Interest Rate Model – Vasicek and Hull White

Content

Ordinary differential equation

ODE type		solution	application
1st order linear *ODE*	$f'(x) = p(x) + q(x)f(x)$	integration factor	Hull White bond price
1st order linear *ODE* no const	$f'(x) = q(x)f(x)$	separation of variables	Hull White bond price
1st order quadratic *ODE*	$f'(x) = p(x) + q(x)f(x) + r(x)f^2(x)$	*Riccati* equation	Heston

Please refer to *Heston.doc* for *Riccati* equation.

1. Short rate models

Instantaneous rate (or short rate) is defined as interest rate accrued for infinitesimal period of time from *t* to *t+dt* as of time *t*.

dr_t	=	$a(b-r_t)dt+\sigma dz_t$	*Vasicek model*	*const parameter : a, b and σ*
dr_t	=	$(\theta_t-ar_t)dt+\sigma dz_t$	*Hull White model*	*const parameter : a, σ and θ_t*
dr_t	=	$(\theta_t-a_tr_t)dt+\sigma dz_t$	*Hull White extended Vasicek model*	*const parameter : σ, a_t and θ_t*
dr_t	=	$\theta_t dt+\sigma dz_t$	*Ho Lee model*	

All above short rate models are ***normal short rate*** model, which implies ***lognormal bond price***.

- *Hull White* model can be degenerated to *Vasicek* by setting $\theta_t = ab$
- *Hull White* model can be degenerated to *Ho Lee* by setting $a=0$
- *Vasicek* fits market data of yield curve and volatility matrix with only 3 constant parameters *a*, *b* and σ
- *Vasicek* does not have enough degree of freedom to achieve a perfect match with market quotes of yield curve
- *Hull White* introduces time dependent θ_t which can achieve a perfect match with market quotes of yield curve
- *Hull White* has extra parameters *a* and σ for fitting the volatility matrix
- both *Hull White* and *Ho Lee* guarantee consistency with the yield curve
- given short rate model, we can derive ***bond price formula*** (which also offers ***discount factor***) as shown in the following …

Vasciek bond price

$$P_t(T) = A(t,T)e^{-r_tB(t,T)} \qquad \textit{where } r_t \textit{ is deterministic as of time } t$$

$$\rightarrow B(t,T) = \frac{1}{a}(1-e^{-a(T-t)}) \qquad \textit{depends on a only (not on b nor } \sigma)$$

$$\rightarrow A(t,T) = \exp\left(\boxed{(b-\frac{\sigma^2}{2a^2})(B(t,T)-(T-t))}-\frac{\sigma^2}{4a}B^2(t,T)\right) \qquad eq(1)$$

Hull White bond price

$$P_t(T) = A(t,T)e^{-r_tB(t,T)} \qquad \textit{where } r_t \textit{ is deterministic as of time } t$$

$$\rightarrow B(t,T) = \frac{1}{a}(1-e^{-a(T-t)}) \qquad \textit{exactly the same as Vasicek}$$

$$\rightarrow A(t,T) = \exp\left(\boxed{-\int_t^T\theta_sB(s,T)ds-\frac{\sigma^2}{2a^2}(B(t,T)-(T-t))}-\frac{\sigma^2}{4a}B^2(t,T)\right) \qquad eq(2)$$

Lets check if we can get *eq(1)* by putting $\theta_t = ab$ into *eq(2)*. We consider *this part* for simplicity.

$$
\begin{aligned}
& -\int_t^T\theta_sB(s,T)ds-\frac{\sigma^2}{2a^2}(B(t,T)-(T-t)) \\
= & -\int_t^T abB(s,T)ds-\frac{\sigma^2}{2a^2}(B(t,T)-(T-t)) && \textit{putting } \theta_t = ab \\
= & -ab\frac{1}{a}\int_t^T(1-e^{-a(T-s)})ds-\frac{\sigma^2}{2a^2}(B(t,T)-(T-t)) \\
= & -b\left[\int_t^T 1ds-\int_t^T e^{-a(T-s)}ds\right]-\frac{\sigma^2}{2a^2}(B(t,T)-(T-t)) \\
= & -b\left[(T-t)-\frac{1}{a}(e^{-a(T-T)}-e^{-a(T-t)})\right]-\frac{\sigma^2}{2a^2}(B(t,T)-(T-t)) \\
= & -b\left[(T-t)-\frac{1}{a}(1-e^{-a(T-t)})\right]-\frac{\sigma^2}{2a^2}(B(t,T)-(T-t)) && \textit{using } B(t,T)=(1-e^{-a(T-t)})/a \\
= & \; b(B(t,T)-(T-t))-\frac{\sigma^2}{2a^2}(B(t,T)-(T-t)) \\
= & \; (b-\frac{\sigma^2}{2a^2})(B(t,T)-(T-t))
\end{aligned}
$$

2. Risk neutral expectation method (*RNE*) to bond price

Consider current time t, and time s is a time point in the future, such that $s>t$:

- *step 1 : solve for short rate* $\quad r_s \quad$ *this is random*
- *step 2 : solve for discount path* $\quad \int_t^T r_s ds \quad$ *this is random*
- *step 3 : expectation and variance of short rate* $\quad E[r_s \mid I_t]$ *and* $V[r_s \mid I_t] \quad$ *deterministic given* r_t
- *step 4 : expectation and variance of discount path* $\quad E[\int_t^T r_s ds \mid I_t]$ *and* $V[\int_t^T r_s ds \mid I_t] \quad$ *deterministic given* r_t
- *step 5 : bond price in form of* $\quad P_t(T) = A(t,T)e^{-r_t B(t,T)} \quad$ *deterministic given* r_t

This method is applicable to both *Vasicek* and *Hull White*. Lets try *RNE* with *Vasicek* $dr_t = a(b-r_t)dt + \sigma dz_t$, the results are :

.

$$
\begin{aligned}
r_s &= r_t e^{-a(s-t)} + b(1-e^{-a(s-t)}) + \sigma\int_t^s e^{-a(s-u)}dz_u && \textit{besides } r_s \textit{ is normal}\\
E[r_s \mid I_t] &= r_t e^{-a(s-t)} + b(1-e^{-a(s-t)})\\
V[r_s \mid I_t] &= \frac{\sigma^2}{2a}(1-e^{-2as})\\
\int_t^T r_s ds &= b(T-t) + \frac{1}{a}(r_t-b)(1-e^{-a(T-t)}) + \sigma\int_t^T\int_u^T e^{-a(s-u)}dsdz_u && \textit{besides } \int_t^T r_s ds \textit{ is normal}\\
E[\int_t^T r_s ds \mid I_t] &= b(T-t) + \frac{1}{a}(r_t-b)(1-e^{-a(T-t)})\\
V[\int_t^T r_s ds \mid I_t] &= \frac{\sigma^2}{a^2}\left((T-t) - \frac{2}{a}(1-e^{-a(T-t)}) + \frac{1}{2a}(1-e^{-2a(T-t)})\right)
\end{aligned}
$$

step 1 : solve for interest rate

Lets remove r_t from *RHS* by considering :

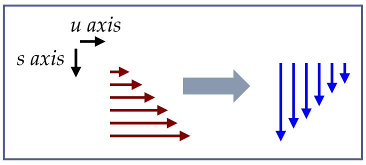
$$
\begin{aligned}
d(r_t e^{at}) &= r_t de^{at} + e^{at}dr_t\\
&= r_t ae^{at}dt + e^{at}(a(b-r_t)dt + \sigma dz_t)\\
&= r_t ae^{at}dt + (abe^{at} - r_t ae^{at})dt + \sigma e^{at}dz_t\\
&= abe^{at}dt + \sigma e^{at}dz_t
\end{aligned}
$$

