Interest Rate Derivatives Pricing

1.1 Definition of bond price

- Spot bond is the *PV* of a riskfree securities at *t*, that guarantees *$1* payment at *T*, given market data at *t*.
- Forward bond is the *PV* of a riskfree securities at *t*, that guarantees *$1* payment at T_1, given market data at T_0.

spot bond	=	$P_t(T)$	with 2 time parameters
forward bond	=	$P_t(T_0, T_1)$	with 3 time parameters
$P_t(T)$	=	$P_t(t, T_1)$	

Both above spot bond and forward bond are fixed at *t*, thus :

- they are stochastic as of time *0*
- they are deterministic as of time *t*
- this is how they are involved in interest rate derivative pricing …

$P_t(T)$	=	prevailing market data, deterministic
$P_{T_0}(T_1)$	=	underlying variable, stochastic, which is usually converted into …
$P_T(T_0, T_1)$	=	transformed underlying variable, which is martingale in forward measure, depends on …
$P_t(T_0, T_1)$	=	transformed market data, again deterministic

Strategy A We can replicate a forward bond by a portfolio of two spot bonds :

- long position in one long-term bond matured at T_1 and
- short position in Δ short-term bond matured at T_0 (such that $T_0 < T_1$) to fully finance the former, initial cashflow is zero

$$\rightarrow \quad \textit{cashflow at } t = -P_t(T_1) + \Delta P_t(T_0)$$

$$0 = -P_t(T_1) + \Delta P_t(T_0)$$

$$\Delta = P_t(T_1) / P_t(T_0)$$

$$\left.\begin{array}{lll} \rightarrow \quad \textit{cashflow at } T_0 & = & -P_t(T_1) / P_t(T_0) \\ \rightarrow \quad \textit{cashflow at } T_1 & = & \$1 \end{array}\right\}$$

Effectively construct a forward bond that costs $P_t(T_1)/P_t(T_0)$ at T_0 and pays $1 at T_1.

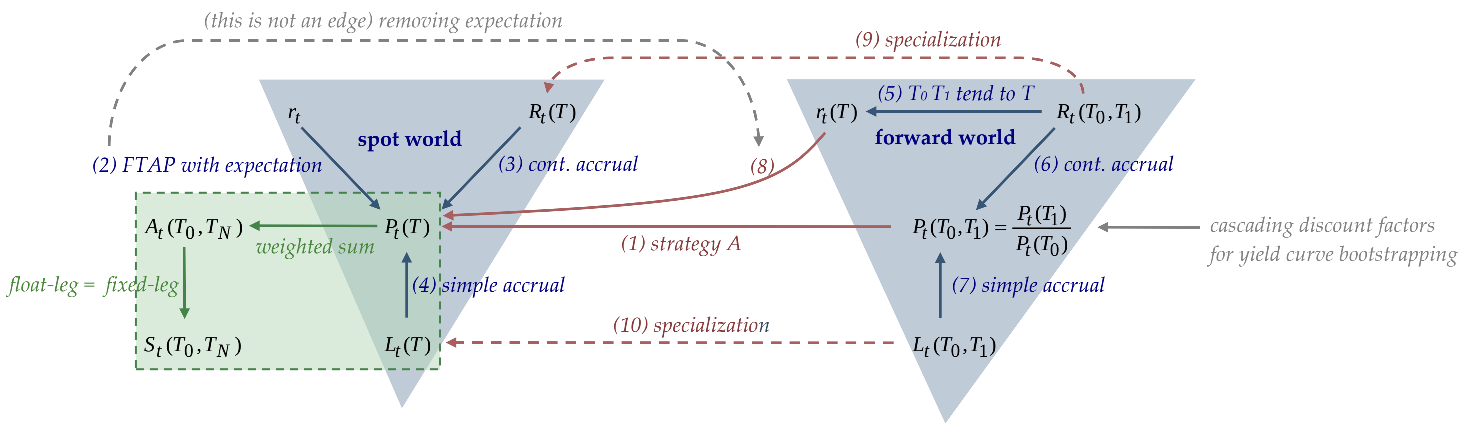
By law of one price, we have :

$$P_t(T_0, T_1) = P_t(T_1) / P_t(T_0) \tag{1}$$

1.2 Definition of rate (short rate / zero rate / LIBOR)

Define 3 different rates for both spot and forward worlds, forming two Y-shape directed acyclic graphs.

- all rate definitions are rooted in bond price, which is traded in market
- short rate is stochastic instantaneous rate for matching with market bond price *(interest accrued over infinitesimal period)*
- zero rate is continuous compounding rate implied by market bond price *(interest accrued over a period)*
- *LIBOR* is simple compounding rate implied by market bond price *(interest accrued over a period)*
- *LIBOR* and zero rate have same definition in spot and forward worlds
- short rate has different definitions in spot and forward worlds
- spot rate starts accrual at t, while forward rate starts accrual at a later time T_0
- for bootstrapping, we work with *spot/forward bond price (for cascading)* and *zero rate (for interpolation)*
- we can model bond price with Black Scholes (which is used as convention of market quote), or
- we can model short rate with Vasicek or Hull White, or
- we can model *LIBOR* is *LIBOR* market model *LMM*, also known as *BGM*



Here are the definitions of the numbered-edges in *DAG*, where δ denotes daycounter which returns a year fraction.

(2)	*spot short rate*	$P_t(T)$	$\equiv$	$E_Q[\$1\times e^{-\int_t^T r_s ds} \mid I_t]$				*FTAP* $\rightarrow$ *different definitions*
(3)	*spot zero rate*	$P_t(T)$	$\equiv$	$e^{-R_t(T)\delta(t,T)}$	$\Rightarrow$	$R_t(T)$	$= -\dfrac{\ln P_t(T)}{\delta(t,T)}$	*continuous accrual*
(4)	*spot LIBOR*	$P_t(T)$	$\equiv$	$\dfrac{1}{1+L_t(T)\delta(t,T)}$	$\Rightarrow$	$L_t(T)$	$= \dfrac{1/P_t(T)-1}{\delta(t,T)}$	*simple accrual*
(5)	*fwd short rate*	$r_t(T)$	$\equiv$	$\lim_{\Delta\to 0} R_t(T,T+\Delta)$				*taking limit*
(6)	*fwd zero rate*	$P_t(T_0,T_1)$	$\equiv$	$e^{-R_t(T_0,T_1)\delta(T_0,T_1)}$	$\Rightarrow$	$R_t(T_0T_1)$	$= -\dfrac{\ln P_t(T_0,T_1)}{\delta(T_0,T_1)}$	*continuous accrual*
(7)	*fwd LIBOR*	$P_t(T_0,T_1)$	$\equiv$	$\dfrac{1}{1+L_t(T_0,T_1)\delta(T_0,T_1)}$	$\Rightarrow$	$L_t(T_0T_1)$	$= \dfrac{1/P_t(T_0,T_1)-1}{\delta(T_0,T_1)}$	*simple accrual*

same definition in spot / forward worlds

Spot short rate is defined by *FTAP* as in *edge(2)*, forward short rate is defined by taking limit on forward zero rate as in *edge(5)*.

$$
\begin{aligned}
r_t(T) &= \lim_{\Delta\to 0} R_t(T,T+\Delta) \\
&= -\lim_{\Delta\to 0} \frac{\ln(P_t(T+\Delta,T)/P_t(T))}{\delta(T,T+\Delta)} \\
&= -\lim_{\Delta\to 0} \frac{\ln P_t(T+\Delta)-\ln P_t(T)}{\Delta} && \text{since } \delta(T,T+\Delta)\sim\Delta \\
&= -\partial_T \ln P_t(T) \\
r_t(s) &= -\partial_s \ln P_t(s) \\
\int_t^T r_t(s)ds &= -\ln P_t(T) + \underbrace{\ln P_t(t)}_{0}
\end{aligned}
$$

(8) $$P_t(T) = \$1\times e^{-\int_t^T r_t(s)ds}$$

Forward short rate is used only in chap5 for proving that …
HW model can perfectly fit market quoted yield curve.

Finally we make three connections between the two worlds, i.e. three dotted arrows in the diagram : *(2vs8), (9) and (10)*.

(2vs8)

$$P_t(T) \equiv E_Q[\$1\times e^{-\int_t^T r_s ds} \mid I_t]$$

$$P_t(T) = \$1\times e^{-\int_t^T r_t(s)ds}$$

when spot short rate is replaced by forward short rate …

…the expectation is removed as $r_t(s)$ are known at t

By substituting $T_0 = t$ and $T_1 = T$ into forward zero rate and *LIBOR*, we obtain the *spot counterparts* :

(9)

$$R_t(t,T) = -\frac{\ln(P_t(T)/P_t(t))}{\delta(t,T)} = -\frac{\ln P_t(T)}{\delta(t,T)} = R_t(T)$$

(10)

$$L_t(t,T) = \frac{1/P_t(t,T)-1}{\delta(t,T)} = L_t(T)$$

Is there any direct connection between r_t and $r_t(T)$?

<u>Forward LIBOR coupon in terms of spot bonds</u>

LIBOR coupon is common in funding and exotic leg of *IRDs*, converting it into bond prices makes risk neutral expectation easy.

$$L_t(T_0T_1) = \frac{1/P_t(T_0,T_1)-1}{\delta(T_0,T_1)}$$

$$\Rightarrow \quad L_t(T_0,T_1)\delta(T_0,T_1) = \frac{P_t(T_0)-P_t(T_1)}{P_t(T_1)}$$

$$\Rightarrow \quad \underbrace{L_t(T_0,T_1)}_{coupon}\underbrace{\delta(T_0,T_1)}_{daycount}\underbrace{P_t(T_1)}_{DF} = P_t(T_0)-P_t(T_1) \qquad (11)$$

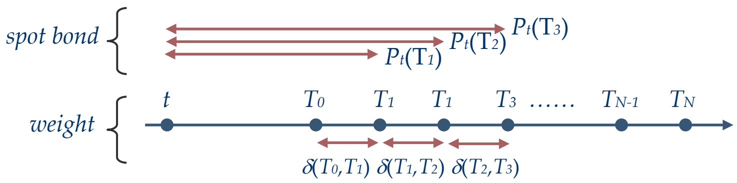
*Equation 11 is useful for **forward LIBOR prediction** given a yield curve. It is used once in strategy B and once in bootstrapping with OIS discounting.*

This is my convention to arrange items in order :
notional, coupon, daycount and DF.

1.3 Definition of annuity

The set of $\{T_n\}$ is called schedule of a leg, leg having N payments should have $N+1$ time points in the schedule. For the n^{th} payment :

coupon fixing day	=	T_{n-1}
accrual start day	=	T_{n-1}
accrual end day	=	T_n
payment day	=	T_n



Annuity is defined as sum of ***spot*** bonds weighted by corresponding year fractions. We usually mean forward annuity.

$$A_t(T_0,T_N) = \sum_{n=1}^N \delta(T_{n-1},T_n)P_t(T_n)$$

this is forward annuity, as the leg starts at future time $T_0 > t$

$$A_{T_0}(T_N) = A_{T_0}(T_0,T_N) = \sum_{n=1}^N \delta(T_{n-1},T_n)P_{T_0}(T_n)$$

this is spot annuity, as the leg starts now $T_0 = t$

There are both *spot annuity* and *forward annuity*, but in general, annuity usually refers to *forward annuity*.

1.4 Cascading forward bond price

Forward bond price $P_t(T_0,T_1)$ is the discount factor from T_1 to T_0 as of market data at t. The following is a common mistake, read the next section for the correct risk neutral expection of cash-deflatted \$1 in *equation(20a)*.

$$E_Q[\$1\times e^{-\int_{T_0}^{T_1} r_s ds} \mid I_t] \neq P_t(T_0,T_1)$$

This is a common mistake.

$$P_t(T_0,T_1) = \frac{P_t(T_1)}{P_t(T_0)} = \frac{E_Q[e^{-\int_t^{T_1} r_s ds} \mid I_t]}{E_Q[e^{-\int_t^{T_0} r_s ds} \mid I_t]}$$

It verifies that $E[A/B] \neq E[A]/E[B]$.

Forward bond price can be cascaeded. We do this a lot in yield curve bootstrapping.

$$P_t(T_0,T_N) = \frac{P_t(T_1)}{P_t(T_0)}\frac{P_t(T_2)}{P_t(T_1)}\frac{P_t(T_3)}{P_t(T_2)}\cdots\frac{P_t(T_N)}{P_t(T_{N-1})}$$

$$= P_t(T_0,T_1)\times P_t(T_1,T_2)\times P_t(T_2,T_3)\times\ldots\times P_t(T_{N-1},T_N)$$

2.1 Risk neutral expectation of cash-deflatted $1

$$(20)\quad E_Q[\$1\times e^{-\int_t^T r_s ds}\,|\,I_t] = P_t(T) \qquad \text{by FTAP, duplicate equation (2) here as equation (20)}$$

$$(20a)\quad E_Q[\$1\times e^{-\int_{T_0}^{T_1} r_s ds}\,|\,I_t] = E_Q[E_Q[\$1\times e^{-\int_{T_0}^{T_1} r_s ds}\,|\,I_{T_0}]\,|\,I_t] \qquad \text{by tower property}$$

$$= E_Q[P_{T_0}(T_1)\,|\,I_t] \qquad \text{generic form of (20), we get back (20) with } T_0 = t \text{ and } T_1 = T$$

2.2 Risk neutral expectation of cash-deflatted spot bond

The counterpart of *equation(20)* for $1 is *equation(21)* for spot bond, result is intuitive enough, cash-deflatted spot bond is martingale under measure Q, thus we can find spot bond price by *RN* expectation. We do not work on the counterpart of *equation(20a)*.

$$(21)\quad E_Q[P_{T_0}(T_1)e^{-\int_t^{T_0} r_s ds}\,|\,I_t] = E_Q[E_Q[e^{-\int_{T_0}^{T_1} r_s ds}\,|\,I_{T_0}]e^{-\int_t^{T_0} r_s ds}\,|\,I_t]$$

$$= E_Q[e^{-\int_{T_0}^{T_1} r_s ds}e^{-\int_t^{T_0} r_s ds}\,|\,I_t] \qquad \text{by tower property}$$

$$= E_Q[e^{-\int_t^{T_1} r_s ds}\,|\,I_t]$$

$$= P_t(T_1)$$

2.3 Risk neutral expectation of cash-deflatted spot LIBOR

The counterpart of *equation(20)* for $1 is *equation(22)* for spot *LIBOR*. The proof involves setting up strategy B.

$$(22)\quad E_Q[L_{T_0}(T_1)e^{-\int_t^{T_1} r_s ds}\,|\,I_t] = L_t(T_0,T_1)P_t(T_1)$$

$$(22a)\quad = \frac{P_t(T_0)-P_t(T_1)}{\delta(T_0,T_1)}$$

Strategy B We can replicate *LIBOR* lending by the following strategy :

