Quantitative Finance - Exotic Option

There are eight different barrier options :

- *up and out call* (we will go through this option here)
- *up and in call*
- *down and out call* (callable bull contract with $K < L$)
- *down and in call*
- *down and out put*
- *down and in put*
- *up and out put* (callable bear contract with $K > L$)
- *up and in put*

The payoff of up and out call with l as knock out price is :

$$f_0 = e^{-rT}\hat{E}[(s_T - k)^+ 1_{\max_{t\in[0,T]}(s_t)<l} \mid s_0]$$

where	s_t	$=$	$s_0 e^{(r-\sigma^2/2)t+\sigma z_t}$	$=$	$s_0 e^{\sigma z'_t}$	
and	z_t	$=$	$\varepsilon(0,\sqrt{T})$			
and	z'_t	$=$	$\varepsilon((1/\sigma)(r-\sigma^2/2)t,\sqrt{t})$	$=$	$\varepsilon(\vartheta t,\sqrt{t})$	*i.e. incorporate drift term into the random variable*
since	z'_t	$=$	$(1/\sigma)(r-\sigma^2/2)t + z_t$	$=$	$\vartheta t + z_t$	*where* $\vartheta = (1/\sigma)(r-\sigma^2/2)$

Why do we define z'_t which incorporate the drift term? This is for future convenience.

- suppose $m_T = \max_{t\in[0,T]} z_t \qquad \tau = \arg\max_{t\in[0,T]} z_t \qquad \Rightarrow \qquad \max_{t\in[0,T]}(s_t) \neq s_0 e^{(r-\sigma^2/2)\tau+\sigma m_T}$ *original definition*
- suppose $m'_T = \max_{t\in[0,T]} z'_t \qquad \tau' = \arg\max_{t\in[0,T]} z'_t \qquad \Rightarrow \qquad \max_{t\in[0,T]}(s_t) = s_0 e^{\sigma m'_T}$ *incorporate drift*

We can then rewrite the knock out condition as :

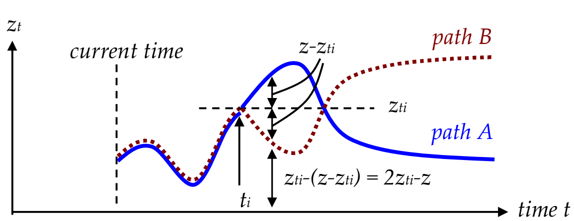
$$\begin{aligned}
\max_{t\in[0,T]}(s_t) &> l \\
s_0 e^{\sigma m'_T} &> l \\
m'_T &> (1/\sigma)\ln(l/s_0)
\end{aligned}$$

The risk neutral pricing of up and out call is a double integration : with integration of m'_T inside, and integration of z'_T outside, the integration of m'_T and z'_T should be refined from : $m_T \geq 0$ and $m_T \geq s_T$.

$$f_0 = e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s_0 e^{\sigma z} - k)^+ 1_{m<(1/\sigma)\ln(l/s_0)}\, p_{m'_T,z'_T}(m,z)\,dm\,dz$$

<u>Reflection principle</u>

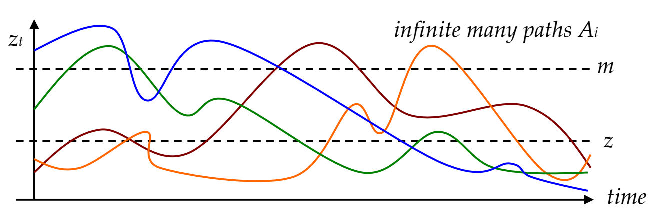
We need to find the joint probability density function $p_{m',z'}$ using (1) reflection principle of Brownian motion and (2) Girsanov theorem. Lets go through the reflection principle : it states that for every path $A = \{z_{t0}, z_{t1}, z_{t2}, \ldots, z_{ti-1}, z_{ti}, z_{ti+1}, z_{ti+2}, \ldots, z_{tN}\}$, if we create a partially-reflected path $B = \{z_{t0}, z_{t1}, z_{t2}, \ldots, z_{ti-1}, z_{ti}, z_{ti}-(z_{ti+1}-z_{ti}), z_{ti}-(z_{ti+2}-z_{ti}), \ldots, z_{ti}-(z_{tN}-z_{ti})\}$ with reflecting point t_i, then probability of path A is the same as that for path B. Note : the reflection point can be any point in the path (i.e. it is nothing related to m_T).



reflection principle : prob(path A) = prob(path B)

Now consider path A satisfying $m_T>m$ and $z_T\leq z$, we can create a partially-reflected path B, by picking reflecting point at time τ, such that z_τ goes above m for the first time in the path, then according to reflecion principle introduced above, path B must satisfy $z_T>2m-z$, and the probability of path A equals to that of path B. In fact, there are infinite many paths A_i and infinite many paths B_i, by counting all paths, we have :

$$\begin{aligned}
prob(m_T > m, z_T \leq z) &= \sum prob(A_i) \\
&= \sum prob(B_i) \\
&= prob(z_T > 2m - z) \\
&\quad \text{(remark 1)}
\end{aligned}$$



$$
\begin{aligned}
\int_0^z\int_0^m p_{m_T,z_T}(x,y)dxdy &= prob(m_T \le m, z_T \le z) \\
p_{m_T,z_T}(m,z) &= \partial_z\partial_m prob(m_T \le m, z_T \le z) \\
&= -\partial_z\partial_m prob(m_T > m, z_T \le z) \\
&= -\partial_z\partial_m prob(z_T > 2m-z) && \text{see remark 1} \\
&= -\partial_z\partial_m (1-prob(z_T \le 2m-z)) && \text{where } z_T \sim \varepsilon(0,\sqrt{T}) \\
&= -\partial_z\partial_m (1-prob(z_1 \le (2m-z)/\sqrt{T})) && \text{where } z_1 \sim \varepsilon(0,1) \\
&= \partial_z\partial_m N((2m-z)/\sqrt{T}) && \text{where } N(x) = (1/\sqrt{2\pi})\int_{-\infty}^x e^{-y^2/2}dy \\
&= \partial_z\partial_m N(u) && \text{where } u = (2m-z)/\sqrt{T} \\
&= \partial_z(\partial_u N(u)\partial_m u) \\
&= 2/\sqrt{T} \times \partial_z\partial_u N(u) && \text{since } \partial_m u = 2/\sqrt{T} \\
&= 2/\sqrt{2\pi T} \times \partial_z(e^{-u^2/2}) && \text{since } \partial_u N(u) = (1/\sqrt{2\pi})\partial_u\int_{-\infty}^u e^{-y^2/2}dy = (1/\sqrt{2\pi})e^{-u^2/2} \\
&= 2/\sqrt{2\pi T} \times \partial_u(e^{-u^2/2})\partial_z u \\
&= -2/(\sqrt{2\pi}T) \times \partial_u(e^{-u^2/2}) && \text{since } \partial_z u = -1/\sqrt{T} \\
&= 2u/(\sqrt{2\pi}T) \times e^{-u^2/2} && \text{since } \partial_u(e^{-u^2/2}) = -ue^{-u^2/2} \\
p_{m_T,z_T}(m,z) &= \frac{2(2m-z)}{T\sqrt{2\pi T}}e^{-(2m-z)^2/(2T)}
\end{aligned}
$$

<u>Girsanov theorem</u>

Now lets firstly consider the Girsanov theorem, which helps to find the probability density function of $z'_t = \theta t + z_t$. In most materials, Girsanov theorem is regarded as a change in probability measure, however, I simpy treat it as change of variable here (is there any problem with this treatment?), then it can be solved easilyby refering "Probability Integral Transform.doc" about changing variable.

$$
\begin{aligned}
\text{Since} \quad p_Y(y) &= p_X(x=f(y))\partial_y f(y) && \text{with change of variable : } x = f(y) \\
\text{Thus} \quad p_{z'_t}(z'_t = z) &= p_{z_t}(z_t = z-\vartheta t)\partial_z(z-\vartheta t) \\
&= p_{z_t}(z_t = z-\vartheta t) && \text{since } \partial_z(z-\vartheta t) = 1 \\
&= 1/\sqrt{2\pi t} \times e^{-(z-\vartheta t)^2/(2t)} \\
&= \underbrace{1/\sqrt{2\pi t} \times e^{-z^2/(2t)}}_{p_{zt}(z)} \times e^{-(-2\vartheta tz+\vartheta^2t^2)/(2t)} \\
&= p_{z_t}(z) \times e^{(2\vartheta tz-\vartheta^2t^2)/(2t)} \\
&= p_{z_t}(z) \times e^{\vartheta z+\vartheta^2 t/2} \quad \text{(equation *)} && \text{where } e^{\vartheta z+\vartheta^2 t/2} \text{ is called Girsanov adjustment factor}
\end{aligned}
$$

In most Girsanov theorem materials, it is usually defined in the following way : let z_t be a Brownian motion that follows probability measure P, and let $z'_t = \theta t + z_t$ be a random process that follows probability meaure Q (note : z'_t is not a Brownian motion, as $dz'_t = \theta dt + dz_t = \varepsilon(\theta dt, \sqrt{dt})$, remind that the differential in Brownian motion is a zero mean normal), then we have :

$$
\begin{aligned}
E_Q(f(z_t)) &= E_P(f(z_t)e^{\theta z_t - \theta^2 t/2}) && \text{when } \theta_t \text{ is deterministic and it is constant in time} \\
\text{or} \quad E_Q(f(z_t)) &= E_P(f(z_t)e^{\int_0^t \theta_s dz_s - (1/2)\int_0^t \theta_s^2 ds}) && \text{when } \theta_t \text{ is stochastic}
\end{aligned}
$$

*The above can be simply derived from equation * as the following, suppose θ is deterministic and constant :*

$$
\begin{aligned}
E_Q(f(z_t)) &= \int_{-\infty}^{\infty} f(z)p_{z'_t}(z)dz \\
&= \int_{-\infty}^{\infty} f(z)p_{z_t}(z)e^{\vartheta z+\vartheta^2 t/2}dz \\
&= E_P(e^{\theta z_t - \theta^2 t/2}f(z_t))
\end{aligned}
$$

<u>Combine Reflection principle with Girsanov theorem</u>

Please note that reflection principle is applicable for the Brownian motion z_T (or other random processes, in which the probability of moving up and moving down are the same), but not for non Brownian motion z'_T. Now how can we combine the result from reflection principle and the result from Girsanov theorem to obtain the density function for transformed variables m'_T and z'_T?

$$
p_{m'_T,z'_T}(m,z)dmdz = \sum_{i'} prob(path_{i'}) \qquad \text{now we consider probability of path, which consists of } \infty \text{ many } dz'_t
$$

where path i is any path that starts with $z'_0 = 0$, ends with $z'_T = z$ and reaches a maximum of $m'_T = m$. The probability of path i in the transformed space (m'_T, z'_T) is related to the probability of the corresponding path in the original space (m_T, z_T) by Girsanov adjustment factor.

