CB2070 Ex 1

(1) Show $(\mathbb{A} \otimes \mathbb{B})(\mathbb{C} \otimes \mathbb{D}) = (\mathbb{A}\mathbb{C}) \otimes (\mathbb{B}\mathbb{D})$

$$\mathbb{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \mathbb{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \quad \mathbb{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \quad \mathbb{D} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$$

$$\mathbb{A} \otimes \mathbb{B} = \begin{pmatrix} A_{11}\mathbb{B} & A_{12}\mathbb{B} \\ A_{21}\mathbb{B} & A_{22}\mathbb{B} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

$$\mathbb{C} \otimes \mathbb{D} = \begin{pmatrix} C_1D_1 \\ C_1D_2 \\ C_2D_1 \\ C_2D_2 \end{pmatrix}$$

$$(\mathbb{A} \otimes \mathbb{B})(\mathbb{C} \otimes \mathbb{D}) = \begin{pmatrix} C_1D_1A_{11}B_{11} + C_1D_2A_{11}B_{12} + C_2D_1A_{12}B_{11} + C_2D_2A_{12}B_{12} \\ C_1D_1A_{11}B_{21} + C_1D_2A_{11}B_{22} + C_2D_1A_{12}B_{21} + C_2D_2A_{12}B_{22} \\ C_1D_1A_{21}B_{11} + C_1D_2A_{21}B_{22} + C_2D_1A_{22}B_{11} + C_2D_2A_{22}B_{12} \\ C_1D_1A_{21}B_{21} + C_1D_2A_{21}B_{22} + C_2D_1A_{22}B_{21} + C_2D_2A_{22}B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} C_1A_{11}(D_1B_{11} + D_2B_{12}) + C_2A_{12}(D_1B_{11} + D_2B_{12}) \\ C_1A_{11}(D_1B_{21} + D_2B_{22}) + C_2A_{12}(D_1B_{21} + D_2B_{22}) \\ C_1A_{21}(D_1B_{11} + D_2B_{22}) + C_2A_{22}(D_1B_{11} + D_2B_{12}) \\ C_1A_{21}(D_1B_{21} + D_2B_{22}) + C_2A_{22}(D_1B_{21} + D_2B_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} C_1A_{11}\begin{pmatrix} D_1B_{11} + D_2B_{12} \\ D_1B_{21} + D_2B_{22} \end{pmatrix} + C_2A_{12}\begin{pmatrix} D_1B_{11} + D_2B_{12} \\ D_1B_{21} + D_2B_{22} \end{pmatrix} \\ C_1A_{21}\begin{pmatrix} D_1B_{11} + D_2B_{22} \\ D_1B_{21} + D_2B_{22} \end{pmatrix} + C_2A_{22}\begin{pmatrix} D_1B_{11} + D_2B_{12} \\ D_1B_{21} + D_2B_{22} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (C_1A_{11} + C_2A_{12})(\mathbb{B}\cdot\mathbb{D}) \\ (C_1A_{21} + C_2A_{22})(\mathbb{B}\cdot\mathbb{D}) \end{pmatrix} = (\mathbb{A}\cdot\mathbb{C}) \otimes (\mathbb{B}\cdot\mathbb{D})$$

QED

CB2070 ex 1 cont'd

(2) Show $e^{\mathbb{A}} \otimes e^{\mathbb{B}} = e^{\mathbb{A}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B}}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{\mathbb{A}} \otimes e^{\mathbb{B}} = \left( \mathbb{I} + \mathbb{A} + \frac{1}{2!}\mathbb{A}^2 + \frac{1}{3!}\mathbb{A}^3 + \dots \right) \otimes \left( \mathbb{I} + \mathbb{B} + \frac{1}{2!}\mathbb{B}^2 + \frac{1}{3!}\mathbb{B}^3 + \dots \right)$$

