Sinc Function

Definition
There are two definitions for sinc function. In mathematics, it is defined as the ratio between sin(x) and x, while in engineering (especially in signal processing), it is defined as the ratio between sin(πx) and πx. The definite integral of sinc function over real numbers equals to π for the mathematics definition, and 1 for the engineering definition.

$$\sin c(x) = \frac{\sin(x)}{x} \qquad \text{(mathematics definition)}$$

$$\sin c(x) = \frac{\sin(\pi x)}{\pi x} \qquad \text{(engineering definition)}$$

We will use the mathematics definition for the rest of this document. Lets derive the limit of sinc function.

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

$$\lim_{x\to \infty}\frac{\sin x}{x} = 0$$

Integration
Integration of sinc functon can be found in two ways : double integration and contour integration (complex analysis).

<u>By double integration</u>
First of all, lets consider another integral : $\int_0^\infty e^{-xt}\sin x dt$.

$$\begin{aligned}
\int_0^\infty e^{-xt}\sin x dt &= \sin x\int_0^\infty e^{-xt} dt \\
&= \frac{\sin x}{-x}[e^{-xt}]_0^\infty \\
&= \frac{\sin x}{x} && \text{(equation 1)}
\end{aligned}$$

Then lets consider the integral : $\int_0^\infty e^{-xt}\sin x dx$. Note : $\int_0^\infty e^{-xt}\sin x dt \neq \int_0^\infty e^{-xt}\sin x dx$.

$$\begin{aligned}
\int e^{-xt}\sin x dx &= \int -e^{-xt} d\cos x \\
&= (-e^{-xt}\cos x) - t\int e^{-xt}\cos x dx && \text{(integration by parts)} \\
&= (-e^{-xt}\cos x) - t\int e^{-xt} d\sin x \\
&= (-e^{-xt}\cos x) - t((e^{-xt}\sin x) + t\int e^{-xt}\sin x dx) && \text{(integration by parts)} \\
\int e^{-xt}\sin x dx \times (1+t^2) &= (-e^{-xt}\cos x) - t(e^{-xt}\sin x) \\
\int e^{-xt}\sin x dx &= \frac{(-e^{-xt}\cos x) - t(e^{-xt}\sin x)}{1+t^2} \\
\int_0^\infty e^{-xt}\sin x dx &= \left[\frac{(-e^{-xt}\cos x) - t(e^{-xt}\sin x)}{1+t^2}\right]_0^\infty \\
&= -\frac{(-e^{-0t}\cos 0) - t(e^{-0t}\sin 0)}{1+t^2} \\
&= -\frac{-1-t\times 0}{1+t^2} \\
&= \frac{1}{1+t^2} && \text{(equation 2)}
\end{aligned}$$

How can we connect these two integrals? By double integration and swapping the integration order.

$$\begin{aligned}
\int_0^\infty (\sin x/x) dx &= \int_0^\infty (\int_0^\infty e^{-xt}\sin x dt) dx && \text{(by equation 1)} \\
&= \int_0^\infty (\int_0^\infty e^{-xt}\sin x dx) dt && \text{(swapping order)} \\
&= \int_0^\infty 1/(1+t^2) dt && \text{(by equation 2)} \\
&= \int_0^\infty 1/(1+\tan^2 x) d\tan x && \text{(let } t = \tan x\text{)}
\end{aligned}$$

$$= \int_0^{\infty} \cos^2 x / (\cos^2 x + \sin^2 x) d\tan x$$

$$= \int_0^{\infty} \cos^2 x d\tan x$$

$$= \int_0^{\pi/2} \cos^2 x (1+\tan^2 x) dx \qquad \text{(see remark, note the integration range)}$$

$$= \int_0^{\pi/2} (\cos^2 x + \sin^2 x) dx$$

$$= \pi/2$$

$$\int_{-\infty}^{+\infty} (\sin x / x) dx = \int_0^{\infty} (\sin x / x) dx + \int_{-\infty}^{0} (\sin x / x) dx$$

$$= \int_0^{\infty} (\sin x / x) dx \times 2 \qquad \text{(since : } \sin x / x = \sin(-x)/(-x)\text{)}$$

