Green Function

Differential operator
Differential operator is a function of differentiation operators, which abstracts a series of differentiaion operations into one. It is a helpful mathematical notation. Examples of differential operator include :

(1)
$$\begin{aligned} Lf &= (\partial_{xx} + \partial_{yy} + \partial_{zz})f \\ &= \partial_{xx}f + \partial_{yy}f + \partial_{zz}f \end{aligned}$$

(2)
$$\begin{aligned} Lf &= (\partial_{xx} - \partial_{yy})f \\ &= \partial_{xx}f - \partial_{yy}f \\ &= \partial_{xx}f - \partial_{xy}f + \partial_{yx}f - \partial_{yy}f && \text{(since } \partial_{xy}f = \partial_{yx}f \text{)} \\ &= \partial_x(\partial_x - \partial_y)f + \partial_y(\partial_x - \partial_y)f \\ &= (\partial_x + \partial_y)(\partial_x - \partial_y)f \\ &= (L_1 L_2)f && \text{(where } L = \partial_{xx} - \partial_{yy},\ L_1 = \partial_x + \partial_y \text{ and } L_2 = \partial_x - \partial_y \text{)} \end{aligned}$$

A differetial operator L is said to be linear iff it fulfills :

$$L(\sum f_i) = \sum Lf_i$$

<u>Wave equation example</u>
Solve the following PDE for f(x,t).

$$\begin{aligned} \partial_{tt}f &= \partial_{xx}f \\ 0 &= \partial_{xx}f - \partial_{tt}f \\ &= (\partial_{xx} - \partial_{tt})f \\ &= (\partial_x + \partial_t)(\partial_x - \partial_t)f \end{aligned}$$

If we substitute $(x',t') = (x+t, x-t)$, this transformation is equivalent to rotating (x,t) space by π/4 and scaling by √2.

$$\begin{aligned} &\partial_{x'}f &&= \partial_x f \partial_{x'}x + \partial_t f \partial_{x'}t &&= \partial_x f \times 1 + \partial_t f \times 1 &&= (\partial_x + \partial_t)f \\ \text{and}\quad &\partial_{t'}f &&= \partial_x f \partial_{t'}x + \partial_t f \partial_{t'}t &&= \partial_x f \times 1 + \partial_t f \times (-1) &&= (\partial_x - \partial_t)f \end{aligned}$$

Hence we have :

$$\begin{aligned} &\partial_{x'}\partial_{t'}f &&= 0 \\ \Rightarrow\quad &f(x',t') &&= c_1(t') + c_2(x') \\ \Rightarrow\quad &f(x,t) &&= c_1(x-t) + c_2(x+t) \end{aligned}$$

Assume that the initial velocity is $\partial_t f(x,0) = 0$ and initial displacement is $h(x)$, then we have :

$$\begin{aligned} &(\partial_t c_1(x-t) + \partial_t c_2(x+t))|_{t=0} &&= 0 &&\text{and} && c_1(x) + c_2(x) &&= h(x) \\ \Rightarrow\quad &(-\partial_z c_1(z) + \partial_z c_2(z))|_{z=x} &&= 0 &&\text{and} && c_1(z) + c_2(z) &&= h(z) \\ \Rightarrow\quad &c_1(z) - c_2(z) &&= const &&\text{and} && c_1(z) + c_2(z) &&= h(z) \end{aligned}$$

By adding the two equations, we have c_1, and subtracting the two equations, we have c_2.

$$\begin{aligned} c_1(z) &= (h(z) + const)/2 \\ c_2(z) &= (h(z) - const)/2 \end{aligned}$$

Finally, we have :

$$\begin{aligned} f(x,t) &= c_1(x-t) + c_2(x+t) \\ &= (h(x-t) + h(x+t))/2 \end{aligned}$$

Intuitively, it is saying that : the solution to wave equation is the sum of a left moving wave and a right moving wave. This is an example of Lagrange D'Alembert's principle.

