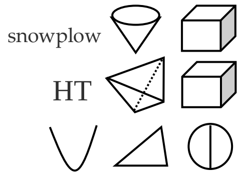
Answer to Quant questions

Classification 13331

- differentiation with Leibniz rule — see 1.6
- differentiation — see 1.1 (snowplow), 1.2 (cone), 1.7 (cube)
- expected trials in Markov chain — see 2.2, 2.3, 2.7 (HT flip), 2.15 (tetrahedron), 2.16 (cube)
- pdf double integration — see 2.1 (quadratic), teaser 7 (triangle), teaser 8 (circle)
- pdf min — see 2.17 (archer)



Leibniz rule

Leibniz rule for differentiating an integral :

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \underbrace{\int_{a(t)}^{b(t)}\partial_t f(x,t)dx}_{\substack{differentiating\\ integrand}} + \underbrace{f(b(t),t)\frac{db(t)}{dt}}_{\substack{differentiating\\ upper-bound}} - \underbrace{f(a(t),t)\frac{da(t)}{dt}}_{\substack{differentiating\\ lower-bound}}$$

Special cases :

$$\frac{d}{dt}\int_{a}^{b} f(x,t)dx = \int_{a}^{b}\partial_t f(x,t)dx \qquad \textit{as a and b do not depend on t}$$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)dx = f(b(t))\frac{db(t)}{dt} - f(a(t))\frac{da(t)}{dt} \qquad \textit{as } \partial_t f(x) = 0$$

$$\frac{d}{dt}\int_{const}^{t} f(x)dx = f(t)$$

Recall that :

$$\frac{d}{dt} f(x,t) = \lim_{\Delta t\to 0}\frac{f(x(t+\Delta t),t+\Delta t) - f(x(t),t)}{\Delta t} \qquad \textit{if x depends on t}$$

$$\frac{\partial}{\partial t} f(x,t) = \lim_{\Delta t\to 0}\frac{f(x,t+\Delta t) - f(x,t)}{\Delta t} \qquad \textit{no matter whether if x depends on t}$$

Proof of the generalized rule :

$$\begin{aligned}
\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx &= \lim_{\Delta t\to 0}\frac{\int_{a(t+\Delta t)}^{b(t+\Delta t)} f(x,t+\Delta t)dx - \int_{a(t)}^{b(t)} f(x,t)dx}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{\int_{a(t)+a'(t)\Delta t+...}^{b(t)+b'(t)\Delta t+...}(f(x,t)+\partial_t f(x,t)\Delta t+...)dx - \int_{a(t)}^{b(t)} f(x,t)dx}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{\begin{bmatrix} +\int_{a(t)}^{b(t)}(f(x,t)+\partial_t f(x,t)\Delta t+...)dx \\ +\int_{b(t)}^{b(t)+b'(t)\Delta t+...}(f(x,t)+\partial_t f(x,t)\Delta t+...)dx \\ -\int_{a(t)}^{a(t)+a'(t)\Delta t+...}(f(x,t)+\partial_t f(x,t)\Delta t+...)dx \\ -\int_{a(t)}^{b(t)} f(x,t)dx \end{bmatrix}}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{\begin{bmatrix} +\cancel{\int_{a(t)}^{b(t)}(f(x,t)}+\partial_t f(x,t)\Delta t+...)dx \\ +f(b(t),t)(b'(t)\Delta t+...) \\ -f(a(t),t)(a'(t)\Delta t+...) \\ -\cancel{\int_{a(t)}^{b(t)} f(x,t)dx} \end{bmatrix}}{\Delta t} \qquad \textit{since } \int_c^{c+\Delta} g(x)dx \sim g(c)(c+\Delta-c) = g(c)\Delta \\
&= \lim_{\Delta t\to 0}\frac{\int_{a(t)}^{b(t)}(\partial_t f(x,t)\Delta t+...)dx}{\Delta t} + \lim_{\Delta t\to 0}\frac{f(b(t),t)(b'(t)\Delta t+...)}{\Delta t} - \lim_{\Delta t\to 0}\frac{f(a(t),t)(a'(t)\Delta t+...)}{\Delta t} \\
&= \int_{a(t)}^{b(t)}\partial_t f(x,t)dx + f(b(t),t)b'(t) - f(a(t),t)a'(t)
\end{aligned}$$

Questions 1.1, 1.2 and 1.7 are about differentiation. The latter two are similar, as we need to do is to set up an equation relating two variables, such as volume and height in 1.2, or volume and surface in 1.7, then take derivative with respect to time in both sides. 1.1 is more complicated as the original setup is a differential equation.