Integrate both sides from t to s :

$$
\begin{aligned}
\int_t^s d(r_u e^{au}) &= \int_t^s abe^{au}du + \int_t^s \sigma e^{au}dz_u\\
r_s e^{as} - r_t e^{at} &= \int_{u=t}^{u=s} be^{au}dau + \sigma\int_t^s e^{au}dz_u\\
r_s e^{as} - r_t e^{at} &= b(e^{as}-e^{at}) + \sigma\int_t^s e^{au}dz_u\\
r_s e^{as} &= r_t e^{at} + b(e^{as}-e^{at}) + \sigma\int_t^s e^{au}dz_u\\
r_s &= r_t e^{-a(s-t)} + b(1-e^{-a(s-t)}) + \sigma\int_t^s e^{-a(s-u)}dz_u\\
r_s &= r_t w_{t,s} + b(1-w_{t,s}) + \sigma\int_t^s w_{u,s}dz_u\\
&= \varepsilon(E[r_s], V[r_s]) && \textit{where } w_{t,s} = e^{-a(s-t)} \textit{ is DF}
\end{aligned}
$$

step 2 : solve for discounting path

$$
\begin{aligned}
\int_t^T r_s ds &= \int_t^T (r_t e^{-a(s-t)} + b(1-e^{-a(s-t)}))ds + \sigma\int_t^T(\int_t^s e^{-a(s-u)}dz_u)ds\\
&= \int_t^T (b + (r_t-b)e^{-a(s-t)})ds + \sigma\int_t^T\int_u^T e^{-a(s-u)}dsdz_u && \textit{swap integration sequence}\\
&= b(T-t) + (r_t-b)\int_{s=t}^{s=T} e^{-a(s-t)}d(s-t) + \sigma\int_t^T\int_u^T e^{-a(s-u)}dsdz_u\\
&= b(T-t) - \frac{1}{a}(r_t-b)(e^{-a(T-t)} - e^{-a(t-t)}) + \sigma\int_t^T\int_u^T e^{-a(s-u)}dsdz_u\\
&= b(T-t) + \frac{1}{a}(r_t-b)(1-e^{-a(T-t)}) + \sigma\int_t^T\int_u^T e^{-a(s-u)}dsdz_u\\
&= \varepsilon(E[\int_t^T r_s ds], V[\int_t^T r_s ds])
\end{aligned}
$$



step 3 : find the expectation and volatility of interest rate

$$
\begin{aligned}
E[r_s \mid I_t] &= r_t e^{-a(s-t)} + b(1-e^{-a(s-t)}) && \textit{expectation of Itos is zero} \\
V[r_s \mid I_t] &= E[(\sigma\int_t^s e^{-a(s-u)} dz_u)^2] \\
&= \sigma^2 \int_t^s E[e^{-2a(s-u)}] du && \textit{variance of Itos becomes time integral} \\
&= \sigma^2 \int_t^s e^{-2a(s-u)} du \\
&= \frac{\sigma^2}{2a}[e^{-2a(s-u)}]_0^s \\
&= \frac{\sigma^2}{2a}(1-e^{-2as})
\end{aligned}
$$

step 4 : find the expectation and volatility of discounting path

$$
\begin{aligned}
E[\int_t^T r_s ds \mid I_t] &= b(T-t) + \frac{1}{a}(r_t - b)(1-e^{-a(T-t)}) && \textit{expectation of Itos is zero} \\
V[\int_t^T r_s ds \mid I_t] &= E[(\sigma\int_t^T \int_u^T e^{-a(s-u)} ds dz_u)^2] \\
&= \sigma^2 \int_t^T E[(\int_u^T e^{-a(s-u)} ds)^2] du \\
&= \sigma^2 \int_t^T (\int_u^T e^{-a(s-u)} ds)^2 du \\
&= \sigma^2 \int_t^T (\frac{1}{-a}[e^{-a(s-u)}]_u^T)^2 du \\
&= \frac{\sigma^2}{a^2} \int_t^T (1-e^{-a(T-u)})^2 du \\
&= \frac{\sigma^2}{a^2} \int_t^T (1-2e^{-a(T-u)} + e^{-2a(T-u)}) du \\
&= \frac{\sigma^2}{a^2}\left((T-t) - \frac{2}{a}(e^{-a(T-T)} - e^{-a(T-t)}) + \frac{1}{2a}(e^{-2a(T-T)} - e^{-2a(T-t)})\right) \\
&= \frac{\sigma^2}{a^2}\left((T-t) - \frac{2}{a}(1-e^{-a(T-t)}) + \frac{1}{2a}(1-e^{-2a(T-t)})\right)
\end{aligned}
$$

step 5 : find spot and forward bond price

$$
\begin{aligned}
P_t(T) &= E\left[\$1 \times e^{-\int_t^T r_s ds} \mid r_t\right] && \textit{FTAP} \\
&= E\left[e^{\varepsilon(-E[\int_t^T r_s ds], V[\int_t^T r_s ds])} \mid r_t\right] \\
&= \exp\left(-E[\int_t^T r_s ds] + \frac{1}{2}V[\int_t^T r_s ds]\right) \\
&= \exp\left(-\overbrace{\left(b(T-t) + \frac{1}{a}(r_t - b)(1-e^{-a(T-t)})\right)}^{E[discounting]} + \frac{1}{2}\overbrace{\frac{\sigma^2}{a^2}\left((T-t) - \frac{2}{a}(1-e^{-a(T-t)}) + \frac{1}{2a}(1-e^{-2a(T-t)})\right)}^{V[discounting]}\right) \\
P_t(T) &= A(t,T)e^{-B(t,T)r_t} && \textit{A and B are known given Vasicek para}
\end{aligned}
$$

Define $B(t,T) = \frac{1}{a}(1-e^{-a(T-t)})$ *eq(3)*

then

$$
\begin{aligned}
e^{-a(T-t)} &= 1 - aB(t,T) \\
e^{-2a(T-t)} &= (1-aB(t,T))^2 \\
\frac{1}{a}(1-e^{-2a(T-t)}) &= \frac{1}{a}(1-(1-aB(t,T))^2) \\
\frac{1}{a}(1-e^{-2a(T-t)}) &= \frac{1}{a}(1-1+2aB(t,T) - a^2B^2(t,T)) \\
\frac{1}{a}(1-e^{-2a(T-t)}) &= 2B(t,T) - aB^2(t,T) && \textit{eq(3')}
\end{aligned}
$$

$$
\begin{aligned}
\textit{then}\quad A(t,T) &= P_t(T)e^{+B(t,T)r_t} && \textit{since}\ \ P_t(T)=A(t,T)e^{-B(t,T)r_t} \\
&= \exp\left(-\left(b(T-t)+\frac{1}{a}(r_t-b)(1-e^{-a(T-t)})\right)+\frac{1}{2}\frac{\sigma^2}{a^2}\left((T-t)-\frac{2}{a}(1-e^{-a(T-t)})+\frac{1}{2a}(1-e^{-2a(T-t)})\right)+r_tB(t,T)\right) \\
&= \exp\left(-b(T-t)-(r_t-b)B(t,T)+\frac{\sigma^2}{2a^2}\left((T-t)-2B(t,T)+B(t,T)-\frac{1}{2}aB^2(t,T)\right)+r_tB(t,T)\right) \\
&= \exp\left(-b(T-t)+bB(t,T)+\frac{\sigma^2}{2a^2}\left((T-t)-B(t,T)-\frac{1}{2}aB^2(t,T)\right)\right) \\
&= \exp\left(b(B(t,T)-(T-t))+\frac{\sigma^2}{2a^2}((T-t)-B(t,T))-\frac{\sigma^2}{4a}B^2(t,T)\right) \\
&= \exp\left((b-\frac{\sigma^2}{2a^2})(B(t,T)-(T-t))-\frac{\sigma^2}{4a}B^2(t,T)\right) && \textit{eq(4)}
\end{aligned}
$$

Ito's integral and Ito's isometry

Itos integral $\int_0^t f_s dz_s$ is an integral in z_t space with deterministic or stochastic integrand f_t. Itos integral is itself stochastic.

	property	*deterministic integrad*	*stochastic integrand*	
(1)	*martingale*	*yes*	*yes*	
(2)	$E[\int_t^T f_s dz_s \mid I_t]$	0	0	
(3)	$E[(\int_t^T f_s dz_s)^2 \mid I_t]$	$\int_t^T f_s^2 ds$	$\int_t^T E[f_s^2 \mid I_t]ds$	*also called Ito's isometry*
(4)	$E[(\int_t^T f_s dz_s)(\int_t^T g_s dz_s) \mid I_t]$	$\int_t^T f_s g_s ds$	$\int_t^T E[f_s g_s \mid I_t]ds$	*also called Ito's isometry*
(5)	*distribution*	$\varepsilon(0,\int_t^T f_s^2 ds)$	*unknown*	

Lets prove. We put $t=0$ for simplicity.

