Stability and Feasibility Proof of DMPC

I. Section I

Consider a discrete-time linear time-invariant system, which consists of M subsystems

$$x_i(k+1) = A_{ii}x_i(k) + \sum_{j=1}^{M} A_{ij}x_j(k) + B_{ii}u_i(k), \tag{1}$$

where $x_i \in \mathbb{R}^{n_i} \subseteq \mathcal{X}_i$ and $u_i \in \mathbb{R}^{m_i} \subseteq \mathcal{U}_i$. The global system is described by

$$x(k+1) = Ax(k) + Bu(k), \tag{2}$$

where $x \triangleq [x_1, ..., x_M] \in \mathbb{R}^n \subseteq \mathcal{X}$ and $u \triangleq [u_1, ..., u_M] \in \mathbb{R}^m \subseteq \mathcal{U}$. $\mathcal{X} = \mathcal{X}_1 \times ... \times \mathcal{X}_M$ and $\mathcal{U} = \mathcal{U}_1 \times ... \times \mathcal{U}_M$.

Assumption 1. *For any subsystem described by Eq.(1), it is assumed that there exists the feedback control law $u_i = K_i x_i$ such that (i) $A_{d_i} = A_{ii} + B_{ii}K_i$ is Schur stable; (ii) $A_d = A + BK$ is Schur stable, where $K_i \in \mathbb{R}^{m_i \times n_i}$, $K = diag(K_1, ..., K_M)$.*

Assumption 2. *For any subsystem i, given the positive definite symmetric matrix Q_i, R_i, we assume that (i) there exists P_i such that $A_{d_i}^T P_i A_{d_i} - P_i \leqslant -(Q_i + K_i^T R_i K_i)$; (ii) and $A_d^T P A_d - \hat{A}_d^T P \hat{A}_d \leqslant (Q + K^T RK)/2$ holds true, where $P = diag(P_1, ..., P_M)$, $R = diag(R_1, ..., R_M)$, $Q = diag(Q_1, ..., Q_M)$ and $\hat{A}_d = diag(A_{d_1}, ..., A_{d_M})$.*

Lemma 1: If Assumption 1 and Assumption 2 hold true, then there exists a small enough ε, and $Kx \in \mathbb{U}$ such that $\phi(\varepsilon) \triangleq \{x \in \mathbb{R}^n : \|x\|_P \leqslant \varepsilon\}$ is an positive definite invariant set of the closed-loop system $x(k+1) = A_d x(k)$.

Proof: Define that $V(x(k)) = \|x(k)\|_P^2$. Since $A_{d_i}^T P_i A_{d_i} - P_i \leqslant -(Q_i + K_i^T R_i K_i)$, there is $\hat{A}_d^T P \hat{A}_d - P \leqslant -(Q + K^T RK)$. Then for the closed-loop system $x(k+1) = A_d x(k)$, there is

$$\begin{aligned} V(x(k+1)) - V(x(k)) &= \|A_d x(k)\|_P^2 - \|x(k)\|_P^2 \\ &= x(k)^T (A_d^T P A_d - P) x(k) \\ &= x(k)^T (A_d^T P A_d - \hat{A}_d^T P \hat{A}_d + \hat{A}_d^T P \hat{A}_d - P) x(k) \\ &\leqslant -\frac{1}{2} x(k)^T (Q + K^T RK) x(k) < 0. \end{aligned}$$

Thus, $x(k+1)\phi(\varepsilon) \in \backslash\{0\}$ holds true, i.e., a state within $\phi(\varepsilon)$ can be controlled to reach the 0 point gradually. Meanwhile, since P is a positive definite matrix, $\phi(\varepsilon)$ can be selected as $\{0\}$. That is, there exists enough small ε and $Kx \in \mathcal{U}$ so that $\phi(\varepsilon)$ is an positive definite invariant set. ∎

Consider the local performance index

$$J_i(x_i(k), u_i(k)) = \sum_{l=0}^{N-1} (\|x_i(k+l|k)\|_{Q_i}^2 + \|u_i(k+1|k)\|_{R_i}^2) + \|x_i(k+N|k)\|_{P_i}^2. \tag{3}$$

The local optimization problem is

$$\min_{U_i(k)} \quad J_i(x_i(k), u_i(k))$$

$$s.t. \quad x_i(k+l+1|k) = A_{ii}x_i(k) + \sum_{j\in\mathcal{N}_i^u} A_{ij}x_j(k+l|k) + B_{ii}u_i(k+l|k) \tag{4a}$$

$$\sum_{h=0}^{l} \pi(l-h)\|x_i(k+1+h|k) - x_i^*(k+1+h|k-1)\| \leqslant \frac{\gamma\kappa\varepsilon}{2N_i^u M}, l=0,...,N-2 \quad (6a, 6b) \tag{4b}$$

$$\|x_i(k+N|k) - \hat{x}_i(k+N|k)\|_{P_i} \leqslant \frac{\kappa\varepsilon}{2M} \quad (4b) \tag{4c}$$

$$\|x_i(k+l+1|k)\|_{P_i} - \|\bar{x}_i(k+l+1|k)\|_{P_i} \leqslant \frac{\varepsilon}{\rho NM}, l=0,...,N-1 \quad (stability) \tag{4d}$$

$$u_i(k+l|k) \in \mathcal{U}_i, l=0,...,N-1 \quad (4b) \tag{4e}$$

$$x_i(k+N|k) \in \phi_i(\frac{\varepsilon}{2M}) \quad (4c, 6c) \tag{4f}$$

$$x_i(k+l+1|k) \in \mathcal{X}_i, l=0,...,N \tag{4g}$$

In which, $x_i(k|k) = x_i(k)$, $\phi_i(\frac{\varepsilon}{2M}) \triangleq \{x_i \in \mathbb{R}^{n_i} : \|x_i\|_{P_i} \leqslant \frac{\varepsilon}{2M}\}$ is the terminal invariant set of subsystem i. $\gamma \in (0,1)$, $\kappa \in (0,1)$ are parameters.