(1.1) We denote time 0 as 8am, time 1 as 9am etc, for mathematical convenience. Three functions, height, location and plow rate are related by the DAG, hence given two of them, the other is known. In this question, height and plow rate are given, solve location.

$$
\begin{tikzcd}
 & \textit{snow height } h(t) \arrow[dr] & \\
\textit{time } t \arrow[ur] \arrow[dr] & & \textit{volume of snow plowed per delta time : } h(t)dx = h(t)x'(t)dt \\
 & \textit{snowplow location } x(t) \arrow[ur] &
\end{tikzcd}
$$

Recall, for continuous differentiation, there are three confusing things :

plow rate at a timepoint :		$= h(t)x'(t)$	$= plowrate(t)$	*no dt*
volume plowed per delta time :	$h(t)dx$	$= h(t)x'(t)dt$	$= plowrate(t)dt$	*with dt*
volume plowed during a period :	$\int h(t)dx$	$= \int h(t)x'(t)dt$	$= \int plowrate(t)dt$	*with dt and integration*

$$
\begin{aligned}
h(t) &= s(t-t_0) \\
h(t)dx(t) &= cdt && \textit{where c is a constant} \\
dx(t) &= \frac{c}{h(t)}dt = \frac{c}{s(t-t_0)}dt \\
x(t) &= \frac{c}{s}\ln(t-t_0) + const \\
&= \frac{c}{s}\ln(t-t_0) - \frac{c}{s}\ln(-t_0) \\
&= \frac{c}{s}\ln(\frac{t-t_0}{-t_0})
\end{aligned}
$$

now solve for the const, put in t = 0, i.e. 8am, where x(0) = 0

$$x(0) = \frac{c}{s}\ln(0-t_0) + const \quad \rightarrow \quad const = -\frac{c}{s}\ln(-t_0)$$

Now we have two unknowns : c/s and t_0, need to setup two equations (at 9am and 10am) :

- $2 = \frac{c}{s}\ln(\frac{1-t_0}{-t_0})$
- $3 = \frac{c}{s}\ln(\frac{2-t_0}{-t_0})$

Divide each other to remove c/s, we have :

$$
\begin{aligned}
\frac{2}{3} &= \ln(\frac{1-t_0}{-t_0}) \Big/ \ln(\frac{2-t_0}{-t_0}) \\
\ln(\frac{2-t_0}{-t_0})^2 &= \ln(\frac{1-t_0}{-t_0})^3 \\
-t_0(2-t_0)^2 &= (1-t_0)^3 \\
-4t_0 + 4t_0^2 - t_0^3 &= 1 - 3t_0 + 3t_0^2 - t_0^3 \\
-4t_0 + 4t_0^2 &= 1 - 3t_0 + 3t_0^2 \\
t_0^2 - t_0 - 1 &= 0 && \textit{pick the negative answer} \\
t_0 &= \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1-\sqrt{5}}{2} = \textit{-0.618} = \textit{7:23am}
\end{aligned}
$$

(1.2) Relate volume v with water level h, then differentiate both sides wrt time :

$$
\begin{aligned}
v &= \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3 && \textit{since } r/h = 1/2 \\
\frac{dv}{dt} &= \frac{1}{4}\pi h^2 \frac{dh}{dt} && \textit{given one rate, we should know the other rate} \\
u &\geq \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dv}{dt} = \frac{4k}{\pi h^2}
\end{aligned}
$$

(1.4) Consider :

$$x^T(-AA)x \quad = \quad (x^T A^T)(Ax) \quad = \quad (xA)^T(Ax) \quad = \quad \sum_n (xA)_n^2 \quad \geq \quad 0$$

