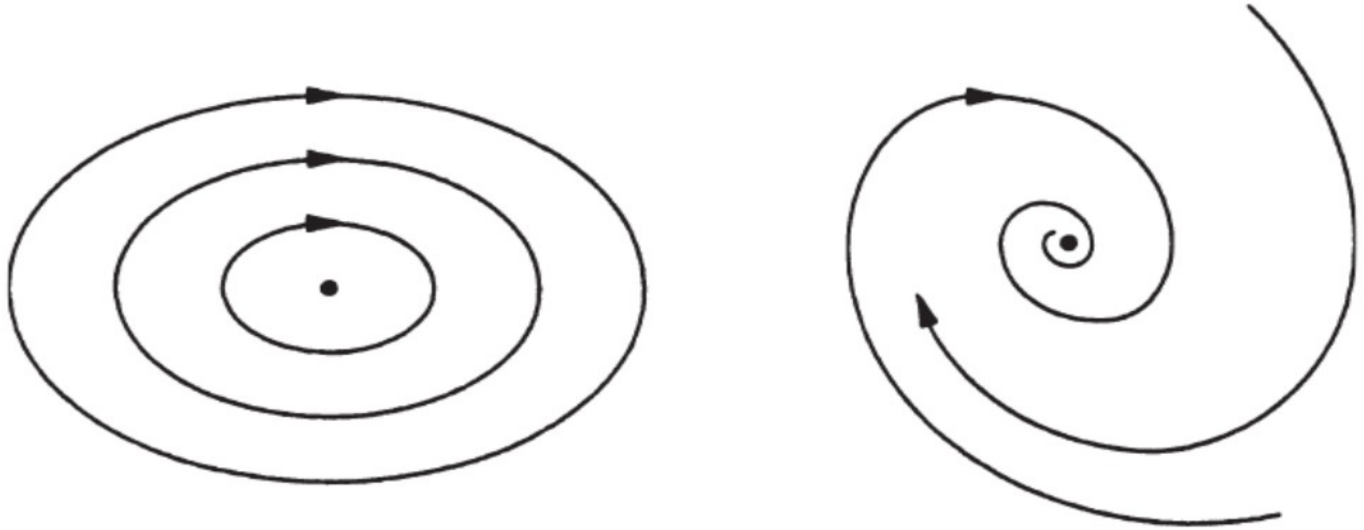
Two dimensional flow

For complex eigen values the phase space diagram would be either circular or spiral.



(a) center (b) spiral

Two dimensional flow

$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right).$$

Thus complex eigenvalues occur when

$$\tau^2 - 4\Delta < 0\ .$$

$$\lambda_{1,2} = \alpha \pm i\omega$$

where

$$\alpha = \tau/2, \quad \omega = \tfrac{1}{2}\sqrt{4\Delta - \tau^2}.$$

Two dimensional flow

exponentially *decaying oscillations* if $\alpha = \mathrm{Re}(\lambda) < 0$ and *growing oscillations* if $\alpha > 0$.

Unstable Spirals

Stable Spirals

If the eigenvalues are pure imaginary ($\alpha = 0$), then all the solutions are periodic with period $T = 2\pi/\omega$. The oscillations have fixed amplitude and the fixed point is a center.

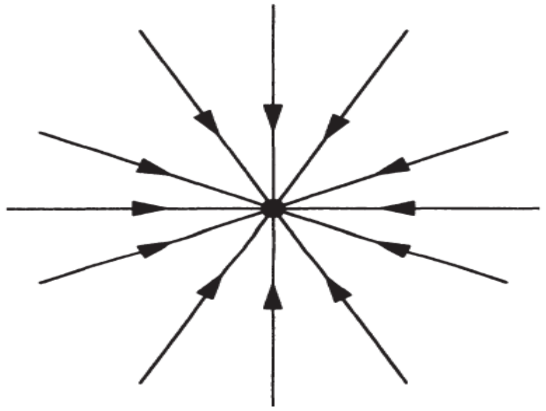
Two dimensional flow

Consider the situation

$$\lambda_1 = \lambda_2 = \lambda$$

There are two possibilities: either there are two independent eigenvectors corresponding to the eigenvalues or there is only one.

Then if $\lambda \neq 0$, all trajectories are straight lines through the origin ($\mathbf{x}(t) = e^{\lambda t}\mathbf{x}_0$)



The fixed point is called the **star node.**

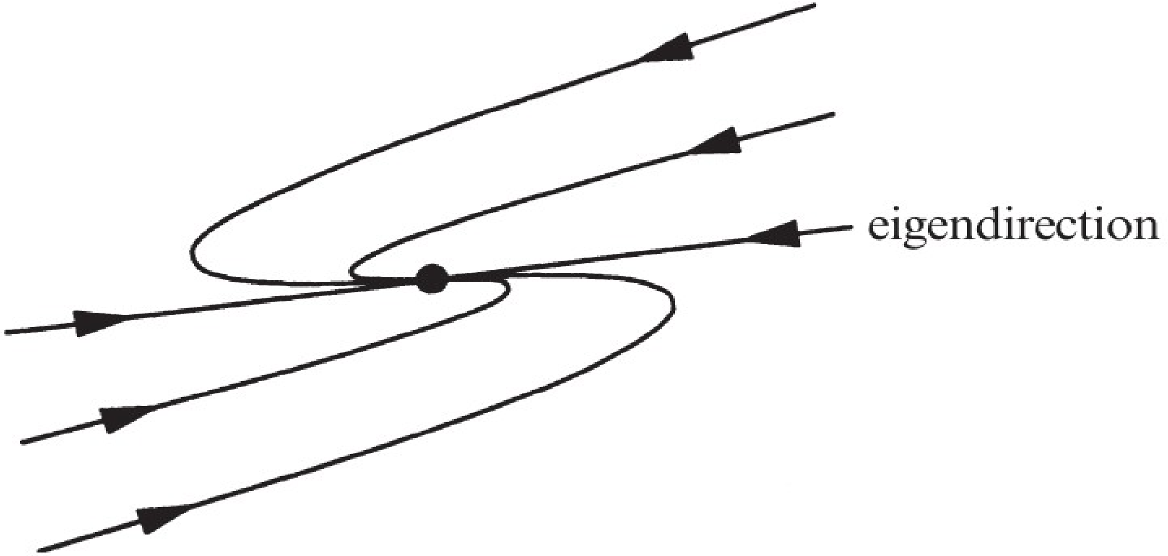
Two dimensional flow

Consider the situation

$$\lambda_1 = \lambda_2 = \lambda$$

The other possibilities will be that there is only one eigenvector.

When there's only one eigendirection, the fixed point is a ***degenerate node.***



Two dimensional flow

Classification of Fixed Points

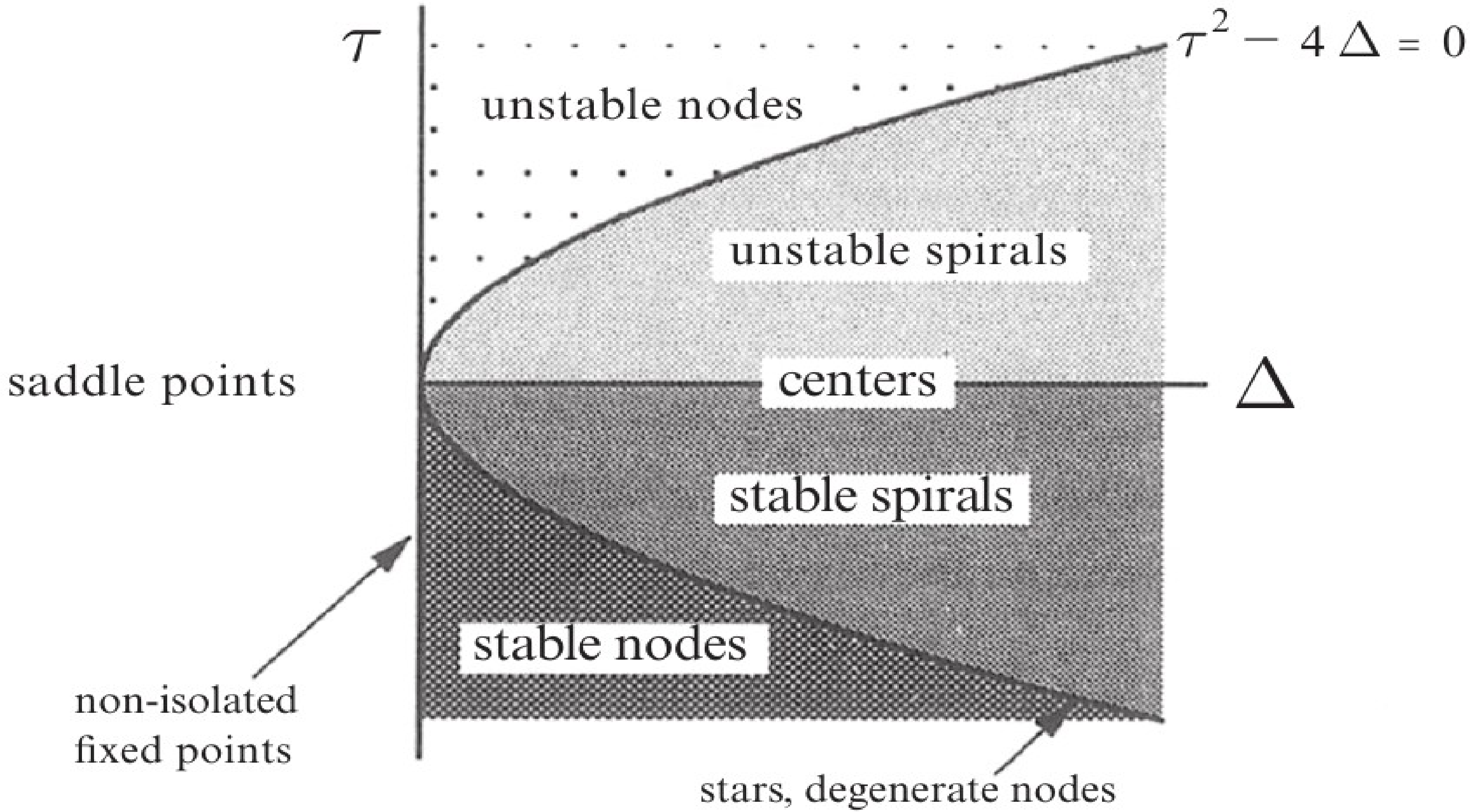
$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1\lambda_2, \qquad \tau = \lambda_1 + \lambda_2 .$$

If $\Delta < 0$, the eigenvalues are real and have opposite signs; hence the fixed point is a *saddle point*.

If $\Delta > 0$, the eigenvalues are either real with the same sign (*nodes*), or complex conjugate (*spirals* and *centers*). Nodes satisfy $\tau^2 - 4\Delta > 0$ and spirals satisfy $\tau^2 - 4\Delta < 0$. The parabola $\tau^2 - 4\Delta = 0$ is the borderline between nodes and spirals; star nodes and degenerate nodes live on this parabola. The stability of the nodes and spirals is determined by τ. When $\tau < 0$, both eigenvalues have negative real parts, so the fixed point is stable. Unstable spirals and nodes have $\tau > 0$. Neutrally stable centers live on the borderline $\tau = 0$, where the eigenvalues are purely imaginary.

Two dimensional flow

Classification of Fixed Points



Two dimensional flow

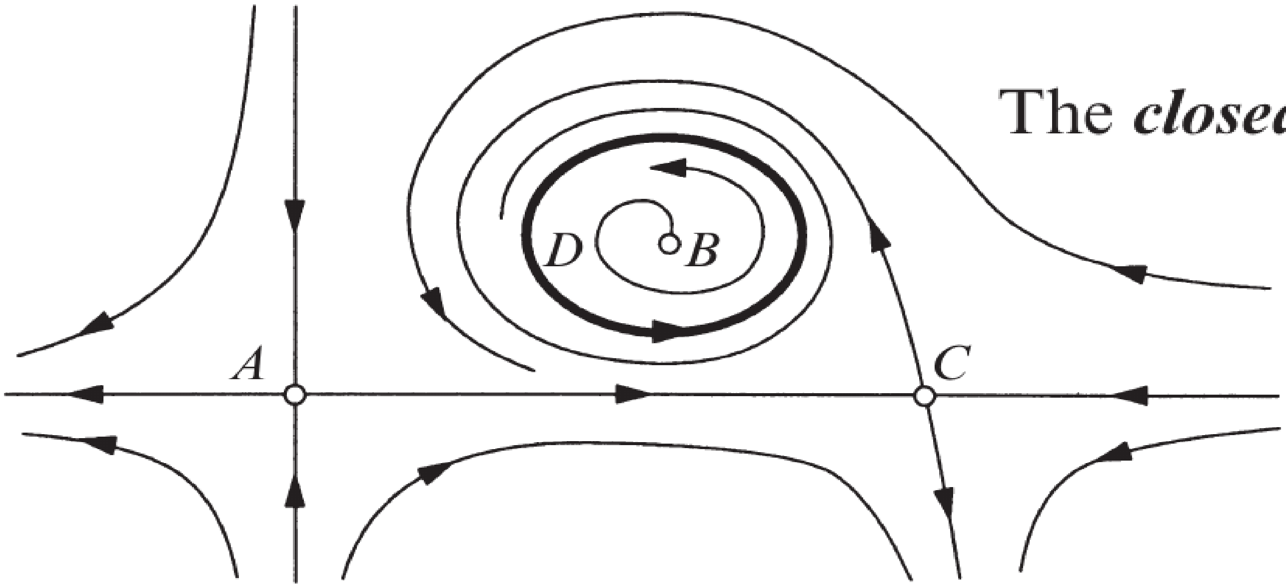
Phase flow and fixed points in two dimensional non-linear system

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

The ***fixed points***, like *A*, *B*, and *C*

The ***closed orbits***, like *D*



Two dimensional flow

Phase flow and fixed points in two dimensional non-linear system

Take an example:

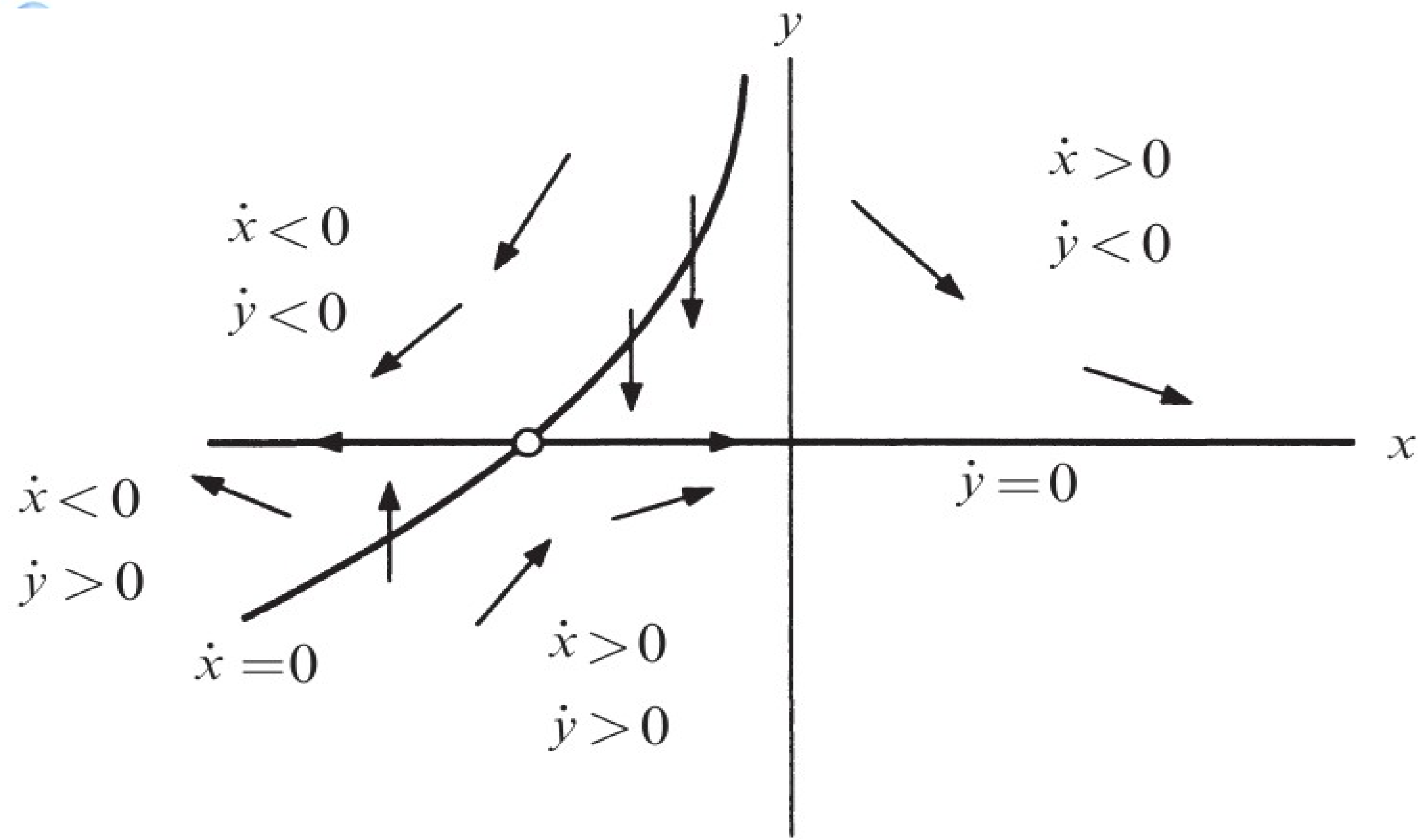
$$\dot{x} = x + e^{-y}, \quad \dot{y} = -y.$$

Fixed points are $(x^*, y^*) = (-1, 0)$

To sketch the phase portrait, it is helpful to plot the ***nullclines***, defined as the curves where either $\dot{x} = 0$ or $\dot{y} = 0$.

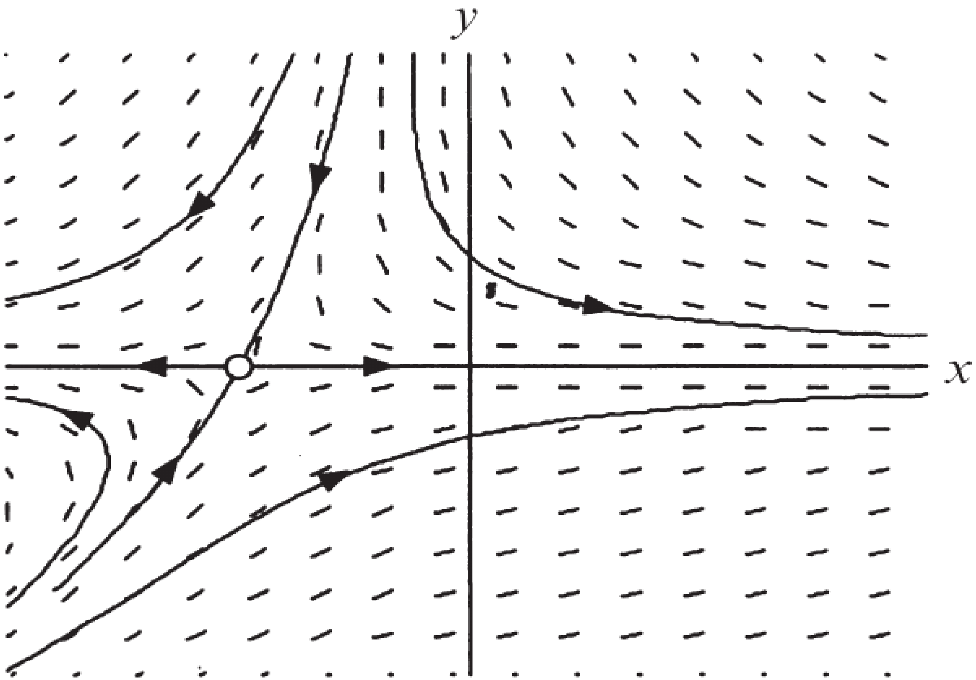
Two dimensional flow

Phase flow and fixed points in two dimensional non-linear system



Two dimensional flow

Phase flow and fixed points in two dimensional non-linear system



Detailed flow using the numerical integration of the dynamical equation using the RK-4 method.

Two dimensional flow

Plot the phase space portrait and classify the fixed points of the following linear systems.

$$\dot{x} = y,\ \dot{y} = -2x - 3y$$

$$\dot{x} = 3x - 4y,\ \dot{y} = x - y$$

$$\dot{x} = 5x + 2y,\ \dot{y} = -17x - 5y$$

$$\dot{x} = 4x - 3y,\ \dot{y} = 8x - 6y$$

$$\dot{x} = 5x + 10y,\ \dot{y} = -x - y$$

$$\dot{x} = -3x + 2y,\ \dot{y} = x - 2y$$

$$\dot{x} = -3x + 4y,\ \dot{y} = -2x + 3y$$

$$\dot{x} = y,\ \dot{y} = -x - 2y.$$

Two dimensional flow

Analysis of fixed points of the nonlinear system:

Consider the system

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

and suppose that (x^*, y^*) is a fixed point, i.e.,

$$f(x^*, y^*) = 0, \qquad g(x^*, y^*) = 0.$$

Let

$$u = x - x^*, \qquad v = y - y^*$$

$$
\begin{aligned}
\dot{u} &= \dot{x} && \text{(since } x^* \text{ is a constant)} \\
&= f(x^* + u, y^* + v) && \text{(by substitution)} \\
&= f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) && \text{(Taylor series expansion)} \\
&= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) && \text{(since } f(x^*, y^*) = 0\text{).}
\end{aligned}
$$

Two dimensional flow

Analysis of fixed points of the nonlinear system:

Similarly we find

$$\dot{v} = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + O(u^2, v^2, uv).$$

Hence the disturbance (u, v) evolves according to

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \text{quadratic terms.}$$

The matrix

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}$$

is called the ***Jacobian matrix*** at the fixed point (x^*, y^*).

Two dimensional flow

Analyis of fixed points of the nonlinear system:

The nature of the system near the fixed point boils to the analysis of the linearized system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Example:

$$\dot{x} = -x + x^3 \, , \;\; \dot{y} = -2y$$

Procedure here is to first find the nature of the fixed point using the lienarized method, i.e. by analyzing the eigen value and eigenvectors of the Jacobian Matrix at the fixed point. Thereafter, we can have the phase portrait for the fully nonlinear system and can have the comparison.

Two dimensional flow

Example:

$$\dot{x} = -x + x^3 , \ \dot{y} = -2y$$

Fixed points are: **(0,0), (1,0) and (-1,0)**

The jacobian of the matrix is given by

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{pmatrix}.$$

At $(0,0)$, we find $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$

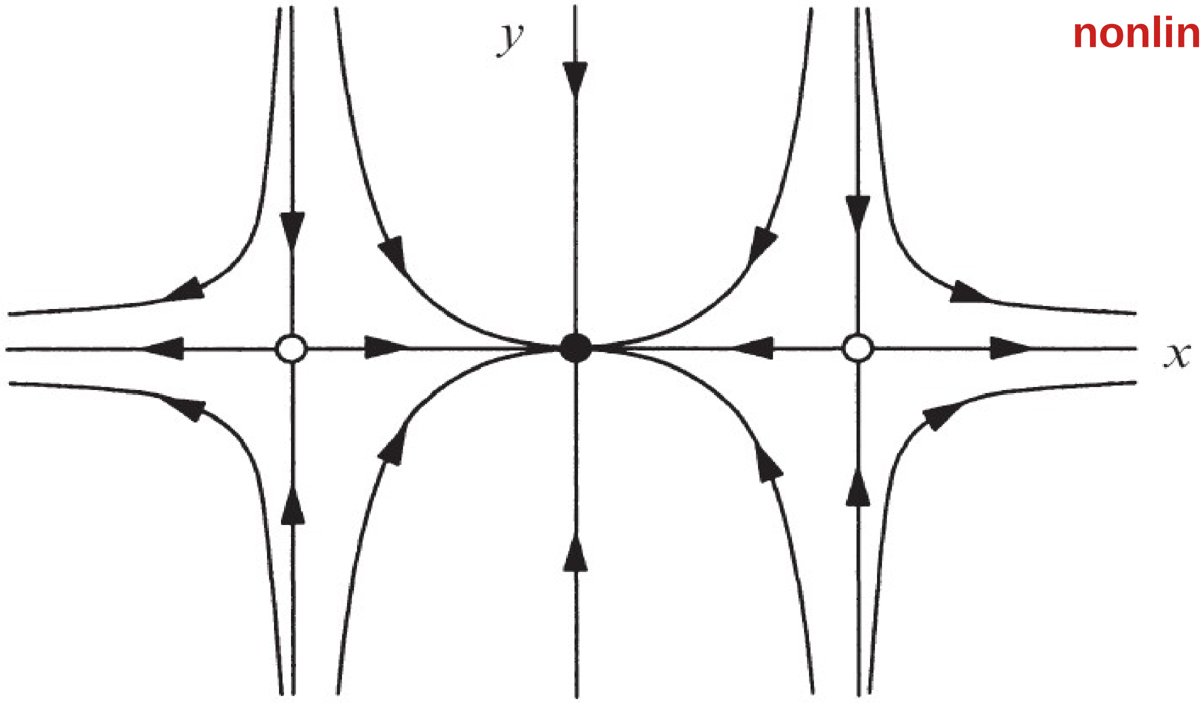
So it is a stable node as both the eigenvalues are negative.

Two dimensional flow

At $(\pm 1, 0)$, $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$,

So, both (1,0) and (-1,0) are saddle points.

Phase Portrait using nonlinear analysis.



Two dimensional flow

Example:

$$\dot{x} = -y + ax(x^2 + y^2)$$
$$\dot{y} = x + ay(x^2 + y^2)$$

where a is a parameter. Show that the linearized system *incorrectly* predicts that the origin is a center for all values of a, whereas in fact the origin is a stable spiral if $a < 0$ and an unstable spiral if $a > 0$.

Linear Analysis:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \dot{x} = -y\,,\ \ \dot{y} = x$$

which has $\tau = 0$, $\Delta = 1 > 0$, so the origin is always a center, according to the linearization.

Two dimensional flow

To analyze the nonlinear system, we change variables to ***polar coordinates***. Let $x = r\cos\theta$, $y = r\sin\theta$. To derive a differential equation for r, we note $x^2 + y^2 = r^2$, so $x\dot{x} + y\dot{y} = r\dot{r}$. Substituting for $\dot{x}$ and $\dot{y}$ yields

$$
\begin{aligned}
r\dot{r} &= x\left(-y + ax(x^2+y^2)\right) + y\left(x + ay(x^2+y^2)\right) \\
&= a(x^2+y^2)^2 \\
&= ar^4.
\end{aligned}
$$

Hence $\dot{r} = ar^3$.

