Fourier Transform

Definition of Fourier transform
There are different definitions of Fourier transform, our objective is to derive their corresponding inverse transform. Lets clarify our definition and notations. Functions in time domain (or spatial domain) are denoted by small letter, while functions in frequency domain are denoted by capital letter. Values and vectors in time demain (or spatial domain) are denoted by x and z, while values and vectors in frequency domain are denoted by u and v (including both ordinary frequency or angular frequency). Fourier transform is defined as :

$$FT_{of}[f](u) = F(u) = \int_{-\infty}^{+\infty} f(x)e^{-j2\pi ux}dx \qquad \text{ordinary frequency (unitary)}$$

$$FT_{af}[f](u) = F(u) = \int_{-\infty}^{+\infty} f(x)e^{-jux}dx \qquad \text{angular frequency (non unitary)}$$

$$FT_{uaf}[f](u) = F(u) = \int_{-\infty}^{+\infty} f(x)e^{-jux}dx\frac{1}{\sqrt{2\pi}} \qquad \text{angular frequency (unitary)}$$

These are all 1D Fourier transform, which can be generalized into N dimensional space. Besides, these three versions of Fourier transform differ in the frequency definition (ordinary frequency vs angular frequency) and normalization (unitary vs non unitary). Unitary means the same normalization factor is used in both forward transform and inverse transform, while non unitary means the normalization factor for forward transform and inverse transform are different.

Fourier transform for Gaussian
Fourier transform for Gaussian is also a Gaussian. We can derive a useful formula in terms of Gaussian, which is useful for the proof of inverse Fourier transform in later section. Assume g(x) be a unit Gaussian, g(σx) be a scaled unit Gaussian and $g_\sigma(x)$ be a Gaussian with zero mean and sigma σ.

$$\text{unit Gaussian} \quad g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad \text{where } \int_{-\infty}^{+\infty} g(x)dx = 1$$

$$\text{scaled unit Gaussian} \quad g(\sigma x) = \frac{1}{\sqrt{2\pi}}e^{-(\sigma x)^2/2} \qquad \text{where } \int_{-\infty}^{+\infty} g(\sigma x)dx = 1/\sigma^2$$

$$\text{zero mean Gaussin} \quad g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x/\sigma)^2/2} \qquad \text{where } \int_{-\infty}^{+\infty} g_\sigma(x)dx = 1$$

$$\Rightarrow \quad \lim_{\sigma\to 0} g(\sigma x) = 1/\sqrt{2\pi} \qquad \text{(equation 1)}$$

$$\Rightarrow \quad \lim_{\sigma\to 0} g_\sigma(x) = \delta(x) \qquad \text{(equation 2)}$$

Lets consider the following integral, involving a Gaussian with in ordinary frequency domain scaled by 2πσ.

$$\begin{aligned}
\int_{-\infty}^{+\infty} g(2\pi(\sigma u))e^{j2\pi ux}du &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-(2\pi\sigma u)^2/2}e^{j2\pi ux}du \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(2\pi\sigma u)^2-j4\pi ux}{2}}du \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(2\pi\sigma u)^2-j4\pi ux+(jx/\sigma)^2}{2}}e^{\frac{(jx/\sigma)^2}{2}}du \\
&= e^{\frac{(jx/\sigma)^2}{2}}\times\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(2\pi\sigma u-jx/\sigma)^2}{2}}du\right) \\
&= \frac{1}{2\pi\sigma}e^{\frac{(jx/\sigma)^2}{2}}\times\left(\frac{1}{\sqrt{2\pi}(2\pi\sigma)^{-1}}\int_{-\infty}^{+\infty} e^{-\frac{(u-j(2\pi\sigma)^{-1}x/\sigma)^2}{2(2\pi\sigma)^{-2}}}du\right) \\
&= \frac{1}{2\pi\sigma}e^{-x^2/(2\sigma^2)} \\
&= \frac{1}{\sqrt{2\pi}}g_\sigma(x)
\end{aligned}$$

$$\Rightarrow \quad g_\sigma(x) = \sqrt{2\pi}\int_{-\infty}^{+\infty} g(2\pi\sigma u)e^{j2\pi ux}du \qquad \text{(equation 3)}$$

$$\Rightarrow \quad g_\sigma(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\sigma u)e^{jux}du \qquad \text{(equation 4)} \quad \text{(substitute u' = 2}\pi\text{u)}$$