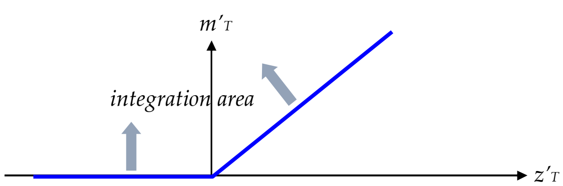
$$
\begin{aligned}
& p_{m'_T,z'_T}(m,z)dmdz \\
= \;& \textstyle\sum_i prob(path_i) \\
= \;& \textstyle\sum_i prob(z'_{\Delta t} = x_1, z'_{2\Delta t} = x_2, ..., z'_{N\Delta t} = x_N) && \textit{where } N\Delta t = T \textit{ and } x_N = z \\
= \;& \textstyle\sum_i prob(z'_{\Delta t} = x_1, z'_{\Delta t} = x_2 - x_1, ..., z'_{\Delta t} = x_N - x_{N-1}) \\
= \;& \textstyle\sum_i (p_{z'_{\Delta t}}(x_1)\Delta t) \times (p_{z'_{\Delta t}}(x_2 - x_1)\Delta t) \times ... \times (p_{z'_{\Delta t}}(x_N - x_{N-1})\Delta t) \\
= \;& \textstyle\sum_i (p_{z_{\Delta t}}(x_1)\Delta t \times e^{\theta(x_1) - \theta^2\Delta t/2}) \times (p_{z_{\Delta t}}(x_2 - x_1)\Delta t \times e^{\theta(x_2 - x_1) - \theta^2\Delta t/2}) \times ... \times (p_{z_{\Delta t}}(x_N - x_{N-1})\Delta t \times e^{\theta(x_N - x_{N-1}) - \theta^2\Delta t/2}) \\
= \;& \textstyle\sum_i (p_{z_{\Delta t}}(x_1)\Delta t) \times (p_{z_{\Delta t}}(x_2 - x_1)\Delta t) \times ... \times (p_{z_{\Delta t}}(x_N - x_{N-1})\Delta t) \times e^{\theta x_N - \theta^2 N\Delta t/2} \\
= \;& \textstyle\sum_i prob(z_{\Delta t} = x_1, z_{\Delta t} = x_2 - x_1, ..., z_{\Delta t} = x_N - x_{N-1}) \times e^{\theta z - \theta^2 T/2} && \textit{since } N\Delta t = T \textit{ and } x_N = z \\
= \;& \textstyle\sum_i prob(z_{\Delta t} = x_1, z_{2\Delta t} = x_2, ..., z'_{N\Delta t} = x_N) \times e^{\theta z - \theta^2 T/2} \\
= \;& p_{m_T,z_T}(m,z)dmdz \times e^{\theta z - \theta^2 T/2} && \textit{thus we have : } p_{m'_T,z'_T}(m,z) = p_{m_T,z_T}(m,z) \times e^{\theta z - \theta^2 T/2}
\end{aligned}
$$

<u>About integration range</u>

Since we need to compare the final stock price with strike price (i.e. s_T vs k, or more precisely z'_T vs k), and to compare the maximum stock price with knock out price (i.e. max(s_t) vs l, or more precisely m'_T vs l), it is more convenient for future calculation if we define :

$$
\begin{aligned}
s_0 e^{\sigma\alpha} &= k & \Rightarrow && \alpha &= (1/\sigma)\ln(k/s_0) && \textit{which is a bound for integrating } z'_T \\
s_0 e^{\sigma\beta} &= l & \Rightarrow && \beta &= (1/\sigma)\ln(l/s_0) && \textit{which is a bound for integrating } m'_T
\end{aligned}
$$

Now we can perform risk neutral pricing using the 2D joint probability density function within the following integration area :



i.e. $m'_T \geq z'_T$ and $m'_T \geq 0$
(if all $z'_t < 0$, then we have : $m'_T = 0$, i.e. max(s_t) = s_0)

figure 1

Yet the integration region should be further refined. For up and out call :

		$k > s_0$	$k < s_0$
case 1 : $l > k$			
	$l > s_0$	out the \$	in the \$
	$l < s_0$	impossible	KO at t = 0
case 2 : $l < k$		$k > s_0$	$k < s_0$
	$l > s_0$	out the \$	impossible
	$l < s_0$	KO at t = 0	KO at t = 0

Corresponding comparison in terms of α and β

	$\alpha > 0$	$\alpha < 0$
$\beta > \alpha$		
$\beta > 0$	out the \$	in the \$
$\beta < 0$	impossible	KO at t = 0
$\beta < \alpha$	$\alpha > 0$	$\alpha < 0$
$\beta > 0$	out the \$	impossible
$\beta < 0$	KO at t = 0	KO at t = 0

Please note that the above bounds are different for different barrier options. As an example, lets try again for up and in call :

		$k > s_0$	$k < s_0$
case 1 : $l > k$			
	$l > s_0$	not knock in	not knock in
	$l < s_0$	impossible	in the \$
case 2 : $l < k$		$k > s_0$	$k < s_0$
	$l > s_0$	not knock in	impossible
	$l < s_0$	out the \$	in the \$

Corresponding comparison in terms of α and β

	$\alpha > 0$	$\alpha < 0$
$\beta > \alpha$		
$\beta > 0$	not knock in	not knock in
$\beta < 0$	impossible	in the \$
$\beta < \alpha$	$\alpha > 0$	$\alpha < 0$
$\beta > 0$	not knock in	impossible
$\beta < 0$	out the \$	in the \$

Before the double integration, lets define the following for convenience : $\tau = \theta + \sigma$

$$\tau = (1/\sigma)(r - \sigma^2/2) + \sigma = (1/\sigma)(r + \sigma^2/2)$$

Besides we have :

$$\tau^2 - \vartheta^2 = (1/\sigma^2)((r + \sigma^2/2)^2 - (r - \sigma^2/2)^2) = (1/\sigma^2)2r\sigma^2 = 2r$$

To summarise

$$\tau, \vartheta = (1/\sigma)(r \pm \sigma^2/2)$$

and

$$\tau^2 - \vartheta^2 = 2r$$

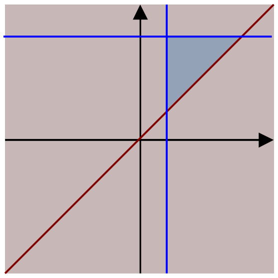
For all the following figures, we have :

- *horizontal axis is $z'_T = z$, horizontal axis is bounded by vertical blue line ($z'_T = \alpha$),*
- *vertical axis is $m'_T = m$, vertical axis is bounded by horizontal blue line ($m'_T = \beta$),*
- *to consider the integration region in figure 1 simultaneously (i.e. AND logic).*

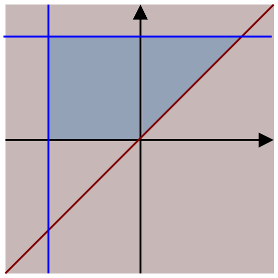
Up and out call (integration region = SE corner of the blue cross)

case 1 : $l > k$ (i.e. $\beta > \alpha$)

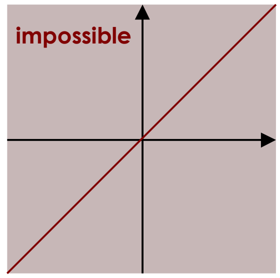
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



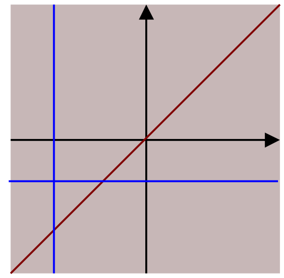
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)

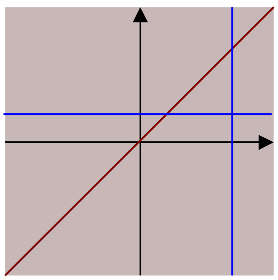


$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)

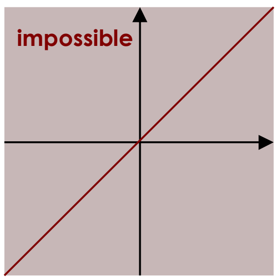


case 2 : $l < k$ (i.e. $\beta < \alpha$)

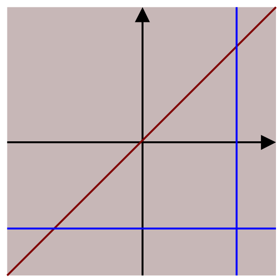
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



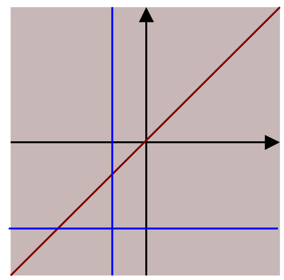
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)



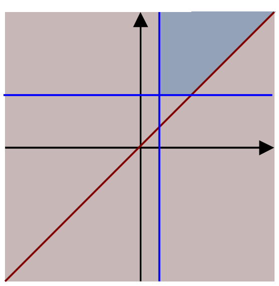
$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)