We assume that optimization problem (4) is feasible initially. Every subsystem solves (4) at time instant k and obtains the optimal control sequence $U_i^*(k) \triangleq [u_i^*(k|k), ..., u_i^*(k+N-1|k)]$, and the optimal state sequence $X_i^*(t) \triangleq [x_i^*(k+1|k), ..., x_i^*(k+N|k)]$.

Then we need to build a feasible control input sequence $\bar{U}_i(k+1)$ of time instant $(k+1)$ such that the global system is stable and the feasible region of the optimization problem (4) is non-empty. The design method is as follows.

Based on the optimal control sequence $U_i^*(k) = [u_i^*(k|k), ..., u_i^*(k+N-1|k)]$, the feasible control input sequence $\bar{U}_i(k+1) \triangleq [\bar{u}_i(k+1|k+1), ...\bar{u}_i(k+N|k+1)]$ is designed as

$$\bar{u}_i(k+1+l|k+1) = \begin{cases} u_i^*(k+l+1|k), & l=0,1,..,N-2 \\ K_i\bar{x}_i(k+l+1|k+1), & l=N-1 \end{cases}.$$

Next, we need to determine the initial feasible state sequence $\bar{X}_i(k+1) \triangleq [\bar{x}_i(k+2|k+1), ..., \bar{x}_i(k+1+N|k+1)]$ of subsystem i. The associated state sequence comes from the time k, i.e., the given information is $X_j^*(k) \triangleq [x_j^*(k|k), x_j^*(k+1|k), ..., x_j^*(k+N|k)]$. To compute $\bar{X}_i(k+1)$, the estimated associated state sequence $\hat{X}_j(k+1) \triangleq [\hat{x}_j(k+1|k+1), ..., \hat{x}_j(k+N+1|k+1)]$ is obtained by

$$\hat{x}_j(k+1+l|k+1) = \begin{cases} x_j^*(k+l+1|k), & l=0,1,..,N-1 \\ A_{d_j}x_j^*(k+l|k), & l=N \end{cases}.$$

Then, according to $\bar{U}_i(k+1)$ and $\hat{X}_j(k+1)$, the feasible state sequence $\bar{X}_i(k+1)$ can be obtained by

For any $\quad l=0,...,N-1$

$$\bar{x}_i(k+2+l|k+1) = A_{ii}\bar{x}_i(k+1+l|k+1) + B_{ii}\bar{u}_i(k+1+l|k+1) + \sum_{j\in\mathcal{N}_i^u} A_{ij}\hat{x}_j(k+1+l|k+1), \tag{5}$$

where $\bar{x}_i(k+1|k+1) = x_i(k+1)$ is the current subsystem state and is obtained by measurement.

A. *Feasibility Proof*

Theorem 1: (feasibility) For the system described by (2), we assume that the problem (4) is initially feasible. If the controller design of every subsystem satisfies the following conditions,

$$\delta_i \leqslant \frac{(1-\gamma)\kappa\varepsilon}{\max\limits_{l\in\{0,\dots,N-2\}} 2M\overline{\lambda}^{\frac{1}{2}}(P_i^{-\frac{1}{2}}(A_{ii}^{l+1})^T P_i A_{ii}^{l+1} P_i^{-\frac{1}{2}})} \tag{6a}$$

$$\sum_{h=0}^{N-2} \pi(N-2-h)\frac{N_i^u}{\gamma \underline{\lambda}^{\frac{1}{2}}(P_i)} \leqslant 1 \tag{6b}$$

$$1 - \min_{i\in\mathcal{M}} \underline{\lambda}(P_i^{-\frac{1}{2}}\hat{Q}_i P_i^{-\frac{1}{2}}) \leqslant (1-\kappa)^2, \quad \hat{Q}_i = Q_i + K_i^T R_i K_i \tag{6c}$$

where $\delta_i(k) = \|e_i(k)\|_{P_i} = \|x_i(k) - x_i^*(k|k-1)\|_{P_i}$, then the feasible region of the optimization problem (4) at any time k is non-empty.

Proof: Now we intend to prove that at any time $(k+1)$, the constructed control input sequence $\bar{U}_i(k+1)$ meets all the constraints of the optimization problem (4), i.e., it is a feasible solution of (4).

(i) constraint (4b): $\sum_{h=0}^{l} \pi(l-h)\|\bar{x}_i(k+2+h|k+1) - x_i^*(k+2+h|k)\| \leqslant \frac{\gamma\kappa\varepsilon}{2N_i^u M}, l = 0, \dots, N-2,$ $(\frac{\kappa\varepsilon}{2M} \rightarrow \frac{\gamma\kappa\varepsilon}{2N_i^u M}$, condition (6a, 6b) is proposed.)

