Optimization

3rd Problem Set

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13/12/2022

Problem 1

(10) Compute the projection of $\mathbf{x_0} \in \mathbb{R}^n$ onto the set $\mathbf{B}(\mathbf{0}, r) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 \leq r\}$.

(a) Draw a scheme of the problem.

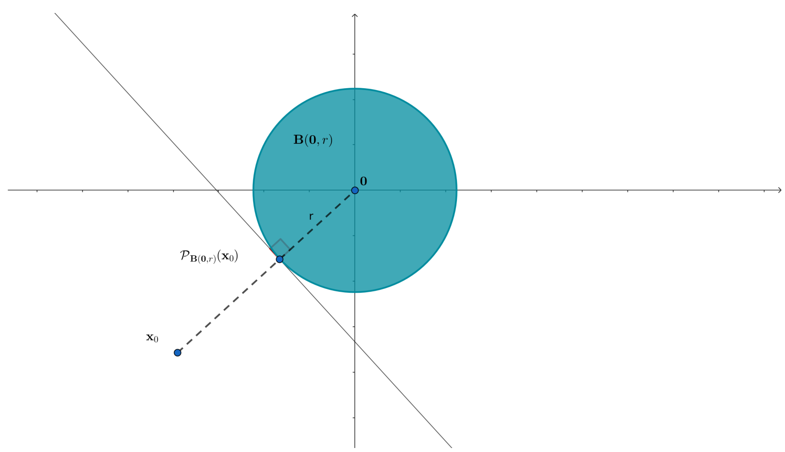

Figure 1: projection of $\mathbf{x_0} \in \mathbb{R}^n$ onto the set $\mathbf{B}(\mathbf{0}, r) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 \leq r\}$.

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(b) Write down the optimization problem you must solve, in terms of differentiable functions.

Solution.

The problem can be rewritten as the constrained convex optimization problem

$$\begin{aligned} &\text{minimize} \quad f_0(\mathbf{x}) := \frac{1}{2}\|\mathbf{x}_0 - \mathbf{x}\|_2^2 \\ &\text{subject to} \quad \|\mathbf{x}\|_2 - r \leq 0 \end{aligned}$$

(c) Write down the KKT conditions, in terms of the optimal parameters $\mathbf{x}_*$ and λ_*.

Solution:

Let $f_1(\mathbf{x}) = \|\mathbf{x}\|_2 - r$. The KKT conditions are:

$$\nabla f_0(\mathbf{x}_*) + \lambda_* \nabla f_1(\mathbf{x}_*) = \mathbf{0} \tag{1}$$

$$\lambda_* \geq 0 \tag{2}$$

$$f_1(\mathbf{x}_*) \leq 0 \tag{3}$$

$$\lambda_* f_1(\mathbf{x}_*) = 0 \tag{4}$$

(d) Consider the case $\lambda_* > 0$. What is the conclusion?

Solution:

Using the equation (4) we obtain that $f_1(\mathbf{x}_*)$ must be equal to 0

$$f_1(\mathbf{x}_*) = 0 \Leftrightarrow \|\mathbf{x}_*\|_2 - r = 0 \Leftrightarrow \|\mathbf{x}_*\|_2 = r \tag{5}$$

$$\nabla f_0(\mathbf{x}) = \nabla(\frac{1}{2}\|\mathbf{x}_0 - \mathbf{x}\|_2^2) = \mathbf{x} - \mathbf{x}_0 \tag{6}$$

$$\nabla f_1(\mathbf{x}) = \nabla(\|\mathbf{x}\|_2 - r) \overset{(*)}{=} \frac{\mathbf{x}}{\|\mathbf{x}\|_2} \overset{(5)}{=} \frac{\mathbf{x}}{r} \tag{7}$$