$$
\begin{aligned}
(1)\quad d(\int_0^t f_s dz_s) &= f_t dz_t && \textit{thus it is martingale} \\
(2)\quad E[\int_0^t f_s dz_s \mid I_0] &= \int_0^0 f_s dz_s && \textit{since it is martingale} \\
&= 0 \\
(3)\quad E[(\int_0^t f_s dz_s)^2 \mid I_0] &= E[\lim_{N\to\infty}(\sum_{n=0}^{N-1} f_{n\Delta t}(z_{(n+1)\Delta t}-z_{n\Delta t}))^2 \mid I_0] && \textit{where}\ \ \Delta t = t/N \\
&= E[\lim_{N\to\infty}(\sum_{n=0}^{N-1} f_{n\Delta t}(z_{(n+1)\Delta t}-z_{n\Delta t}))\times(\sum_{m=0}^{N-1} f_{m\Delta t}(z_{(m+1)\Delta t}-z_{m\Delta t})) \mid I_0] \\
&= E[\lim_{N\to\infty}\sum_{n=0}^{N-1} f_{n\Delta t}^2(z_{(n+1)\Delta t}-z_{n\Delta t})^2 + 2\sum_{n=0}^{N-1}\sum_{m=0}^{n-1} f_{n\Delta t}f_{m\Delta t}(z_{(n+1)\Delta t}-z_{n\Delta t})(z_{(m+1)\Delta t}-z_{m\Delta t}) \mid I_0] \\
&= \lim_{N\to\infty}\sum_{n=0}^{N-1} E[f_{n\Delta t}^2(z_{(n+1)\Delta t}-z_{n\Delta t})^2 \mid I_0] + 2\sum_{n=0}^{N-1}\sum_{m=0}^{n-1} E[f_{n\Delta t}f_{m\Delta t}(z_{(n+1)\Delta t}-z_{n\Delta t})(z_{(m+1)\Delta t}-z_{m\Delta t}) \mid I_0] \\
&= \lim_{N\to\infty}\sum_{n=0}^{N-1} E[f_{n\Delta t}^2(z_{(n+1)\Delta t}-z_{n\Delta t})^2 \mid I_0] + 2\times 0 \\
&= \lim_{N\to\infty}\sum_{n=0}^{N-1} E[f_{n\Delta t}^2 \mid I_0]\Delta t \\
&= \int_0^t E[f_s^2 \mid I_0]ds && \textit{for stochastic fs} \\
&= \int_0^t f_s^2 ds && \textit{for deterministic fs}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
E[f_t^2(z_{t'}-z_t)^2 \mid I_0] &= E[E[f_t^2(z_{t'}-z_t)^2 \mid I_t] \mid I_0] && \textit{assume } t < t' \\
&= E[f_t^2 E[(z_{t'}-z_t)^2 \mid I_t] \mid I_0] \\
&= E[f_t^2(t'-t)] \mid I_0] \\
&= E[f_t^2 \mid I_0](t'-t) \\
E[f_t f_s(z_{t'}-z_t)(z_{s'}-z_s) \mid I_0] &= E[E[f_t f_s(z_{t'}-z_t)(z_{s'}-z_s) \mid I_t] \mid I_0] && \textit{assume } s < s' < t < t' \\
&= E[f_t f_s(z_{s'}-z_s)E[z_{t'}-z_t \mid I_t] \mid I_0] \\
&= E[f_t f_s(z_{s'}-z_s)\times 0 \mid I_0] \\
&= 0
\end{aligned}
}
$$

$$
\begin{aligned}
(4)\quad E[(\int_0^t f_s dz_s)(\int_0^t g_s dz_s) \mid I_0] &= E[\lim_{N\to\infty} (\sum_{n=0}^{N-1} f_{n\Delta t}(z_{(n+1)\Delta t} - z_{n\Delta t})) \times (\sum_{m=0}^{N-1} g_{m\Delta t}(z_{(m+1)\Delta t} - z_{m\Delta t})) \mid I_0] \\
&= E[\lim_{N\to\infty} \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} f_{n\Delta t} g_{m\Delta t}(z_{(n+1)\Delta t} - z_{n\Delta t})(z_{(m+1)\Delta t} - z_{m\Delta t}) \mid I_0] \\
&= \lim_{N\to\infty} \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} E[f_{n\Delta t} g_{m\Delta t}(z_{(n+1)\Delta t} - z_{n\Delta t})(z_{(m+1)\Delta t} - z_{m\Delta t}) \mid I_0] \\
&= \lim_{N\to\infty} \sum_{n=0}^{N-1} E[f_{n\Delta t} g_{n\Delta t}(z_{(n+1)\Delta t} - z_{n\Delta t})(z_{(n+1)\Delta t} - z_{m\Delta t}) \mid I_0] \\
&= \lim_{N\to\infty} \sum_{n=0}^{N-1} E[f_{n\Delta t} g_{n\Delta t} \mid I_0]\Delta t \\
&= \int_0^t E[f_s g_s \mid I_0] ds
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
E[f_t g_t (z_{t'} - z_t)^2 \mid I_0] &= E[E[f_t g_t (z_{t'} - z_t)^2 \mid I_t] \mid I_0] \\
&= E[f_t g_t E[(z_{t'} - z_t)^2 \mid I_t] \mid I_0] \\
&= E[f_t g_t (t' - t)] \mid I_0] \\
&= E[f_t g_t \mid I_0](t' - t) \\
E[f_t g_s (z_{t'} - z_t)(z_{s'} - z_s) \mid I_0] &= E[E[f_t g_s (z_{t'} - z_t)(z_{s'} - z_s) \mid I_t] \mid I_0] \\
&= E[f_t g_s (z_{s'} - z_s) E[z_{t'} - z_t \mid I_t] \mid I_0] \\
&= E[f_t g_s (z_{s'} - z_s) \times 0 \mid I_0] \\
&= 0
\end{aligned}
}
$$

If f_t is deterministic, then Ito's integral must be normal (because the sum of normal is also normal).

$$
\begin{aligned}
(5)\quad \int_0^t f_s dz_s &= \lim_{N\to\infty} \sum_{n=0}^{N-1} f_{n\Delta t}(z_{(n+1)\Delta t} - z_{n\Delta t}) \\
&= \lim_{N\to\infty} \sum_{n=0}^{N-1} f_{n\Delta t}(\varepsilon \Delta t) \\
&= \varepsilon(\mu = 0, \sigma^2 = \int_0^t f_s{}^2 ds)
\end{aligned}
$$

3. Partial differential equation method (*PDE*) to bond price

Let's try *PDE* approach with *Hull White*, which does not involve solving short rate and discounting path :

- *step 1 : given SDE of underlying short rate*
- *step 2 : derive PDE of contingent claim using Dr Yan's fastlane*
- *step 3 : break into two ODEs*
- *step 4 : solve first order linear ODE by integrating factor*
- *step 5 : solve first order linear ODE by separation of variables*

step 2 : fastlane to PDE

With Dr Yan's fastlane, we can write the *PDE* of any contingent claim $P_t(T)$ with underlyings x_t and y_t in single step.

$$
\begin{array}{lcll}
dx_t & = & \alpha_{xt}dt + \beta_{xt}dz_{1t} & \\
dy_t & = & \alpha_{yt}dt + \beta_{yt}dz_{2t} & \\
dz_{1t}dz_{2t} & = & \rho dt & \\
 & & & \\
r_t P_t(T) & = & \partial_t P_t + \alpha_{xt}(\partial_x P_t) + \alpha_{yt}(\partial_y P_t) + \frac{1}{2}\beta_{xt}^2(\partial_{xx}P_t) + \rho\beta_{xt}\beta_{yt}(\partial_{xy}P_t) + \frac{1}{2}\beta_{yt}^2(\partial_{yy}P_t) + ... & \textit{for 2 risk factors} \\
 & = & \partial_t P_t + \alpha_{xt}(\partial_x P_t) + \frac{1}{2}\beta_{xt}^2(\partial_{xx}P_t) & \textit{for 1 risk factor} \\
 & = & \partial_t P_t + (\theta_t - ar_t)(\partial_r P_t) + \frac{\sigma^2}{2}(\partial_{rr}P_t) & x_t = r_t,\ \ \alpha_t = \theta_t - ar_t,\ \ \beta_t = \sigma \\
P_T(T) & = & 1 & \textit{boundary condition}
\end{array}
$$

step 3 : breakdown into two ODEs

We put an ansatz (educated guess) into the *PDE* :

$$
P_t(T) = A(t,T)e^{-r_t B(t,T)}
$$

$$
\begin{array}{lcl}
r_t A(t,T)e^{-r_t B(t,T)} & = & \partial_t(A(t,T)e^{-r_t B(t,T)}) + (\theta_t - ar_t)\partial_r(A(t,T)e^{-r_t B(t,T)}) + \frac{\sigma^2}{2}\partial_{rr}(A(t,T)e^{-r_t B(t,T)}) \\
 & = & (\partial_t A(t,T))e^{-r_t B(t,T)} - r_t(\partial_t B(t,T))A(t,T)e^{-r_t B(t,T)} - (\theta_t - ar_t)A(t,T)B(t,T)e^{-r_t B(t,T)} + \frac{\sigma^2}{2}A(t,T)B^2(t,T)e^{-r_t B(t,T)} \\
r_t A(t,T) & = & \partial_t A(t,T) - r_t(\partial_t B(t,T))A(t,T) - (\theta_t - ar_t)A(t,T)B(t,T) + \frac{\sigma^2}{2}A(t,T)B^2(t,T)
\end{array}
$$