- long position in one short-term bond matured at T_0 and
- short position in one long-term bond matured at T_1 (such that $T_0 < T_1$) both at time t
- as T_0 bond price is higher than T_1 bond price, there is positive cash outflow, which is the cost of *LIBOR*
- at time T_0, we get back $1 from T_0 bond, which earns *LIBOR* in money market
- at time T_1, we pay back $1 for T_1 bond, net *LIBOR* interest is the profit
- discounted payoff at time T_1 should equal to the cost of *LIBOR* at time t

$$\to \quad \text{cash outflow at } t = P_t(T_0)-P_t(T_1)$$

$$\to \quad \text{cashflow at } T_0 = 0$$

$$\to \quad \text{cashflow at } T_1 = 1+L_{T_0}(T_1)\delta(T_0,T_1)-1$$

Thus we have :

$$E_Q[L_{T_0}(T_1)\delta(T_0,T_1)e^{-\int_t^{T_1} r_s ds}\,|\,I_t] = P_t(T_0)-P_t(T_1)$$

$$E_Q[L_{T_0}(T_1)e^{-\int_t^{T_1} r_s ds}\,|\,I_t] = \frac{P_t(T_0)-P_t(T_1)}{\delta(T_0,T_1)} \qquad \text{thus we have equation(22a)}$$

$$= \frac{L_t(T_0,T_1)\delta(T_0,T_1)P_t(T_1)}{\delta(T_0,T_1)} \qquad \text{by equation (11)}$$

$$= L_t(T_0,T_1)P_t(T_1) \qquad \text{thus we have equation(22)}$$

2.4 Risk neutral expectation of cash-deflatted floating leg

Consider a swap with *$1* notional, floating leg (funding leg) binded to *LIBOR*, and schedule $\{T_{n-1}\ T_n\}\ \forall n \in [1,N]$, then the risk neutral expectation of cash-deflatted floating leg is :

$$
\begin{aligned}
& E_Q[\textstyle\sum_{n=1}^N notional_n \times rate_n \times daycount_n \times DF_n \mid I_t] \\
= \;& E_Q[\textstyle\sum_{n=1}^N \$1 \times L_{T_{n-1}}(T_n)\delta(T_{n-1},T_n)e^{-\int_t^{T_n} r_s ds} \mid I_t] && \textit{by FTAP} \\
= \;& \textstyle\sum_{n=1}^N \delta(T_{n-1},T_n)E_Q[L_{T_{n-1}}(T_n)e^{-\int_t^{T_n} r_s ds} \mid I_t] \\
= \;& \textstyle\sum_{n=1}^N \delta(T_{n-1},T_n)L_t(T_{n-1},T_n)P_t(T_n) && \textit{by equation(22)} && \leftarrow (swap1) \\
= \;& \textstyle\sum_{n=1}^N \delta(T_{n-1},T_n)\dfrac{P_t(T_{n-1})-P_t(T_n)}{\delta(T_{n-1},T_n)} && \textit{by equation(22a)} \\
= \;& \textstyle\sum_{n=1}^N (P_t(T_{n-1})-P_t(T_n)) \\
= \;& P_t(T_0)-P_t(T_N) && \textit{by telescoping sum} && \leftarrow (swap2)
\end{aligned}
$$

2.5 Risk neutral expectation of cash-deflatted fixed leg

Consider the same swap, fixed leg binded to constant swap rate, and schedule $\{\Gamma_{m-1}\ \Gamma_m\}\ \forall m \in [1,M]$, then the risk neutral expectation of cash-deflatted fixed leg is (both legs should have same start time and end time, i.e. $T = T_0 = \Gamma_0$ and $T_N = \Gamma_M$) :

$$
\begin{aligned}
& E_Q[\textstyle\sum_{m=1}^M notional_m \times rate_m \times daycount_m \times DF_m \mid I_t] \\
= \;& E_Q[\textstyle\sum_{m=1}^M \$1 \times C\delta(\Gamma_{m-1},\Gamma_m)e^{-\int_t^{\Gamma_m} r_s ds} \mid I_t] && \textit{by FTAP} \\
= \;& C\textstyle\sum_{m=1}^M \delta(\Gamma_{m-1},\Gamma_m)E_Q[e^{-\int_t^{\Gamma_m} r_s ds} \mid I_t] \\
= \;& C\textstyle\sum_{m=1}^M \delta(\Gamma_{m-1},\Gamma_m)P_t(\Gamma_m) && \textit{by equation(20)} && \leftarrow (swap3) \\
= \;& CA_t(\Gamma_0,\Gamma_M) && && \leftarrow (swap4)
\end{aligned}
$$

2.6 Risk neutral expectation of cash-deflatted spot annuity

In general, we have $T \neq T_0$.

$$
\begin{aligned}
(23) \quad & E_Q[A_T(T_0,T_N)e^{-\int_t^T r_s ds} \mid I_t] \\
= \;& E_Q[\textstyle\sum_{n=1}^N \delta(T_{n-1},T_n)P_T(T_n)e^{-\int_t^T r_s ds} \mid I_t] && \textit{definition of annuity} \\
= \;& \textstyle\sum_{n=1}^N \delta(T_{n-1},T_n)E_Q[P_T(T_n)e^{-\int_t^T r_s ds} \mid I_t] && \textit{becomes expectation of cash-deflatted spot bond} \\
= \;& \textstyle\sum_{n=1}^N \delta(T_{n-1},T_n)P_t(T_n) && \textit{by equation(21)} \\
= \;& A_t(T_0,T_N) && \textit{definition of annuity}
\end{aligned}
$$

Annuity is analogous to bond price, while swap rate S is analogous to *LIBOR*. The former pair is price, the latter pair is index.

price	$A_t(T_0,T_N)$	$\Leftrightarrow$	$P_t(T_{n-1},T_n)$
index	$S_t(T_0,T_N)$	$\Leftrightarrow$	$L_t(T_{n-1},T_n)$

What is swap rate then? Let's see …

2.7 Swap rate and swap pricing

- Payer swap can be considered as buying a of stream of *LIBOR* coupons at the expense of fixed rate coupons *C*.
- Receiver swap can be considered as selling a of stream of *LIBOR* coupons at the price of fixed rate coupons *C*.

Spot swap rate and forward swap rate

- The fixed coupon rate at which a swap is at par (or equivalently, both legs have same *PV*) is called *swap rate*.
- Its fixing and prediction are abstracted as *constant maturity swap CMS index*.
- It is regarded as average of *LIBOR*, it rises when yield curve moves up and vice versa.
- Grouping *(swap1-4)* we have :

$$\begin{array}{lcll} P_t(T_0)-P_t(T_N) & = & \sum_{n=1}^{N} L_t(T_{n-1},T_n)\delta(T_{n-1},T_n)P_t(T_n) & \textit{floating leg} \\ CA_t(\Gamma_0,\Gamma_M) & = & C\sum_{m=1}^{M}\delta(\Gamma_{m-1},\Gamma_m)P_t(\Gamma_m) & \textit{fixed leg} \end{array}$$

- we define *forward swap rate* as : *it depends on forward annuity*

$$S_t(\Gamma_0,\Gamma_M) = \frac{P_t(T_0)-P_t(T_N)}{A_t(\Gamma_0,\Gamma_M)} \quad \text{eq(24')} \qquad \textit{by comparing LHS of floating leg with LHS of fixed leg}$$

$$S_t(\Gamma_0,\Gamma_M) = \frac{\sum_{n=1}^{N} L_t(T_{n-1},T_n)\delta(T_{n-1},T_n)P_t(T_n)}{\sum_{m=1}^{M}\delta(\Gamma_{m-1},\Gamma_m)P_t(\Gamma_m)} \quad (24) \qquad \textit{by comparing RHS of floating leg with RHS of fixed leg}$$

Equation(24) is bootstrapping formula. See later section.

- then *spot swap rate* is just a specialized *forward swap rate*, where $t = T = T_0 = \Gamma_0$:

$$S_T(T,\Gamma_M) = \frac{P_T(T)-P_T(T_N)}{A_T(\Gamma_M)} = \frac{1-P_T(T_N)}{A_T(\Gamma_M)} \qquad \textit{it depends on spot annuity}$$

Swap pricing

► Swap must be traded at par. Suppose today *t* is trade day :

$$C = S_t(\Gamma_0,\Gamma_M) \qquad \textit{suppose this is a forward swap, i.e. } t < T = T_0 = \Gamma_0$$

► Swap with *predetermined* coupon $C = S_t(\Gamma_0,\Gamma_M)$ is not at par after trade day. Suppose today *t'* is after trade day *t* :

$$\begin{array}{lcll} C = S_t(\Gamma_0,\Gamma_M) & \neq & S_{t'}(\Gamma_0,\Gamma_M) & \textit{suppose this is the same forward swap, i.e. } t < t' < T = T_0 = \Gamma_0 \\ & = & \dfrac{P_{t'}(T_0)-P_{t'}(T_N)}{A_{t'}(\Gamma_0,\Gamma_M)} & \end{array}$$

► If yield curve goes up at *t'*, so that $S_{t'}(\Gamma_0, \Gamma_M) > S_t(\Gamma_0, \Gamma_M) = C$

- payer-side makes a profit, as it bought a *LIBOR* stream at $S_t(\Gamma_0, \Gamma_M)$ which worths $S_{t'}(\Gamma_0, \Gamma_M)$ in the market today
- payer-side speculates a rise in *LIBOR* curve or a rise in swap rate
- vice versa for yield curve going down (or receiver-side)

Par value

Swap is par if *PV* is zero. Floating rate bond *FRN* is at par if its *PV* is notional. Let's verify by considering floating leg as a *FRN*.

$$\begin{array}{lcll} \textit{floating leg PV} & = & P_t(T_0)-P_t(T_N) & \textit{by equation(swap2)} \\ \textit{FRN PV} & = & \textit{floating leg PV} + \textit{notional} \times DF & \textit{don't forget the notional on maturity} \\ & = & P_t(T_0)-P_t(T_N)+P_t(T_N) & \\ & = & P_t(T_0) & \\ & = & 1 & \textit{when } t = T_0\textit{, i.e. on the start day of } 1^{st} \textit{ interval} \end{array}$$

2.8 Construction of martingale

One of the most important objectives in quant finance is to ***construct martingale***. It involves finding :

- what variable *(spot bond vs forward bond)* and …
- under what measures *(cash RN meaure vs forward RN measure)* so that …
- the *variable itself* or *numeraire-deflatted variable* is martingale.

Equation(20-23) give martingale properties of *cash deflatted* \$1, spot bond, spot *LIBOR* and annuity. By change of measure, we obtain martingale properties of *non-deflatted* forward counterparts. Different change of measures are involved below :

Bond

$$P_t(T_1) = E_Q[P_{T_0}(T_1)e^{-\int_t^{T_0} r_s ds} \mid I_t]$$

by equation(21)

$$= E_{Q_{T_0}}\left[\frac{e^{+\int_t^{T_0} r_s ds}}{P_{T_0}(T_0)/P_t(T_0)} P_{T_0}(T_1)e^{-\int_t^{T_0} r_s ds} \mid I_t\right]$$

Radon Nikodym derivative $\dfrac{gain-in-cash}{gain-in-P_t(T_0)}$

$$= P_t(T_0)E_{Q_{T_0}}[P_{T_0}(T_1) \mid I_t]$$

$P_t(T_0)$ *is deterministic and* $P_{T_0}(T_0)=1$

(25) $$P_t(T_0,T_1) = E_{Q_{T_0}}[P_{T_0}(T_1) \mid I_t]$$

yet this is not in martingale form …

(25a) $$P_t(T_0,T_1) = E_{Q_{T_0}}[P_{T_0}(T_0,T_1) \mid I_t]$$

forward bond accruing in (T_0T_1) *is martingale under* Q_{T0}

LIBOR

$$P_t(T_1)L_t(T_0,T_1) = E_Q[L_{T_0}(T_1)e^{-\int_t^{T_1} r_s ds} \mid I_t]$$

by equation(22)

$$= E_{Q_{T_1}}\left[\frac{e^{+\int_t^{T_1} r_s ds}}{P_{T_1}(T_1)/P_t(T_1)} L_{T_0}(T_1)e^{-\int_t^{T_1} r_s ds} \mid I_t\right]$$

Radon Nikodym derivative $\dfrac{gain-in-cash}{gain-in-P_t(T_1)}$

$$= P_t(T_1)E_{Q_{T_1}}[L_{T_0}(T_1) \mid I_t]$$

$P_t(T_1)$ *is deterministic and* $P_{T_1}(T_1)=1$

(26) $$L_t(T_0,T_1) = E_{Q_{T_1}}[L_{T_0}(T_1) \mid I_t]$$

yet this is not in martingale form …

(26a) $$L_t(T_0,T_1) = E_{Q_{T_1}}[L_{T_0}(T_0,T_1) \mid I_t]$$

forward LIBOR accruing in (T_0T_1) *is martingale under* Q_{T1}

Annuity

$$A_t(T_0,T_N) = E_Q[A_T(T_0,T_N)e^{-\int_t^{T} r_s ds} \mid I_t]$$

by equation(23)

$$= E_{Q_{T_0T_N}}\left[\frac{e^{+\int_t^{T} r_s ds}}{A_T(T_0,T_N)/A_t(T_0,T_N)} A_T(T_0,T_N)e^{-\int_t^{T} r_s ds} \mid I_t\right]$$

$$= A_t(T_0,T_N)$$

Unfortunately we get nothing useful.

Swap rate

$$S_t(T_0,T_N) = \frac{P_t(T_0)-P_t(T_N)}{A_t(T_0,T_N)}$$

by equation(24)

$$= \frac{E_Q[(P_T(T_0)-P_T(T_N))e^{-\int_t^{T} r_s ds} \mid I_t]}{A_t(T_0,T_N)}$$

by equation(21)

$$= \frac{1}{A_t(T_0,T_N)} E_{Q_{T_0T_N}}\left[\frac{e^{+\int_t^{T} r_s ds}}{A_T(T_0,T_N)/A_t(T_0,T_N)} (P_T(T_0)-P_T(T_N))e^{-\int_t^{T} r_s ds} \mid I_t\right]$$

$$= E_{Q_{T_0T_N}}\left[\frac{P_T(T_0)-P_T(T_N)}{A_T(T_0,T_N)} \mid I_t\right]$$

Radon Nikodym derivative

(27) $$S_t(T_0,T_N) = E_{Q_{T_0T_N}}[S_T(T_0,T_N) \mid I_t]$$

by equation(24)

2.9 LIBOR futures

John Crosby discusses *LIBOR* futures in his lecture. Futures is defined as :

$$F_T(T_0,T_1) = 100(1-L_T(T_0,T_1)) \qquad \textit{It is quoted as figure like 99.75, 98.53 …}$$

Futures is traded using margin account, marked to market daily, so its PV can be represented as the following sum where t is today, Δt is one day, and start day $T_0 = t+N\Delta t$.

$$\begin{aligned} PV_t(T_0,T_1) &= E_Q[\textstyle\sum_{n=1}^N [100(1-L_{t+n\Delta t}(T_0,T_1))-100(1-L_{t+(n-1)\Delta t}(T_0,T_1))]e^{-\int_t^{t+n\Delta t} r_s ds} \,|\, I_t] \\ &= \left[\begin{array}{l} +100\sum_{n=1}^N E_Q[L_{t+(n-1)\Delta t}(T_0,T_1)e^{-\int_t^{t+n\Delta t} r_s ds} \,|\, I_t] \\ -100\sum_{n=1}^N E_Q[L_{t+n\Delta t}(T_0,T_1)e^{-\int_t^{t+n\Delta t} r_s ds} \,|\, I_t] \end{array}\right. \\ &= \textit{please prove it, Crosby proof is not correct} \end{aligned}$$

Differences between *LIBOR* forward and *LIBOR* futures :

- *OTC* vs exchange
- collateral vs margin for protection against counterparty risk
- *LIBOR* forward is martingale under Q_{T1}, see *equation(26a)*
- *LIBOR* futures is martingale under cash measure (please check)?