$$= \mathbb{I}\otimes\mathbb{I} + \mathbb{A}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B} + \frac{1}{2!}\mathbb{A}^2\otimes\mathbb{I} + \frac{1}{2!}\mathbb{I}\otimes\mathbb{B}^2 + \frac{1}{3!}\mathbb{A}^3\otimes\mathbb{I} + \frac{1}{3!}\mathbb{I}\otimes\mathbb{B}^3 + \dots$$
$$+ \mathbb{A}\otimes\mathbb{B} + \frac{1}{2!}\mathbb{A}^2\otimes\mathbb{B} + \frac{1}{3!}\mathbb{A}^3\otimes\mathbb{B} + \dots$$
$$+ \frac{1}{2!}\mathbb{A}\otimes\mathbb{B}^2 + \frac{1}{3!}\mathbb{A}\otimes\mathbb{B}^3 + \dots$$
$$+ \left(\frac{1}{2!}\right)^2 \mathbb{A}^2\otimes\mathbb{B}^2 + \frac{1}{2!}\frac{1}{3!}\mathbb{A}^2\otimes\mathbb{B}^3 + \frac{1}{3!}\frac{1}{2!}\mathbb{A}^3\otimes\mathbb{B}^2 + \left(\frac{1}{3!}\right)^2 \mathbb{A}^3\otimes\mathbb{B}^3 + \dots$$

Do $(\mathbb{A}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B})^2 = (\mathbb{A}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B})(\mathbb{A}\otimes\mathbb{I} + \mathbb{I}\,\mathbb{B})$

$$= \mathbb{A}^2\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B}^2 + \underbrace{2\,\mathbb{A}\otimes\mathbb{I}\,\mathbb{I}\otimes\mathbb{B}}_{\mathbb{A}\otimes\mathbb{B}}$$

$$\times\left(\frac{1}{2!}\right) = \frac{1}{2!}\left(\mathbb{A}^2\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B}^2\right) + \mathbb{A}\otimes\mathbb{B}$$

$$\frac{3}{3!} = \frac{1}{2!}$$

Do $\frac{1}{3!}(\mathbb{A}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B})^3 = \frac{1}{3!}\left(\mathbb{A}^3\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B}^3 + 3\mathbb{A}^2\mathbb{B} + 3\mathbb{A}\mathbb{B}^2\right)$

$$= \frac{1}{3!}\left(\mathbb{A}^3\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B}^3\right) + \frac{1}{2!}\mathbb{A}^2\mathbb{B} + \frac{1}{2!}\mathbb{A}\mathbb{B}^2$$

$$\Rightarrow \mathbb{I}\otimes\mathbb{I} + \underbrace{(\mathbb{A}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B})}^{n=0} + \frac{1}{2!}\left(\mathbb{A}^2\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B}^2\right) + \mathbb{A}\otimes\mathbb{B} \quad \leftarrow \frac{1}{2!}(\mathbb{A}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B})^2$$
$$+ \frac{1}{3!}\left(\mathbb{A}^3\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B}^3\right) + \frac{1}{2!}\mathbb{A}^2\otimes\mathbb{B} + \frac{1}{2!}\mathbb{A}\otimes\mathbb{B}^2 \quad \leftarrow \frac{1}{3!}(\mathbb{A}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B})^3$$
$$+ \dots$$

$$\Rightarrow \sum_n \frac{1}{n!}(\mathbb{A}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B})^n = e^{\mathbb{A}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{B}}$$

QED

CB2070 Ex 1 cont'd

(3) $R(\phi, \bar{n}) = e^{-i\phi(\bar{n}\cdot\bar{\sigma})/2}$ , show $R(\phi,\bar{n}) = \cos(\phi/2)\mathbb{I} - i\sin(\phi/2)(\bar{n}\cdot\bar{\sigma})$

We have:

$$(\bar{n}\cdot\bar{\sigma})^2 = \mathbb{I}^2 \overset{?}{=} \mathbb{I}$$

$(+1)$ $+1$
$(-i)(-i)(-i)(-i)$
$(-i)(-i)(-i) = -i^3$
$= -i(-1) = +i$

$$e^x = \sum_n \frac{1}{n!} x^n$$

$$\cos(x) = \sum_n \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\sin(x) = \sum_n \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Expand R