Later, we will understand that the above equation 3 and 4 are in fact the inverse Fourier transform of Gaussian with scaled frequency (equation 3 for ordinary frequency and equation 4 for angular frequency). At this moment, we simply call them equation 3 and 4, which will be useful in the proof of inverse Fourier transform.

Derivation of inverse Fourier transform
Inverse Fourier tranform is defined as the counter part of Fourier transform, with which the original function can be retrieved in time domain or spatial domain.

$$
\begin{aligned}
FT_{of}^{-1} FT_{of}[f](x) &= f(x) \\
FT_{af}^{-1} FT_{af}[f](x) &= f(x) \\
FT_{afn}^{-1} FT_{afn}[f](x) &= f(x)
\end{aligned}
$$

Inverse Fourier transform can be derived based on dirac delta function. Since delta function is defined as a limit of function sequence, there are many ways to derive inverse Fourier transform. Here I name two ways : (approach 1) using Gaussian as the function sequence (together with equation 3 or 4), and (approach 2) using sinc function as the function sequence (together with Euler formula).

<u>Approach 1 for ordinary frequency Fourier transform</u>
Lets consider the convolution of function f with Gaussian.

$$
\begin{aligned}
(f * g_\sigma)(x) &= \int_{-\infty}^{+\infty} f(x-z) g_\sigma(z) dz \\
&= \sqrt{2\pi} \int_{-\infty}^{+\infty} f(x-z) \int_{-\infty}^{+\infty} g(2\pi\sigma u) e^{j2\pi uz} dudz && \text{(using equation 3)} \\
&= \sqrt{2\pi} \int_{-\infty}^{+\infty} g(2\pi\sigma u) \int_{-\infty}^{+\infty} f(x-z) e^{j2\pi uz} dzdu && \text{(swap integration order)} \\
&= \sqrt{2\pi} \int_{-\infty}^{+\infty} g(2\pi\sigma u) e^{j2\pi ux} \int_{-\infty}^{+\infty} f(x-z) e^{-j2\pi u(x-z)} dzdu && \text{(construct Fourier transform for f)} \\
&= \sqrt{2\pi} \int_{-\infty}^{+\infty} g(2\pi\sigma u) e^{j2\pi ux} \int_{+\infty}^{-\infty} f(z') e^{-j2\pi uz'} (-1) dz'du && \text{(substitute z' = x – z)} \\
&= \sqrt{2\pi} \int_{-\infty}^{+\infty} g(2\pi\sigma u) e^{j2\pi ux} \int_{-\infty}^{+\infty} f(z') e^{-j2\pi uz'} dz'du \\
&= \sqrt{2\pi} \int_{-\infty}^{+\infty} g(2\pi\sigma u) e^{j2\pi ux} F_{of}(u) du \\
\lim_{\sigma\to 0} (f * g_\sigma)(x) &= \lim_{\sigma\to 0} \sqrt{2\pi} \int_{-\infty}^{+\infty} g(2\pi\sigma u) e^{j2\pi ux} F_{of}(u) du \\
(f * \delta)(x) &= \sqrt{2\pi} \int_{-\infty}^{+\infty} (1/\sqrt{2\pi}) e^{j2\pi ux} F_{of}(u) du && \text{(using equation 1)} \\
f(x) &= \int_{-\infty}^{+\infty} e^{j2\pi ux} F_{of}(u) du && \text{(IFT for ordinary frequency)}
\end{aligned}
$$

