1. Prime numbers

Definition 1.1: A natural number is called a *prime number* if it is greater than 1 and cannot be written as the product of two smaller natural numbers.

Example: The numbers 2, 3, and 17 are prime. Corollary 1.1.1 shows that this list is not exhaustive!

Theorem 1.1 (Euclid): There are infinitely many primes.

Proof: Suppose to the contrary that $p_1, p_2, \ldots, p_n$ is a finite enumeration of all primes. Set $P = p_1 p_2 \ldots p_n$. Since $P + 1$ is not in our list, it cannot be prime. Thus, some prime factor p_j divides $P + 1$. Since p_j also divides P, it must divide the difference $(P + 1) - P = 1$, a contradiction. □

Corollary 1.1.1: There is no largest prime number.

Corollary 1.1.2: There are infinitely many composite numbers.

Theorem 1.2: There are arbitrarily long stretches of composite numbers.

Proof: For any $n > 2$, consider

$$n! + 2, \quad n! + 3, \quad \ldots, \quad n! + n$$ □