On point $(*)$ we calculated the $\nabla(\|\mathbf{x}\|_{\mathbf{2}})$ as follows:

$$\nabla(\|\mathbf{x}\|_{\mathbf{2}}) = \begin{bmatrix} \frac{\partial(\|\mathbf{x}\|_2)}{\partial x_1} \\ \frac{\partial(\|\mathbf{x}\|_2)}{\partial x_2} \\ \ldots \\ \frac{\partial(\|\mathbf{x}\|_2)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{2x_1}{2\sqrt{x_1^2+x_2^2+\ldots+x_n^2}} \\ \frac{2x_2}{2\sqrt{x_1^2+x_2^2+\ldots+x_n^2}} \\ \ldots \\ \frac{2x_n}{2\sqrt{x_1^2+x_2^2+\ldots+x_n^2}} \end{bmatrix} = \frac{1}{\|\mathbf{x}\|_2} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \frac{\mathbf{x}}{\|\mathbf{x}\|_2}$$

Using (6) and (7) in equation (1)

$$\mathbf{x}_* - \mathbf{x}_0 + \lambda_* \frac{\mathbf{x}_*}{r} = \mathbf{0} \Leftrightarrow \frac{r + \lambda_*}{r}\mathbf{x}_* = \mathbf{x}_0 \Leftrightarrow \mathbf{x}_* = \frac{r}{r + \lambda_*}\mathbf{x}_0$$

Finally we use this result in equation (5).

$$\|\frac{r}{r + \lambda_*}\mathbf{x}_0\|_2 = r \Leftrightarrow \frac{1}{r + \lambda_*}\|\mathbf{x}_0\|_2 = 1 \Leftrightarrow \lambda_* = \|\mathbf{x}_0\|_2 - r \tag{8}$$

Hence the final result is

$$\mathbf{x}_* = \frac{r}{r + \lambda_*}\mathbf{x}_0 = \frac{r}{r + \|\mathbf{x}_0\|_2 - r}\mathbf{x}_0 = r\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|_2}$$

Note that due to equation (8), the final result can only be true if $\|\mathbf{x}_0\|_2 - r > 0 \Leftrightarrow \|\mathbf{x}_0\|_2 > r$

(e) Consider the case $\lambda_* = 0$. What is the conclusion?

Solution:

Using the equation (1) we obtain

$$\nabla f_0(\mathbf{x}_*) = 0 \overset{(5)}{\Longleftrightarrow} \mathbf{x}_0 - \mathbf{x} = 0 \Leftrightarrow \mathbf{x} = \mathbf{x}_0$$

Due to (3) the final result can only be true if

$$f_1(\mathbf{x}_*) \leq 0 \Leftrightarrow \|\mathbf{x}_*\|_2 - r \leq 0 \Leftrightarrow \|\mathbf{x}_*\|_2 \leq r \Leftrightarrow \|\mathbf{x}_0\|_2 \leq r$$

Problem 2

(10) Repeat the steps of the previous question and compute the projection of $\mathbf{x}_0 \in \mathbb{R}^n$ onto the set $\mathbf{B}(\mathbf{y}, r) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{y}\|_2 \leq r\}$ (for given $\mathbf{y} \in \mathbb{R}^n$ and $r \in \mathbb{R}_{++}$).

(a) Draw a scheme of the problem.

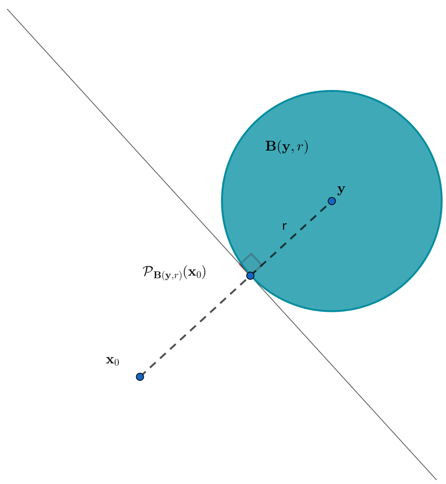

Figure 2: projection of $\mathbf{x_0} \in \mathbb{R}^n$ onto the set $\mathbf{B}(\mathbf{y}, r) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{y}\|_2 \leq r\}$.

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(b) Write down the optimization problem you must solve, in terms of differentiable functions.

Solution.