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}.$$

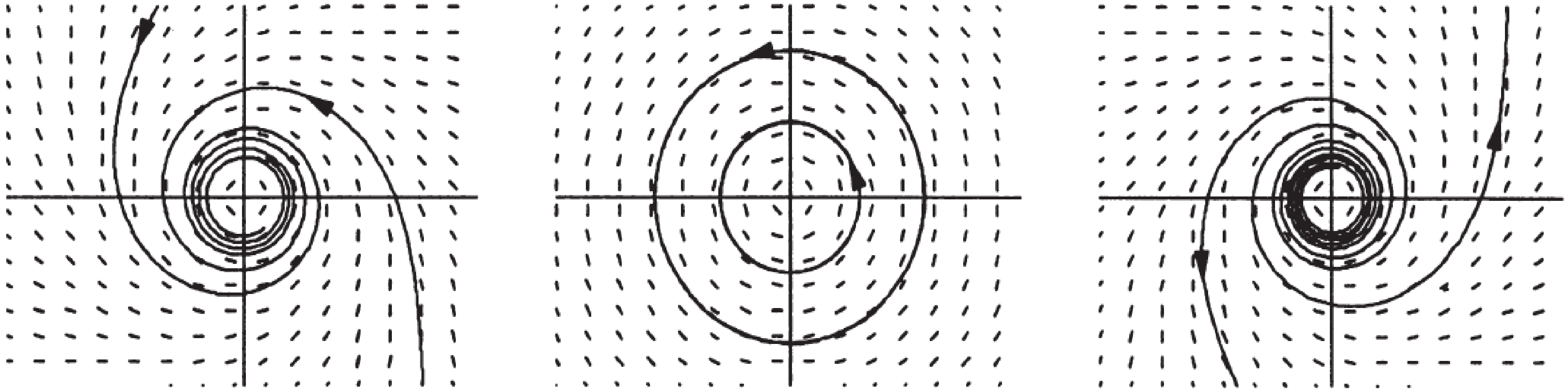
After substituting for $\dot{x}$ and $\dot{y}$ we find $\dot{\theta} = 1$

Two dimensional flow

In polar coordinate the system becomes:

$$\dot{r} = ar^3$$
$$\dot{\theta} = 1.$$

The system is easy to analyze in this form, because the radial and angular motions are independent. All trajectories rotate about the origin with constant ang



$a < 0$ $a = 0$ $a > 0$

Two dimensional flow

Robust cases:

Repellers (also called *sources*): both eigenvalues have positive real part.
Attractors (also called *sinks*): both eigenvalues have negative real part.
Saddles: one eigenvalue is positive and one is negative.

Marginal cases:

Centers: both eigenvalues are pure imaginary.
Higher-order and non-isolated fixed points: at least one eigenvalue is zero.

Thus, from the point of view of stability, the marginal cases are those where at least one eigenvalue satisfies $\mathrm{Re}(\lambda) = 0$.

Two dimensional flow

Example:

Rabbits versus Sheep

(Lotka-Volterra model of competition between the two species)

$$\dot{x} = x(3 - x - 2y)$$

$$\dot{y} = y(2 - x - y)$$

where

$$x(t) = \text{population of rabbits},$$

$$y(t) = \text{population of sheep}$$

Two dimensional flow

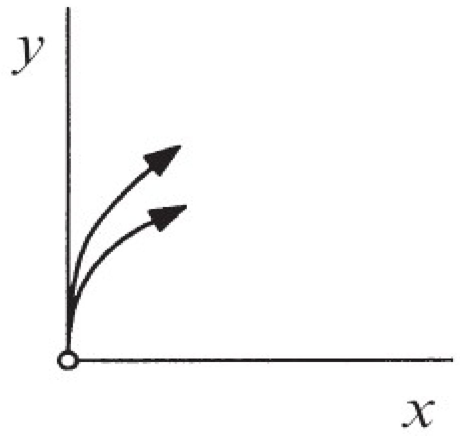
To find the fixed points for the system, we solve $\dot{x}=0$ and $\dot{y}=0$ simultaneously. Four fixed points are obtained: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. To classify them, we compute the Jacobian:

$$A=\begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix}=\begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}.$$

$$(0,0): \text{Then } A=\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues are $\lambda=3,\ 2$ so $(0,0)$ is an *unstable node*.

Two dimensional flow



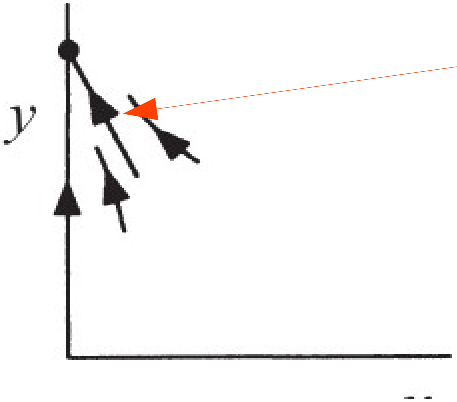
General observation that the trajectory is tangential to the slowest direction which is (0,1) in this case.

For

$$(0,2): \text{Then } A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}.$$

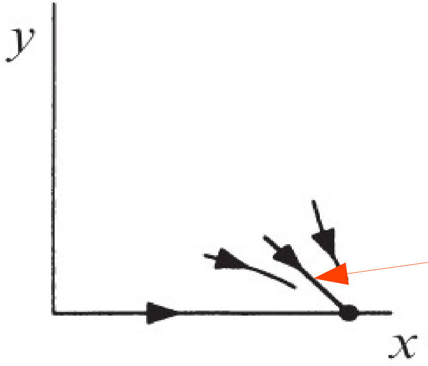
This matrix has eigenvalues $\lambda = -1, -2$

Two dimensional flow



V=(1,-2) with eigen value -1.

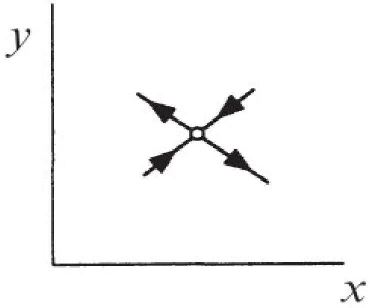
$$(3,0)\text{: Then } A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \text{ and } \lambda = -3, -1.$$



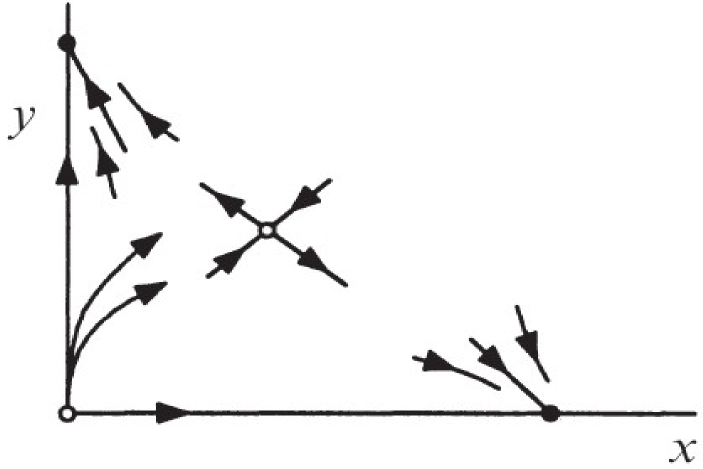
Trajectory is tangent to the slow direction) V=(3,-1) and eigen value -1.

Two dimensional flow

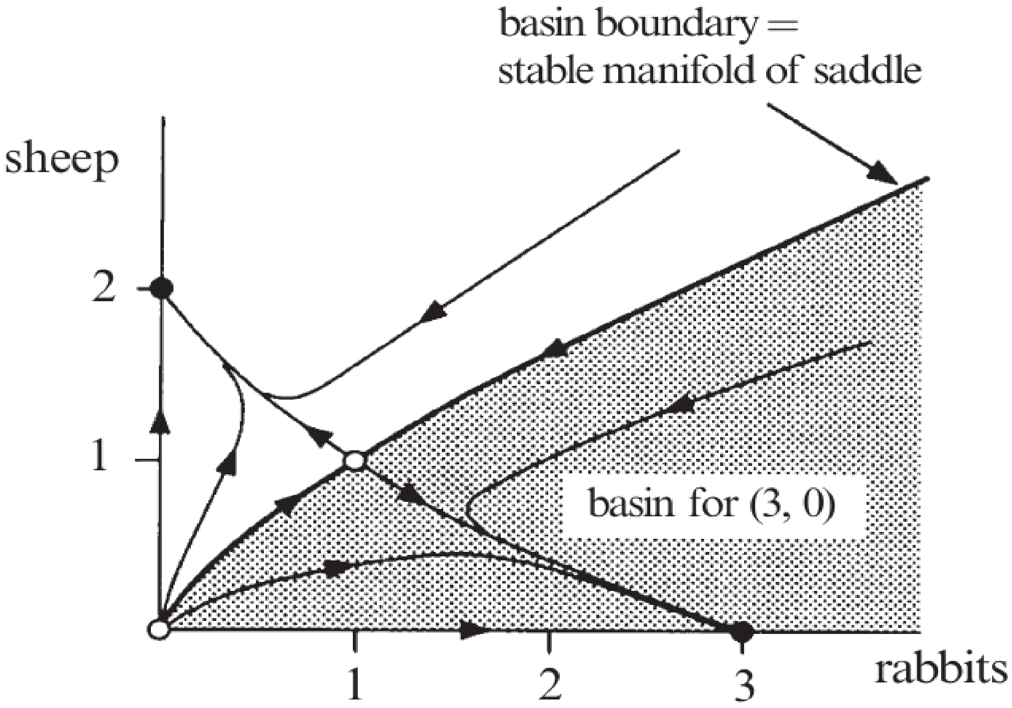
(1, 1): Then $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, which has $\tau = -2$, $\Delta = -1$, and $\lambda = -1 \pm \sqrt{2}$.



Cobining all the picture together we have



Two dimensional flow



Note that the trajectory starting below the stable manifold leads to the extinction of the sheep, while the trajectory which start from above leads to the extinction of the rabbits. All the points below the stable manifold if chosen as an initial condition finally ends up at the fixed point (3,0). All these points are called the basin of attraction of the fixed point (3,0).

Two dimensional flow

Assignment 3:

For each of the following systems, find the fixed points. Then sketch the null-clines, the vector field, and a plausible phase portrait.

$$\dot{x} = x - y, \quad \dot{y} = 1 - e^{x}$$

$$\dot{x} = x - x^3, \quad \dot{y} = -y$$

$$\dot{x} = x(x - y), \quad \dot{y} = y(2x - y)$$

$$\dot{x} = y, \quad \dot{y} = x(1 + y) - 1$$

$$\dot{x} = x(2 - x - y), \quad \dot{y} = x - y$$

$$\dot{x} = x^2 - y, \quad \dot{y} = x - y$$

Using numerical simulation plot the phase portrait of

(van der Pol oscillator) $\dot{x} = y, \quad \dot{y} = -x + y(1 - x^2)$

Two dimensional flow

Conservative Systems

$$m\ddot{x} = F(x).$$

$V(x)$ denote the ***potential energy***, defined by $F(x) = -dV/dx$.

$$m\ddot{x} + \frac{dV}{dx} = 0.$$

$$m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = 0 \Rightarrow \frac{d}{dt}\left[\tfrac{1}{2}m\dot{x}^2 + V(x)\right] = 0$$

Chain rule

$$\frac{d}{dt}V(x(t)) = \frac{dV}{dx}\frac{dx}{dt}$$

Two dimensional flow

Conserved energy is

$$E = \frac{1}{2} m\dot{x}^2 + V(x)$$

Show that *a conservative system cannot have any attracting fixed points.*

If x* is the attracting fixed point, all the point on the fixed point will have the same energy E(x*). Therefore, it is required that the E(x) must be constant for x in the basin. But this will contradict our definition of a conservative system, in wich we demand that E(x) must be nonconstant on all open set.

If this the situation then what kind of fixed point we should expect for the conservative system?

EXAMPLE

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 .$$

Two dimensional flow

The force is $-dV/dx = x - x^3$, so the equation of motion is

$$\ddot{x} = x - x^3 .$$ m=1

This can be rewritten as the vector field

$$\dot{x} = y$$

$$\dot{y} = x - x^3$$

Fixed points are

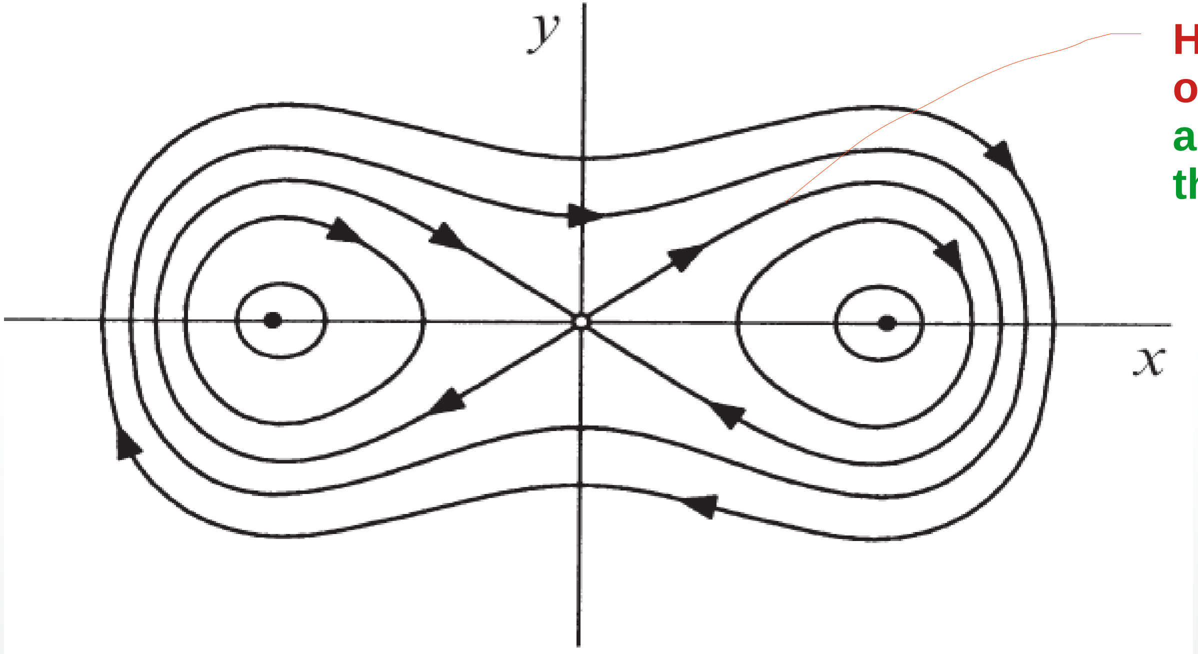
$$(x^*,\ y^*) = (0,0) \text{ and } (\pm 1, 0).$$

Two dimensional flow

Jacobian matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}.$$

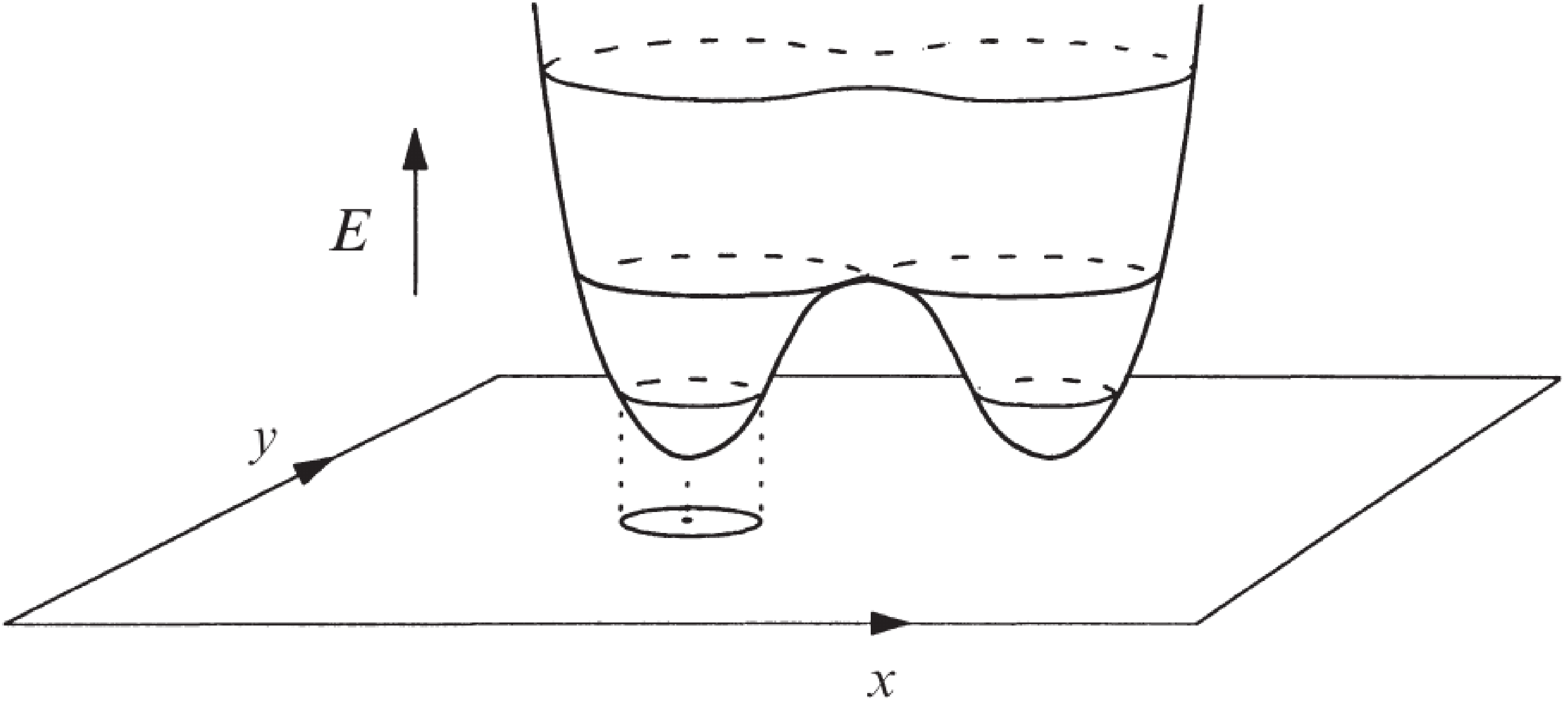
At (0,0), we have $\Delta = -1$, so the origin is a saddle point. But when $(x^*, y^*) = (\pm 1, 0)$, we find $\tau = 0$, $\Delta = 2$; hence these equilibria are predicted to be centers.



Homoclinic orbits: starts and ends at the origin

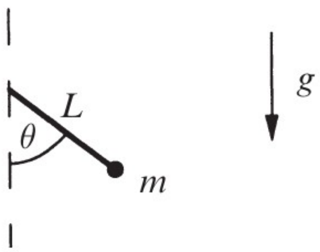
Two dimensional flow

Energy Portrait corresponding to the particle motion in the double well potential.



Two dimensional flow

Simple Pendulum:



$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

$\omega = \sqrt{g/L}$ and a dimensionless time $\tau = \omega t$.

$$\ddot{\theta} + \sin\theta = 0$$

$$\dot{\theta} = v$$

$$\dot{v} = -\sin\theta$$

The fixed points are $(\theta^*, v^*) = (k\pi, 0)$, where k is any integer. There's no physical difference between angles that differ by 2π, so we'll concentrate on the two fixed points $(0,0)$ and $(\pi, 0)$. At $(0,0)$, the Jacobian is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Two dimensional flow

System is conservative. The energy constant is

$$\dot{\theta}(\ddot{\theta} + \sin\theta) = 0 \Rightarrow \tfrac{1}{2}\dot{\theta}^2 - \cos\theta = \text{constant}.$$

The energy function

$$E(\theta, v) = \tfrac{1}{2}v^2 - \cos\theta$$

has a local minimum at (0,0), since $E \approx \frac{1}{2}(v^2 + \theta^2) - 1$ for small (θ, v).

It is also quite evident that the trajectory near the orgin is circular is nature.

$$\theta^2 + v^2 \approx 2(E + 1).$$

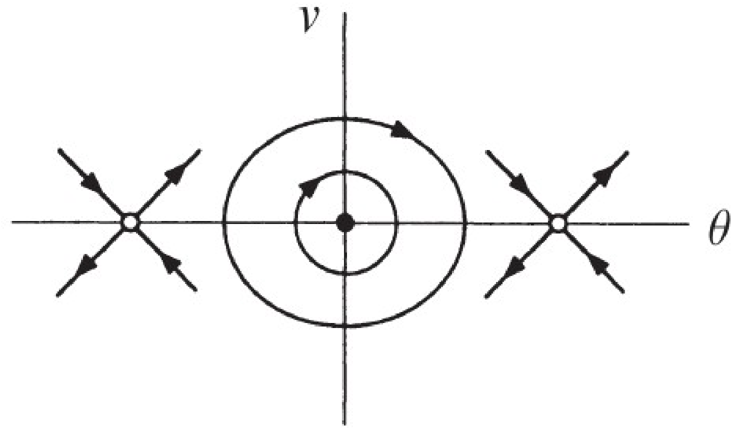
Two dimensional flow

The nature of trajector at $(\pi, 0)$.

Jacobian is

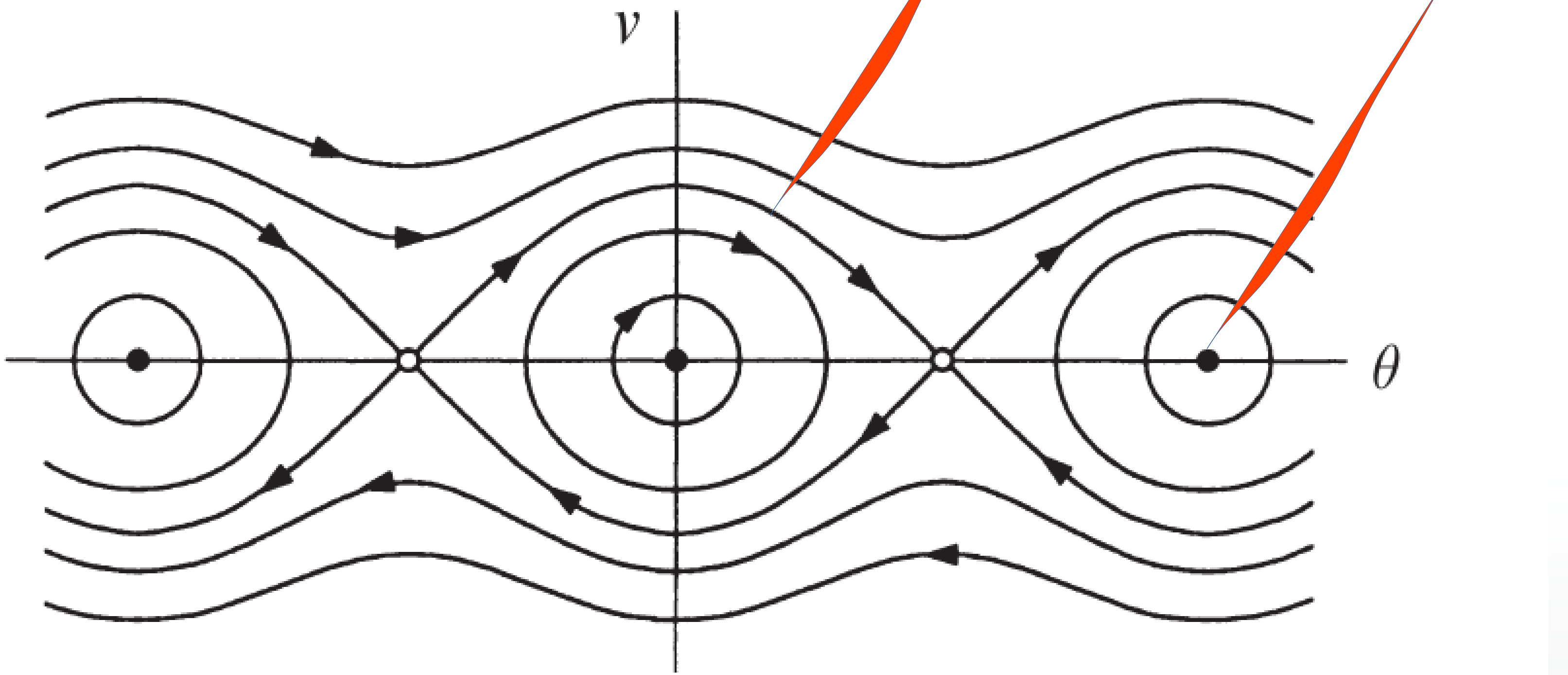
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 - 1 = 0$. Therefore $\lambda_1 = -1$, $\lambda_2 = 1$; the fixed point is a saddle. The corresponding eigenvectors are $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (1, 1)$.