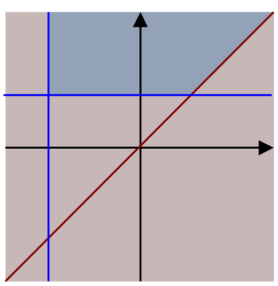
Up and in call (integration region = NE corner of the blue cross)

case 1 : $l > k$ (i.e. $\beta > \alpha$)

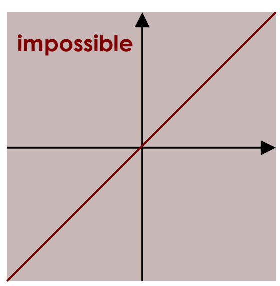
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



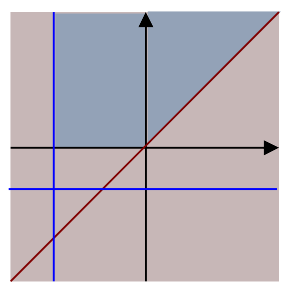
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)

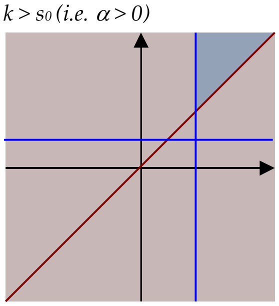


$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)

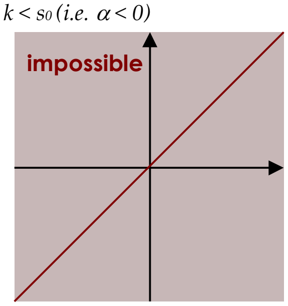


case 2 : $l < k$ (i.e. $\beta < \alpha$)

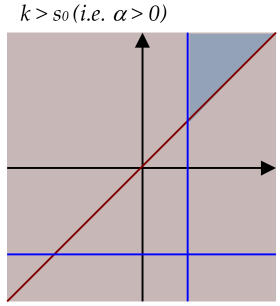
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



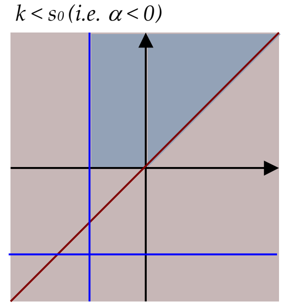
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)



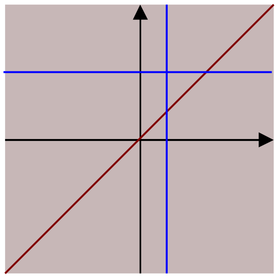
$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)



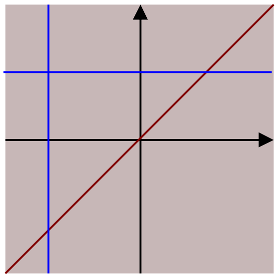
Down and out call (integration region = NE corner of the blue cross)

case 1 : $l > k$ (i.e. $\beta > \alpha$)

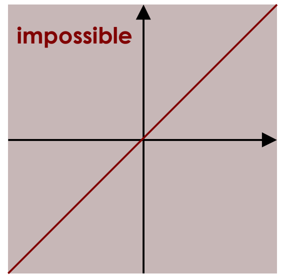
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



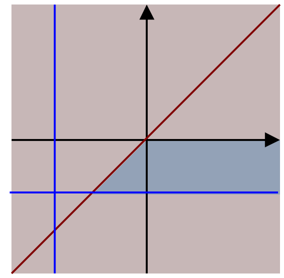
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)

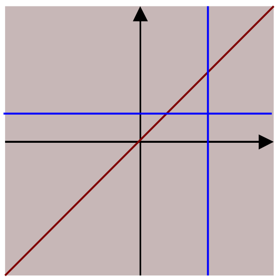


$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)

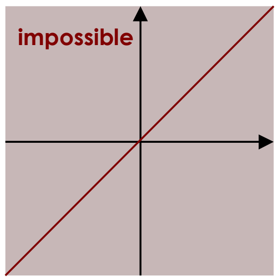


case 2 : $l < k$ (i.e. $\beta < \alpha$)

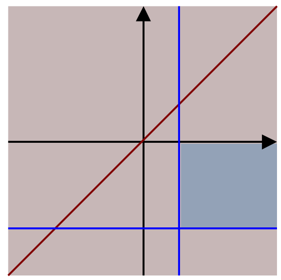
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



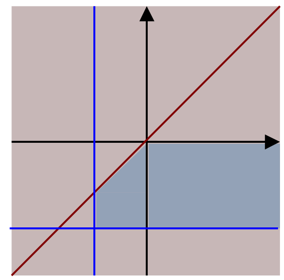
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)



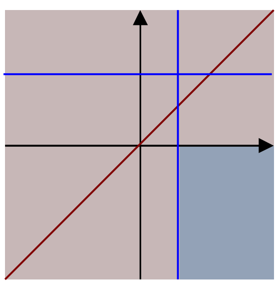
$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)



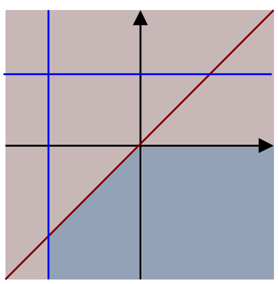
Down and in call (integration region = SE corner of the blue cross)

case 1 : $l > k$ (i.e. $\beta > \alpha$)

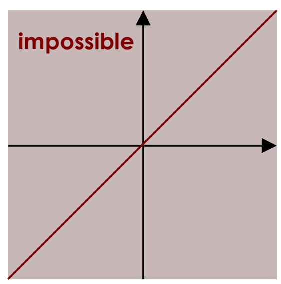
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



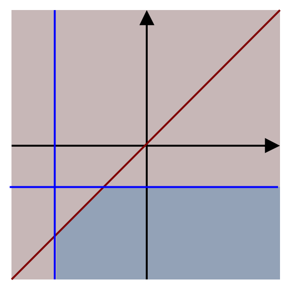
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)

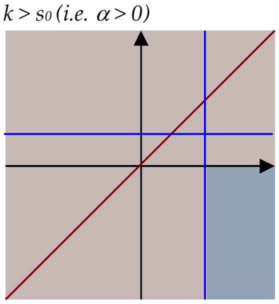


$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)

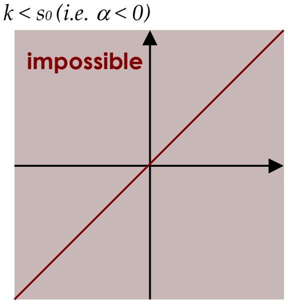


case 2 : $l < k$ (i.e. $\beta < \alpha$)

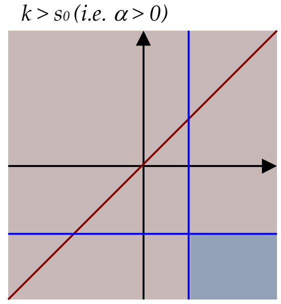
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



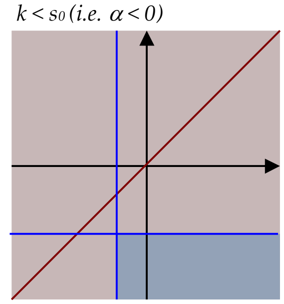
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)



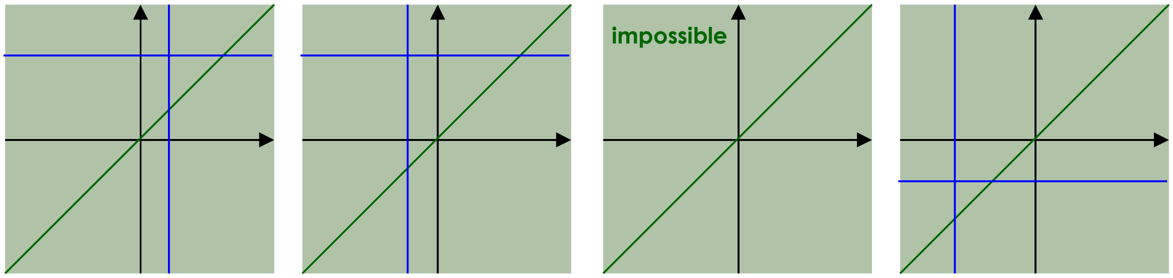
$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)



Down and out put (integration region = NW corner of the blue cross)

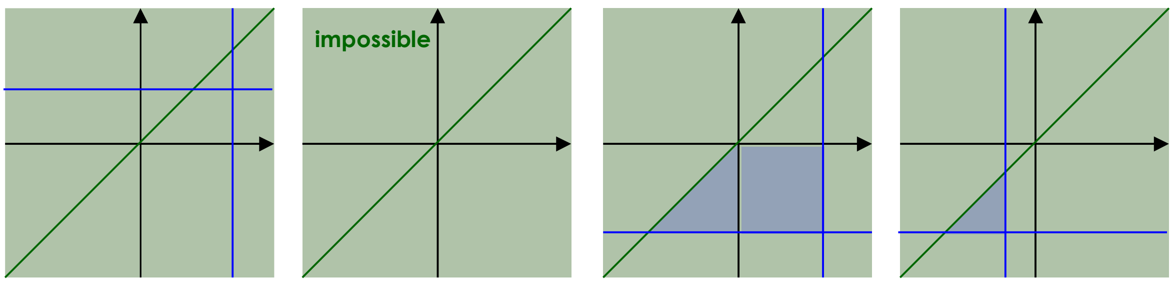
case 1 : $l > k$ *(i.e.* $\beta > \alpha$*)*