The problem can be rewritten as the constrained convex optimization problem

$$\begin{aligned} &\text{minimize} \quad f_0(\mathbf{x}) := \frac{1}{2}\|\mathbf{x}_0 - \mathbf{x}\|_2^2 \\ &\text{subject to} \quad \|\mathbf{x} - \mathbf{y}\|_2 - r \leq 0 \end{aligned}$$

(c) Write down the KKT conditions, in terms of the optimal parameters $\mathbf{x}_*$ and λ_*.

Solution:

Let $f_1(\mathbf{x}) = \|\mathbf{x} - \mathbf{y}\|_2 - r$. The KKT conditions are:

$$\nabla f_0(\mathbf{x}_*) + \lambda_* \nabla f_1(\mathbf{x}_*) = \mathbf{0} \tag{1}$$

$$\lambda_* \geq 0 \tag{2}$$

$$f_1(\mathbf{x}_*) \leq 0 \tag{3}$$

$$\lambda_* f_1(\mathbf{x}_*) = 0 \tag{4}$$

(d) Consider the case $\lambda_* > 0$. What is the conclusion?

Solution:

Using the equation (4) we obtain that $f_1(\mathbf{x}_*)$ must be equal to 0

$$f_1(\mathbf{x}_*) = 0 \Leftrightarrow \|\mathbf{x}_* - \mathbf{y}\|_2 - r = 0 \Leftrightarrow \|\mathbf{x}_* - \mathbf{y}\|_2 = r \tag{5}$$

$$\nabla f_0(\mathbf{x}) = \nabla(\frac{1}{2}\|\mathbf{x}_0 - \mathbf{x}\|_2^2) = \mathbf{x} - \mathbf{x}_0 \tag{6}$$

$$\nabla f_1(\mathbf{x}) = \nabla(\|\mathbf{x}\|_2 - r) = \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|_2} \overset{(5)}{=} \frac{\mathbf{x} - \mathbf{y}}{r} \tag{7}$$

Using (6) and (7) in equation (1)

$$\mathbf{x}_* - \mathbf{x}_0 + \lambda_* \frac{\mathbf{x}_* - \mathbf{y}}{r} = \mathbf{0} \Leftrightarrow \frac{r + \lambda_*}{r}\mathbf{x}_* - \frac{\lambda_*}{r}\mathbf{y} = \mathbf{x}_0 \Leftrightarrow \mathbf{x}_* = \frac{r}{r + \lambda_*}\mathbf{x}_0 + \frac{\lambda_*}{r + \lambda_*}\mathbf{y}$$

Finally we use this result in equation (5).

$$\begin{aligned} &\|\frac{r}{r + \lambda_*}\mathbf{x}_0 + \frac{\lambda_*}{r + \lambda_*}\mathbf{y} - \mathbf{y}\|_2 = r \Leftrightarrow \|\frac{r}{r + \lambda_*}\mathbf{x}_0 - \frac{r}{r + \lambda_*}\mathbf{y}\|_2 = r \Leftrightarrow \\ &\Leftrightarrow \frac{1}{r + \lambda_*}\|\mathbf{x}_0 - \mathbf{y}\|_2 = 1 \Leftrightarrow \lambda_* = \|\mathbf{x}_0 - \mathbf{y}\|_2 - r \end{aligned} \tag{8}$$

Hence the final result is

$$\mathbf{x}_* = \frac{r}{r + \lambda_*}\mathbf{x}_0 + \frac{\lambda_*}{r + \lambda_*}\mathbf{y} = \frac{r}{r + \|\mathbf{x}_0 - \mathbf{y}\|_2 - r}\mathbf{x}_0 + \frac{\|\mathbf{x}_0 - \mathbf{y}\|_2 - r}{r + \|\mathbf{x}_0 - \mathbf{y}\|_2 - r}\mathbf{y} = r\frac{\mathbf{x}_0 - \mathbf{y}}{\|\mathbf{x}_0 - \mathbf{y}\|_2} + \mathbf{y}$$

Note that due to equation (8), the final result can only be true if $\|\mathbf{x}_0 - \mathbf{y}\|_2 - r > 0 \Leftrightarrow \|\mathbf{x}_0 - \mathbf{y}\|_2 > r$

(e) Consider the case $\lambda_* = 0$. What is the conclusion?