Two dimensional flow

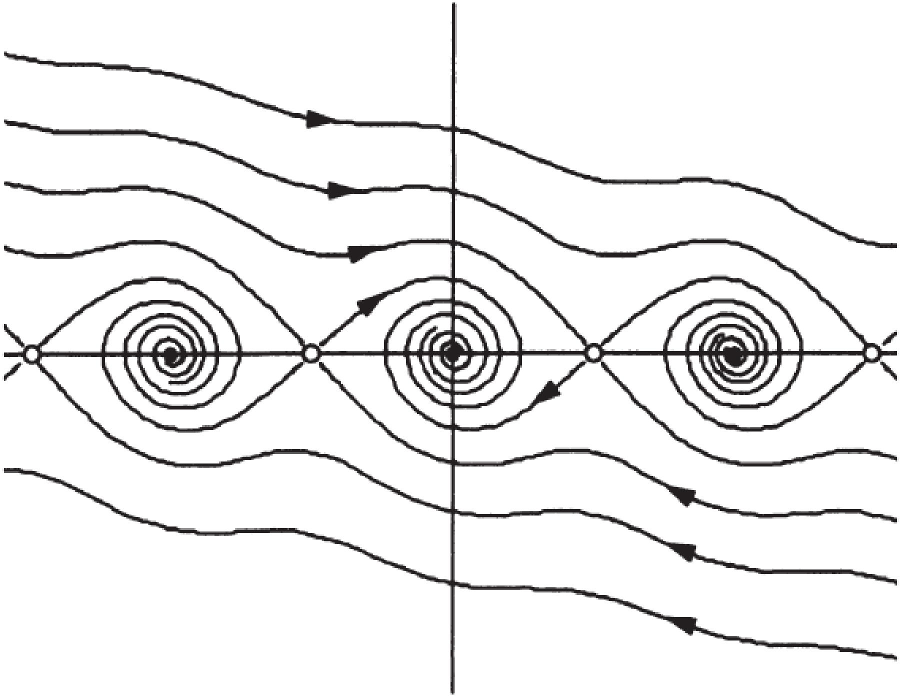
$$E = \frac{1}{2}v^2 - \cos\theta$$



Two dimensional flow

Damped Oscillator:

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$$



Two dimensional flow

Poincaré-Bendixson Theorem

Although $N = 2$ systems are much richer than $N = 1$ systems, they are still ultimately rather impoverished in terms of their long-time behavior. If an orbit does not flow off to infinity or asymptotically approach a stable fixed point (node or spiral or nongeneric example), the only remaining possibility is limit cycle behavior. This is the content of the *Poincaré-Bendixson theorem*, which states:

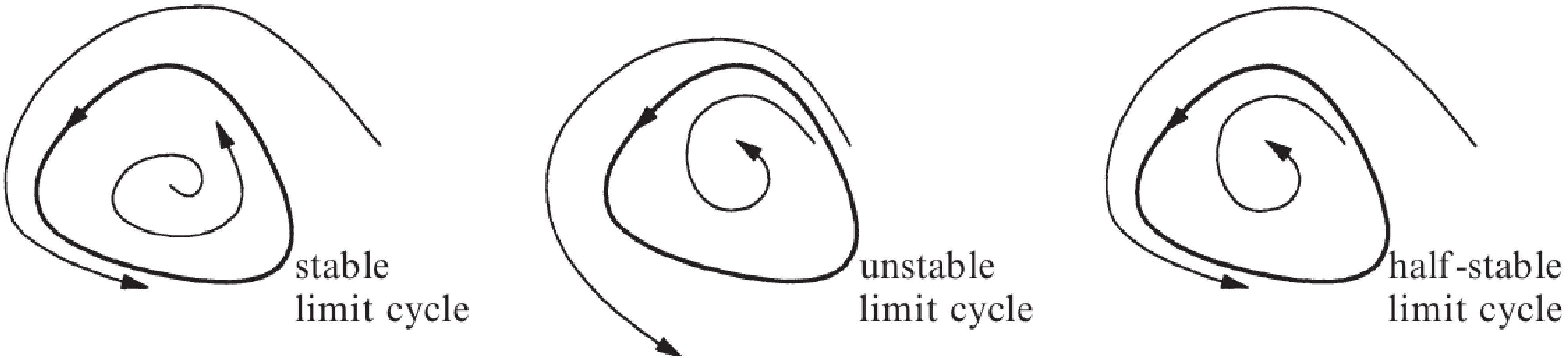
- *IF* Ω is a compact (*i.e.* closed and bounded) subset of phase space,
- *AND* $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ is continuously differentiable on Ω,
- *AND* Ω contains no fixed points (*i.e.* $\boldsymbol{V}(\boldsymbol{\varphi})$ never vanishes in Ω),
- *AND* a phase curve $\boldsymbol{\varphi}(t)$ is always confined to Ω,

◇ *THEN* $\boldsymbol{\varphi}(t)$ is either closed or approaches a closed trajectory in the limit $t \to \infty$.

Thus, under the conditions of the theorem, Ω must contain a closed orbit.

Two dimensional flow

A ***limit cycle*** is an isolated closed trajectory. *Isolated* means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle



If all neighboring trajectories approach the limit cycle, we say the limit cycle is *stable* or *attracting*. Otherwise the limit cycle is *unstable*, or in exceptional cases, *half-stable*.

Two dimensional flow

Stable limit cycles are very important scientifically—they model systems that exhibit self-sustained oscillations. In other words, these systems oscillate even in the absence of external periodic forcing. Of the countless examples that could be given, we mention only a few: the beating of a heart; the periodic firing of a pace-maker neuron; daily rhythms in human body temperature and hormone secretion; chemical reactions that oscillate spontaneously; and dangerous self-excited vibrations in bridges and airplane wings. In each case, there is a standard oscillation of some preferred period, waveform, and amplitude. If the system is perturbed slightly, it always returns to the standard cycle.

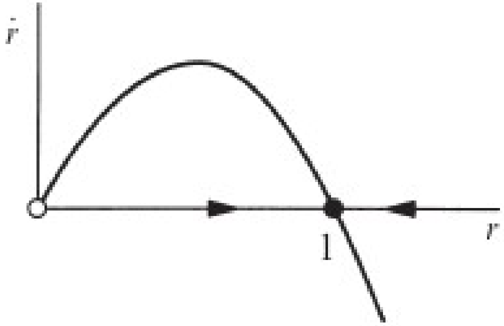
Limit Cycle is nonlinear phenomena; it can not occur in the linear system. As for an example, the closed cycle in around the center is not a limit cycle.

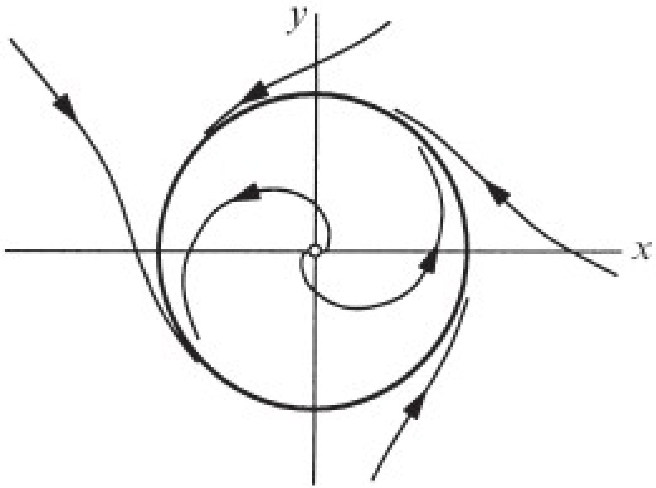
Two dimensional flow

Example:

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1$$

All trajectories will approach to r=1 orbit monotinically as angular velocity is constant.





Two dimensional flow

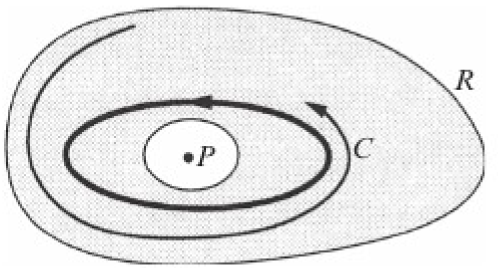
Poincaré–Bendixson Theorem: Suppose that:

(1) R is a closed, bounded subset of the plane;

(2) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R;

(3) R does not contain any fixed points; and

(4) There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time



Two dimensional flow

Consider the system

$$\dot{r} = r(1 - r^2) + \mu r \cos\theta$$
$$\dot{\theta} = 1.$$

When $\mu = 0$, there's a stable limit cycle at $r = 1$

a closed orbit still exists for $\mu > 0$, as long as μ is sufficiently small.

Two dimensional flow

Solution: We seek two concentric circles with radii $r_{\min}$ and $r_{\max}$, such that $\dot{r} < 0$ on the outer circle and $\dot{r} > 0$ on the inner circle. Then the annulus $0 < r_{\min} \le r \le r_{\max}$ will be our desired trapping region. Note that there are no fixed points in the annulus since $\dot{\theta} > 0$; hence if $r_{\min}$ and $r_{\max}$ can be found, the Poincaré-Bendixson theorem will imply the existence of a closed orbit.

To find $r_{\min}$, we require $\dot{r} = r(1 - r^2) + \mu r \cos\theta > 0$ for all θ. Since $\cos\theta \ge -1$, a sufficient condition for $r_{\min}$ is $1 - r^2 - \mu > 0$. Hence any $r_{\min} < \sqrt{1-\mu}$ will work, as long as $\mu < 1$ so that the square root makes sense. We should choose $r_{\min}$ as large as possible, to hem in the limit cycle as tightly as we can. For instance, we could pick $r_{\min} = 0.999\sqrt{1-\mu}$. (Even $r_{\min} = \sqrt{1-\mu}$ works, but more careful reasoning is required.) By a similar argument, the flow is inward on the outer circle if $r_{\max} = 1.001\sqrt{1+\mu}$.

Therefore a closed orbit exists for all $\mu < 1$, and it lies somewhere in the annulus $0.999\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$. ■

Two dimensional flow

No Chaos in the Phase Plane

The Poincaré-Bendixson theorem is one of the central results of nonlinear dynamics. It says that the dynamical possibilities in the phase plane are very limited: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. Nothing more complicated is possible.

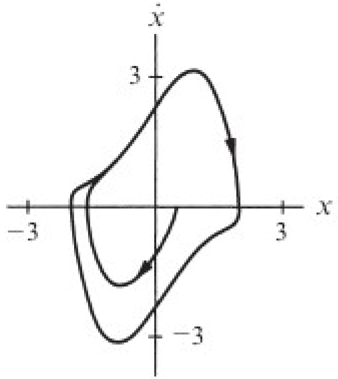
Two dimensional flow

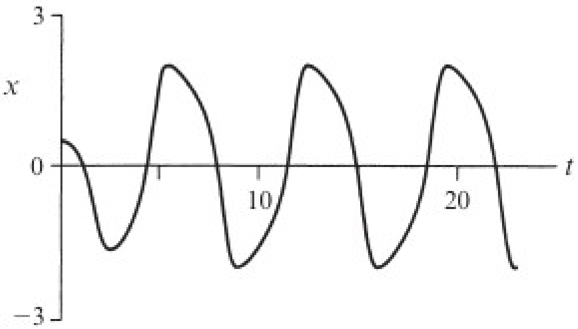
VAN DER POL OSCILLATOR

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \qquad \text{where } \mu \geq 0 \text{ is a parameter.}$$

For |x| <1 it is negative dissipation while for other values of x it is positive damping.

$$\mu = 1.5,$$





Two dimensional flow

Ruling out limit cycle in a dynamical system.

Gradient Systems

Suppose the system can be written in the form $\dot{\mathbf{x}} = -\nabla V$, for some continuously differentiable, single-valued scalar function $V(\mathbf{x})$. Such a system is called a ***gradient system*** with ***potential function*** V.

Theorem: It is impossible to have closed orbits for the gradient system.

For closed orbit we have $\Delta V = 0$

$$\begin{aligned}\Delta V &= \int_0^T \frac{dV}{dt}\,dt \\ &= \int_0^T (\nabla V \cdot \dot{\mathbf{x}})dt \\ &= -\int_0^T \|\dot{\mathbf{x}}\|^2\,dt \\ &< 0\end{aligned}$$

(unless $\dot{\mathbf{x}} \equiv \mathbf{0}$, in which case the trajectory is a fixed point, not a closed orbit).

Two dimensional flow

Show that there are no closed orbits for the system $\dot{x} = \sin y, \quad \dot{y} = x\cos y$.

$V(x,y) = -x \sin y$, since $\dot{x} = -\partial V/\partial x$ and $\dot{y} = -\partial V/\partial y$.

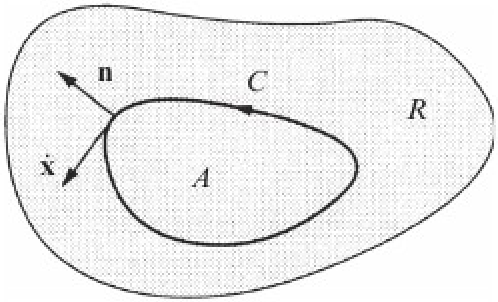
Dulac's Criterion: Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. If there exists a continuously differentiable, real-valued function $g(\mathbf{x})$ such that $\nabla \cdot (g\dot{\mathbf{x}})$ has one sign throughout R, then there are no closed orbits lying entirely in R.

Two dimensional flow

Ruling out limit cycle in a dynamical system.

Suppose there were a closed orbit C lying entirely in the region R.

$$\iint_A \nabla \cdot (g\dot{\mathbf{x}})\,dA = \oint_C g\dot{\mathbf{x}} \cdot \mathbf{n}\, d\ell$$



where $\mathbf{n}$ is the outward normal and $d\ell$ is the element of arc length along C. Look first at the double integral on the left: it must be *nonzero*, since $\nabla \cdot (g\dot{\mathbf{x}})$ has one sign in R. On the other hand, the line integral on the right equals *zero* since $\dot{\mathbf{x}} \cdot \mathbf{n} = 0$ everywhere, by the assumption that C is a trajectory (the tangent vector $\dot{\mathbf{x}}$ is orthogonal to $\mathbf{n}$). This contradiction implies that no such C can exist. ■

There is no particular algorithm to find the g(x).

Two dimensional flow

Candidates that occasionally work are $g = 1$, $1/x^a y^b$, e^{ax}, and e^{ay}.

Show that the system $\dot{x} = x(2 - x - y)$, $\dot{y} = y(4x - x^2 - 3)$ has no closed orbits in the positive quadrant $x, y > 0$.

$$g = 1/xy.$$

$$\begin{aligned}
\nabla \cdot (g\dot{\mathbf{x}}) &= \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y}) \\
&= \frac{\partial}{\partial x}\left(\frac{2 - x - y}{y}\right) + \frac{\partial}{\partial y}\left(\frac{4x - x^2 - 3}{x}\right) \\
&= -1/y \\
&< 0.
\end{aligned}$$

Since the region $x, y > 0$ is simply connected and g and $\mathbf{f}$ satisfy the required smoothness conditions, Dulac's criterion implies there are no closed orbits in the positive quadrant. ■

Two dimensional flow

Show that the system $\dot{x} = y, \;\; \dot{y} = -x - y + x^2 + y^2$ has no closed orbits.

Solution: Let $g = e^{-2x}$. Then $\nabla \cdot (g\dot{\mathbf{x}}) = -2e^{-2x}y + e^{-2x}(-1+2y) = -e^{-2x} < 0$. By Dulac's criterion, there are no closed orbits. ■

Two dimensional flow

Liénard Systems

In the early days of nonlinear dynamics, say from about 1920 to 1950, there was a great deal of research on nonlinear oscillations. The work was initially motivated by the development of radio and vacuum tube technology, and later it took on a mathematical life of its own. It was found that many oscillating circuits could be modeled by second-order differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

now known as ***Liénard's equation.***

It is a generalization of Vanderpol Oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

Two dimensional flow

Liénard Systems

Liénard's equation is equivalent to the system

$$\dot{x} = y$$
$$\dot{y} = -g(x) - f(x)y.$$

Liénard's Theorem: Suppose that $f(x)$ and $g(x)$ satisfy the following conditions:

Liénard Systems

(1) $f(x)$ and $g(x)$ are continuously differentiable for all x;

(2) $g(-x) = -g(x)$ for all x (i.e., $g(x)$ is an *odd* function);

(3) $g(x) > 0$ for $x > 0$;

(4) $f(-x) = f(x)$ for all x (i.e., $f(x)$ is an *even* function);

(5) The odd function $F(x) = \int_0^x f(u)\,du$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.

If all the criteria are fulfilled system will have unique stable limit cycle.

Two dimensional flow

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

for $\mu >> 1$. In this *strongly nonlinear* limit, we'll see that the limit cycle consists of an extremely slow buildup followed by a sudden discharge, followed by another slow buildup, and so on. Oscillations of this type are often called ***relaxation oscillations***, because the "stress" accumulated during the slow buildup is "relaxed" during the sudden discharge.

$$\ddot{x} + \mu\dot{x}(x^2 - 1) = \frac{d}{dt}\left(\dot{x} + \mu\left[\tfrac{1}{3}x^3 - x\right]\right).$$

Two dimensional flow

$$F(x) = \tfrac{1}{3}x^3 - x, \qquad w = \dot{x} + \mu F(x),$$

the van der Pol equation implies that

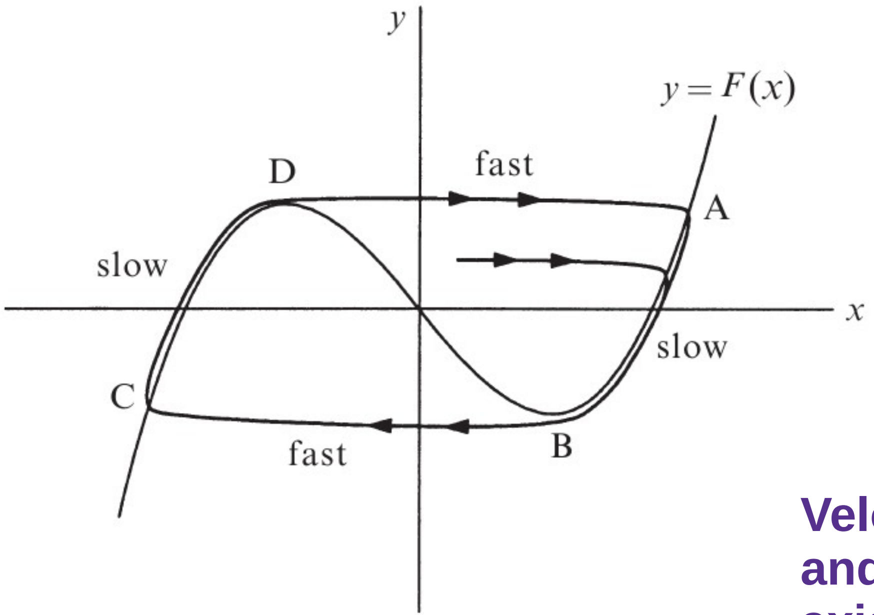
$$\dot{w} = \ddot{x} + \mu\dot{x}(x^2 - 1) = -x.$$

$$\dot{x} = w - \mu F(x)$$
$$\dot{w} = -x.$$

$$y = \frac{w}{\mu}$$

$$\dot{x} = \mu\left[y - F(x)\right]$$
$$\dot{y} = -\tfrac{1}{\mu}x.$$

Two dimensional flow



Initial condition is not too close to the nullclines.

suppose $y - F(x) \sim O(1)$.

$$|\dot{x}| \sim O(\mu) >> 1$$

whereas $|\dot{y}| \sim O(\mu^{-1}) << 1$;

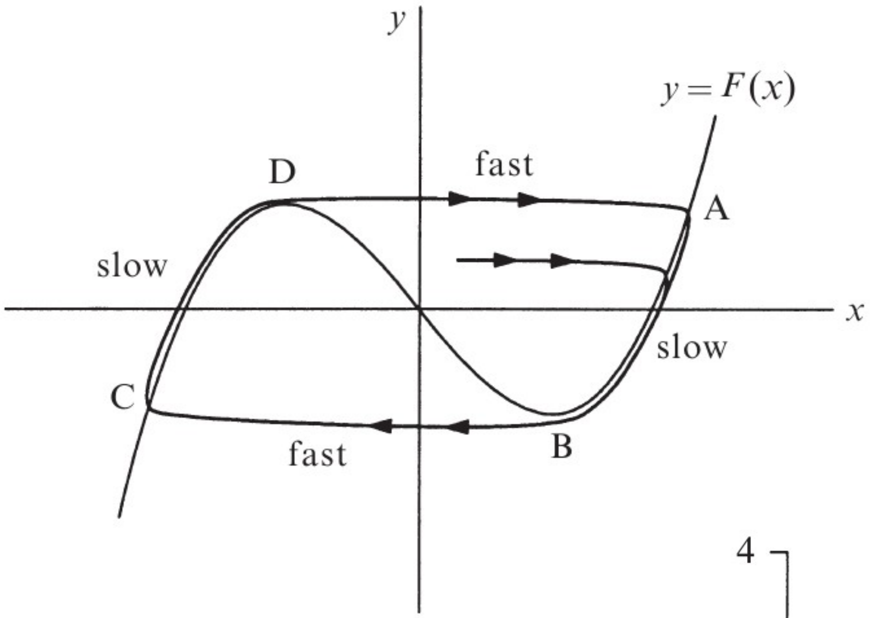
Velocity is huge in the horizontal direction and trajectroy will move parallel to the x-axis.

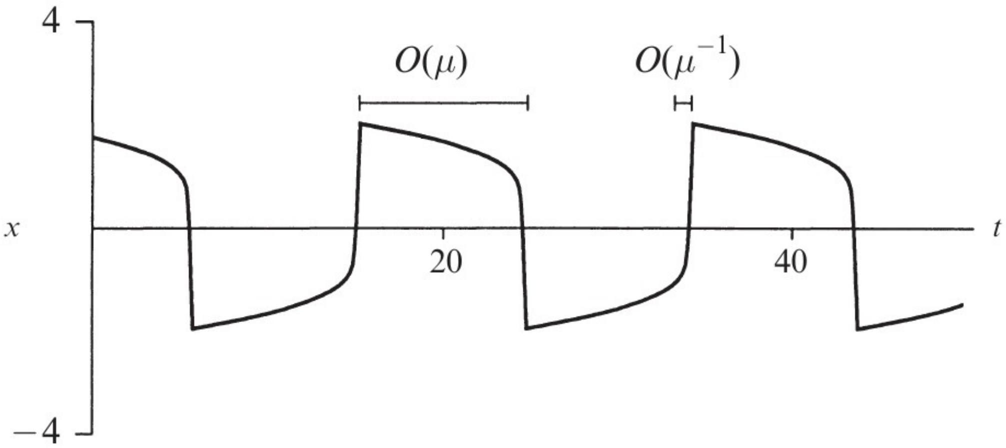
If the initial condition is *above* the nullcline, then $y - F(x) > 0$

$$y - F(x) > 0 \longrightarrow \dot{x} > 0$$

This analysis shows that the limit cycle has two ***widely separated time scales***: the crawls require $\Delta t \sim O(\mu)$ and the jumps require $\Delta t \sim O(\mu^{-1})$.

Two dimensional flow





Two dimensional flow

Weakly Nonlinear Oscillators

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$$

where $0 \le \varepsilon << 1$ and $h(x, \dot{x})$ is an arbitrary smooth function.

Vander Pol Oscillator

$$\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0,$$

Duffing Oscillator

$$\ddot{x} + x + \varepsilon x^3 = 0.$$

Two dimensional flow

Weakly Nonlinear Oscillators

Purturbative Scheme:

$$x(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots,$$

Damped Oscillator

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0,$$

with initial conditions

$$x(0) = 0,\ \dot{x}(0) = 1.$$

Exact Solution:

$$x(t,\varepsilon) = (1-\varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin\left[(1-\varepsilon^2)^{1/2} t\right].$$

Two dimensional flow

Weakly Nonlinear Oscillators

$$\frac{d^2}{dt^2}(x_0+\varepsilon x_1+\ldots)+2\varepsilon\frac{d}{dt}(x_0+\varepsilon x_1+\ldots)+(x_0+\varepsilon x_1+\ldots)=0.$$

$$O(1): \quad \ddot{x}_0+x_0=0$$

$$O(\varepsilon): \quad \ddot{x}_1+2\dot{x}_0+x_1=0.$$

Ignoring higher orders of the puturbative parameter....

Using the initial condition we have

$$0=x_0(0)+\varepsilon x_1(0)+\ldots$$

$$x_0(0)=0,\ x_1(0)=0.$$

Two dimensional flow

$$\dot{x}_0(0) = 1, \quad \dot{x}_1(0) = 0.$$

$$x_0(t) = \sin t.$$

$$\ddot{x}_1 + x_1 = -2\cos t.$$

subject to $x_1(0) = 0, \ \dot{x}_1(0) = 0$ is

$$x_1(t) = -t \sin t,$$

which is a ***secular*** term, i.e., a term that *grows* without bound as $t \to \infty$.