First, we derive that:

$$l = 0, \bar{x}_i(k+2|k+1) = A_{ii}\bar{x}_i(k+1|k+1) + B_{ii}\bar{u}_i(k+1|k+1) + \sum_{j\in\mathcal{N}_i^u} A_{ij}\hat{x}_j(k+1|k+1)$$

$$\begin{aligned} l = 1, \bar{x}_i(k+3|k+1) &= A_{ii}\bar{x}_i(k+2|k+1) + B_{ii}\bar{u}_i(k+2|k+1) + \sum_{j\in\mathcal{N}_i^u} A_{ij}\hat{x}_j(k+2|k+1) \\ &= A_{ii}^2\bar{x}_i(k+1|k+1) + A_{ii}B_{ii}\bar{u}_i(k+1|k+1) + \sum_{j\in\mathcal{N}_i^u} A_{ii}A_{ij}\hat{x}_j(k+1|k+1) \\ &\quad + B_{ii}\bar{u}_i(k+2|k+1) + \sum_{j\in\mathcal{N}_i^u} A_{ij}\hat{x}_j(k+2|k+1) \end{aligned}$$

$$\begin{aligned} l = 2, \bar{x}_i(k+4|k+1) &= A_{ii}\bar{x}_i(k+3|k+1) + B_{ii}\bar{u}_i(k+3|k+1) + \sum_{j\in\mathcal{N}_i^u} A_{ij}\hat{x}_j(k+3|k+1) \\ &= A_{ii}^3\bar{x}_i(k+1|k+1) + A_{ii}^2B_{ii}\bar{u}_i(k+1|k+1) + \sum_{j\in\mathcal{N}_i^u} A_{ii}^2A_{ij}\hat{x}_j(k+1|k+1) + A_{ii}B_{ii}\bar{u}_i(k+2|k+1) \\ &\quad + \sum_{j\in\mathcal{N}_i^u} A_{ii}A_{ij}\hat{x}_j(k+2|k+1) + B_{ii}\bar{u}_i(k+3|k+1) + \sum_{j\in\mathcal{N}_i^u} A_{ij}\hat{x}_j(k+3|k+1) \end{aligned}$$

....

For any $l = 0, \dots, N-1$

$$\bar{x}_i(k+l+2|k+1) = A_{ii}^{l+1}x_i(k+1) + \sum_{h=0}^{l} A_{ii}^{l-h}B_{ii}\bar{u}_i(k+1+h|k+1) + \sum_{j\in\mathcal{N}_i^u}\sum_{h=0}^{l} A_{ii}^{l-h}A_{ij}\hat{x}_j(k+1+h|k+1) \tag{7}$$

where $\bar{x}_i(k+1|k+1) = x_i(k+1)$. Similarly, based on the optimal control input sequence $U_i^*(k)$, the optimal state $x_i^*(k+l+2|k)$ is denoted.

For any $l = 0, ..., N-2$

$$x_i^*(k+l+2|k) = A_{ii}^{l+1}x_i^*(k+1|k) + \sum_{h=0}^{l} A_{ii}^{l-h}B_{ii}u_i^*(k+1+h|k) + \sum_{j\in\mathcal{N}_i^u}\sum_{h=0}^{l} A_{ii}^{l-h}A_{ij}\hat{x}_j(k+1+h|k) \tag{8}$$

Then for any $l = 0, ..., N-2$, there is

$$\begin{aligned}
&\bar{x}_i(k+l+2|k+1) - x_i^*(k+l+2|k)\\
&= A_{ii}^{l+1}x_i(k+1) + \sum_{h=0}^{l} A_{ii}^{l-h}B_{ii}\bar{u}_i(k+1+h|k+1) + \sum_{j\in\mathcal{N}_i^u}\sum_{h=0}^{l} A_{ii}^{l-h}A_{ij}\hat{x}_j(k+1+h|k+1)\\
&\quad - A_{ii}^{l+1}x_i^*(k+1|k) - \sum_{h=0}^{l} A_{ii}^{l-h}B_{ii}u_i^*(k+1+h|k) - \sum_{j\in\mathcal{N}_i^u}\sum_{h=0}^{l} A_{ii}^{l-h}A_{ij}\hat{x}_j(k+1+h|k)
\end{aligned}$$

By definition, $\bar{u}_i(k+1+h|k+1) = u_i^*(k+1+h|k)$, for any $h = 0, ..., N-2$, we have

$$\begin{aligned}
&\bar{x}_i(k+l+2|k+1) - x_i^*(k+l+2|k)\\
&= A_{ii}^{l+1}[x_i(k+1) - x_i^*(k+1|k)] + \sum_{j\in\mathcal{N}_i^u}\sum_{h=0}^{l} A_{ii}^{l-h}A_{ij}[\hat{x}_j(k+1+h|k+1) - \hat{x}_j(k+1+h|k)]\\
&\triangleq A_{ii}^{l+1}e_i(k+1) + \sum_{j\in\mathcal{N}_i^u}\Gamma_j(\eta_j(k+1), l),
\end{aligned} \tag{9}$$

where we define that $e_i(k+1) = x_i(k+1) - x_i^*(k+1|k)$, $\eta_j(k+1) = \hat{x}_j(k+1+h|k+1) - \hat{x}_j(k+1+h|k)$, $\Gamma_j(\eta_j(k+1), l) = \sum_{h=0}^{l} A_{ii}^{l-h}A_{ij}\eta_j(k+1)$.