$$= \pi$$

Remark :

$$\frac{d\tan x}{dx} = \frac{d}{dx}\frac{\sin x}{\cos x}$$

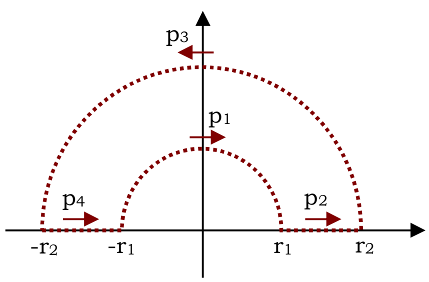
$$= \frac{\cos x}{\cos x} - \frac{\sin x}{\cos^2 x}(-\sin x)$$

$$= 1 + \tan^2 x$$

By contour integration

Please check whether the following is correct. Consider the path p = $p_1 + p_2 + p_3 + p_4$. We define :

z	=	$re^{j\theta}$	(for any complex number)
dz	=	$e^{j\theta}dr + jre^{j\theta}d\theta$	(for any contour)
dz	=	$e^{j\theta}dr$	(for contour on x axis, since there is no change in angle)
dz	=	$jre^{j\theta}d\theta$	(for contour on arc, since there is no change in magnitude)



Lets consider the following integral :

$$\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_p \frac{\exp(jz)}{z}dz$$

$$= \left(\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_{p_1} \frac{\exp(jz)}{z}dz\right) + \left(\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_{p_2} \frac{\exp(jz)}{z}dz\right) + \left(\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_{p_3} \frac{\exp(jz)}{z}dz\right) + \left(\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_{p_4} \frac{\exp(jz)}{z}dz\right)$$

$$= \left(\lim_{r_1\to 0}\int_{\pi}^{0} \frac{\exp(jr_1e^{j\theta})}{r_1e^{j\theta}} jr_1e^{j\theta}d\theta\right) + \left(\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_{r_1}^{r_2} \frac{\exp(jre^{j0})}{re^{j0}} e^{j0}dr\right) + \left(\lim_{r_2\to\infty}\int_{0}^{\pi} \frac{\exp(jr_2e^{j\theta})}{r_2e^{j\theta}} jr_2e^{j\theta}d\theta\right) + \left(\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_{r_2}^{r_1} \frac{\exp(jre^{j\pi})}{re^{j\pi}} e^{j\pi}dr\right)$$

$$= \left(\lim_{r_1\to 0}\int_{\pi}^{0} j\exp(jr_1e^{j\theta})d\theta\right) + \left(\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_{r_1}^{r_2} \frac{\exp(jr)}{r}dr\right) + \left(\lim_{r_2\to\infty}\int_{0}^{\pi} j\exp(jr_2e^{j\theta})d\theta\right) + \left(\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_{r_2}^{r_1} \frac{\exp(-jr)}{r}dr\right)$$

The 1st term :

$$\lim_{r_1\to 0}\int_{\pi}^{0} j\exp(jr_1e^{j\theta})d\theta = \int_{\pi}^{0} j\exp(0)d\theta = -j\pi$$

The 3rd term :

$$\lim_{r_2\to\infty}\int_{0}^{\pi} j\exp(jr_2e^{j\theta})d\theta = 0 \qquad \text{(Jordan's lemma, please check)}$$

The 2nd and 4th term :

$$\int_{0}^{\infty} \frac{\exp(jr)}{r}dr + \int_{\infty}^{0} \frac{\exp(-jr)}{r}dr = \int_{0}^{\infty} \frac{\exp(jr)}{r}dr + \int_{-\infty}^{0} \frac{\exp(jr')}{r'}dr' \qquad \text{(put } r' = -r\text{)}$$

$$= \int_{-\infty}^{\infty} \frac{\exp(jr)}{r}dr$$

Hence we have :

$$\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_p \frac{\exp(jz)}{z}dz = -j\pi + \int_{-\infty}^{\infty}\frac{\exp(jr)}{r}dr$$

but $$\lim_{r_1\to 0}\lim_{r_2\to\infty}\int_p \frac{\exp(jz)}{z}dz = 0$$ (Cauchy theorem, please check : p doesn't enclose pole.)