Move all instantaneous rate r_t terms to *LHS :*

$$
\begin{array}{lcl}
r_t A(t,T) + r_t A(t,T)\partial_t B(t,T) - ar_t A(t,T)B(t,T) & = & \partial_t A(t,T) - \theta_t A(t,T)B(t,T) + \frac{\sigma^2}{2}A(t,T)B^2(t,T) \\
r_t A(t,T)(1 + \partial_t B(t,T) - aB(t,T)) & = & \partial_t A(t,T) - \theta_t A(t,T)B(t,T) + \frac{\sigma^2}{2}A(t,T)B^2(t,T)
\end{array}
$$

Since equality holds for all short rate r_t, we can breakdown the *ODE* into two linear *ODE* :

$$
\begin{array}{llcll}
eq(5) & \partial_t B(t,T) - aB(t,T) + 1 & = & 0 \ \ and \ \boxed{B(T,T) = 0} & \textit{linear ODE solved by integrating factor} \\
eq(6) & \partial_t A(t,T) + \left(-\theta_t B(t,T) + \frac{\sigma^2}{2}B^2(t,T)\right)A(t,T) & = & 0 \ \ and \ \boxed{A(T,T) = 1} & \textit{linear ODE solved by separation of variables}
\end{array}
$$

Why boundary condition can be broken down in this way?

$$
\begin{array}{llcl}
 & P_T(T) & = & 1 \\
\Rightarrow & A(T,T)e^{-r_T B(T,T)} & = & 1e^{-r_T 0} \\
\Rightarrow & \begin{bmatrix} A(T,T) \\ B(T,T) \end{bmatrix} & = & \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{array}
$$

step 4 : solve first order ODE by integrating factor

The following ***first order linear ODE*** can be solved by scaling both sides with an integrating factor $e^{-\int q(x)dx}$.

$$
\begin{array}{rcll}
f'(x) & = & p(x)+q(x)f(x) & \\
(\frac{df(x)}{dx}-q(x)f(x))e^{-\int q(x)dx} & = & p(x)e^{-\int q(x)dx} & \textit{no f in RHS} \\
\frac{d}{dx}(f(x)e^{-\int q(x)dx}) & = & p(x)e^{-\int q(x)dx} & \textit{no f in both sides, only } f' \textit{ exists} \\
f(x)e^{-\int q(x)dx} & = & \int (p(x)e^{-\int q(x)dx})dx & \\
f(x) & = & e^{\int q(x)dx}\int (p(x)e^{-\int q(x)dx})dx & \\
f(x) & = & Q(x)\int \frac{p(x)}{Q(x)}dx & \textit{where } Q(x)=e^{\int q(x)dx}
\end{array}
$$

Now solve *eq(5)* with integrating factor technique.

$$
\begin{array}{rclllll}
\frac{dB(t,T)}{dt}-aB(t,T) & = & -1 & & \textit{such that } B(T,T)=0 & \textit{recap eq(5)} \\
(\frac{dB(t,T)}{dt}-aB(t,T))e^{-at} & = & -e^{-at} & & & \\
\frac{d}{dt}(B(t,T)e^{-at}) & = & -e^{-at} & & & \\
B(t,T)e^{-at} & = & -\int e^{-at}dt & = & \frac{1}{a}e^{-at}+C & \\
B(t,T) & = & \frac{1}{a}+Ce^{at} & & & \\
C & = & [B(t,T)-\frac{1}{a}]e^{-at} & = & [\underbrace{B(T,T)}_{0}-\frac{1}{a}]e^{-aT} & = \; -\frac{1}{a}e^{-aT} \\
B(t,T) & = & \frac{1}{a}-\frac{1}{a}e^{-aT}e^{at} & = & \frac{1}{a}(1-e^{-a(T-t)}) & \textit{exactly the same as Vasicek}
\end{array}
$$

step 5 : solving first order ODE by separation of variables

The following ***first order linear ODE*** can be solved by separation of variables.

$$
\begin{array}{rcll}
f'(x) & = & q(x)f(x) & \textit{such that } f(x_{bc})=y_{bc} \\
\frac{dy}{dx} & = & q(x)y & \textit{denote } y=f(x) \\
y^{-1}dy & = & q(x)dx & \\
\int y^{-1}dy & = & \int q(x)dx & \\
\ln y-\ln y_{bc} & = & \int_{x_{bc}}^{x} q(z)dz & \textit{why? see this remark} \\
y & = & y_{bc}\exp(\int_{x_{bc}}^{x} q(z)dz) & \textit{eq(7)}
\end{array}
$$

Remark

denote $Q(x)=\int q(x)dx$

$$
\begin{array}{rcl}
\ln y & = & Q(x)+C \\
\ln y_{bc} & = & Q(x_{bc})+C \\
\ln y & = & Q(x)+(\ln y_{bc}-Q(x_{bc}))
\end{array}
$$

Now solve *eq(6)* with separation of variables technique.

$$\frac{dA(t,T)}{dt}+\overbrace{(-\theta_t B(t,T)+\frac{\sigma^2}{2}B^2(t,T))}^{-q(t)}A(t,T) = 0 \qquad \text{such that } A(T,T)=1 \qquad \textit{recap eq(6)}$$

$$\frac{dA(t,T)}{dt} = q(t)A(t,T)$$

$$A(t,T) = A(T,T)\exp\left(-\int_t^T q(s)ds\right) \qquad \textit{using eq(7)}$$

$$= \exp\left(-\int_t^T \theta_s B(s,T)ds+\frac{\sigma^2}{2}\int_t^T B^2(s,T)ds\right) \qquad \textit{using } A(T,T)=1$$

$$= \exp\left(-\int_t^T \theta_s B(s,T)ds-\frac{\sigma^2}{2}\left(\frac{1}{a^2}(B(t,T)-(T-t))+\frac{1}{2a}B^2(t,T)\right)\right) \qquad \textit{using eq(3")}$$

$$= \exp\left(-\int_t^T \theta_s B(s,T)ds-\frac{\sigma^2}{2a^2}(B(t,T)-(T-t))-\frac{\sigma^2}{4a}B^2(t,T)\right)$$

B(t,T) related formulae

We recap *eq(3)* and *eq(3')* and derive *eq(3")* below, as they are useful in simplifying the messy *A(t,T)* expression.

$$\frac{1}{a}(1-e^{-a(T-t)}) = B(t,T) \qquad \textit{eq(3)}$$

$$\frac{1}{a}(1-e^{-2a(T-t)}) = 2B(t,T)-aB^2(t,T) \qquad \textit{eq(3')}$$

$$\int_t^T B^2(s,T)ds = \frac{1}{a^2}\int_t^T (1-e^{-a(T-s)})^2 ds$$

$$= \frac{1}{a^2}\left[\int_t^T ds-2\int_t^T e^{-a(T-s)}ds+\int_t^T e^{-2a(T-s)}ds\right]$$

$$= \frac{1}{a^2}\left[(T-t)-\frac{2}{a}[e^{-a(T-s)}]_t^T+\frac{1}{2a}[e^{-2a(T-s)}]_t^T\right]$$

$$= \frac{1}{a^2}\left[(T-t)-\frac{2}{a}[1-e^{-a(T-t)}]+\underbrace{\frac{1}{2a}[1-e^{-2a(T-t)}]}_{apply-eq(3')}\right]$$

$$= \frac{1}{a^2}\left[(T-t)-2B(t,T)+\frac{1}{2}(2B(t,T)-aB^2(t,T))\right]$$

$$= -\frac{1}{a^2}(B(t,T)-(T-t))-\frac{1}{2a}B^2(t,T) \qquad \textit{eq(3")}$$

Various derivatives of *B(t,T)* for the sake of ***part4***.

$$\partial_t B(t,T) = -e^{-a(T-t)} = aB(t,T)-1 \Rightarrow aB(t,T)-\partial_t B(t,T)=1 \qquad \textit{eq(3a)}$$

$$\partial_{tt} B(t,T) = a\partial_t B(t,T) = a(aB(t,T)-1) \Rightarrow a\partial_t B(t,T)-\partial_{tt} B(t,T)=0 \qquad \textit{eq(3b)}$$

$$\partial_T B(t,T) = e^{-a(T-t)} = 1-aB(t,T) \Rightarrow aB(t,T)+\partial_T B(t,T)=1 \qquad \textit{eq(3c)}$$

$$\partial_{TT} B(t,T) = -a\partial_T B(t,T) = a(aB(t,T)-1) \Rightarrow a\partial_T B(t,T)+\partial_{TT'} B(t,T)=0 \qquad \textit{eq(3d)}$$

4. Perfect match with yield curve

Calibrate Hull White with term structure

In this section, we are going to prove that Hull White parameters can be decoupled into two groups :

- θ_t for perfect match with yield curve
- a and σ for matching volatility matrix

The proof makes use of forward short rate $r_t(s)$. We recap its definition in *eq(8)* of **chap4.doc**.

$$P_t(T) = A(t,T)e^{-r_t B(t,T)}$$

$$\text{and} \quad P_t(T) = e^{-\int_t^T r_t(s)ds} \qquad \textit{eq(8) of chap4.doc}$$

step1

We have not derived relation between r_t and $r_t(s)$ in **chap4.doc**, we are going to do it now. Comparing 2 equations above :

$$\begin{aligned}
e^{-\int_t^T r_t(s)ds} &= A(t,T)e^{-r_t B(t,T)} \\
\int_t^T r_t(s)ds &= -\ln A(t,T) + r_t B(t,T) \\
&= -\ln\exp\left(-\int_t^T \theta_s B(s,T)ds - \frac{\sigma^2}{2a^2}(B(t,T)-(T-t)) - \frac{\sigma^2}{4a}B^2(t,T)\right) + r_t B(t,T) \\
\Rightarrow \quad \int_t^T r_t(s)ds &= \int_t^T \theta_s B(s,T)ds + \frac{\sigma^2}{2a^2}(B(t,T)-(T-t)) + \frac{\sigma^2}{4a}B^2(t,T) + r_t B(t,T) \qquad eq(8)
\end{aligned}$$

step2

We differentiate both *LHS* and *RHS* of *eq(8) wrt* to maturity T using Leibniz rule. We have *eq(9)*.