<u>Approach 1 for angular frequency Fourier transform</u>
Lets consider the convolution of function f with Gaussian.

$$
\begin{aligned}
(f * g_\sigma)(x) &= \int_{-\infty}^{+\infty} f(x-z) g_\sigma(z) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-z) \int_{-\infty}^{+\infty} g(\sigma u) e^{juz} dudz && \text{(using equation 4)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\sigma u) \int_{-\infty}^{+\infty} f(x-z) e^{juz} dzdu && \text{(swap integration order)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\sigma u) e^{jux} \int_{-\infty}^{+\infty} f(x-z) e^{-ju(x-z)} dzdu && \text{(construct Fourier transform for f)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\sigma u) e^{jux} \int_{+\infty}^{-\infty} f(z') e^{-juz'} (-1) dz'du && \text{(substitute z' = x – z)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\sigma u) e^{jux} \int_{-\infty}^{+\infty} f(z') e^{-juz'} dz'du \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\sigma u) e^{jux} F_{af}(u) du \\
\lim_{\sigma\to 0} (f * g_\sigma)(x) &= \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\sigma u) e^{jux} F_{af}(u) du \\
(f * \delta)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (1/\sqrt{2\pi}) e^{jux} F_{af}(u) du && \text{(using equation 1)} \\
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j2\pi ux} F_{af}(u) du && \text{(IFT for angular frequency, non unitary)}
\end{aligned}
$$

Approach 2 for ordinary frequency Fourier transform

Evaluate the following integral :

$$
\begin{aligned}
\int_{-\infty}^{+\infty} e^{j2\pi ux} F(u)du &= \int_{-\infty}^{+\infty} e^{j2\pi ux} \int_{-\infty}^{+\infty} f(z)e^{-j2\pi uz} dzdu \\
&= \int_{-\infty}^{+\infty} f(z) \int_{-\infty}^{+\infty} e^{j2\pi u(x-z)} dudz && \text{(swap integration order)} \\
&= \int_{-\infty}^{+\infty} f(z) \lim_{n\to\infty} \int_{-n}^{n} e^{j2\pi u(x-z)} dudz \\
&= \int_{-\infty}^{+\infty} f(z) \lim_{n\to\infty} \left. \frac{e^{j2\pi u(x-z)}}{j2\pi(x-z)} \right|_{u=-n}^{u=n} dz \\
&= \int_{-\infty}^{+\infty} f(z) \lim_{n\to\infty} \frac{e^{j2\pi n(x-z)} - e^{-j2\pi n(x-z)}}{j2\pi(x-z)} dz \\
&= \int_{-\infty}^{+\infty} f(z) \lim_{n\to\infty} \frac{\sin(2\pi n(x-z))}{\pi(x-z)} dz \\
&= \int_{-\infty}^{+\infty} f(z) \lim_{n'\to\infty} \frac{\sin(n'(x-z))}{\pi(x-z)} dz && \text{(put } n' = 2\pi n\text{)} \\
&= \int_{-\infty}^{+\infty} f(z)\delta(x-z)dz && \text{(since } \lim_{n\to\infty} \frac{\sin(nx)}{\pi x} = \delta(x)\text{)} \\
&= (f * \delta)(x) \\
&= f(x) && \text{(IFT for ordinary frequency)}
\end{aligned}
$$

Approach 2 for angular frequency Fourier transform

Evaluate the following integral :