(1.5) Part a is a linear regression with regularization, while part b is ordinary regression.

$$\begin{array}{llcl}
a. & L & = & (AX-B)^T(AX-B) - \lambda X^T X \\
 & L_X & = & 2A^T(AX-B) - 2\lambda X \\
 & X* & = & (A^T A - \lambda)^{-1}(A^T B) \\
b. & L & = & (AX-B)^T(AX-B) - \lambda X^T X \\
 & L_X & = & 2A^T(AX-B) \\
 & X* & = & (A^T A)^{-1}(A^T B)
\end{array}$$

(1.6) Apply Leibniz integral rule, which states :

$$\begin{array}{rcl}
\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt & = & f(x,b(x))\frac{db(x)}{dx} - f(x,a(x))\frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt \\
\frac{d}{dx}\int_{-\ln x}^{x^2} \Phi(x,t)dt & = & \Phi(x-x^2)\frac{dx^2}{dx} - \Phi(x+\ln x)\frac{-d\ln x}{dx} + \int_{-\ln x}^{x^2} \frac{d}{dx}\Phi(x-t)dt \\
 & = & 2x\Phi(x-x^2) + \frac{1}{x}\Phi(x+\ln x) + \int_{-\ln x}^{x^2} N(x-t)dt \\
 & = & 2x\Phi(x-x^2) + \frac{1}{x}\Phi(x+\ln x) + [\Phi(x-t)]_{t=-\ln x}^{t=x^2} \\
 & = & (1+2x)\Phi(x-x^2) + (1+\frac{1}{x})\Phi(x+\ln x)
\end{array}$$

(1.7) Relate volume v with surface s, then differentiate both sides wrt time :

$$\begin{array}{rclcrcl}
s & = & 6x^2 & \rightarrow & x & = & \sqrt{\frac{s}{6}} \\
v & = & x^3 & \rightarrow & v & = & (\frac{s}{6})^{3/2} \\
\frac{dv}{dt} & = & \frac{3}{2}(\frac{s}{6})^{1/2}\frac{1}{6}\frac{ds}{dt} & & \multicolumn{3}{l}{\textit{when } v = 125, \textit{ we have } x = 5 \textit{ and } s = 6\times5\times5 = 150} \\
4 & = & \frac{3}{2}(\frac{150}{6})^{1/2}\frac{1}{6}\frac{ds}{dt} & \rightarrow & \frac{ds}{dt} & = & \frac{4}{\frac{3}{2}(\frac{150}{6})^{1/2}\frac{1}{6}} = \frac{4}{\frac{3}{2}5\frac{1}{6}} = \frac{16}{5}
\end{array}$$

$$\begin{array}{lrcl}
(1.8) & \sqrt{x+x} & = & x \\
 & x(x-2) & = & 0 \\
\Rightarrow & x & = & 0 \text{ or } 2
\end{array}$$

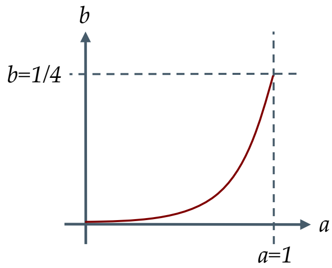
$$\begin{array}{lrcl}
(1.9) & y & = & \frac{1}{1+y} \\
 & y^2+y-1 & = & 0 \\
\Rightarrow & y & = & \frac{-1\pm\sqrt{1-4(-1)}}{2} \quad = \quad \frac{-1+\sqrt{5}}{2} \qquad \textit{pick the positive answer}
\end{array}$$

(2.1) Real root happens when :

$$\begin{aligned} a^2 - 4b &\geq 0 \\ b &\leq a^2/4 \end{aligned}$$

This is a double integration, with b limits depending on a, that is using a as outer integration, b as inner integration :

$$\begin{aligned} \Pr(b \leq a^2/4) &= \int_0^1 \int_0^{a^2/4} 1 db da \\ &= \int_0^1 a^2/4 da \\ &= [a^3/12]_0^1 \\ &= 1/12 \end{aligned}$$



(2.2 / 2.3 / 2.7 / 2.16)
In geometric distribution document, we derive the pmf, followed by its expectation. However for questions in here, its simpler if we plot and markov chain (or tree) and directly calculate the expectation (without pmf) by exploring its recursive nature. Prob is marked on the ***edges***, while ***values*** are bracketed. Its like finding ***value function*** in ***reinforcement learning Markov chain***.