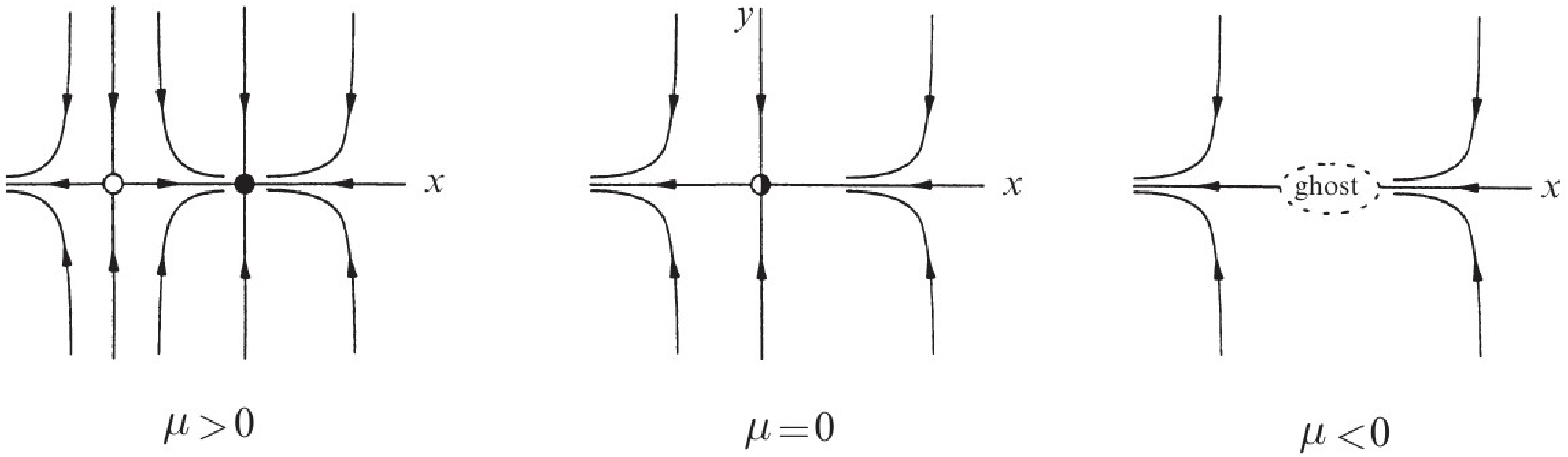
$$x(t, \varepsilon) = \sin t - \varepsilon t \sin t + O(\varepsilon^2).$$

Bifurcations in two dimension flow

Saddle-Node Bifurcation

$$\dot{x} = \mu - x^2$$

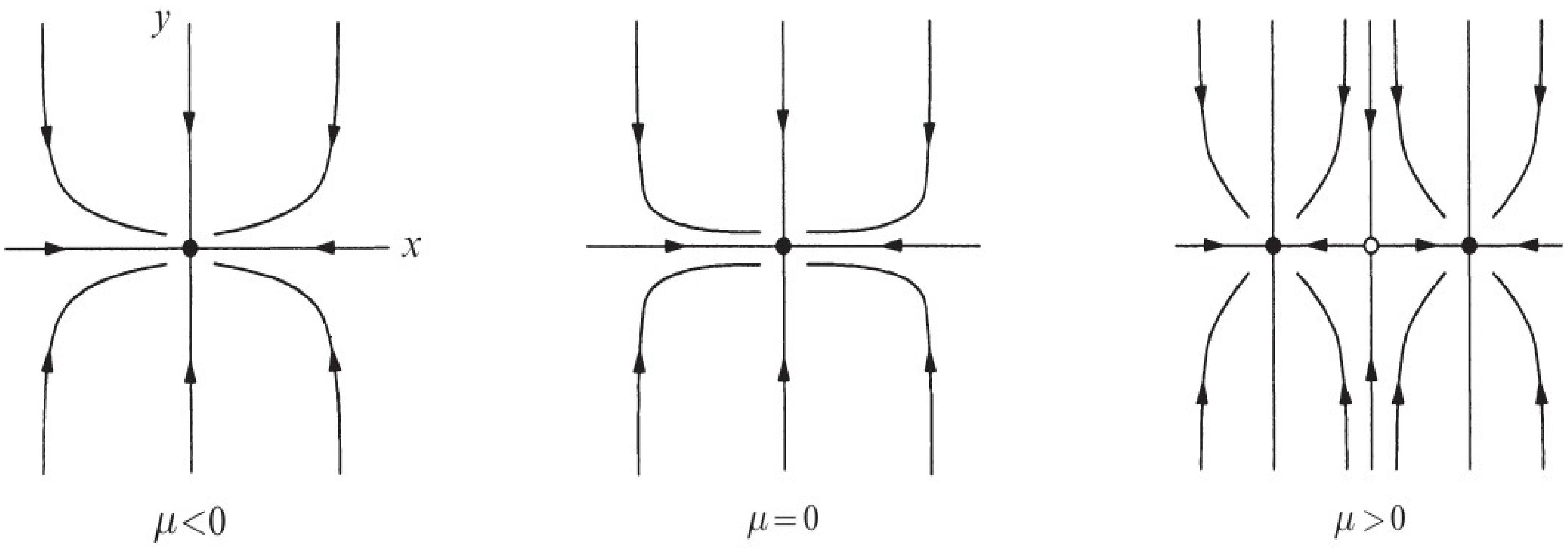
$$\dot{y} = -y.$$



Bifurcations in two dimension flow

Supercritcial Pitchfork bifurcation:

Plot the phase portraits for the supercritical pitchfork system $\dot{x} = \mu x - x^3$, $\dot{y} = -y$, for $\mu < 0$, $\mu = 0$, and $\mu > 0$.



Bifurcations in two dimension flow

Exercise:

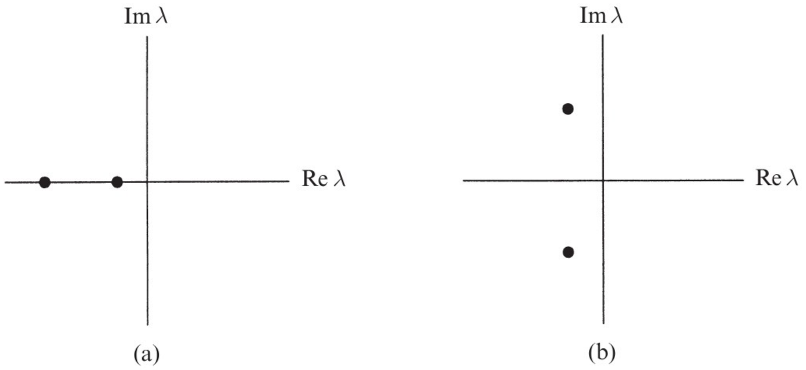
$$\dot{x} = \mu x - x^2, \qquad \dot{y} = -y \qquad \text{(transcritical)}$$

$$\dot{x} = \mu x - x^3, \qquad \dot{y} = -y \qquad \text{(supercritical pitchfork)}$$

$$\dot{x} = \mu x + x^3, \qquad \dot{y} = -y \qquad \text{(subcritical pitchfork)}$$

Bifurcations in two dimension flow

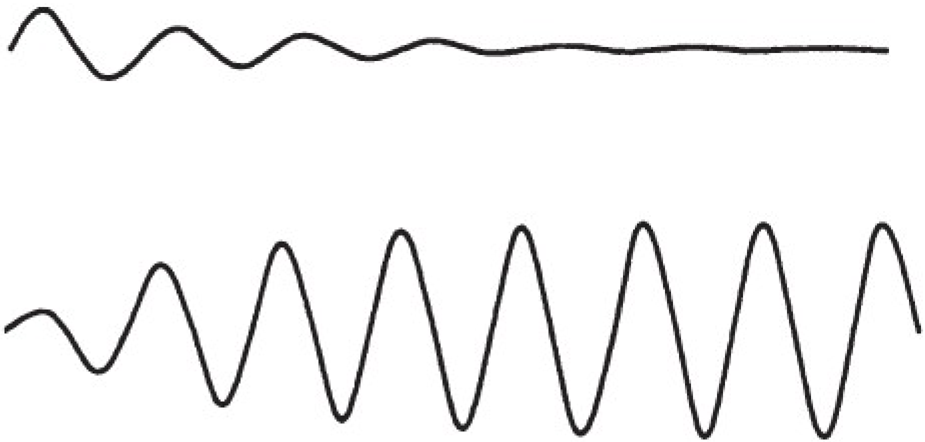
One of the interesting bifurcation that come in the two dimensional flow is the Hopf bifurcation.



We have two kind of situations: (a) both the eigen values are stable and crosses the zero eigen value and becomes positive upon changing the parameters. This leads to destabilization of the system. We have encouneterd several such examples of this bifurcations like supercritical pithcfork, transcritical, saddle node bifurcations, etc.. (b) The eigen values are complex conjugate which real part is negative. So it forms stable spiral node. Now upon varying the parameter the eigen value crosses the imaginary and become positive. This sort of bifurcation is known as Hopf bifurcation.

Bifurcations in two dimension flow

Supercritical Hopf Bifurcation



(a) $\mu < \mu_c$

The flow decays to a fixed point.

(b) $\mu > \mu_c$

The equilibrium point looses its stability and a limit cycle is born.

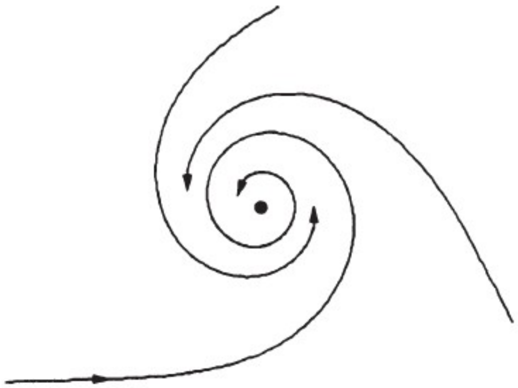
At the critical point the the equilibrium state has limit cycle.

Supercritical Hopf Bifurcation

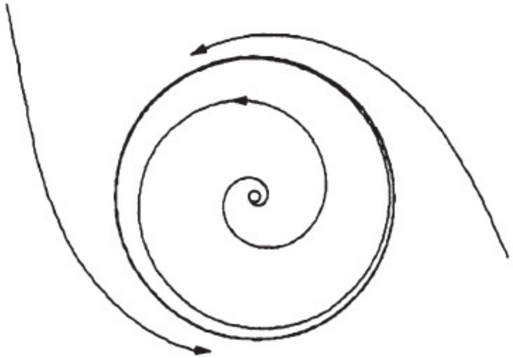
$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = \omega + br^2.$$

There are three parameters: μ controls the stability of the fixed point at the origin, ω gives the frequency of infinitesimal oscillations, and b determines the dependence of frequency on amplitude for larger amplitude oscillations.

Bifurcations in two dimension flow



$\mu < 0$



$\mu > 0$

r=0: Spiral node fixed point.

Stable limit cycle with radius $r = \sqrt{\mu}$.

Bifurcations in two dimension flow

To see how the eigenvalues behave during the bifurcation, we rewrite the system in Cartesian coordinates; this makes it easier to find the Jacobian. We write $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\begin{aligned}
\dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\
&= (\mu r - r^3)\cos\theta - r(\omega + br^2)\sin\theta \\
&= \left(\mu - [x^2 + y^2]\right)x - \left(\omega + b[x^2 + y^2]\right)y \\
&= \mu x - \omega y + \text{cubic terms}
\end{aligned}$$

and similarly

$$\dot{y} = \omega x + \mu y + \text{cubic terms.}$$

So the Jacobian at the origin is

$$A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix},$$

which has eigenvalues

$$\lambda = \mu \pm i\omega.$$

Bifurcations in two dimension flow

Characteristics of supercritical Hopf bifurcation.

1. The size of the limit cycle grows continuously from zero, and increases proportional to $\sqrt{\mu - \mu_c}$, for μ close to μ_c.
2. The frequency of the limit cycle is given approximately by $\omega = \text{Im}\ \lambda$, evaluated at $\mu = \mu_c$. This formula is exact at the birth of the limit cycle, and correct within $O(\mu - \mu_c)$ for μ close to μ_c. The period is therefore $T = (2\pi/\text{Im}\,\lambda) + O(\mu - \mu_c)$.

Bifurcations in two dimension flow

Subcritical Hopf Bifurcation

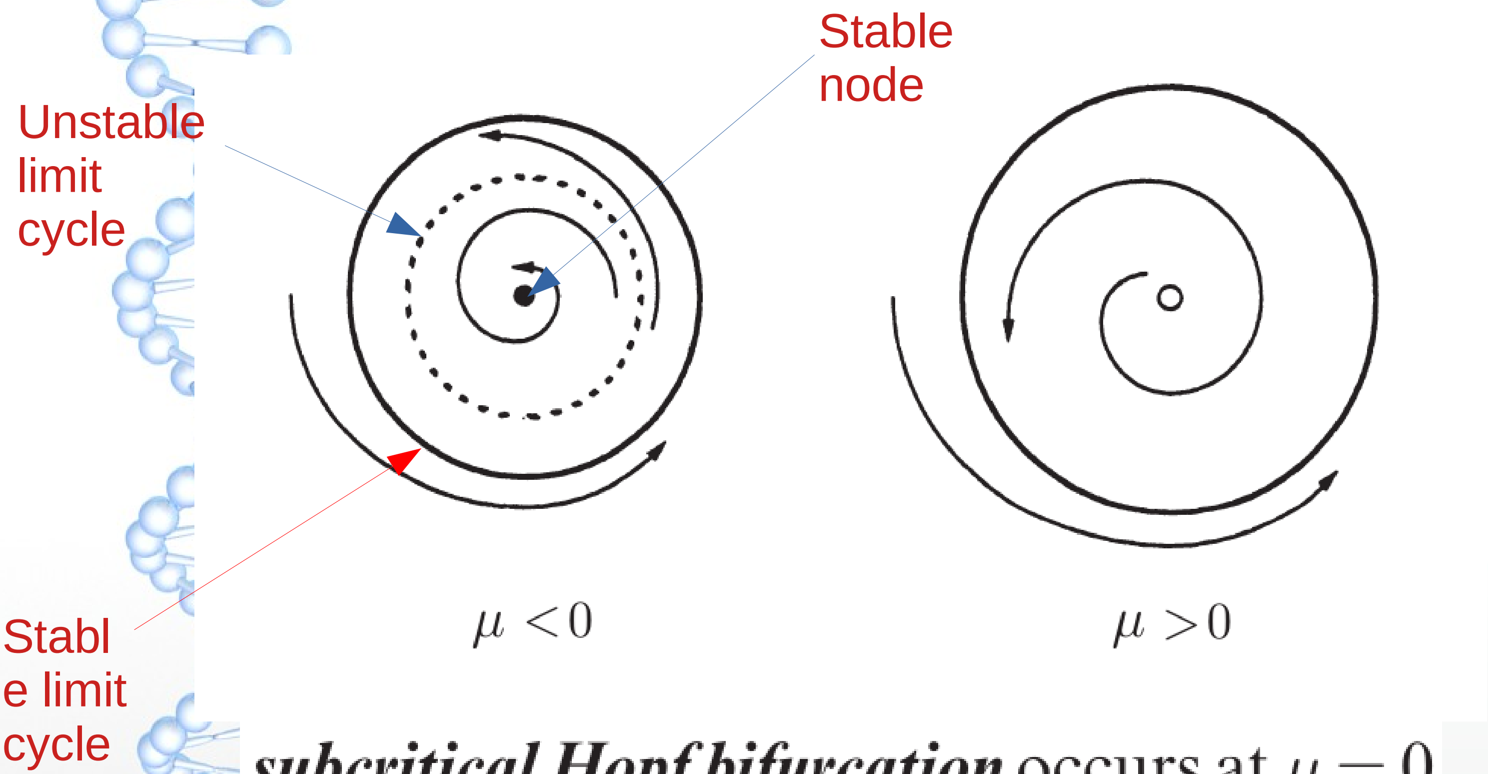
Like pitchfork bifurcations, Hopf bifurcations come in both super- and subcritical varieties. The subcritical case is always much more dramatic, and potentially dangerous in engineering applications. After the bifurcation, the trajectories must *jump* to a *distant* attractor, which may be a fixed point, another limit cycle, infinity, or—in three and higher dimensions—a chaotic attractor. We'll see a concrete example of this last, most interesting case when we study the Lorenz equations

Example:

$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega + br^2.$$

Bifurcations in two dimension flow

The important difference from the earlier supercritical case is that the cubic term r^3 is now *destabilizing*; it helps to drive trajectories away from the origin.



The unstable limit cycle shrinks upon increasing the parameter.

subcritical Hopf bifurcation occurs at $\mu = 0$.

Bifurcations in two dimension flow

Example of Hopf bifurcation

$$\dot{x} = ax - by - C(x^2 + y^2)\,x$$
$$\dot{y} = bx + ay - C(x^2 + y^2)\,y\ ,$$

where a, b, and C are real. Clearly the origin is a fixed point, at which one finds the eigenvalues $\lambda = a \pm ib$. Thus, the fixed point is a stable spiral if $a < 0$ and an unstable spiral if $a > 0$.

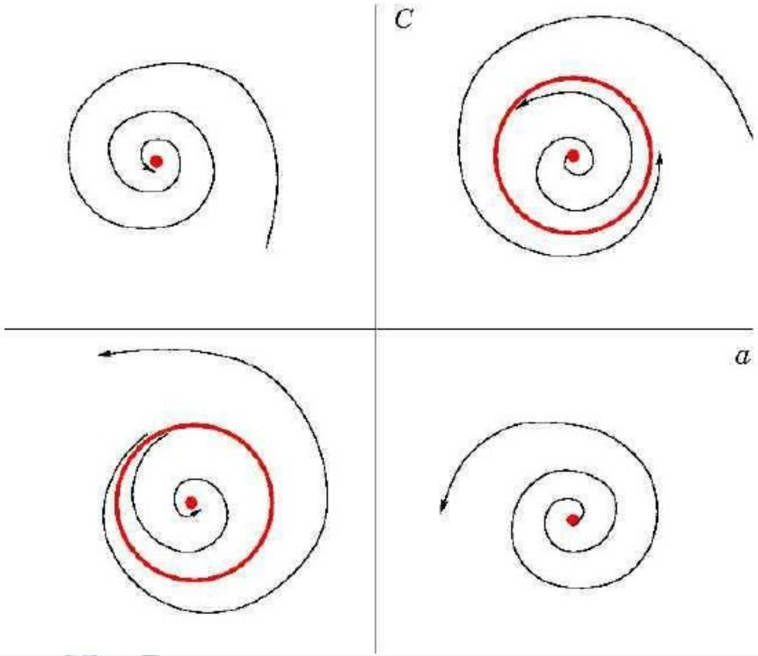
Written in terms of the complex variable $z = x + iy$, these two equations collapse to the single equation

$$\dot{z} = (a + ib)\,z - C\,|z|^2\,z\ .$$

Bifurcations in two dimension flow

Example of Hopf bifurcation

The dynamics are also simple in polar coordinates $r = |z|$, $\theta = \arg(z)$:



$$\dot{r} = ar - Cr^3$$
$$\dot{\theta} = b\ .$$

Hopf bifurcation: for $C > 0$ the bifurcation is supercritical

For $C < 0$ the bifurcation is subcritical

Bifurcations in two dimension flow

Oscillating Chemical oscillation an example of Hopf bifurcation

Belousov's "Supposedly Discovered Discovery"

In the early 1950s the Russian biochemist Boris Belousov was trying to create a test tube caricature of the Krebs cycle, a metabolic process that occurs in living cells. When he mixed citric acid and bromate ions in a solution of sulfuric acid, and in the presence of a cerium catalyst, he observed to his astonishment that the mixture became yellow, then faded to colorless after about a minute, then returned to yellow a minute later, then became colorless again, and continued to oscillate dozens of times before finally reaching equilibrium after about an hour.

Today it comes as no surprise that chemical reactions can oscillate spontaneously—such reactions have become a standard demonstration in chemistry classes, and you may have seen one yourself. (For recipes, see Winfree (1980).) But in Belousov's day, his discovery was so radical that he couldn't get his work published. It was thought that all solutions of chemical reagents must go *monotonically* to equilibrium, because of the laws of thermodynamics. Belousov's paper was rejected by one journal after another. According to Winfree (1987b, p.161), one editor even added a snide remark about Belousov's "supposedly discovered discovery" to the rejection letter.

Bifurcations in two dimension flow

Oscillating Chemical oscillation an example of Hopf bifurcation

$$MA + I_2 \rightarrow IMA + I^- + H^+; \quad \frac{d[I_2]}{dt} = -\frac{k_{1a}[MA][I_2]}{k_{1b} + [I_2]} \qquad (1)$$

$$ClO_2 + I^- \rightarrow ClO_2^- + \tfrac{1}{2} I_2; \quad \frac{d[ClO_2]}{dt} = -k_2[ClO_2][I^-] \qquad (2)$$

$$ClO_2^- + 4I^- + 4H^+ \rightarrow Cl^- + 2I_2 + 2H_2O;$$

$$\frac{d[ClO_2^-]}{dt} = -k_{3a}[ClO_2^-][I^-][H^+] - k_{3b}[ClO_2^-][I_2]\frac{[I^-]}{u + [I^-]^2} \qquad (3)$$

Bifurcations in two dimension flow

Oscillating Chemical reaction an example of Hopf bifurcation

Numerical integrations of (1)–(3) show that the model exhibits oscillations that closely resemble those observed experimentally. However this model is still too complicated to handle analytically. To simplify it, Lengyel et al. (1990) use a result found in their simulations: Three of the reactants (MA, I_2, and ClO_2) vary much more slowly than the intermediates I^- and ClO_2^-, which change by several orders of magnitude during an oscillation period. By approximating the concentrations of the slow reactants as *constants* and making other reasonable simplifications, they reduce the system to a two-variable model. (Of course, since this approximation neglects the slow consumption of the reactants, the model will be unable to account for the eventual approach to equilibrium.) After suitable nondimensionalization, the model becomes

Bifurcations in two dimension flow

Oscillating Chemical oscillation an example of Hopf bifurcation

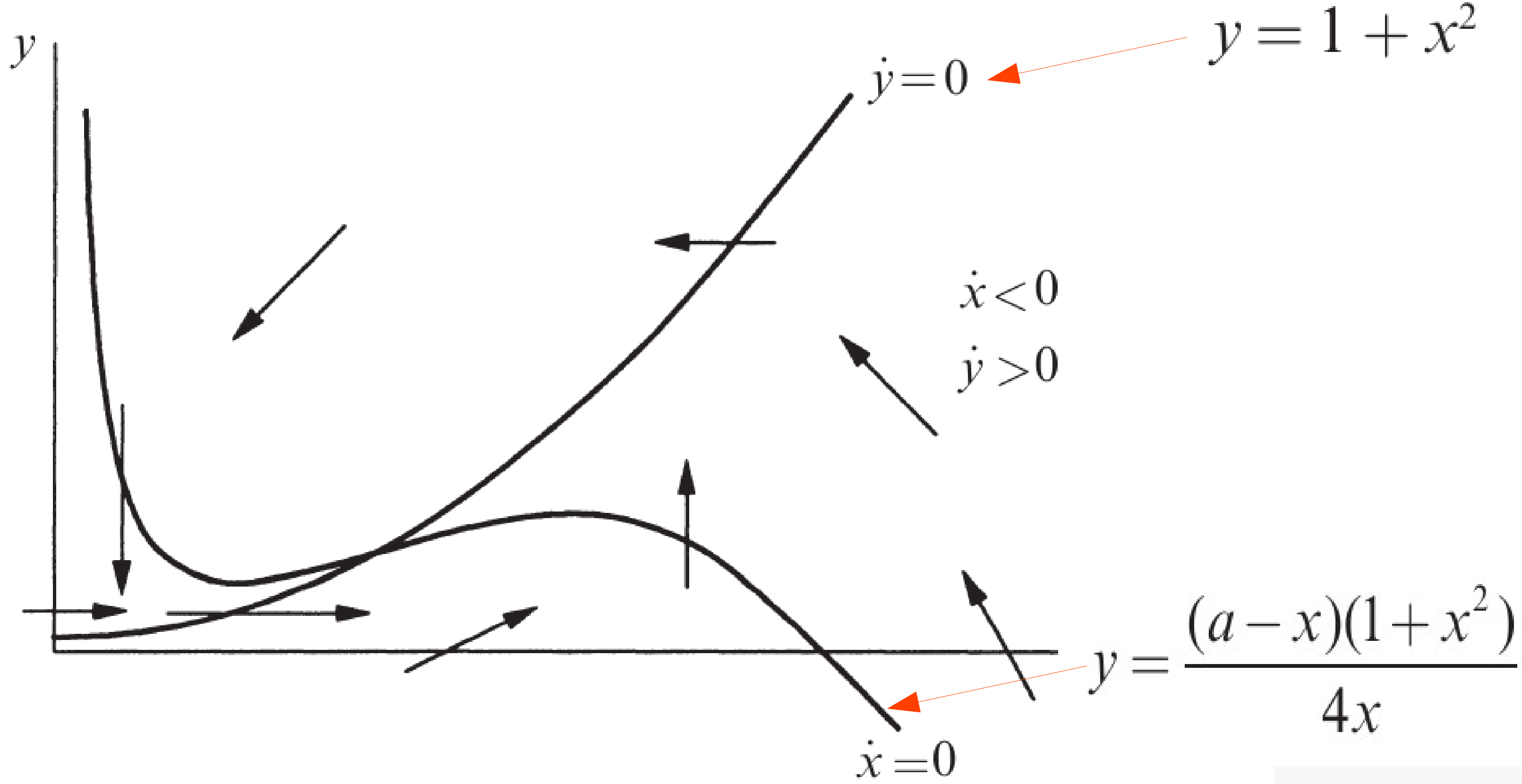
$$\dot{x} = a - x - \frac{4xy}{1 + x^2} \qquad (4)$$

$$\dot{y} = bx\left(1 - \frac{y}{1 + x^2}\right) \qquad (5)$$

where x and y are the dimensionless concentrations of I^- and ClO_2^-. The parameters $a, b > 0$ depend on the empirical rate constants and on the concentrations assumed for the slow reactants.

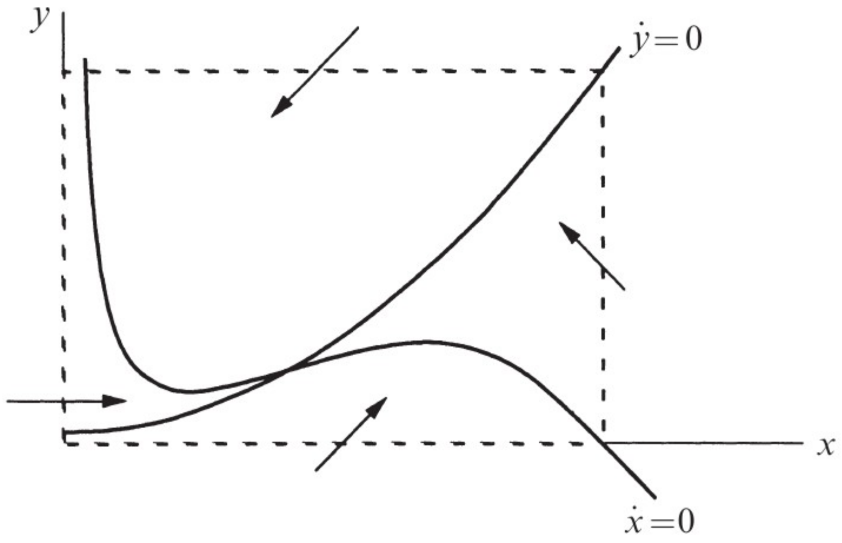
Bifurcations in two dimension flow

Oscillating Chemical oscillation an example of Hopf bifurcation



Bifurcations in two dimension flow

Oscillating Chemical oscillation an example of Hopf bifurcation



We can't apply the Poincaré–Bendixson theorem yet, because there's a fixed point

$$x^* = a/5, \qquad y^* = 1 + (x^*)^2 = 1 + (a/5)^2$$

However, if the fixed point becomes repeller the Poincare-Bendixson theorem can be applied in the punctured box by removing the fixed point!

