Reihen- Aufgaben

A1 a) $\sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k = \frac{1}{1-\frac{1}{10}} = \frac{1}{\frac{9}{10}} = \frac{10}{9}$

b) $\sum_{k=1}^{\infty} \frac{2^k+(-3)^k}{5^k} = \sum_{k=1}^{\infty}\left(\frac{2}{5}\right)^k + \sum_{k=1}^{\infty}\left(-\frac{3}{5}\right)^k = \frac{1}{1-\frac{2}{5}} - 1 + \frac{1}{\left(1+\frac{3}{5}\right)} - 1 = \frac{5}{3} + \frac{5}{8} - 2$

$$= \frac{40+15}{24} - \frac{48}{24} = \frac{7}{24}$$

c) $\sum_{k=2}^{\infty} \frac{2^{k+1}}{7\cdot 5^k} = \frac{2}{7}\cdot\sum_{k=2}^{\infty}\left(\frac{2}{5}\right)^k = \frac{2}{7}\left(\frac{1}{1-\frac{2}{5}} - \left(1+\frac{2}{5}\right)\right)$

$$= \frac{2}{7}\left(\frac{5}{3} - \frac{7}{5}\right) = \frac{2}{7}\left(\frac{25-21}{15}\right)$$

$$= \frac{8}{105}$$

A2 a) $0.\overline{48} = 0.48 + 0.48\cdot\frac{1}{100} + 0.48\frac{1}{100^2} + \dots$

$$= 0.48\cdot\left(1+\frac{1}{100}+\frac{1}{100^2}+\dots\right) = 0.48\,\frac{1}{1-\frac{1}{100}} = 0.48\cdot\frac{100}{99} = \frac{48}{99}$$

b) $3.1\overline{48} = 3.1 + 0.0\overline{48} = \frac{31}{10} + \frac{1}{10}\cdot 0.\overline{48} = \frac{31}{10} + \frac{48}{990} = \frac{99\cdot 31+48}{990} = \frac{3117}{990}$

c) $0.\overline{1234} = 0.1234\cdot\sum_{k=0}^{\infty}\left(\frac{1}{10000}\right)^k = 0.1234\,\frac{1}{1-\frac{1}{10000}} = 0.1234\cdot\frac{10000}{9999} = \frac{1234}{9999}$

A3 a) $\sum_{k=1}^{\infty} \frac{1-\frac{1}{\pi}}{\pi^k} = \frac{1}{\pi}$

$$\frac{1}{\pi} \underbrace{- \frac{1}{\pi^2} + \frac{1}{\pi^2}}_{0} \underbrace{- \frac{1}{\pi^3} + \frac{1}{\pi^3}}_{0} \underbrace{- \frac{1}{\pi^4}}_{0} \longrightarrow \frac{1}{\pi}$$

b) $\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+3}\right) = 1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + \frac{1}{5} - \frac{1}{8} + \frac{1}{6} - \frac{1}{9} + \frac{1}{7} - \frac{1}{10} + \dots$

$$= 1 + \frac{1}{2} + \frac{1}{3} = \frac{6+3+2}{6} = \frac{11}{6}$$

A4 $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \infty$

Bew: Für $k \geq 1$ gilt: $\sqrt{k} < k \Rightarrow \frac{1}{k} < \frac{1}{\sqrt{k}}$

Da $\sum_n \frac{1}{n}$ divergiert, divergiert auch $\sum_n \frac{1}{\sqrt{n}}$

A5

a) $\left|\frac{\cos n}{3}\right| \leq \frac{1}{3} \Rightarrow$ konvergiert absolut nach Majorantenkriterium

b) $\left|\frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2}\right| = \frac{1}{3} \frac{(n+1)^2}{n^2} \xrightarrow[n\to\infty]{} \frac{1}{3}$ konvergiert nach Quotientenkriterium

c) $\frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} = \frac{x}{n+1} \xrightarrow[n\to\infty]{} 0$ konvergiert für jedes x (Quotientenkriterium)

A6

a) $\sum_{k=0}^{\infty} \frac{k+2}{2^k} x^k$

Wurzelkriterium: $\sqrt[k]{|a_k|} = \frac{\sqrt[k]{k+2}}{2} |x| \xrightarrow[k\to\infty]{} \frac{1}{2}|x|$

$\frac{1}{2}|x| < 1 \Rightarrow |x| < 2 \Rightarrow$ Konvergenzradius $R = 2$.

b) $\sum \frac{(2+x)^{2k}}{(2+\frac{1}{k})^k} \Rightarrow \sqrt{|a_k|} = \frac{(2+x)^2}{2+\frac{1}{k}} \xrightarrow[k\to\infty]{} \frac{(2+x)^2}{2}$

$(2+x)^2 < 2 \Leftrightarrow |2+x| < \sqrt{2}$

$\Rightarrow$ Konvergenzradius ist $R = \sqrt{2}$

c) $\sum_{k=0}^{\infty} \frac{3^{k+2}}{2^k} x^k \quad \sqrt[k]{|a_k|} = \sqrt[k]{\frac{3^k \cdot 3^2}{2^k} |x^k|} = \sqrt[k]{9} \cdot \frac{3}{2} |x| \xrightarrow[k\to\infty]{} \frac{3}{2}|x|$

$\frac{3}{2}|x| < 1 \Leftrightarrow |x| < \frac{2}{3} \Rightarrow$ Konvergenzradius $R = \frac{2}{3}$.

A7

a) $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ mit $R = \infty \Rightarrow e^{-x} = \sum \frac{(-1)^k}{k!} x^k$ mit $R = \infty$

b) " $\Rightarrow e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}$ mit $R = \infty$

c) Es gilt: $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ für $|x| < 1$

$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-1)^k x^k$ für $|x| < 1$, also $R = 1$.

A8

a) $f(x) = \sqrt{1+x}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\qquad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ $\qquad f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$ $\qquad f'''(0) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-\frac{7}{2}}$ $\qquad f^{(4)}(0) = -\frac{15}{16}$

$f^{(5)}(x) = \frac{105}{32}(1+x)^{-\frac{9}{2}}$ $\qquad f^{(5)}(0) = \frac{105}{32}$

$$p_5(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{15}{16\cdot 4!}x^4 + \frac{105}{32\cdot 5!}x^5$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5$$

b) $f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{2}(e^x + e^{-x})$ $(= \cosh(x))$ $\qquad f'(0) = 1$

$f''(x) = \frac{1}{2}(e^x - e^{-x}) = f(x)$ $\qquad f''(0) = 0$

$$\Rightarrow \quad p_5(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} = x + \frac{x^3}{6} + \frac{x^5}{120}$$

A9

a) $f(x) = x \cdot e^x$

$f'(x) = e^x + x \cdot e^x = e^x(1+x)$

$f''(x) = e^x + (1+x)e^x = e^x(2+x)$

$f'''(x) = e^x(2+x) + e^x = e^x(3+x)$

$f^{(k)}(x) = e^x(k+x)$

$$f(x) = \sum_{k=0}^{\infty} \frac{e(1+k)}{k!}(x-1)^k$$

b) $f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$ $\qquad x_0 = 0$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$ $\qquad f'(0) = 0$

$f''(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$ $\qquad f''(0) = 1$

$\vdots$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

nur die geraden Exponenten haben Faktor $\neq 0$