On this basis, now we begin to prove that, for any $l = 0, ..., N-2$, since $\|x+y\|_P \leqslant \|x\|_P + \|y\|_P$, there is

$$\begin{aligned}
&\|\bar{x}_i(k+l+2|k+1) - x_i^*(k+l+2|k)\|_{P_i}\\
&= \|A_{ii}^{l+1}e_i(k+1) + \sum_{j\in\mathcal{N}_i^u}\Gamma_j(\eta_j(k+1), l)\|_{P_i}\\
&\leqslant \|A_{ii}^{l+1}e_i(k+1)\|_{P_i} + \sum_{j\in\mathcal{N}_i^u}\|\Gamma_j(\eta_j(k+1), l)\|_{P_i}\\
&= \|e_i(k+1)\|_{(A_{ii}^{l+1})^T P_i A_{ii}^{l+1}} + \sum_{j\in\mathcal{N}_i^u}\sum_{h=0}^{l}\|A_{ii}^{l-h}A_{ij}\eta_j(k+1)\|_{P_i}\\
&= \|P_i^{\frac{1}{2}}e_i(k+1)\|_{P_i^{-\frac{1}{2}}(A_{ii}^{l+1})^T P_i A_{ii}^{l+1}P_i^{-\frac{1}{2}}} + \sum_{j\in\mathcal{N}_i^u}\sum_{h=0}^{l}\|\eta_j(k+1)\|_{(A_{ii}^{l-h}A_{ij})^T P_i (A_{ii}^{l-h}A_{ij})}\\
&\leqslant \overline{\lambda}^{\frac{1}{2}}(P_i^{-\frac{1}{2}}(A_{ii}^{l+1})^T P_i A_{ii}^{l+1}P_i^{-\frac{1}{2}})\|e_i(k+1)\|_{P_i} + \sum_{j\in\mathcal{N}_i^u}\sum_{h=0}^{l}\overline{\lambda}^{\frac{1}{2}}((A_{ii}^{l-h}A_{ij})^T P_i (A_{ii}^{l-h}A_{ij}))\|\eta_j(k+1)\|\\
&\leqslant \overline{\lambda}^{\frac{1}{2}}(P_i^{-\frac{1}{2}}(A_{ii}^{l+1})^T P_i A_{ii}^{l+1}P_i^{-\frac{1}{2}})\delta_i(k+1) + \sum_{j\in\mathcal{N}_i^u}\sum_{h=0}^{l}\pi(l-h)\|\eta_j(k+1)\|\\
&\leqslant \overline{\lambda}^{\frac{1}{2}}(P_i^{-\frac{1}{2}}(A_{ii}^{l+1})^T P_i A_{ii}^{l+1}P_i^{-\frac{1}{2}})\delta_i(k+1) + N_i^u\sum_{h=0}^{l}\pi(l-h)\|\eta_j(k+1)\|,
\end{aligned} \tag{10}$$

where we define that $\delta_i(k) = \|e_i(k)\|_{P_i}$, $\pi(s) = \max_{i,j\in\mathcal{M}, i\neq j} \overline{\lambda}^{\frac{1}{2}}((A_{ii}^s A_{ij})^T P_i (A_{ii}^s A_{ij}))$ and $N_i^u = \max_{i\in\mathcal{M}} |\mathcal{N}_i^u|$ is the biggest number of the upriver neighbors among all subsystems i. Because for any $h = 0, ..., N-2$, $\hat{x}_j(k+1+h|k+1) = x_j^*(k+1+h|k)$ holds true, then there is

$$\begin{aligned}\eta_j(k+1) &= \hat{x}_j(k+1+h|k+1) - \hat{x}_j(k+1+h|k)\\ &= x_j^*(k+1+h|k) - x_j^*(k+1+h|k-1),\end{aligned}$$

which must meets the constraint (4b). Thus, for any $j \in \mathcal{N}_i^u$, we have

$$\sum_{h=0}^{l} \pi(l-h)\|\eta_j(k+1)\| \leqslant \frac{\gamma\kappa\varepsilon}{2N_i^u M} \tag{11}$$

Substitute (11) to (10) and by condition (6a), for any $l = 0, ..., N-2$, we have

$$\begin{aligned}&\|\bar{x}_i(k+l+2|k+1) - x_i^*(k+l+2|k)\|_{P_i}\\ &\leqslant \overline{\lambda}^{\frac{1}{2}}(P_i^{-\frac{1}{2}}(A_{ii}^{l+1})^T P_i A_{ii}^{l+1} P_i^{-\frac{1}{2}})\delta_i(k+1) + N_i^u \sum_{h=0}^{l} \pi(l-h)\|\eta_j(k+1)\|\\ &\leqslant \frac{(1-\gamma)\kappa\varepsilon}{2M} + \frac{\gamma\kappa\varepsilon}{2M} = \frac{\kappa\varepsilon}{2M}\end{aligned} \tag{12}$$

By definition of $\pi(s)$, we know that for any $l, h = 0, ..., N-2$, there is $\pi(l-h) \geqslant 0$. Meanwhile, according to $\underline{\lambda}(F)\|x\|^2 \leqslant \|x\|_F^2 \leqslant \overline{\lambda}(F)\|x\|^2$, $(F \geqslant 0)$, we have $\|x\| \leqslant \frac{\|x\|_F}{\underline{\lambda}^{\frac{1}{2}}(F)}$. Then, by condition (6b), there is

$$\begin{aligned}&\sum_{h=0}^{l} \pi(l-h)\|\bar{x}_i(k+h+2|k+1) - x_i^*(k+h+2|k)\|\\ &\leqslant \sum_{h=0}^{l} \pi(l-h)\frac{\|\bar{x}_i(k+h+2|k+1) - x_i^*(k+h+2|k)\|_{P_i}}{\underline{\lambda}^{\frac{1}{2}}(P_i)}\\ &\leqslant \sum_{h=0}^{l} \pi(l-h)\frac{\kappa\varepsilon}{2M\underline{\lambda}^{\frac{1}{2}}(P_i)}\\ &\leqslant \sum_{h=0}^{N-2} \pi(N-2-h)\frac{N_i^u}{\gamma\underline{\lambda}^{\frac{1}{2}}(P_i)} \cdot \frac{\gamma\kappa\varepsilon}{2N_i^u M} \leqslant \frac{\gamma\kappa\varepsilon}{2N_i^u M}\end{aligned} \tag{13}$$

Therefore, $\sum_{h=0}^{l} \pi(l-h)\|\bar{x}_i(k+2+h|k+1) - x_i^*(k+2+h|k)\| \leqslant \frac{\gamma\kappa\varepsilon}{2N_i^u M}, l = 0, ..., N-2$ is proved.