- $\partial_T \int_t^T r_t(s)ds = r_t(T)$
- $\partial_T \int_t^T \theta_s B(s,T)ds = \int_t^T \partial_T(\theta_s B(s,T))ds + \theta_T \underbrace{B(T,T)}_{0}\underbrace{\partial_T T}_{1} - \theta_t B(t,T)\underbrace{\partial_T t}_{0}$

  $= \int_t^T \theta_s \partial_T B(s,T)ds$
- $\partial_T(B(t,T)-(T-t)) = \partial_T B(t,T) - 1$

  $= 1 - aB(t,T) - 1$ $\qquad$ *using eq(3c)*

  $= -aB(t,T)$
- $\partial_T\left[\frac{\sigma^2}{4a}B^2 + r_t B\right] = \frac{\sigma^2}{2a}B(t,T)\partial_T B(t,T) + r_t \partial_T B(t,T)$

$$\Rightarrow \quad r_t(T) = \int_t^T \theta_s \partial_T B(s,T)ds - \frac{\sigma^2}{2a}B(t,T) + \frac{\sigma^2}{2a}B(t,T)\partial_T B(t,T) + r_t \partial_T B(t,T) \qquad eq(9)$$

step3

We differentiate ***again*** both *LHS* and *RHS* of *eq(9) wrt* maturity T using Leibniz rule. We have *eq(10)*.

- $\partial_T \int_t^T \theta_s \partial_T B(s,T)ds = \int_t^T \partial_T(\theta_s \partial_T B(s,T))ds + \theta_T \underbrace{\partial_T B(T,T)}_{1-aB(T,T)=1}\underbrace{\partial_T T}_{1} - \theta_t \partial_T B(t,T)\underbrace{\partial_T t}_{0}$

  $= \int_t^T \theta_s \partial_{TT} B(s,T)ds + \theta_T$

$$\Rightarrow \quad \partial_T r_t(T) = \int_t^T \theta_s \partial_{TT} B(s,T)ds + \theta_T - \frac{\sigma^2}{2a}\partial_T B(t,T) + \frac{\sigma^2}{2a}[(\partial_T B(t,T))^2 + B(t,T)\partial_{TT} B(t,T)] + r_t \partial_{TT} B(t,T) \qquad eq(10)$$

step4

Since derivative of B gives a linear function of B, see *eq(3a-d)* just like exponential function, so we repeatedly differentiate *eq(8)* so as to obtain *eq(9,10)*, whose sum does help to remove many terms by applying *eq(3c,d)* :

$$
\begin{aligned}
ar_t(T)+\partial_T r_t(T) &= \left[\begin{array}{l} \int_t^T \theta_s\,\cancel{(a\partial_T B(s,T))}ds-\frac{\sigma^2}{2a}aB(t,T)+\frac{\sigma^2}{2a}B(t,T)\cancel{(a\partial_T B(t,T))}+r_t\,\cancel{(a\partial_T B(t,T))}+ \\ \int_t^T \theta_s\cancel{\partial_{TT}B(s,T)}ds+\theta_T-\frac{\sigma^2}{2a}\partial_T B(t,T)+\frac{\sigma^2}{2a}[(\partial_T B(t,T))^2+B(t,T)\cancel{\partial_{TT}B(t,T)}]+r_t\cancel{\partial_{TT}B(t,T)} \end{array}\right. \\
&= \theta_T-\frac{\sigma^2}{2a}\underbrace{(aB(t,T)+\partial_T B(t,T))}_{1}+\frac{\sigma^2}{2a}(\underbrace{\partial_T B(t,T)}_{e^{-a(T-t)}})^2 \\
&= \theta_T-\frac{\sigma^2}{2a}(1-e^{-2a(T-t)}) \\
\theta_t &= ar_t(T)+\partial_T r_t(T)+\frac{\sigma^2}{2a}(1-e^{-2at}) \qquad \textit{putting } t=0 \textit{ and } T=t \qquad eq(11)
\end{aligned}
$$

We write θ_t in terms of *forward rate term structure* and the other two *Hull White* parameters a and σ.

- as opposed to Vasciek, *Hull White* has enough freedom to fit market yield curve perfectly
- in Quantlib, we construct term structure using rate-helpers and then …
- *Hull White* model can ask for *forward short rate* from term structure to calculate θ_t
- *Hull White* model parameters a and σ are then calibrated to fit volatility matrix

Recall Leibniz rule of differentiation

General and specialized versions

$$
\begin{aligned}
\partial_T \int_{a(T)}^{b(T)} f(s,T)ds &= \int_{a(T)}^{b(T)}\partial_T f(s,T)ds+f(b(T),T)\partial_T b(T)-f(a(T),T)\partial_T a(T) \\
\partial_T \int_{a(T)}^{b(T)} f(s)ds &= f(b(T))\partial_T b(T)-f(a(T))\partial_T a(T) \\
\partial_T \int_c^T f(s)ds &= f(T)
\end{aligned}
$$

5. Hull White tree

Construction of ***Hull White tree*** *(from market data)* is done in two steps :

(1) construction of a tree for *symmetric process* x_t

(2) apply time dependent *shift process* y_t on the tree

- *step 1* finds branching probabilities which are uniquely determined by *Hull White* parameters (a,σ)
- *step 2* finds shifts by pricing bonds with the tree and compared to *market quoted yield curve* without knowing θ_t
- *Hull White* parameter θ_t is not needed in both steps

Step 1 : Construction of symmetric process tree

Markov process x_t (***NOT*** the short rate process) is defined as a process that fluctuates around zero symmetrically :

$$\begin{aligned} dx_t &= -ax_t dt + \sigma dz_t \\ &= -ax_t dt + \sigma\varepsilon\sqrt{dt} \\ \textit{and} \quad x_0 &= 0 \qquad \textit{boundary condition} \end{aligned}$$

Discretization in time domain (with constant time resolution Δt) :

$$\begin{aligned} t &= n\Delta t && \textit{where } n \textit{ is time index} \\ n &\in [0,N] && \textit{where } N = T/\Delta t \\ \textit{hence} \quad x_{(n+1)\Delta t} &= x_{n\Delta t} - ax_{n\Delta t}\Delta t + \sigma\varepsilon\sqrt{\Delta t} && eq(12) \end{aligned}$$

Discretization in price domain (constant price resolution Δx) :

$$\begin{aligned} x_{n\Delta t} &= m\Delta x \quad \textit{denoted as state } (n,m) && \textit{where } m \textit{ is price index} \\ m &\in [-\min(n,M), +\min(n,M)] && \textit{where } M \textit{ is price limit} \\ \textit{hence} \quad x_{(n+1)\Delta t} &= x_{n\Delta t} + \varepsilon_m \Delta x && \textit{where } \varepsilon_m \textit{ is RNG} \\ &= m\Delta x + \varepsilon_m \Delta x \\ &= (m+\varepsilon_m)\Delta x && eq(13) \end{aligned}$$

Given current state *(n,m)*, transition probability matrix is :

$$\begin{bmatrix} \varepsilon_m = +1 & with & p_{up}(m) \\ \varepsilon_m = +0 & with & p_{mid}(m) \\ \varepsilon_m = -1 & with & p_{down}(m) \end{bmatrix} \qquad for \;\; m \in (-\min(n,M), +\min(n,M))$$

$$\begin{bmatrix} \varepsilon_m = +0 & with & p_{up}(m) \\ \varepsilon_m = -1 & with & p_{mid}(m) \\ \varepsilon_m = -2 & with & p_{down}(m) \end{bmatrix} \qquad for \;\; m = +\min(n,M) \;\; \textit{exhibiting mean reversion}$$

$$\begin{bmatrix} \varepsilon_m = +2 & with & p_{up}(m) \\ \varepsilon_m = +1 & with & p_{mid}(m) \\ \varepsilon_m = +0 & with & p_{down}(m) \end{bmatrix} \qquad for \;\; m = -\min(n,M) \;\; \textit{exhibiting mean reversion}$$

