Gamma distribution

Gamma distribution is a continuous distribution, which is usually used to model the waiting time for the k[th] event in a Poisson process, i.e. it is the sum of k independent exponential distributions having the same event rate. For details of the proof, please refer to the document of Poisson distribution. Gamma distribution can take three different forms :

(1) shape parameter k and scale parameter θ $\quad f(x;k,\theta) = \dfrac{x^{k-1}e^{-x/\theta}}{\theta^k\Gamma(k)} \quad x>0, k>0, \theta>0$

(2) shape parameter k and rate parameter λ = 1/θ $\quad f(x;k,\lambda) = \dfrac{\lambda^k x^{k-1}e^{-\lambda x}}{\Gamma(k)} \quad x>0, k>0, \lambda>0$

(3) shape parameter k and mean parameter μ = kθ = k/λ $\quad f(x;k,\mu) = \dfrac{k^k x^{k-1}e^{-kx/\mu}}{\mu^k\Gamma(k)} \quad x>0, k>0, \mu>0$

where gamma function is defined as : $\quad \Gamma(k) = \int_0^\infty r^{k-1}e^{-r}dr$

Expected value and variance

Here are the derivations of the expected value and the variance for gamma distribution in k and θ. Most derivations are based on the recursive property of gamma function.

$$
\begin{aligned}
E(X) &= \int_0^\infty \frac{x^{k-1}e^{-x/\theta}}{\theta^k\Gamma(k)}x\,dx \\
&= \frac{1}{\theta^k\Gamma(k)}\int_0^\infty x^k e^{-x/\theta}dx \\
&= \frac{1}{\theta^k\Gamma(k)}\int_0^\infty (\theta y)^k e^{-y}\theta dy && \text{(substitute } x/\theta = y\text{)} \\
&= \frac{\theta}{\Gamma(k)}\int_0^\infty y^k e^{-y}dy \\
&= \frac{\Gamma(k+1)}{\Gamma(k)}\theta \\
&= k\theta
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \int_0^\infty \frac{x^{k-1}e^{-x/\theta}}{\theta^k\Gamma(k)}x^2dx \\
&= \frac{1}{\theta^k\Gamma(k)}\int_0^\infty x^{k+1}e^{-x/\theta}dx \\
&= \frac{1}{\theta^k\Gamma(k)}\int_0^\infty (\theta y)^{k+1}e^{-y}\theta dy && \text{(substitute } x/\theta = y\text{)} \\
&= \frac{\theta^2}{\Gamma(k)}\int_0^\infty y^{k+1}e^{-y}dy \\
&= \frac{\Gamma(k+2)}{\Gamma(k)}\theta^2 \\
&= k(k+1)\theta^2
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \\
&= k(k+1)\theta^2 - (k\theta)^2 \\
&= k^2\theta^2 + k\theta^2 - k^2\theta^2 \\
&= k\theta^2
\end{aligned}
$$

Characteristic function

Characterisitic function of gamma distribution is found as the following.

$$
\begin{aligned}
\varphi_X(t) &= E[e^{itX}] \\
&= \int_0^{\infty} \frac{x^{k-1}e^{-x/\theta}}{\theta^k \Gamma(k)} e^{itx}\,dx \\
&= \frac{1}{\theta^k \Gamma(k)} \int_0^{\infty} x^{k-1} e^{-(1/\theta - it)x}\,dx \\
&= \frac{1}{\theta^k \Gamma(k)} \int_0^{\infty} \frac{y^{k-1}}{(1/\theta - it)^k} e^{-y}\,dy \qquad \text{(substitute } (it - 1/\theta)x = y \text{)} \\
&= \frac{1}{\theta^k (1/\theta - it)^k} \frac{\Gamma(k)}{\Gamma(k)} \\
&= (1 - it\theta)^{-k}
\end{aligned}
$$

The characteristic function of gamma distribution is useful in proving that sum of multiple independent exponential distributions with same event rate gives the gamma distribution.