Gamma Function and Beta Function

This document introduces the gamma function, which can be regarded as a continuous (and even complex) extension of the factorial function (i.e. n!). We will then discuss about the Bohr Mollerup Theorem, which plays an important role in the derivation of gamma function. Besides, due to the recursive property of gamma function, it can be applied in solving recursive problems, such as (1) the calculation of volume of N dimensional hypersphere and (2) the gamma distribution, which is the sum of N exponential distributions. Finally, a closely related function, the beta function will be introduced.

Convex function
There are different definitions for convexity, the following are three of them, which will be proved to be equivalent. Suppose $f:(a,b)\to\Re$ (i.e. function with input $x\in(a,b)$), then it is said to be convex if :

Definition 1 : Graphically

$$f((1-\lambda)x+\lambda y)\le(1-\lambda)f(x)+\lambda f(y)\ ,\ \forall x,y\in(a,b)\ \text{ and }\ \forall\lambda\in(0,1)$$

Definition 2 : Difference quotient $\phi(x,y)$

$\phi(x,y)$ is monotonic increasing function $\forall x,y\in(a,b)$, where $\phi(x,y)=\phi(y,x)=\dfrac{f(x)-f(y)}{x-y}$

Definition 3 : Second derivative

$\dfrac{d^2f(x)}{dx^2}\ge 0$, $\forall x\in(a,b)$, if f is twice differentiable

<u>Derive from definition 1 to definition 2</u>
Suppose $x,y\in(a,b)$, $x<y$ and $x+\Delta x\le y$

$$\begin{aligned}
f(x+\Delta x) &\le (1-\lambda)f(x)+\lambda f(y) && \text{where } \lambda=\frac{\Delta x}{y-x} \text{ and } 1-\lambda=\frac{y-(x+\Delta x)}{y-x}\\
f(y)-f(x+\Delta x) &\ge f(y)-((1-\lambda)f(x)+\lambda f(y)) && \text{taking negative on both sides}\\
\frac{f(y)-f(x+\Delta x)}{y-(x+\Delta x)} &\ge \frac{(1-\lambda)(f(y)-f(x))}{y-(x+\Delta x)}\\
\frac{f(y)-f(x+\Delta x)}{y-(x+\Delta x)} &\ge \frac{(f(y)-f(x))}{y-(x+\Delta x)}\frac{y-(x+\Delta x)}{y-x}\\
\frac{f(y)-f(x+\Delta x)}{y-(x+\Delta x)} &\ge \frac{f(y)-f(x)}{y-x}\\
\phi(x+\Delta x,y) &\ge \phi(x,y) && \Rightarrow \text{difference quotient is an increasing function of x}\\
\text{similarly,}\quad \phi(x,y+\Delta y) &\ge \phi(x,y) && \Rightarrow \text{difference quotient is an increasing function of y}
\end{aligned}$$

<u>Derive from definition 2 to definition 3</u>
Since $\phi(x,y)$ is increasing for all $x,y\in(a,b)$, now if $f(x)$ is differentiable, let $y=x+\Delta x$, taking limit $\Delta x\to 0$ we have :

$$\lim_{\Delta x\to 0}\phi(x,x+\Delta x) = \lim_{\Delta x\to 0}\frac{f(x)-f(x+\Delta x)}{x-(x+\Delta x)} = f'(x)$$

Hence $f'(x)$ is increasing if $f(x)$ is differentiable. If $f'(x)$ is also differentiable (i.e. $f(x)$ is twice differentiable), then :

$$f''(x)\ge 0$$

Log convex function
Log convex function is a function that is convex after taking log. If a function is log convex, and if its log is twice differentiable, then we have :

$$\begin{aligned}
& \frac{d^2\log f(x)}{dx^2} \ge 0\\
\Rightarrow\ & \frac{d}{dx}\frac{f'(x)}{f(x)} \ge 0\\
\Rightarrow\ & -\frac{f'(x)f'(x)}{f^2(x)}+\frac{f''(x)}{f(x)} \ge 0\\
\Rightarrow\ & f(x)f''(x)-(f'(x))^2 \ge 0 && \text{(equation 1)}
\end{aligned}$$

Gamma function
Gamma function is a generalization of factorial function to the continuous domain and even the complex domain. Gamma function is defined as an integral in the complex space, however **we consider the real space for simplicity**.

$$\Gamma(z) = \int_0^\infty r^{z-1}e^{-r}dr \quad \text{where } z \in C$$

$$\Gamma(x) = \int_0^\infty r^{x-1}e^{-r}dr \quad \text{where } x \in \Re$$

It is a function that fulfills the following three conditions simultaneously. The first and second conditions are analog to the factorial function for integers.