- each edge is a feasible state transition
- each node in timestep n connects to 3 nodes in timestep $n+1$, with probabilities $p_u(m)$, $p_m(m)$ and $p_d(m)$
- topology for nodes in the middle is different from those at both ends in price domain
- with the constraint that sum of transition probabilites is *one*, degree of freedom is *two* only
- probabilities are found by matching *1st* and *2nd* moments of *eq(12)* and *eq(13)*
- probabilities are *price dependent* but *time independent*, thus they bear index m, as there is no t index in moment matching :

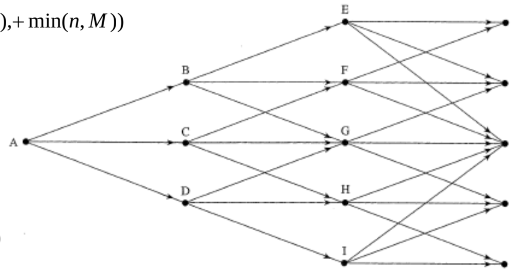
$$\left.\begin{aligned} & \textit{moments of eq(13)} && && \textit{moments of eq(12)} \\ eq(14a) \quad & p_{up}(m) + p_{mid}(m) + p_{down}(m) && = && 1 \\ eq(14b) \quad & p_{up}(m)(+\Delta x) + p_{mid}(m)(0) + p_{down}(m)(-\Delta x) && = && -ax_{n\Delta t}\Delta t \\ eq(14c) \quad & \underbrace{p_{up}(m)(+\Delta x)^2 + p_{mid}(m)(0)^2 + p_{down}(m)(-\Delta x)^2}_{E[(\varepsilon_m \Delta x)^2]} && = && \underbrace{(-a(m\Delta x)\Delta t)^2 + \sigma^2\Delta t}_{E^2[\varepsilon_m \Delta x] + V[\varepsilon_m \Delta x]} \end{aligned}\right\} \textit{only m index, no t index, thus probabilities are price dependent but time independent}$$

By solving the moment matching, we have answers for all probabilities.

$$
\begin{aligned}
p_{up}(m) &= \frac{1}{2}(am\Delta t)^2 - \frac{1}{2}(am\Delta t) + \frac{1}{6} \\
p_{mid}(m) &= -(am\Delta t)^2 + \frac{2}{3} && \text{for } m \in (-\min(n,M), +\min(n,M)) \\
p_{down}(m) &= \frac{1}{2}(am\Delta t)^2 + \frac{1}{2}(am\Delta t) + \frac{1}{6}
\end{aligned}
$$

$$
\begin{aligned}
p_{up}(m) &= \frac{1}{2}(am\Delta t)^2 - \frac{3}{2}(am\Delta t) + \frac{7}{6} \\
p_{mid}(m) &= -(am\Delta t)^2 + 2(am\Delta t) - \frac{1}{3} && \text{for } m = +\min(n,M) \\
p_{down}(m) &= \frac{1}{2}(am\Delta t)^2 - \frac{1}{2}(am\Delta t) + \frac{1}{6}
\end{aligned}
$$

$$
\begin{aligned}
p_{up}(m) &= \frac{1}{2}(am\Delta t)^2 + \frac{1}{2}(am\Delta t) + \frac{1}{6} \\
p_{mid}(m) &= -(am\Delta t)^2 - 2(am\Delta t) - \frac{1}{3} && \text{for } m = -\min(n,M) \\
p_{down}(m) &= \frac{1}{2}(am\Delta t)^2 + \frac{3}{2}(am\Delta t) + \frac{7}{6}
\end{aligned}
$$



Let's verify the answer for case $m \in (-\min(n,M), +\min(n,M))$.

$$
\begin{aligned}
& p_{up}(m) + p_{mid}(m) + p_{down}(m) \\
= \; & \frac{1}{2}(am\Delta t)^2 - \frac{1}{2}(am\Delta t) + \frac{1}{6} - (am\Delta t)^2 + \frac{2}{3} + \frac{1}{2}(am\Delta t)^2 + \frac{1}{2}(am\Delta t) + \frac{1}{6} \\
= \; & 1 && \Rightarrow \textit{verified eq(14a)}
\end{aligned}
$$

$$
\begin{aligned}
& p_{up}(m)(+\Delta x) + p_{mid}(m)(0) + p_{down}(m)(-\Delta x) \\
= \; & \left(\frac{1}{2}(am\Delta t)^2 - \frac{1}{2}(am\Delta t) + \frac{1}{6}\right)(+\Delta x) + \left(\frac{1}{2}(am\Delta t)^2 + \frac{1}{2}(am\Delta t) + \frac{1}{6}\right)(-\Delta x) \\
= \; & -am\Delta x\Delta t && \Rightarrow \textit{verified eq(14b)}
\end{aligned}
$$

$$
\begin{aligned}
& p_{up}(m)(+\Delta x)^2 + p_{mid}(m)(0)^2 + p_{down}(m)(-\Delta x)^2 \\
= \; & \left(\frac{1}{2}(am\Delta t)^2 - \frac{1}{2}(am\Delta t) + \frac{1}{6}\right)(+\Delta x)^2 + \left(\frac{1}{2}(am\Delta t)^2 + \frac{1}{2}(am\Delta t) + \frac{1}{6}\right)(-\Delta x)^2 \\
= \; & ((am\Delta t)^2 + \frac{1}{3})(\Delta x)^2 \\
= \; & (am\Delta x\Delta t)^2 + \frac{1}{3}(\Delta x)^2 && \textit{by putting we set } (\Delta x)^2 = 3\sigma^2\Delta t \\
= \; & (am\Delta x\Delta t)^2 + \sigma^2\Delta t && \Rightarrow \textit{verified eq(14c)}
\end{aligned}
$$

Therefore, with the second order moment matching, we effectively determine the resolution of price domain :

$$
\Delta x = \sigma\sqrt{3\Delta t}
$$

Besides, this price resolution gives *kurtosis* $3(\sigma^2\Delta t)^2$ *(please prove it yourself)*, which is consistent with the *kurtosis* of Gaussian.

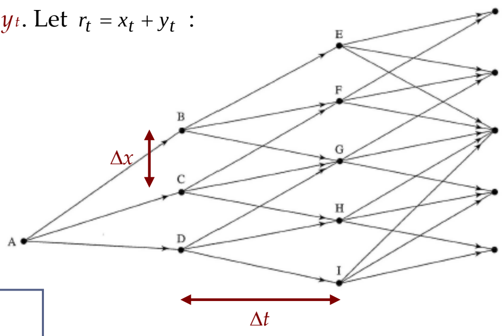
Step 2 : Deformation of tree

Deformation is done by adding *time dependent* but *price independent* shift to each node in the original tree. Shift process is :

$$dy_t = (\theta_t - a y_t)dt$$

Hull White short rate process is the sum of *symmetric process x_t* and *shift process y_t*. Let $r_t = x_t + y_t$:

$$\begin{aligned} dr_t &= d(x_t + y_t) \\ &= dx_t + dy_t \\ &= -a x_t dt + \sigma dz_t + (\theta_t dt - a y_t)dt \\ &= (\theta_t - a(x_t + y_t))dt + \sigma dz_t \\ &= (\theta_t - a r_t)dt + \sigma dz_t \end{aligned}$$