Inner product and orthogonality of function
Inner product and orthogonality of function are defined in a similar way as vector. Unlike correlation and convolution, inner product (also called dot product) returns a value.

$$< f(x), g(x) > \quad = \quad \int_{-\infty}^{+\infty} f(x)g(x)dx$$

If the inner product is zero, then the two functions are said to be orthogonal.

$$< f(x), g(x) > \quad = \quad \int_{-\infty}^{+\infty} f(x)g(x)dx \quad = \quad 0 \quad \Leftrightarrow \quad f(x) \perp g(x)$$

A function is said to be normalized if inner product between two copies of itself is 1, denoted by :

$$| g(x) > \quad = \quad \frac{g(x)}{< g(x), g(x) >} \quad = \quad \frac{g(x)}{\int_{-\infty}^{+\infty} g(x)g(x)dx}$$

Inner product of a function f(x) with a normalized funcion g(x) is :

$$< f(x) | g(x) > \quad = \quad \int_{-\infty}^{+\infty} f(x) | g(x) > dx \quad = \quad \frac{\int_{-\infty}^{+\infty} f(x)g(x)dx}{\int_{-\infty}^{+\infty} g(x)g(x)dx}$$

Eigen value and eigen function
Eigen value and eigen function of a linear differential operator L are defined as values and functions satisfying :

$$L | f > \quad = \quad \lambda | f > \qquad \text{i.e. eigen functions are normalized}$$

Suppose there are N eigen values and eigen functions, then the N eigen functions must form an orthogonal basis set.

i.e. $\quad L | f_n > \quad = \quad \lambda_n | f_n > \qquad \forall n \in [1, N]$

$\Rightarrow \quad < f_n | f_m > \quad = \quad \begin{bmatrix} 1 & n = m \\ 0 & n \neq m \end{bmatrix}$

$\Rightarrow \quad < f_n | f_m > \quad = \quad \delta_{n,m}$

Differential operator L is called Hermitian if it has no zero eigen value.

Green function
Given (1) a linear differential operator L and (2) a set of boundary conditions, then Green function is defined as the solution to the following differential equation (ODE if there is one variable, PDE if there are more than one variables) :

$$LG(x, x_0) \quad = \quad \delta(x - x_0)$$
$$LG(\vec{x}, \vec{x}_0) \quad = \quad \delta(\vec{x} - \vec{x}_0)$$

For example, given a linear differential operator which acts on a function f(x, y, z) :

$$L \quad = \quad -(\partial_{xx} + \partial_{yy} + \partial_{zz})$$

and
$$\partial_x f(x, y, z)|_{x=0} \quad = \quad \partial_x f(x, y, z)|_{x=X} \quad = \quad c_1(y, z)$$
$$\partial_y f(x, y, z)|_{y=0} \quad = \quad \partial_y f(x, y, z)|_{y=Y} \quad = \quad c_2(x, z)$$
$$\partial_z f(x, y, z)|_{z=0} \quad = \quad \partial_z f(x, y, z)|_{z=Z} \quad = \quad c_3(x, y) \quad \ldots \text{ etc}$$

are boundary conditions, then Green function is defined as the solution to the following PDE while satisfying all the above boundary conditions :

$$-(\partial_{xx} + \partial_{yy} + \partial_{zz}) f(x, x_0, y, y_0, z, z_0) \quad = \quad \delta(x - x_0, y - y_0, z - z_0)$$

In physics, f(x,y,z,t) is the quantity of interest, such as displacement, electric potential, magnetic field, heat etc, which are usually spatial dependent and time dependent. The laws of physics tells us about how the quantity of interest behaves according to a differential equation, such as :

$$Lf(x, y, z, t) \quad = \quad s(x, y, z, t)$$

where s(x,y,z,t) is known as the source, when s is a delta function, then f becomes the Green function. In other words, Green function characterizes how the system response to point source. Since any source can be written as a sum (or integral) of point sources, once we know how a system responses to a point source, we can find out how the system responses to any source by summing up (or integrating) individual contributions, i.e. convolution of Green function with source distribution. Thus, the **Green function** for linear differential operator plays a very similar role as the **impulse response** in 'signal and system'. Once we know the impulse response of a system, we can find out how the system responses to an input signal by convolution of the impulse response with the input signal. Lets prove it, we assume the differential operator to be linear in the proof. Given that :

$$LG(x, x_0) = \delta(x - x_0) \qquad \text{(for point source)}$$

and

$$Lf(x) = s(x) \qquad \text{(for any source)}$$

find $f(x)$ in terms of $G(x)$.