Bifurcations in two dimension flow

Oscillating Chemical oscillation an example of Hopf bifurcation

All that remains is to see under what conditions (if any) the fixed point is a repeller. The Jacobian at (x^*, y^*) is

$$\frac{1}{1+(x^*)^2}\begin{pmatrix} 3(x^*)^2-5 & -4x^* \\ 2b(x^*)^2 & -bx^* \end{pmatrix}.$$

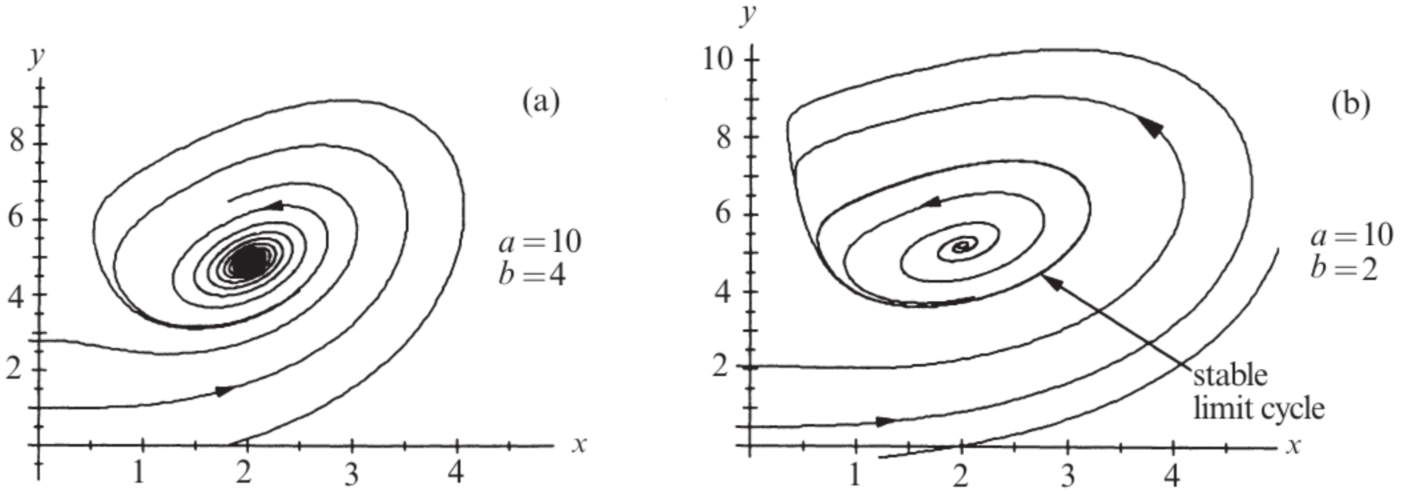
$$\Delta = \frac{5bx^*}{1+(x^*)^2} > 0, \qquad \tau = \frac{3(x^*)^2-5-bx^*}{1+(x^*)^2}.$$

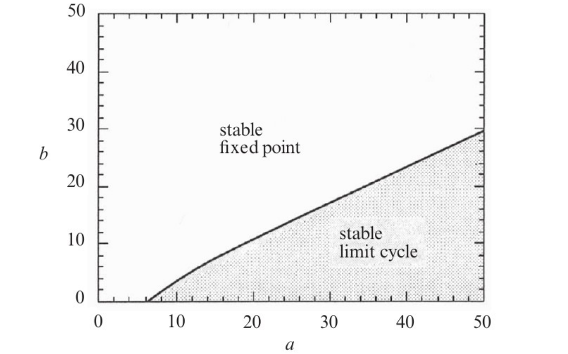
(x^*, y^*) is a repeller if $\tau > 0$, i.e., if

$$b < b_c \equiv 3a/5 - 25/a.$$

Bifurcations in two dimension flow

Oscillating Chemical oscillation an example of Hopf bifurcation





Bifurcations in two dimension flow

Approximate the period of the limit cycle for b slightly less than b_c.

Solution: The frequency is approximated by the imaginary part of the eigenvalues at the bifurcation. As usual, the eigenvalues satisfy $\lambda^2 - \tau\lambda + \Delta = 0$. Since $\tau = 0$ and $\Delta > 0$ at $b = b_c$, we find

$$\lambda = \pm i\sqrt{\Delta}.$$

But at b_c,

$$\Delta = \frac{5b_c x^*}{1+(x^*)^2} = \frac{5\left(\frac{3a}{5} - \frac{25}{a}\right)\left(\frac{a}{5}\right)}{1+(a/5)^2} = \frac{15a^2 - 625}{a^2+25}.$$

Hence $\omega \approx \Delta^{1/2} = \left[(15a^2 - 625)/(a^2+25)\right]^{1/2}$ and therefore

$$T = 2\pi/\omega$$
$$= 2\pi\left[(a^2+25)/(15a^2-625)\right]^{1/2}.$$

Bifurcations in two dimension flow

Assignment 4:

Consider the system $\dot{x} = y - 2x, \;\; \dot{y} = \mu + x^2 - y.$

a) Sketch the nullclines.

b) Find and classify the bifurcations that occur as μ varies.

c) Sketch the phase portrait as a function of μ.

For each of the following systems, a Hopf bifurcation occurs at the origin when $\mu = 0$. Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical.

$$\dot{x} = y + \mu x, \quad \dot{y} = -x + \mu y - x^2 y$$

$$\dot{x} = \mu x + y - x^3, \quad \dot{y} = -x + \mu y - 2y^3$$

$$\dot{x} = \mu x + y - x^2, \quad \dot{y} = -x + \mu y - 2x^2$$

Bifurcations in two dimension flow

Assignment 4:

(Predator-prey model) Odell (1980) considered the system

$$\dot{x} = x\left[x(1-x) - y\right], \quad \dot{y} = y(x-a),$$

where $x \geq 0$ is the dimensionless population of the prey, $y \geq 0$ is the dimensionless population of the predator, and $a \geq 0$ is a control parameter.

a) Sketch the nullclines in the first quadrant $x, y \geq 0$.

b) Show that the fixed points are $(0, 0)$, $(1, 0)$, and $(a, a - a^2)$, and classify them.

c) Sketch the phase portrait for $a > 1$, and show that the predators go extinct.

d) Show that a Hopf bifurcation occurs at $a_c = \frac{1}{2}$. Is it subcritical or supercritical?

Bifurcations in two dimension flow

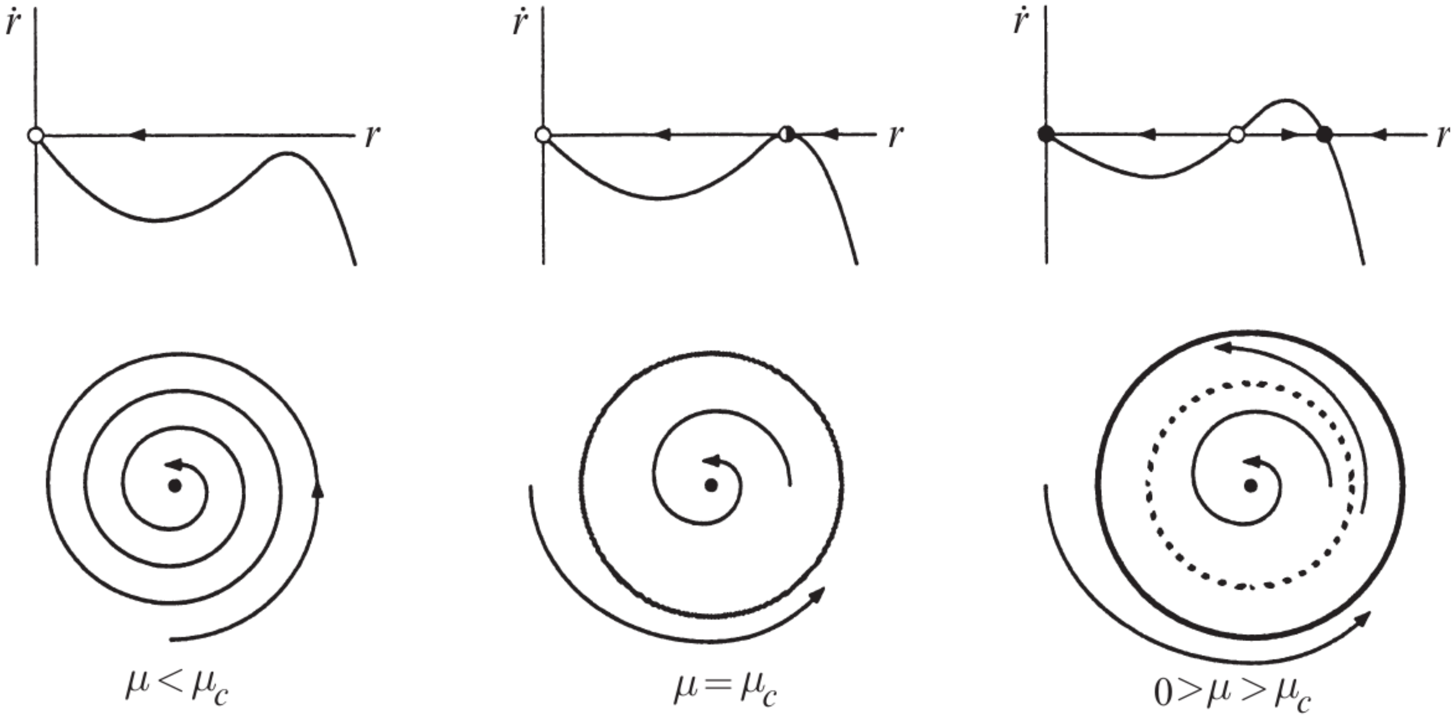
In two dimensions there are four ways through which the limit cycle can be created and destroyed:

(a) Hopf Bifurcation

(b) Saddle node bifurcation of cycles: A bifurcation in which two limit cycles coalesce and annihilate is called saddle-node bifurcation of cycles.

$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega + br^2$$

Bifurcations in two dimension flow



At μ_c a half-stable cycle is born out

As μ increases it splits into a pair of limit cycles, one stable, one unstable.

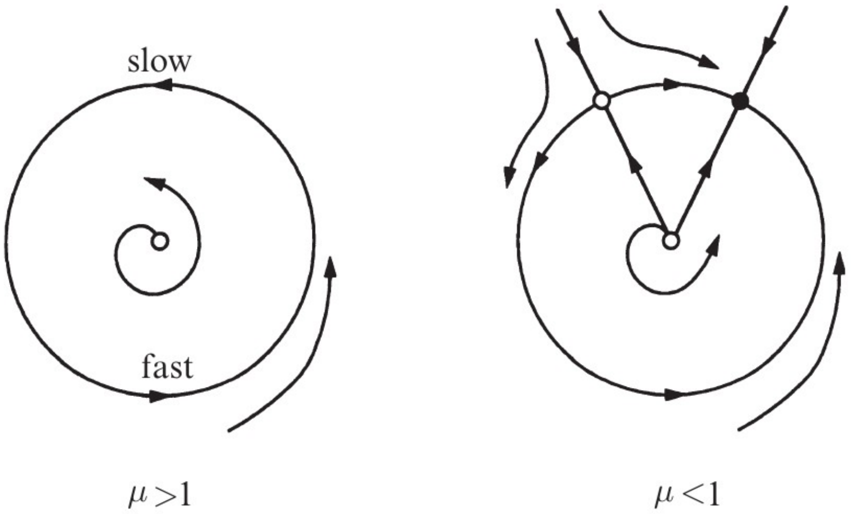
(c) Infinite-period bifurcations

$$\dot{r} = r(1-r^2) \qquad \mu \geq 0.$$

$$\dot{\theta} = \mu - \sin\theta$$

In the radial direction, all trajectories (except $r^* = 0$) approach the unit circle monotonically as $t \to \infty$. In the angular direction, the motion is everywhere counterclockwise if $\mu > 1$, whereas there are two invariant rays defined by $\sin\theta = \mu$ if $\mu < 1$. Hence as μ decreases through

Bifurcations in two dimension flow



As μ decreases, the limit cycle $r = 1$ develops a bottleneck at $\theta = \pi/2$ that becomes increasingly severe as $\mu \to 1^+$. The oscillation period lengthens and finally becomes infinite at $\mu_c = 1$, when a fixed point appears on the circle; hence the term ***infinite-period bifurcation***. For $\mu < 1$, the fixed point splits into a saddle and a node.

Bifurcations in two dimension flow

(d) Homoclinic bifurcation:

In this scenario, part of a limit cycle moves closer and closer to a saddle point. At the bifurcation the cycle touches the saddle point and becomes a homoclinic orbit. This is another kind of infinite-period bifurcation; to avoid confusion, we'll call it a *saddle-loop* or ***homoclinic bifurcation***.

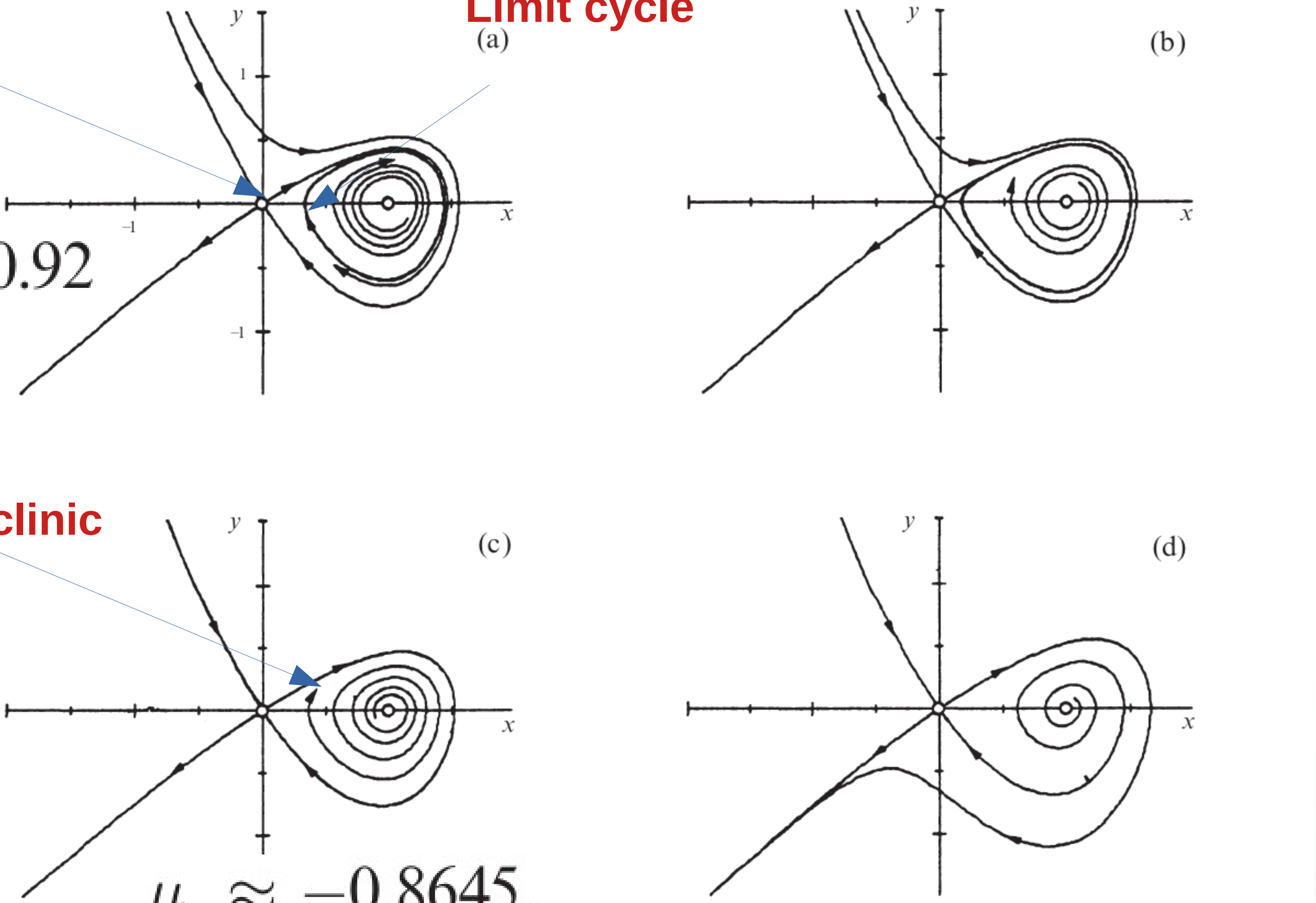
$$\dot{x} = y$$
$$\dot{y} = \mu y + x - x^2 + xy.$$

Bifurcations in two dimension flow

Saddle point

Limit cycle

$\mu = -0.92$



Homoclinic orbit

$\mu_c \approx -0.8645.$

Bifurcations in two dimension flow

	Amplitude of stable limit cycle	Period of cycle
Supercritical Hopf	$O(\mu^{1/2})$	$O(1)$
Saddle-node bifurcation of cycles	$O(1)$	$O(1)$
Infinite-period	$O(1)$	$O(\mu^{-1/2})$
Homoclinic	$O(1)$	$O(\ln \mu)$

Three dimension flow

Lorenz's equation:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz.$$

where σ ("Prandtl number"), r ("Rayleigh number") and b are parameters (> 0).

These equations also arise in studies of convection and instabilities in planetary atmospheres, modesl of lasers and dynamos etc.

Three dimension flow

Simple properties of the Lorenz equations:

- *Nonlinearity* - the two nonlinearities are xy and xz

- *Symmetry* - Equations are invariant under $(x, y) \rightarrow (-x, -y)$. Hence if $(x(t), y(t), z(t))$ is a solution, so is $(-x(t), -y(t), z(t))$

Three dimension flow

The fixed points, or steady solutions, occur where

$$\dot{x} = \dot{y} = \dot{z} = 0.$$

An obvious fixed point is

$$(x^*, y^*, z^*) = (0, 0, 0)$$

which corresponds, respectively, to a fluid at rest, pure conduction, and a temperature distribution consistent with conductive equilibrium.

Another steady solution is

$$x^* = y^* = \pm\sqrt{b(r-1)},$$

$$z^* = r - 1.$$

This solution corresponds to flow around the loop at constant speed; the $\pm$ signs arise because the circulation can be in either sense. That $\mathrm{sgn}(x) = \mathrm{sgn}(y)$ implies that hot fluid rises and cold fluid falls.

Three dimension flow

Summary: The rest state, $x^* = y^* = z^* = 0$, is

$$\begin{array}{rll} \text{stable} & \text{for} & 0 < r < 1 \\ \text{unstable} & \text{for} & r > 1. \end{array}$$

The convective state (steady circulation), $x^* = y^* = \pm\sqrt{r-1}$, $Z^* = r - 1$, is

$$\begin{array}{rll} \text{stable} & \text{for} & 1 < r < r_c \\ \text{unstable} & \text{for} & r > r_c. \end{array}$$

What happens for $r > r_c$?

Three dimension flow

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \sum_i \frac{\partial \dot{\phi}_i}{\partial \phi_i}, \qquad i = 1, 2, 3, \qquad \phi_1 = x, \phi_2 = y, \phi_3 = z$$

Thus

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -(\sigma + 1 + b)V$$

which may be solved to yield

$$V(t) = V(0)e^{-(\sigma+1+b)t}.$$

The system is clearly dissipative, since $\sigma > 0$ and $b > 0$.

Three dimension flow

$$\sigma = 10$$
$$b = 8/3 \qquad \text{(corresponding to the first wavenumber to go unstable).}$$

For these parameters,

$$V(t) = V(0)e^{-\frac{41}{3}t}.$$

Thus after 1 time unit, volumes are reduced by a factor of $e^{-\frac{41}{3}} \sim 10^{-6}$. The system is therefore *highly* dissipative.

Three dimension flow

Simple properties of the Lorenz equations:

- *Volume contraction* - The Lorenz system is *dissipative* i.e. volumes in phase-space contract under the flow

- *Fixed points* - $(x^*, y^*, z^*) = (0, 0, 0)$ is a fixed point for *all* values of the parameters. For $r > 1$ there is also a pair of fixed points $C^{\pm}$ at $x^* = y^* = \pm\sqrt{b(r-1)}$, $z^* = r - 1$. These coalesce with the origin as $r \to 1^+$ in a *pitchfork bifurcation*

Linear stability of the origin

Linearization of the original equations about the origin yields

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - y\\ \dot{z} &= -bz\end{aligned}$$

Three dimension flow

Hence, the z-motion decouples, leaving

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with trace $\tau = -\sigma - 1 < 0$ and determinant $\Delta = \sigma(1 - r)$.

For $r > 1$, origin is a *saddle point* since $\Delta < 0$

Three dimension flow

For $r < 1$, origin is a *sink* since $\tau^2 - 4\Delta = (\sigma + 1)^2 - 4\sigma(1 - r) = (\sigma - 1)^2 + 4\sigma\tau > 0 \rightarrow$ a stable node.

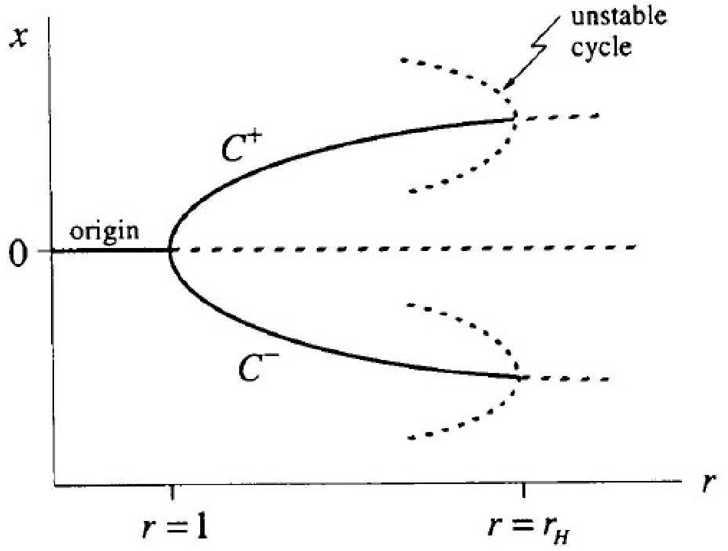
Actually for $r < 1$ it can be shown that *every* trajectory approaches the origin as $t \rightarrow \infty$ the origin is *globally stable*, hence there can be *no limit cycles or chaos for* $r < 1$.

Three dimension flow

Subcritical Hopf bifurcation occurs at

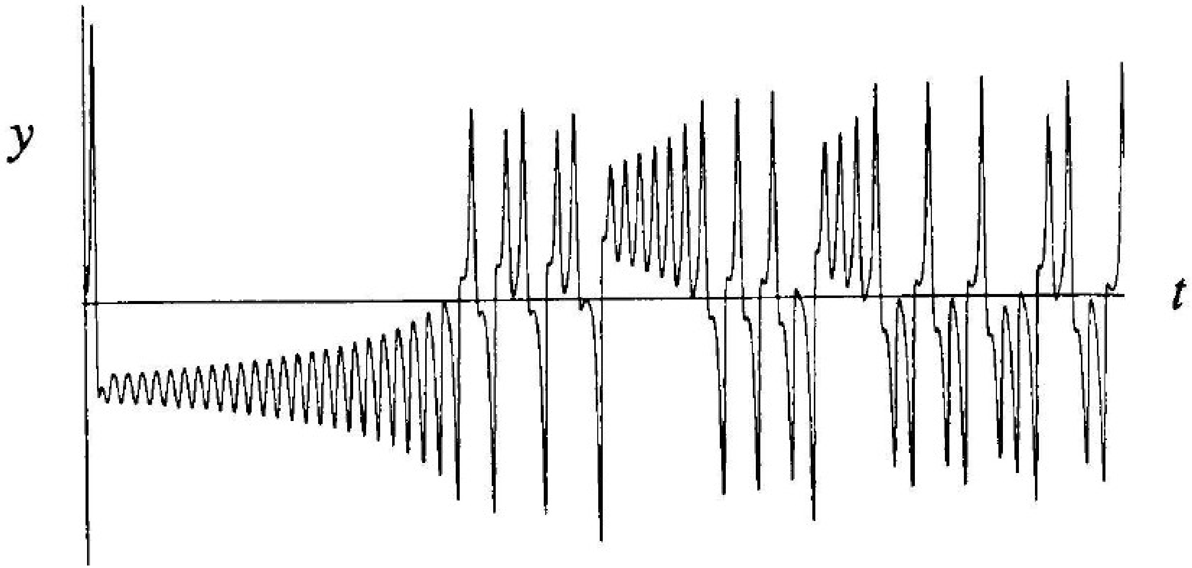
$$r = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \equiv r_H > 1$$

assuming that $\sigma - b - 1 > 0$.



Three dimension flow

Lorenz considered the case $\sigma = 10, b = 8/3, r = 28$ with $(x_0, y_0, z_0) = (0, 1, 0)$.

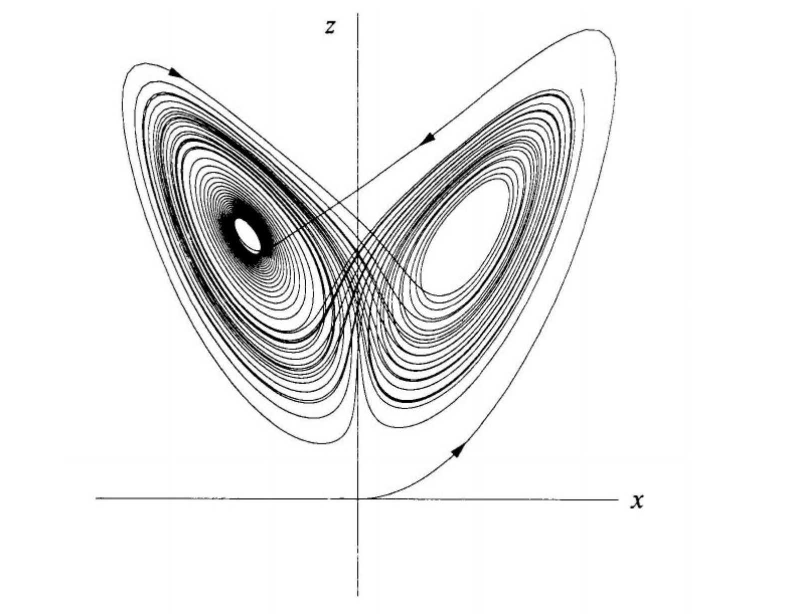


After an initial transient, the solution settles into an irregular oscillation that persist for infinite time but never repeats exactly. The motion is called aperiodic.

$$r_H = \sigma(\sigma + b + 3)/(\sigma - b - 1) \simeq 24.74,$$

Three dimension flow

Lorenz discovered that a wonderful structure emerges if the solution is visualized as a *trajectory in phase space*. For instance, when $x(t)$ is plotted against $z(t)$, the famous *butterfly wing pattern* appears

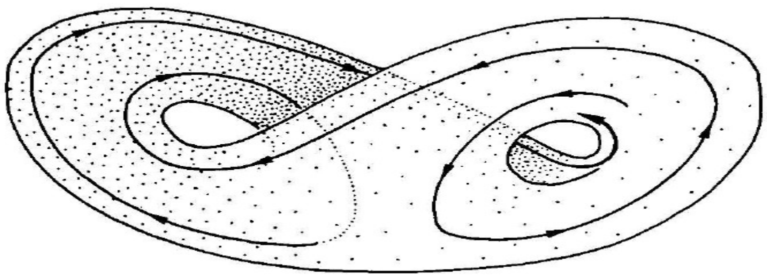


The trajectory appears to cross iteself repeatedly, but that's just an artifact of projecting the 3-dimensional trajectory onto a 2-dimensional plane. In 3-D no crossings occur.