(condition 1) $\Gamma(1) = 1$
(condition 2) $\Gamma(x+1) = x\Gamma(x)$ i.e. $\Gamma(x-1) = \Gamma(x)/(x-1)$
(condition 3) $\Gamma(x)$ is log convex $\forall x \in (0,\infty]$ and piecewisely $\forall x \in (n-1,n)$ where $n \in \aleph$ (integer), s.t. $n \le 0$.

In fact, as stated by Bohr Mullerup Theorem, the gamma function is the only function that fulfills all the above three conditions. We will look into the proof of Bohr Mullerup Theorem in next section, before that, lets verify how gamma function fulfills the three conditions in real space (how to extend the idea to complex space?).

<u>proof of condition 1</u>

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty r^{1-1}e^{-r}dr \\
&= \int_0^\infty e^{-r}dr \\
&= [-e^{-r}]_0^\infty \\
&= \frac{1}{e^0} - \frac{1}{e^\infty} \\
&= 1
\end{aligned}$$

<u>proof of condition 2</u>

$$\begin{aligned}
\Gamma(x) &= \int_0^\infty r^{x-1}e^{-r}dr && \forall x \in \Re \\
&= \frac{1}{x}\int_0^\infty e^{-r}dr^x && \forall x \in \Re, x \ne 0 \\
&= \frac{1}{x}\left([r^x e^{-r}]_0^\infty - \int_0^\infty r^x de^{-r}\right) \\
&= \frac{[r^x e^{-r}]_0^\infty}{x} - \frac{\Gamma(x+1)}{x} && \forall x \in \Re, x \ne 0 \text{ (recursive relation is created)}
\end{aligned}$$

$$\begin{aligned}
\left[\frac{r^x e^{-r}}{x}\right]_{r=0}^{r=\infty} &= \lim_{r\to\infty}\frac{r^x e^{-r}}{x} - \lim_{r\to 0}\frac{r^x e^{-r}}{x} \\
&= \lim_{r\to\infty}\frac{r^x e^{-r}}{x} - \lim_{r\to 0}\frac{0^x e^{-0}}{x} \\
&= \lim_{r\to\infty}\frac{r^x}{xe^r} \\
&= \lim_{r\to\infty}\frac{xr^{x-1}}{xe^r} && \text{Applying L Hospital rule} \\
&= \lim_{r\to\infty}\frac{(x-1)r^{x-2}}{e^r} && \text{Applying L Hospital rule} \\
&= \lim_{r\to\infty}\frac{(x-1)(x-2)r^{x-3}}{e^r} && \text{Applying L Hospital rule} \\
&= \ldots \\
&= \lim_{r\to\infty}\frac{(x-1)!}{e^r} \\
&= 0
\end{aligned}$$

$$\Rightarrow \quad \Gamma(x) = \frac{\Gamma(x+1)}{x} \qquad \forall x \in \Re, x \ne 0, x \ne -1, x \ne -2, \ldots etc$$

$$\Gamma(x+1) = x\Gamma(x)$$

proof of condition 3 (method 1 - failed)

We try to show that gamma is log convex using equation 1 in page 1, i.e. proving $\Gamma(x)\Gamma''(x)-(\Gamma'(x))^2 \ge 0$

$$
\begin{aligned}
\Gamma(x) &= \int_0^\infty r^{x-1}e^{-r}dr \\
\Gamma'(x) &= \int_0^\infty (dr^{x-1}/dx)e^{-r}dr && \text{(from remark 1)} \\
&= \int_0^\infty r^{x-1}e^{-r}(\ln r)dr && \text{(from remark 2)} \\
\Gamma''(x) &= \int_0^\infty r^{x-1}e^{-r}(\ln r)^2 dr
\end{aligned}
$$