2.10 Summary

Comparison between :

- cash-deflatted spot variable under cash numeraire vs
- non-deflatted spot variable under forward numeraire

	under cash measure in eq(21-23)		*under forward measure or annuity measure in eq(25-27)*		
bond	$E_Q[P_{T_0}(T_1)e^{-\int_t^{T_0} r_s ds} \,|\, I_t]$	$= P_t(T_1)$	$E_{Q_{T_0}}[P_{T_0}(T_1) \,|\, I_t]$	$= P_t(T_0,T_1)$	*(T_0 forward measure)*
LIBOR	$E_Q[L_{T_0}(T_1)e^{-\int_t^{T_1} r_s ds} \,|\, I_t]$	$= L_t(T_0,T_1)P_t(T_1)$	$E_{Q_{T_1}}[L_{T_0}(T_1) \,|\, I_t]$	$= L_t(T_0,T_1)$	*(T_1 forward measure)*
annuity	$E_Q[A_T(T_0,T_N)e^{-\int_t^{T} r_s ds} \,|\, I_t]$	$= A_t(T_0,T_N)$		×	
swap rate		×	$E_{Q_{T_0T_N}}[S_T(T_0,T_N) \,|\, I_t]$	$= S_t(T_0,T_N)$	*(T_0T_1 annuity measure)*

Equation(25-27) are not in martingale form, yet *equation(25a-27a)* are in martingale form :

$$P_t(T_0,T_1) = E_{Q_{T_0}}[P_{T_0}(T_0,T_1) \,|\, I_t] \tag{25a}$$

$$L_t(T_0,T_1) = E_{Q_{T_1}}[L_{T_0}(T_0,T_1) \,|\, I_t] \tag{26a}$$

$$S_t(T_0,T_N) = E_{Q_{T_0T_N}}[S_T(T_0,T_N) \,|\, I_t] \tag{27}$$

FTAP claims that numeraire-deflatted contingent claims are martingale under numeraire risk neutral measure. Yet the expectations may not be easy to solve directly. Instead, with suitable change of numeraire, we can end up with a easier calculation.

Redundant securities	*primitive securities*	*possible numeraires*
equity derivatives	*cash + stock*	*cash numeraire, stock numeraire*
FX derivatives	*ccy1 + ccy2*	*ccy1 cash, ccy1 forward, ccy2 cash, ccy2 forward*
IR derivatives	*T_0 bond + T_1 bond*	*cash numeraire, forward T_0 measure, forward T_1 measure*

3.1 Bond price under cash/forward risk neutral numeraire

In interest rate derivative pricing, we usually express risk neutral expectation of payoff in terms of spot/forward bond prices under various measures. Thus we are going to derive the following 6 formulae. *BP* stands for bond price.

spot *BPSDE* under cash numeraire *(28a)* $\longrightarrow$ spot *BP* under cash numeraire *(28b)*

$\downarrow$

spot *BPSDE* under forward numeraire *(28c)* $\longrightarrow$ spot *BP* under forward numeraire *(28d)*

forward *BP* under cash numeraire *(28e)*

forward *BP* under forward numeraire *(28f)*

<u>*Spot bond price under cash numeraire*</u>

Assume that bonds with different tenors follow geometric Brownian motions having the same risk factor z_t but different bond price volatilities under risk neutral measure of cash numeraire (recall that we replace physical drift by risk free drift) :

$$
\begin{array}{llll}
 & dB_t = r_t B_t dt & & \textit{cash SDE under risk neutral measure} \\
 & dP_{n,t} = r_t P_{n,t} dt + \sigma_{n,t} P_{n,t} dz_t & & \textit{(28a)} \\
 & P_{n,T} = P_{n,t} \exp\left(\int_t^T (r_s - \frac{1}{2}\sigma_{n,s}^2) ds + \int_t^T \sigma_{n,s} dz_s\right) & & \textit{(28b) standard solution to geometric Brownian} \\
\textit{where} & P_{n,T} = \textit{spot bond with fixed maturity } T_n & &
\end{array}
$$

<u>*Spot bond price under forward numeraire*</u>

By *FTAP*, price of T_n bond deflatted securities must be martingale under risk neutral measure with T_n bond as numeraire, known as forward T_n measure. In *IR* market, the set of primitive securities are *cash* and *bonds*, thus T_n bond deflatted cash must be driftless.

$$
\begin{array}{lll}
d\dfrac{B_t}{P_{n,t}} & = \dfrac{1}{P_{n,t}} dB_t - \dfrac{B_t}{P_{n,t}^2} dP_{n,t} - \dfrac{1}{2}(-2)\dfrac{B_t}{P_{n,t}^3}(dP_{n,t})^2 & \\
 & = \dfrac{1}{P_{n,t}}(r_t B_t dt) - \dfrac{B_t}{P_{n,t}^2}(r_t P_{n,t} dt + \sigma_{n,t} P_{n,t} dz_t) + \dfrac{B_t}{P_{n,t}^3}(\sigma_{n,t} P_{n,t})^2 dt & \\
 & = \dfrac{B_t}{P_{n,t}}(r_t dt - (r_t dt + \sigma_{n,t} dz_t) + \sigma_{n,t}^2 dt) & \\
 & = \dfrac{B_t}{P_{n,t}}(\sigma_{n,t}^2 dt - \sigma_{n,t} dz_t) & \textit{under Q measure, } z_t \textit{ is Brownian and } B_t/P_{n,t} \textit{ is drifted} \\
 & = \dfrac{B_t}{P_{n,t}}(-\sigma_{n,t} dz_t^n) & \textit{under } T_n \textit{ measure, } z^n_t \textit{ is Brownian and } B_t/P_{n,t} \textit{ is driftless} \\
\textit{implies} \quad dz_t^n & = \dfrac{\sigma_{n,t}^2 dt - \sigma_{n,t} dz_t}{-\sigma_{n,t}} & \\
dz_t^n & = -\sigma_{n,t} dt + dz_t & \textit{identical to Burgess eq(31)}
\end{array}
$$

We obtain T_m bond price *SDE* under T_n forward measure :

$$
\begin{array}{lll}
dP_{m,t} & = r_t P_{m,t} dt + \sigma_{m,t} P_{m,t} dz_t & \\
 & = r_t P_{m,t} dt + \sigma_{m,t} P_{m,t} (\sigma_{n,t} dt + dz_t^n) & \\
 & = (r_t + \sigma_{m,t}\sigma_{n,t}) P_{m,t} dt + \sigma_{m,t} P_{m,t} dz_t^n & \textit{(28c)}
\end{array}
$$

Again by *standard solution to geometric Brownian*, bond price formula under T_n forward measure is :

$$
\begin{array}{lll}
P_{m,T} & = P_{m,t} \exp\left(\int_t^T (r_s + \sigma_{m,s}\sigma_{n,s}) ds - \frac{1}{2}\int_t^T \sigma_{m,s}^2 ds + \int_t^T \sigma_{m,s} dz_s^n\right) & \\
 & = P_{m,t} \exp\left(\int_t^T (r_s - \frac{1}{2}(\sigma_{m,s}^2 - 2\sigma_{m,s}\sigma_{n,s})) ds + \int_t^T \sigma_{m,s} dz_s^n\right) & \textit{start to do completing square} \\
 & = P_{m,t} \exp\left(\int_t^T (r_s - \frac{1}{2}(\sigma_{m,s}^2 - 2\sigma_{m,s}\sigma_{n,s} + \sigma_{n,s}^2) + \frac{1}{2}\sigma_{n,s}^2) ds + \int_t^T \sigma_{m,s} dz_s^n\right) & \\
 & = P_{m,t} \exp\left(\int_t^T (r_s - \frac{1}{2}(\sigma_{m,s} - \sigma_{n,s})^2 + \frac{1}{2}\sigma_{n,s}^2) ds + \int_t^T \sigma_{m,s} dz_s^n\right) & \textit{(28d)}
\end{array}
$$

There is no need to derive forward *BPPDE*, we can directly make use of spot bond price formula *(28b)* and *(28d)*.

Forward bond price under cash numeraire

$$P_T(T_0,T_1) = \frac{P_T(T_1)}{P_T(T_0)}$$

This is stochastic as of time t.

$$= \frac{P_{1,t}}{P_{0,t}} \frac{\exp(\int_t^T (r_s - \frac{1}{2}\sigma_{1,s}{}^2)ds + \int_t^T \sigma_{1,s} dz_s)}{\exp(\int_t^T (r_s - \frac{1}{2}\sigma_{0,s}{}^2)ds + \int_t^T \sigma_{0,s} dz_s)}$$

using eq(28b) with n = (0,1)

$$= P_t(T_0,T_1)\exp(-\frac{1}{2}\int_t^T (\sigma_{1,s}{}^2 - \sigma_{0,s}{}^2)ds + \int_t^T (\sigma_{1,s} - \sigma_{0,s})dz_s) \qquad (28e)$$

Forward bond price under forward numeraire

$$P_T(T_0,T_1) = \frac{P_T(T_1)}{P_T(T_0)}$$

This is stochastic as of time t.

$$= \frac{P_{1,t}}{P_{0,t}} \frac{\exp(\int_t^T (r_s - \frac{1}{2}(\sigma_{1,s} - \sigma_{n,s})^2 + \frac{1}{2}\sigma_{n,s}{}^2)ds + \int_t^T \sigma_{1,s} dz_s^n)}{\exp(\int_t^T (r_s - \frac{1}{2}(\sigma_{0,s} - \sigma_{n,s})^2 + \frac{1}{2}\sigma_{n,s}{}^2)ds + \int_t^T \sigma_{0,s} dz_s^n)}$$

using eq(28d) with m = (0,1)

$$= P_t(T_0,T_1)\exp(-\frac{1}{2}\int_t^T ((\sigma_{1,s} - \sigma_{n,s})^2 - (\sigma_{0,s} - \sigma_{n,s})^2)ds + \int_t^T (\sigma_{1,s} - \sigma_{0,s})dz_s^n)$$

$$= P_t(T_0,T_1)\exp(-\frac{1}{2}\int_t^T (\sigma_{1,s} - \sigma_{0,s})^2 ds + \int_t^T (\sigma_{1,s} - \sigma_{0,s})dz_s^0)$$

(28f) when n = 0, i.e. using T_0 numeraire

$$= P_t(T_0,T_1)\exp(+\frac{1}{2}\int_t^T (\sigma_{0,s} - \sigma_{1,s})^2 ds + \int_t^T (\sigma_{1,s} - \sigma_{0,s})dz_s^1)$$

(28g) when n = 1, i.e. using T_1 numeraire

(28f) is more useful in IRD pricing

3.2 Bond price expectation and variance

We are going to derive mean and variance of *equation(28b,d,e,f)* :

$$P_T(T_0) = P_t(T_0)\exp(\int_t^T (r_s - \frac{1}{2}\sigma_{0,s}{}^2)ds + \int_t^T \sigma_{0,s} dz_s) \qquad (28b)$$

$$P_T(T_0) = P_t(T_0)\exp(\int_t^T (r_s - \frac{1}{2}(\sigma_{0,s} - \sigma_{n,s})^2 + \frac{1}{2}\sigma_{n,s}{}^2)ds + \int_t^T \sigma_{0,s} dz_s^n) \qquad (28d)$$

$$P_T(T_0,T_1) = P_t(T_0,T_1)\exp(-\frac{1}{2}\int_t^T (\sigma_{1,s}{}^2 - \sigma_{0,s}{}^2)ds + \int_t^T (\sigma_{1,s} - \sigma_{0,s})dz_s) \qquad (28e)$$

$$P_T(T_0,T_1) = P_t(T_0,T_1)\exp(-\frac{1}{2}\int_t^T (\sigma_{1,s} - \sigma_{0,s})^2 ds + \int_t^T (\sigma_{1,s} - \sigma_{0,s})dz_s^0)$$

(28f) using T_0 numeraire, where $t < T < T_0 < T_1$

very similar, only minor difference

Black Scholes formula

$$S_T = S_t \exp(\int_t^T (r_s - \sigma_s^2/2)ds + \int_t^T \sigma_s dz_s)$$

$$f(S_t) = (F_{BS} N(d_1) - KN(d_2)) \times DF$$

$$d_{1,2} = (\ln(F_{BS}/K) \pm \Sigma_{BS}/2)/\sqrt{\Sigma_{BS}}$$

$$F_{BS} = E_Q[S_T \mid I_t]$$

$$\Sigma_{BS} \neq V_Q[S_T \mid I_t]$$

$$\Sigma_{BS} = V_Q[\ln S_T \mid I_t] = V_Q[\int_t^T \sigma_s dz_s \mid I_t] = \int_t^T \sigma_s^2 ds$$

Some useful formulae

$$E[e^{\varepsilon(\mu,\sigma)}] = e^{\mu+\sigma^2/2}$$

$$E[\int_t^T f_s dz_s] = 0$$

$$V[\int_t^T f_s dz_s] = \int_t^T f_s{}^2 ds$$

We will not go through each of them. Let's consider F_{BS} and Σ_{BS} of *(28f)* under T_0 measure *(it will be used in bond option)* :

$$F_{BS} = E_{Q_{T_0}}[P_T(T_0,T_1) \mid I_t]$$

$$= P_t(T_0,T_1)\exp(-\frac{1}{2}\int_t^T (\sigma_{1,s} - \sigma_{0,s})^2 ds + \frac{1}{2}\int_t^T (\sigma_{1,s} - \sigma_{0,s})^2 ds)$$

$$= P_t(T_0,T_1)$$

(29a) See, martingale under forward measure

$$\Sigma_{BS} = V_{Q_{T_0}}[\ln P_T(T_0,T_1) \mid I_t]$$

$$= V_{Q_{T_0}}[\int_t^T (\sigma_{1,s} - \sigma_{0,s})dz_s^0 \mid I_t]$$

$$= \int_t^T (\sigma_{1,s} - \sigma_{0,s})^2 ds$$

(29b) known as forward bond price variance

3.3 Bond Price under Hull White

<u>*Bond price volatility vs short rate volatility*</u>

Upto this moment we work with lognormal bond model only, no short rate model has been involved. As *equation(28b,d,e,f)* are all bond price volatility dependent, it is useful if relationship between bond price volatility and short rate volatility is known. Given a short rate model, we can derive its bond price formula, being expressed as :

$$\begin{array}{lrcll} & P_t(T_m) & = & A(t,T_m)e^{-r_t B(t,T_m)} & \textit{This is IR version } S_t = S_0 e^{(r-\sigma^2/2)t+\sigma z_t}. \\ & d\ln P_t(T_m) & = & d(\ln A(t,T_m)) - d(r_t B(t,T_m)) & \textit{where A and B are deterministic} \\ & \dfrac{dP_t(T_m)}{P_t(T_m)} & = & \underbrace{\dfrac{dA(t,T_m)}{A(t,T_m)}}_{all-dt-terms} - r_t dB(t,T_m) - B(t,T_m)dr_t & \textit{plug lognormal bond price (28a) into LHS} \\ & r_t dt + \sigma_{m,t}dz_t & = & (...)dt - B(t,T_m)(\mu_{r,t}dt + \sigma_{r,t}dz_t) & \textit{plug short rate } dr_t = \mu_{r,t}dt + \sigma_{r,t}dz_t \textit{ into RHS} \\ & r_t dt + \sigma_{m,t}dz_t & = & (...)dt - B(t,T_m)\sigma_{r,t}dz_t & \\ \Rightarrow & \sigma_{m,t} & = & -B(t,T_m)\sigma_{r,t} & \textit{(30), identical to equation (38) in Burgess} \end{array}$$