$$
\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{jux} F(u)du &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{jux} \int_{-\infty}^{+\infty} f(z)e^{-juz} dzdu \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(z) \int_{-\infty}^{+\infty} e^{ju(x-z)} dudz && \text{(swap integration order)} \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(z) \lim_{n\to\infty} \int_{-n}^{n} e^{ju(x-z)} dudz \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(z) \lim_{n\to\infty} \left. \frac{e^{ju(x-z)}}{j(x-z)} \right|_{u=-n}^{u=n} dz \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(z) \lim_{n\to\infty} \frac{e^{jn(x-z)} - e^{-jn(x-z)}}{j(x-z)} dz \\
&= \frac{1}{\pi} \int_{-\infty}^{+\infty} f(z) \lim_{n\to\infty} \frac{\sin(n(x-z))}{(x-z)} dz \\
&= \int_{-\infty}^{+\infty} f(z) \lim_{n\to\infty} \frac{\sin(n(x-z))}{\pi(x-z)} dz \\
&= \int_{-\infty}^{+\infty} f(z)\delta(x-z)dz && \text{(since } \lim_{n\to\infty} \frac{\sin(nx)}{\pi x} = \delta(x)\text{)} \\
&= (f * \delta)(x) \\
&= f(x) && \text{(IFT for ordinary frequency)}
\end{aligned}
$$

Here we summarize the Fourier transform and inverse Fourier transform. Please note that the Fourier transform of a function in x returns a function in u, while the inverse Fourier transform of a function in u returns a function in x.

$$
\begin{aligned}
FT_{of}[f](u) &= \int_{-\infty}^{+\infty} f(x)e^{-j2\pi ux} dx &\Leftrightarrow&& FT_{of}^{-1}[F](x) &= \int_{-\infty}^{+\infty} F(u)e^{+j2\pi ux} du \\
FT_{af}[f](u) &= \int_{-\infty}^{+\infty} f(x)e^{-jux} dx &\Leftrightarrow&& FT_{af}^{-1}[F](x) &= \int_{-\infty}^{+\infty} F(u)e^{+jux} du \frac{1}{2\pi} \\
FT_{uaf}[f](u) &= \int_{-\infty}^{+\infty} f(x)e^{-jux} dx \frac{1}{\sqrt{2\pi}} &\Leftrightarrow&& FT_{uaf}^{-1}[F](x) &= \int_{-\infty}^{+\infty} F(u)e^{+jux} du \frac{1}{\sqrt{2\pi}}
\end{aligned}
$$

Hence no normalization is needed in both forward transform and inverse transform if ordinary frequency is used. For angular frequency, we should normalize either forward transform or inverse transform by a factor of $1/2\pi$, or normalize both forward transform and inverse transform by a factor $1/\sqrt{(2\pi)}$, so as to obtain the original function f(x) after a round trip transformation.

Property of Fourier transform
Only some basic properties are listed here, please refer to wiki for the complete list.

Time shift property

$$FT_{of}[f(x-x_0)] = \int_{-\infty}^{+\infty} f(x-x_0)e^{-j2\pi ux}dx = \int_{-\infty}^{+\infty} f(y)e^{-j2\pi uy}e^{-j2\pi ux_0}dy = e^{-j2\pi ux_0}F(u)$$
$$FT_{af}[f(x-x_0)] = \int_{-\infty}^{+\infty} f(x-x_0)e^{-jux}dx = \int_{-\infty}^{+\infty} f(y)e^{-juy}e^{-jux_0}dy = e^{-jux_0}F(u)$$
$$FT_{uaf}[f(x-x_0)] = \int_{-\infty}^{+\infty} f(x-x_0)e^{-jux}dx\frac{1}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty} f(y)e^{-juy}e^{-jux_0}dy\frac{1}{\sqrt{2\pi}} = e^{-jux_0}F(u)$$

Frequency shift property

$$FT_{of}[f(x)e^{j2\pi u_0 x}] = \int_{-\infty}^{+\infty} f(x)e^{-j2\pi(u-u_0)x}dx = F(u-u_0)$$
$$FT_{af}[f(x)e^{j2\pi u_0 x}] = \int_{-\infty}^{+\infty} f(x)e^{-j(u-u_0)x}dx = F(u-u_0)$$
$$FT_{uaf}[f(x)e^{j2\pi u_0 x}] = \int_{-\infty}^{+\infty} f(x)e^{-j(u-u_0)x}dx\frac{1}{\sqrt{2\pi}} = F(u-u_0)$$

