CHARACTERISTIC CYCLES FOR K-ORBITS ON GRASSMANNIANS

Kostiantyn Timchenko

March 26, 2020

Let $X = \mathbb{C}^n$ with variables $(z_1, \dots, z_n)$. Then the ring of differential operators D_X on X is

$$D_X = \mathbb{C}[z_1, \dots, z_n, \partial_1, \dots, \partial_n],$$

with relations

$$[z_i, z_j] = 0, [\partial_i, \partial_j] = 0, [\partial_i, z_j] = \delta_{ij}.$$

There is an order filtration F on D_X such that

$$F_l D_X = \sum_{\alpha_1 + \cdots + \alpha_n \leq l} \mathbb{C}[z_1, \dots, z_n] \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}.$$

The associated graded ($\bigoplus_l F_l D_X / F_{l-1} D_X$) ring is

$$gr^F D_X = \mathbb{C}[z_1, \dots, z_n, \xi_1, \dots, \xi_n] = \mathcal{O}(T^* X).$$

Let M be a finitely generated D_X-module with a good filtration F. Then the graded module $gr^F M$ is finitely generated over $gr^F D_X = \mathcal{O}(T^*X)$.

Definition

The support of the finitely generated $\mathcal{O}(T^*X)$-module $gr^F M$

$$Ch(M) = supp_{\mathcal{O}(T^*X)} gr^F M = Z_{\mathcal{O}(T^*X)}\left(\sqrt{Ann_{\mathcal{O}(T^*X)} gr^F M}\right)$$

is called the characteristic variety of M.

The characteristic variety $Ch(M)$ doesn't depend on the choice of a good filtration F of M and is a closed conic subset of T^*X.

Example

Let $X = \mathbb{C}^1$ then $D_X = \mathbb{C}[x, \partial]$ together with the relation $[\partial, x] = 1$. The coordinates on the cotangent bundle T^*X are (x, ξ).

- If $M = \mathbb{C}[x] = D_X/D_X\partial$, then $Ch(M) = \{(x, \xi) \in T^*X | \xi = 0\} = T^*_X X$.
- Suppose $\lambda \in \mathbb{C} - \mathbb{Z}$ and consider $M_\lambda = D_X/D_X(x\partial - \lambda)$. Then $Ch(M_\lambda) = \{x\xi = 0\} = T^*_{\{0\}}X \cup T^*_X X$.
- Let $M_\delta = D_X/D_X x$. Then $Ch(M_\delta) = \{x = 0\} = T^*_{\{0\}}X$.

In all of these examples, M is simple (and holonomic), but $Ch(M)$ is not always irreducible.

By taking multiplicities into account, we have the following

Definition

For a finitely generated D_X-module M we define the characteristic cycle of M to be the formal sum

$$CC(M) = \sum_{V \subset Ch(M)} mult_V(gr^F M) \cdot V, \tag{1}$$

where V runs along the connected components of $Ch(M)$.

Example

As before, take $X = \mathbb{C}^1$.

- $CC(\mathbb{C}[x]^{\oplus k}) = k \cdot T^*_X X$,
- $CC(M_\lambda) = T^*_{\{0\}} X + T^*_X X$.

Let G be an algebraic group acting on a smooth variety X with a finite number of orbits. There is a bijection

$$\begin{array}{c}\text{Simple} \\ G\text{-equivariant} \\ D_X\text{-modules}\end{array} \longleftrightarrow \Big\{ (\ Q\ ,\ L\ ) \Big\}$$

Let G be an algebraic group acting on a smooth variety X with a finite number of orbits. There is a bijection

$$\begin{array}{c}\text{Simple}\\ G\text{-equivariant}\\ D_X\text{-modules}\end{array} \longleftrightarrow \Big\{ (\underset{\substack{\uparrow\\ G\text{-orbit}}}{Q} \,,\, L) \Big\}$$

L: G-equivariant irreducible local system on Q

D-MODULES ASSOCIATED TO G-ORBITS

Let G be an algebraic group acting on a smooth variety X with a finite number of orbits. There is a bijection

$$\begin{array}{c}\text{Simple}\\ G\text{-equivariant}\\ D_X\text{-modules}\end{array} \longleftrightarrow \left\{\left(\underset{\substack{\uparrow\\ G\text{-orbit}}}{Q}\ ,\ L \right)\right\}$$

L: G-equivariant irreducible local system on Q

Notation: $\mathcal{L}_Q$ is the D-module on X corresponding to a pair $(Q, \underline{\mathbb{C}}_Q)$.

$$CC(\mathcal{L}_Q) = \overline{T^*_Q X} + \sum_{Q \neq Q' \subset \overline{Q}} m_{Q',Q}\, \overline{T^*_{Q'} X},$$

for some non-negative numbers $m_{Q',Q}$.

