CHARACTERISTIC CYCLES FOR K-ORBITS ON GRASSMANNIANS

Kostiantyn Timchenko March 26, 2020

REMINDER ON D-MODULES

Let $X=\mathbb{C}^n$ with variables $(z_1,\dots,z_n).$ Then the ring of differential operators D_X on X is

$$D_X=\mathbb{C}[z_1,\dots,z_n,\partial_1,\dots,\partial_n],$$

with relations

$$[z_i, z_j] = 0, [\partial_i, \partial_j] = 0, [\partial_i, z_j] = \delta_{ij}.$$

There is an order filtration F on D_X such that

$$F_l D_X = \sum_{\alpha_1 + \dots + \alpha_n \leq l} \mathbb{C}[z_1, \dots, z_n] \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}.$$

The associated graded $(\bigoplus_l F_l D_X/F_{l-1} D_X)$ ring is

$$gr^FD_X=\mathbb{C}[z_1,\dots,z_n,\xi_1,\dots,\xi_n]=\mathcal{O}(T^*X).$$

CHARACTERISTIC VARIETY

Let M be a finitely generated D_X -module with a good filtration F. Then the graded module gr^FM is finitely generated over $gr^FD_X=\mathcal{O}(T^*X)$.

Definition

The support of the finitely generated $\mathcal{O}(T^*X)$ -module gr^FM

$$Ch(M) = supp_{\mathcal{O}(T^*X)}gr^FM = Z_{\mathcal{O}(T^*X)}\left(\sqrt{Ann_{\mathcal{O}(T^*X)}gr^FM}\right)$$

is called the characteristic variety of M.

The characteristic variety Ch(M) doesn't depend on the choice of a good filtration F of M and is a closed conic subset of T^*X .

CHARACTERISTIC VARIETY

Example

Let $X=\mathbb{C}^1$ then $D_X=\mathbb{C}[x,\partial]$ together with the relation $[\partial,x]=1$. The coordinates on the cotangent bundle T^*X are (x,ξ) .

- · If $M=\mathbb{C}[x]=D_X/D_X\partial$, then $Ch(M)=\{(x,\xi)\in T^*X|\xi=0\}=T_X^*X.$
- · Suppose $\lambda\in\mathbb{C}-\mathbb{Z}$ and consider $M_\lambda=D_X/D_X(x\partial-\lambda)$. Then $Ch(M_\lambda)=\{x\xi=0\}=T^*_{\{0\}}X\cup T^*_XX$.
- · Let $M_\delta = D_X/D_X x$. Then $Ch(M_\delta) = \{x=0\} = T^*_{\{0\}} X$.

In all of these examples, M is simple (and holonomic), but Ch(M) is not always irreducible.

By taking multiplicities into account, we have the following

Definition

For a finitely generated $D_X\mbox{-}\mathrm{module}\ M$ we define the characteristic cycle of M to be the formal sum

$$CC(M) = \sum_{V \subset Ch(M)} mult_V(gr^FM) \cdot V, \tag{1}$$

where V runs along the connected components of Ch(M).

Example

As before, take $X = \mathbb{C}^1$.

- $\cdot \ CC(\mathbb{C}[x]^{\oplus k}) = k \cdot T_X^* X$
- $\cdot \ CC(M_\lambda) = T^*_{\{0\}}X + T^*_XX.$

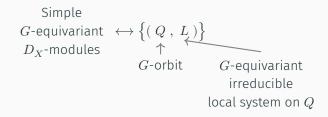
D-MODULES ASSOCIATED TO G-ORBITS

Let G be an algebraic group acting on a smooth variety X with a finite number of orbits. There is a bijection

$$\begin{array}{ll} \text{Simple} \\ G\text{-equivariant} &\longleftrightarrow \left\{ \left(\right.Q \,,\; L \left. \right) \right\} \\ D_X\text{-modules} \end{array}$$

D-MODULES ASSOCIATED TO G-ORBITS

Let G be an algebraic group acting on a smooth variety X with a finite number of orbits. There is a bijection



D-MODULES ASSOCIATED TO G-ORBITS

Let G be an algebraic group acting on a smooth variety X with a finite number of orbits. There is a bijection

Notation: \mathcal{L}_Q is the D-module on X corresponding to a pair $(Q,\underline{\mathbb{C}}_Q)$.

$$CC(\mathcal{L}_Q) = \overline{T_Q^*X} + \sum_{Q \neq Q' \subset \overline{Q}} m_{Q',Q} \, \overline{T_{Q'}^*X},$$

for some non-negative numbers $m_{Q^\prime,Q}$.