<u>*Forward bond price formula under Hull White (2 versions)*</u>

We are going to enhance *equation(28f)* by plugging Hull White model into *equation(30)* :

$$\begin{array}{lrcll} \textit{Recall HW} & dr_t & = & (\theta_t - ar_t)dt + \sigma dz_t & \\ \textit{bond price} & P_t(T_m) & = & A(t,T_m)e^{-B(t,T_m)r_t} & \\ \textit{where} & B(t,T_m) & = & \dfrac{1}{a}(1-e^{-a(T_m-t)}) & \\ (30) \Rightarrow & \sigma_{m,t} & = & -B(t,T_m)\sigma & \\ & & = & -\dfrac{1}{a}(1-e^{-a(T_m-t)})\sigma & (31) \end{array}$$

Equation(28f) can be enhanced, which ends up in two representations :

- *QuantLib* version, which contains $4a^3$
- *DrYan* version, which is in terms of $\beta(T_0,T_1)$, please note that $\beta(T_0,T_1) \neq B(T_0,T_1)$

QuantLib version

Consider the integrals inside *exp* of *equation(28f)* :

$$\begin{array}{llcll} \textit{1st} & \int_t^T (\sigma_{1,s}-\sigma_{0,s})^2 ds & = & \dfrac{\sigma^2}{a^2}\int_t^T (-(1-e^{-a(T_1-s)}) + (1-e^{-a(T_0-s)}))^2 ds & \textit{using eq(31) with m = 0 and m = 1} \\ & & = & \dfrac{\sigma^2}{a^2}\int_t^T (e^{-a(T_1-s)} - e^{-a(T_0-s)})^2 ds & \\ & & = & \dfrac{\sigma^2}{a^2}(e^{-aT_1} - e^{-aT_0})^2 \int_t^T e^{2as} ds & (32) \\ & & = & \dfrac{\sigma^2}{2a^3}(e^{-aT_1} - e^{-aT_0})^2 (e^{2aT} - e^{2at}) & \textit{using } \int_t^T e^{2as}ds = \dfrac{1}{2a}(e^{2aT} - e^{2at}) \\ \textit{2nd} & \int_t^T (\sigma_{1,s}-\sigma_{0,s})dz_s^0 & = & \dfrac{\sigma}{a}\int_t^T (-(1-e^{-a(T_1-s)}) + (1-e^{-a(T_0-s)}))dz_s^0 & \textit{using eq(31) with m = 0 and m = 1} \\ & & = & \dfrac{\sigma}{a}\int_t^T (e^{-a(T_1-s)} - e^{-a(T_0-s)})dz_s^0 & \\ & & = & \dfrac{\sigma}{a}(e^{-aT_1} - e^{-aT_0})\int_t^T e^{as} dz_s^0 & \\ & & = & \dfrac{\sigma}{a}(e^{-aT_1} - e^{-aT_0})Z_{t,T}^0 & \textit{where } Z_{t,T}^0 = \int_t^T e^{as}dz_s^0 \\ (28f) \Rightarrow & P_T(T_0,T_1) & = & P_t(T_0,T_1)\exp(-\frac{1}{2}\int_t^T (\sigma_{1,s}-\sigma_{0,s})^2 ds + \int_t^T (\sigma_{1,s}-\sigma_{0,s})dz_s^0) & \textit{which is stochastic as of time t} \\ & & = & P_t(T_0,T_1)\exp\left(-\dfrac{\sigma^2}{4a^3}(e^{-aT_1} - e^{-aT_0})^2(e^{2aT} - e^{2at}) + \dfrac{\sigma}{a}(e^{-aT_1} - e^{-aT_0})Z_{t,T}^0\right) & \end{array}$$

Dr Yan version

However *DrYan* offers an elegant version. We have *QuantLib* version :

$$
\begin{aligned}
P_T(T_0,T_1) &= P_t(T_0,T_1)\exp\left(-\frac{\sigma^2}{4a^3}(e^{-aT_1}-e^{-aT_0})^2(e^{2aT}-e^{2at})+\frac{\sigma}{a}(e^{-aT_1}-e^{-aT_0})Z^0_{t,T}\right) && \textit{(33a) QuantLib version} \\
&= P_t(T_0,T_1)\exp\left(-\frac{\sigma^2}{2a^2}(e^{-aT_1}-e^{-aT_0})^2\int_t^T e^{2as}ds+\frac{\sigma}{a}(e^{-aT_1}-e^{-aT_0})Z^0_{t,T}\right) \\
&= P_t(T_0,T_1)\exp\left(-\frac{\sigma^2}{2a^2}(e^{-aT_1}-e^{-aT_0})^2 Var[\int_t^T e^{as}dz^0_s]+\frac{\sigma}{a}(e^{-aT_1}-e^{-aT_0})Z^0_{t,T}\right) && \textit{since } Var[\int_t^T e^{as}dz^0_s]=\int_t^T e^{2as}ds \\
&= P_t(T_0,T_1)\exp\left(-\frac{\sigma^2}{2a^2}(e^{-aT_1}-e^{-aT_0})^2 Var[Z^0_{t,T}]+\frac{\sigma}{a}(e^{-aT_1}-e^{-aT_0})Z^0_{t,T}\right) && \textit{since } Z^0_{t,T}=\int_t^T e^{as}dz^0_s \\
&= P_t(T_0,T_1)\exp(-\frac{1}{2}\beta^2(T_0,T_1)Var[Z^0_{t,T}]+\beta(T_0,T_1)Z^0_{t,T}) && \textit{(33b) DrYan version} \\
\textit{where}\quad \beta(T_0,T_1) &= \frac{\sigma}{a}(e^{-aT_1}-e^{-aT_0}) \\
&= -e^{-aT_0}\sigma B(T_0,T_1) && \textit{which is t-independent} \\
Z_t &= \int_t^T e^{as}dz_s && \textit{which is the only stochastic term}
\end{aligned}
$$

<u>*Forward bond price SDE under Hull White*</u>

Going back to *DAG* in *section 3.1*, we haven't derived forward bond price *SDE* as it is not needed in *IRD* pricing. Yet we are looking into it for a comprehensive understanding. Let's start with Hull White model :

$$
\begin{aligned}
dr_t &= (\theta_t - ar_t)dt + \sigma dz_t && \textit{under cash numeraire} \\
&= (\theta_t - ar_t)dt + \sigma(\sigma_{n,t}dt + dz^n_t) && \textit{under forward numeraire} \\
&= (\theta_t - ar_t)dt + \sigma(-\frac{\sigma}{a}(1-e^{-a(T_n-t)})dt + dz^n_t) && \textit{using eq(31)} \\
&= (\theta_t - ar_t - \frac{\sigma^2}{a}(1-e^{-a(T_n-t)}))dt + \sigma dz^n_t
\end{aligned}
$$

$$
\begin{aligned}
dP_{m,t} &= r_t P_{m,t}dt + \sigma_{m,t}P_{m,t}dz_t && \textit{under cash numeraire} \\
&= (r_t + \sigma_{m,t}\sigma_{n,t})P_{m,t}dt + \sigma_{m,t}P_{m,t}dz^n_t && \textit{under forward numeraire} \\
&= \left[\begin{array}{l} (r_t + (\frac{\sigma}{a})^2(1-e^{-a(T_m-t)})(1-e^{-a(T_n-t)}))P_{m,t}dt \\ -\frac{\sigma}{a}(1-e^{-a(T_m-t)})P_{m,t}dz^n_t \end{array}\right. && \textit{using eq(31)}
\end{aligned}
$$

We have derived forward bond price *SDE* in terms of Hull White parameters.

4.1 Introduction to Interest Rate Derivative Pricing

Main problem with *IRD* pricing is that both underlying risk factor X_T and discount path are stochastic and depend on short rate :

$$E_Q[payoff(X_T)e^{-\int_t^T r_s ds} \mid I_t]$$

which can be simplified under a different measure. Here are notations for different measures :

$E_Q[...]$	=	*risk neutral measure of cash numeraire*
$E_{Q_T}[...]$	=	*risk neutral measure of T forward numeraire*
$E_{Q_n}[...]$	=	*risk neutral measure of T_n forward numeraire*
$E_{Q_{0N}}[...]$	=	*risk neutral measure of annuity T_0 T_N numeraire*

Classification of *IRDs* :

linear product	*nonlinear product*	*callable exotic product*
bond	*bond option*	
FRA	*capfloor*	*callable LIBOR range accrual*
swap	*swaption*	*callable swap*

bond(T_0 T_1)	=	*interest accrual from T_0 to T_1*	
LIBOR(T_0 T_1)	=	*interest accrual from T_0 to T_1*	
bond option	=	*option on bond(T_0 T_1)*	*optionality ends at T_0, bond is settled at T_0*
LIBOR option	=	*option on LIBOR(T_0 T_1)*	*optionality ends at T_0, LIBOR payment is settled at T_1*

- Why bond option and *LIBOR* option have different settlement days? It is because …
- forward bond price (T_0 T_1) is T_0 forward martingale
- forward *LIBOR* (T_0 T_1) is T_1 forward martingale
- hence the main difference between bond and *LIBOR* is the numeraires that make them martingale

- forward contract on *spot LIBOR* is FRA
- option contract on *spot LIBOR* caplet (call) or floorlet (put)
- forward contract on *stream of LIBOR* at fixed swap rate is swap
- option contract on *stream of LIBOR* at fixed swap rate is swaption

► cap (floor) a sequence of caplets (floorlets), the first payment is deterministic on trading day and is thus omitted
► payer swap longs *LIBOR* (speculates a rise in curve), receiver swap shorts *LIBOR* (speculates a drop in curve)
► payer swaption can be considered either as :
- call option on floating leg at the expense of fixed leg (with *strike = swap rate*) or
- call option to enter a payer swap at zero cost (with *strike = \$0*)
- in other words, at maturity, if swap rate is higher than the one specified *here*, you make profit

Suppose swap floating schedule and fixed schedule are :

floating	=	$[T_0, T_1, T_2, T_3, ..., T_N]$	=	*N+1 timepoints and N payments*
fixed	=	$[\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_M]$	=	*M+1 timepoints and M payments*

$$T_0 = \Gamma_0 = T$$
$$T_N = \Gamma_M$$

Here are *IRD* payoffs. Please note :

- Payoff must be written such that it is ***stochastic*** before maturity T, but ***deterministic*** afteward*s*.
- Discounting must be done starting from settlement day (it is different across *IRDs*) to reference day.

Please ensure the payoff is deterministic at maturity, otherwise the RN expectation is incorrect.

$$\begin{aligned}
&\textit{bond} && = && E_Q[\$1\times e^{-\int_t^T r_s ds} \mid I_t] && \\
&\textit{bond option} && = && E_Q[(E_Q[e^{-\int_{T_0}^{T_1} r_s ds} \mid I_{T_0}]-K)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t] && \textit{where } P_{T_0}(T_1)=E_Q[e^{-\int_{T_0}^{T_1} r_s ds} \mid I_{T_0}] \\
& && = && E_Q[(P_{T_0}(T_1)-K)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t] && \textit{fixed@}T_0 \textit{ and pay@}T_0 \\
&\textit{FRA} && = && E_Q[(L_{T_0}(T_1)-K) \times e^{-\int_t^{T_1} r_s ds} \mid I_t] && \textit{fixed@}T_0 \textit{ and pay@}T_1 \\
&\textit{caplet} && = && E_Q[(L_{T_0}(T_1)-K)^+ \times e^{-\int_t^{T_1} r_s ds} \mid I_t] && \textit{fixed@}T_0 \textit{ and pay@}T_1
\end{aligned}$$

$$\textit{swap} \quad = \quad E_Q\left[\left(E_Q\left[\sum_{n=1}^N L_{T_{n-1}}(T_n)\delta(T_{n-1},T_n)e^{-\int_T^{T_n} r_s ds} - \sum_{m=1}^M K\delta(\Gamma_{m-1},\Gamma_m)e^{-\int_T^{\Gamma_m} r_s ds} \mid I_T\right]\right)\times e^{-\int_t^T r_s ds} \mid I_t\right]$$

$$\textit{swaption} \quad = \quad E_Q\left[\underbrace{\left(E_Q\left[\sum_{n=1}^N L_{T_{n-1}}(T_n)\delta(T_{n-1},T_n)e^{-\int_T^{T_n} r_s ds} - \sum_{m=1}^M K\delta(\Gamma_{m-1},\Gamma_m)e^{-\int_T^{\Gamma_m} r_s ds} \mid I_T\right]\right)}_{\text{This payoff is deterministic as of time } T.}^+ \times e^{-\int_t^T r_s ds} \mid I_t\right]$$

Just a minor difference between swap and swaption

We remove the inner expectation inside the RN expectation of swap by Tower property :

remove inner expectation

$$\textit{swap} \quad = \quad E_Q\left[\left(\sum_{n=1}^N L_{T_{n-1}}(T_n)\delta(T_{n-1},T_n)e^{-\int_T^{T_n} r_s ds} - \sum_{m=1}^M K\delta(\Gamma_{m-1},\Gamma_m)e^{-\int_T^{\Gamma_m} r_s ds}\right)\times e^{-\int_t^T r_s ds} \mid I_t\right]$$

However we cannot do the same thing on swaption, because $(E[f])^+ \neq E[(f)^+]$*, so Tower property does not apply.*

$$\textit{swaption} \quad \neq \quad E_Q\left[\underbrace{\left(\sum_{n=1}^N L_{T_{n-1}}(T_n)\delta(T_{n-1},T_n)e^{-\int_T^{T_n} r_s ds} - \sum_{m=1}^M K\delta(\Gamma_{m-1},\Gamma_m)e^{-\int_T^{\Gamma_m} r_s ds}\right)^+}_{\text{This payoff is not deterministic as of time } T.} \times e^{-\int_t^T r_s ds} \mid I_t\right]$$

Besides this payoff is incorrect as it is not deterministic as of time T.