(2.2) *start* $\xrightarrow{1/2}$ *head* $\rightarrow$ *head(2)*; *start* $\rightarrow$ *tail (1+N)*; *head* $\rightarrow$ *tail (2+N)*

$$\begin{aligned} N &= \frac{1}{2}(1+N) + \frac{1}{4}(2+N) + \frac{1}{4}2 \\ 4N &= (2+2N) + (2+N) + 2 \\ N &= 6 \end{aligned}$$

(2.3) *start* $\xrightarrow{1/3}$ *A (3)*; *start* $\rightarrow$ *B (2+x)*; *start* $\rightarrow$ *C (1+x)*

$$\begin{aligned} x &= \frac{1}{3}(3) + \frac{1}{3}(2+x) + \frac{1}{3}(1+x) \\ 3x &= 3 + (2+x) + (1+x) \\ x &= 6 \end{aligned}$$

(2.5)

$$\begin{aligned} E[N(X)] &= \int_{-\infty}^{\infty} N(x)p(x)dx \\ &= \int_{-\infty}^{0} N(x)p(x)dx + \int_0^{\infty} N(x)p(x)dx \\ &= \int_{y=\infty}^{y=0} N(-y)p(-y)d(-y) + \int_0^{\infty} N(x)p(x)dx && \textit{put } y = -x \\ &= \int_{y=0}^{y=\infty} N(-y)p(y)dy + \int_0^{\infty} N(x)p(x)dx && \textit{since } p(y) = p(-y) \textit{ for zero mean Gaussian} \\ &= \int_{y=0}^{y=\infty} (1-N(y))p(y)dy + \int_0^{\infty} N(x)p(x)dx \\ &= \int_0^{\infty} p(y)dy && \textit{since } -\int_{y=0}^{y=\infty} N(y)p(y)dy + \int_0^{\infty} N(x)p(x)dx = 0 \\ &= 1/2 \end{aligned}$$

(2.7) *start* $\xrightarrow{p}$ *head* $\rightarrow$ *head* $\rightarrow$ *head ...*; *start* $\xrightarrow{1-p}$ *tail (1+N)*; *head* $\rightarrow$ *tail (2)*; *head* $\rightarrow$ *tail (3)*

$$\begin{aligned} N &= q(1+N) + 2pq + 3pq^2 + 4pq^3 + 5pq^4 + ... \\ pN &= q + p(2q + 3q^2 + 4q^3 + 5q^4 + ...) \\ &= q + p(\frac{d}{dx}(x^2 + x^3 + x^4 + x^5 + ...)_{x=q}) \\ &= q + p\frac{d}{dx}\frac{x^2}{1-x}\bigg|_{x=q} \\ &= q + p\left(\frac{2x}{1-x} + \frac{x^2}{(1-x)^2}\right)\bigg|_{x=q} \\ &= q + p\left(\frac{2p}{q} + \frac{p^2}{q^2}\right) \quad = \quad 1/q \\ N &= 1/(pq) \end{aligned}$$

(2.8)
$$\begin{aligned}
\arg\min_c E[(X-c)^2] &= \arg\min_c E[X^2]-2cE[X]+c^2 \\
\frac{d(E[X^2]-2cE[X]+c^2)}{dc} &= -2E[X]+2c = 0 \\
\arg\min_c E[(X-c)^2] &= E[X]
\end{aligned}$$

(2.9) Given
$$\begin{aligned}
\frac{f(x)}{\int_{-\infty}^{x} f(s)ds} &= \lambda \\
f(x) &= \lambda\int_{-\infty}^{x} f(s)ds \\
f'(x) &= \lambda f(x) && \textit{differentiate both sides, using Leibniz rule} \\
df &= \lambda f(x)dx && \textit{this is ODE for riskfree asset} \\
d\ln f &= (1/f)df \\
&= (1/f)\lambda f(x)dx \\
&= \lambda dx \\
\ln\frac{f(x)}{f(0)} &= \lambda x \\
f(x) &= f(0)e^{\lambda x}
\end{aligned}$$

(2.12)
$$E[F(X)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{x} f(s)ds\, f(x)dx \qquad \textit{can we proceed further?}$$

(2.14) This is a naïve version American option. The underling is not Wiener process, but a sequence of independent RVs. The value outside bracket is dice realization, while the value inside bracket is contingent claim price. The answer is 4.6667.

