Game Theory

I. Concepts

A. Example (Ice-Cream)

- 3 kids: Adam (6$), Bilal (4$), Chang (3$)
- 3 ice-cream types: 500g (7$), 750g (9$), 1000g (11$)

B. Comparison:

Non-cooperative Game Theory:

- Players can't make binding agreements
- Competition between individual players
- Details of strategic interaction
- More "descriptive" (or positive) – How they should play it

Cooperative Game Theory:

- Binding agreements are possible
- Competition between coalitions
- "Black box" approach
- More "prescriptive" (or normative) – Focus more on the outcome of the game. The outcome would be like

II. Coalitional Games

A. Concepts:

– Games with transferable utility (TU)

Players can transfer or distribute their utilities (which are divisible) among them.

- $N = \{1, 2, \dots, n\}$ the set of players
- $C, D, S \subseteq N$ are called coalition
- N is called a Grand Coalition
- $v : 2^N \to \mathbb{R}$: v is called characteristic (or worth, or value) function

  The function v maps a subset of N into a real number, where 2^N is the total number of subsets of N. For example: there are $2^2 = 4$ subsets of $\{1, 2\}$, which are $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$
- For any $C \subseteq N$, $v(C)$: value that the members of C can get from this game $G = (N, v)$

Definition 1: ***A transferable utility coalitional game*** *is a pair $G = (N, v)$ where*

(a) $N = \{1, 2, \dots, n\}$ is the set of players;

(b) $v : 2^N \to \mathbb{R}$ is the characteristic (value) function

– Games with nontransferable utility (NTU)

The utilities cannot be divided perfectly, like a car.

B. Example:

Compute the worth function of the Ice-Cream example.

- $v(\emptyset) = 0$
- $v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$
- $v(\{A, B\}) = 750 = v(\{A, C\})$
- $v(\{B, C\}) = 500$
- $v(\{A, B, C\}) = 1000$

Note: We assume the payoff within a coalition will not be affected by the external coalitions or the environment.

III. Outcomes

A. Concept:

*Definition 2: **An outcome of a coalitional game** $G = (N, v)$ is a pair $(\mathcal{C}, x)$ where*

(1) $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ is the coalition structure, i.e., partition of N, satisfying

(a) $\cup_{i=1}^{k} C_i = N$, and

(b) $C_i \cap C_j = \emptyset$, $\forall i \neq j$: A player cannot be a member of two different coalitions

(2) $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is a payoff vector (indicating the value each player i gets), satisfying

(a) $x_i \geq v(\{i\})$, for all $i \in N$ – [Individual Rationality]

(b) $\sum_{i \in C} x_i = v(C)$, for all $C \in \mathcal{C}$ – [Feasibility]

B. Example:

Ice-Cream example.

- $((\{A, B\}, \{C\}), (400, 350, 0))$ One possible outcome of the ice-cream game
- $((\{A, B, C\}), (500, 300, 200))$ One possible outcome of the ice-cream game
- $((\{A, B\}, \{C\}), (500, 300, 200))$ Not an outcome of the ice-cream

  Because: $v(\{A, B\}) = 750$, they cannot divide 1000 as the payoff

IV. Superadditive games

A. Concept:

Definition 3: A game $G = (N, v)$ is superadditive if $v(C \cup D) \geq v(C) + v(D)$ for all $C, D \subseteq N$ with $C \cap D = \emptyset$

- The bigger, the better, meaning that the bigger coalition always creates no less values (could be equal)
- We will assume superadditivity (for notation simplicity)
- So we do not need to worry about $\mathcal{C}$ (coalition structure), i.e., the optimal coalition will be the grand coalition.

  We can ignore $\mathcal{C}$ and just focus on the payoff vector
- Ice-cream game is superadditive
- Hence, for those superadditive games, the outcome is just the payoff vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ satisfying

  (a) $x_i \geq v(\{i\})$, $\forall i \in N$

  (b) $\sum_{i=1}^{n} x_i = v(N)$

V. Stability (Core)

A. Concept

Definition 4: Core of a game G is the set of all stable outcomes

$$\textit{Core}(G) = \Big\{x | \sum_{i \in C} x_i \geq v(C),\ \textit{for all } C \subseteq N\Big\} \tag{1}$$

– Meaning that, if (1) is not satisfied, players in the coalition C have incentives to leave the coalition C and form their own coalitions. For example, if $C = N$, and (1) is not satisfied, it means that the grand coalition is not best for every player. Therefore, they may form some other smaller coalitions.