$[-i\frac{\phi}{2}(\bar{n}\cdot\bar{\sigma})]$ $\qquad (\bar{n}\cdot\bar{\sigma})^2(\bar{n}\cdot\bar{\sigma}) = \mathbb{I}(\bar{n}\cdot\bar{\sigma})$

$$R(\phi,\bar{n}) = \mathbb{I} + \frac{1}{2!}\left[-i\frac{\phi}{2}(\bar{n}\cdot\bar{\sigma})\right]^2 + \frac{1}{3!}\left[-i\frac{\phi}{2}(\bar{n}\cdot\bar{\sigma})\right]^3 + \frac{1}{4!}\left[-i\frac{\phi}{2}(\bar{n}\cdot\bar{\sigma})\right]^4 + \frac{1}{5!}\left[-i\frac{\phi}{2}(\bar{n}\cdot\bar{\sigma})\right]^5 + \dots$$

$$= \mathbb{I} - i\frac{\phi}{2}(\bar{n}\cdot\bar{\sigma}) - \frac{1}{2!}\left(\frac{\phi}{2}\right)^2\mathbb{I} + i\frac{1}{3!}\left(\frac{\phi}{2}\right)^3(\bar{n}\cdot\bar{\sigma}) + \frac{1}{4!}\left(\frac{\phi}{2}\right)^4\mathbb{I} - i\frac{1}{5!}\left(\frac{\phi}{2}\right)^5(\bar{n}\cdot\bar{\sigma}) + \dots$$

$$= \mathbb{I} - \frac{1}{2!}\left(\frac{\phi}{2}\right)^2\mathbb{I} + \frac{1}{4!}\left(\frac{\phi}{2}\right)^4\mathbb{I} - \left(i\frac{\phi}{2}(\bar{n}\cdot\bar{\sigma}) - i\frac{1}{3!}\left(\frac{\phi}{2}\right)(\bar{n}\cdot\bar{\sigma}) - i\frac{1}{5!}\left(\frac{\phi}{2}\right)^5(\bar{n}\cdot\bar{\sigma})\right) + \dots$$

$$= \left[\sum_n \frac{1}{(2n)!}\left(\frac{\phi}{2}\right)^{2n}\right]\mathbb{I} - i\left[\sum_n \frac{1}{(2n+1)!}\left(\frac{\phi}{2}\right)^{2n+1}\right](\bar{n}\cdot\bar{\sigma})$$

$$= \cos\left(\frac{\phi}{2}\right)\mathbb{I} - i\sin\left(\frac{\phi}{2}\right)(\bar{n}\cdot\bar{\sigma})$$

QED

CB2070 Ex 1 cont'd

4) We have

$$|\Psi\rangle = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 \\ \alpha_1 \beta_2 \\ \beta_1 \alpha_2 \\ \beta_1 \beta_2 \end{pmatrix} , \qquad \hat{s}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) direct product:

$$\hat{S}_z = \hat{s}_z^{(1)} \otimes \mathbb{I}^{(2)} + \mathbb{I}^{(1)} \otimes \hat{s}_z^{(2)} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

b) 1-elec

c)

$$\hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha_1 \alpha_2 \\ \alpha_1 \beta_2 \\ \beta_1 \alpha_2 \\ \beta_1 \beta_2 \end{pmatrix} = \hbar\begin{pmatrix} \alpha_1 \alpha_2 \\ 0 \\ 0 \\ -\beta_1 \beta_2 \end{pmatrix}$$

d) $\hat{S}_x |\Psi\rangle \neq \lambda |\Psi\rangle$

e) with Python, numpy

$\lambda_1 = \hbar$ , $\lambda_2 = \lambda_3 = 0$ , $\lambda_4 = -\hbar$

$v_1 = [1, 0, 0, 0]$ , $v_2 = [0, 1, 0, 0]$ , $v_3 = [0, 0, 1, 0]$ , $v_4 = [0, 0, 0, 1]$