We say that $CC(\mathcal{L}_Q)$ is irreducible if $CC(\mathcal{L}_Q)=\overline{T_Q^*X}$, i.e., all the $m_{Q',Q}=0$.

These $m_{Q^{\prime},Q}$'s are hard to compute.

Conjecture by Kazhdan and Lusztig '80: Let Q be a Schubert cell in the flag variety of $G = GL(n, \mathbb{C})$. Then, $CC(\mathcal{L}_Q)$ is irreducible.

These $m_{Q',Q}$'s are hard to compute.

Conjecture by Kazhdan and Lusztig '80: Let Q be a Schubert cell in the flag variety of $G=GL(n,\mathbb{C})$. Then, $CC(\mathcal{L}_Q)$ is irreducible.

· Bressler-Finkelberg-Lunts '90: conjecture is true for Gr(k, n).

These $m_{Q',Q}$'s are hard to compute.

Conjecture by Kazhdan and Lusztig '80: Let Q be a Schubert cell in the flag variety of $G=GL(n,\mathbb{C})$. Then, $CC(\mathcal{L}_Q)$ is irreducible.

- · Bressler-Finkelberg-Lunts '90: conjecture is true for Gr(k, n).
- · Kashiwara and Saito '97: counterexample for $G = GL(8, \mathbb{C})$.

These $m_{Q',Q}$'s are hard to compute.

Conjecture by Kazhdan and Lusztig '80: Let Q be a Schubert cell in the flag variety of $G=GL(n,\mathbb{C})$. Then, $CC(\mathcal{L}_Q)$ is irreducible.

- · Bressler-Finkelberg-Lunts '90: conjecture is true for Gr(k, n).
- · Kashiwara and Saito '97: counterexample for $G = GL(8, \mathbb{C})$.
- · Williamson 15: counterexample for $G = GL(12, \mathbb{C})$.

These $m_{Q',Q}$'s are hard to compute.

Conjecture by Kazhdan and Lusztig '80: Let Q be a Schubert cell in the flag variety of $G=GL(n,\mathbb{C})$. Then, $CC(\mathcal{L}_Q)$ is irreducible.

- · Bressler-Finkelberg-Lunts '90: conjecture is true for Gr(k, n).
- · Kashiwara and Saito '97: counterexample for $G = GL(8, \mathbb{C})$.
- · Williamson 15: counterexample for $G = GL(12, \mathbb{C})$.

Barcini, Somberg and Trapa '19: an example of a $K=GL(6)\times GL(6)$ -orbit on a flag variety of $G=GL(12,\mathbb{C})$ with a reducible characteristic cycle.

K-ORBITS ON GRASSMANNIANS

Let θ be an involution on $GL(n,\mathbb{C})$. Take K to be the identity component of $GL(n)^{\theta}$. There are three cases:

$$\cdot \ K = GL(p) \times GL(q) \text{, } p + q = n$$

$$\cdot K = Sp(n)$$
, n is even

$$\cdot K = SO(n)$$

Each K acts on Gr(k, n) with a finite number of orbits.

Question: What are the characteristic cycles in this case? Are they irreducible?

GL-ORBITS

Case of $K=GL(p)\times GL(q)$. Here p+q=n, $p\geq q$, $n-k\geq k$. Write $\mathbb{C}^n=\mathbb{C}^p\oplus\mathbb{C}^q$. The K - orbits on Gr(k,n) are

$$Q(s,t) = \{U \in Gr(k,n) \, | \, \dim U \cap \mathbb{C}^p = s, \dim U \cap \mathbb{C}^q = t\},$$

where $s+t \leq k$, $s \leq p$ and $t \leq q$.