Solution:

Using the equation (1) we obtain

$$\nabla f_0(\mathbf{x}_*) = 0 \overset{(5)}{\Longleftrightarrow} \mathbf{x}_0 - \mathbf{x} = 0 \Leftrightarrow \mathbf{x} = \mathbf{x}_0$$

Due to (3) the final result can only be true if

$$f_1(\mathbf{x}_*) \leq 0 \Leftrightarrow \|\mathbf{x}_* - \mathbf{y}\|_2 - r \leq 0 \Leftrightarrow \|\mathbf{x}_* - \mathbf{y}\|_2 \leq r \Leftrightarrow \|\mathbf{x}_0 - \mathbf{y}\|_2 \leq r$$

Problem 3

(10) Let $\mathbf{a} \in \mathbb{R}^n$. Compute the projection of $\mathbf{x_0} \in \mathbb{R}^\mathbf{n}$ onto set $\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^\mathbf{n} \mid \mathbf{a} \leq \mathbf{x}\}$.

Solution:

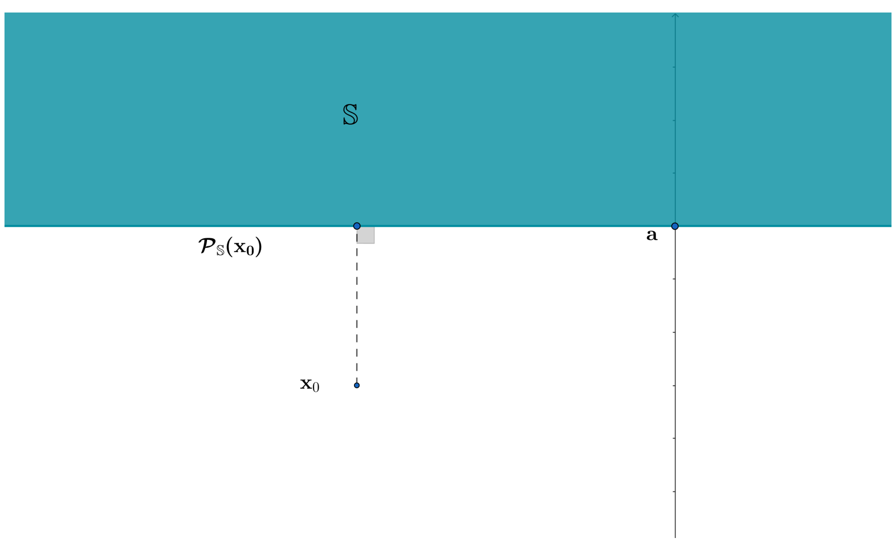

Figure 3: projection of $\mathbf{x_0} \in \mathbb{R}^\mathbf{n}$ onto set $\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^\mathbf{n} \mid \mathbf{a} \leq \mathbf{x}\}$

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Assuming that $\mathbf{a} = (a_1, a_2, ..., a_n)^T$ then the constrained optimization problem, we have to solve is:

$$\begin{aligned} &\text{minimize} \quad f_0(\mathbf{x}) := \frac{1}{2}\|\mathbf{x}_0 - \mathbf{x}\|_2^2 \\ &\text{subject to} \quad f_i(\mathbf{x}) = a_i - x_i \leq 0 \end{aligned}$$

The KKT conditions are the following:

$$\nabla f_0(\mathbf{x}_*) + \sum_{i=1}^{n} \lambda_i \nabla f_i(\mathbf{x}_*) = \mathbf{0} \tag{1}$$

$$\lambda_i \geq 0 \tag{2}$$

$$f_i(\mathbf{x}_*) \leq 0 \Leftrightarrow a_i - x_i^* \leq 0 \tag{3}$$

$$\lambda_i f_i(\mathbf{x}_*) = 0 \Leftrightarrow \lambda_i(a_i - x_i^*) = 0 \tag{4}$$