$$\begin{aligned}
\textit{swap} \quad &= E_Q\left[\left(\sum_{n=1}^N \delta(T_{n-1},T_n)E_Q[L_{T_{n-1}}(T_n)e^{-\int_T^{T_n} r_s ds} \mid I_T] - K\sum_{m=1}^M \delta(\Gamma_{m-1},\Gamma_m)E_Q[e^{-\int_T^{\Gamma_m} r_s ds} \mid I_T]\right)\times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= E_Q\left[\left(\sum_{n=1}^N \delta(T_{n-1},T_n)L_T(T_{n-1},T_n)P_T(T_n) - K\sum_{m=1}^M \delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m)\right)\times e^{-\int_t^T r_s ds} \mid I_t\right] && \textit{using eq(22)} \\
&= E_Q\left[(P_T(T_0)-P_T(T_N)-KA_T(\Gamma_0,\Gamma_M))\times e^{-\int_t^T r_s ds} \mid I_t\right] && \textit{using eq(swap1-4)} \\
&= P_t(T_0)-P_t(T_N)-KA_t(\Gamma_0,\Gamma_M) && \textit{using eq(21,23)}
\end{aligned}$$

$$\begin{aligned}
\textit{swaption} \quad &= E_Q\left[\left(\sum_{n=1}^N \delta(T_{n-1},T_n)E_Q[L_{T_{n-1}}(T_n)e^{-\int_T^{T_n} r_s ds} \mid I_T] - K\sum_{m=1}^M \delta(\Gamma_{m-1},\Gamma_m)E_Q[e^{-\int_T^{\Gamma_m} r_s ds} \mid I_T]\right)^+\times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= E_Q\left[\left(\sum_{n=1}^N L_T(T_{n-1},T_n)\delta(T_{n-1},T_n)P_T(T_n) - \sum_{m=1}^M K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m)\right)^+\times e^{-\int_t^T r_s ds} \mid I_t\right] && \textit{follow above steps}
\end{aligned}$$

Just a minor difference between swap and swaption

4.2 Bond option

Zero coupon bond option

We are going to price :

- a call option with maturity T_0 and strike K
- with zero coupon bond that accrues from T_0 to T_1 as underlying
- using ***lognormal bond model*** in *eq(28a,b)* and using ***Hull White short rate model***

$$
\begin{aligned}
\textit{bond option} &= E_Q[(E_Q[e^{-\int_{T_0}^{T_1} r_s ds} \mid I_{T_0}]-K)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t] \\
&= E_Q[(P_{T_0}(T_1)-K)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t] && \textit{using eq(2) or eq(20)} \\
&= E_{Q_{T_0}}\left[\frac{e^{+\int_t^{T_0} r_s ds}}{P_{T_0}(T_0)/P_t(T_0)}(P_{T_0}(T_1)-K)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t\right] \\
&= P_t(T_0)E_{Q_{T_0}}[(P_{T_0}(T_0,T_1)-K)^+ \mid I_t] \\
&= P_t(T_0)E_{Q_{T_0}}\left[\left(P_t(T_0,T_1)\exp(-\frac{1}{2}\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds+\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})dz_s^0)-K\right)^+ \mid I_t\right] && \textit{using eq(28f)} \\
&= E_{Q_{T_0}}\left[\left(\underbrace{P_t(T_1)\exp(-\frac{1}{2}\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds+\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})dz_s^0)-P_t(T_0)K}\right)^+ \mid I_t\right] && \textit{using eq(1)} \\
&= F_{BS}N(d_1)-P_t(T_0)KN(d_2) \qquad \textit{This is lognormal under } T_0 \textit{ forward measure.} \\
&= P_t(T_1)N(d_1)-P_t(T_0)KN(d_2)
\end{aligned}
$$

Note $\sigma_{n,s}$ is different from σ as the former is volatility, i.e. rate of `stddev`*, while the latter is simple a* `stddev`*.*

$$
\begin{aligned}
\textit{where} \quad F_{BS} &= E_{Q_{T_0}}\left[P_t(T_1)\exp\left(-\frac{1}{2}\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds+\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})dz_s^0\right) \mid I_t\right] \\
&= P_t(T_1)\exp\left(-\frac{1}{2}\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds+\frac{1}{2}V_{Q_{T_0}}\left[\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})dz_s^0 \mid I_t\right]\right) && \textit{where } E[e^{\varepsilon(\mu,\sigma)}]=e^{\mu+\sigma^2/2} \\
&= P_t(T_1)\exp\left(-\frac{1}{2}\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds+\frac{1}{2}\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds\right) \\
&= P_t(T_1) \\
\Sigma_{BS} &= V_{Q_{T_0}}\left[\ln\left(P_t(T_1)\exp\left(-\frac{1}{2}\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds+\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})dz_s^0\right)\right) \mid I_t\right] \\
&= V_{Q_{T_0}}[\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})dz_s^0 \mid I_t] && \textit{deterministic terms removed} \\
&= \int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds \\
d_{1,2} &= \frac{\ln\left(\frac{F_{BS}}{P_t(T_0)K}\right)\pm\frac{1}{2}\Sigma_{BS}}{\sqrt{\Sigma_{BS}}} && \textit{eq(40a)} \\
&= \frac{\ln\left(\frac{P_t(T_0,T_1)}{K}\right)\pm\frac{1}{2}\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds}{\sqrt{\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds}} && \textit{eq(40b)}
\end{aligned}
$$

- Bond option price is quoted by $\sigma_{bondopt,mkt}$ which can be plugged into *eq(40a)* and gives *PV* (no $\sigma_{0,s}$ and $\sigma_{1,s}$ involved).

$$
\sigma_{bondopt,mkt} = \sqrt{\frac{\Sigma_{BS}}{T_0-t}} = \sqrt{\frac{\int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds}{T_0-t}}
$$

- Plug Hull White model into *eq(40b),* so that we have a bond option formula purely in terms of Hull White parameters :

$$
\begin{aligned}
F_{BS} &= P_t(T_1) &&= A(t,T_1)e^{-r_t B(t,T_1)} \\
\Sigma_{BS} &= \int_t^{T_0}(\sigma_{1,s}-\sigma_{0,s})^2 ds &&= \frac{\sigma^2}{2a^3}(e^{-aT_1}-e^{-aT_0})^2(e^{2aT}-e^{2at}) && \textit{using eq(32)} \\
d_{1,2} &= \frac{\ln\left(\frac{F_{BS}}{P_t(T_0)K}\right)\pm\frac{1}{2}\Sigma_{BS}}{\sqrt{\Sigma_{BS}}} &&= \frac{\ln\left(\frac{A(t,T_1)e^{-r_t B(t,T_1)}}{A(t,T_0)e^{-r_t B(t,T_0)}K}\right)\pm\frac{1}{2}\Sigma_{BS}}{\sqrt{\Sigma_{BS}}}
\end{aligned}
$$

Coupon bond option – Jamshidian's trick

Consider an option on *$1* notional bond :

- with N coupons, coupon rate are c_n for $n \in [1,N]$
- T_n is coupon payment date and
- T_0 is option maturity

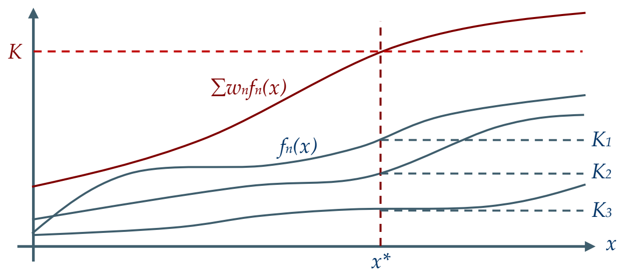
$$\textit{bond} \quad = \quad P_{T_0}(T_N) + \sum_{n=1}^{N} c_n P_{T_0}(T_n)$$

$$\textit{bond option} \quad = \quad E_Q\left[(P_{T_0}(T_N) + \sum_{n=1}^{N} c_n P_{T_0}(T_n) - K)^+ \times e^{-\int_t^{T_0} r_s ds} \,|\, I_t \right]$$

Main challenge for calculating this expectation is that sum of lognormal is not lognormal, thus Black Scholes doesnt apply. In order to solve the expectation, we apply Jamshidian's trick, breakdown the ***coupon bond option*** into sum of ***zero coupon bond options***.

For any monotonic increasing functions $f_n(x)$ $\forall n \in [1,N]$, we have :

$$
\begin{aligned}
& [\sum_n w_n f_n(x) - K]^+ & & \\
= \; & [\sum_n w_n f_n(x) - \sum_n w_n f_n(x^*)]^+ & & \textit{where we have } K = \sum_n w_n f_n(x^*) \\
= \; & \sum_n w_n [f_n(x) - f_n(x^*)]^+ & & \textit{for mono-increasing } f_n(x),\; f_n(x) > f_n(x^*) \textit{ iff } x > x^* \\
= \; & \sum_n w_n [f_n(x) - K_n]^+ & & \textit{where } K_n = f_n(x^*), \textit{ hence } K = \sum_n w_n K_n
\end{aligned}
$$



By putting $f_n(x)$ as spot bond price $P_{T_0}(T_n \,|\, r_{T_0})$ and putting x as short rate r_{T_0} at T_0, then we solve for r_{T_0} such that :

$$
\begin{aligned}
K \; & = \; P_{T_0}(T_N \,|\, r_{T_0} = r^*) + \sum_{n=1}^{N} c_n P_{T_0}(T_n \,|\, r_{T_0} = r^*) & & \textit{where } P_{T_0}(T_n \,|\, r_{T_0}) = A(T_0, T_n) e^{-r_{T_0} B(T_0, T_n)} \\
& = \; \sum_{n=1}^{N} w_n P_{T_0}(T_n \,|\, r_{T_0} = r^*) & & \textit{where } w_n = \begin{bmatrix} c_n & \textit{for} & n \neq N \\ 1 + c_N & \textit{for} & n = N \end{bmatrix} \\
& = \; \sum_{n=1}^{N} w_n K_n & & \textit{define } K_n = P_{T_0}(T_n \,|\, r_{T_0} = r^*)
\end{aligned}
$$

Therefore we can price each sub-option separately as :

$$
\begin{aligned}
\textit{bond option} \; & = \; E_Q\left[(P_{T_0}(T_N) + \sum_{n=1}^{N} c_n P_{T_0}(T_n) - K)^+ \times e^{-\int_t^{T_0} r_s ds} \,|\, I_t \right] \\
& = \; E_Q\left[(\sum_{n=1}^{N} w_n P_{T_0}(T_n) - \sum_{n=1}^{N} w_n K_n)^+ \times e^{-\int_t^{T_0} r_s ds} \,|\, I_t \right] \\
& = \; E_Q\left[\sum_{n=1}^{N} w_n (P_{T_0}(T_n) - K_n)^+ \times e^{-\int_t^{T_0} r_s ds} \,|\, I_t \right] \\
& = \; \sum_{n=1}^{N} w_n E_Q\left[(P_{T_0}(T_n) - K_n)^+ \times e^{-\int_t^{T_0} r_s ds} \,|\, I_t \right] \\
& = \; \sum_{n=1}^{N} w_n (P_t(T_n) N(d_{1,n}) - P_t(T_0) K N(d_{2,n})) & & \textit{using bond call-option formula directly}
\end{aligned}
$$

$$\textit{where} \quad d_{1/2,n} \; = \; \frac{\ln\left(\frac{P_t(T_0, T_n)}{K}\right) \pm \frac{1}{2}\int_t^{T_0} (\sigma_{n,s} - \sigma_{0,s})^2 ds}{\sqrt{\int_t^{T_0} (\sigma_{n,s} - \sigma_{0,s})^2 ds}}$$

4.3 Caplet

Caplet - Blacks formula

Forward *LIBOR* is proved to be forward martingale, yet there is no implication about its distribution. In this part, we derive Black's formula for caplet simply by assuming *(driftless)* log-normal forward *LIBOR* under T_1 forward measure (not T_0 forward measure) :

$$
\begin{array}{llll}
& dL_t(T_0,T_1) & = & \sigma_{L,t} L_t(T_0,T_1) dz_t^1 \\
\Rightarrow & L_T(T_0,T_1) & = & L_t(T_0,T_1)\exp(-\frac{1}{2}\int_t^T \sigma_{L,s}{}^2 ds + \int_t^T \sigma_{L,s} dz_s^1)
\end{array}
$$

We are going to price :

- a call option with maturity $T = T_0$ and strike K
- with forward *LIBOR* $L_T(T_0,T_1)$ that accrues from T_0 to T_1 as underlying
- using ***lognormal bond model*** (as shown above) and using ***Hull White short rate model*** (in next section)

$$
\begin{array}{lll}
caplet & = & E_Q[(L_{T_0}(T_1)-K)^+ \times e^{-\int_t^{T_1} r_s ds} \mid I_t] \\
& = & E_{Q_{T_1}}\left[\dfrac{e^{+\int_t^{T_1} r_s ds}}{P_{T_1}(T_1)/P_t(T_1)} \times (L_{T_0}(T_1)-K)^+ \times e^{-\int_t^{T_1} r_s ds} \mid I_t\right] \\
& = & P_t(T_1) E_{Q_{T_1}}[(L_{T_0}(T_0,T_1)-K)^+ \mid I_t] \\
& = & P_t(T_1)(F_{BS}N(d_1)-KN(d_2)) \\
& = & P_t(T_1)(L_t(T_0,T_1)N(d_1)-KN(d_2))
\end{array}
$$

$$
\begin{array}{llll}
where & F_{BS} & = & E_{Q_{T_1}}\left[L_t(T_0,T_1)\exp\left(-\frac{1}{2}\int_t^{T_0}\sigma_{L,s}{}^2 ds + \int_t^{T_0}\sigma_{L,s} dz_s^1\right) \mid I_t\right] \\
& & = & L_t(T_0,T_1)\exp\left(-\frac{1}{2}\int_t^{T_0}\sigma_{L,s}{}^2 ds + \frac{1}{2} V_{Q_{T_1}}\left[\int_t^{T_0}\sigma_{L,s} dz_s^1 \mid I_t\right]\right) \\
& & = & L_t(T_0,T_1)\exp\left(-\frac{1}{2}\int_t^{T_0}\sigma_{L,s}{}^2 ds + \frac{1}{2}\int_t^{T_0}\sigma_{L,s}{}^2 ds\right) \\
& & = & L_t(T_0,T_1) \\
& \Sigma_{BS} & = & E_{Q_{T_1}}\left[\ln\left(L_t(T_0,T_1)\exp\left(-\frac{1}{2}\int_t^{T_0}\sigma_{L,s}{}^2 ds + \int_t^{T_0}\sigma_{L,s} dz_s^1\right)\right) \mid I_t\right] \\
& & = & V_{Q_{T_1}}[\int_t^{T_0}\sigma_{L,s} dz_s^1 \mid I_t] \qquad \textit{deterministic terms removed} \\
& & = & \int_t^{T_0}\sigma_{L,s}{}^2 ds \\
& d_{1,2} & = & \dfrac{\ln\left(\dfrac{F_{BS}}{K}\right) \pm \frac{1}{2}\Sigma_{BS}}{\sqrt{\Sigma_{BS}}} \qquad eq(41a) \\
& & = & \dfrac{\ln\left(\dfrac{L_t(T_0,T_1)}{K}\right) \pm \frac{1}{2}\int_t^{T_0}\sigma_{L,s}{}^2 ds}{\sqrt{\int_t^{T_0}\sigma_{L,s}{}^2 ds}} \qquad eq(41b)
\end{array}
$$

- Caplet price is quoted by $\sigma_{caplet,mkt}$ which can be plugged into *eq(41a)* and gives *PV* (no $\sigma_{L,s}$ involved).

$$
\sigma_{caplet,mkt} \quad = \quad \sqrt{\frac{\Sigma_{BS}}{T_0 - t}} \quad = \quad \sqrt{\frac{\int_t^{T_0}\sigma_{L,s}{}^2 ds}{T_0 - t}}
$$

- Yet the Blacks formula is not in terms of bond prices, we cannot plug in Hull White model like what we do for bond option.