Scale property

$$FT_{of}[f(x/s)] = \int_{-\infty}^{+\infty} f(x/s)e^{-j2\pi ux}dx = \int_{-\infty}^{+\infty} f(y)e^{-j2\pi(su)y}sdy = sF(su)$$
$$FT_{af}[f(x/s)] = \int_{-\infty}^{+\infty} f(x/s)e^{-jux}dx = \int_{-\infty}^{+\infty} f(y)e^{-j(su)y}sdy = sF(su)$$
$$FT_{uaf}[f(x/s)] = \int_{-\infty}^{+\infty} f(x/s)e^{-jux}dx\frac{1}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty} f(y)e^{-j(su)y}sdy\frac{1}{\sqrt{2\pi}} = sF(su)$$

Convolution property

$$\begin{aligned}
FT_{of}[f(x)*g(x)] &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(s)g(x-s)ds\right)e^{-j2\pi ux}dx \\
&= \int_{-\infty}^{+\infty} f(s)\left(\int_{-\infty}^{+\infty} g(x-s)e^{-j2\pi ux}dx\right)ds \\
&= \int_{-\infty}^{+\infty} f(s)FT_{of}[g(x-s)]ds \\
&= \int_{-\infty}^{+\infty} f(s)G(u)e^{-j2\pi us}ds \\
&= \int_{-\infty}^{+\infty} f(s)e^{-j2\pi us}ds\times G(u) \\
&= F(u)\times G(u)
\end{aligned}$$

Recall : $f(x)*g(x) = \int_{-\infty}^{+\infty} f(s)g(x-s)ds$

$$\begin{aligned}
FT_{af}[f(x)*g(x)] &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(s)g(x-s)ds\right)e^{-jux}dx \\
&= \int_{-\infty}^{+\infty} f(s)\left(\int_{-\infty}^{+\infty} g(x-s)e^{-jux}dx\right)ds \\
&= \int_{-\infty}^{+\infty} f(s)FT_{af}[g(x-s)]ds \\
&= \int_{-\infty}^{+\infty} f(s)G(u)e^{-jus}ds \\
&= \int_{-\infty}^{+\infty} f(s)e^{-jus}ds\times G(u) \\
&= F(U)\times G(u)
\end{aligned}$$

$$\begin{aligned}
FT_{uaf}[f(x)*g(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(s)g(x-s)ds\right)e^{-jux}dx \\
&= \int_{-\infty}^{+\infty} f(s)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(x-s)e^{-jux}dx\right)ds \\
&= \int_{-\infty}^{+\infty} f(s)FT_{uaf}[g(x-s)]ds \\
&= \int_{-\infty}^{+\infty} f(s)G(u)e^{-jus}ds \\
&= \int_{-\infty}^{+\infty} f(s)e^{-jus}ds\times G(u) \\
&= \sqrt{2\pi}F(U)\times G(u)
\end{aligned}$$

Remark : differ by a normalization factor

Differentiation property

$$FT_{of}[f_x(x)] = FT_{of}\left[\frac{d}{dx} FT_{of}^{-1} F_{of}(u)\right]$$
$$= FT_{of}\left[\frac{d}{dx}\int_{-\infty}^{+\infty} F_{of}(u)e^{j2\pi ux}du\right]$$
$$= FT_{of}\left[\int_{-\infty}^{+\infty} F_{of}(u)\frac{d}{dx}e^{j2\pi ux}du\right]$$
$$= FT_{of}\left[\int_{-\infty}^{+\infty} F_{of}(u)e^{j2\pi ux}du\right]j2\pi u$$
$$= j2\pi u F_{of}(u)$$