Let $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_n(\mathbf{x}))^T = \mathbf{a} - \mathbf{x}$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n)^T$

Then

$$\sum_{i=1}^{n} \lambda_i \nabla f_i(\mathbf{x}) = \nabla(\boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x})) = \nabla(\boldsymbol{\lambda}^T(\mathbf{a} - \mathbf{x})) = -\boldsymbol{\lambda}$$

Hence equation (1) becomes :

$$\mathbf{x}_* - \mathbf{x}_0 - \boldsymbol{\lambda} = \mathbf{0} \Leftrightarrow x_i^* = x_{0,i} + \lambda_i \tag{5}$$

1. If $\lambda_i > 0$, then because of (4) we have $a_i - x_i^* = 0 \Leftrightarrow x_i^* = a_i$ (6) and because of (5) we conclude that

$$\lambda_i = x_i^* - x_{0,i} > 0 \overset{(6)}{\Longleftrightarrow} x_{0,i} < a_i$$

2. If $\lambda_i = 0$, then because of (5) we have $x_i^* = x_{0,i}$ and because of (3) we have $x_i^* \geq a_i \Leftrightarrow x_{0,i} \geq a_i$

Hence $\mathcal{P}_{\mathbb{S}}(\mathbf{x}_0) = \mathbf{max}\{\mathbf{a}, \mathbf{x}_0\}$

Problem 4

Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$ and consider the problem

$$(\mathbf{P}) \quad \min_{\mathbf{x}} f_0(\mathbf{x}) := \frac{1}{2}\|\mathbf{x}\|_2^2, \text{ subject to } \mathbf{x} \in \mathbb{H} := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} = b\}.$$

(a) (10) Write and solve the KKT for problem (P).

Solution:

The KKT conditions are

$$\nabla f_0(\mathbf{x}_*) + v_*\mathbf{a} = \mathbf{0} \Leftrightarrow \mathbf{x}_* = -v_*\mathbf{a} \tag{1}$$

$$\mathbf{a}^T\mathbf{x}_* - b = 0 \tag{2}$$

By combining the 2 equations we get:

$$-v_*\mathbf{a}^T\mathbf{a} - b = 0 \Leftrightarrow v_* = -\frac{b}{\|\mathbf{a}\|_2^2}$$

Hence, by replacing v_* in equation (1) we have

$$\mathbf{x}_* = b\frac{\mathbf{a}}{\|\mathbf{a}\|_2^2}$$

The solution of problem (P) is

$$p_* = \frac{1}{2}\|\mathbf{x}_*\|_2^2 = \frac{1}{2}\|b\frac{\mathbf{a}}{\|\mathbf{a}\|_2^2}\|_2^2 = \frac{b}{2\|\mathbf{a}\|_2^2}$$

(b) (10) Compute the solution of problem (P) using the projected gradient descent method

$$\mathbf{x}_{k+1} = \mathcal{P}_{\mathbb{H}}\left(\mathbf{x}_k - \frac{1}{L}\nabla f_0(\mathbf{x}_k)\right), \tag{2}$$

where $L := \max(\text{eig}(\nabla^2 f_0(\mathbf{x})))$. What do you observe?

Solution:

Initialy we have to calculate the $\mathcal{P}_{\mathbb{H}}(\mathbf{x}_0)$. In order to do so we have to solve the problem

$$\min_{\mathbf{x}} f_0(\mathbf{x}) := \frac{1}{2}\|\mathbf{x}_0 - \mathbf{x}\|_2^2, \text{ subject to } \mathbf{x} \in \mathbb{H} := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} = b\}.$$

The KKT conditions are

$$\nabla f_0(\mathbf{x}_*) + v_*\mathbf{a} = \mathbf{0} \Leftrightarrow \mathbf{x}_* - \mathbf{x}_0 + v_*\mathbf{a} = \mathbf{0} \Leftrightarrow \mathbf{x}_* = \mathbf{x}_0 - v_*\mathbf{a} \tag{1}$$

$$\mathbf{a}^T\mathbf{x}_* - b = 0 \tag{2}$$

By combining the 2 equations we get:

$$\mathbf{a}^T\mathbf{x}_0 - v_*\mathbf{a}^T\mathbf{a} - b = 0 \Leftrightarrow v_* = \frac{\mathbf{a}^T\mathbf{x}_0}{\|\mathbf{a}\|_2^2} - \frac{b}{\|\mathbf{a}\|_2^2}$$