$$\Rightarrow \quad -j\pi + \int_{-\infty}^{\infty}\frac{\exp(jr)}{r}dr = 0$$

$$\Rightarrow \quad \int_{-\infty}^{\infty}\frac{\exp(jr)}{r}dr = j\pi$$

$$\Rightarrow \quad \int_{-\infty}^{\infty}\frac{\exp(jx)}{x}dx = j\pi$$ (changing dummy variable)

Comparing real part and imaginary part of both sides, we have :

$$\int_{-\infty}^{+\infty}(\cos x/x)dx = 0$$

and $$\int_{-\infty}^{+\infty}(\sin x/x)dx = \pi$$

Remark : By the time I wrote this document, I am still not very familiar with complex analysis, thus please check Jordan's lemma and Cauchy theorem later.

Fourier transform

Fourier transform is defined as :

$$FT_{of}[f(x)] = F(u) = \int_{-\infty}^{+\infty}f(x)e^{-j2\pi ux}dx$$ (ordinary frequency, unitary)

$$FT_{af}[f(x)] = F(u) = \int_{-\infty}^{+\infty}f(x)e^{-jux}dx$$ (angular frequency, non unitary)

$$FT_{uaf}[f(x)] = F(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(x)e^{-jux}dx$$ (angular frequency, unitary)

<u>Ordinary frequency (unitary)</u>

Since sinc function is an entire function **(please check the definition of entire function in complex analysis)**, and sinc function decays with x, hence we slightly **shift the contour of integration** to avoid singularity.

$$\begin{aligned}
FT_{of}[\sin x/x] &= \int_{-\infty}^{+\infty}(\sin x/x)e^{-j2\pi ux}dx \\
&= \int_{-\infty+\varepsilon j}^{+\infty+\varepsilon j}(\sin x/x)e^{-j2\pi ux}dx \\
&= \int_{-\infty+\varepsilon j}^{+\infty+\varepsilon j}((e^{jx}-e^{-jx})/(2jx))e^{-j2\pi ux}dx \\
&= \int_{-\infty+\varepsilon j}^{+\infty+\varepsilon j}e^{jx(1-2\pi u)}/(2jx)dx - \int_{-\infty+\varepsilon j}^{+\infty+\varepsilon j}e^{-jx(1+2\pi u)}/(2jx)dx
\end{aligned}$$

Substitute $x' = x(1-2\pi u)$ for first integral and substitute $x'' = x(1+2\pi u)$ for second integral, we have :

$$FT_{of}[\sin x/x] = \int_{x'_0}^{x'_1}e^{jx'}/(2jx')dx' - \int_{x''_0}^{x''_1}e^{jx''}/(2jx'')dx''$$

The key is to figure out the integration range (x'_0, x'_1) and (x''_0, x''_1), which depend on location of u in complex space.

- if u lies outside the rectangle, then the sign in substitution x' and x" are the same **(why? what rectangle?)**
- if u lies inside the rectangle, then the sign in substitution x' and x" are different **(why? what rectangle?)**

$$\begin{aligned}
FT_{of}[\sin x/x] &= \begin{bmatrix} \int_{x'_0}^{x'_1}e^{jx'}/(2jx')dx' - \int_{x'_0}^{x'_1}e^{jx''}/(2jx'')dx'' & \text{if} \quad x\notin rect \\ \int_{x'_0}^{x'_1}e^{jx'}/(2jx')dx' + \int_{x'_0}^{x'_1}e^{jx''}/(2jx'')dx'' & \text{if} \quad x\in rect \end{bmatrix} \\
&= \begin{bmatrix} 0 & \text{if} \quad x\notin rect \\ \oint_{rect}e^{jx}/(2jx)dx & \text{if} \quad x\in rect \end{bmatrix} \\
&= \begin{bmatrix} 0 & \text{if} \quad x\notin rect \\ 2\pi & \text{if} \quad x\in rect \end{bmatrix}
\end{aligned}$$

Since $\oint_{rect}e^{jx}/(2jx)dx$ encloses the pole (at origin), hence the value of $\oint_{rect}e^{jx}/(2jx)dx$ is 2π **(why?)**.