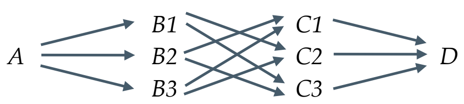
time1	*time2*	*time3*
S_1=6, V_1= max(6,4.25) = 6	*S_2=6, V_2= max(6,3.5) = 6*	*S_3=6, V_3=6*
S_1=5, V_1= max(5,4.25) = 5	*S_2=5, V_2= max(5,3.5) = 5*	*S_3=5, V_3=5*
S_1=4, V_1= max(4,4.25) = 4.25	*S_2=4, V_2= max(4,3.5) = 4*	*S_3=4, V_3=4*
S_1=3, V_1= max(3,4.25) = 4.25	*S_2=3, V_2= max(3,3.5) = 3.5*	*S_3=3, V_3=3*
S_1=2, V_1= max(2,4.25) = 4.25	*S_2=2, V_2= max(2,3.5) = 3.5*	*S_3=2, V_3=2*
S_1=1, V_1= max(1,4.25) = 4.25	*S_2=1, V_2= max(1,3.5) = 3.5*	*S_3=1, V_3=1*
*$E[V_1]$ = (4.25*4+5+6)/6 = 4.667*	*$E[V_2]$ = (3.5*3+4+5+6)/6 = 4.25*	*$E[V_3]$ = (1+2+3+4+5+6)/6 = 3.5*

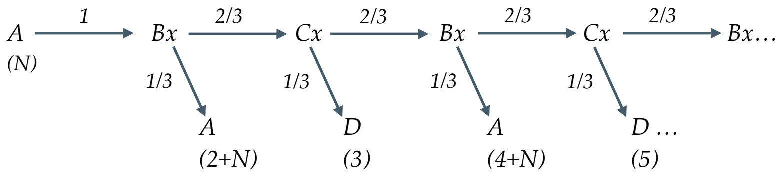
(2.15) The required path is A → B/C/D → B/C/D → B/C/D → B/C/D → B/C/D → B/C/D → A

$$\Pr(N=7) = 1\times(\frac{2}{3})^5\times\frac{1}{3}$$

(2.16) We have a topology graph *(LHS)* and a state-transition graph *(RHS)* :

A = starting node, $B_1B_2B_3$ are its neighbours
D = ending node, $C_1C_2C_3$ are its neighbours
this is a multipartite graph, i.e. no connection within the same layer