$$FT_{af}[f_x(x)] = juF_{af}(u)$$
$$FT_{uaf}[f_x(x)] = juF_{uaf}(u)$$

Duality property

$$f(x) = \int_{-\infty}^{+\infty} F_{of}(u)e^{j2\pi ux}du$$
$$f(-v) = \int_{-\infty}^{+\infty} F_{of}(u)e^{-j2\pi uv}du \qquad \text{(put } v = -x\text{)}$$
$$= FT_{of}[F_{of}](v)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_{af}(u)e^{jux}du$$
$$f(-v) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_{af}(u)e^{-juv}du \qquad \text{(put } v = -x\text{)}$$
$$= \frac{1}{2\pi}FT_{af}[F_{af}](v)$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F_{uaf}(u)e^{jux}du$$
$$f(-v) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F_{uaf}(u)e^{-juv}du \qquad \text{(put } v = -x\text{)}$$
$$= FT_{uaf}[F_{uaf}](v)$$

Hence we have duality :

$$FT_{of}[F_{of}](v) = f(-v) \qquad \text{(hence this is unitary)}$$
$$FT_{af}[F_{af}](v) = 2\pi f(-v) \qquad \text{(hence this is non unitary)}$$
$$FT_{uaf}[F_{uaf}](v) = f(-v) \qquad \text{(hence this is unitary)}$$

Application : Solving heat equation
Heat equation is a PDE that characterizes how heat propagates. f(x,t) denotes the heat at location x and time t, initial heat at any points is given as g(x). The objective is to solve for f(x,t). By applying Fourier transform on x (not on t) and making use of the differentiation property of Fourier transform, we can convert the PDE (with variables x and t) into ODE (with variable u), which can be easily solved with standard solution.

$$\partial_t f(x,t) = \partial_{xx} f(x,t)$$

and $$f(x,0) = g(x)$$

Solution
Apply Fourier transform (unitary, angular frequency version) on x for both PDE and initial condition :

$$FT_{uaf}[\partial_t f(x,t)](u) = FT_{uaf}[\partial_{xx} f(x,t)](u) \quad \Rightarrow \quad \partial_t F(u,t) = (ju)^2 F(u,t) = -u^2 F(u,t) \qquad \text{(becomes ODE in t)}$$

and $$FT_{uaf}[f(x,0)] = FT_{uaf}[g(x)] \quad \Rightarrow \quad F(u,0) = G(u)$$

The standard solution of ODE is (c is a function of u, since this is an ODE in t only) :

$$F(u,t) = c(u)e^{-u^2 t}$$

(1) Lets check the standard solution.

$$\begin{aligned}\partial_t F(u,t) &= \partial_t c(u)e^{-u^2 t} \\ &= -u^2 c(u)e^{-u^2 t} \\ &= -u^2 F(u,t)\end{aligned}$$

(2) Lets find the value of c in the standard solution.

$$\begin{aligned}F(u,0) &= G(u) \\ c(u)e^0 &= G(u) \\ c(u) &= G(u)\end{aligned}$$

Hence we have :

$$\begin{aligned}F(u,t) &= G(u)e^{-u^2 t} \\ \Rightarrow \quad f(x,t) &= \left[FT_{uaf}^{-1}[G(u)e^{-u^2 t}]\right](x,t) \\ &= \frac{1}{\sqrt{2\pi}}\left[FT_{uaf}^{-1}[G(u)] * FT_{uaf}^{-1}[e^{-u^2 t}]\right](x,t) && \text{(remark : don't forget the factor } \tfrac{1}{\sqrt{2\pi}}\text{)} \\ &= [g * \Phi](x,t) && \text{(remark : inverse Fourier transform of Gaussian)}\end{aligned}$$

