Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Method 1 : Using Taylor series with pivot at zero

$$\begin{aligned}
e^{i\theta} &= \textstyle\sum_{n=0}^{\infty} i^n e^{i\theta}\,|_{\theta=0}\,(\theta-0)^n &&= \textstyle\sum_{n=0}^{\infty} i^n \theta^n \\
\cos\theta &= \textstyle\sum_{n=0}^{\infty} \cos(n\pi/2+\theta)\,|_{\theta=0}\,(\theta-0)^n &&= \textstyle\sum_{n=0}^{\infty} \cos(n\pi/2)\theta^n \\
\sin\theta &= \textstyle\sum_{n=0}^{\infty} \sin(n\pi/2+\theta)\,|_{\theta=0}\,(\theta-0)^n &&= \textstyle\sum_{n=0}^{\infty} \sin(n\pi/2)\theta^n
\end{aligned}$$

$$\begin{aligned}
\cos\theta + i\sin\theta &= \textstyle\sum_{n=0}^{\infty} \cos(n\pi/2)\theta^n + i\sum_{n=0}^{\infty} \sin(n\pi/2)\theta^n \\
&= \textstyle\sum_{n=0}^{\infty} [\cos(n\pi/2) + i\sin(n\pi/2)]\theta^n
\end{aligned}$$

Since : for $n = 4m$ $\quad \cos(4m\pi/2) + i\sin(4m\pi/2) = 1 = i^{4m}$

for $n = 4m+1$ $\quad \cos((4m+1)\pi/2) + i\sin((4m+1)\pi/2) = i = i^{4m+1}$

for $n = 4m+2$ $\quad \cos((4m+2)\pi/2) + i\sin((4m+2)\pi/2) = -1 = i^{4m+2}$

for $n = 4m+3$ $\quad \cos((4m+3)\pi/2) + i\sin((4m+3)\pi/2) = -i = i^{4m+3}$

Hence finally we have : $\cos\theta + i\sin\theta = e^{i\theta}$

Method 2 : Consider the following ratio

$$\begin{aligned}
f(\theta) &= (\cos\theta + i\sin\theta)e^{-i\theta} \\
f'(\theta) &= -i(\cos\theta + i\sin\theta)e^{-i\theta} + (-\sin\theta + i\cos\theta)e^{-i\theta} \\
&= (-i\cos\theta + \sin\theta - \sin\theta + i\cos\theta)e^{-i\theta} \\
&= 0
\end{aligned}$$

Since $f'(\theta) = 0$ for all θ, thus $f(\theta)$ is a constant. Lets find the value of the constant by putting θ with any value.

$$\begin{aligned}
& (\cos 0 + i\sin 0)e^{-i0} = 1 \\
\Rightarrow\quad & (\cos\theta + i\sin\theta)e^{-i\theta} = 1 \\
\Rightarrow\quad & \cos\theta + i\sin\theta = e^{i\theta}
\end{aligned}$$

Method 3 : Consider the following differential equation

$f'(\theta) = if(\vartheta)$ with initial condition $f(0) = 1$

We are going to prove that $\cos\theta + i\sin\theta$ is a specific solution.

$$\begin{aligned}
f'(\theta) &= -\sin\theta + i\cos\theta = i(i\sin\theta + \cos\theta) = if(\vartheta) \\
f(0) &= \cos 0 + i\sin 0 = 1
\end{aligned}$$

We are going to prove that $e^{i\theta}$ is a specific solution.

$$\begin{aligned}
f'(\theta) &= ie^{i\theta} = if(\vartheta) \\
f(0) &= e^{i0} = 1
\end{aligned}$$

Specific solution is not a general solution. Specific solution satisfies initial condition, and thus it is unique. Hence :

$$\Rightarrow\quad \cos\theta + i\sin\theta = e^{i\theta}$$