Three dimension flow

- The number of circuits made on either side varies *unpredictably* from one cycle to the next. The sequence of the number of circuits in each lobe has many of the characteristics of a *random sequence!*

- When the trajectory is viewed in all 3 dimensions, it appears to settle onto a thin set that looks like a pair of butterfly wings. We call this attractor a *strange attractor* and it can be shown schematically as...



Three dimension flow

What is the *geometric structure* of the strange attractor?

The uniqueness theorem means that trajectories *cannot cross* or merge, hence the two surfaces of the strange attractor can only *appear* to merge.

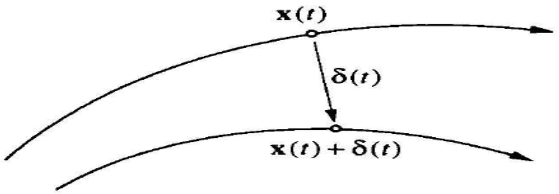
Lorenz concluded that “there is an infinite complex of surfaces” where they *appear* to merge. Today this “infinite complex of surfaces” would be called a **FRACTAL**.

A *fractal* is a set of points with zero volume but infinite surface area.

Exponential divergence of nearby trajectories

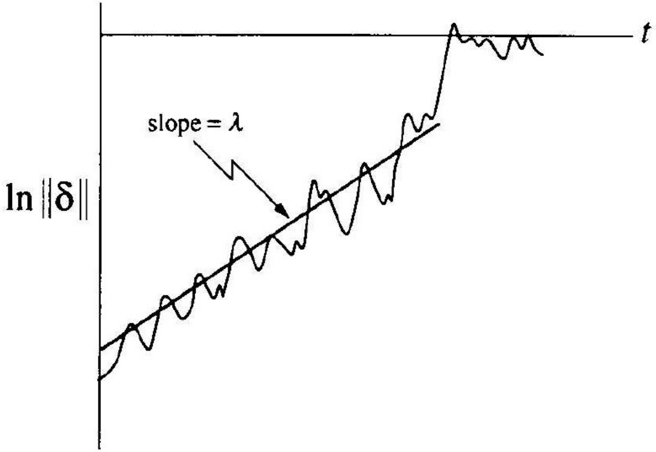
The motion on the attractor exhibits *sensitive dependence on initial conditions*. Two trajectories starting very close together will rapidly diverge from each other, and thereafter have totally different futures. The practical implication is that long-term prediction becomes impossible in a system like this, where small uncertainties are amplified enormously fast.

Suppose we let transients decay so that the trajectory is "on" the attractor. Suppose $\mathbf{x}(t)$ is a point on the attractor at time t, and consider a nearby point, say $\mathbf{x}(t) + \delta(t)$, where δ is a tiny separation vector of initial length $||\delta_0|| = 10^{-15}$, say



In numerical studies of the Lorenz attractor, one finds that $||\delta(t)|| \sim ||\delta_{\mathbf{0}}||e^{\lambda t}$, where $\lambda \simeq 0.9$.

Hence *neighbouring trajectories separate exponentially fast!*



Note...

- The curve is never straight, but has wiggles since the strength of exponential divergence varies somewhat along the attractor.

- The exponential divergence must stop when the separation is comparable to the "diameter" of the attractor - the trajectories cannot get any further apart! (curve saturates for large t)

- The number λ is often called the *Lyapunov exponent*, though this is somewhat sloppy terminology....

(i) There are actually n different Lyapunov exponents for an n-dimensional system, defined as follows...

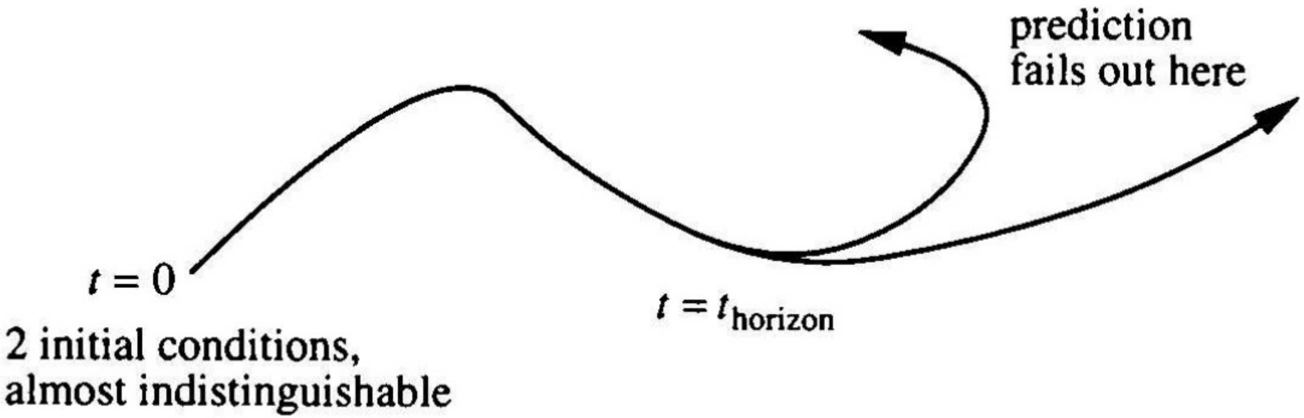
Consider the evolution of an infinitesimal sphere (in phase space) of perturbed initial conditions. During its evolution the sphere becomes distorted into an infinitesimal ellipsoid. Let $\delta_k(t), k = 1, 2, 3...n$ denote the length of the kth principal axis of the ellipsoid. Then

$$\delta_k(t) \sim \delta_k(0)e^{\lambda_k t},$$

where the λ_k are the Lyapunov exponents. For large t, the diameter of the ellipsoid is controlled by the most positive λ_k. Thus our λ above is actually the *largest Lyapunov exponent*.

(ii) λ depends (slightly) on which trajectory we study. We should really *average* over many different points on the same trajectory to get the true value of λ.

When a system has a positive Lyapunov exponent, there is a time horizon beyond which prediction will break down....



Suppose we measure the initial conditions of an experimental system very accurately. Of course no measurement is perfect - there is always some error $||\delta_0||$ between our estimate and the true initial state. After a time t the discrepancy grows to $||\delta(t)|| \sim ||\delta_0||e^{\lambda t}$.

Let a be a measure of our tolerance, i.e. if a prediction is within a of the true state, we consider it acceptable. Then our prediction becomes intolerable when $||\delta(t)|| \geq a$, i.e. after a time

$$t_{horizon} \sim O\left(\frac{1}{\lambda} \ln \frac{a}{||\delta||}\right)$$

Example Suppose $a = 10^{-3}$, $||\delta_0|| = 10^{-7}$.

$$\Rightarrow t_{horizon} = \frac{4 \ln 10}{\lambda}$$

If we improve the initial error to $||\delta_0|| = 10^{-13}$,

$$\Rightarrow t_{horizon} = \frac{10 \ln 10}{\lambda}$$

i.e. only $10/4 = 2.5$ times longer!

Defining Chaos

No definition of the term "chaos" is universally accepted - even now! - but almost everyone would agree on the three ingredients used in the following working definition:

Chaos is **aperiodic long-term behaviour** *in a* **deterministic** *system that exhibits* **sensitive dependence on initial conditions**

1. *Aperiodic long-term behaviour* means that there are trajectories which do not settle down to fixed points, periodic or quasi-periodic orbits as $t \to \infty$.

2. *Deterministic* means that the system has no random or noisy inputs or parameters. Irregular behaviour arises solely from the system's nonlinearity.

3. *Sensitive dependence on initial conditions* means that nearby trajectories diverge exponentially fast, i.e. the system has at least one positive Lyapunov exponent.

Some people think that chaos is just a fancy word for instability. For example, the system $\dot{x} = x$ is deterministic and shows exponential separation of nearby trajectories. However, we should not consider this system to be chaotic!

Defining "attractor" and "strange attractor"

The term *attractor* is also difficult to define in a rigorous way. Loosely, an attractor is a set of points to which all neighbouring trajectories converge. Stable fixed points and stable limit cycles are examples.

More precisely, we define an attractor to be a closed set A with the following properties:

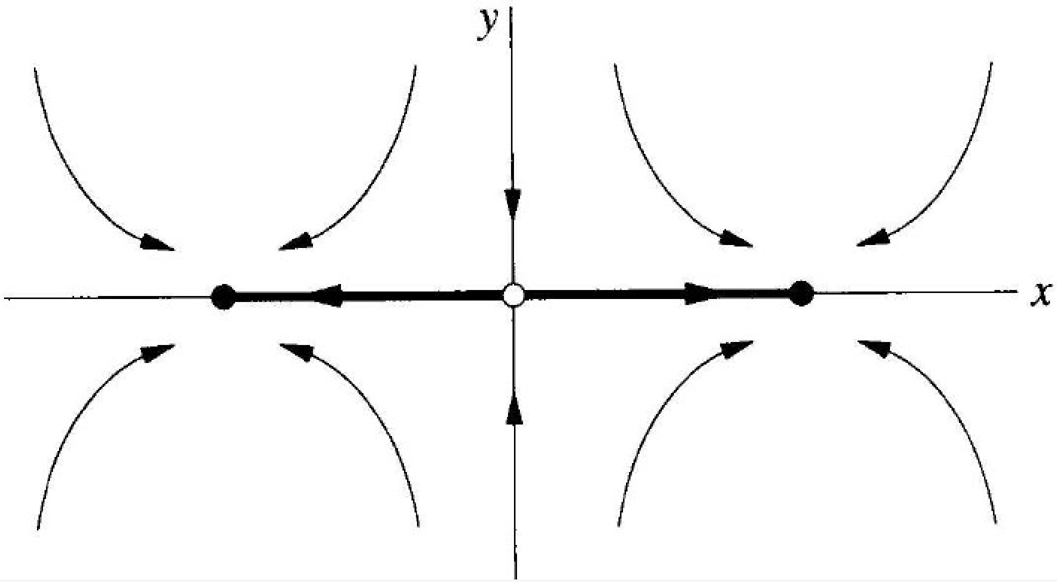
1. A is an *invariant set*: any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.

2. A *attracts an open set of initial conditions*: there is an open set U containing A such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$. Hence A attracts all trajectories that start sufficiently close to it. The largest such U is called the *basin of attraction* of A.

3. A is *minimal*: there is no proper subset of A that satisfies conditions 1. and 2.

Example $\begin{cases} \dot{x} &= \mu x - x^3 \\ \dot{y} &= -y \end{cases}$

Let I denote the interval $-1 \leq x \leq 1, y = 0$.
Is I an attractor?



So, I is an invariant set (condition 1.). Also, I attracts an open set of initial conditions - it attracts *all* trajectories in the xy-plane. But I is *not* an attractor because it is not minimal. The stable fixed points $(\pm 1, 0)$ are proper subsets of I that also satisfy conditions 1. and 2. These points are the *only* attractors for the system.

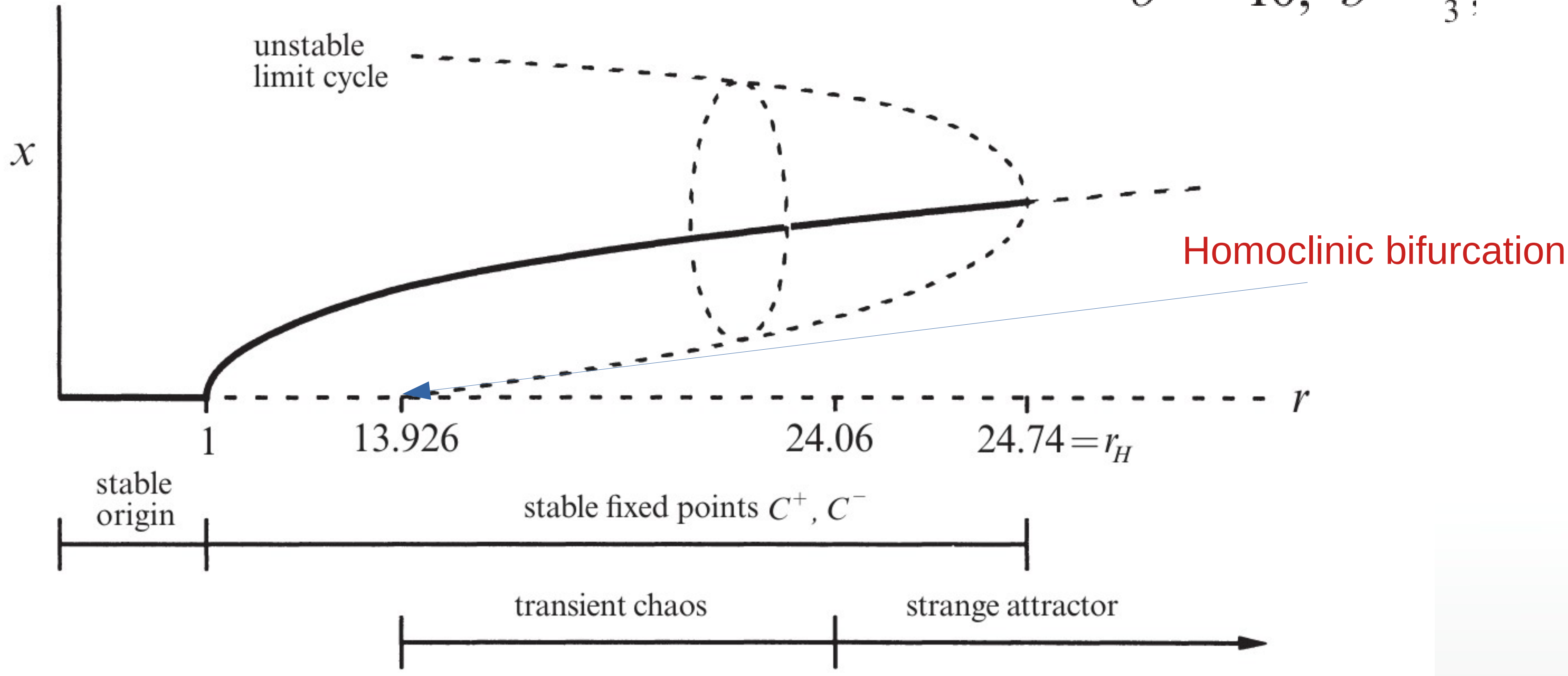
Maybe the same is true for the Lorenz Equations? - Nobody has yet proved that the Lorenz attractor is *truly* an attractor!!

Finally we define a *strange attractor* to be *an attractor that exhibits sensitive dependence on initial conditions*.

Strange attractors were originally called strange because they are often (but not always...) fractal sets. Nowadays, this geometric property is regarded as less important than the dynamical property of sensitive dependence on initial conditions. The terms “chaotic attractor” and “fractal attractor” are used when one wishes to emphasize one or other of these aspects.

A detailed bifurcation digram of the Lorenz attractor

$$\sigma = 10,\ b = \frac{8}{3};$$



A detailed bifurcation digram of the Lorenz attractor

(Rikitake model of geomagnetic reversals) Consider the system

$$\dot{x} = -vx + zy$$
$$\dot{y} = -vy + (z-a)x$$
$$\dot{z} = 1 - xy$$

where $a, v > 0$ are parameters.

a) Show that the system is dissipative.

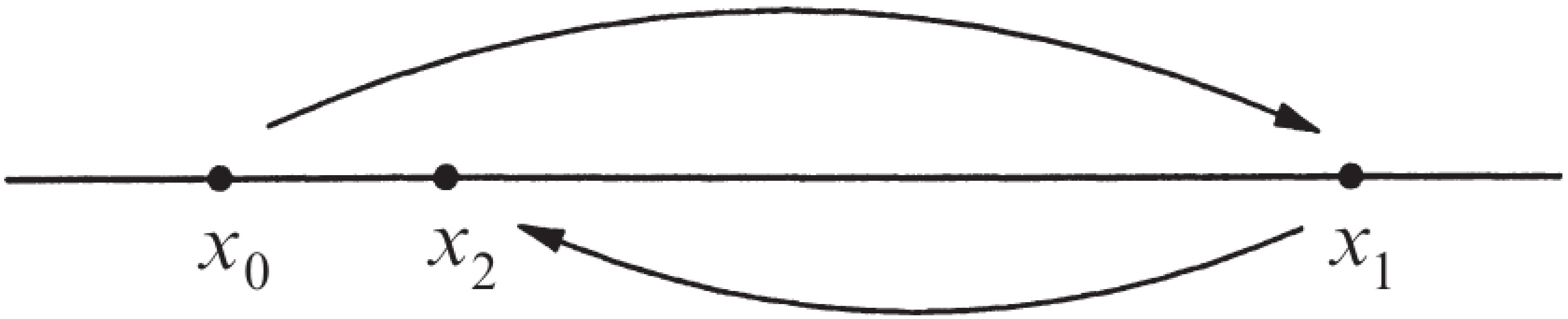
b) Show that the fixed points may be written in parametric form as $x^* = \pm k$, $y^* = \pm k^{-1}$, $z^* = vk^2$, where $v(k^2 - k^{-2}) = a$.

c) Classify the fixed points.

One-dimensional Map

As the time is discrete the dynamical system may be represented as

$$x_{n+1} = f(x_n)$$

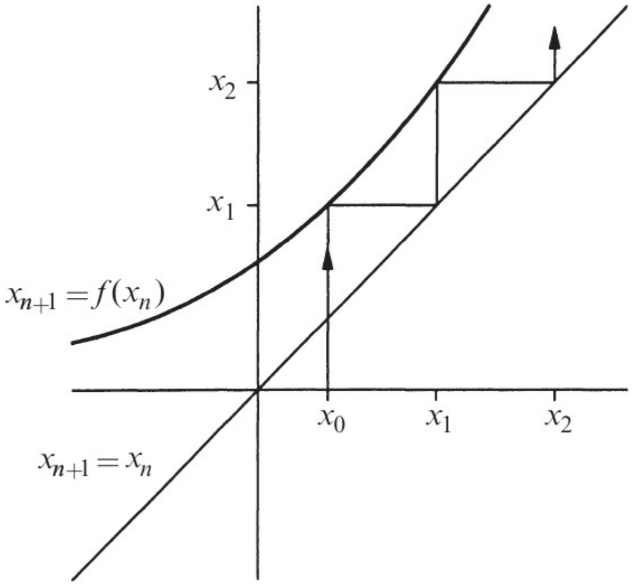


Where f is a smooth function from the line to itself. These are also popular with a name recurssion relation, iterated maps or simply maps.

One-dimensional Map

Example:

$$x_{n+1} = f(x_n)$$



$x_0, x_1, x_2, \ldots$ is called the ***orbit*** starting from x_0.

One-dimensional Map

Necessity of Maps:

As models of natural phenomena. In some scientific contexts it is natural to regard time as discrete. This is the case in digital electronics, in parts of economics and finance theory, in impulsively driven mechanical systems, and in the study of certain animal populations where successive generations do not overlap.

As a simple example of chaos.

Maps are helpful in analyzing the solution of differential equation. Like whether the solution is periodic or aperiodic. Example: Lorenz maps and Poincare map (Will be discussed in the upcoming classes.)

One-dimensional Map

Fixed points and linear stability analysis:

$$x_{n+1} = f(x_n)$$

Suppose x^* satisfies $f(x^*) = x^*$. Then x^* is a ***fixed point***, for if $x_n = x^*$ then $x_{n+1} = f(x_n) = f(x^*) = x^*$; hence the orbit remains at x^* for all future iterations.

To determine the stability of x^*, we consider a nearby orbit $x_n = x^* + \eta_n$ and ask whether the orbit is attracted to or repelled from x^*. That is, does the deviation η_n grow or decay as n increases? Substitution yields

$$x^* + \eta_{n+1} = x_{n+1} = f(x^* + \eta_n) = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2).$$

But since $f(x^*) = x^*$, this equation reduces to

$$\eta_{n+1} = f'(x^*)\eta_n + O(\eta_n^2).$$

One-dimensional Map

$\eta_{n+1} = f'(x^*)\eta_n$ with *eigenvalue* or ***multiplier*** $\lambda = f'(x^*)$.

$$\eta_1 = \lambda\eta_0,\ \eta_2 = \lambda\eta_1 = \lambda^2\eta_0,$$

If $|\lambda| = |f'(x^*)| < 1$, then $\eta_n \to 0$ as $n \to \infty$

fixed point x^* is ***linearly stable***.

Conversely, if $|f'(x^*)| > 1$ the fixed point is Unstable.

Find the fixed points for the map $x_{n+1} = x_n{}^2$ and determine their stability.

The fixed points satisfy $x^* = (x^*)^2$. Hence $x^* = 0$ or $x^* = 1$.

$$\lambda = f'(x^*) = 2x^*.$$

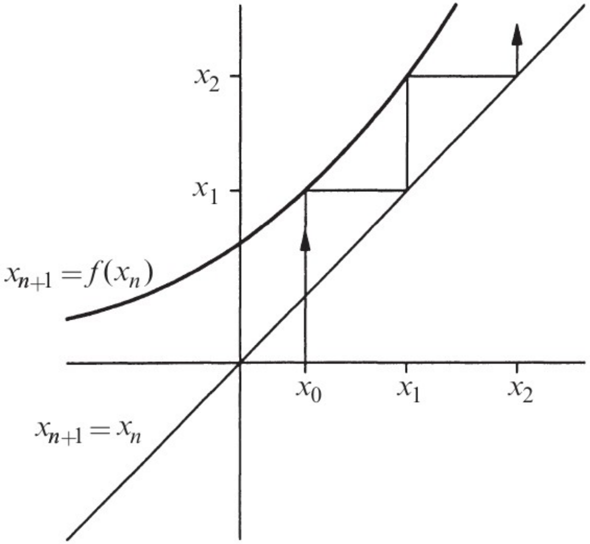
The fixed point $x^* = 0$ is stable since $|\lambda| = 0 < 1$,

and $x^* = 1$ is unstable since $|\lambda| = 2 > 1$.

One-dimensional Map

Cobwebs:

Given $x_{n+1} = f(x_n)$ and an initial condition x_0, draw a vertical line until it intersects the graph of f; that height is the output x_1. At this stage we could return to the horizontal axis and repeat the procedure to get x_2 from x_1, but it is more convenient simply to trace a horizontal line till it intersects the diagonal line $x_{n+1} = x_n$, and then move vertically to the curve again. Repeat the process n times to generate the first n points in the orbit.



One-dimensional Map

Example:

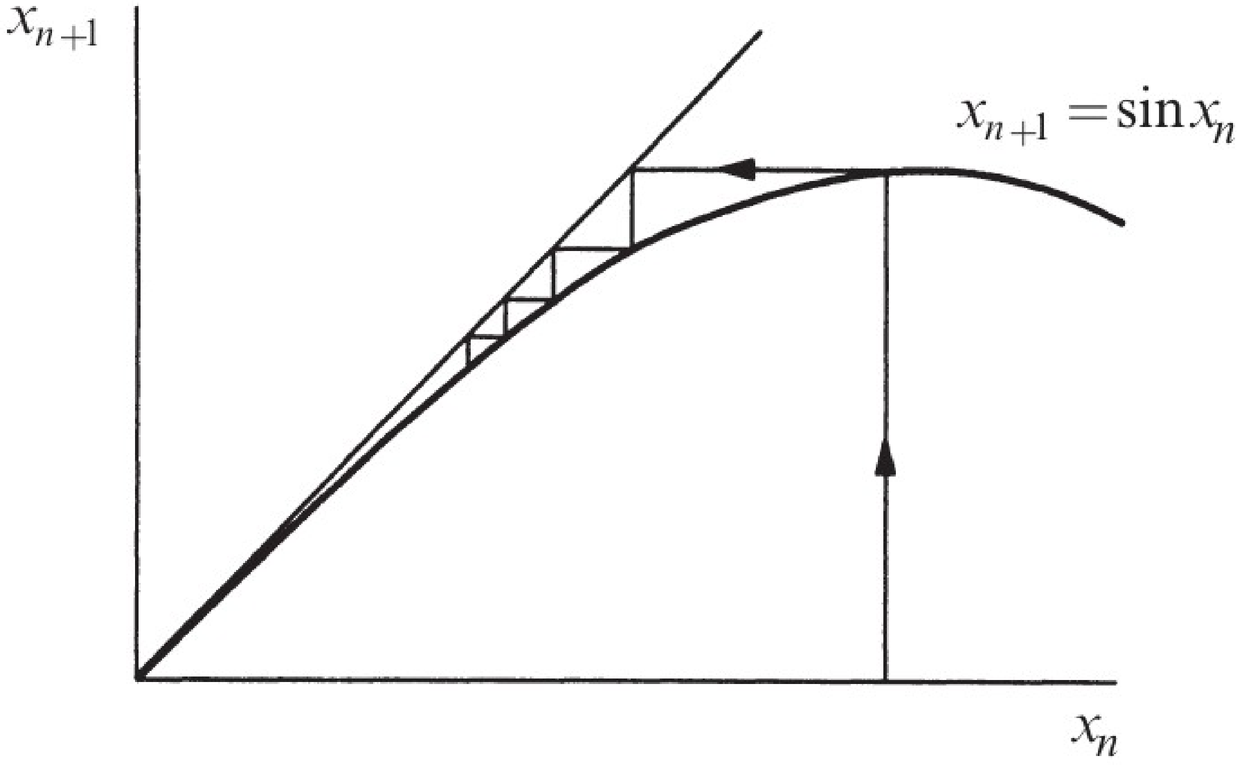
Consider the map $x_{n+1} = \sin x_n$. Show that the stability of the fixed point $x^* = 0$ is not determined by the linearization. Then use a cobweb to show that $x^* = 0$ is stable—in fact, *globally* stable.

The multiplier at $x^* = 0$ is $f'(0) = \cos(0) = 1$,

This is an example of marginal case. The linear stability analysis is inconclusive. However, we can show using the cobweb digram that all the initial condition finally converges to x=0 solution. Hence x=0 is locally stable.