We say that $CC(\mathcal{L}_Q)$ is irreducible if $CC(\mathcal{L}_Q) = \overline{T^*_Q X}$, i.e., all the $m_{Q',Q} = 0$.

These $m_{Q',Q}$'s are hard to compute.

Conjecture by Kazhdan and Lusztig '80: Let Q be a Schubert cell in the flag variety of $G = GL(n, \mathbb{C})$. Then, $CC(\mathcal{L}_Q)$ is irreducible.

These $m_{Q',Q}$'s are hard to compute.

Conjecture by Kazhdan and Lusztig '80: Let Q be a Schubert cell in the flag variety of $G = GL(n, \mathbb{C})$. Then, $CC(\mathcal{L}_Q)$ is irreducible.

- Bressler-Finkelberg-Lunts '90: conjecture is true for $Gr(k, n)$.

These $m_{Q',Q}$'s are hard to compute.

Conjecture by Kazhdan and Lusztig '80: Let Q be a Schubert cell in the flag variety of $G = GL(n, \mathbb{C})$. Then, $CC(\mathcal{L}_Q)$ is irreducible.

- Bressler-Finkelberg-Lunts '90: conjecture is true for $Gr(k, n)$.
- Kashiwara and Saito '97: counterexample for $G = GL(8, \mathbb{C})$.

These $m_{Q',Q}$'s are hard to compute.

Conjecture by Kazhdan and Lusztig '80: Let Q be a Schubert cell in the flag variety of $G = GL(n, \mathbb{C})$. Then, $CC(\mathcal{L}_Q)$ is irreducible.

- Bressler-Finkelberg-Lunts '90: conjecture is true for $Gr(k, n)$.
- Kashiwara and Saito '97: counterexample for $G = GL(8, \mathbb{C})$.
- Williamson '15: counterexample for $G = GL(12, \mathbb{C})$.

These $m_{Q',Q}$'s are hard to compute.

Conjecture by Kazhdan and Lusztig '80: Let Q be a Schubert cell in the flag variety of $G = GL(n, \mathbb{C})$. Then, $CC(\mathcal{L}_Q)$ is irreducible.

- Bressler-Finkelberg-Lunts '90: conjecture is true for $Gr(k, n)$.
- Kashiwara and Saito '97: counterexample for $G = GL(8, \mathbb{C})$.
- Williamson '15: counterexample for $G = GL(12, \mathbb{C})$.

Barcini, Somberg and Trapa '19: an example of a $K = GL(6) \times GL(6)$-orbit on a flag variety of $G = GL(12, \mathbb{C})$ with a reducible characteristic cycle.

Let θ be an involution on $GL(n, \mathbb{C})$. Take K to be be the identity component of $GL(n)^\theta$. There are three cases:

- $K = GL(p) \times GL(q)$, $p + q = n$
- $K = Sp(n)$, n is even
- $K = SO(n)$

Each K acts on $Gr(k, n)$ with a finite number of orbits.

Question: What are the characteristic cycles in this case? Are they irreducible?

Case of $K = GL(p) \times GL(q)$. Here $p + q = n$, $p \geq q$, $n - k \geq k$. Write $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^q$. The K - orbits on $Gr(k, n)$ are

$$Q(s,t) = \{U \in Gr(k,n) \mid \dim U \cap \mathbb{C}^p = s, \dim U \cap \mathbb{C}^q = t\},$$

where $s + t \leq k$, $s \leq p$ and $t \leq q$.

Note that $Q(s', t') \subset \overline{Q(s,t)}$ if and only if $s' \geq s$ and $t' \geq t$.