$l > s_0$ (i.e. $\beta > 0$)	$l > s_0$ (i.e. $\beta > 0$)	$l < s_0$ (i.e. $\beta < 0$)	$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)	$k < s_0$ (i.e. $\alpha < 0$)	$k > s_0$ (i.e. $\alpha > 0$)	$k < s_0$ (i.e. $\alpha < 0$)



case 2 : $l < k$ *(i.e.* $\beta < \alpha$*)*

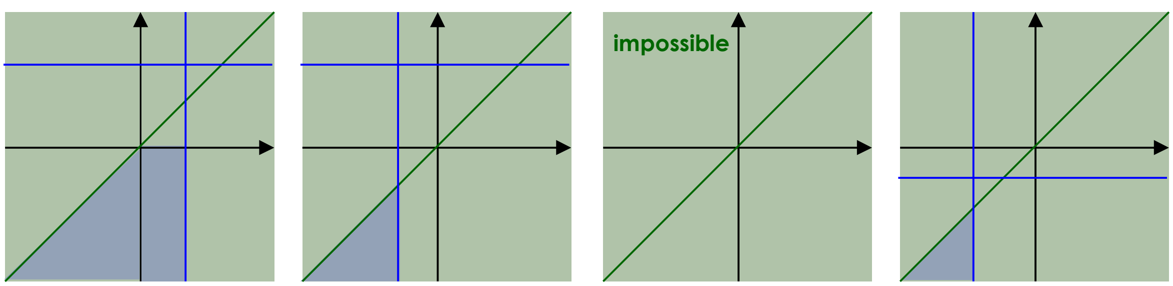
$l > s_0$ (i.e. $\beta > 0$)	$l > s_0$ (i.e. $\beta > 0$)	$l < s_0$ (i.e. $\beta < 0$)	$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)	$k < s_0$ (i.e. $\alpha < 0$)	$k > s_0$ (i.e. $\alpha > 0$)	$k < s_0$ (i.e. $\alpha < 0$)



Down and in put (integration region = SW corner of the blue cross)

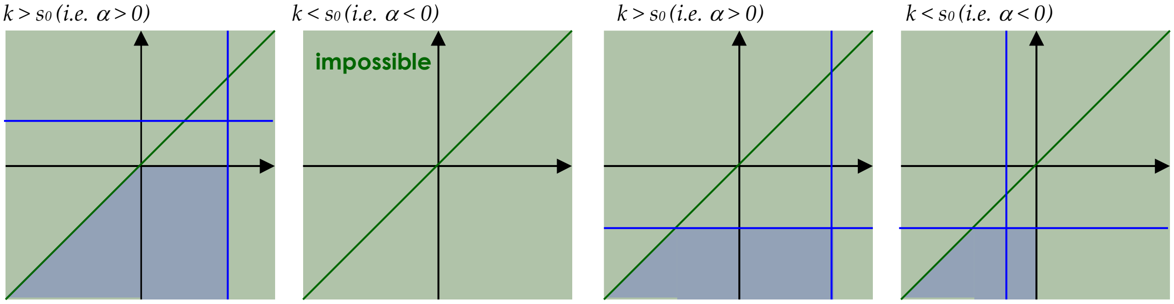
case 1 : $l > k$ *(i.e.* $\beta > \alpha$*)*

$l > s_0$ (i.e. $\beta > 0$)	$l > s_0$ (i.e. $\beta > 0$)	$l < s_0$ (i.e. $\beta < 0$)	$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)	$k < s_0$ (i.e. $\alpha < 0$)	$k > s_0$ (i.e. $\alpha > 0$)	$k < s_0$ (i.e. $\alpha < 0$)



case 2 : $l < k$ *(i.e.* $\beta < \alpha$*)*

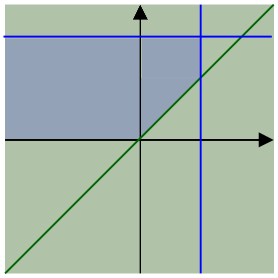
$l > s_0$ (i.e. $\beta > 0$)	$l > s_0$ (i.e. $\beta > 0$)	$l < s_0$ (i.e. $\beta < 0$)	$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)	$k < s_0$ (i.e. $\alpha < 0$)	$k > s_0$ (i.e. $\alpha > 0$)	$k < s_0$ (i.e. $\alpha < 0$)



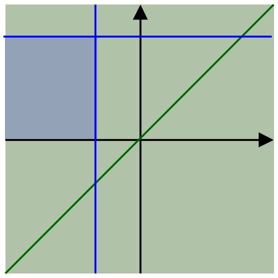
Up and out put (integration region = SW corner of the blue cross)

case 1 : $l > k$ *(i.e.* $\beta > \alpha$*)*

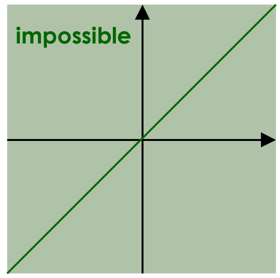
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



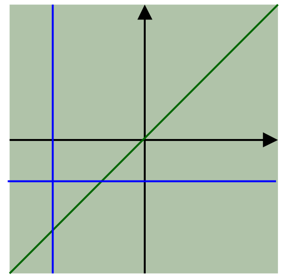
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)

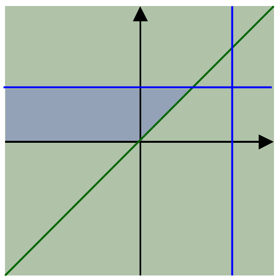


$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)

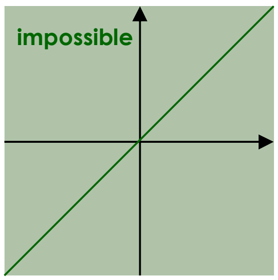


case 2 : $l < k$ *(i.e.* $\beta < \alpha$*)*

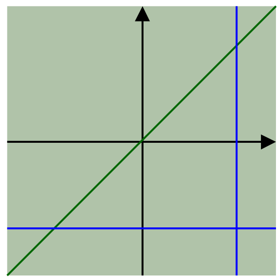
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



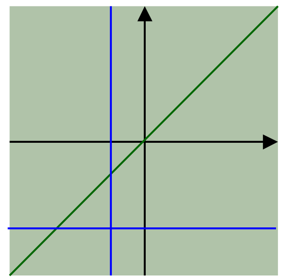
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)



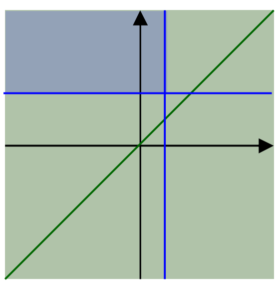
$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)



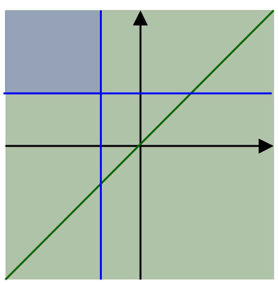
Up and in put (integration region = NW corner of the blue cross)

case 1 : $l > k$ *(i.e.* $\beta > \alpha$*)*

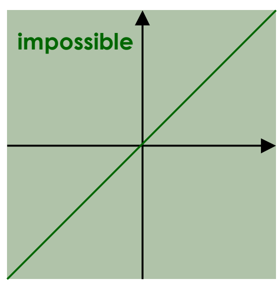
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



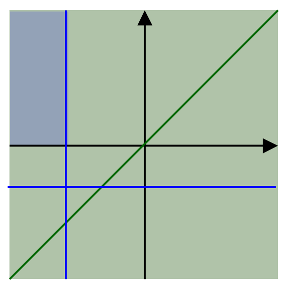
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)

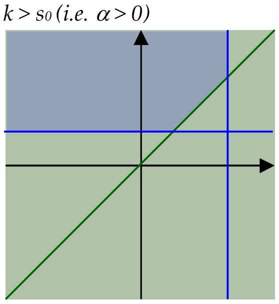


$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)

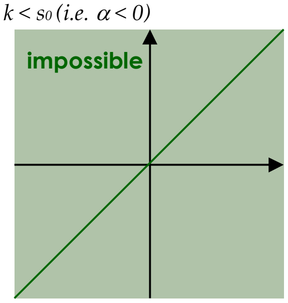


case 2 : $l < k$ *(i.e.* $\beta < \alpha$*)*

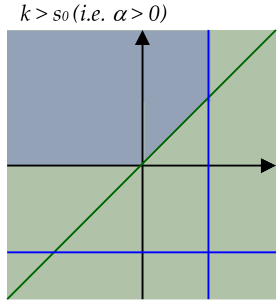
$l > s_0$ (i.e. $\beta > 0$)
$k > s_0$ (i.e. $\alpha > 0$)



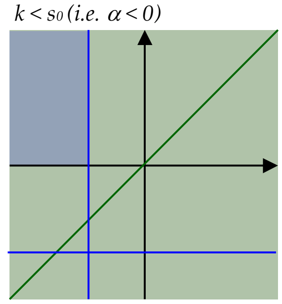
$l > s_0$ (i.e. $\beta > 0$)
$k < s_0$ (i.e. $\alpha < 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k > s_0$ (i.e. $\alpha > 0$)



$l < s_0$ (i.e. $\beta < 0$)
$k < s_0$ (i.e. $\alpha < 0$)



Risk neutral pricing – Up and out call

Please refer to the attached figures for integration range. For up and out call, only 2 cases out of 8 cases have valid integration region. We need to handle these 2 cases separately, they are different only in the integration range.

$$f_{0,\alpha>0} \quad = \quad e^{-rT}[\int_\alpha^\beta\int_z^\beta (s_0 e^{\sigma z}-k)\underbrace{\underbrace{\frac{2(2m-z)}{T\sqrt{2\pi T}}e^{-(2m-z)^2/(2T)}}_{p_{m,z}}e^{\theta z-\theta^2 T/2}}_{p_{m',z'}}dmdz]$$

$$f_{0,\alpha<0} \quad = \quad e^{-rT}\begin{bmatrix}\int_\alpha^0\int_0^\beta (s_0 e^{\sigma z}-k)\frac{2(2m-z)}{T\sqrt{2\pi T}}e^{-(2m-z)^2/(2T)}e^{\theta z-\theta^2 T/2}dmdz+ \\ \int_0^\beta\int_z^\beta (s_0 e^{\sigma z}-k)\frac{2(2m-z)}{T\sqrt{2\pi T}}e^{-(2m-z)^2/(2T)}e^{\theta z-\theta^2 T/2}dmdz\end{bmatrix}$$