Caplet –Hull White formula

We have to work out a caplet formula in terms of bond prices using alternative approach, so that we can plug in Hull White model. In this alternative approach, we do not assume lognormal forward *LIBOR*, instead we assume lognormal bond price.

$$
\begin{aligned}
\textit{caplet} &= E_Q[(L_{T_0}(T_1)-K)^+ \times e^{-\int_t^{T_1} r_s ds} \mid I_t] \\
&= E_Q[E_Q[(L_{T_0}(T_1)-K)^+ \times e^{-\int_{T_0}^{T_1} r_s ds} \mid I_{T_0}] \times e^{-\int_t^{T_0} r_s ds} \mid I_t] \\
&= E_Q\left[E_{Q_{T_1}}\left[\frac{\times e^{+\int_{T_0}^{T_1} r_s ds}}{P_{T_1}(T_1)/P_{T_0}(T_1)} (L_{T_0}(T_1)-K)^+ \times e^{-\int_{T_0}^{T_1} r_s ds} \mid I_{T_0} \right] \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] \\
&= E_Q\left[E_{Q_{T_1}}[P_{T_0}(T_1)(L_{T_0}(T_1)-K)^+ \mid I_{T_0}] \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] \\
&= E_Q\left[P_{T_0}(T_1)(L_{T_0}(T_1)-K)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] && \textit{all items inside inner E[...] are known at } T_0 \\
&= E_Q\left[\left(P_{T_0}(T_1)\frac{1/P_{T_0}(T_1)-1}{\delta(T_0,T_1)} - P_{T_0}(T_1)K \right)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] && \textit{using eq(4) which is } L_t(T)=\frac{1/P_t(T)-1}{\delta(t,T)} \\
&= \frac{1}{\delta(T_0,T_1)} E_Q\left[(1-P_{T_0}(T_1)-P_{T_0}(T_1)\delta(T_0,T_1)K)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] \\
&= \frac{1}{\delta(T_0,T_1)} E_Q\left[(1-(1+\delta(T_0,T_1)K)P_{T_0}(T_1))^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] \\
&= \frac{1+\delta(T_0,T_1)K}{\delta(T_0,T_1)} E_Q\left[\left(\frac{1}{1+\delta(T_0,T_1)K} - P_{T_0}(T_1) \right)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] && \textit{This discounting path also exists in bond option.} \\
&= \frac{1+\delta(T_0,T_1)K}{\delta(T_0,T_1)} \times [P_t(T_0)K^* N(-d_2) - P_t(T_1)N(-d_1)] && \textit{using bond put-option formula directly}
\end{aligned}
$$

where

$$
\begin{aligned}
F_{BS} &= P_t(T_1) && \textit{using bond put-option formula directly} \\
\Sigma_{BS} &= \int_t^{T_0} (\sigma_{1,s}-\sigma_{0,s})^2 ds && \textit{using bond put-option formula directly} \\
K^* &= \frac{1}{1+\delta(T_0,T_1)K} \\
d_{1,2} &= \frac{\ln\left(\frac{F_{BS}}{P_t(T_0)K^*} \right) \pm \frac{1}{2}\Sigma_{BS}}{\sqrt{\Sigma_{BS}}}
\end{aligned}
$$

This approach will be used in swaption too. Now, caplet can be considered as bond option with :

- different shares $\frac{1+\delta(T_0,T_1)K}{\delta(T_0,T_1)}$
- different strike $K^* = \frac{1}{1+\delta(T_0,T_1)K}$
- different sides

- caplet speculates a rise in *LIBOR*, bond put-option speculates a drop in bond price, or equivalently, a rise in *LIBOR*
- floorlet speculates a drop in *LIBOR*, bond call-option speculates a rise in bond price, or equivalently, a drop in *LIBOR*

Plug Hull White model into the above, so that we have a caplet formula purely in terms of Hull White parameters :

$$
\begin{aligned}
F_{BS} &= P_t(T_1) &&= A(t,T_1)e^{-r_t B(t,T_1)} \\
\Sigma_{BS} &= \int_t^{T_0} (\sigma_{1,s}-\sigma_{0,s})^2 ds &&= \frac{\sigma^2}{2a^3}(e^{-aT_1}-e^{-aT_0})^2(e^{2aT}-e^{2at})
\end{aligned}
$$

which are exactly the same as counterparts in bond option.

Remark 1 – Caplet formula in terms of LIBOR

Caplet is an option on *LIBOR*, hence it is better to express caplet price in terms of *LIBOR* rather than in terms of bond price.

$$
\begin{aligned}
caplet &= \frac{1+\delta(T_0,T_1)K}{\delta(T_0,T_1)}\times[P_t(T_0)\frac{1}{1+\delta(T_0,T_1)K}N(-d_2)-P_t(T_1)N(-d_1)] && \textit{do manipulation so as to apply eq(11)}\\
&= \frac{1}{\delta(T_0,T_1)}\times[P_t(T_0)N(-d_2)-P_t(T_1)(1+\delta(T_0,T_1)K)N(-d_1)]\\
&= \frac{P_t(T_1)}{\delta(T_0,T_1)}\times[\frac{1}{P_t(T_0,T_1)}N(-d_2)-(1+\delta(T_0,T_1)K)N(-d_1)] && \textit{using eq(11)}\;\; L_t(T_0,T_1)=\frac{1/P_t(T_0,T_1)-1}{\delta(T_0,T_1)}\\
&= \frac{P_t(T_1)}{\delta(T_0,T_1)}\times[(1+\delta(T_0,T_1)L_t(T_0,T_1))N(-d_2)-(1+\delta(T_0,T_1)K)N(-d_1)]\\
&= \frac{P_t(T_1)}{\delta(T_0,T_1)}\times[w_2N(-d_2)-w_1N(-d_1)]
\end{aligned}
$$

$$
\begin{aligned}
\textit{where}\quad w_1 &= 1+\delta(T_0,T_1)K\\
w_2 &= 1+\delta(T_0,T_1)L_t(T_0,T_1)
\end{aligned}
$$

Remark 2 - Caplet stripping and BS2HW calibration

In practice market price of *LIBOR* option is quoted in *cap-volatility* in lieu of *caplet-volatility*. We need a two-step conversion :

- Blacks *cap-volatility* to Blacks *caplet-volatility* stripping (also known as bootstrapping)
- Blacks *caplet-volatility* to Hull White *short rate volatility* calibration

Stripping (bootstrapping)

Cap(T$_N$*)* is consisted of *N-1* caplets namely *caplet(T*$_1$*) caplet(T*$_2$*) … caplet(T*$_{N-1}$*)*, whereas *caplet(0)* is excluded as *L*$_0$*(0,T*$_1$*)* is already fixed.

Suppose current time is *0*, consider cap and caplet on *3* months *LIBOR* (lets denote *3* months as Δ) :

excluded

$$
\begin{aligned}
cap(N\Delta) &= \cancel{caplet(0\Delta)} + caplet(1\Delta) + caplet(2\Delta) + \ldots + caplet((N-1)\Delta)\\
caplet(n\Delta) &= caplet(T_0 = n\Delta,\ T_1 = (n+1)\Delta)\\
\sigma_{caplet}(t) &= \textit{bootstrapped } \textbf{caplet-volatility} \textit{ term structure, which is a curve } \forall t\in[0,(N-1)\Delta]\\
\sigma_{cap,mkt}(T) &= \textit{market quoted } \textbf{cap-volatility} \textit{ with tenor T, which is a scalar}
\end{aligned}
$$

We extend range of $\sigma_{caplet}(t)$ to $t\in[0,N]$ by adding market quote $\sigma_{cap,mkt}((N+1)\Delta)$. Solve for *x* below and append it to $\sigma_{caplet}(t)$.

$$
\boxed{
\begin{aligned}
\sum_{n\in[1,N]} f(n,\sigma_{cap,mkt}((N+1)\Delta)) &= \sum_{n\in[1,N-1]} f(n,\sigma_{caplet}(n\Delta)) + f(N,x)\\
\textit{where}\quad f(n,\sigma) &= P_0((n+1)\Delta)\times[L_0(n\Delta,(n+1)\Delta)N(d_1(n,\sigma))-KN(d_2(n,\sigma))]\\
d_{1,2}(n,\sigma) &= \frac{\ln\left(\frac{L_0(n\Delta,(n+1)\Delta)}{K}\right)\pm\frac{1}{2}\sigma^2 n\Delta}{\sqrt{\sigma^2 n\Delta}}
\end{aligned}
}
$$

Calibration

BS2HW calibration refers to solving for volatility x below for each tenor point :

$$
\begin{aligned}
& P_t(T_1)[L_t(T_0,T_1)N(d_{BS1}(\sigma_{caplet}(T_0)))-KN(d_{BS2}(\sigma_{caplet}(T_0)))]\\
=\ & \frac{1+\delta(T_0,T_1)K}{\delta(T_0,T_1)}\times\left[P_t(T_0)\frac{1}{1+\delta(T_0,T_1)K}N(-d_{HW2}(x))-P_t(T_1)N(-d_{HW1}(x))\right]
\end{aligned}
$$

typical BS volatility	$\sigma_{cap,mkt}$	~	*[0.10 to 0.50]*
typical HW volatility	σ	~	*[0.0040 to 0.0080]*

4.4 Swaption

Swaption – Blacks formula

Forward *swaprate* is proved to be annuity martingale, yet there is no implication about its distribution. In this part, we derive Blacks formula for swaption simply by assuming *(driftless)* log-normal forward *swaprate* under $\Gamma_0\,\Gamma_M$ annuity measure :

$$
\begin{aligned}
dS_t(\Gamma_0,\Gamma_M) &= \sigma_{S,t} S_t(\Gamma_0,\Gamma_M) dz_t^A \\
\Rightarrow \quad S_T(\Gamma_0,\Gamma_M) &= S_t(\Gamma_0,\Gamma_M)\exp(-\frac{1}{2}\int_t^T \sigma_{S,s}{}^2 ds + \int_t^T \sigma_{S,s} dz_s^A)
\end{aligned}
$$

We are going to price :

- a call option with maturity T and strike $\$0$
- with forward swap having floating schedule $T_n\ \forall n\in[0,N]$ and fixed schedule $\Gamma_m\ \forall m\in[0,M]$ as underlying
- using ***lognormal swaprate model*** (as shown above) and using ***Hull White short rate model*** (in next section)

$$
\begin{aligned}
swaption &= E_Q\left[\left(\sum_{n=1}^N L_T(T_{n-1},T_n)\delta(T_{n-1},T_n)P_T(T_n) - \sum_{m=1}^M K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m)\right)^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= E_Q\left[\left(\sum_{m=1}^M S_T(\Gamma_0,\Gamma_M)\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m) - \sum_{m=1}^M K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m)\right)^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \qquad \textit{using eq(24)} \\
&= E_Q\left[((S_T(\Gamma_0,\Gamma_M)-K)\times\sum_{m=1}^M \delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m))^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \qquad \textit{only fixed schedule matters} \\
&= E_Q\left[\underbrace{((S_T(\Gamma_0,\Gamma_M)-K)\times A_T(\Gamma_0,\Gamma_M))^+}\times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= E_{Q_{\Gamma_0,\Gamma_M}}\left[\frac{e^{+\int_t^T r_s ds}}{A_T(\Gamma_0,\Gamma_M)/A_t(\Gamma_0,\Gamma_M)}((S_T(\Gamma_0,\Gamma_M)-K)\times A_T(\Gamma_0,\Gamma_M))^+ e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= A_t(\Gamma_0,\Gamma_M)E_{Q_{\Gamma_0,\Gamma_M}}[(S_T(\Gamma_0,\Gamma_M)-K)^+ \mid I_t] \\
&= A_t(\Gamma_0,\Gamma_M)[F_{BS}N(d_1)-KN(d_2)] \\
&= A_t(\Gamma_0,\Gamma_M)[S_t(\Gamma_0,\Gamma_M)N(d_1)-KN(d_2)] \qquad S_t(\Gamma_0,\Gamma_M)=\frac{P_t(T_0)-P_t(T_N)}{A_t(\Gamma_0,\Gamma_M)} \\
&= (P_t(\Gamma_0)-P_t(\Gamma_M))N(d_1)-KA_t(\Gamma_0,\Gamma_M)N(d_2) \qquad \textit{using eq(24')}
\end{aligned}
$$

$$
\begin{aligned}
\textit{where} \quad F_{BS} &= E_{Q_{\Gamma_0\Gamma_M}}\left[S_t(\Gamma_0,\Gamma_M)\exp(-\frac{1}{2}\int_t^T \sigma_{S,s}{}^2 ds + \int_t^T \sigma_{S,s} dz_s^A) \mid I_t\right] \\
&= S_t(\Gamma_0,\Gamma_M)\exp\left((-\frac{1}{2}\int_t^T \sigma_{S,s}{}^2 ds + \frac{1}{2}V_{Q_{\Gamma_0\Gamma_M}}\left[\int_t^T \sigma_{S,s} dz_s^A \mid I_t\right]\right) \\
&= S_t(\Gamma_0,\Gamma_M)\exp\left((-\frac{1}{2}\int_t^T \sigma_{S,s}{}^2 ds + \frac{1}{2}\int_t^T \sigma_{S,s}{}^2 ds\right) \\
&= S_t(\Gamma_0,\Gamma_M) \\
\Sigma_{BS} &= V_{Q_{\Gamma_0\Gamma_M}}\left[\ln\left(S_t(\Gamma_0,\Gamma_M)\exp(-\frac{1}{2}\int_t^T \sigma_{S,s}{}^2 ds + \int_t^T \sigma_{S,s} dz_s^A)\right) \mid I_t\right] \\
&= V_{Q_{\Gamma_0\Gamma_M}}\left[\int_t^T \sigma_{S,s} dz_s^A \mid I_t\right] \\
&= \int_t^T \sigma_{S,s}{}^2 ds \\
d_{1,2} &= \frac{\ln\frac{F_{BS}}{K} \pm \frac{1}{2}\Sigma_{BS}}{\sqrt{\Sigma_{BS}}} \\
&= \frac{\ln\frac{S_t(\Gamma_0,\Gamma_M)}{K} \pm \frac{1}{2}\int_t^T \sigma_{S,s}{}^2 ds}{\sqrt{\int_t^T \sigma_{S,s}{}^2 ds}}
\end{aligned}
$$

payoff

$$
\begin{aligned}
&= ((S_T(\Gamma_0,\Gamma_M)-K)\times A_t(\Gamma_0,\Gamma_M))^+ \\
&= (S_T(\Gamma_0,\Gamma_M)A_t(\Gamma_0,\Gamma_M)-KA_t(\Gamma_0,\Gamma_M))^+ \\
&= (P_T(\Gamma_0)-P_T(\Gamma_M)-KA_t(\Gamma_0,\Gamma_M))^+
\end{aligned}
$$

- Caplet price is quoted by $\sigma_{caplet,mkt}$ which can be plugged into *eq(41a)* and gives *PV* (no $\sigma_{L,s}$ involved).

$$
\sigma_{swpt,mkt} = \sqrt{\frac{\Sigma_{BS}}{T-t}} = \sqrt{\frac{\int_t^T \sigma_{S,s}{}^2 ds}{T-t}}
$$

- Yet the Blacks formula is not in terms of bond prices, we cannot plug in Hull White model like what we do for bond option.