There are *two* steps in deforming the *Hull White* tree :
(1) define Q and derive a recursive formula for Q, update it by *forward induction*
(2) match with market quoted yield curve
i.e. calibrating the tree (*or adjusting y_t*) so that it describes the market data

step1 : update Q value recursively

$$Q_{n,m} \stackrel{def}{=} \textit{PV at time0 of a security that pays \$1 on reaching state (n,m) and nothing otherwise, by FTAP} \ldots$$

$$= E\left[1(r_{n\Delta t} = m\Delta x + y_{n\Delta t}) \times \exp(-\int_0^{n\Delta t} r_t dt) \mid I_0\right] \qquad \textit{expectation of a random path (not a random variable)}$$

$$= E\left[1(x_{n\Delta t} = m\Delta x) \times \exp(-\int_0^{n\Delta t} r_t dt) \mid I_0\right]$$

eq(15)→ $$= E[1(x_{n\Delta t} = m\Delta x) \times \exp(-\textstyle\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t) \mid I_0] \qquad \textit{short rate } r_{(n-1)\Delta t} \textit{ accrues in } [(n\text{-}1)\Delta t, n\Delta t]$$

$$= E[E[1(x_{n\Delta t} = m\Delta x) \times \exp(-\textstyle\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t) \mid I_{(n-1)\Delta t}] \mid I_0]$$

$$= E[E[1(x_{n\Delta t} = m\Delta x) \mid I_{(n-1)\Delta t}] \times \exp(-\textstyle\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t) \mid I_0]$$

deterministic as of time (n-1)Δt

$$= E\left[E\left[\begin{pmatrix} +1(x_{(n-1)\Delta t} = m\Delta x) & \times 1(\varepsilon_m = 0) \\ +1(x_{(n-1)\Delta t} = (m-1)\Delta x) & \times 1(\varepsilon_{m-1} = +1) \\ +1(x_{(n-1)\Delta t} = (m+1)\Delta x) & \times 1(\varepsilon_{m+1} = -1) \end{pmatrix} \mid I_{(n-1)\Delta t}\right] \times \exp(-\textstyle\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t) \mid I_0\right] \qquad \textit{by eq(16)}$$

$$= E\left[\begin{pmatrix} +1(x_{(n-1)\Delta t} = m\Delta x) & \times E[1(\varepsilon_m = 0) \mid I_{(n-1)\Delta t}] \\ +1(x_{(n-1)\Delta t} = (m-1)\Delta x) & \times E[1(\varepsilon_{m-1} = +1) \mid I_{(n-1)\Delta t}] \\ +1(x_{(n-1)\Delta t} = (m+1)\Delta x) & \times E[1(\varepsilon_{m+1} = -1) \mid I_{(n-1)\Delta t}] \end{pmatrix} \times \exp(-\textstyle\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t) \mid I_0\right]$$

$$\begin{aligned} &E[1(\varepsilon_m = 0) \mid I_{(n-1)\Delta t}] \\ &= \textstyle\sum_{\varepsilon_m} 1(\varepsilon_m = 0)\, p(\varepsilon_m) \\ &= p(\varepsilon_m = 0) \\ &= p_{mid}(m) \end{aligned}$$

$$= E\left[\begin{pmatrix} +1(x_{(n-1)\Delta t} = m\Delta x) & \times p_{mid}(m) \\ +1(x_{(n-1)\Delta t} = (m-1)\Delta x) & \times p_{up}(m-1) \\ +1(x_{(n-1)\Delta t} = (m+1)\Delta x) & \times p_{down}(m+1) \end{pmatrix} \times \exp(-\textstyle\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t) \mid I_0\right]$$

$$= \begin{bmatrix} +E[1(x_{(n-1)\Delta t} = m\Delta x) & \times \exp(-\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t) \mid I_0] & \times p_{mid}(m) \\ +E[1(x_{(n-1)\Delta t} = (m-1)\Delta x) & \times \exp(-\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t) \mid I_0] & \times p_{up}(m-1) \\ +E[1(x_{(n-1)\Delta t} = (m+1)\Delta x) & \times \exp(-\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t) \mid I_0] & \times p_{down}(m+1) \end{bmatrix}$$

$$= \begin{bmatrix} +E[1(x_{(n-1)\Delta t} = m\Delta x) & \times \exp(-\sum_{k=0}^{n-2} r_{k\Delta t}\Delta t) \mid I_0] & \times \exp(-(m\Delta x + y_{(n-1)\Delta t})t) & \times p_{mid}(m) \\ +E[1(x_{(n-1)\Delta t} = (m-1)\Delta x) & \times \exp(-\sum_{k=0}^{n-2} r_{k\Delta t}\Delta t) \mid I_0] & \times \exp(-((m-1)\Delta x + y_{(n-1)\Delta t})\Delta t) & \times p_{up}(m-1) \\ +E[1(x_{(n-1)\Delta t} = (m+1)\Delta x) & \times \exp(-\sum_{k=0}^{n-2} r_{k\Delta t}\Delta t) \mid I_0] & \times \exp(-((m+1)\Delta x + y_{(n-1)\Delta t})\Delta t) & \times p_{down}(m+1) \end{bmatrix} \qquad \textit{by eq(17)}$$

`fctA→` $$= \begin{bmatrix} +Q_{n-1,m} & \times p_m(m) & \times \exp(-(m\Delta x + y_{(n-1)\Delta t})\Delta t) \\ +Q_{n-1,m-1} & \times p_u(m-1) & \times \exp(-((m-1)\Delta x + y_{(n-1)\Delta t})\Delta t) \\ +Q_{n-1,m+1} & \times p_d(m+1) & \times \exp(-((m+1)\Delta x + y_{(n-1)\Delta t})\Delta t) \end{bmatrix} \qquad \textit{very intuitive}$$

where $Q_{0,0} = 1$ *boundary condition*

Useful formulae involving delta function
In order to derive recursive formula for Q, we need a recursive formula for its payoff :

$$1(x_{n\Delta t}=m\Delta x) \quad \equiv \quad \left[\begin{array}{l} 1(x_{(n-1)\Delta t}=m\Delta x)\times 1(\varepsilon_m=0)+ \\ 1(x_{(n-1)\Delta t}=(m-1)\Delta x)\times 1(\varepsilon_{m-1}=+1)+ \\ 1(x_{(n-1)\Delta t}=(m+1)\Delta x)\times 1(\varepsilon_{m+1}=-1) \end{array}\right. \qquad eq(16)$$

Suppose x is a random variable with x_n *as possible realization.*

$$\begin{aligned} E[1(x=x_0)f(x)] &= \int 1(x=x_0)f(x)p(x)dx \\ &= f(x_0)p(x_0) \\ &= f(x_0)\int 1(x=x_0)p(x)dx \\ &= f(x_0)E[1(x=x_0)] \qquad eq(17) \end{aligned}$$

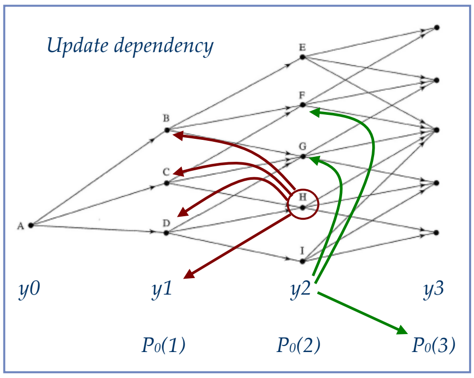
Conversion from random x (a tree layer at a specific time) to realized value x_n *(a specific layer node)*

$$f(x) = \sum_n f(x_n)1(x=x_n) \qquad eq(18)$$

step2 : calibration to market yield curve
Consider bond price *(not Q contract)* with maturity *(n+1)Δt* :

$$\begin{aligned} P_0((n+1)\Delta t) &= E[\exp(-\int_0^{(n+1)\Delta t} r_t dt)\,|\,I_0] \\ &= E[\exp(-\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t)\exp(-r_{n\Delta t}\Delta t)\,|\,I_0] && \textit{note } r_{n\Delta t} \textit{ is random w} \\ &= E[\exp(-\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t)[\sum_m \exp(-(m\Delta x+y_{n\Delta t})\Delta t)1(r_{n\Delta t}=m\Delta x+y_{n\Delta t})]\,|\,I_0] && \textit{by eq(18)} \\ &= \sum_m \left(\underbrace{E[1(r_{n\Delta t}=m\Delta x+y_{n\Delta t})\times \exp(-\sum_{k=0}^{n-1} r_{k\Delta t}\Delta t)\,|\,I_0]}_{Q_{n,m}} \times \exp(-(m\Delta x+y_{n\Delta t})\Delta t) \right) && \textit{rearrange terms, m is deterministic} \\ &= \sum_m Q_{n,m} e^{-(m\Delta x+y_{n\Delta t})\Delta t} \\ &= (\sum_m Q_{n,m} e^{-m\Delta x\Delta t})\times e^{-y_{n\Delta t}\Delta t} \end{aligned}$$

$$\boxed{\texttt{fctB}\rightarrow \quad y_{n\Delta t} = -\frac{1}{\Delta t}\ln\left(\frac{P_0((n+1)\Delta t)}{\sum_m Q_{n,m}e^{-m\Delta x\Delta t}}\right)}$$



Here is the algorithm, forward induction of *Q* and *y* should be done simultaneously.

```
Q[0,0] = 1;
y[0] = yieldcurve.zero_rate(dt); // y[0] = short rate at t=0
for (n=1:N)
{
    // step1 - Q[n,:] depends on Q[n-1,:] and y[n-1]
    for(m=-n:+n) Q[n,m] = fctA(Q[n-1,:], y[n-1]);
    // step2 - y[n] depends on Q[n,:] and bond_price[n+1]
    y[n] = fctB(Q[n,:], bond_price[n+1]);
}
```

Pricing with the tree
Pricing of *IRD* is done by marking possible coupons on appropriate nodes of the tree and discount :

$$\boxed{\texttt{fctC}\rightarrow \quad f(n,m) = \sum_{n,m} coupon_{n,m}Q_{n,m}}$$

$$or \qquad f(n,m) = coupon_{n,m} + (p_{up}(m)f(n+1,m+1)+p_{mid}(m)f(n+1,m)+p_{down}(m)f(n+1,m-1))\times e^{-(m\Delta x+y_{n\Delta t})\Delta t}$$

`fctA`/`fctB`/`fctC` are quite intuitive, we can jump to the answer directly, like a fastlane.