(ii) constraint (4c): $\|\bar{x}_i(k+N+1|k+1) - \hat{x}_i(k+N+1|k+1)\|_{P_i} \leqslant \frac{\kappa\varepsilon}{2M}$. (constraint (4b) is required.)

First, we prove that $\bar{x}(k+N|k+1) \in \phi(\varepsilon)$.

We know that $x_i^*(k+N|k) \in \phi_i(\frac{\varepsilon}{2M})$, by (12), there is

$$\|\bar{x}_i(k+N|k+1)\|_{P_i} - \|x_i^*(k+N|k)\|_{P_i} \leqslant \|\bar{x}_i(k+N|k+1) - x_i^*(k+N|k)\|_{P_i} < \frac{\kappa\varepsilon}{2M} \tag{14}$$

Thus, there is

$$\|\bar{x}_i(k+N|k+1)\|_{P_i} < \frac{\kappa\varepsilon}{2M} + \frac{\varepsilon}{2M} = \frac{(1+\kappa)\varepsilon}{2M} < \frac{\varepsilon}{M} \tag{15}$$

Then we have

$$\|\bar{x}(k+N|k+1)\|_P = (\sum_{i=1}^{M} \|\bar{x}_i(k+N|k+1)\|_{P_i}^2)^{\frac{1}{2}} < (\frac{\varepsilon^2}{4M^2} \cdot M)^{\frac{1}{2}} < \varepsilon$$

So $\bar{x}(k+N|k+1) \in \phi_i(\varepsilon)$. By Lemma 1, $\bar{u}_i(k+N|k+1) = K_i\bar{x}_i(k+N|k+1)$ holds true. In the invariant set, we use linear feedback control, thus, there is

$$\begin{aligned}\bar{x}_i(k+N+1|k+1) &= A_{ii}\bar{x}_i(k+N|k+1) + B_{ii}K_i\bar{x}_i(k+N|k+1) \\ &= A_{d_i}\bar{x}_i(k+N|k+1) \qquad (16)\end{aligned}$$

Meanwhile, by the definition of $\hat{x}_i(k+1+l|k+1)$, for $l=N$, there is

$$\hat{x}_i(k+N+1|k+1) = A_{d_i}x_i^*(k+N|k) \qquad (17)$$

Use (16)-(17). Since $\|x\|_F \leqslant \|x\|_G$ holds if $F \leqslant G$ (if $F-G \leqslant 0$, $\|x\|_F - \|x\|_G = x^T(F-G)x \leqslant 0$), and by Lemma 1, i.e., $A_{d_i}^T P_i A_{d_i} - P_i \leqslant -\hat{Q}_i$, by (14), it derives

$$\begin{aligned}&\|\bar{x}_i(k+N+1|k+1) - \hat{x}_i(k+N+1|k+1)\|_{P_i} \\ &= \|A_{d_i}(\bar{x}_i(k+N|k+1) - x_i^*(k+N|k))\|_{P_i} \\ &= \|(\bar{x}_i(k+N|k+1) - x_i^*(k+N|k))\|_{A_{d_i}^T P_i A_{d_i}} \\ &\leqslant \|(\bar{x}_i(k+N|k+1) - x_i^*(k+N|k))\|_{P_i} \\ &< \frac{\kappa\varepsilon}{2M}, \qquad (18)\end{aligned}$$

which ends the proof.

(iii) constraint (4d): $\|\bar{x}_i(k+l+2|k+1)\|_{P_i} - \|\bar{x}_i(k+l+2|k+1)\|_{P_i} \leqslant \frac{\varepsilon}{\rho NM}, l = 0, ..., N-1$.

Since $\frac{\varepsilon}{\rho NM} \geqslant 0$, so $\|\bar{x}_i(k+l+2|k+1)\|_{P_i} \leqslant \|\bar{x}_i(k+l+2|k+1)\|_{P_i} + \frac{\varepsilon}{\rho NM}$ is always true.

(iv) constraint (4e): $\bar{u}_i(k+1+l|k+1) \in \mathcal{U}_i, l = 0, ..., N-1$. ($\varepsilon \to \frac{\varepsilon}{2M}$, inequality (12) is required).

(Note that, in (4b), there should include $\alpha\varepsilon$, here we select $\kappa\varepsilon$ to decrease the number of parameters).

By definition, since $u_i^*(k+l|k) \in \mathcal{U}_i$, $l = 0, ..., N-1$, there is $\bar{u}_i(k+1+l|k+1) \in \mathcal{U}_i$, for any $l = 0, ..., N-2$. That is, we need to prove that $\bar{u}_i(k+N|k+1) \in \mathcal{U}_i$.