$$\begin{aligned}
N &= \frac{1}{3}(2+N)+\frac{2}{3}\frac{1}{3}3+\left(\frac{2}{3}\right)^2\frac{1}{3}(4+N)+\left(\frac{2}{3}\right)^3\frac{1}{3}5+\left(\frac{2}{3}\right)^4\frac{1}{3}(6+N)+\left(\frac{2}{3}\right)^5\frac{1}{3}7+\ldots \\
&= \left[\frac{1}{3}2+\frac{2}{3}\frac{1}{3}3+\left(\frac{2}{3}\right)^2\frac{1}{3}4+\left(\frac{2}{3}\right)^3\frac{1}{3}5+\left(\frac{2}{3}\right)^4\frac{1}{3}6+\left(\frac{2}{3}\right)^5\frac{1}{3}7+\ldots\right]+\left[\frac{1}{3}N+\left(\frac{2}{3}\right)^2\frac{1}{3}N+\left(\frac{2}{3}\right)^4\frac{1}{3}N+\ldots\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\left[2+\frac{2}{3}3+\left(\frac{2}{3}\right)^2 4+\left(\frac{2}{3}\right)^3 5+\left(\frac{2}{3}\right)^4 6+\left(\frac{2}{3}\right)^5 7+\ldots\right]+\frac{1}{3}N\left[1+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^4+\ldots\right] \\
&= \frac{1}{3}\frac{3}{2}\left[\frac{2}{3}2+\left(\frac{2}{3}\right)^2 3+\left(\frac{2}{3}\right)^3 4+\left(\frac{2}{3}\right)^4 5+\left(\frac{2}{3}\right)^5 6+\left(\frac{2}{3}\right)^6 7+\ldots\right]+\frac{1}{3}N\frac{1}{1-(2/3)^2} \\
&= \frac{1}{3}\frac{3}{2}\left[\frac{d}{dx}(x^2+x^3+x^4+\ldots)\right]_{x=2/3}+\frac{1}{3}N\frac{9}{5} \\
&= \frac{1}{3}\frac{3}{2}\left[\frac{d}{dx}\frac{x^2}{1-x}\right]_{x=2/3}+\frac{3}{5}N \\
&= \frac{1}{3}\frac{3}{2}\left[\frac{2x}{1-x}+\frac{x^2}{(1-x)^2}\right]_{x=2/3}+\frac{3}{5}N \\
\frac{2}{5}N &= \frac{1}{3}\frac{3}{2}\left[\frac{4/3}{1/3}+\frac{4/9}{1/9}\right] = \frac{1}{3}\frac{3}{2}8 = 4 \qquad \textit{hence } N = 10
\end{aligned}
$$

(2.17)

$$
\begin{aligned}
\Pr(\min(X_1,X_2)<x) &= 1-\Pr(\min(X_1,X_2)>x) \\
&= 1-\Pr(X_1>x)\Pr(X_2>x) \\
&= 1-(1-\Pr(X_1<x))(1-\Pr(X_2<x)) \\
&= 1-(1-x)^2 \\
&= 2x-x^2 \\
p(\min(X_1,X_2)=x) &= 2-x
\end{aligned}
$$

About option pricing

(4.2)

$$
\begin{aligned}
z_t &= \varepsilon(0,\sqrt{t}) \\
\Pr(z_t<0) &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{0} -x^2/(2t)dx \\
&= 0.5
\end{aligned}
$$

(4.3-4.4) From Quant-chapter1.doc, as T increases, the Ito's integral is :

- *non-martingale, Gaussian with zero mean and increasing variance*
- *zero expectation regardless of previous value, hence it is somehow mean reverting*

$$
\begin{aligned}
\int_0^T z_t dt &= \varepsilon(0, \text{var}=T^3/3) \\
E\left[\left(\frac{1}{T}\int_0^T z_t dt\right)^2\right] &= V\left[\frac{1}{T}\int_0^T z_t dt\right]-E^2\left[\frac{1}{T}\int_0^T z_t dt\right] = \frac{1}{T^2}\frac{T^3}{3}-0 = \frac{T}{3}
\end{aligned}
$$

(4.9)

$$
\begin{aligned}
dX^n &= nX^{n-1}dX+n(n-1)X^{n-2}(dX)^2 \\
&= nX^{n-1}(\mu Xdt+\sigma Xdz_t)+n(n-1)X^{n-2}(\sigma X)^2 dt \\
&= (n\mu+n(n-1)\sigma^2)X^n dt+n\sigma X^n dz_t \\
n\mu+n(n-1)\sigma^2 &= 0 \\
\mu+(n-1)\sigma^2 &= 0 \qquad \textit{since } n\neq 0 \\
n &= 1-\mu/\sigma^2
\end{aligned}
$$

(5.1, 5.2, 5.4) They are the same, find the implied probability and apply risk neutral expectation, regardless of physical probability.
(5.3) Plot the two payoffs and (b) is always higher than (a), by FTAP, (b) is more expensive for whatever underlying model.