$$
\begin{aligned}
& \Gamma(x)\Gamma''(x)-(\Gamma'(x))^2 \\
= & \left(\int_0^\infty r^{x-1}e^{-r}dr\right)\times\left(\int_0^\infty r^{x-1}e^{-r}(\ln r)^2 dr\right)-\left(\int_0^\infty r^{x-1}e^{-r}(\ln r)dr\right)\times\left(\int_0^\infty r^{x-1}e^{-r}(\ln r)dr\right) \\
= & \int_0^\infty\int_0^\infty (r_1r_2)^{x-1}e^{-(r_1+r_2)}(\ln r_1)^2 dr_1dr_2-\int_0^\infty\int_0^\infty (r_1r_2)^{x-1}e^{-(r_1+r_2)}(\ln r_1)(\ln r_2)dr_1dr_2 \\
= & \int_0^\infty\int_0^\infty (r_1r_2)^{x-1}e^{-(r_1+r_2)}((\ln r_1)^2-\ln r_1\ln r_2)dr_1dr_2 \\
= & \ldots \qquad \text{(however, it cannot proceed, so lets try method 2)}
\end{aligned}
$$

Remark 1 :

$$
\begin{aligned}
\frac{d}{dx}\int_{u_0}^{u_1} f(x,u)du &= \lim_{\Delta x\to 0}\frac{\int_{u_0}^{u_1} f(x+\Delta x,u)du-\int_{u_0}^{u_1} f(x,u)du}{\Delta x} && \text{(known as Leibniz rule)} \\
&= \lim_{\Delta x\to 0}\frac{\int_{u_0}^{u_1} (f(x+\Delta x,u)-f(x,u))du}{\Delta x} \\
&= \lim_{\Delta x\to 0}\int_{u_0}^{u_1}\frac{f(x+\Delta x,u)-f(x,u)}{\Delta x}du \\
&= \int_{u_0}^{u_1} f'(x,u)du && \boxed{\frac{d}{dx}\int_{u_0}^{u_1} f(x,u)du = \int_{u_0}^{u_1} f'(x,u)du}
\end{aligned}
$$

Remark 2 : Find $\frac{d}{dx}c^{f(x)}$. Lets consider $\frac{d}{dx}\ln c^{f(x)}$:

$$
\begin{aligned}
\frac{d}{dx}\ln c^{f(x)} &= \frac{d}{dx}f(x)\ln c \\
\frac{1}{c^{f(x)}}\frac{dc^{f(x)}}{dx} &= f'(x)\ln c \\
\frac{d}{dx}c^{f(x)} &= c^{f(x)}f'(x)\ln c && \boxed{\frac{d}{dx}c^{f(x)} = c^{f(x)}f'(x)\ln c}
\end{aligned}
$$

proof of condition 3 (method 2)

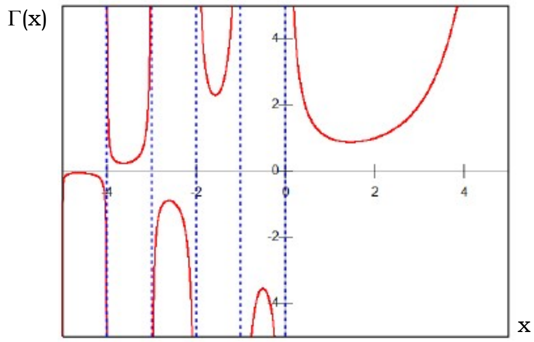
We then try to show that gamma is log convex using definition 1 of convexity in page 1. Consider :

$$
\begin{aligned}
\ln\Gamma((1-\lambda)x+\lambda y) &= \ln\Gamma((1-\lambda)x+\lambda y) && \text{for } \lambda\in[0,1] \\
&= \ln\int_0^\infty r^{(1-\lambda)x+\lambda y-1}e^{-r}dr \\
&= \ln(\int_0^\infty r^{(1-\lambda)x-(1-\lambda)+\lambda y-\lambda}e^{-r+r\lambda-r\lambda}dr) \\
&= \ln(\int_0^\infty r^{(x-1)(1-\lambda)+(y-1)\lambda}e^{-r(1-\lambda)-r\lambda}dr) \\
&= \ln(\int_0^\infty r^{(x-1)(1-\lambda)}e^{-r(1-\lambda)}r^{(y-1)\lambda}e^{-r\lambda}dr) \\
&= \ln((\int_0^\infty (r^{x-1}e^{-r})^{1-\lambda}(r^{y-1}e^{-r})^{\lambda}dr)) \\
&\le \ln((\int_0^\infty r^{x-1}e^{-r}dr)^{1-\lambda}(\int_0^\infty r^{y-1}e^{-r}dr)^{\lambda}) && \text{(from remark 3 : holder's inequality)} \\
&= \ln(\Gamma^{1-\lambda}(x)\Gamma^{\lambda}(y)) \\
&= (1-\lambda)\ln\Gamma(x)+\lambda\ln\Gamma(y)
\end{aligned}
$$

Remark 3 : Holder's inequality states that $\int_{x_0}^{x_1} f(x)^a g(x)^b dx \le \left(\int_{x_0}^{x_1} f(x)dx\right)^a \times\left(\int_{x_0}^{x_1} g(x)dx\right)^b$.