By definition, we have $\bar{u}_i(k+N|k+1) = K_i\bar{x}_i(k+N|k+1)$. Since $x_i^*(k+N|k) \in \phi_i(\frac{\varepsilon}{2M})$, $\|x_i^*(k+N|k)\|_{P_i} \leqslant \frac{\varepsilon}{2M}$. Then by (12), there is

$$\begin{aligned}\|\bar{x}_i(k+N|k+1)\|_{P_i} &\leqslant \|\bar{x}_i(k+N|k+1) - x_i^*(k+N|k)\|_{P_i} + \|x_i^*(k+N|k)\|_{P_i} \\ &\leqslant \frac{\kappa\varepsilon}{2M} + \frac{\varepsilon}{2M} < \frac{\varepsilon}{2M} \qquad (19)\end{aligned}$$

Therefore, $\bar{x}_i(k+N|k+1) \in \phi_i(\frac{\varepsilon}{2M})$. Since

$$\|\bar{x}(k+N|k+1)\|_P = (\sum_{i=1}^{M} \|\bar{x}_i(k+N|k+1)\|_{P_i}^2)^{\frac{1}{2}} = (\frac{\varepsilon^2}{4M^2} \cdot M)^{\frac{1}{2}} = \frac{\varepsilon}{2\sqrt{M}} < \varepsilon \tag{20}$$

By Lemma 1, $K\bar{x} \in \mathcal{U}$, thus, $\bar{u}_i(k+N|k+1) = K_i\bar{x}_i(k+N|k+1) \in \mathcal{U}_i$ holds true.

(v) constraint (4f): $\bar{x}_i(k+N+1|k+1) \in \phi_i(\frac{\varepsilon}{2M})$. ($\frac{\varepsilon}{2M} \to \frac{\kappa\varepsilon}{2M}$, constraint (4c) is required, condition (6c) is proposed)

In light of the triangle inequality $\|x+y\| \leqslant \|x\| + \|y\|$, and by (4c), we have

$$\begin{aligned}\|\bar{x}_i(k+N+1|k+1)\|_{P_i} &\leqslant \|\bar{x}_i(k+N+1|k+1) - \hat{x}_i(k+N+1|k+1)\|_{P_i} + \|\hat{x}_i(k+N+1|k+1)\|_{P_i} \\ &\leqslant \frac{\kappa\varepsilon}{2M} + \|\hat{x}_i(k+N+1|k+1)\|_{P_i}\end{aligned} \tag{21}$$

Since $\hat{x}_i(k+N+1|k+1) = A_{d_i}x_i^*(k+N|k)$ and by Lemma 1, $A_{d_i}^T P_i A_{d_i} - P_i \leqslant -\hat{Q}_i$ holds true, where $\hat{Q}_i = Q_i + K_i^T R_i K_i$, we can obtain that

$$\begin{aligned}\|\hat{x}_i(k+N+1|k+1)\|_{P_i}^2 - \|x_i^*(k+N|k)\|_{P_i}^2 &= \|x_i^*(k+N|k)\|_{A_{d_i}^T P_i A_{d_i}}^2 - \|x_i^*(k+N|k)\|_{P_i}^2 \\ &= \|x_i^*(k+N|k)\|_{A_{d_i}^T P_i A_{d_i} - P_i}^2 \\ &\leqslant \|x_i^*(k+N|k)\|_{-\hat{Q}_i}^2 = -\|x_i^*(k+N|k)\|_{\hat{Q}_i}^2\end{aligned} \tag{22}$$

According to the inequality $\underline{\lambda}(F)\|x\|^2 \leqslant \|x\|_F^2 \leqslant \overline{\lambda}(F)\|x\|^2$, where $\underline{\lambda}(F)$ is the smallest eigenvalue of matrix F and $\overline{\lambda}(F)$ is the largest eigenvalue of F. Meanwhile, because $x_i^*(k+N|k) \in \phi_i(\frac{\varepsilon}{2M})$ holds true, i.e., $\|x_i^*(k+N|k)\|_{P_i} \leqslant \frac{\varepsilon}{2M}$, and $\|Fx\| = \|x\|_{F^TF}$, then there is

$$\begin{aligned}-\|x_i^*(k+N|k)\|_{\hat{Q}_i}^2 &= -x_i^*(k+N|k)^T\hat{Q}_i x_i^*(k+N|k) \\ &= -x_i^*(k+N|k)^T P_i^{\frac{1}{2}}(P_i^{-\frac{1}{2}}\hat{Q}_i P_i^{-\frac{1}{2}})P_i^{\frac{1}{2}}x_i^*(k+N|k) \\ &= -(P_i^{\frac{1}{2}}x_i^*(k+N|k))^T(P_i^{-\frac{1}{2}}\hat{Q}_i P_i^{-\frac{1}{2}})(P_i^{\frac{1}{2}}x_i^*(k+N|k)) \\ &= -\|P_i^{\frac{1}{2}}x_i^*(k+N|k)\|_{P_i^{-\frac{1}{2}}\hat{Q}_i P_i^{-\frac{1}{2}}}^2 \\ &\leqslant -\underline{\lambda}(P_i^{-\frac{1}{2}}\hat{Q}_i P_i^{-\frac{1}{2}})\|P_i^{\frac{1}{2}}x_i^*(k+N|k)\|^2 \\ &= -\underline{\lambda}(P_i^{-\frac{1}{2}}\hat{Q}_i P_i^{-\frac{1}{2}})\|x_i^*(k+N|k)\|_{P_i}^2\end{aligned} \tag{23}$$