Swaption –Hull White formula

In this alternative approach, we do not assume lognormal forward swaprate, instead we assume lognormal bond price.

$$\begin{aligned}
swaption &= E_Q\left[\left(\sum_{n=1}^{N} L_T(T_{n-1},T_n)\delta(T_{n-1},T_n)P_T(T_n) - \sum_{m=1}^{M} K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m)\right)^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= E_Q\left[\left(P_T(T_0) - P_T(T_N) - \sum_{m=1}^{M} K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m)\right)^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= E_Q\left[\left(P_T(\Gamma_0) - P_T(\Gamma_M) - \sum_{m=1}^{M} K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m)\right)^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= E_Q\left[(1 - \sum_{m=1}^{M} K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m) - P_T(\Gamma_M))^+ \times e^{-\int_t^T r_s ds} \mid I_t\right]
\end{aligned}$$

Thus payer swaption can be regarded as a coupon-bond put-option with :

- maturity T, strike $\$1$ and notional $\$1$
- coupon schedule equivalent to fixed leg schedule
- coupon rate equivalent to predetermined swap rate $K\delta(\Gamma_{m-1},\Gamma_m)$
- thus we have to apply Jamshidian's trick

Firstly, we solve for r^* such that :

$$\begin{aligned}
1 &= \sum_{m=1}^{M} K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m \mid r^*) + P_T(\Gamma_M \mid r^*) && \textit{where } P_T(\Gamma_m \mid r) = A(\Gamma_0,\Gamma_m)e^{-rB(\Gamma_0,\Gamma_m)} \textit{ and } \Gamma_0 = T \\
&= \sum_{m=1}^{M} w_m P_T(\Gamma_m \mid r^*) && \textit{where } w_m = \begin{bmatrix} K\delta(\Gamma_{m-1},\Gamma_m) & m \neq M \\ K\delta(\Gamma_{M-1},\Gamma_M)+1 & m = M \end{bmatrix} \\
&= \sum_{m=1}^{M} w_m \kappa_m && \textit{define } \kappa_m = P_T(\Gamma_m \mid r^*)
\end{aligned}$$

$$\begin{aligned}
swaption &= E_Q\left[(1 - \sum_{m=1}^{M} K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m) - P_T(\Gamma_M))^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= E_Q\left[(\sum_{m=1}^{M} w_m\kappa_m - \sum_{m=1}^{M} w_m P_T(\Gamma_m))^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= E_Q\left[\sum_{m=1}^{M} w_m(\kappa_m - P_T(\Gamma_m))^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= \sum_{m=1}^{M} w_m E_Q[(\kappa_m - P_T(\Gamma_m))^+ e^{-\int_t^T r_s ds} \mid I_t] \\
&= \sum_{m=1}^{M} w_m (P_t(T)\kappa_m N(d_{2,m}) - P_t(\Gamma_m)N(d_{1,m}))
\end{aligned}$$

$$\begin{aligned}
\textit{where} \quad F_{BS,m} &= P_t(\Gamma_m) \\
\Sigma_{BS,m} &= \int_t^T (\sigma_{m,s} - \sigma_{0,s})^2 ds \\
d_{1/2,m} &= \frac{\ln\left(\dfrac{F_{BS,m}}{P_T(\Gamma_m \mid r^*)}\right) \pm \frac{1}{2}\Sigma_{BS,m}}{\sqrt{\Sigma_{BS,m}}}
\end{aligned}$$

Plug Hull White model into the above, so that we have a swaption formula purely in terms of Hull White parameters :

$$\begin{aligned}
F_{BS,m} &= P_t(\Gamma_m) &&= A(t,\Gamma_m)e^{-r_t B(t,\Gamma_m)} \\
\Sigma_{BS} &= \int_t^T (\sigma_{m,s} - \sigma_{0,s})^2 ds &&= \frac{\sigma^2}{2a^3}(e^{-a\Gamma_m} - e^{-a\Gamma_0})^2(e^{2a\Gamma_0} - e^{2at})
\end{aligned}$$

which are similar to counterparts in bond option.

Swaption – bounded by sum of caplet
This part is optional.

Privault does also present an upper bound on swaption price as a weighted sum, making use of $(\sum x_n)^+ \le \sum (x_n{}^+)$. Unlike swaption Hull White which depends on fixed leg annuity, here we focus on floating annuity.

$$\begin{aligned}
swaption &= E_Q\left[\left(\textstyle\sum_{n=1}^N L_T(T_{n-1},T_n)\delta(T_{n-1},T_n)P_T(T_n) - \sum_{m=1}^M K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m)\right)^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= E_Q\left[\left(\textstyle\sum_{n=1}^N (L_T(T_{n-1},T_n) - K^*)\delta(T_{n-1},T_n)P_T(T_n)\right)^+ e^{-\int_t^T r_s ds} \mid I_t\right] \\
&\le E_Q\left[\textstyle\sum_{n=1}^N (L_T(T_{n-1},T_n) - K^*)^+ \delta(T_{n-1},T_n)P_T(T_n) \times e^{-\int_t^T r_s ds} \mid I_t\right] \qquad \textit{swaption is bounded by a weighted sum} \\
&= \sum_{n=1}^N \delta(T_{n-1},T_n) E_Q\left[(L_T(T_{n-1},T_n) - K^*)^+ P_T(T_n) \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= \sum_{n=1}^N \delta(T_{n-1},T_n) E_Q\left[(L_T(T_{n-1},T_n) - K^*)^+ E_Q\left[e^{-\int_T^{T_n} r_s ds} \mid I_T\right] \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
&= \sum_{n=1}^N \delta(T_{n-1},T_n) E_Q\left[E_Q\left[(L_T(T_{n-1},T_n) - K^*)^+ \times e^{-\int_t^T r_s ds} e^{-\int_T^{T_n} r_s ds} \mid I_T\right] \mid I_t\right] \\
&= \sum_{n=1}^N \delta(T_{n-1},T_n) E_Q\left[(L_T(T_{n-1},T_n) - K^*)^+ \times e^{-\int_t^{T_n} r_s ds} \mid I_t\right] \\
&\ne \sum_{n=1}^N \delta(T_{n-1},T_n) E_Q\left[(L_{T_{n-1}}(T_n) - K^*)^+ \times e^{-\int_t^{T_n} r_s ds} \mid I_t\right] \\
&= \sum_{n=1}^N \delta(T_{n-1},T_n) caplet(option = T_{n-1}, payment = T_n, strike = K^*)
\end{aligned}$$

$$\begin{aligned}
where \quad K^* &= \frac{\sum_{m=1}^M K\delta(\Gamma_{m-1},\Gamma_m)P_T(\Gamma_m)}{\sum_{n=1}^N \delta(T_{n-1},T_n)P_T(T_n)} \\
&= \frac{KA_t(\Gamma_0,\Gamma_M)}{A_t(T_0,T_N)}
\end{aligned}$$

Please note that the above is *NOT* a weighted sum of caplets, as caplet is defined on spot *LIBOR*, but not on forward *LIBOR*.

4.5 Comparison among bond option, cap and swaption

Speculation

- Bond option speculates a rise in bond price (a drop in *LIBOR* curve).
- Cap and caplet speculate a rise in *LIBOR* curve.
- Swaption speculates a rise in *LIBOR* curve.

Dissecting each items

For Blacks formula of bond option and caplet, which assumes lognormal bond price and lognormal *LIBOR* respectively :

bond option

$$\underbrace{P_t(T_1)}_{\text{discount factor of bond notional}}\underbrace{N(d_1)}_{\text{ITM prob under forward } T_0 \text{ measure}}-\underbrace{P_t(T_0)}_{\text{discount factor of bond price}}K\underbrace{N(d_2)}_{\text{ITM prob under cash measure}}$$

caplet

$$\underbrace{P_t(T_1)}_{\text{discount factor}}[\underbrace{L_t(T_0,T_1)}_{\text{LIBOR coupon}}\underbrace{N(d_1^L)}_{\text{ITM prob under forward } T_1 \text{ measure}}-\underbrace{K}_{\text{price of coupon}}\underbrace{N(d_2^L)}_{\text{ITM prob under cash measure}}]$$

Comparison

optionality	*lognormal underlying model*	*normal short rate model*
bond option	$P_t(T_1)N(d_1)-P_t(T_0)KN(d_2)$	*same as LHS*
coupon bond option	$\sum_{n=1}^{N} w_n(P_t(T_n)N(d_{1,n})-P_t(T_0)KN(d_{2,n}))$ *Jamshidian trick*	*same as LHS*
		becomes bond put-option
caplet	$P_t(T_1)[L_t(T_0,T_1)N(d_1^L)-KN(d_2^L)]$	$\frac{1+\delta(T_0,T_1)K}{\delta(T_0,T_1)}\times[P_t(T_0)K*N(-d_2)-P_t(T_1)N(-d_1)]$
swaption	$(P_t(\Gamma_0)-P_t(\Gamma_M))N(d_1^S)-KA_t(\Gamma_0,\Gamma_M)N(d_2^S)$	$\sum_{m=1}^{M} w_m[P_t(T)\kappa_m N(d_{2,m})-P_t(\Gamma_m)N(d_{1,m})]$ *Jamshidian trick*

Various volatilities

HW short rate volatility	σ	*used as model*
bond price volatility	$\sigma_{n,t}$	*used only for derivation of BS (cannot be observed)*
LIBOR volatility	$\sigma_{L,t}$	*used only for derivation of BS (cannot be observed)*
swap rate volatility	$\sigma_{S,t}$	*used only for derivation of BS (cannot be observed)*
bond option BS volatility	$\sigma_{bondopt,mkt}$	*quoted in market, directly gives PV*
cap BS volatility	$\sigma_{cap,mkt}$	*quoted in market, directly gives PV*
swaption BS volatility	$\sigma_{swpt,mkt}$	*quoted in market, directly gives PV*

- When *IRD* is nonlinear, volatility kicks in, however which market volatility should we use? Cap-vol and swaption-vol?
- For nonlinear *IRD* depending on *LIBOR* such as :
  - vanilla option on *LIBOR*, such as caplet / floorlet
  - digital option on *LIBOR*, such as *LIBOR* range accrual
- For nonlinear *IRD* depending on swap-rate such as :
  - swaption
  - *CMS* range accrual
  - *CMS* spread swap, *CMS* digital swap
- All callable swap depends on swaption-vol because of the call-feature, thus :
  - callable *LIBOR* range accrual depends on both cap-vol and swaption-vol

4.6 LIBOR range accrual *[not completed, please verify]*

LIBOR range accrual is a swap with funding leg tied to *LIBOR* and exotic leg as a series of coupons with conditional rate depending on the number of days when *LIBOR* is within the range $[L,U]$. Suppose there are M days during interval $[T_{n-1}, T_n]$, coupon n is fixed daily from T_{n-1} until T_n, payment is done at T_n. Each coupon can be approximated as $4M$ caplets / floorlets.

$$
\begin{aligned}
c_n &= \frac{c}{M}\sum_{m=1}^{M} 1(L_{T_{n-1}+m}(T_{n-1}+m+3M) \in [L,U]) && \textit{where } T_n = T_{n-1}+3M \\
&= \frac{c}{M}\sum_{m=1}^{M} [1(L_{T_{n-1}+m}(T_{n-1}+m+3M) > L) - 1(L_{T_{n-1}+m}(T_{n-1}+m+3M) > U)] && \begin{array}{l}\textit{1 range option decomposed as 2 digital options} \\ \textit{1 digital option decomposed as 2 vanilla options}\end{array} \\
&= \frac{c}{M}\sum_{m=1}^{M} \begin{bmatrix} +[L_{T_{n-1}+m}(T_{n-1}+m+3M)-(L-\Delta)]^+ - [L_{T_{n-1}+m}(T_{n-1}+m+3M)-(L+\Delta)]^+ \\ -[L_{T_{n-1}+m}(T_{n-1}+m+3M)-(U-\Delta)]^+ + [L_{T_{n-1}+m}(T_{n-1}+m+3M)-(U+\Delta)]^+ \end{bmatrix}
\end{aligned}
$$

Coupon n depends on numerous *LIBOR* fixings on different days, yet settlement of coupon n is done on only one day T_n.

$$
\begin{aligned}
\textit{range accrual} &= E_Q[\textstyle\sum_n c_n \times e^{-\int_t^{T_n} r_s ds} \mid I_t] \\
&= E_Q\left[\sum_n \frac{c}{M}\sum_{m=1}^{M} \begin{bmatrix} +[L_{T_{n-1}+m}(T_{n-1}+m+3M)-(L-\Delta)]^+ - [L_{T_{n-1}+m}(T_{n-1}+m+3M)-(L+\Delta)]^+ \\ -[L_{T_{n-1}+m}(T_{n-1}+m+3M)-(U-\Delta)]^+ + [L_{T_{n-1}+m}(T_{n-1}+m+3M)-(U+\Delta)]^+ \end{bmatrix} \times e^{-\int_t^{T_n} r_s ds} \mid I_t \right] \\
&= \sum_n \frac{c}{M}\sum_{m=1}^{M} E_Q\left[\begin{array}{l} +[L_{T_{n-1}+m}(T_{n-1}+m+3M)-(L-\Delta)]^+ \times e^{-\int_t^{T_n} r_s ds} - [L_{T_{n-1}+m}(T_{n-1}+m+3M)-(L+\Delta)]^+ \times e^{-\int_t^{T_n} r_s ds} \\ -[L_{T_{n-1}+m}(T_{n-1}+m+3M)-(U-\Delta)]^+ \times e^{-\int_t^{T_n} r_s ds} + [L_{T_{n-1}+m}(T_{n-1}+m+3M)-(U+\Delta)]^+ \times e^{-\int_t^{T_n} r_s ds} \end{array} \mid I_t \right]
\end{aligned}
$$