<u>Proof 1</u>

$$\begin{aligned}
Lf(x) &= s(x) \\
&= \int_{-\infty}^{+\infty} s(x_0)\delta(x - x_0)dx_0 && \text{(since } (s*\delta)(x) = s(x)\text{, property of delta function)} \\
&= \int_{-\infty}^{+\infty} s(x_0)LG(x, x_0)dx_0 \\
&= L\int_{-\infty}^{+\infty} s(x_0)G(x, x_0)dx_0 && \text{(linear differential operator)} \\
f(x) &= \int_{-\infty}^{+\infty} s(x_0)G(x, x_0)dx_0
\end{aligned}$$

<u>Proof 2</u>

Plug in the answer :

$$\begin{aligned}
Lf(x) &= L\int_{-\infty}^{+\infty} s(x_0)G(x, x_0)dx_0 \\
&= \int_{-\infty}^{+\infty} s(x_0)LG(x, x_0)dx_0 && \text{(linear differential operator on x only, it doesn't care about } x_0\text{)} \\
&= \int_{-\infty}^{+\infty} s(x_0)\delta(x - x_0)dx_0 \\
&= s(x)
\end{aligned}$$

How to find Green function?

Green function can be obtained with the help of eigen functions and eigen values, this method is applicable only to Hermitian linear differential operator. With Hermitian linear differential operator, we can apply the spectral theorem : any function satisfying the differential equation LG = δ and the given boundary conditions can be written as a sum of eigen functions of the linear differential operator L. Suppose we have solved for the set of eigen functions and eigen values for L :

$$L|\phi_n(x)\rangle = \lambda_n|\phi_n(x)\rangle$$

then we write both the green function and delta function as sum of eigen functions,

i.e.

$$\begin{aligned}
LG(x, x_0) &= \delta(x - x_0) \\
|G(x, x_0)\rangle &= \sum g_n|\phi_n(x)\rangle && \text{(equation 1)} \\
|\delta(x - x_0)\rangle &= \sum d_n|\phi_n(x)\rangle && \text{(equation 2)}
\end{aligned}$$

where g_n and d_n are constant and x – independent. Lets solve for d_n.

$$\begin{aligned}
d_n &= \sum_m d_m\delta_{nm} \\
&= \sum_m d_m\langle\phi_n(x)|\phi_m(x)\rangle && \text{(property of eigen function : } \langle\phi_n(x)|\phi_m(x)\rangle = \delta_{n,m}\text{)} \\
&= \sum_m d_m\langle\phi_n(x)|\phi_m(x)\rangle \\
&= \langle\phi_n(x)|\sum_m d_m|\phi_m(x)\rangle && \text{(moving summation inwards)} \\
&= \langle\phi_n(x)|\delta(x - x_0)\rangle && \text{(using equation 2)} \\
&= \int_{-\infty}^{+\infty}\phi_n^*(x)\delta(x - x_0)dx && (\phi_n^*(x) \text{ means conjugate of } \phi_n(x)) \\
&= \phi_n^*(x_0)
\end{aligned}$$

Note : δ_{nm} is delta function in discrete domain.
$\delta(x)$ is delta function in continuous domain.

Then we solve for g_n.

$$
\begin{aligned}
LG(x,x_0) &= \delta(x-x_0) \\
L\sum g_n \,|\phi_n(x)> &= \sum d_n \,|\phi_n(x)> && \text{(using equation 1 and 2)} \\
\sum g_n L\,|\phi_n(x)> &= \sum d_n \,|\phi_n(x)> \\
\sum g_n \lambda_n \,|\phi_n(x)> &= \sum d_n \,|\phi_n(x)> \\
\Rightarrow \qquad g_n\lambda_n &= d_n && \text{(since } \phi_n(x) \text{ form orthogonal basis)} \\
g_n &= d_n / \lambda_n \\
&= \phi_n^*(x_0)/\lambda_n && \text{(using the result } d_n = \phi_n^*(x_0)\text{)}
\end{aligned}
$$

Hence the Green function is given by :

$$
\begin{aligned}
|G(x,x_0)> &= \sum g_n \,|\phi_n(x)> \\
&= \frac{1}{\lambda_n}\sum \phi_n^*(x_0)\,|\phi_n(x)> \\
G(x,x_0) &= \frac{1}{\lambda_n}\sum \phi_n^*(x_0)\phi_n(x)
\end{aligned}
$$

Green function is a linear combination of eigen function.