Case of $K = GL(p) \times GL(q)$. Here $p + q = n$, $p \geq q$, $n - k \geq k$. Write $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^q$. The K - orbits on $Gr(k, n)$ are

$$Q(s,t) = \{U \in Gr(k,n) \mid \dim U \cap \mathbb{C}^p = s, \dim U \cap \mathbb{C}^q = t\},$$

where $s + t \leq k$, $s \leq p$ and $t \leq q$.

Note that $Q(s', t') \subset \overline{Q(s,t)}$ if and only if $s' \geq s$ and $t' \geq t$.

$GL(2) \times GL(2)$
acting on
$Gr(2,4)$

$$\begin{tikzcd}
 & & Q(2,0) \\
 & Q(1,0) \arrow[ur, dash] & \\
Q(0,0) \arrow[ur, dash] \arrow[dr, dash] & & Q(1,1) \arrow[ul, dash] \\
 & Q(0,1) \arrow[ur, dash] \arrow[dr, dash] & \\
 & & Q(0,2)
\end{tikzcd}$$

Barbasch and Evens:

$$Z(s,t) = \{(U,V,W) \in Gr(k,n) \times Gr(s,p) \times Gr(t,q) | V \subset \mathbb{C}^p \cap U, W \subset \mathbb{C}^q \cap U\}.$$

Theorem (BE)
The map $\theta : Z(s,t) \to \overline{Q(s,t)}, (U,V,W) \mapsto U$ is a resolution of singularities. For $n-k \geq p$ the resolution $\theta : Z(s,t) \to \overline{Q(s,t)}$ is small.

Barbasch and Evens:

$$Z(s,t) = \{(U,V,W) \in Gr(k,n) \times Gr(s,p) \times Gr(t,q) | V \subset \mathbb{C}^p \cap U, W \subset \mathbb{C}^q \cap U\}.$$

Theorem (BE)

The map $\theta : Z(s,t) \to \overline{Q(s,t)}, (U,V,W) \mapsto U$ is a resolution of singularities. For $n-k \geq p$ the resolution $\theta : Z(s,t) \to \overline{Q(s,t)}$ is small. For $n-k \leq p$, there is a similar small resolution $\tilde{\theta} : \tilde{Z}(s,t) \to \overline{Q(s,t)}$.

Theorem
Let $Q(s,t)$ be a $GL(p) \times GL(q)$-orbit in $Gr(k,n)$. Then, the characteristic cycle $CC(\mathcal{L}_{Q(s,t)})$ is irreducible.

Theorem
Let $Q(s,t)$ be a $GL(p) \times GL(q)$-orbit in $Gr(k,n)$. Then, the characteristic cycle $CC(\mathcal{L}_{Q(s,t)})$ is irreducible.

Proof.
Consider $\theta : Z(s,t) \to \overline{Q(s,t)}$.

- θ is a proper and small map: $\theta_* \mathcal{O}_Z = \mathcal{L}_{Q(s,t)}$
- Write $CC(\mathcal{L}_Q) = \overline{T_Q^* X} + \sum_{Q \neq Q' \subset \overline{Q}} m_{Q',Q} \overline{T_{Q'}^* X}$.

  For a K-orbit $Q' \subset \overline{Q(s,t)}$ and a general (x,ξ) in $T_{Q'}^* Gr(k,n)$ the microlocal fiber of θ over (x,ξ) is empty $\implies m_{Q',Q} = 0$.

$\square$

$Sp(n,\mathbb{C})$ or $SO(n,\mathbb{C})$ acts on $Gr(k,n)$. Let B be a nondegenerate symmetric/skew-symmetric bilinear form on $\mathbb{C}^n$. Let K denote the isometry group of B. The K-orbits on $Gr(k,n)$ are

$$Q(i) = \{U \in Gr(k,n) | \dim(rad(U)) = i\}. \tag{2}$$

$Q(i) \subset \overline{Q(j)}$ if and only if $i \geq j$.