One-dimensional Map

Example:

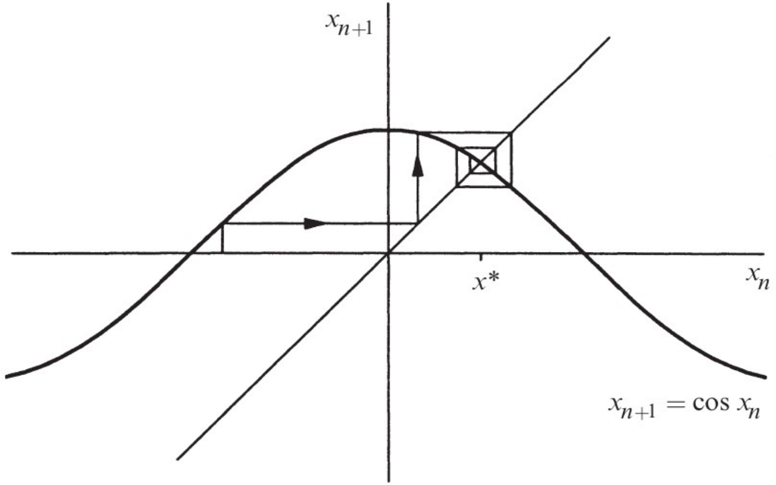


One-dimensional Map

Example:

Given $x_{n+1} = \cos x_n$, how does x_n behave as $n \to \infty$?

$$x^* = 0.739\ldots \text{ as } n \to \infty.$$



The spiraling motion implies that x_n converges to x^* through *damped oscillations*. That is characteristic of fixed points with $\lambda < 0$. In contrast, at stable fixed points with $\lambda > 0$ the convergence is monotonic.

One-dimensional Map

We begin with the simplest model of population growth. Suppose, for example, a population increases by 15 percent each year. Let p_n be the population at the end of year n, and assume that p_0 is given. Then an increase of 15 percent each year gives

$$p_{n+1} = 1.15 p_n.$$

If $p_0 = 100$, then $p_1 = 1.15(100) = 115$, $p_2 = 1.15(115) = 132.3$, $p_3 = 1.15(132.3) = 152.1$,

n	p_n	n	p_n
0	100.0	5	201.1
1	115.0	6	231.3
2	132.3	7	266.0
3	152.1	8	305.9
4	174.9	9	351.8

One-dimensional Map

We derive a formula for p_n by noting that

$$
\begin{aligned}
p_1 &= 1.15p_0 \\
p_2 &= 1.15p_1 = (1.15)^2 p_0 \\
p_3 &= 1.15p_2 = (1.15)^3 p_0 \\
&\vdots \\
p_n &= (1.15)^n p_0
\end{aligned}
$$

Thus we have the familiar "exponential growth" of the population.

More generally, the same argument shows that the solution to

$$p_{n+1} = a p_n,$$

given p_0, is

$$p_n = a^k p_0.$$

One-dimensional Map

Continuous model of population growth:

$$\frac{dp}{dt} = rp, \qquad p(0) = p_0,$$

which has the solution

$$p(t) = p_0 e^{rt} = (e^r)^t p_0.$$

The term e^r is analogous to a, and t is analogous to n.

Another way the discrete model can be written as

$$p_{n+1} = (1 + 0.15)p_n = p_n + 0.15p_n$$

$$p_{n+1} - p_n = 0.15p_n.$$

One-dimensional Map

Logistic Map:

In a fascinating and influential review article, Robert May (1976) emphasized that even simple nonlinear maps could have very complicated dynamics. The article ends memorably with "an evangelical plea for the introduction of these difference equations into elementary mathematics courses, so that students' intuition may be enriched by seeing the wild things that simple nonlinear equations can do."

May illustrated his point with the ***logistic map***

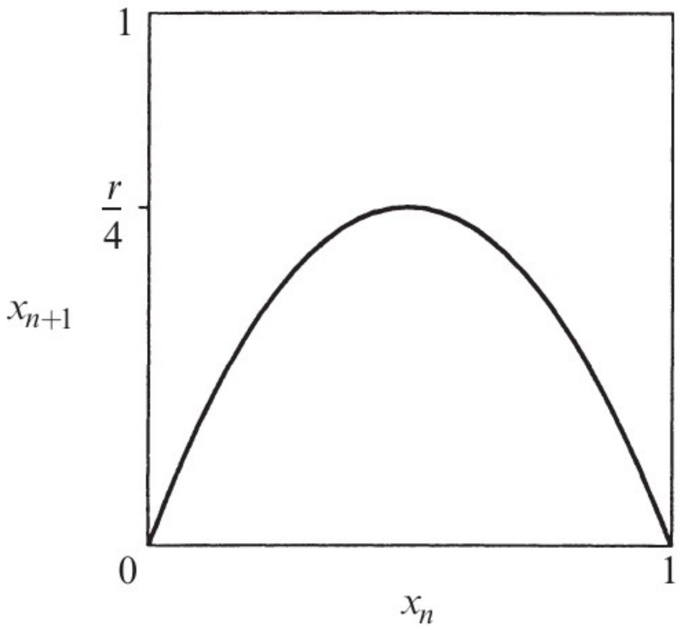
$$x_{n+1} = r x_n (1 - x_n),$$

a discrete-time analog of the logistic equation for population growth

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

One-dimensional Map

Logistic Map:



$$0 \le r \le 4$$

maps the interval $0 \le x \le 1$

The fixed points satisfy $x^* = f(x^*) = rx^*(1 - x^*)$. Hence $x^* = 0$ or $1 = r(1 - x^*)$, i.e., $x^* = 1 - \frac{1}{r}$. The origin is a fixed point for all r, whereas $x^* = 1 - \frac{1}{r}$ is in the range of allowable x only if $r \ge 1$.

One-dimensional Map

Logistic Map:

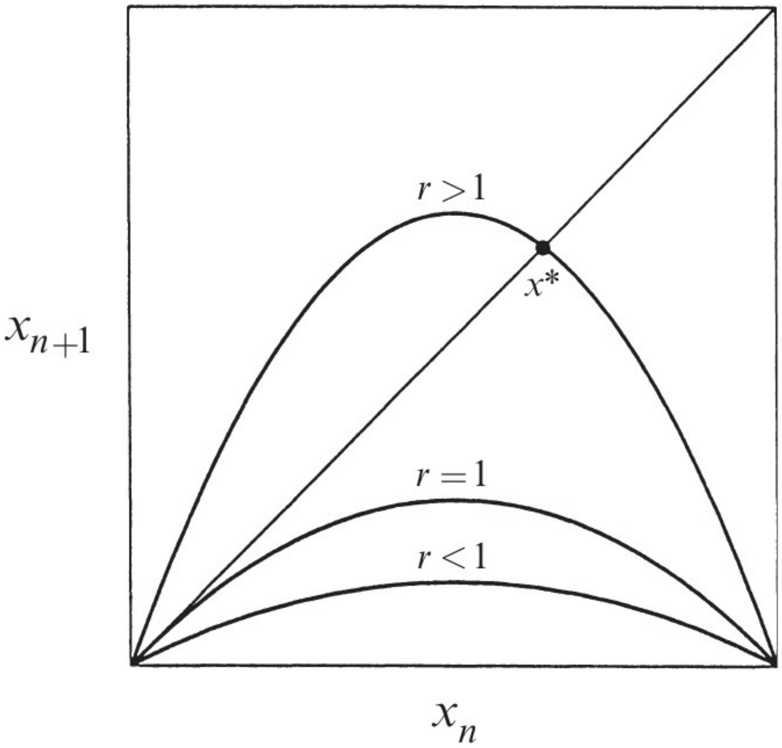
$$f'(x^*) = r - 2rx^*.$$

Since $f'(0) = r$, the origin is stable for $r < 1$ and unstable for $r > 1$. At the other fixed point,

$f'(x^*) = r - 2r(1 - \frac{1}{r}) = 2 - r$. Hence $x^* = 1 - \frac{1}{r}$ is stable for $-1 < (2 - r) < 1$, i.e., for $1 < r < 3$. It is unstable for $r > 3$.

One-dimensional Map

Logistic Map:



For $r > 1$

$x^* = 1 - \frac{1}{r}$, while the origin loses stability.

transcritical bifurcation at $r = 1$

$f'(x^*) = -1$ is attained when $r = 3$.

Flip bifurcation (period-doubling!)

One-dimensional Map

Logistic Map:

What happens if we fix r, choose some initial condition and generate some susequent x_n

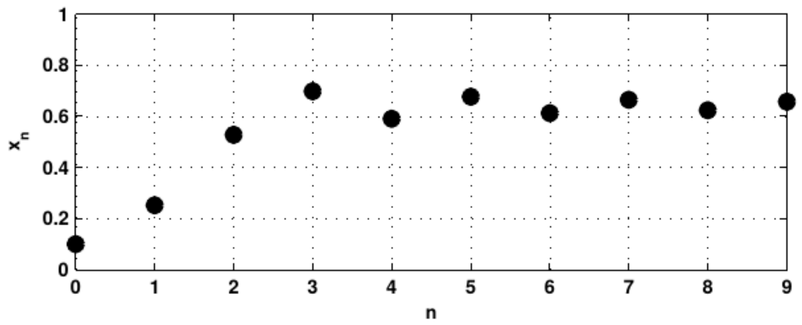
For small growth rate $r < 1$, the population always goes extinct: $x_n \to 0$ as

$$n \to \infty.$$

For $1 < r < 3$ the population grows and eventually reaches a nonzero steady state

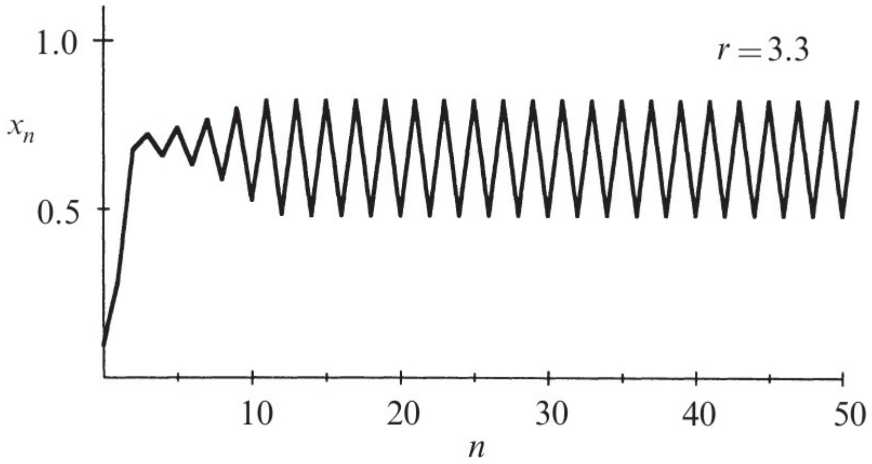
$$r = 2.8 \text{ and } x_0 = 0.1.$$

n	x_n	n	x_n
0	0.1	5	0.6771
1	0.2520	6	0.6122
2	0.5278	7	0.6648
3	0.6978	8	0.6240
4	0.5904	9	0.6570



One-dimensional Map

Logistic Map:

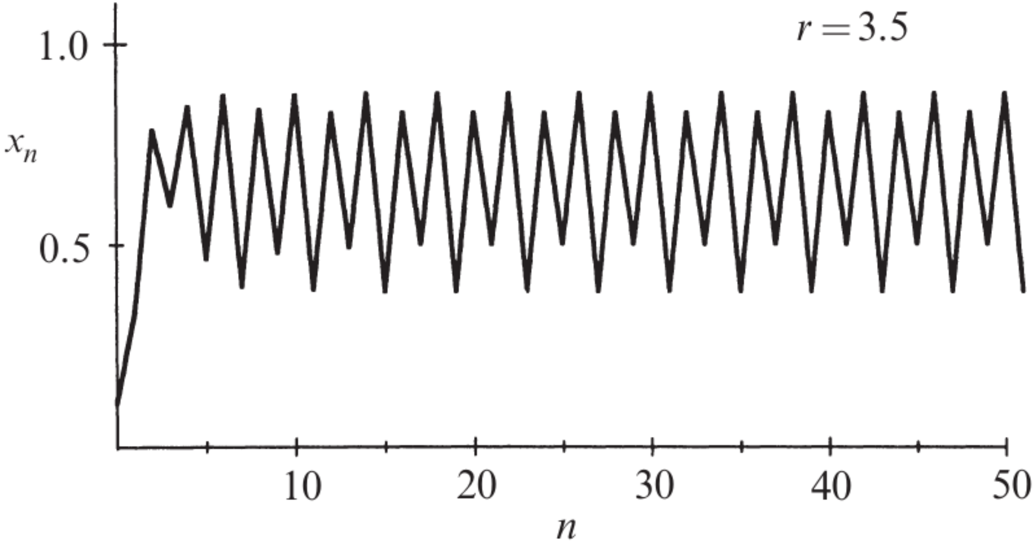


the population builds up again but now *oscillates* about the former steady state, alternating between a large population in one generation and a smaller population in the next

x_n repeats every *two* iterations, is called a ***period-2 cycle***.

One-dimensional Map

Logistic Map:



period-4

In the steady state repetition happens after four iteration.

One-dimensional Map

Logistic Map:

$$
\begin{array}{ll}
r_1 = 3 & \text{(period 2 is born)} \\
r_2 = 3.449\ldots & 4 \\
r_3 = 3.54409\ldots & 8 \\
r_4 = 3.5644\ldots & 16 \\
r_5 = 3.568759\ldots & 32 \\
\vdots & \vdots \\
r_\infty = 3.569946\ldots & \infty
\end{array}
$$

Represents the chaotic state.

One-dimensional Map

Logistic Map:

Show that the logistic map has a 2-cycle for all $r > 3$.

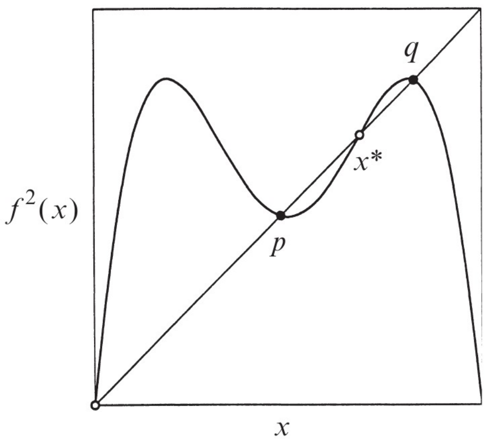
A 2-cycle exists if and only if there are two points p and q such that $f(p) = q$ and $f(q) = p$. Equivalently, such a p must satisfy $f(f(p)) = p$, where $f(x) = rx(1 - x)$. Hence p is a fixed point of the ***second-iterate map***

$f^2(x) \equiv f(f(x))$. Since $f(x)$ is a quadratic polynomial,

$f^2(x)$ **is a quartic polynomial.**

One-dimensional Map

Logistic Map:



We outline the algebra involved in the rest of the solution. Expansion of the equation $f^2(x) - x = 0$ gives $r^2x(1-x)[1-rx(1-x)] - x = 0$. After factoring out x and $x-(1-\frac{1}{r})$ by long division, and solving the resulting quadratic equation, we obtain a pair of roots

$$p, q = \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r},$$

which are real for $r > 3$. Thus a 2-cycle exists for all $r > 3$

One-dimensional Map

Logistic Map:

Stability of the 2-cycle solution: We compute the stability multiplier as

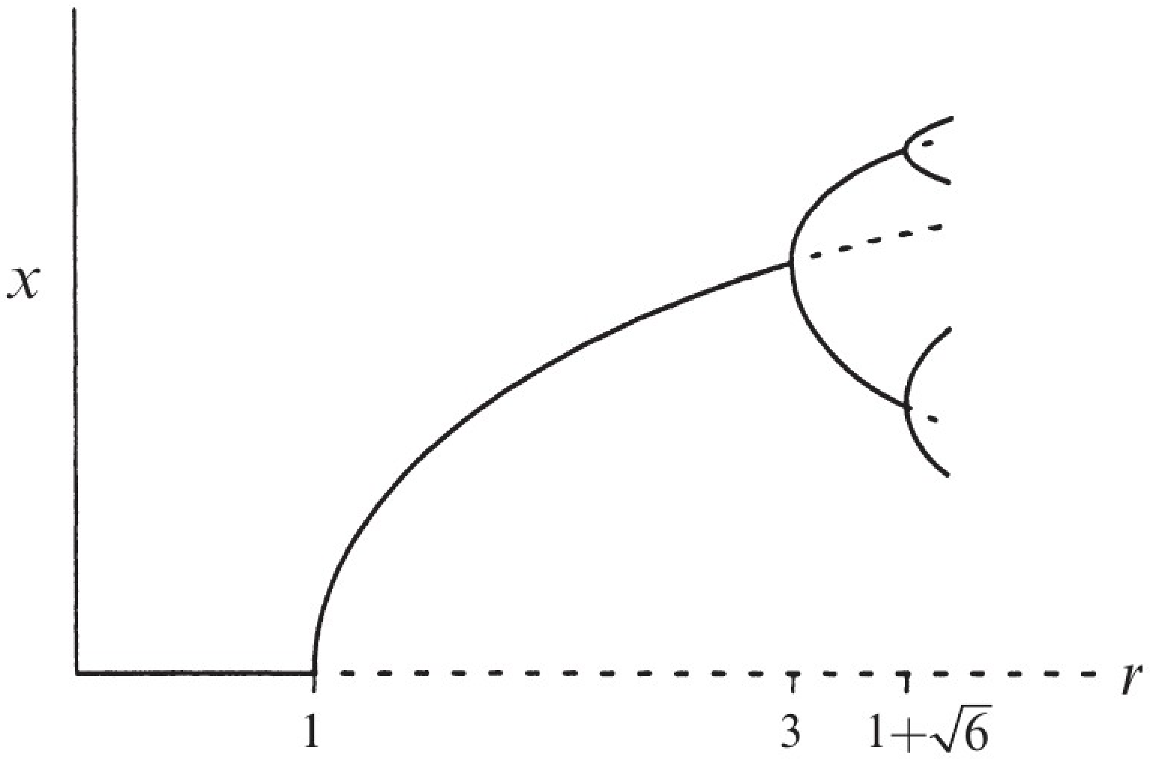
$$\lambda = \frac{d}{dx}(f(f(x)))_{x=p} = f'(f(p))f'(p) = f'(q)f'(p).$$

$$\begin{aligned}
\lambda &= r(1-2q)\,r(1-2p) \\
&= r^2\left[1-2(p+q)+4pq\right] \\
&= r^2\left[1-2(r+1)/r+4(r+1)/r^2\right] \\
&= 4+2r-r^2.
\end{aligned}$$

Therefore the 2-cycle is linearly stable for $\left|4+2r+r^2\right|<1$, i.e., for $3<r<1+\sqrt{6}$.

One-dimensional Map

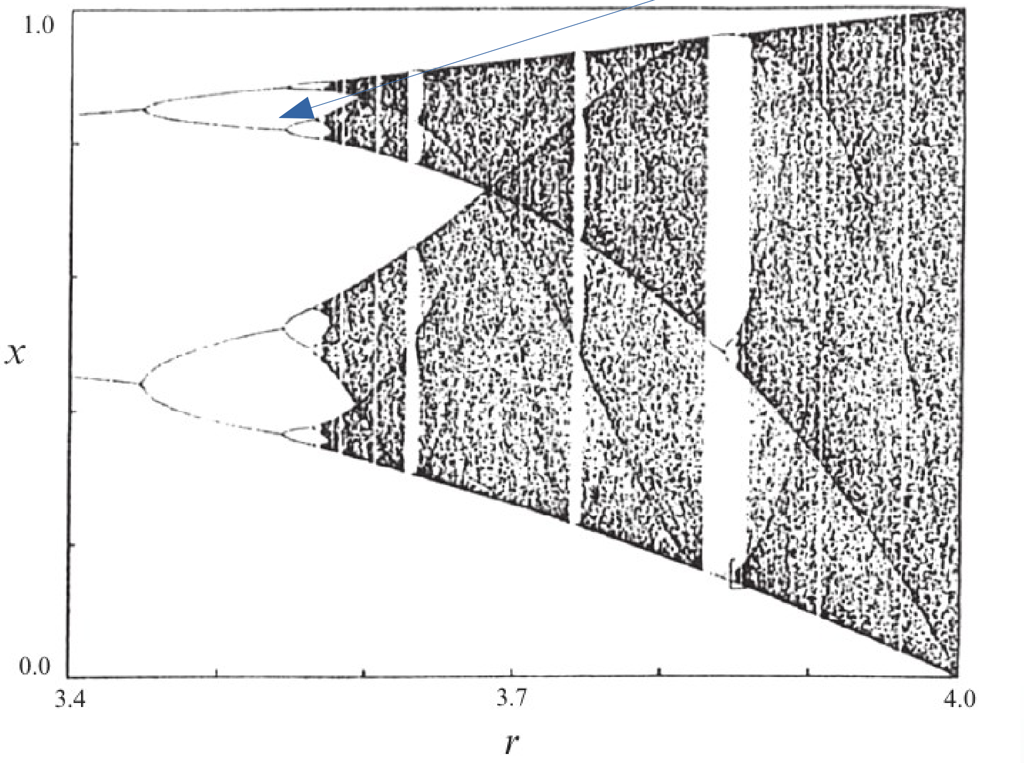
Logistic Map:



One-dimensional Map

Logistic Map:

Period four branch



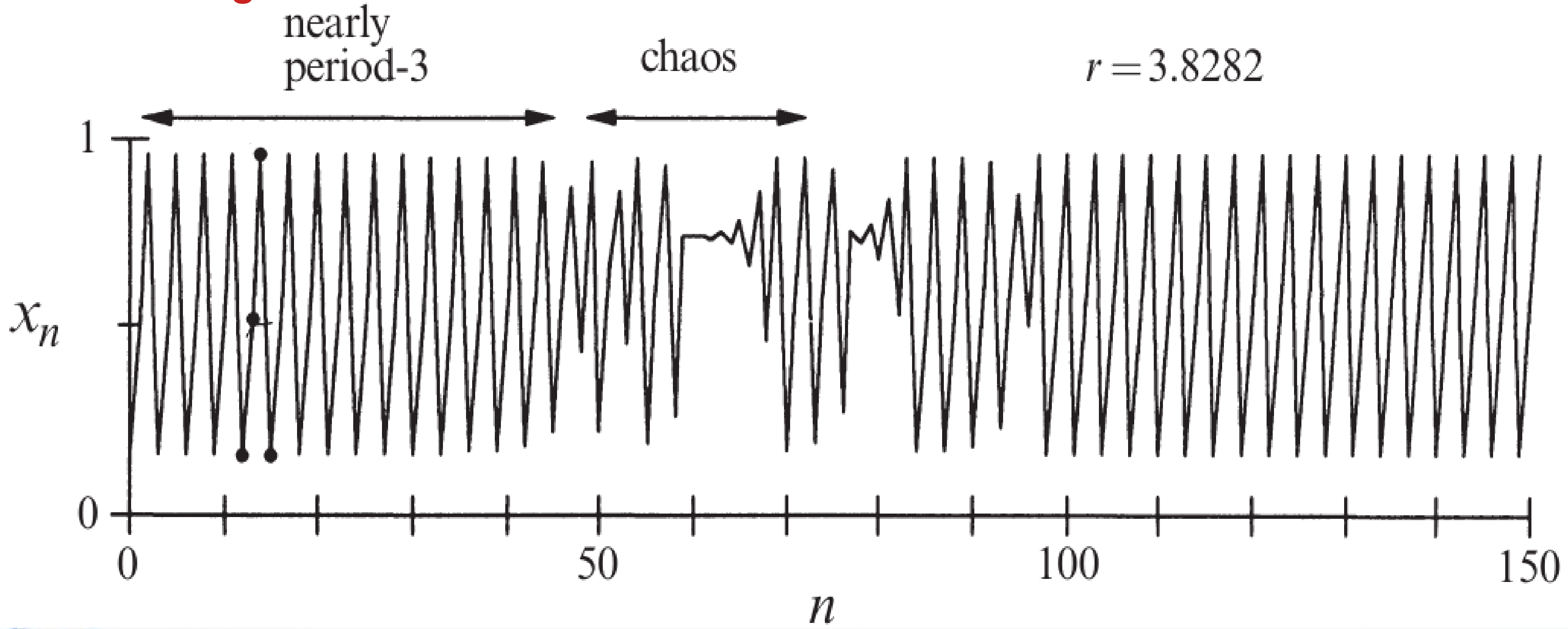
$$3.8284\ldots \le r \le 3.8415 .$$

Period-3 windows!!

One-dimensional Map

Logistic Map:

For r just below the period-3 window, the system exhibits an interesting kind of chaos!



One-dimensional Map

Lyapunov exponent: A tool to characterize the chaotic bhevaiour.

We have seen that the logistic map can exhibits the aperiodic behaviour in some parameter range. **The question arises how will we know that it is chaotic regime?**

One way for this would be to show the system inherits the dependence on the initial condition. The same idea can be quantified using the Lyapunov exponents!

Given some initial condition x_0, consider a nearby point $x_0 + \delta_0$, where the initial separation δ_0 is extremely small. Let δ_n be the separation after n iterates. If $|\delta_n| \approx |\delta_0| e^{n\lambda}$, then λ is called the Liapunov exponent. A positive Liapunov exponent is a signature of chaos.

$$\delta_n = f^n(x_0 + \delta_0) - f^n(x_0)$$

One-dimensional Map

Lyapunov exponent: A tool to characterize the chaotic bhevaiour.

$$\lambda \approx \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right|$$

$$= \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right|$$

$$= \frac{1}{n} \ln \left| (f^n)'(x_0) \right|$$

where we've taken the limit $\delta_0 \to 0$ in the last step. The term inside the logarithm can be expanded by the chain rule:

$$(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i) \quad .$$

One-dimensional Map

Lyapunov exponent: A tool to characterize the chaotic bhevaiour.

$$\lambda \approx \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right|$$
$$= \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|.$$

If this expression has a limit as $n \to \infty$, we define that limit to be the ***Liapunov exponent*** for the orbit starting at x_0:

$$\lambda = \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \right\}.$$

Note that λ depends on x_0. However, it is the same for all x_0 in the basin of attraction of a given attractor. For stable fixed points and cycles, λ is negative; for chaotic attractors, λ is positive.

Example:

Suppose that f has a stable p-cycle containing the point x_0. Show that the Liapunov exponent $\lambda < 0$. If the cycle is superstable, show that $\lambda = -\infty$.