Conclusion

From proof 3, given that if $\Gamma(x)$ is valid within range $x\in[x_0,x_1]$, then $\Gamma(x)$ is log convex $\forall x\in[x_0,x_1]$. From proof 2, $\Gamma(x)$ is invalid when $x=0,-1,-2,-3,\ldots$, thus $\Gamma(x)$ is piecewise log convex within range $x\in(0,\infty]$ or $x\in(-1,0)$ or $x\in(-2,-1)$ and so on. Here is a plot of the gamma function for real number $x\in\Re$. Please refer to wiki for gamma function in complex space.



Consider the integer case :

$$\begin{array}{lclcl} \Gamma(1) & = & 1 & = & 0! \\ \Gamma(2) & = & \Gamma(1)\times 1 & = & 1! \\ \Gamma(3) & = & \Gamma(2)\times 2 & = & 2! \\ \Gamma(4) & = & \Gamma(3)\times 3 & = & 3! \end{array}$$

Bohr Mullerup Theorem

We have proved that gamma function fulfills all the three conditions. Bohr Mullerup Theorem further states that the only function that can fulfill all three conditions is the gamma function. Here is the proof, which firstly considers the range $x\in(0,1]$, and we will extend the proof to all real x (except non positive integer x) in the next part of the proof.

<u>Part 1 : consider range $x\in(0,1]$</u>

Let $x\in(0,1]$ and $n\in\aleph$ (integer) s.t. $n\geq 2$. Suppose $f(\lambda)$ is a function that fulfills the 3 conditions, i.e. $f(\lambda+1)=\lambda f(\lambda)$, $f(1)=1$ and $f(\lambda)$ is log convex for $\lambda>0$. We are going to prove that $f(\lambda)$ is unique and it is the gamma function. Using definition 2 of convex function, the difference quotients $\phi(\lambda, n)$ is an increasing function of λ. Thus we have :

$$\begin{array}{llllll} & \lambda_1 & = & n-1 & & \text{recall : } n\in\aleph \text{ s.t. } n\geq 2 \\ & \lambda_2 & = & n+x & & \text{recall : } x\in(0,1] \\ & \lambda_3 & = & n+1 & & \\ & \lambda_1 & < & \lambda_2 & \leq & \lambda_3 \\ \Rightarrow & \phi(\lambda_1,n) & \leq & \phi(\lambda_2,n) & \leq & \phi(\lambda_3,n) \end{array}$$

Since $n\in\aleph$, thus $f(n+1)=n!$, we have :

$$\begin{array}{lllll} \dfrac{\ln f(\lambda_1)-\ln f(n)}{\lambda_1-n} & \leq & \dfrac{\ln f(\lambda_2)-\ln f(n)}{\lambda_2-n} & \leq & \dfrac{\ln f(\lambda_3)-\ln f(n)}{\lambda_3-n} \\ \dfrac{\ln f(n-1)-\ln f(n)}{n-1-n} & \leq & \dfrac{\ln f(n+x)-\ln f(n)}{n+x-n} & \leq & \dfrac{\ln f(n+1)-\ln f(n)}{n+1-n} \\ \ln\dfrac{f(n)}{f(n-1)} & \leq & \dfrac{\ln f(n+x)-\ln f(n)}{x} & \leq & \ln\dfrac{f(n+1)}{f(n)} \\ \ln(n-1) & \leq & \dfrac{\ln f(n+x)-\ln(n-1)!}{x} & \leq & \ln n \\ x\ln(n-1)+\ln(n-1)! & \leq & \ln f(n+x) & \leq & x\ln n+\ln(n-1)! \qquad \text{since } x\in(0,1]\text{, i.e. } x>0 \\ \ln((n-1)^x(n-1)!) & \leq & \ln f(n+x) & \leq & \ln(n^x(n-1)!) \\ (n-1)^x(n-1)! & \leq & f(n+x) & \leq & n^x(n-1)! \\ (n-1)^x(n-1)! & \leq & f(x)x(x+1)(x+2)...(x+n-1) & \leq & n^x(n-1)! \\ \dfrac{(n-1)^x(n-1)!}{x(x+1)(x+2)...(x+n-1)} & \leq & f(x) & \leq & \dfrac{n^x(n-1)!}{x(x+1)(x+2)...(x+n-1)} = \dfrac{n^x n!}{x(x+1)(x+2)...(x+n)}\dfrac{x+n}{n} \end{array}$$