$Sp(n,\mathbb{C})$ or $SO(n,\mathbb{C})$ acts on $Gr(k,n)$. Let B be a nondegenerate symmetric/skew-symmetric bilinear form on $\mathbb{C}^n$. Let K denote the isometry group of B. The K-orbits on $Gr(k,n)$ are

$$Q(i) = \{U \in Gr(k,n) | \dim(rad(U)) = i\}. \tag{2}$$

$Q(i) \subset \overline{Q(j)}$ if and only if $i \geq j$. There is a resolution of singularities

$$Z_i = \{(U,V) \in Gr(k,n) \times Gr(i,n) | V \subset rad(B|_U)\} \tag{3}$$

with $\theta : Z_i \to \overline{Q(i)}$ projection to $Gr(k,n)$. However, θ is not small.

Theorem

1. Consider the $Sp(n)$-orbits $Q(i)$ on $Gr(k,n)$. Then, the characteristic cycle of $\mathcal{L}_{Q(i)}$ is irreducible.
2. Consider the $SO(n)$-orbits $Q(0), Q(1), \dots, Q(k)$ on $Gr(k,n)$
   - i even or $i = k$: the characteristic cycle of $\mathcal{L}_{Q(i)}$ is irreducible
   - i odd:
     $$CC(\mathcal{L}_{Q(i)}) = \overline{T^*_{Q(i)}Gr(k,n)} + \overline{T^*_{Q(i+1)}Gr(k,n)},$$
     (when n is even and $k = n/2$, the set $Q(k)$ is a union of two closed $SO(n)$-orbits. Hence the characteristic cycle of $\mathcal{L}_{Q(k-1)}$ has three irreducible components when k is even)

Proof. Let $\mathcal{S}$ be the tautological bundle over a Grassmannian $Gr(k,n)$. Then $\overline{Q(i)} = s^{-1}(\text{rank } \leq (k-i)$ matrices) is a degeneracy locus.

$$
\begin{tikzcd}
Hom(\mathcal{S}, \mathcal{S}^*) \arrow[d] \\
Gr(k,n) \arrow[u, bend right, "s"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Hom(U, U^*) \arrow[d] \\
U \arrow[u, bend right, "s(U)=(B|_U : U \to U^*)"']
\end{tikzcd}
$$

Proof. Let $\mathcal{S}$ be the tautological bundle over a Grassmannian $Gr(k,n)$. Then $\overline{Q(i)} = s^{-1}(\text{rank } \leq (k-i) \text{ matrices})$ is a degeneracy locus.

$$
\begin{tikzcd}
Hom(\mathcal{S},\mathcal{S}^*) \arrow[d] & Hom(U,U^*) \arrow[d] \\
Gr(k,n) \arrow[u, bend right, "s"'] & U \arrow[u, bend right, "s(U)=(B|_U : U\to U^*)"']
\end{tikzcd}
$$

Theorem (Raicu)

- Let O_i denote the set of n-by-n skew-symmetric matrices of rank i. Then each $CC(\mathcal{L}_{O_i})$ is irreducible, so that $CC(\mathcal{L}_{O_i}) = [\overline{T^*_{O_i}X}]$.
- Let O_i denote the set of n-by-n symmetric matrices of rank i. Then $CC(\mathcal{L}_{O_i}) = [\overline{T^*_{O_i}X}]$ for $n-i$ even or $i=0$, and $CC(\mathcal{L}_{O_i}) = [\overline{T^*_{O_i}X}] + [\overline{T^*_{O_{i-1}}X}]$ for $n-i$ odd.

Joint with Scott Larson.

Let $G = Sp(2n, \mathbb{C})$ and $K = GL(n, \mathbb{C})$. Let $X = Gr^0(k, 2n)$ be the space of k-dimensional isotropic subspaces of $\mathbb{C}^{2n}$. Let w be a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^{2n}$. Fix $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^{-n}$. The K-orbits are as follows:

$$\begin{aligned} Q(a,b,c) = \{U \in Gr^0(k,2n) \mid \dim U \cap \mathbb{C}^p = a, \dim U \cap \mathbb{C}^q = b,\\ \dim(rad\epsilon|_U) = a+b+c\}, \end{aligned}$$

where $\epsilon(x,y) = w(pr_{\mathbb{C}^n}(x), pr_{\mathbb{C}^{-n}})$.

There is a resolution of singularities $Z(a,b,c) \to \overline{Q(a,b,c)}$.