Substituting (23) into (22), we have

$$\|\hat{x}_i(k+N+1|k+1)\|_{P_i}^2 \leqslant (1 - \underline{\lambda}(P_i^{-\frac{1}{2}}\hat{Q}_i P_i^{-\frac{1}{2}}))\|x_i^*(k+N|k)\|_{P_i}^2$$

Therefore,

$$\begin{aligned}\|\hat{x}_i(k+N+1|k+1)\|_{P_i} &\leqslant \sqrt{1 - \underline{\lambda}(P_i^{-\frac{1}{2}}\hat{Q}_i P_i^{-\frac{1}{2}})}\|x_i^*(k+N|k)\|_{P_i} \\ &\leqslant \sqrt{1 - \underline{\lambda}(P_i^{-\frac{1}{2}}\hat{Q}_i P_i^{-\frac{1}{2}})} \cdot \frac{\varepsilon}{2M}\end{aligned} \tag{24}$$

Then, substitute (24) into (21), since the condition (6c) holds true, i.e., $1 - \min_{i\in\mathcal{M}} \underline{\lambda}(P_i^{-\frac{1}{2}} \hat{Q}_i P_i^{-\frac{1}{2}}) \leqslant (1-\kappa)^2$, there is

$$\begin{aligned}\|\bar{x}_i(k+N+1|k+1)\|_{P_i} &\leqslant \frac{\kappa\varepsilon}{2M} + \|\hat{x}_i(k+N+1|k+1)\|_{P_i} \\ &\leqslant \frac{\kappa\varepsilon}{2M} + \sqrt{1-\underline{\lambda}(P_i^{-\frac{1}{2}}\hat{Q}_i P_i^{-\frac{1}{2}})} \cdot \frac{\varepsilon}{2M} \\ &\leqslant \frac{\kappa\varepsilon}{2M} + \frac{(1-\kappa)\varepsilon}{2M} = \frac{\varepsilon}{2M}\end{aligned} \tag{25}$$

Thus, $\bar{x}_i(k+N+1|k+1) \in \phi_i(\frac{\varepsilon}{2M})$ is proved.

Taken the above together, $\bar{U}_i(k+1)$ is a feasible solution of the optimization problem (4) at time instant $k+1$, so the feasible region is non-empty.

■

B. Stability Proof

Theorem 2: (stability) For the system described by (2), we assume that conditions (6a)-(6c) are satisfied. If the parameters δ_i, κ, ρ still meet the following condition

$$\frac{\varepsilon}{\rho} + \frac{(N-1)\kappa\varepsilon}{2} + \sum_{i=1}^{M} \delta_i \leqslant \frac{\varepsilon}{2}, \quad \rho > 2, \tag{26}$$

then the global closed-loop system is stable.

Proof: When the system state is outside the terminal invariant set, we use DMPC algorithm to compute the optimal control input, the candidate Lyapunov function is denoted by $V_{out}(X(k)) = \sum_{i=1}^{M}\sum_{l=0}^{N} \|x_i(k+l|k)\|_{P_i}$; while when the system state is within the terminal invariant set, we use the feedback control law, the candidate Lyapunov function is denoted by $V_{in}(x(k)) = \|x(k)\|_P$. According to the Theorem 2.7 of [1], if $V_{out}(X(k))$ and $V_{in}(X(k))$ are the Lyapunov functions, i.e., positive and monotonically decreasing functions, meanwhile, at the switching time instant τ, there is $V_{in}(x(\tau)) - V_{out}(x(\tau-1)) < 0$, then the global closed-loop system is stable.

(i) Prove $V_{in}(x(k+1)) - V_{in}(x(k)) < 0$

By Lemma 1, there is

$$\|x(k+1)\|_P^2 - \|x(k)\|_P^2 = \|A_d x(k)\|_P^2 - \|x(k)\|_P^2 = \|x(k)\|^2_{A_d^T P A_d - P} = \sum_{i=1}^{M} \|x_i(k)\|_{A_{d_i}^T P_i A_{d_i} - P_i} < 0 \tag{27}$$

Meanwhile, since $V_{in}^2(x(k+1)) - V_{in}^2(x(k)) = [V_{in}(x(k+1)) + V_{in}(x(k))][V_{in}(x(k+1)) - V_{in}(x(k))]$, and $V_{in}(x(k+1)) + V_{in}(x(k)) > 0$, so there is

$$V_{in}(x(k+1)) - V_{in}(x(k)) < 0 \tag{28}$$

Therefore, for any $x(k) \in \phi(\varepsilon)$, $V_{in}(x(k))$ is a Lyapunov function.

(ii) Prove $V_{in}(x(\tau)) - V_{out}(X(\tau-1)) < 0$

Since $\|x(\tau)\|_P \leqslant \varepsilon$ and $\|x(\tau-1)\|_P > \varepsilon$, meanwhile, $\|x_1\|_{P_1} + \|x_2\|_{P_2} + ... + \|x_M\|_{P_M} \geqslant \|x\|_P = \|x_1^T P_1 x_1 + x_2^T P_2 x_2 + ... + x_M^T P_M x_M\|$, so we have

$$
\begin{aligned}
V_{in}(x(\tau)) - V_{out}(x(\tau-1)) &= \|x(\tau)\|_P - \sum_{i=1}^{M}\sum_{l=0}^{N} \|x_i(\tau+l-1|\tau-1)\|_{P_i} \\
&= \|x(\tau)\|_P - \sum_{i=1}^{M} \|x_i(\tau-1|\tau-1)\|_{P_i} - \sum_{i=1}^{M}\sum_{l=1}^{N} \|x_i(\tau+l-1|\tau-1)\|_{P_i} \\
&\leqslant \|x(\tau)\|_P - \|x(\tau-1)\|_P - \sum_{i=1}^{M}\sum_{l=1}^{N} \|x_i(\tau+l-1|\tau-1)\|_{P_i} \\
&< -\sum_{i=1}^{M}\sum_{l=1}^{N} \|x_i(\tau+l-1|\tau-1)\|_{P_i} < 0 \qquad (29)
\end{aligned}
$$