$$\begin{array}{lllllll} \Rightarrow & & f(x) & \geq & \dfrac{(n-1)^x(n-1)!}{x(x+1)(x+2)...(x+n-1)} = \dfrac{m^x m!}{x(x+1)(x+2)...(x+m)} & m\in\aleph \text{ s.t. } m\geq 1 & \text{(inequality 1)} \\ & \text{and} & f(x) & \leq & \dfrac{n^x n!}{x(x+1)(x+2)...(x+n)}\dfrac{x+n}{n} & n\in\aleph \text{ s.t. } n\geq 2 & \text{(inequality 2)} \end{array}$$

$$\begin{array}{lllllll} \Rightarrow & & f(x) & \geq & \dfrac{n^x n!}{x(x+1)(x+2)...(x+n)} \quad \text{(change of notation)} & n\in\aleph \text{ s.t. } n\geq 1 & \text{(inequality 1)} \\ & \text{and} & \dfrac{nf(x)}{x+n} & \leq & \dfrac{n^x n!}{x(x+1)(x+2)...(x+n)} \quad \text{(moving } \dfrac{x+n}{n} \text{ to LHS)} & n\in\aleph \text{ s.t. } n\geq 2 & \text{(inequality 2)} \end{array}$$

Consider $n \geq 2$, both inequality 1 and inequality 2 hold, combining together, we have :

$$\frac{nf(x)}{x+n} \leq \frac{n^x n!}{x(x+1)(x+2)...(x+n)} \leq f(x)$$

Taking limit n tends to infinity, we have :

$$\lim_{n\to\infty} \frac{nf(x)}{x+n} = f(x)$$
$$\lim_{n\to\infty} f(x) = f(x)$$

By sandwich rule, we have :

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)(x+2)...(x+n)} \qquad \text{(equation 2)}$$

which is unique. In other words, if f(x) is a function that fulfulls the three conditions, then it is unique, and it equals to the infinite product as stated in RHS of equation 2. Now, gamma function is a function that fulfills the three conditions, hence gamma function is the unique function that fulfills the three conditions, and we have :

$$\Gamma(x) = \int_0^\infty r^{x-1}e^{-r}dr \qquad \text{(definition 1 – Euler integral)}$$
$$= \lim_{n\to\infty} \frac{n^x n!}{x(x+1)(x+2)...(x+n)} \qquad \text{(definition 2 – Euler's infinite product)}$$

<u>Part 2 : extend to all real x (except non positive integer x)</u>
Now lets extend the proof for all real x by simple trick. Lets denote :

$$\Gamma(x) = \lim_{n\to\infty} \Gamma_n(x)$$

where $\Gamma_n(x) = \dfrac{n^x n!}{x(x+1)(x+2)...(x+n)}$

$$\Gamma_n(x+1) = \frac{n^{x+1} n!}{(x+1)(x+2)...(x+n)(x+n+1)} = \Gamma_n(x)\frac{nx}{x+n+1}$$

and therefore :

$\Rightarrow \quad \Gamma_n(x+1) = \Gamma_n(x)\dfrac{nx}{x+n+1}$ (equation 3) $\Rightarrow$ if $\lim_{n\to\infty}\Gamma_n(x)$ exists, then $\lim_{n\to\infty}\Gamma_n(x+1)$ exists

$\Rightarrow \quad \Gamma_n(x) = \Gamma_n(x+1)\dfrac{x+n+1}{nx}$ (equation 4) $\Rightarrow$ if $\lim_{n\to\infty}\Gamma_n(x+1)$ exists, then $\lim_{n\to\infty}\Gamma_n(x)$ exists if $x \neq 0$

From part 1, Bohr Mullerup theorem is valid for $x \in (0,1]$ (note that 0 is excluded). By forward induction using equation 3, Bohr Mullerup theorem is valid for all $x > 0$. For backward induction using equation 4, Bohr Mullerup theorem is valid for all $x < 0$ except $x \neq 0$, $x \neq -1$, $x \neq -2$ and so on.