Hence , by replacing v_* in equation (1) we have

$$\mathbf{x}_* = \mathcal{P}_{\mathbb{H}}(\mathbf{x}_0) = \mathbf{x}_0 - (\mathbf{a}^T\mathbf{x}_0 - b)\frac{\mathbf{a}}{\|\mathbf{a}\|_2^2}$$

The gradient of $f_0(\mathbf{x})$ is $\nabla f_0(\mathbf{x}) = \frac{1}{2}\nabla\|\mathbf{x}\|_2^2 = \mathbf{x}$

The Hessian is $\nabla^2 f_0(\mathbf{x}) = \nabla\mathbf{x} = \mathbf{I}$

Thus, the maximum eigenvalue of $\nabla^2 f_0(\mathbf{x})$ is $L = 1$

The result after implementing the projected gradient descent method in matlab is the following

```
iter_number = 1, pstar computed by projected gradient descent = 0.037090
pstar exact  = 0.037090
```

We observe that the projected gradient method requires only one iteration to find the optimal value. This is because:

$$\mathbf{x}_{k+1} = \mathcal{P}_{\mathbb{H}}\left(\mathbf{x}_k - \frac{1}{L}\nabla f_0(\mathbf{x}_k)\right) = \mathcal{P}_{\mathbb{H}}\left(\mathbf{x}_k - \frac{1}{1}\mathbf{x}_k\right) = \mathcal{P}_{\mathbb{H}}(\mathbf{0})$$

But calculating $\mathcal{P}_{\mathbb{H}}(\mathbf{0})$ is equivalent to solving the initial problem $(\mathbf{P})$, which justifies why the gradient descent method reaches the optimal value on the first iteration

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Problem 5

Let $\mathbf{A} \in \mathbb{R}^{p\times n}$, with $\operatorname{rank}(\mathbf{A}) = p$, and $\mathbf{b} \in \mathbb{R}^p$.

(a) (10) Find the distance of a point $\mathbf{x}_0 \in \mathbb{R}^n$ from the set $\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$.

Solution:

We have to calculate the $\mathcal{P}_{\mathbb{S}}(\mathbf{x}_0)$.

$$\min_{\mathbf{x}} f_0(\mathbf{x}) := \frac{1}{2}\|\mathbf{x}_0 - \mathbf{x}\|_2^2, \text{ subject to } \mathbf{x} \in \mathbb{S} := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}.$$

Since $\text{rank}(\mathbf{A}) = p$ we know that the rows of the matrix $\mathbf{A}$ are linear independent which means that the gradiends of all constraints on a feasible point $\mathbf{x}$ are linear independent. Hence we can apply the KKT conditions which are

$$\nabla f_0(\mathbf{x}_*) + \mathbf{A}^T\mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{x}_* - \mathbf{x}_0 + \mathbf{A}^T\mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{x}_* = \mathbf{x}_0 - \mathbf{A}^T\mathbf{v} \quad \text{for } \mathbf{v} \in \mathbb{R}^p \tag{1}$$

$$\mathbf{A}\mathbf{x}_* - \mathbf{b} = \mathbf{0} \tag{2}$$

By combining the 2 equations we get:

$$\mathbf{A}\mathbf{x}_0 - \mathbf{A}\mathbf{A}^T\mathbf{v} - \mathbf{b} = \mathbf{0} \overset{(*)}{\Longleftrightarrow} \mathbf{v} = (\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x}_0 - \mathbf{b})$$

Hence, by replacing $\mathbf{v}$ in equation (1) we have

$$\mathbf{x}_* = \mathcal{P}_{\mathbb{H}}(\mathbf{x}_0) = \mathbf{x}_0 - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x}_0 - \mathbf{b})$$