More brain teaser

1. Find all roots of $x^6 = 64$
2. Find x in equation $x^{x^{x\cdots}} = 2$
3. Find $\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}$
4. It is now *12* oclock, when will be the first time hour hand meet with minute hand?
5. Given *14* balls one of which is heavier, what is the minimum times weighting by a balance to identify it? Generalize to case *N*.
6. Break a segment of unit length into *3* randomly, what is the prob of them forming a triangle?
7. Draw *3* points on a circle randomly, what is the prob of them being on the same semicircle?
8. Given *N* people in a circle (indiced with *0* to *N-1*), keep removing alternative person from the circle (starting from *1*), who will be the last one who remains? Write a C++ function. Do not loop a vector of size *N*, this is considered as brute force *O(NlogN)*.

Answers

1. $x = \cos(n\pi/3) + i\sin(n\pi/3)$ *where* $n = 0,1,\ldots,5$
2. $x^2 = 2$ *hence* $x = \sqrt{2}$
3. $A = \sqrt{x+A}$ *hence* $A = (1 \pm \sqrt{1+4x})/2$
4. $x = (1+x)/12$ *where x is the number of hours after 1 oclock when they meet again*
5. $n = \log_3 N$ *the 3 groups are* `{ceiling(N/3), ceiling(N/3), N-2*ceiling(N/3)}`
6. Suppose breakpoints are *x* and *y*, where *x,y*$\in$*[0,1]* and *p(x,y)* = $1_{x,y\in[0,1]}$.

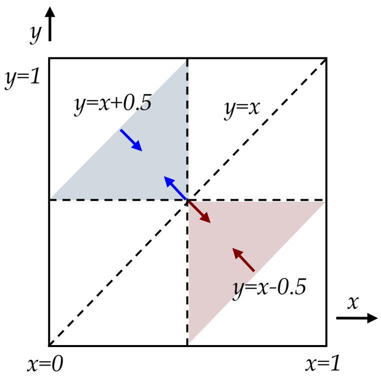
case 1 when x < 0.5
- *y-x < 0.5*
- *1-y < 0.5*

→ *0.5 < y < x + 0.5 (see blue region)*

case 2 when x > 0.5 (implying 1-x < 0.5)
- *y < 0.5*
- *x-y < 0.5*

→ *x - 0.5 < y < 0.5 (see red region)*

$$\int_0^{0.5}\int_{0.5}^{x+0.5} p(x,y)dydx + \int_{0.5}^{1}\int_{x-0.5}^{0.5} p(x,y)dydx = \frac{1}{4}$$



Both question 6,7 are two-step Markov chain. As they are continuous, we can't represent them as tree.

In double integration :
- *outer integration is the first decision x*
- *inner integration is the second decision y which depends on the first decision x*

7. Suppose the first point is $\theta = 0$, the other two points are $\theta = x$ *and y* in polar coordinate, where *x,y*$\in[\pm\pi]$ and $p(x,y) = 1/(2\pi)^2$.

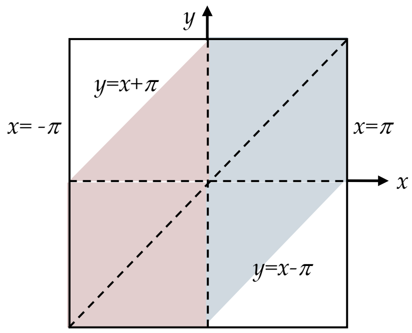
case 1 when x > 0
- $x-\pi < y < 0+\pi$

→ *we have the blue triangular region*

case 2 when x < 0
- $0-\pi < y < x+\pi$

→ *we have the red triangular region*

answer = 3/4



8. Represent indices remained as an arithmetic series *(a, d, N) = {a, a+d, a+2d, …, a+(N-1)d}*, we implement *O(logN)* recursion as :

```
unsigned long f(unsigned long a, unsigned long d, unsigned long N, bool first_alive_in_this_round)
{
    if (N==1) return a;
    if (N%2==0)
    {
        if (first_alive_in_this_round)    return f(a,   2*d, N/2, true);
        else                              return f(a+d, 2*d, N/2, false);
    }
    else
    {
        if (first_alive_in_this_round)    return f(a,   2*d, (N+1)/2, false);
        else                              return f(a+d, 2*d, (N-1)/2, true);
    }
}
unsigned long answer(unsigned long N) { return f(0,1,N,true); }
```