Three definitions for gamma function

$$\Gamma(x) = \int_0^\infty r^{x-1}e^{-r}dr \qquad \text{(definition 1 – Euler integral)}$$
$$= \lim_{n\to\infty} \frac{n^x n!}{x(x+1)(x+2)...(x+n)} \qquad \text{(definition 2 – Euler's infinite product)}$$
$$= \left[xe^{\gamma x}\prod_{n=1}^\infty (1+x/n)e^{-x/n}\right]^{-1} \qquad \text{(definition 3 – Weierstra}\beta\text{'s infinite product)}$$

where $\gamma = \lim_{N\to\infty}(1+1/2+1/3+...+1/N - \ln N)$ (known as Euler's constant)

Proof of definition 3

From defintion of $\Gamma_N(x)$, we have :

$$\begin{aligned}
\Gamma_N(x) &= \frac{N^x N!}{x(x+1)(x+2)...(x+N)} \\
&= \frac{N^x}{x(1+x)(1+x/2)(1+x/3)...(1+x/N)} \\
&= \left[\frac{x\prod_{n=1}^{N}(1+x/n)}{N^x}\right]^{-1} \\
&= \left[\frac{x\prod_{n=1}^{N}e^{x/n}\times\prod_{n=1}^{N}(1+x/n)e^{-x/n}}{N^x}\right]^{-1} \\
&= \left[\frac{xe^{x(1/1+1/2+1/3+...+1/N)}}{e^{\ln N^x}}\times\prod_{n=1}^{N}(1+x/n)e^{-x/n}\right]^{-1} \\
&= \left[xe^{x(1/1+1/2+1/3+...+1/N-\ln N)}\times\prod_{n=1}^{N}(1+x/n)e^{-x/n}\right]^{-1}
\end{aligned}$$

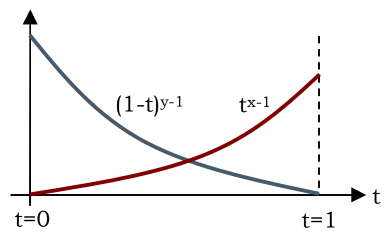
$$\Gamma(x) = \lim_{N\to\infty}\left[xe^{x(1/1+1/2+1/3+...+1/N-\ln N)}\times\prod_{n=1}^{N}(1+x/n)e^{-x/n}\right]^{-1} = \left[xe^{\gamma x}\prod_{n=1}^{\infty}(1+x/n)e^{-x/n}\right]^{-1}$$

where $\gamma = \lim_{N\to\infty}(1+1/2+1/3+...+1/N-\ln N)$

Beta function

Beta function is defined as correlation between t^{x-1} and $(t-1)^{y-1}$ in range $t\in[0,1]$, where $x,y\in\Re$ s.t. $x,y\geq 0$. With different order x and y, a wide varieties of curves can be generated. Besides, beta function is proved to be closely related to gamma function, this is why beta function is useful in different types of modelling.

$$\begin{aligned}
B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}dt \\
&= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\end{aligned}$$



(proof)

$$\begin{aligned}
B(x+1,y) &= \int_0^1 t^x(1-t)^{y-1}dt \\
&= \int_0^1 (t/(1-t))^x(1-t)^{x+y-1}dt \\
&= -\frac{1}{x+y}\int_{t=0}^{t=1}(t/(1-t))^x d(1-t)^{x+y} \\
&= -\frac{1}{x+y}\left[[(t/(1-t))^x(1-t)^{x+y}]_{t=0}^{t=1}-\int_{t=0}^{t=1}(1-t)^{x+y}d(t/(1-t))^x\right] \\
&= -\frac{1}{x+y}\left[[t^x(1-t)^y]_{t=0}^{t=1}-x\int_{t=0}^{t=1}(1-t)^{x+y}(t/(1-t))^{x-1}d(t/(1-t))\right] \\
&= -\frac{1}{x+y}\left[[t^x(1-t)^y]_{t=0}^{t=1}-x\int_{t=0}^{t=1}(1-t)^{x+y}(t/(1-t))^{x-1}(1-t)^{-2}dt\right] \\
&= -\frac{1}{x+y}\left[[t^x(1-t)^y]_{t=0}^{t=1}-x\int_{t=0}^{t=1}t^{x-1}(1-t)^{y-1}dt\right] \\
&= -\frac{1}{x+y}(0-xB(x,y)) \\
B(x+1,y) &= \frac{x}{x+y}B(x,y) \qquad \text{(equation 5)}
\end{aligned}$$

since we have :

$$\begin{aligned}
d(t/(1-t)) &= (1/(1-t)+t/(1-t)^2)dt \\
&= ((1-t+t)/(1-t)^2)dt \\
&= (1-t)^{-2}dt
\end{aligned}$$