– Can we say there always exists stable outcomes for a game? Or can the set of core allocations be empty?

(Yes, it can be empty)

B. Example

Calculate the set of stable outcomes: $N = \{1,2,3\}$, $v(C) = 1$ if $\#C > 1$ (i.e., the number of players in coalition C is larger than 1) and $v(C) = 0$ otherwise.

- A potential payoff vector: $x = (x_1, x_2, x_3)$, the conditions to meet:
- (1) Individual Rationality: $x_1 \geq v(\{x_1\}) = 0$; $x_2 \geq v(\{x_2\}) = 0$; $x_3 \geq v(\{x_3\}) = 0$
- (2) Feasibility: $x_1 + x_2 + x_3 = v(\{x_1, x_2, x_3\}) = 1$

Conditions to meet as a core:

- The potential coalitions:
- $\emptyset$
- $\{1\} \rightarrow x_1 \geq 0$
- $\{2\} \rightarrow x_2 \geq 0$
- $\{3\} \rightarrow x_3 \geq 0$
- $\{1,2\} \rightarrow x_1 + x_2 \geq v(\{x_1, x_2\}) = 1 \longrightarrow 1 - x_3 \geq 1 \rightarrow x_3 \leq 0 \rightarrow x_3 = 0$
- $\{1,3\} \rightarrow x_1 + x_3 \geq v(\{x_1, x_3\}) = 1 \longrightarrow x_1 \geq 1$
- $\{2,3\} \rightarrow x_2 + x_3 \geq v(\{x_2, x_3\}) = 1 \longrightarrow x_2 \geq 1$, i.e., $x_1 + x_2 \geq 2$ which is conflict with $x_1 + x_2 = 1$
- $\{1,2,3\} \rightarrow x_1 + x_2 + x_3 \geq v(\{x_1, x_2, x_3\}) = 1$ (feasibility captures this requirement)

No solution satisfies the seven requirements so that x is a stable outcome! That is, Core$(G) = \emptyset$

VI. ϵ-Core

A. Concept

Definition 5: For any $\epsilon > 0$,

$$\epsilon\textit{-Core}(G) = \Big\{x | \sum_{i \in C} x_i \geq v(C) - \epsilon,\ \textit{for all } C \subseteq N\Big\} \tag{2}$$

Definition 6:

$$\epsilon^*(G) = inf\{\epsilon > 0 | \epsilon - \textit{core of G is non-empty}\} \tag{3}$$

$\epsilon^*(G)$ is called least-core of G.

VII. Shapley Value

A. Concept

Given $G=(N,v)$, the Shapley value of player i is

$$\phi_i(N,v)=\frac{1}{n!}\Big(\sum_{S\subseteq N\backslash\{i\}}|S|!\big(n-1-|S|\big)!\Big[v(S\cup\{i\})-v(S)\Big]\Big) \tag{4}$$

$$=\frac{1}{n}\Big(\sum_{S\subseteq N\backslash\{i\}}\frac{1}{C_{n-1}^{|S|}}\Big[v(S\cup\{i\})-v(S)\Big]\Big), \tag{5}$$

where $n=|N|$ is the number of players in the set N.

Understanding:

(a). First, we would explain that (4) is equivalent to (5).

For any coalition S where i is not included, we have

$$\frac{|S|!(n-1-|S|)!}{n!}=\frac{1}{n}\frac{|S|!(n-1-|S|)!}{(n-1)!}=\frac{1}{n}\frac{1}{C_{n-1}^{|S|}}, \tag{6}$$

where $C_{n-1}^{|S|}=\frac{(n-1)!}{|S|!(n-1-|S|)!}$, denoting the number of all subsets (i.e., coalitions) of the set $N\backslash\{i\}$, where $|S|\in\{0,1,\dots,n-1\}$. Note that $0!=1$ and $1!=1$.