Integrate m first, since :

$$f_{0,\alpha>0} \quad = \quad e^{-rT}\left[\int_\alpha^\beta \frac{2}{T}(s_0 e^{\sigma z}-k)e^{\theta z-\theta^2 T/2}\int_z^\beta \frac{2m-z}{\sqrt{2\pi T}}e^{-(2m-z)^2/(2T)}dmdz\right]$$

$$f_{0,\alpha<0} \quad = \quad e^{-rT}\begin{bmatrix}\int_\alpha^0 \frac{2}{T}(s_0 e^{\sigma z}-k)e^{\theta z-\theta^2 T/2}\int_0^\beta \frac{2m-z}{\sqrt{2\pi T}}e^{-(2m-z)^2/(2T)}dmdz+ \\ \int_0^\beta \frac{2}{T}(s_0 e^{\sigma z}-k)e^{\theta z-\theta^2 T/2}\int_z^\beta \frac{2m-z}{\sqrt{2\pi T}}e^{-(2m-z)^2/(2T)}dmdz\end{bmatrix}$$

Consider the internal integration :

$$\begin{aligned}
\int_p^q \frac{2m-z}{\sqrt{2\pi T}}e^{-(2m-z)^2/(2T)}dm &= \int_{m=p}^{m=q}(\sqrt{T}/2)(u/\sqrt{2\pi})e^{-u^2/2}du && \textit{by putting} \quad u=(2m-x)/\sqrt{T} \quad \Rightarrow \quad du=(2/\sqrt{T})dm \\
&= \int_{m=p}^{m=q}(\sqrt{T}/(4\sqrt{2\pi}))e^{-v/2}dv && \textit{by putting} \quad v=u^2 \quad \Rightarrow \quad dv=2udu \\
&= -(\sqrt{T}/(2\sqrt{2\pi}))e^{-v/2}\Big|_{m=p}^{m=q} \\
&= -(\sqrt{T}/(2\sqrt{2\pi}))e^{-(2m-x)^2/(2T)}\Big|_{m=p}^{m=q} \\
&= \sqrt{T}/(2\sqrt{2\pi})\times(e^{-(2p-x)^2/(2T)}-e^{-(2q-x)^2/(2T)})
\end{aligned}$$

By the way, you can find the above integration by change of variable once : u = (2m-x)²/(2T) instead of doing it twice : u = (2m-x)/ √T and v = u². Try to do it if you are interested. With the above result, we can simplify the double integration as :

$$f_{0,\alpha>0} \quad = \quad e^{-rT}\left[\int_\alpha^\beta \frac{2}{T}(s_0 e^{\sigma z}-k)e^{\theta z-\theta^2 T/2}\frac{\sqrt{T}}{2\sqrt{2\pi}}(e^{-(2z-z)^2/(2T)}-e^{-(2\beta-z)^2/(2T)})dz\right]$$

$$f_{0,\alpha<0} \quad = \quad e^{-rT}\begin{bmatrix}\int_\alpha^0 \frac{2}{T}(s_0 e^{\sigma z}-k)e^{\theta z-\theta^2 T/2}\frac{\sqrt{T}}{2\sqrt{2\pi}}(e^{-(0-z)^2/(2T)}-e^{-(2\beta-z)^2/(2T)})dz+ \\ \int_0^\beta \frac{2}{T}(s_0 e^{\sigma z}-k)e^{\theta z-\theta^2 T/2}\frac{\sqrt{T}}{2\sqrt{2\pi}}(e^{-(2z-z)^2/(2T)}-e^{-(2\beta-z)^2/(2T)})dz\end{bmatrix}$$

Both cases are the same. From now on, we can consider them together.

$$\begin{aligned}
f_0 &= e^{-rT}\left[\int_\alpha^\beta \frac{2}{T}(s_0 e^{\sigma z}-k)e^{\theta z-\theta^2 T/2}\frac{\sqrt{T}}{2\sqrt{2\pi}}(e^{-z^2/(2T)}-e^{-(2\beta-z)^2/(2T)})dz\right] \\
&= e^{-rT}\left[\int_\alpha^\beta \frac{1}{\sqrt{2\pi T}}(s_0 e^{\sigma z}-k)e^{\theta z-\theta^2 T/2}(e^{-z^2/(2T)}-e^{-(z^2-4\beta z+4\beta^2)/(2T)})dz\right] \\
&= e^{-rT}[\int_\alpha^\beta (1/\sqrt{2\pi T})(n_0-n_1-n_2+n_3)dz] \qquad (1)
\end{aligned}$$

$$
\begin{aligned}
\textit{where}\quad n_0 &= s_0 e^{\sigma z} \times e^{\theta z-\theta^2 T/2} \times e^{-z^2/(2T)} &&= s_0 e^{-(z^2-2(\vartheta+\sigma)Tz+\theta^2T^2)/(2T)}\\
n_1 &= k \times e^{\theta z-\theta^2 T/2} \times e^{-z^2/(2T)} &&= k e^{-(z^2-2\theta Tz+\theta^2T^2)/(2T)}\\
n_2 &= s_0 e^{\sigma z} \times e^{\theta z-\theta^2 T/2} \times e^{-(z^2-4\beta z+4\beta^2)/(2T)} &&= s_0 e^{-(z^2-4\beta z-2(\theta+\sigma)Tz+4\beta^2+\theta^2T^2)/(2T)}\\
n_3 &= k \times e^{\theta z-\theta^2 T/2} \times e^{-(z^2-4\beta z+4\beta^2)/(2T)} &&= k e^{-(z^2-4\beta z-2\theta Tz+4\beta^2+\theta^2T^2)/(2T)}
\end{aligned}
$$

They can be simplified by completing square in z and substitution of $\tau = \theta + \sigma$:

$$
\begin{aligned}
n_0 &= s_0 e^{-(z-\tau T)^2/(2T)} \times e^{-(-\tau^2T^2+\theta^2T^2)/(2T)} &&= s_0 e^{-(z-\tau T)^2/(2T)} e^{rT} && (2a)\\
n_1 &= k e^{-(z-\theta T)^2/(2T)} \times e^{(-\theta^2T^2+\theta^2T^2)/(2T)} &&= k e^{-(z-\theta T)^2/(2T)} && (2b)\\
n_2 &= s_0 e^{-(z-(2\beta+\tau T))^2/(2T)} e^{-(-(2\beta+\tau T)^2+4\beta^2+\theta^2T^2)/(2T)} &&= s_0 e^{-(z-(2\beta+\tau T))^2/(2T)} e^{2\beta\tau} e^{rT} && (2c)\\
n_3 &= k e^{-(z-(2\beta+\theta T))^2/(2T)} e^{-(-(2\beta+\theta T)^2+4\beta^2+\theta^2T^2)/(2T)} &&= k e^{-(z-(2\beta+\theta T))^2/(2T)} e^{2\beta\theta} && (2d)
\end{aligned}
$$

Please note that n_0 and n_1 differ by $s_0 \Leftrightarrow k$ and $\tau \Leftrightarrow \theta$, while n_2 and n_3 differ by $s_0 \Leftrightarrow k$ and $\tau \Leftrightarrow \theta$. Now, what are $e^{2\beta\tau}$ and $e^{2\beta\theta}$?

$$
\begin{aligned}
e^{2\beta\tau} &= e^{2\times(1/\sigma)\ln(l/s_0)\times(1/\sigma)(r+\sigma^2/2)} = (e^{\ln(l/s_0)})^{(2/\sigma^2)(r+\sigma^2/2)} = (l/s_0)^{(2/\sigma^2)(r+\sigma^2/2)} = (l/s_0)^{2\tau/\sigma} && (3a)\\
e^{2\beta\theta} &= e^{2\times(1/\sigma)\ln(l/s_0)\times(1/\sigma)(r-\sigma^2/2)} = (e^{\ln(l/s_0)})^{(2/\sigma^2)(r-\sigma^2/2)} = (l/s_0)^{(2/\sigma^2)(r-\sigma^2/2)} = (l/s_0)^{2\vartheta/\sigma} && (3b)\\
\textit{and}\quad \tau/\sigma-\vartheta/\sigma &= (1/\sigma^2)(r+\sigma^2/2)-(1/\sigma^2)(r-\sigma^2/2) = (1/\sigma^2)\sigma^2 = 1 && (4)
\end{aligned}
$$

By putting results in (2a) – (2d), (3a) and (3b) into (1), we have :

$$
\begin{aligned}
f_0 &= e^{-rT}\begin{bmatrix}
+s_0e^{rT}\times(2\pi T)^{-1/2}\int_\alpha^\beta e^{-(z-\tau T)^2/(2T)}dz\\
-k\times(2\pi T)^{-1/2}\int_\alpha^\beta e^{(z-\theta T)^2/(2T)}dz\\
-s_0e^{rT}(l/s_0)^{2\tau/\sigma}\times(2\pi T)^{-1/2}\int_\alpha^\beta e^{(z-(2\beta+\tau T))^2/(2T)}dz\\
+k(l/s_0)^{2\vartheta/\sigma}\times(2\pi T)^{-1/2}\int_\alpha^\beta e^{-(z-(2\beta+\theta T))^2/(2T)}dz
\end{bmatrix}\\
&= e^{-rT}\begin{bmatrix}
+s_0e^{rT}\times(N((\beta-\tau T)/\sqrt{T})-N((\alpha-\tau T)/\sqrt{T}))\\
-k\times(N((\beta-\vartheta T)/\sqrt{T})-N((\alpha-\vartheta T)/\sqrt{T}))\\
-s_0e^{rT}(l/s_0)^{2\tau/\sigma}\times(N((\beta-(2\beta+\tau T))/\sqrt{T})-N((\alpha-(2\beta+\tau T))/\sqrt{T}))\\
+k(l/s_0)^{2\vartheta/\sigma}\times(N((\beta-(2\beta+\vartheta T))/\sqrt{T})-N((\alpha-(2\beta+\vartheta T))/\sqrt{T}))
\end{bmatrix} \qquad \textit{see remark \#}
\end{aligned}
$$