(ii) Prove $V_{out}(X^*(k+1)) - V_{out}(X^*(k)) < 0$, ($\frac{\varepsilon}{\rho N\sqrt{M}} -> \frac{\varepsilon}{\rho NM}$, $\frac{\kappa\varepsilon}{2\sqrt{M}} -> \frac{\kappa\varepsilon}{2M}$, constraint (4d) is required, condition (26) is proposed.)

By constraint (4d), i.e., $\|x_i^*(k+l+1|k)\|_{P_i} \leqslant \|\bar{x}_i(k+l+1|k)\|_{P_i} + \frac{\varepsilon}{\rho NM}$, $l = 0, ..., N-1$, there is

$$
\begin{aligned}
V_i(X_i^*(k+1)) - V_i(X_i^*(k)) &= \sum_{l=0}^{N} \|x_i^*(k+l+1|k+1)\|_{P_i} - \sum_{l=0}^{N} \|x_i^*(k+l|k)\|_{P_i} \\
&= \sum_{l=0}^{N-1} \|x_i^*(k+l+2|k+1)\|_{P_i} + \|x_i^*(k+1|k+1)\|_{P_i} - \sum_{l=0}^{N} \|x_i^*(k+l|k)\|_{P_i} \\
&\leqslant \frac{\varepsilon}{\rho NM}\cdot N + \sum_{l=0}^{N-1} \|\bar{x}_i(k+l+2|k+1)\|_{P_i} - \sum_{l=0}^{N} \|x_i^*(k+l|k)\|_{P_i} + \|x_i^*(k+1)\|_{P_i}
\end{aligned}
$$

According to (12), i.e., for any $l = 0, ..., N-2$, $\|\bar{x}_i(k+l+2|k+1) - x_i^*(k+l+2|k)\|_{P_i} \leqslant \frac{\kappa\varepsilon}{2M}$, and by (25), we have

$$
\begin{aligned}
V_i(X_i^*(k+1)) - V_i(X_i^*(k)) &\leqslant \frac{\varepsilon}{\rho M} + \sum_{l=0}^{N-2} \|\bar{x}_i(k+l+2|k+1)\|_{P_i} - \sum_{l=0}^{N-2} \|x_i^*(k+l+2|k)\|_{P_i} + \|x_i^*(k+1)\|_{P_i} \\
&\quad + \|\bar{x}_i(k+N+1|k+1)\|_{P_i} - \|x_i^*(k+1|k)\|_{P_i} - \|x_i^*(k)\|_{P_i} \\
&\leqslant \frac{\varepsilon}{\rho M} + \sum_{l=0}^{N-2} \|\bar{x}_i(k+l+2|k+1) - x_i^*(k+l+2|k)\|_{P_i} + \|\bar{x}_i(k+N+1|k+1)\|_{P_i} \\
&\quad + \|x_i(k+1) - x_i^*(k+l|k)\|_{P_i} - \|x_i(k)\|_{P_i} \\
&\leqslant \frac{\varepsilon}{\rho M} + \frac{(N-1)\kappa\varepsilon}{2M} + \frac{\varepsilon}{2M} + \delta_i - \|x_i(k)\|_{P_i} \qquad (30)
\end{aligned}
$$

Since $\sum_{i=1}^{M} \|x_i(k)\|_{P_i} \geqslant \|x\|_P$, moreover, $x(k) \notin \phi(\varepsilon)$, i.e., $\|x\|_P > \varepsilon$. If condition (26) holds true, then we can obtain that

$$\begin{aligned}
V_{out}(X^*(k+1)) - V_{out}(X^*(k)) &= \sum_{i=1}^{M}[V_i(X_i^*(k+1)) - V_i(X_i^*(k))] \\
&\leqslant \sum_{i=1}^{M}[\frac{\varepsilon}{\rho M} + \frac{(N-1)\kappa\varepsilon}{2M} + \frac{\varepsilon}{2M} + \delta_i - \|x_i(k)\|_{P_i}] \\
&\leqslant \frac{\varepsilon}{\rho} + \frac{\varepsilon}{2} + \frac{(N-1)\kappa\varepsilon}{2} + \sum_{i=1}^{M}\delta_i - \|x\|_P \\
&< -\frac{\varepsilon}{2} + \frac{\varepsilon}{\rho} + \frac{(N-1)\kappa\varepsilon}{2} + \sum_{i=1}^{M}\delta_i \leqslant 0.
\end{aligned} \tag{31}$$

By condition (4d), i.e., $\frac{(N-1)\kappa\varepsilon}{2} + \sum_{i=1}^{M}\delta_i \leqslant \frac{\varepsilon}{2}$, so there is

$$V_{out}(X^*(k+1)) - V_{out}(X^*(k)) < 0. \tag{32}$$

Thus, we prove that $V_{out}(X(k))$ is the Lyapunov when $x(k) \notin \phi(\varepsilon)$.

Taken the above together, if the condition (4d) is fulfilled, the stability of the global system is guaranteed. ∎

References

[1] Branicky, M. S. (1998). Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hybrid Systems. IEEE Transactions on Automatic Control, 43(4), 475-482.