similar argument as before*

$$\frac{\partial p'}{\partial S'} = -N(-d'_1)e^{-r_{JPY}T}{}_{\$}$$ *following similar argument as before*

For calculating forward delta, it is prefered to slightly reformulate Black Scholes as Black76 :

$$c' = (F'N(d'_1) - K'N(d'_2)) \times e^{-r_{USD}T}{}_{\$}$$ *since* $F' = S'e^{(r_{USD}-r_{JPY})T} = 1/F$

and $$p' = (K'N(-d'_2) - F'N(-d'_1))e^{-r_{USD}T}{}_{\$}$$

where $$d'_{1,2} = \frac{\ln(F'/K') \pm v'/2}{\sqrt{v'}}$$ *where* $v' = \sigma'^2 T = v$ *and again* $d'_2 = d'_1 - \sigma'\sqrt{T}$

$$\frac{\partial d'_{1,2}}{\partial S'} = \frac{1}{\sqrt{v'}}\frac{1}{F'}$$

$$\frac{\partial c'}{\partial F'} = N(d'_1)e^{-r_{USD}T}{}_{\$}$$

$$\frac{\partial p'}{\partial F'} = -N(-d'_1)e^{-r_{USD}T}{}_{\$}$$

Connection between two perspectives

First of all, lets check if USDJPY call is equivalent to JPYUSD put in the same currency :

$$\begin{aligned}
c &= (Se^{-r_{USD}T})N(d_1) - (Ke^{-r_{JPY}T})N(d_2)_{yen} \\
&= (e^{-r_{USD}T})N(d_1) - ((K/S)e^{-r_{JPY}T})N(d_2)_{\$} \\
&= (e^{-r_{USD}T})N(-d'_2) - ((K/S)e^{-r_{JPY}T})N(-d'_1)_{\$} \\
&= (e^{-r_{USD}T})N(-d'_2) - ((S'/K')e^{-r_{JPY}T})N(-d'_1)_{\$} \\
&= [(K'e^{-r_{USD}T})N(-d'_2) - (S'e^{-r_{JPY}T})N(-d'_1)_{\$}] \times (1/K') \\
&= p' \times (1/K') \\
\textit{or} \quad &= p' \times K
\end{aligned}$$

Therefore, one USDJPY call contract is equivalent to K JPYUSD put contracts.

The following shows how these deltas are related to each other :

$$
\begin{aligned}
\frac{\partial c}{\partial S} &= N(d_1)e^{-r_{USD}T}{}_{yen} & \frac{\partial c'}{\partial S'} &= N(d'_1)e^{-r_{JPY}T}{}_{\$} \\
\frac{\partial p}{\partial S} &= -N(-d_1)e^{-r_{USD}T}{}_{yen} & \frac{\partial p'}{\partial S'} &= -N(-d'_1)e^{-r_{JPY}T}{}_{\$} \\
\frac{\partial c}{\partial F} &= N(d_1)e^{-r_{JPY}T}{}_{yen} & \frac{\partial c'}{\partial F'} &= N(d'_1)e^{-r_{USD}T}{}_{\$} \\
\frac{\partial p}{\partial F} &= -N(-d_1)e^{-r_{JPY}T}{}_{yen} & \frac{\partial p'}{\partial F'} &= -N(-d'_1)e^{-r_{USD}T}{}_{\$}
\end{aligned}
$$