Note that $Q(s',t')\subset \overline{Q(s,t)}$ if and only if $s'\geq s$ and $t'\geq t$.

GL-ORBITS

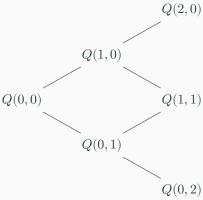
Case of $K=GL(p)\times GL(q)$. Here p+q=n, $p\geq q,$ $n-k\geq k$. Write $\mathbb{C}^n=\mathbb{C}^p\oplus\mathbb{C}^q$. The K - orbits on Gr(k,n) are

$$Q(s,t) = \{ U \in Gr(k,n) \mid \dim U \cap \mathbb{C}^p = s, \dim U \cap \mathbb{C}^q = t \},$$

where $s+t \le k$, $s \le p$ and $t \le q$.

Note that $Q(s',t') \subset \overline{Q(s,t)}$ if and only if $s' \geq s$ and $t' \geq t$.





RESOLUTIONS OF SINGULARITIES

Barbasch and Evens:

$$Z(s,t) = \{(U,V,W) \in Gr(k,n) \times Gr(s,p) \times Gr(t,q) | V \subset \mathbb{C}^p \cap U, W \subset \mathbb{C}^q \cap U\}.$$

Theorem (BE)

The map $\theta: Z(s,t) \to \overline{Q(s,t)}, (U,V,W) \mapsto U$ is a resolution of singularities. For $n-k \geq p$ the resolution $\theta: Z(s,t) \to \overline{Q(s,t)}$ is small.

RESOLUTIONS OF SINGULARITIES

Barbasch and Evens:

$$Z(s,t) = \{(U,V,W) \in Gr(k,n) \times Gr(s,p) \times Gr(t,q) | V \subset \mathbb{C}^p \cap U, W \subset \mathbb{C}^q \cap U\}.$$

Theorem (BE)

The map $\theta: Z(s,t) \to \overline{Q(s,t)}, (U,V,W) \mapsto U$ is a resolution of singularities. For $n-k \geq p$ the resolution $\theta: Z(s,t) \to \overline{Q(s,t)}$ is small. For $n-k \leq p$, there is a similar small resolution $\widetilde{\theta}: \widetilde{Z}(s,t) \to \overline{Q(s,t)}$.

Theorem

Let Q(s,t) be a $GL(p)\times GL(q)$ -orbit in Gr(k,n). Then, the characteristic cycle $CC(\mathcal{L}_{Q(s,t)})$ is irreducible.

Theorem

Let Q(s,t) be a $GL(p)\times GL(q)$ -orbit in Gr(k,n). Then, the characteristic cycle $CC(\mathcal{L}_{Q(s,t)})$ is irreducible.

Proof.

Consider $\theta: Z(s,t) \to \overline{Q(s,t)}$.

- · θ is a proper and small map: $\theta_*\mathcal{O}_Z=\mathcal{L}_{Q(s,t)}$
- $\cdot \text{ Write } CC(\mathcal{L}_Q) = \overline{T_Q^*X} + \sum_{Q \neq Q' \subset \overline{Q}} m_{Q',Q} \, \overline{T_{Q'}^*X}.$

For a K-orbit $Q'\subset \overline{Q(s,t)}$ and a general (x,ξ) in $T_{Q'}^*Gr(k,n)$ the microlocal fiber of θ over (x,ξ) is empty $\implies m_{Q',Q}=0$.

SP AND SO ORBITS

 $Sp(n,\mathbb{C})$ or $SO(n,\mathbb{C})$ acts on Gr(k,n). Let B be a nondegenerate symmetric/skew-symmetric bilinear form on \mathbb{C}^n . Let K denote the isometry group of B. The K-orbits on Gr(k,n) are

$$Q(i) = \{ U \in Gr(k, n) | \dim(rad(U)) = i \}.$$
 (2)

 $Q(i) \subset \overline{Q(j)}$ if and only if $i \geq j$.