(b). Next, we would explain the meaning of (5).

For each type of coalition S with a certain size of $|S|$, $v(S\cup\{i\})-v(S)$ denotes the marginal contribution of player i to the coalition S. The number of different coalitions with the same size $|S|$ is $C_{n-1}^{|S|}$. Thus,

$$\sum_{S\subseteq N\backslash\{i\}}\frac{1}{C_{n-1}^{|S|}}\Big[v(S\cup\{i\})-v(S)\Big] \tag{7}$$

represents the average marginal contribution of player i to one type of coalition S of size $|S|$. In total, there are n types of coalitions, i.e., $|S|=0,1,\dots,n-1$, where $|S|=0$ corresponds to $S=\emptyset$. Consequently, even out player i's marginal contribution further to each type of coalition, its Shapley value is denoted as

$$\phi_i(N,v)=\frac{1}{n}\Big(\sum_{S\subseteq N\backslash\{i\}}\frac{1}{C_{n-1}^{|S|}}\Big[v(S\cup\{i\})-v(S)\Big]\Big). \tag{8}$$

B. Example

Compute the Shapley value of each player in the inc-cream game, where $N=\{1,2,3\}$ and $n=3$.

(1) Shapley value of player 1:

Coalitions without including 1: $S=\emptyset,\{2\},\{3\},\{2,3\}$.

For $|S|=0$, we have $\frac{1}{C_2^0}[v(\{1\})-v(\emptyset)]=0$;

For $|S|=1$, we have $\frac{1}{C_2^1}[v(\{1,2\})-v(\{1\})+v(\{1,3\})-v(\{1\})]=\frac{1}{2}(750+750)=750$;

For $|S|=2$, we have $\frac{1}{C_2^2}[v(\{1,2,3\})-v(\{2,3\})]=1000-500=500$;

Thus, $\phi_1(N,v)=\frac{1}{3}(0+750+500)=\frac{1250}{3}$

(2) Shapley value of player 2:

Coalitions without including 2: $S=\emptyset,\{1\},\{3\},\{1,3\}$.

For $|S|=0$, we have $\frac{1}{C_2^0}[v(\{2\})-v(\emptyset)]=0$;

For $|S| = 1$, we have $\frac{1}{C_2^1}[v(\{1,2\}) - v(\{1\}) + v(\{2,3\}) - v(\{3\})] = \frac{1}{2}(750 + 500) = 625$;
For $|S| = 2$, we have $\frac{1}{C_2^2}[v(\{1,2,3\}) - v(\{1,3\})] = 1000 - 750 = 250$;
Thus, $\phi_2(N, v) = \frac{1}{3}(0 + 625 + 250) = \frac{875}{3}$

(3) Shapley value of player 3:

Coalitions without including 3: $S = \emptyset, \{1\}, \{2\}, \{1,2\}$.
For $|S| = 0$, we have $\frac{1}{C_2^0}[v(\{3\}) - v(\emptyset)] = 0$;
For $|S| = 1$, we have $\frac{1}{C_2^1}[v(\{1,3\}) - v(\{1\}) + v(\{2,3\}) - v(\{2\})] = \frac{1}{2}(750 + 500) = 625$;
For $|S| = 2$, we have $\frac{1}{C_2^2}[v(\{1,2,3\}) - v(\{1,2\})] = 1000 - 750 = 250$;
Thus, $\phi_3(N, v) = \frac{1}{3}(0 + 625 + 250) = \frac{875}{3}$

Therefore,

$$(\phi_1, \phi_2, \phi_3) = (\frac{1250}{3}, \frac{875}{3}, \frac{875}{3}), \tag{9}$$

where we can see $\phi_1 + \phi_2 + \phi_3 = 1000$, meaning that it is feasible. In addition, $\phi_1, \phi_2, \phi_3 \geq 0$, so all of them are individually rational (see III. (2)). Therefore, (9) is an outcome of the ice-cream game. But note that, it is not a stable outcome.