Now that we know the projection of $\mathbf{x}_0$ on $\mathbb{S}$ we can calculate the distance.

$$\|\mathbf{x}_* - \mathbf{x}_0\|_2 = \|\mathbf{x}_0 - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x}_0 - \mathbf{b}) - \mathbf{x}_0\|_2 = \|\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x}_0 - \mathbf{b})\|_2$$

On point $(*)$ we have to prove tha the matrix $\mathbf{A}\mathbf{A}^T$ is invertible. We know that $\mathbf{A}$ has a full row rank hence $\mathbf{A}^T$ has a full column rank. Suppose that that the matrix $\mathbf{A}\mathbf{A}^T$ is not invertible and there is some non-zero vector $\mathbf{x} \in \mathbb{R}^p$ such that $\mathbf{A}\mathbf{A}^T\mathbf{x} = \mathbf{0}$. But this implies

$$\mathbf{x}^T\mathbf{A}\mathbf{A}^T\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0 \Leftrightarrow (\mathbf{A}^T\mathbf{x})^T\mathbf{A}^T\mathbf{x} = \|\mathbf{A}^T\mathbf{x}\|_2^2 = 0$$

That is $\mathbf{A}^T\mathbf{x} = \mathbf{0}$ which violates the full column rank of $\mathbf{A}^T$ (the columns must be linear independent). Hence $\mathbf{A}\mathbf{A}^T$ must be invertible

(b) Let the $(n \times n)$ positive definite matrix $\mathbf{P} = \mathbf{P}^T \succ \mathbf{0},\ \mathbf{q} \in \mathbb{R}^n$ and

$$f_0(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x} + \mathbf{q}^T\mathbf{x}.$$

Consider the problem

$$(Q) \quad \min_{\mathbf{x} \in \mathbb{S}} f_0(\mathbf{x}).$$

i. Solve problem (Q) using cvx.

Solution:

The result of the cvx is the following:

```
pstar computed by cvx  = 4.092005
```

ii. (10) Write the KKT for problem (Q) and compute the optimal solution by solving them using matlab (no cvx).

Solution:

The KKT conditions are :

$$\begin{aligned}
&\nabla f_0(\mathbf{x}_*) + \mathbf{A}^T\mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{P}\mathbf{x}_* + \mathbf{q} + \mathbf{A}^T\mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{P}\mathbf{x}_* + \mathbf{A}^T\mathbf{v} = -\mathbf{q} \\
&\mathbf{A}\mathbf{x}_* = \mathbf{b}
\end{aligned}$$

for $\mathbf{v} \in \mathbb{R}^p$.

These can be rewritten as the linear equations system

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{x}_* \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} -\mathbf{q} \\ \mathbf{b} \end{bmatrix}$$

This can be easily solved in matlab using the **linsolve** function. The result is the following:

```
pstar computed by linsolve  = 4.092005
```

Which is exactly the the same as the previous one computed by the cvx

iii. (30) Compute the optimal solution via the projected gradient method

$$\mathbf{x}_{k+1} = \mathcal{P}_{\mathbb{S}}\left(\mathbf{x}_k - \frac{1}{L}\nabla f_0(\mathbf{x}_k)\right),$$

where $L := \max(\mathrm{eig}(\nabla^2 f_0(\mathbf{x})))$

Solution:

The gradient of $f_0(\mathbf{x})$, $\nabla f_0(\mathbf{x})$ is

$$\nabla f_0(\mathbf{x}) = \mathbf{P}\mathbf{x} + \mathbf{q}$$

The Hessian of $f_0(\mathbf{x})$, $\nabla^2 f_0(\mathbf{x})$

$$\nabla^2 f_0(\mathbf{x}) = \nabla(\mathbf{P}\mathbf{x} + \mathbf{q}) = \mathbf{P}$$

The result of the projected gradient method in matlab is the following:

```
iter_number = 17, pstar computed by projected gradient descent = 4.092011
```

Using $\epsilon = 0.00001$ as tolerance