Now, lets assume function f(x) to be B(x,y)Γ(x+y)/Γ(y) for a fixed y, hence Γ(y) is simply a constant, and we are going to show that f(x) fulfills the three conditions of gamma function, therefore f(x) is Γ(x).

$$f(x) = B(x,y)\Gamma(x+y)/\Gamma(y) \qquad \text{(given that y is fixed, } \Gamma(y) \text{ is simply a constant)}$$

(condition 1)

$$\begin{aligned} f(1) &= B(1,y)\Gamma(1+y)/\Gamma(y) \\ &= B(1,y)y\Gamma(y)/\Gamma(y) \\ &= \int_0^1 (1-t)^{y-1} dt \times y \\ &= -[(1-t)^y]_0^1 \times (1/y) \times y \\ &= -[(1-t)^y]_0^1 \\ &= 1 \end{aligned}$$

(condition 2)

$$\begin{aligned} f(x) &= B(x,y) \times \frac{\Gamma(x+y)}{\Gamma(y)} \\ &= \frac{x-1}{x+y-1} B(x-1,y) \times (x+y-1) \frac{\Gamma(x+y-1)}{\Gamma(y)} \\ &= (x-1)B(x-1,y) \frac{\Gamma(x+y-1)}{\Gamma(y)} \\ &= (x-1)f(x-1) \end{aligned}$$

(condition 3) If $g(x)$ is convex, $h(x)$ is convex, then $g(x)+h(x)$ is convex.

$$\begin{aligned} & g((1-\lambda)x+\lambda y) && \le && (1-\lambda)g(x)+\lambda g(y) \\ & h((1-\lambda)x+\lambda y) && \le && (1-\lambda)h(x)+\lambda h(y) \\ \Rightarrow\ & g((1-\lambda)x+\lambda y)+h((1-\lambda)x+\lambda y) && \le && (1-\lambda)(g(x)+h(x))+\lambda(g(y)+h(y)) \end{aligned}$$ (QED)

If $g(x)$ is log convex, $h(x)$ is log convex, then $g(x)h(x)$ is log convex.

$$\ln(g(x)h(x)) = \ln(g(x)) + \ln(h(x))$$

since $\ln(g(x))$ is convex, and $\ln(h(x))$ is convex, thus $\ln(g(x)h(x))$ is convex. (QED)

Now $f(x) = B(x,y)\Gamma(x+y)/\Gamma(y)$ is a product of log convex function, hence $f(x)$ is log convex.
Remark : The proof that $B(x,y)$ is log convex is too long, and it is omitted here.

Since $f(x) = B(x,y)\Gamma(x+y)/\Gamma(y)$ fulfills all three conditions of gamma function, and according to Bohr Mullerup theorem, this function is unique, therefore $f(x) = \Gamma(x)$. Finally, we have :

$$\begin{aligned} & \Gamma(x) &&= B(x,y)\Gamma(x+y)/\Gamma(y) \\ \Rightarrow\ & B(x,y) &&= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \end{aligned}$$

Finding Γ(1/2)
Γ(1/2) can be easily found with the help of beta function.

$$\begin{aligned} & \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1/2+1/2)} &&= B(1/2,1/2) \\ \Rightarrow\ & \Gamma^2(1/2) &&= \int_0^1 t^{-1/2}(1-t)^{-1/2} dt \\ & &&= \int_0^{\pi/2} (\sin^2\theta)^{-1/2}(1-\sin^2\theta)^{-1/2} d\sin^2\theta \\ & &&= \int_0^{\pi/2} \frac{1}{\sin\theta\cos\theta} d\sin^2\theta \\ & &&= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} d\theta \\ & &&= \pi \\ & \Gamma(1/2) &&= \sqrt{\pi} \end{aligned}$$

(since $t \in [0,1]$, it is feasible to let $t = \sin^2\theta$)