SP AND SO ORBITS

 $Sp(n,\mathbb{C})$ or $SO(n,\mathbb{C})$ acts on Gr(k,n). Let B be a nondegenerate symmetric/skew-symmetric bilinear form on \mathbb{C}^n . Let K denote the isometry group of B. The K-orbits on Gr(k,n) are

$$Q(i) = \{ U \in Gr(k, n) | \dim(rad(U)) = i \}.$$
 (2)

 $Q(i) \subset \overline{Q(j)}$ if and only if $i \geq j$. There is a resolution of singularities

$$Z_i = \{(U, V) \in Gr(k, n) \times Gr(i, n) | V \subset rad(B|_U)\}$$
 (3)

with $\theta:Z_i \to \overline{Q(i)}$ projection to Gr(k,n). However, θ is not small.

Theorem

- 1. Consider the Sp(n)-orbits Q(i) on Gr(k,n). Then, the characteristic cycle of $\mathcal{L}_{Q(i)}$ is irreducible.
- 2. Consider the SO(n)-orbits $Q(0), Q(1), \dots, Q(k)$ on Gr(k, n)
 - · i even or i = k: the characteristic cycle of $\mathcal{L}_{Q(i)}$ is irreducible
 - · *i* odd:

$$CC(\mathcal{L}_{Q(i)}) = \overline{T_{Q(i)}^*Gr(k,n)} + \overline{T_{Q(i+1)}^*Gr(k,n)},$$

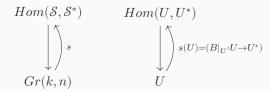
(when n is even and k=n/2, the set Q(k) is a union of two closed SO(n)-orbits. Hence the characteristic cycle of $\mathcal{L}_{Q(k-1)}$ has three irreducible components when k is even)

CHARACTERISTIC CYCLES AND DEGENERACY LOCI

Proof. Let $\mathcal S$ be the tautological bundle over a Grassmannian Gr(k,n). Then $\overline{Q(i)}=s^{-1}({\rm rank}\ \leq (k-i)\ {\rm matrices})$ is a degeneracy locus.

CHARACTERISTIC CYCLES AND DEGENERACY LOCI

Proof. Let $\mathcal S$ be the tautological bundle over a Grassmannian Gr(k,n). Then $\overline{Q(i)}=s^{-1}({\rm rank}\ \leq (k-i)\ {\rm matrices})$ is a degeneracy locus.



Theorem (Raicu)

- · Let O_i denote the set of n-by-n skew-symmetric matrices of rank i. Then each $CC(\mathcal{L}_{O_i})$ is irreducible, so that $CC(\mathcal{L}_{O_i}) = [\overline{T_{O_i}^*X}]$.
- · Let O_i denote the set of n-by-n symmetric matrices of rank i. Then $CC(\mathcal{L}_{O_i}) = [\overline{T_{O_i}^*X}]$ for n-i even or i=0, and $CC(\mathcal{L}_{O_i}) = [\overline{T_{O_i}^*X}] + [\overline{T_{O_{i-1}}^*X}]$ for n-i odd.

WORK IN PROGRESS

Joint with Scott Larson.

Let $G=Sp(2n,\mathbb{C})$ and $K=GL(n,\mathbb{C})$. Let $X=Gr^0(k,2n)$ be the space of k-dimensional isotropic subspaces of \mathbb{C}^{2n} . Let w be a nondegenerate skew-symmetric bilinear form on \mathbb{C}^{2n} . Fix $\mathbb{C}^{2n}=\mathbb{C}^n\oplus\mathbb{C}^{-n}$. The K-orbits are as follows:

$$Q(a,b,c) = \{U \in Gr^0(k,2n) \mid \dim U \cap \mathbb{C}^p = a, \dim U \cap \mathbb{C}^q = b, \\ \dim(rad\epsilon|_U) = a + b + c\},$$

where $\epsilon(x,y) = w(pr_{\mathbb{C}^n}(x), pr_{\mathbb{C}^{-n}}).$

There is a resolution of singularities $Z(a,b,c) \to \overline{Q(a,b,c)}$.