Remark #
$$
\begin{aligned}
(2\pi T)^{-1/2}\int_a^b e^{(x-k)^2/(2T)}dx &= (2\pi)^{-1/2}\int_{(a-k)/\sqrt{T}}^{(b-k)/\sqrt{T}} e^{-y^2/2}dy\\
&= N((b-k)/\sqrt{T})-N((a-k)/\sqrt{T})
\end{aligned}
$$

In order to make it simple, we define variables :

$$
\begin{aligned}
(\alpha-\tau T)/\sqrt{T} &= (\ln(k/s_0)-(r+\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -(\ln(s_0/k)+(r+\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -d_1\\
(\alpha-\vartheta T)/\sqrt{T} &= (\ln(k/s_0)-(r-\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -(\ln(s_0/k)+(r-\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -d_2\\
(\beta-\tau T)/\sqrt{T} &= (\ln(l/s_0)-(r+\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -(\ln(s_0/l)+(r+\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -x_1\\
(\beta-\vartheta T)/\sqrt{T} &= (\ln(l/s_0)-(r-\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -(\ln(s_0/l)+(r-\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -x_2\\
(\beta-(2\beta+\tau T))/\sqrt{T} &= (-\ln(l/s_0)-(r+\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -(\ln(l/s_0)+(r+\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -y_1\\
(\beta-(2\beta+\vartheta T))/\sqrt{T} &= (-\ln(l/s_0)-(r-\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -(\ln(l/s_0)+(r-\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -y_2\\
(\alpha-(2\beta+\tau T))/\sqrt{T} &= (\ln(k/s_0)-2\ln(l/s_0)-(r+\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -(\ln(l^2/(ks_0))+(r+\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -z_1\\
(\alpha-(2\beta+\vartheta T))/\sqrt{T} &= (\ln(k/s_0)-2\ln(l/s_0)-(r-\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -(\ln(l^2/(ks_0))+(r-\sigma^2/2)T)/(\sigma\sqrt{T}) &&= -z_2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad f_0 &= \begin{bmatrix}
s_0e^{rT}\times(N(-x_1)-N(-d_1))-k\times(N(-x_2)-N(-d_2))\\
-s_0e^{rT}(l/s_0)^{2\tau/\sigma}\times(N(-y_1)-N(-z_1))+k(l/s_0)^{2\vartheta/\sigma}\times(N(-y_2)-N(-z_2))
\end{bmatrix}e^{-rT}\\
&= \begin{bmatrix}
s_0\times(N(d_1)-N(x_1))-k'\times(N(d_2)-N(x_2))\\
-s_0(l/s_0)^{2\tau/\sigma}\times(N(z_1)-N(y_1))+k'(l/s_0)^{2\vartheta/\sigma}\times(N(z_2)-N(y_2))
\end{bmatrix} \qquad \textit{since}\quad N(-x)=1-N(x)
\end{aligned}
$$

Summary of all barrier options

$$\begin{bmatrix} +s_0 & -k' & -s_0 & +k' & +s_0 a & -k'b & -s_0 a & +k'b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +s_0 & -k' & -s_0 a & +k'b & +s_0 a & -k'b \\ +s_0 & -k' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +s_0 & -k' & -s_0 a & +k'b & 0 & 0 \\ +s_0 & -k' & 0 & 0 & 0 & 0 & -s_0 a & +k'b \\ +s_0 & -k' & -s_0 & +k' & +s_0 a & -k'b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +s_0 a & -k'b \end{bmatrix} \times \begin{bmatrix} N(d_1) \\ N(d_2) \\ N(x_1) \\ N(x_2) \\ N(y_1) \\ N(y_2) \\ N(z_1) \\ N(z_2) \end{bmatrix} = \begin{bmatrix} c_{up\_and\_out} & l > k \\ c_{up\_and\_out} & l \le k \\ c_{up\_and\_in} & l > k \\ c_{up\_and\_in} & l \le k \\ c_{down\_and\_out} & l > k \\ c_{down\_and\_out} & l \le k \\ c_{down\_and\_in} & l > k \\ c_{down\_and\_in} & l \le k \end{bmatrix} \qquad \textit{matrix A}$$

$$\begin{bmatrix} -s_0 & +k' & +s_0 & -k' & -s_0 a & +k'b & +s_0 a & -k'b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -s_0 & +k' & +s_0 a & -k'b & -s_0 a & +k'b \\ -s_0 & +k' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -s_0 & +k' & +s_0 a & -k'b & 0 & 0 \\ -s_0 & +k' & 0 & 0 & 0 & 0 & +s_0 a & -k'b \\ -s_0 & +k' & +s_0 & -k' & -s_0 a & +k'b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -s_0 a & +k'b \end{bmatrix} \times \begin{bmatrix} N(-d_1) \\ N(-d_2) \\ N(-x_1) \\ N(-x_2) \\ N(-y_1) \\ N(-y_2) \\ N(-z_1) \\ N(-z_2) \end{bmatrix} = \begin{bmatrix} p_{down\_and\_out} & l < k \\ p_{down\_and\_out} & l \ge k \\ p_{down\_and\_in} & l < k \\ p_{down\_and\_in} & l \ge k \\ p_{up\_and\_out} & l < k \\ p_{up\_and\_out} & l \ge k \\ p_{up\_and\_in} & l < k \\ p_{up\_and\_in} & l \ge k \end{bmatrix} \qquad \textit{matrix B}$$

$$\begin{aligned} \textit{where} \quad k' &= ke^{-rT} \\ a &= (l/s_0)^{2\tau/\sigma} = (l/s_0)^{(2/\sigma^2)(r+\sigma^2/2)} = (l/s_0)^{2r/\sigma^2+1} \\ b &= (l/s_0)^{2\vartheta/\sigma} = (l/s_0)^{(2/\sigma^2)(r-\sigma^2/2)} = (l/s_0)^{2r/\sigma^2-1} \end{aligned}$$

Here are some observations :

- *Coefficients in matrix A are just negative of those in matrix B.*
- *The 4th row of matrix corresponds to vanilla call.*
- *The 4th row of matrix corresponds to vanilla put.*
- *Please note that for both l>k and l≤k, we have :*

$$\begin{aligned} c_{up\_and\_out} + c_{up\_and\_in} &= c_{vanilla} & \qquad c_{up\_and\_out} &= p_{down\_and\_out} \\ c_{down\_and\_out} + c_{down\_and\_in} &= c_{vanilla} & c_{up\_and\_in} &\ne p_{down\_and\_in} \\ p_{down\_and\_out} + p_{down\_and\_in} &= p_{vanilla} & c_{down\_and\_out} &\ne p_{up\_and\_out} \\ p_{up\_and\_out} + p_{up\_and\_in} &= p_{vanilla} & c_{down\_and\_in} &\ne p_{up\_and\_in} \end{aligned}$$

Look back option

Look back option is defined by its payoff structure : *Note : ()+ is redundant as it must be always fulfilled.*

$$\begin{aligned} c_T &= (s_T - s_{\min,T_{issue},T})^+ = s_T - s_{\min,T_{issue},T} & \textit{where} \quad s_{\min,T_A,T_B} &= \min_{t\in[T_A,T_B]} s_t \quad \textit{(call lock back option)} \\ p_T &= (s_{\max,T_{issue},T} - s_T)^+ = s_{\max,T_{issue},T} - s_T & \textit{where} \quad s_{\max,T_A,T_B} &= \max_{t\in[T_A,T_B]} s_t \quad \textit{(put look back option)} \end{aligned}$$

Suppose current time is t=0, option issue time is t=T_{issue}, option maturity time is t=T, where T_{issue} < 0 < T, then risk neutral pricing is :

$$\begin{aligned} c_0 &= e^{-rT}\hat{E}[s_T - s_{\min,T_{issue},T}] = e^{-rT}\hat{E}[s_T - \min(s_{\min,T_{issue},0}, s_{\min,0,T})] \\ p_0 &= e^{-rT}\hat{E}[s_{\max,T_{issue},T} - s_T] = e^{-rT}\hat{E}[\max(s_{\max,T_{issue},0}, s_{\max,0,T}) - s_T] \end{aligned}$$

$s_{\min,T_{issue},0}$ *is deterministic, for simplicity, denoted as :* $s_{\min} = s_0 e^{\sigma k}$ *where* $k = (1/\sigma)\ln(s_{\min}/s_0)$ *is a* ***known const***

$s_{\min,0,T}$ *is stochastic, which is modelled as :* $s_{\min,0,T} = s_0 e^{\sigma w'_T}$ *where* $w'_T = \min_{t\in(0,T)}((1/\sigma)(r-\sigma^2/2)t + z_t)$

$s_{\max,T_{issue},0}$ *is deterministic, for simplicity, denoted as :* $s_{\max} = s_0 e^{\sigma h}$ *where* $h = (1/\sigma)\ln(s_{\max}/s_0)$ *is a* ***known const***

$s_{\max,0,T}$ *is stochastic, which is modelled as :* $s_{\max,0,T} = s_0 e^{\sigma m'_T}$ *where* $m'_T = \max_{t\in(0,T)}((1/\sigma)(r-\sigma^2/2)t + z_t)$