6. Black Derman Toy (BDT) tree

BDT tree is a binary recombining tree modelling short rate :

- up-prob equals to down-prob (which is ½)
- up-scale and down-scale are *time dependent* but *price independent*
- root vertex in layer *0* representing current time *0*
- *n+1* vertices in layer *n* representing forward time $n\Delta t$
- vertex *(N, :)* in the last layer do not bear any short rate
- vertex *(n,m)* denotes short rate $r_{n,m}$ within $[n\Delta t,(n+1)\Delta t]$

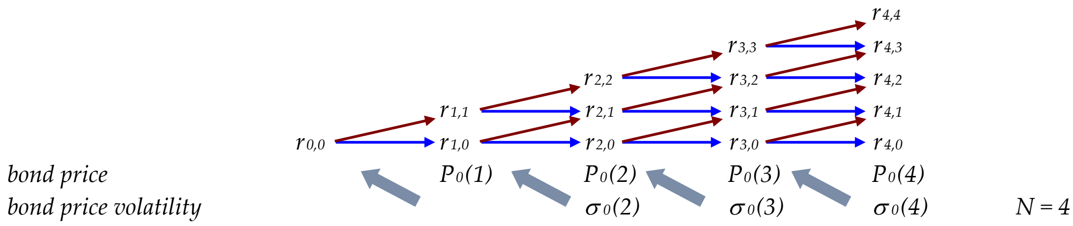
$$r_{n,m} = \alpha_n \beta_n^m \qquad \forall n \in [0, N-1] \text{ and } \forall m \in [0, n]$$

$$u_n = \frac{r_{n+1,m+1}}{r_{n,m}} = \frac{\alpha_{n+1}\beta_{n+1}^{m+1}}{\alpha_n \beta_n^m}$$

time dependent but *price independent*

$$d_n = \frac{r_{n+1,m}}{r_{n,m}} = \frac{\alpha_{n+1}\beta_{n+1}^{m}}{\alpha_n \beta_n^m}$$

time dependent but *price independent*



Calibration

There are *2N-1* parameters (α_0) $(\alpha_1\beta_1)$ $(\alpha_2\beta_2)$ …$(\alpha_{N-1}\beta_{N-1})$ calibration needs *2N-1* data :

- *N* data from bond price term structure : $P_0(1)$ $P_0(2)$ … $P_0(N)$
- *N-1* data from bond price volatility term structure : $\sigma_0(2)$ $\sigma_0(3)$ …$\sigma_0(N)$
- calibration is done by forward propagation, which repeatedly calls Newton Raphson
- the *nth* Newton Raphson solves two nonlinear equations for parameters α_{n-1} and β_{n-1}
- one equation is constructed from *bond price* with maturity $n\Delta t$ and
- one equation is constructed from *bond price volatility* with maturity $n\Delta t$
- for *n=1*, β_0 is redundant, *bond price volatility* with maturity $1\Delta t$ is not needed
- lets derive the system of nonlinear equation …

How to read bond price from a tree?

Fill the payoff with all one, then propagate the value using the similar (but simplier) equation as in *HW* tree :

$$f(n,m) = coupon_{n,m} + (p_{up}(m) f(n+1,m+1) + p_{mid}(m) f(n+1,m) + p_{down}(m) f(n+1,m-1)) \times e^{-(m\Delta x + y_{n\Delta t})\Delta t} \qquad \text{from HW}$$

$$P_{n,m} = \left(\frac{1}{2}P_{n+1,m+1} + \frac{1}{2}P_{n+1,m}\right) \times \exp(-\alpha_n \beta_n^m \Delta t) \qquad \text{for BDT}$$

How to read bond price volatility from a tree?

$$\begin{aligned}
dP_t(T) &= \mu_t(T)P_t(T)dt + \sigma_t(T)P_t(T)dz_t \\
d\ln P_t(T) &= (\mu_t(T) - \sigma_t^2(T)/2)dt + \sigma_t(T)dz_t \\
\sigma_t^2(T)\Delta t &= V[\Delta \ln P_t(T)] \\
&= V[\ln P_{t+\Delta t}(T) - \ln P_t(T)] \\
&= V\left[\ln \frac{P_{t+\Delta t}(T)}{P_t(T)}\right]
\end{aligned}$$

Now we have *reference day t = 0*, hence we have bond price volatility in terms of *current bond price* and *forward bond price* :

$$
\begin{aligned}
\sigma_0(T) &= \sqrt{\frac{1}{\Delta t} V\left[\ln\frac{P_{\Delta t}(T)}{P_0(T)}\right]} \\
\sigma_0(n\Delta t) &= \sqrt{\frac{1}{\Delta t} V\left[\ln\frac{P_{\Delta t}(n\Delta t)}{P_0(n\Delta t)}\right]} \\
&= \frac{1}{\sqrt{\Delta t}}\sqrt{E\left[\ln\frac{P_{\Delta t}(n\Delta t)}{P_0(n\Delta t)}\right]^2 - E^2\left[\ln\frac{P_{\Delta t}(n\Delta t)}{P_0(n\Delta t)}\right]} \\
&= \frac{1}{\sqrt{\Delta t}}\sqrt{\frac{1}{2}(\ln\frac{P_{1,1}}{P_0})^2 + \frac{1}{2}(\ln\frac{P_{1,0}}{P_0})^2 - \left(\frac{1}{2}\ln\frac{P_{1,1}}{P_0} + \frac{1}{2}\ln\frac{P_{1,0}}{P_0}\right)^2} \qquad \textit{substitute BDT tree nodes} \\
&= \frac{1}{\sqrt{\Delta t}}\sqrt{\frac{1}{2}(\ln\frac{P_{1,1}}{P_0})^2 + \frac{1}{2}(\ln\frac{P_{1,0}}{P_0})^2 - \frac{1}{4}(\ln\frac{P_{1,1}}{P_0})^2 - \frac{1}{4}(\ln\frac{P_{1,0}}{P_0})^2 - \frac{1}{2}(\ln\frac{P_{1,1}}{P_0})(\ln\frac{P_{1,0}}{P_0})} \\
&= \frac{1}{2\sqrt{\Delta t}}\sqrt{(\ln\frac{P_{1,1}}{P_0})^2 + (\ln\frac{P_{1,0}}{P_0})^2 - 2(\ln\frac{P_{1,1}}{P_0})(\ln\frac{P_{1,0}}{P_0})} \\
&= \frac{1}{2\sqrt{\Delta t}}\sqrt{(\ln\frac{P_{1,1}}{P_0} - \ln\frac{P_{1,0}}{P_0})^2} \\
&= \frac{1}{2\sqrt{\Delta t}}\ln\frac{P_{1,1}}{P_{1,0}}
\end{aligned}
$$

Algorithm

Suppose `bdt` is in the middle of forward propagation, the following are known :

```
bdt.a[0] ... bdt.a[n-2]
bdt.b[0] ... bdt.b[n-2]
```

Now given $P_0(T)$ and $\sigma_0(T)$ such that :

$$
\begin{aligned}
P_0(T) &= \textit{market bond price of maturity } n\Delta t \\
\sigma_0(T) &= \textit{market bond price volatility of maturity } n\Delta t
\end{aligned}
$$

we setup the system of two nonlinear equation solve for `bdt.a[n-1]` and `bdt.b[n-1]` :

$P_0(T)$ = `bdt.node(0)`

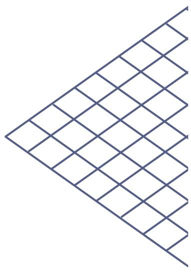
$\sigma_0(T)$ = `ln(bdt.node(1,1)/ bdt.node(1,0))/(2*sqrt(bdt.dt))`

Short summary

Comparison of common trees

1	binomial *CRR* tree	*two degree of freedom : u and p*	*constraining d = 1/u*	*multiplication topology*
2	binomial *1/2 prob* tree	*two degree of freedom : u and d*	*constraining p = 1/2*	*multiplication topology*
3	trinomial *HW* tree	*two degree of freedom :* p_u *and* p_d	*constraining* $p_m = 1 - p_u - p_d$	*addition topology*
4	*BDT* tree	*two degree of freedom :*		*multiplication topology*

multiplication topology



addition topology