We attempt to apply the method used in *caplet Hull White formula* here, however the main challenge is that the *LIBOR* end date does not match with the payment date. We have to cope it with change of measure :

unmatched

$$
\begin{aligned}
\textit{caplet}_{\textit{unmatched}} &= E_Q\left[(\underbrace{L_{T_{n-1}+m}}_{\to T_0}\underbrace{(T_{n-1}+m+3M)}_{\to T_1} - K)^+ \times \underbrace{e^{-\int_t^{T_n} r_s ds}}_{T_n \to T} \mid I_t \right] && \textit{we simplify the time point notation} \\
&= E_Q\left[(L_{T_0}(T_1)-K)^+ \times e^{-\int_t^{T} r_s ds} \mid I_t \right] && \textit{where } t < T_0 < T < T_1 \\
&= E_Q\left[E_Q\left[(L_{T_0}(T_1)-K)^+ \times e^{-\int_{T_0}^{T} r_s ds} \mid I_{T_0} \right] \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] \\
&= E_Q\left[E_{Q_T}\left[\frac{e^{+\int_{T_0}^{T} r_s ds}}{P_T(T)/P_{T_0}(T)} (L_{T_0}(T_1)-K)^+ \times e^{-\int_{T_0}^{T} r_s ds} \mid I_{T_0} \right] \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] \\
&= E_Q\left[E_{Q_T}[P_{T_0}(T)(L_{T_0}(T_1)-K)^+ \mid I_{T_0}] \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] \\
&= E_Q\left[P_{T_0}(T)(L_{T_0}(T_1)-K)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] && \textit{all items are known as of } T_0 \\
&= E_Q\left[\left(P_{T_0}(T)\frac{1/P_{T_0}(T_1)-1}{\delta(T_0,T_1)} - P_{T_0}(T)K \right)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] && \textit{using eq(4)} \;\; L_t(T) = \frac{1/P_t(T)-1}{\delta(t,T)} \\
&= \frac{1}{\delta(T_0,T_1)} E_Q\left[(P_{T_0}{}^{-1}(T,T_1) - P_{T_0}(T) - P_{T_0}(T)\delta(T_0,T_1)K)^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t \right] \\
&= \ldots \\
&= \textit{it may contain } E_{Q_T}[P_{T_0}(T_1) \mid I_t]
\end{aligned}
$$

As the underlying fixing time T_0 does not match with forward T_1 measure of the expectation, convexity adjustment kicks in.

$$
E_{Q_T}[L_{T_0}(T_1) \mid I_t] = E_{Q_{T_1}}\left[L_{T_0}(T_1)\frac{dQ_T}{dQ_{T_1}} \mid I_t \right]
$$

4.7 CMS digital option and CMS range option

CMS digital

CMS digital is a digital call with optionality T on forward swap rate from T_0 to T_N. Risk neutral pricing for *CMS* digital is :

$$\begin{array}{lll}
CMS\ vanilla & = & E_Q\left[(S_T(T_0,T_N)-K)^+ \times e^{-\int_t^T r_s ds} \mid I_t\right] \\
CMS\ digital & = & E_Q\left[1(S_T(T_0,T_N)>K)\times e^{-\int_t^T r_s ds} \mid I_t\right] \qquad \textit{normally, we have } T=T_0 \\
 & = & E_{Q_{T_0}}\left[\dfrac{e^{+\int_t^T r_s ds}}{P_{T_0}(T_0)/P_t(T_0)} 1(S_{T_0}(T_0,T_N)>K)\times e^{-\int_t^T r_s ds} \mid I_t\right] \\
 & = & P_t(T_0)\Pr_{Q_{T_0}}(S_{T_0}(T_0,T_N)>K \mid I_t) \\
 & = & P_t(T_0)\Pr_{Q_T}\left(\dfrac{P_{T_0}(T_0)-P_{T_0}(T_N)}{A_{T_0}(T_0,T_N)}>K \mid I_t\right) \qquad \textit{using eq(24')} \\
 & = & P_t(T_0)\Pr_{Q_T}(KA_{T_0}(T_0,T_N)+P_{T_0}(T_N)<1 \mid I_t) \\
 & = & P_t(T_0)\Pr_{Q_T}(\sum_{n=1}^N w_n P_{T_0}(T_n)<1 \mid I_t) \qquad \textit{where } w_n = \left[\begin{array}{ll} K\delta(T_{n-1},T_n) & n<N \\ K\delta(T_{n-1},T_n)+1 & n=N \end{array}\right. \\
 & = & \textit{coupon-bond digital-option (we haven't go through before)}
\end{array}$$

We then convert the *ITM* probability into integration of a standard normal. Recall Dr Yan's bond price formula *eq(33b)* :

$$\begin{array}{lll}
P_{T_0}(T_0,T_n) & = & P_t(T_0,T_n)\exp(-\frac{1}{2}\beta^2(T_0,T_n)Var[Z_t]+\beta(T_0,T_n)Z_t) \\
 & = & P_t(T_0,T_n)\exp(-\frac{1}{2}\beta^2(T_0,T_n)Var[Z_t]+\beta(T_0,T_n)\sqrt{Var[Z_t]}x) \qquad \textit{where } x\sim \varepsilon(0,1) \textit{ under forward measure} \\
 & = & \underbrace{P_t(T_0,T_1)\exp(-\frac{1}{2}\beta^2(T_0,T_n)Var[Z_t])}_{F(T_n)}\exp(\underbrace{\beta(T_0,T_n)\sqrt{Var[Z_t]}}_{\Sigma(T_n)}x) \qquad \textit{all bond prices share the same } x \\
 & = & F(T_n)e^{\Sigma(T_n)x}
\end{array}$$

$$\begin{array}{lll}
CMS\ digital & = & P_t(T_0)\Pr_{Q_T}(\sum_{n=1}^N w_n P_{T_0}(T_n)<1 \mid I_t) \qquad \textit{We will apply Jamshidian's trick again.} \\
 & = & P_t(T_0)\Pr_{Q_T}(\sum_{n=1}^N w_n F(T_n)e^{\Sigma(T_n)x}<1 \mid I_t) \\
 & = & P_t(T_0)\Pr_{Q_T}(\underbrace{\sum_{n=1}^N w_n F(T_n)e^{\Sigma(T_n)x}}_{f(x)}<\sum_{n=1}^N w_n F(T_n)e^{\Sigma(T_n)x^*} \mid I_t) \qquad \textit{where } f \textit{ is increasing in } x \\
 & = & P_t(T_0)\Pr_{Q_T}(f(x)<f(x^*) \mid I_t) \qquad \textit{where } f(x^*)=1 \\
 & = & P_t(T_0)\Pr_{Q_T}(x<x^* \mid I_t) \\
 & = & P_t(T_0)N(x^*)
\end{array}$$

Implementation of *CMS* digital price is simple :

- solve x^* in $f(x) = 1$
- then apply $P_t(T_0)N(x^*)$

$$\begin{array}{llll}
\textit{where} & f(x) & = & \sum_{n=1}^N\left[w_n \times F(T_n)e^{\Sigma(T_n)x}\right] \\
 & & = & \sum_{n=1}^N\left[w_n \times P_t(T_0,T_n)\exp(-\frac{1}{2}\beta^2(T_0,T_n)Var[Z_t]+\beta(T_0,T_n)\sqrt{Var[Z_t]}x)\right]
\end{array}$$

CMS range

CMS range is a digital option on swap rate with both upper and lower bounds :

$$\begin{array}{lll}
CMS\ range & = & E_Q\left[1(S_T(T_0,T_N)\in[L,U])\times e^{-\int_t^T r_s ds} \mid I_t\right] \\
 & = & E_Q\left[1(S_T(T_0,T_N)>L)\times e^{-\int_t^T r_s ds} \mid I_t\right]-E_Q\left[1(S_T(T_0,T_N)>U)\times e^{-\int_t^T r_s ds} \mid I_t\right] \qquad \textit{which is decomposed into 2 CMS digitals} \\
 & = & P_t(T_0)(N(x^*)-N(y^*)) \qquad \textit{where } f_L(x^*)=1 \textit{ and } f_U(y^*)=1
\end{array}$$

5 How to price IRD under OIS discounting?

After bootstrapping *LIBOR prediction curve* and *OIS discounting curve*, how can they be used to price swap and *IRD*? For brevity, we name swap having floating leg pegged to *LIBOR index* (and *OIS index*) as *LIBOR swap* (and *OIS swap* respectively). Recall :

$$\sum_{n=1}^{N} L_t(T_{n-1},T_n)\delta(T_{n-1},T_n)P_t(T_n) \equiv P_t(T_0)-P_t(T_N) \quad \textit{always true} \quad \textit{recap eq(swap1,2)}$$

$$C\sum_{m=1}^{M} \delta(\Gamma_{m-1},\Gamma_m)P_t(\Gamma_m) \cong CA_t(\Gamma_0,\Gamma_M) \quad \textit{by definition} \quad \textit{recap eq(swap3,4)}$$

LIBOR swap

Pricing of *LIBOR* swap is straight forward :

$$\begin{aligned} \textit{payer LIBOR swap} &= \sum_{n=1}^{N} L_{LIBOR,t}(T_{n-1},T_n)\delta(T_{n-1},T_n)P_{OIS,t}(T_n) - C\sum_{m=1}^{M}\delta(\Gamma_{m-1},\Gamma_m)P_{OIS,t}(\Gamma_m) && \textit{eq(53)} \\ &\neq P_{OIS,t}(T_0)-P_{OIS,t}(T_N)-CA_{OIS,t}(\Gamma_0,\Gamma_M) \end{aligned}$$

- predict $L_{LIBOR,t}(T_{n-1},T_n)$ using *LIBOR* curve
- predict $P_{OIS,t}(T_0)$ using *OIS* curve
- cannot apply *eq(swap1,2)* to *eq(53)* as prediction index and discounting index are different

LIBOR swap to adjusted OIS swap

The idea is to convert *LIBOR* swap with *constant fixed-leg coupons* to OIS swap with *non-constan fixed-leg coupons* (the latter is known as adjusted *OIS* swap) so that they have identical *PV* at *t*. This is done by finding *adjustment x* via solving *eq(55)* with given *C* :

$$\begin{aligned} \textit{adjusted OIS swap(x)} &= \sum_{n=1}^{N} L_{OIS,t}(T_{n-1},T_n)\delta(T_{n-1},T_n)P_{OIS,t}(T_n) - (C+x)\sum_{m=1}^{M}\delta(\Gamma_{m-1},\Gamma_m)P_{OIS,t}(\Gamma_m) && \textit{eq(54)} \\ &= P_{OIS,t}(T_0)-P_{OIS,t}(T_N)-(C+x)A_{OIS,t}(\Gamma_0,\Gamma_M) && \textit{remove OIS index prediction} \\ \textit{payer LIBOR swap} &= \textit{adjusted OIS swap(x)} && \textit{eq(55)} \\ \Rightarrow \quad x &= -C-\frac{1}{A_{OIS,t}(\Gamma_0,\Gamma_M)}\left[\begin{array}{l} \sum_{n=1}^{N} L_{OIS,t}(T_{n-1},T_n)\delta(T_{n-1},T_n)P_{OIS,t}(T_n) \\ -C\sum_{m=1}^{M}\delta(\Gamma_{m-1},\Gamma_m)P_{OIS,t}(\Gamma_m)-P_{OIS,t}(T_0)+P_{OIS,t}(T_N) \end{array}\right] \end{aligned}$$

- predict $L_{LIBOR,t}(T_{n-1},T_n)$ using *LIBOR* curve
- predict $L_{OIS,t}(T_{n-1},T_n)$ and $P_{OIS,t}(T_0)$ using *OIS* curve

LIBOR swap to adjusted OIS swap 2

Yet eq(55) cannot guarantee LIBOR swap and adjusted OIS swap are equivalent as time proceeds. Therefore, instead of performing global adjustment x we perform different adjustments x_n for different coupons. The implementation is like bootstrapping, we firstly solve the nearest x_n, then proceed to solve the next x_{n+1} based on all previous results [x_1 x_2 x_3 x_{n-1} x_n]. Suppose :

$$\begin{aligned} \textit{floating leg period} &= \textit{3 months} \\ \textit{fixed leg period} &= \textit{6 months} \\ \textit{thus} \quad N &= 2\times M \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{2K} L_{LIBOR,t}(T_{n-1},T_n)\delta(T_{n-1},T_n)P_{OIS,t}(T_n) - C\sum_{m=1}^{K}\delta(\Gamma_{m-1},\Gamma_m)P_{OIS,t}(\Gamma_m) \\ = \; & \sum_{n=1}^{2K} L_{OIS,t}(T_{n-1},T_n)\delta(T_{n-1},T_n)P_{OIS,t}(T_n) - \sum_{m=1}^{K}(C+x_m)\delta(\Gamma_{m-1},\Gamma_m)P_{OIS,t}(\Gamma_m) \qquad \textit{eq(56)} \end{aligned}$$

The keys are highlighted above. Here is the algorithm :

K = 1   *solve eq(56) for* x_1
K = 2   *solve eq(56) for* x_2 *given previous result* x_1
K = 3   *solve eq(56) for* x_3 *given previous result* $x_1 x_2$
and so on, until K = M

LIBOR swaption
Option on a *LIBOR* swap is equivalent to an option on an adjusted OIS swap with same maturity and same strike (i.e. $0). Thus the implementation is simple, just a replacement of underlying swap.

For caplet, range accrual (i.e. bounds on index)
There are two stochastic indices *OIS* and *LIBOR*, we do not model them with two separate *HW* models, instead we model *OIS* with *HW* model, assume there exists deterministic but time dependent ratio between *LIBOR* discount factor and *OIS* discount factor. *Cap* pricing can be done by adjusting the strike *K*.

From *caplet Hull White formula*, we have :

$$caplet = \frac{1+\delta(T_0,T_1)K}{\delta(T_0,T_1)} E_Q\left[(K^*-P_{T_0}(T_1))^+ \times e^{-\int_t^{T_0} r_s ds} \mid I_t\right]$$

For *caplet* with *OIS* discounting (τ_s is short rate for *OIS*) :

$$caplet = \frac{1+\delta(T_0,T_1)K}{\delta(T_0,T_1)} E_Q\left[(adj \times K^*-P_{T_0}(T_1))^+ \times e^{-\int_t^{T_0} \tau_s ds} \mid I_t\right] \quad \text{we adjust } K^* \text{ only, no adjustment on } K$$

$$\text{where} \quad adj = \frac{DF_{OIS}(0,T_1)/DF_{OIS}(0,T_0)}{DF_{LIBOR}(0,T_1)/DF_{LIBOR}(0,T_0)} \quad \text{adj factor is conversion from LIBOR to OIS}$$

Reference
Written by Nicholas Burgess :
- An overview of the Vasicek short rate model *(covered in MSFE)*
- Martingale measures and change of measure explained *(covered in next paper)*
- Bond option pricing using the Vasicek short rate model *(change of measure and Jamshidian)*
- Interest rate swaptions – review and derivation of swaption pricing formulae

Written by others *(I read these references during 18-25 Oct 2018)* :
- Caps and floors — *John Crosby, Glasgow University*
- Pricing of interest rate derivatives, chapter 14 — *Nicolas Privault*
- Why can a swap option be regarded as a type of bond option? — *StackExchange*
- The Hull White swaption formula *(a two pages note)* — *Mark Davis*

The thread in *StackExchange* together with *Mark Davis* note lead us to the Jamshidian pricing engine for swaption.