Remark : inverse Fourier transform of Gaussian

$$\begin{aligned}\Phi(x,t) &= \frac{1}{\sqrt{2\pi}} FT_{uaf}^{-1}[e^{-u^2 t}] \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-u^2 t} e^{jux} du \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-t(u^2 - jux/t + j^2x^2/(4t^2))} e^{j^2x^2/(4t)} du \\ &= \frac{1}{2\pi}\frac{\sqrt{\pi/t}}{\sqrt{\pi/t}}\int_{-\infty}^{+\infty} e^{-t(u-jx/(2t))^2} e^{j^2x^2/(4t)} du && \text{(substitute } 2\sigma^2 = 1/t \text{ and hence } 2\pi\sigma^2 = \pi/t\text{)} \\ &= \frac{\sqrt{\pi/t}}{2\pi} e^{j^2x^2/(4t)} \\ &= \frac{1}{2\sqrt{\pi t}} e^{-x^2/(4t)} \\ &= Gaussian(mean = 0, sigma = \sqrt{2t})\end{aligned}$$

The intuition of the solution : heat transfer is equivalent to Gaussian smoothing of the initial heat condition and the smoothing filter Φ(x,t) has a spread increases with time square root of t. Φ(x,t) is also known as the heat kernel.

Application : Solving wave equation

Wave equation is a PDE that characterizes how wave propagates. f(x,t) denotes the displacement at location x and time t, initial velocity and initial displacement at any points is given as g(x) and h(x). The objective is to solve for f(x,t). By applying Fourier transform on x (not on t) and making use of the differentiation property of Fourier transform, we can convert the PDE (with variables x and t) into ODE (with variable u), which can be easily solved with standard solution.

$$\partial_{tt} f(x,t) = \partial_{xx} f(x,t)$$

and $$\partial_t f(x,0) = g(x)$$

and $$f(x,0) = h(x)$$

<u>Solution 1</u>

Apply Fourier transform (unitary, angular frequency version) on x for both PDE and initial condition :

$$FT_{uaf}[\partial_{tt} f(x,t)](u) = FT_{uaf}[\partial_{xx} f(x,t)](u) \Rightarrow \partial_{tt} F(u,t) = (ju)^2 F(u,t) = -u^2 F(u,t)$$ (becomes ODE in t)

and $$FT_{uaf}[\partial_t f(x,0)] = FT_{uaf}[g(x)] \Rightarrow \partial_t F(u,0) = G(u)$$

and $$FT_{uaf}[f(x,0)] = FT_{uaf}[h(x)] \Rightarrow F(u,0) = H(u)$$

The standard solution of ODE is (c_1 and c_2 are functions of u, since this is an ODE in t only) :

$$F(u,t) = c_1(u)\cos(ut) + c_2(u)\sin(ut)$$

(1) Lets check the standard solution.

$$\begin{aligned}
\partial_{tt} F(u,t) &= \partial_{tt}(c_1(u)\cos(ut) + c_2(u)\sin(ut)) \\
&= \partial_t(-c_1(u)u\sin(ut) + c_2(u)u\cos(ut)) \\
&= -c_1(u)u^2\cos(ut) - c_2(u)u^2\sin(ut)) \\
&= -u^2 F(u,t)
\end{aligned}$$

(2) Lets find the value of c_1 and c_2 in the standard solution.

$$\begin{aligned}
\partial_t F(u,0) &= G(u) \quad \Rightarrow \quad -c_1(u)u\sin(u0) + c_2(u)u\cos(u0) = G(u) \quad \Rightarrow \quad c_2(u)u = G(u) \\
F(u,0) &= H(u) \quad \Rightarrow \quad c_1(u)\cos(u0) + c_2(u)\sin(u0) = H(u) \quad \Rightarrow \quad c_1(u) = H(u)
\end{aligned}$$

Hence we have :

$$\begin{aligned}
F(u,t) &= H(u)\cos(ut) + G(u)\sin(ut)/u \\
\Rightarrow \quad f(x,t) &= \left[FT_{uaf}^{-1}[H(u)\cos(ut) + G(u)\sin(ut)/u]\right](x,t) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} H(u)\cos(ut)e^{jux}du + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} G(u)\sin(ut)e^{jux}/u\,du
\end{aligned}$$

<u>Solution 2</u>

Solution 2 of wave equation involves direct solving PDE, please read the document for Green function.