(1) Call delta and put delta are related by call-put parity :

$$
\begin{aligned}
\frac{\partial c}{\partial S} - \frac{\partial p}{\partial S} &= N(d_1)e^{-r_{USD}T}{}_{yen} + N(-d_1)e^{-r_{USD}T}{}_{yen} \\
&= N(d_1)e^{-r_{USD}T}{}_{yen} + (1-N(d_1))e^{-r_{USD}T}{}_{yen} = e^{-r_{USD}T}{}_{yen}
\end{aligned}
$$

similar result can be derived for other call-put pairs

(2) Spot delta and forward delta are related by ratio of discount factors :

$$
\frac{\partial c}{\partial S} \Big/ \frac{\partial c}{\partial F} = \frac{N(d_1)e^{-r_{USD}T}{}_{yen}}{N(d_1)e^{-r_{JPY}T}{}_{yen}} = \frac{e^{-r_{USD}T}}{e^{-r_{JPY}T}}
$$

similar result can be derived for other call-put pairs

(3) USDJPY call delta and JPYUSD put delta are related by :

$$
\begin{aligned}
\frac{\partial c'}{\partial S'} &= N(d'_1)e^{-r_{JPY}T}{}_{\$} = N(-d_2)e^{-r_{JPY}T}{}_{\$} = \frac{\partial p}{\partial K}{}_{\$} \\
\frac{\partial p'}{\partial S'} &= -N(-d'_1)e^{-r_{JPY}T}{}_{\$} = -N(d_2)e^{-r_{JPY}T}{}_{\$} = \frac{\partial c}{\partial K}{}_{\$} \\
\frac{\partial c'}{\partial F'} &= N(d'_1)e^{-r_{USD}T}{}_{\$} = N(-d_2)e^{-r_{USD}T}{}_{\$} = \frac{\partial p}{\partial K}\frac{e^{-r_{USD}T}}{e^{-r_{JPY}T}}{}_{\$} \\
\frac{\partial p'}{\partial F'} &= -N(-d'_1)e^{-r_{USD}T}{}_{\$} = -N(d_2)e^{-r_{USD}T}{}_{\$} = \frac{\partial c}{\partial K}\frac{e^{-r_{USD}T}}{e^{-r_{JPY}T}}{}_{\$}
\end{aligned}
$$

Delta representation

There are two ways to represent FX delta : by currency pair or currency. The former is more intuitive. Suppose there exists portfolio of USDJPY calls and puts with different strikes and maturities, having a net delta of *1* million USDJPY, it implies market risk can be fully hedged by shorting *1* million USDJPY spot contract, i.e. selling *1* million USD for *S* millions JPY, where *S* is spot rate. On other hand, if there exists portfolio of EURJPY options with a net delta of *-0.77* million EURJPY, it means that the portfolio can be hedged separately with spot contract on EURJPY. In currency-pair representation, delta is always represented as a notional of spot contract, the ***positions of different FX pairs are hedged separately*** (recall that the FX underling *S* is a ratio between a currency pair, hence the unit is not a single currency). However, FX hedging with spot can be optimized (with less transaction and smaller spread) if delta is considered as in the perspective of a portfolio. Suppose USDJPY spot rate is *100*, where EURJPY spot rate is *130*, instead of hedging by selling *1* million USDJPY spot contract plus longing *0.77* million EURJPY spot contract, delta risk can be fully hedged by longing *0.77* million EURUSD spot contract only, as the JPY position is net off :

$$
1.0 million_{yen} \times 100 - 0.77 million_{yen} \times 130 \sim 0_{yen}
$$

In order to facilitate delta-hedging in the perspective of the whole portfolio, we need to breakdown the deltas of currency-pairs into deltas of currencies, like this :

currency pair representation			*portfolio 1*	*portfolio 2*	*total*
1 million USDJPY		*USD*	*1 million*		*1 million*
-0.77 million EURJPY	$\Leftrightarrow$	*EUR*		*-0.77 million*	*-0.77 million*
	equivalent	*YEN*	*-100 × 1 million*	*130 × 0.77 million*	*0*

Please note that this representation of delta for individual currency is valid for delta-hedging using spot contract only, it is invalid for gamma nor vega. Besides, hedging like this must be monitored dynamically, as a change in USDJPY (whereas EURJPY remains unchanged) introduces risk exposure in YEN. This is why some traders prefer to hedge FX pairs individually using former method.