In contrast to barrier option : (1) look back option needs to consider s_{min} and s_{max}, while barrier option does not depends on s_{min} nor s_{max}, because if barrier option fulfilled knock out condition before t=0, then its value is zero, if barrier option fulfilled knock in condition before t=0, then it is just *a vanilla option, and (2) look back option executes double integration in different order, i.e. integrate in z first, followed by integration in m, hence the integration range is simply defined by figure 1, which is different from that in barrier option. The reason for reversing the integration order is that there is no s_T (and hence no z term) inside the max or min function, therefore it is easier to integrate in z first. Lets consider the put look back option.*

$$
\begin{aligned}
p_0 &= e^{-rT}\hat{E}[\max(s_{\max,T_{issue},0}, s_{\max,0,T}) - s_T] \\
&= e^{-rT}\hat{E}[\max(s_0e^{\sigma h}, s_0e^{\sigma m'_T})] - s_0 \\
&= e^{-rT}\int_0^\infty\int_{-\infty}^m \max(s_0e^{\sigma h}, s_0e^{\sigma m}) p_{m'_T,s'_T}(m,z)dzdm - s_0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})[\int_{-\infty}^m p_{m'_T,s'_T}(m,z)dz]dm - s_0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})\left[\int_{-\infty}^m \frac{2(2m-z)}{T\sqrt{2\pi T}} e^{-(2m-z)^2/(2T)} e^{\theta z - \theta^2 T/2} dz\right]dm - s_0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})\left[\int_{-\infty}^m \frac{2(2m-z)}{T\sqrt{2\pi T}} e^{-(z^2-4mz-2\vartheta Tz+4m^2+\vartheta^2T^2)/(2T)} dz\right]dm - s_0 && \text{then completing square in z ...} \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})\left[\int_{-\infty}^m \frac{2(2m-z)}{T\sqrt{2\pi T}} e^{-(z-(2m+\vartheta T))^2)/(2T)} e^{-(4m^2+\vartheta^2T^2-(2m+\vartheta T)^2)/(2T)} dz\right]dm - s_0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})e^{2\vartheta m}\left[\int_{-\infty}^m \frac{2(2m-z)}{T\sqrt{2\pi T}} e^{-(z-(2m+\vartheta T))^2)/(2T)} dz\right]dm - s_0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})e^{2\vartheta m}\left[\int_{z=-\infty}^{z=m} \frac{-2(u\sqrt{T}+\vartheta T)}{T\sqrt{2\pi}} e^{-u^2/2} du\right]dm - s_0 && \text{see remark 1} \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})e^{2\vartheta m}\left[-\int_{z=-\infty}^{z=m} \frac{2u}{\sqrt{2\pi T}} e^{-u^2/2} du - \int_{x=-\infty}^{x=m} \frac{2\vartheta}{\sqrt{2\pi}} e^{-u^2/2} du\right]dm - s_0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})e^{2\vartheta m}\left[-\int_{z=-\infty}^{z=m} \frac{1}{\sqrt{2\pi T}} \underbrace{e^{-v/2}}_{v=u^2} dv - 2\vartheta(N((m-(2m+\vartheta T))/\sqrt{T}) - \underbrace{N((-\infty-(2m+\vartheta T))/\sqrt{T})}_{0})\right]dm - s_0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})e^{2\vartheta m}\left[\frac{2}{\sqrt{2\pi T}} e^{-v/2}\Big|_{z=-\infty}^{z=m} - 2\vartheta N(-(m+\vartheta T)/\sqrt{T})\right]dm - s_0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})e^{2\vartheta m}\left[\frac{2}{\sqrt{2\pi T}} e^{-(m-(2m+\vartheta T))^2/(2T)} - 2\vartheta N(-(m+\vartheta T)/\sqrt{T})\right]dm - s_0 && \text{since gaussian}(-\infty) = 0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})\left[\frac{2}{\sqrt{2\pi T}} e^{2\vartheta m} e^{-(m+\vartheta T)^2/(2T)} - 2\vartheta e^{2\vartheta m} N(-(m+\vartheta T)/\sqrt{T})\right]dm - s_0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})\left[\frac{2}{\sqrt{2\pi T}} e^{-(m-\vartheta T)^2/(2T)} - 2\vartheta e^{2\vartheta m} N(-(m+\vartheta T)/\sqrt{T})\right]dm - s_0 \\
&= e^{-rT}\int_0^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})\underbrace{\left[\frac{2}{\sqrt{2\pi T}} e^{-w'^2/2} - 2\vartheta e^{2\vartheta m} N(w)\right]}_{term\#}dm - s_0 && \text{where} \quad \begin{aligned} w &= -(m+\vartheta T)/\sqrt{T} \\ w' &= +(m-\vartheta T)/\sqrt{T} \end{aligned} \\
&= e^{-rT}\int_0^h \max(s_0e^{\sigma h}, s_0e^{\sigma m})[term\#]dm + e^{-rT}\int_h^\infty \max(s_0e^{\sigma h}, s_0e^{\sigma m})[term\#]dm - s_0 \\
&= e^{-rT}\int_0^h s_0e^{\sigma h}[term\#]dm + e^{-rT}\int_h^\infty s_0e^{\sigma m}[term\#]dm - s_0 \\
&= s_{\max}e^{-rT}\left[\int_0^h \frac{2}{\sqrt{2\pi T}} e^{-w'^2/2} - 2\vartheta e^{2\vartheta m}N(w)dm\right] + s_0e^{-rT}\left[\int_h^\infty \frac{2}{\sqrt{2\pi T}} e^{\sigma m}e^{-w'^2/2} - 2\vartheta e^{\sigma m}e^{2\vartheta m}N(w)dm\right] - s_0 \\
&= s_{\max}e^{-rT}(n_0 - n_1) + s_0e^{-rT}(n_2 - n_3) - s_0
\end{aligned}
$$

where

$$
n_0 = \frac{2}{\sqrt{2\pi T}}\int_0^h e^{-w'^2/2}dm \qquad n_1 = \int_0^h 2\vartheta e^{2\vartheta m}N(w)dm
$$

$$
n_2 = \frac{2}{\sqrt{2\pi T}}\int_h^\infty e^{\sigma m}e^{-w'^2/2}dm \qquad n_3 = \int_h^\infty 2\vartheta e^{(2\vartheta+\sigma)m}N(w)dm
$$

(Remark 1) By putting $u = (z-(2m+\vartheta T))/\sqrt{T}$ *, we have :* $du = dx/\sqrt{T}$ *and*
$2m - z = -u\sqrt{T} - \vartheta T$

Lets further simplify the four terms. Firstly, we have n_0 and n_2, which are integrals of normal.

$$
\begin{aligned}
n_0 &= 2/\sqrt{2\pi T}\times\int_0^h e^{-w'^2/2}dm &&= 2/\sqrt{2\pi}\times\int_{m=0}^{m=h} e^{-w'^2/2}dw' \\
&&&= 2[N(w')]_{m=0}^{m=h} \\
&&&= 2[N((m-\vartheta T)/\sqrt{T})]_{m=0}^{m=h} \\
&&&= 2N((h-\vartheta T)/\sqrt{T})-2N((-\vartheta T)/\sqrt{T}) \\
n_2 &= 2/\sqrt{2\pi T}\times\int_h^\infty e^{\sigma m}e^{-w'^2/2}dm &&= 2/\sqrt{2\pi T}\times\int_h^\infty e^{\sigma m}e^{-(m-\vartheta T)^2/(2T)}dm \\
&&&= 2/\sqrt{2\pi T}\times\int_h^\infty e^{-(m^2-2\vartheta Tm-2\sigma Tm+\vartheta^2T^2)/(2T)}dm \qquad \textit{then completing square in } m \\
&&&= 2/\sqrt{2\pi T}\times\int_h^\infty e^{-(m-(\vartheta+\sigma)T)^2/(2T)}e^{-(-(\vartheta+\sigma)^2T^2+\vartheta^2T^2)/(2T)}dm \\
&&&= 2/\sqrt{2\pi T}\times\int_h^\infty e^{-(m-\tau T)^2/(2T)}e^{(\tau^2-\vartheta^2)T/2}dm \\
&&&= 2e^{rT}/\sqrt{2\pi T}\times\int_h^\infty e^{-(m-\tau T)^2/(2T)}dm \\
&&&= 2e^{rT}/\sqrt{2\pi}\times\int_{m=h}^{m=\infty} e^{-w''^2/2}dw'' \qquad \textit{where} \quad w''=+(m-\tau T)/\sqrt{T} \\
&&&= 2e^{rT}[N(w'')]_{m=h}^{m=\infty} \\
&&&= 2e^{rT}[N((m-\tau T)/\sqrt{T})]_{m=h}^{m=\infty} \\
&&&= 2e^{rT}(1-N((h-\tau T)/\sqrt{T}))
\end{aligned}
$$

Then we have n_1 and n_3, which should proceed with integration by parts.

