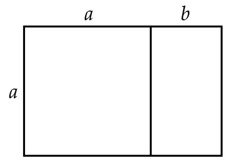
From Golden Ratio to Soccer

Golden ratio is defined as the aspect ratio of a rectangle, such that when a square with sides equal to the shorter side of rectangle is cut from the rectangle, leaves behind a smaller similar rectangle. In short :

$$
\begin{aligned}
\varphi &= a/b = (a+b)/a \\
a^2 &= ab+b^2 \\
0 &= (a/b)^2-(a/b)-1 \\
a/b &= (1\pm\sqrt{1-4(-1)})/2 \\
&= (1+\sqrt{5})/2 \qquad \textit{must be +ve} \\
&\sim 1.618
\end{aligned}
$$

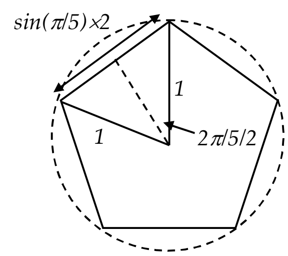


Then we find that $cos(\pi/5)$ equals to half of the golden ratio, i.e. $\varphi/2$.

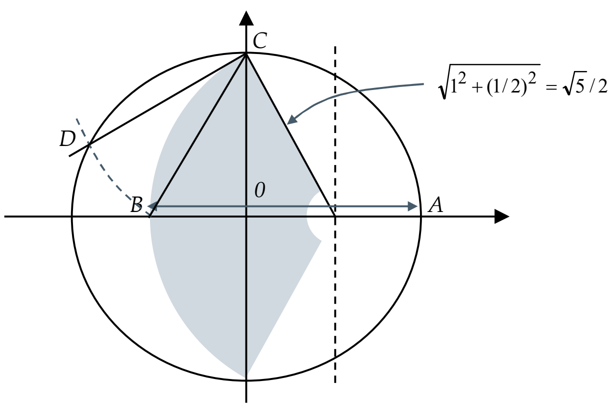
$$
\begin{aligned}
\theta &= \pi/5 \\
5\theta &= \pi \\
\sin(3\theta) &= \sin(\pi-2\theta) = \sin(2\theta) \\
3\sin\theta-4\sin^3\theta &= 2\sin\theta\cos\theta \\
3-4\sin^2\theta &= 2\cos\theta \qquad \textit{sin}\,\theta \textit{ must be} > 0 \\
3-4+4\cos^2\theta &= 2\cos\theta \\
4\cos^2\theta-2\cos\theta-1 &= 0 \\
\cos\theta &= (2\pm\sqrt{2^2-4\cdot4\cdot(-1)})/(2\cdot4) \\
&= (1+\sqrt{5})/4 \qquad \textit{must be +ve} \\
\cos(\pi/5) &= \varphi/2
\end{aligned}
$$

Regular pentagon inside unit circle has length :

$$\textit{length} = \sin(\pi/5)\times 2$$



Recall that $sin3\theta = 3sin\theta - 4sin^3\theta$, its proof is done by expanding $sin(\theta+2\theta) = sin\theta cos2\theta + cos\theta sin2\theta$, the detail is omitted. How can we generate the golden ratio with compass? The answer is to draw a unit circle centred at origin, bisect the positive x-axis, join the mid point $(x,y) = (1/2,0)$ with $(0,1)$, this line has length $\sqrt{5}/2$, thus by drawing an arc centred at the mid point of positive x-axis, a line with length $(1+\sqrt{5})/2$ is constructed as segment AB. How can we construct a pentagon with line AB?



Since $AB/OC = \varphi$, by the property of golden ratio, we have $OC/OB = \varphi$, hence $OB = 1/\varphi$

$$
\begin{aligned}
BC &= \sqrt{(1/\varphi)^2+1} \\
&= \sqrt{(2/\varphi)^2/4+1} \\
&= ??? \\
&= \sin(\pi/5)\times 2
\end{aligned}
$$

Finally, soccer is a *"Truncated Icosahedron"* composed of 12 pentagons and 20 hexagon, arranged as the following *graph* $\times 2$.