$$
\begin{aligned}
n_1 &= \int_0^h 2\vartheta e^{2\vartheta m}N(w)dm &&= \int_{m=0}^{m=h} N(w)de^{2\vartheta m} \\
&&&= [e^{2\vartheta m}N(w)]_{m=0}^{m=h}-\int_0^h e^{2\vartheta m}(\partial_w N(w)\partial_m w)dm \\
&&&= [e^{2\vartheta m}N(w)]_{m=0}^{m=h}-\int_0^h e^{2\vartheta m}((1/\sqrt{2\pi}\times e^{-w^2/2})\times(-1/\sqrt{T}))dm \\
&&&= [e^{2\vartheta m}N(w)]_{m=0}^{m=h}+1/\sqrt{2\pi T}\times\int_0^h e^{-((m+\vartheta T)^2-4\vartheta Tm)/(2T)}dm \\
&&&= [e^{2\vartheta m}N(w)]_{m=0}^{m=h}+1/\sqrt{2\pi T}\times\int_0^h e^{-(m-\vartheta T)^2/(2T)}dm \\
&&&= [e^{2\vartheta m}N(w)]_{m=0}^{m=h}+1/\sqrt{2\pi}\times\int_{m=0}^{m=h} e^{-w'^2/2}dw' \\
&&&= [e^{2\vartheta m}N(w)]_{m=0}^{m=h}+[N(w')]_{m=0}^{m=h} \\
&&&= [e^{2\vartheta m}N(-(m+\vartheta T)/\sqrt{T})]_{m=0}^{m=h}+[N((m-\vartheta T)/\sqrt{T})]_{m=0}^{m=h} \\
&&&= e^{2\vartheta h}N(-(h+\vartheta T)/\sqrt{T})-e^0N(-\vartheta T/\sqrt{T})+N((h-\vartheta T)/\sqrt{T})-N(-\vartheta T/\sqrt{T}) \\
&&&= e^{2\vartheta h}N(-(h+\vartheta T)/\sqrt{T})+N((h-\vartheta T)/\sqrt{T})-2N(-\vartheta T/\sqrt{T}) \\
n_3 &= \int_h^\infty 2\vartheta e^{(2\vartheta+\sigma)m}N(w)dm &&= 2\vartheta/(2\vartheta+\sigma)\times\int_{m=h}^{m=\infty} N(w)de^{(2\vartheta+\sigma)m} \\
&&&= 2\vartheta/(2\vartheta+\sigma)\times\left[[e^{(2\vartheta+\sigma)m}N(w)]_{m=h}^{m=\infty}-\int_h^\infty e^{(2\vartheta+\sigma)m}(\partial_w N(w)\partial_m w)dm\right] \\
&&&= 2\vartheta/(2\vartheta+\sigma)\times\left[[e^{(2\vartheta+\sigma)m}N(w)]_{m=h}^{m=\infty}-\int_h^\infty e^{(2\vartheta+\sigma)m}((1/\sqrt{2\pi}\times e^{-w^2/2})\times(-1/\sqrt{T}))dm\right] \\
&&&= 2\vartheta/(2\vartheta+\sigma)\times\left[[e^{(2\vartheta+\sigma)m}N(w)]_{m=h}^{m=\infty}+1/\sqrt{2\pi T}\times\int_h^\infty e^{-((m+\vartheta T)^2-2(2\vartheta+\sigma)Tm)/(2T)}dm\right] \\
&&&= 2\vartheta/(2\vartheta+\sigma)\times\left[[e^{(2\vartheta+\sigma)m}N(w)]_{m=h}^{m=\infty}+1/\sqrt{2\pi T}\times\int_h^\infty e^{-(m^2-2(\vartheta+\sigma)Tm+\vartheta^2T^2)/(2T)}dm\right] \\
&&&= 2\vartheta/(2\vartheta+\sigma)\times\left[[e^{(2\vartheta+\sigma)m}N(w)]_{m=h}^{m=\infty}+1/\sqrt{2\pi T}\times\int_h^\infty e^{-(m-\tau T)^2/(2T)}e^{-(-\tau^2T^2+\vartheta^2T^2)/(2T)}dm\right] \\
&&&= 2\vartheta/(2\vartheta+\sigma)\times\left[[e^{(2\vartheta+\sigma)m}N(w)]_{m=h}^{m=\infty}+e^{rT}/\sqrt{2\pi T}\times\int_h^\infty e^{-(m-\tau T)^2/(2T)}dm\right] \\
&&&= 2\vartheta/(2\vartheta+\sigma)\times\left[[e^{(2\vartheta+\sigma)m}N(w)]_{m=h}^{m=\infty}+e^{rT}/\sqrt{2\pi}\times\int_{m=h}^{m=\infty} e^{-w''^2/2}dw''\right] \\
&&&= 2\vartheta/(2\vartheta+\sigma)\times[[e^{(2\vartheta+\sigma)m}N(w)]_{m=h}^{m=\infty}+e^{rT}[N(w'')]_{m=h}^{m=\infty}] \\
&&&= 2\vartheta/(2\vartheta+\sigma)\times[[e^{(2\vartheta+\sigma)m}N(-(m+\vartheta T)/\sqrt{T})]_{m=h}^{m=\infty}+e^{rT}[N((m-\tau T)/\sqrt{T})]_{m=h}^{m=\infty}] \\
&&&= 2\vartheta/(2\vartheta+\sigma)\times\left[\begin{array}{l}\lim\limits_{m\to\infty} e^{(2\vartheta+\sigma)m}N(-(m+\vartheta T)/\sqrt{T}) \\ -e^{(2\vartheta+\sigma)h}N(-(h+\vartheta T)/\sqrt{T})+e^{rT}(1-N((h-\tau T)/\sqrt{T}))\end{array}\right]
\end{aligned}
$$

since

(1) $1 - N(x) = N(-x)$

(2)
$$\lim_{m\to\infty} e^{(2\vartheta+\sigma)m} N(-(m+\vartheta T)/\sqrt{T}) = \lim_{w\to-\infty} e^{-(2\vartheta+\sigma)(\sqrt{T}w+\vartheta T)} N(w) \qquad \textit{recall } w = -(m+\vartheta T)/\sqrt{T}\textit{, thus } m = -\sqrt{T}w - \vartheta T$$
$$= e^{-(2\vartheta+\sigma)\vartheta T} \lim_{w\to-\infty} e^{-(2\vartheta+\sigma)\sqrt{T}w} N(w)$$
$$= e^{-(2\vartheta+\sigma)\vartheta T} \lim_{w\to-\infty} N(w)/e^{kw} \qquad \textit{where } k = (2\vartheta+\sigma)\sqrt{T}$$
$$= e^{-(2\vartheta+\sigma)\vartheta T} \lim_{w\to-\infty} (2\pi)^{-1} e^{-w^2/2}/(ke^{kw}) \qquad \textit{applying L Hospital rule}$$
$$= e^{-(2\vartheta+\sigma)\vartheta T} \lim_{w\to-\infty} (2\pi k)^{-1} e^{-(w^2+2kw)/2}$$
$$= e^{-(2\vartheta+\sigma)\vartheta T} \underbrace{\lim_{w\to-\infty} (2\pi k)^{-1} e^{-(w+k)^2/2}}_{0} e^{k^2/2} = 0$$

thus we have : $$n_3 = 2\vartheta/(2\vartheta+\sigma)\times[-e^{(2\vartheta+\sigma)h} N(-(h+\vartheta T)/\sqrt{T}) + e^{rT} N(-(h-\tau T)/\sqrt{T})]$$

For convenience, lets define the following terms. Recall that $h = (1/\sigma)\ln(s_{\max}/s_0)$.

$$(h-\tau T)/\sqrt{T} = (\ln(s_{\max}/s_0) - (r+\sigma^2/2)T)/(\sigma\sqrt{T}) = b_1$$
$$(h-\vartheta T)/\sqrt{T} = (\ln(s_{\max}/s_0) - (r-\sigma^2/2)T)/(\sigma\sqrt{T}) = b_2$$
$$(h+\vartheta T)/\sqrt{T} = (\ln(s_{\max}/s_0) + (r-\sigma^2/2)T)/(\sigma\sqrt{T}) = b_3$$
$$2\vartheta/(2\vartheta+\sigma) = \frac{2(\mathrm{r}-\sigma^2/2)/\sigma}{2(\mathrm{r}-\sigma^2/2)/\sigma+\sigma} = \frac{2\mathrm{r}-\sigma^2}{2\mathrm{r}-\sigma^2+\sigma^2} = 1-\sigma^2/(2r)$$
$$e^{2\vartheta h} = e^{(2/\sigma)(\mathrm{r}-\sigma^2/2)\times(1/\sigma)\ln(s_{\max}/s_0)} = e^a$$
$$e^{(2\vartheta+\sigma)h} = e^{((2/\sigma)(\mathrm{r}-\sigma^2/2)+\sigma)\times(1/\sigma)\ln(s_{\max}/s_0)} = e^{a+\sigma\times(1/\sigma)\ln(s_{\max}/s_0)} = e^{\ln(s_{\max}/s_0)}e^a = (s_{\max}/s_0)e^a$$

where $$a = (2/\sigma^2)(\mathrm{r}-\sigma^2/2)\ln(s_{\max}/s_0)$$

We then have :
$$n_0 = 2N((h-\vartheta T)/\sqrt{T}) - 2N((-\vartheta T)/\sqrt{T}) = 2N(b_2) - 2N(-\vartheta\sqrt{T})$$
$$n_1 = e^{2\vartheta h} N(-(h+\vartheta T)/\sqrt{T}) + N((h-\vartheta T)/\sqrt{T}) - 2N(-\vartheta T/\sqrt{T}) = e^a N(-b_3) + N(b_2) - 2N(-\vartheta\sqrt{T})$$
$$n_2 = 2e^{rT}(1 - N((h-\tau T)/\sqrt{T})) = 2e^{rT} N(-(h-\tau T)/\sqrt{T}) = 2e^{rT} N(-b_1)$$
$$n_3 = 2\vartheta/(2\vartheta+\sigma)\times[-e^{(2\vartheta+\sigma)h} N(-(h+\vartheta T)/\sqrt{T}) + e^{rT} N(-(h-\tau T)/\sqrt{T})]$$
$$= (1-\sigma^2/(2r))\times(-(s_{\max}/s_0)e^a N(-b_3) + e^{rT} N(-b_1))$$
$$= -(1-\sigma^2/(2r))(s_{\max}/s_0)e^a N(-b_3) + (1-\sigma^2/(2r))e^{rT} N(-b_1)$$
$$(n_0 - n_1) = N(b_2) - e^a N(-b_3)$$
$$(n_2 - n_3)s_0 = (1-\sigma^2/(2r))s_{\max}e^a N(-b_3) + (1+\sigma^2/(2r))s_0 e^{rT} N(-b_1)$$

Finally, look back European put option price is found as :

$$p_0 = s_{\max}e^{-rT}(n_0 - n_1) + s_0 e^{-rT}(n_2 - n_3) - s_0$$
$$= s_{\max}e^{-rT}(N(b_2) - e^a N(-b_3)) + (1-\sigma^2/(2r))s_{\max}e^{-rT}e^a N(-b_3) + (1-\sigma^2/(2r))s_0 N(-b_1) - s_0$$
$$= s_{\max}e^{-rT}(N(b_2) - \sigma^2/(2r)\times e^a N(-b_3)) + s_0(1-\sigma^2/(2r))N(-b_1) - s_0$$