By assumption, the cycle is stable; hence the multiplier $\left|(f^p)'(x_0)\right| < 1$.

$$\ln\left|(f^p)'(x_0)\right| < \ln(1) = 0$$

Next observe that for a p-cycle,

$$\lambda = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\ln\left|f'(x_i)\right|\right\}$$
$$= \frac{1}{p}\sum_{i=0}^{p-1}\ln\left|f'(x_i)\right|$$

One-dimensional Map

$$\frac{1}{p}\sum_{i=0}^{p-1}\ln\left|f'(x_i)\right| = \frac{1}{p}\ln\left|(f^p)'(x_0)\right| < 0,$$

If the cycle is superstable, then $\left|(f^p)'(x_0)\right| = 0$ by definition

and thus $$\lambda = \frac{1}{p}\ln(0) = -\infty.$$

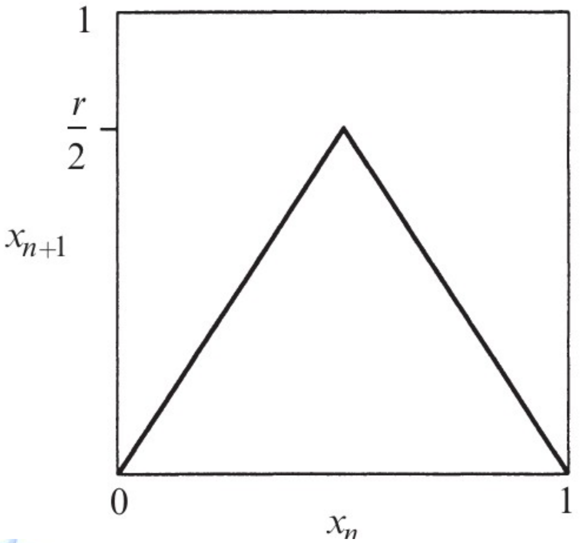
Example:

Tent Map

$$f(x) = \begin{cases} rx, & 0 \le x \le \frac{1}{2} \\ r - rx, & \frac{1}{2} \le x \le 1 \end{cases}$$

for $0 \le r \le 2$ and $0 \le x \le 1$

One-dimensional Map



$$f(x) = \begin{cases} rx, & 0 \le x \le \frac{1}{2} \\ r - rx, & \frac{1}{2} \le x \le 1 \end{cases}$$

Because it is piecewise linear, the tent map is far easier to analyze than the logistic map.

Since $f'(x) = \pm r$ for all x, we find $\lambda = \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln\left| f'(x_i) \right| \right\} = \ln r.$

The ***Rössler system*** is

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

where a, b, and c are parameters.

This system contains only one nonlinear term, zx, and is even simpler than the Lorenz system

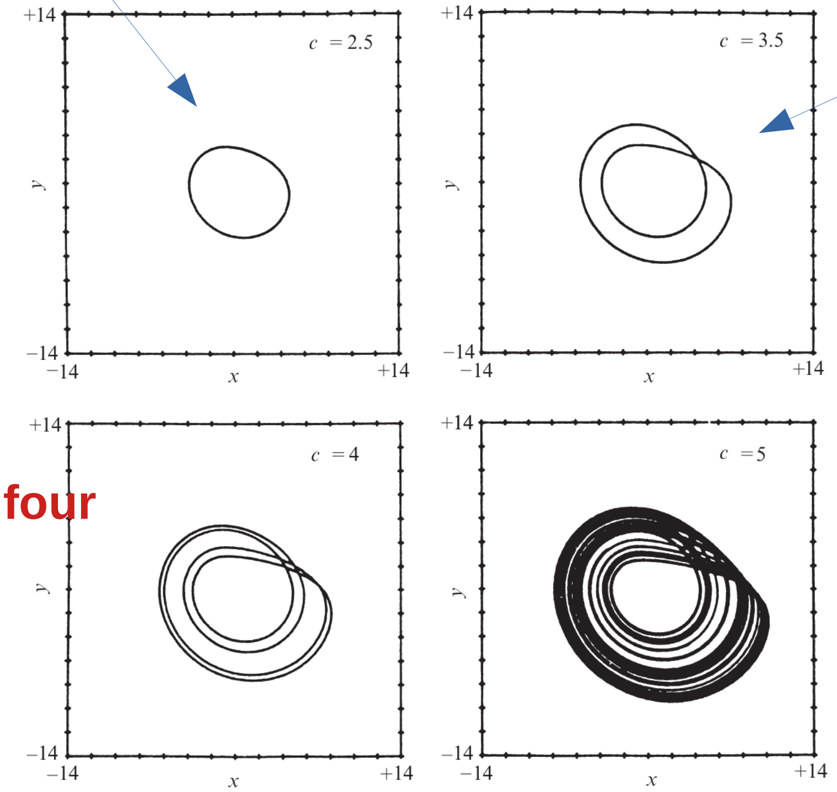
One-dimensional Map

Limit cycle

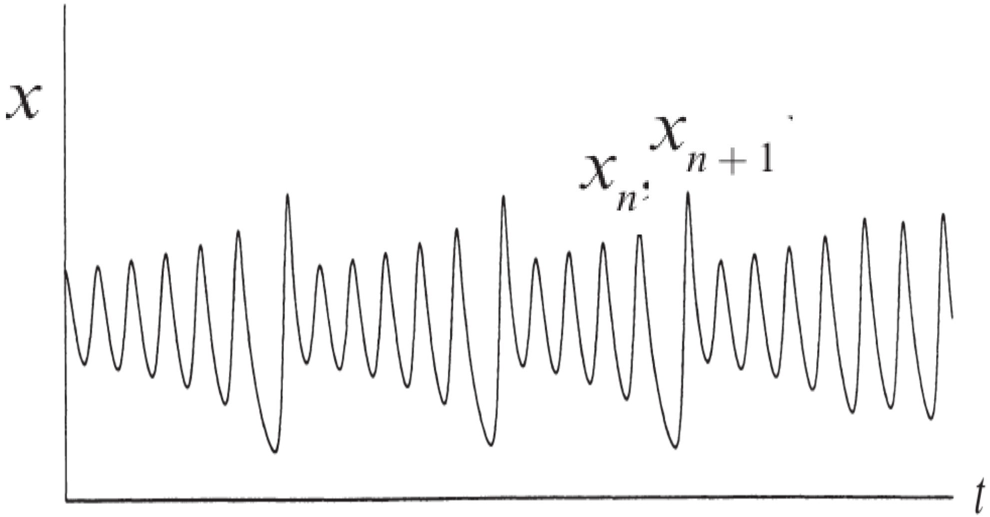
$$a = b = 0.2$$

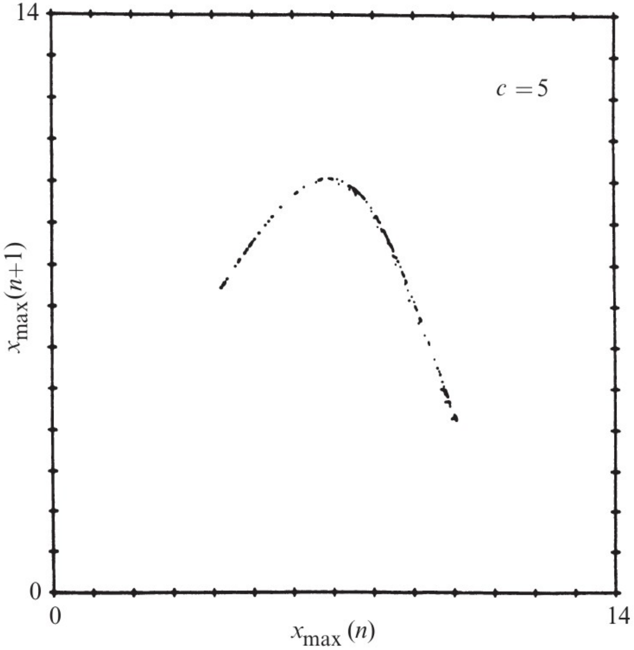
Period doubling

Period four



One-dimensional Map

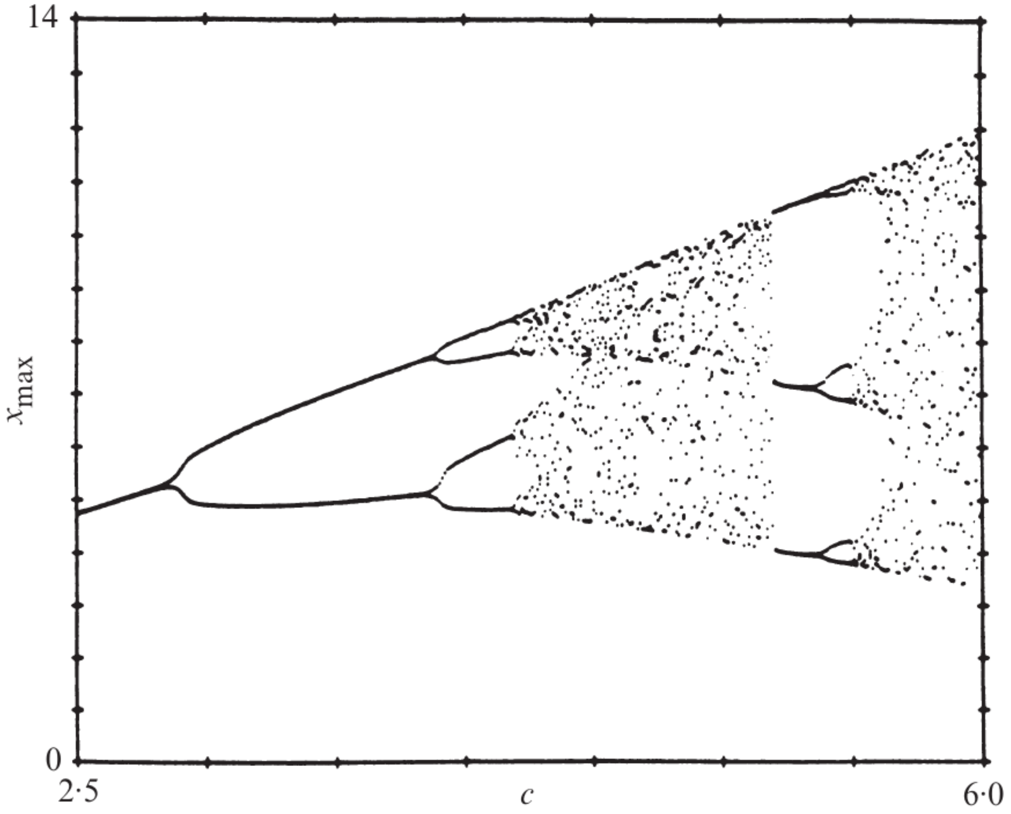




Looks very much like logistic map!

One-dimensional Map

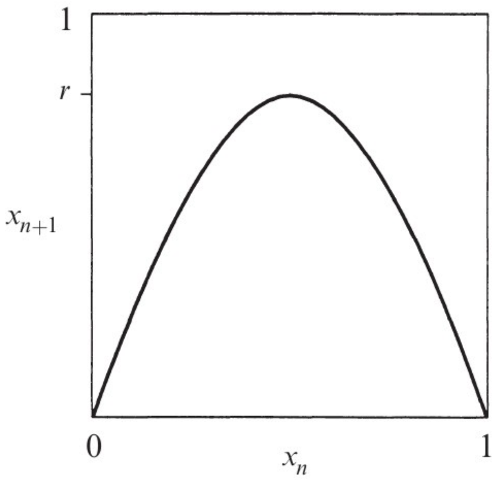
Bifurcation diagram of Rossler’s system.



One-dimensional Map

sine map

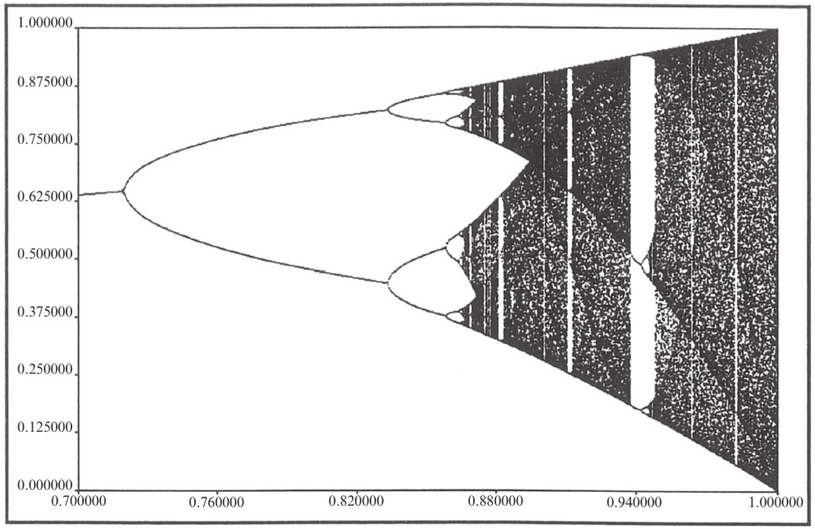
$$x_{n+1} = r\sin\pi x_n \text{ for } 0 \le r \le 1 \text{ and } 0 \le x \le 1$$

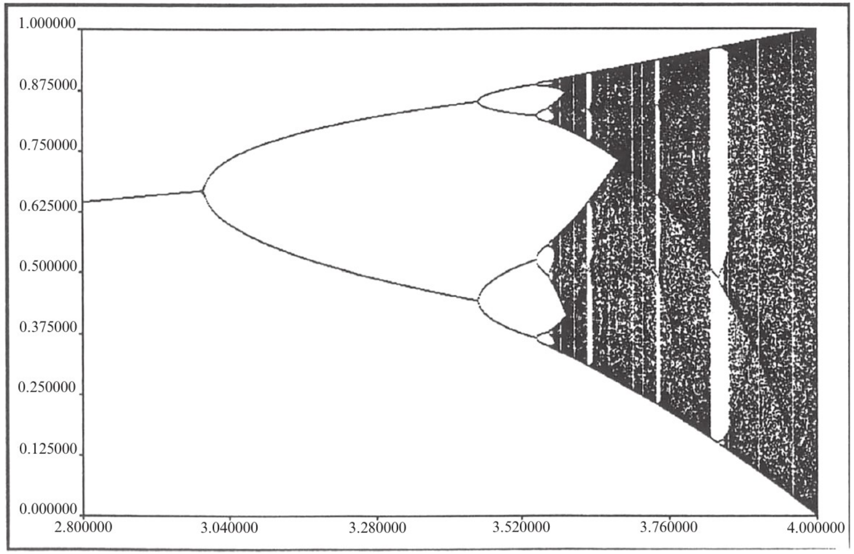


It has the same shape as the graph of the logistic map. Both curves are smooth, concave down, and have a single maximum. Such maps are called ***unimodal***.

Resemblance in the bifurcation between the Sin map and logistic map

Sin Map

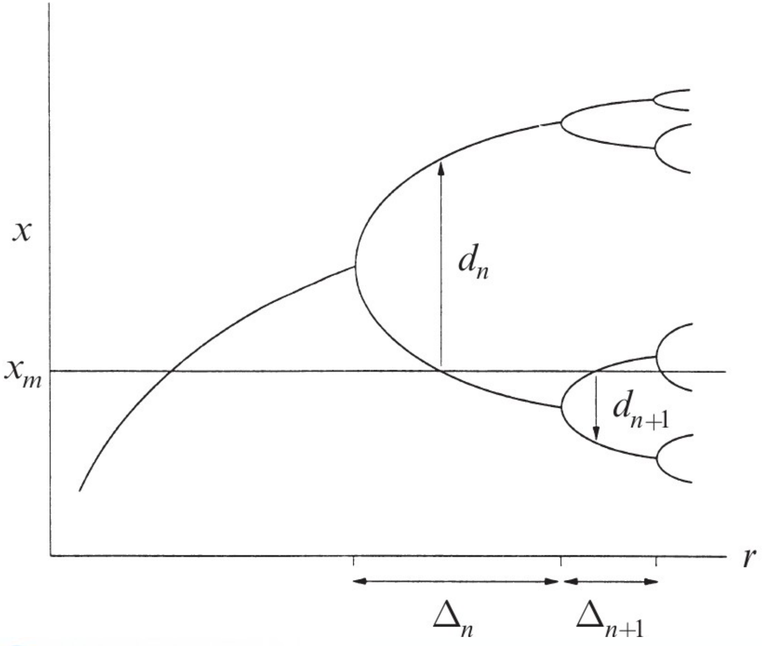




Logistic map

They both undergo period-doubling routes to chaos, followed by periodic windows interwoven with chaotic bands. Even more remarkably, the periodic windows occur in the same order, and with the same relative sizes. For instance, the period-3 window is the largest in both cases, and the next largest windows preceding it are period-5 and period-6.

Feigenbaum universal theory for the period doubling route to chaos



$$\Delta_n = r_n - r_{n+1}$$

Distance between two consecutive bifurcations

Universal number

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

$$\Delta_n / \Delta_{n+1} \to \delta$$

$$\frac{d_n}{d_{n+1}} \to \alpha = -2.5029\ldots,$$

Feigenbaum universal theory for the period doubling route to chaos

Feigenbaum went on to develop a beautiful theory that explained why α and δ are universal (Feigenbaum 1979). He borrowed the idea of renormalization from statistical physics, and thereby found an analogy between α, δ and the universal exponents observed in experiments on second-order phase transitions in magnets.

Renormalization

First we introduce some notation. Let $f(x,r)$ denote a unimodal map that undergoes a period-doubling route to chaos as r increases, and suppose that x_m is the maximum of f. Let r_n denote the value of r at which a 2^n-cycle is born, and let R_n denote the value of r at which the 2^n-cycle is superstable.

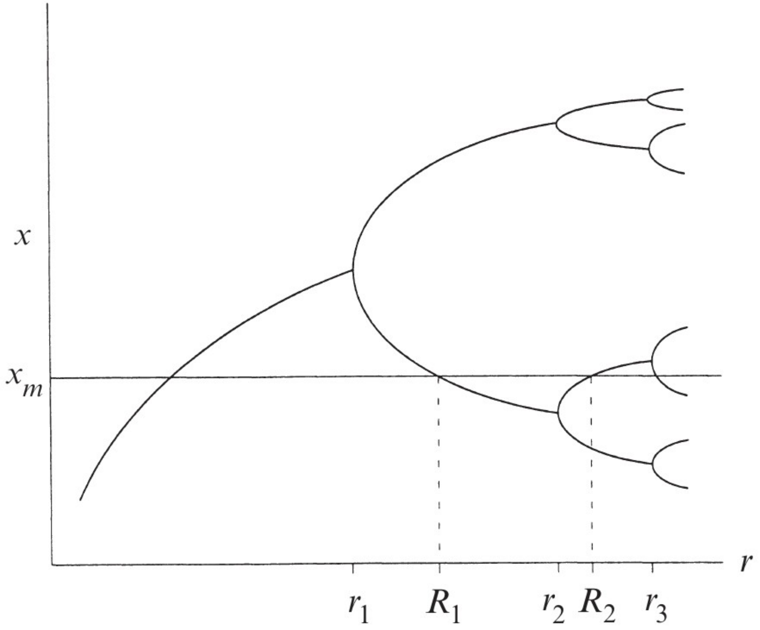
Feigenbaum universal theory for the period doubling route to chaos

Find R_0 and R_1 for the map $f(x,r) = r - x^2$.

Solution: At R_0 the map has a superstable fixed point, by definition. The fixed point condition is $x^* = R_0 - (x^*)^2$ and the superstability condition is $\lambda = (\partial f/\partial x)_{x=x^*} = 0$. Since $\partial f/\partial x = -2x$, we must have $x^* = 0$, i.e., the fixed point is the maximum of f. Substituting $x^* = 0$ into the fixed point condition yields $R_0 = 0$.

At R_1 the map has a superstable 2-cycle. Let p and q denote the points of the cycle. Superstability requires that the multiplier $\lambda = (-2p)(-2q) = 0$, so the point $x = 0$ must be one of the points in the 2-cycle. Then the period-2 condition $f^2(0, R_1) = 0$ implies $R_1 - (R_1)^2 = 0$. Hence $R_1 = 1$ (since the other root gives a fixed point, not a 2-cycle). ■

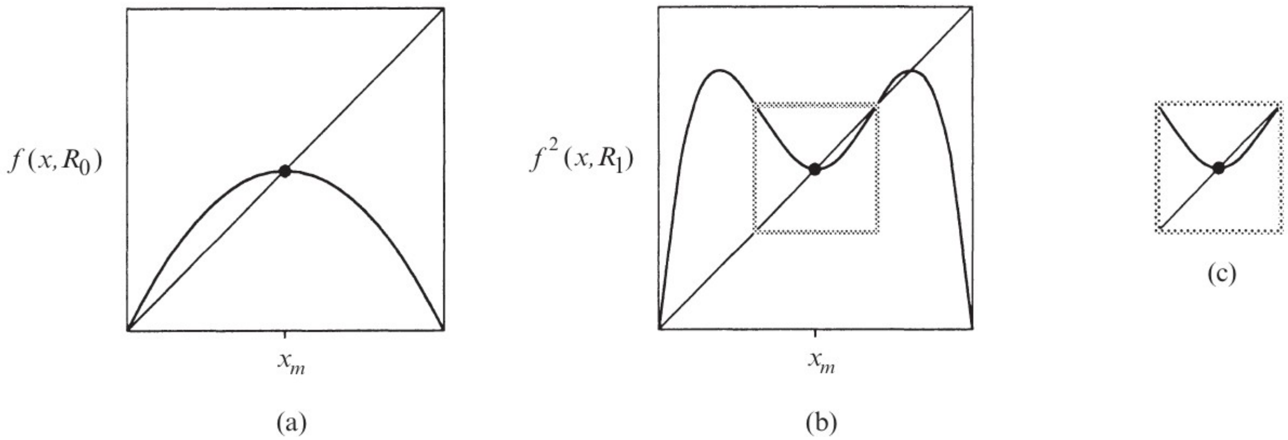
Feigenbaum universal theory for the period doubling route to chaos



A superstable cycle of a unimodal map always contains x_m as one of its points. Draw a horizontal line at x_m and find out its intersection with the bifurcation curve.

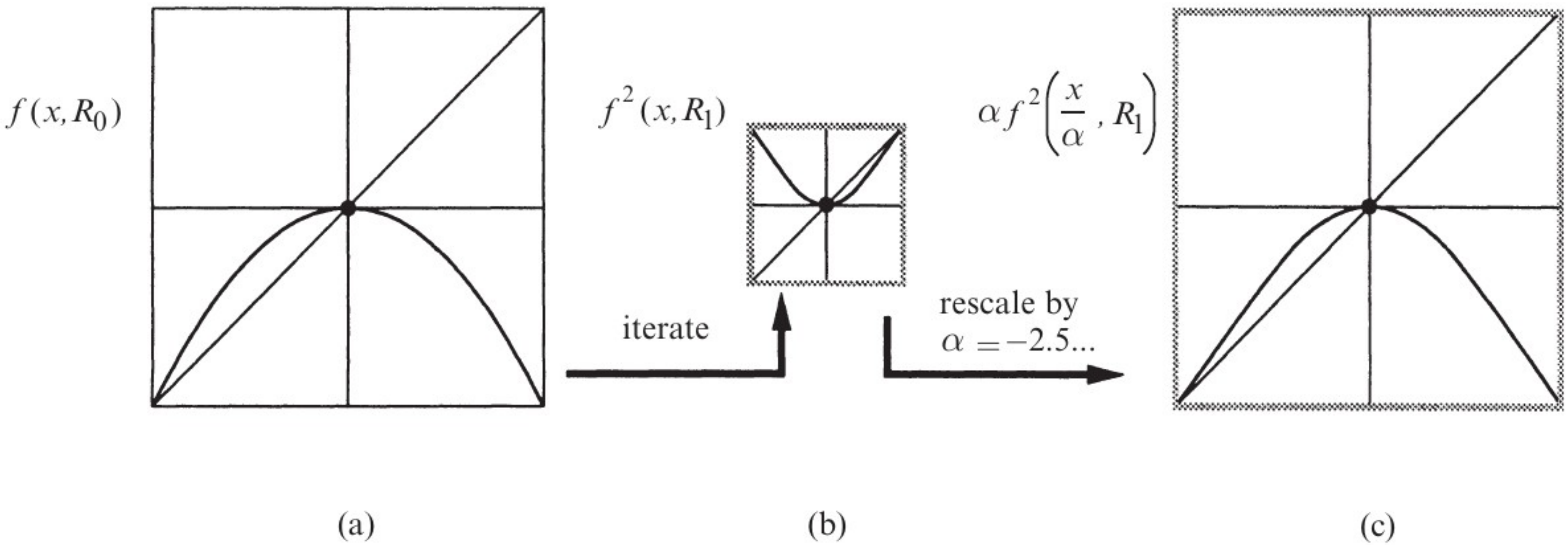
Renormalization group is based on the self similarity of the bifurcation diagram. To express the self similarity mathematically we comapre f with its second iterate f^2 At corresponding values of r, and then renormalize one map into the other.

Feigenbaum universal theory for the period doubling route to chaos



The figure (a) and figure (c) looks same only there is change in the scale of R.
Now task is to make the figure (c) look like figure (a).

Feigenbaum universal theory for the period doubling route to chaos



Next, to make Figure b look like Figure a, we blow it up by a factor $|\alpha| > 1$ in both directions, and also invert it by replacing (x, y) by $(-x, -y)$. Both operations can be accomplished in one step if we define the ***scale factor*** α to be *negative.*

Feigenbaum universal theory for the period doubling route to chaos

The resemblance between the figure a and c suggests that

$$f(x, R_0) \approx \alpha f^2\left(\frac{x}{\alpha}, R_1\right).$$

In summary, f has been ***renormalized*** by taking its second iterate, rescaling $x \to x/\alpha$, and shifting r to the next superstable value.

$$f^2\left(\frac{x}{\alpha}, R_1\right) \approx \alpha f^4\left(\frac{x}{\alpha^2}, R_2\right).$$

Continuing in the similar way we get

$$f(x, R_0) \approx \alpha^n f^{(2^n)}\left(\frac{x}{\alpha^n}, R_n\right).$$

Feigenbaum universal theory for the period doubling route to chaos

Feigenbaum found numerically that

$$\lim_{n\to\infty} \alpha^n f^{(2^n)}\left(\frac{x}{\alpha^n}, R_n\right) = g_0(x),$$

where $g_0(x)$ is a ***universal function*** with a superstable fixed point. The limiting function exists only if α is chosen correctly, specifically, $\alpha = -2.5029\ldots$

Here "universal" means that the limiting function $g_0(x)$ is independent of the original f (almost). This seems incredible at first, but the form of (1) suggests the explanation: $g_0(x)$ depends on f only through its behavior near $x = 0$, since that's all that survives in the argument x/α^n as $n \to \infty$. With each renormalization, we're blowing up a smaller and smaller neighborhood of the maximum of f, so practically all information about the global shape of f is lost.

Fractals

A fractal is a mathematical object that is both self-similar and chaotic.

self-similar: As you magnify, you see the object over and over again in its parts.

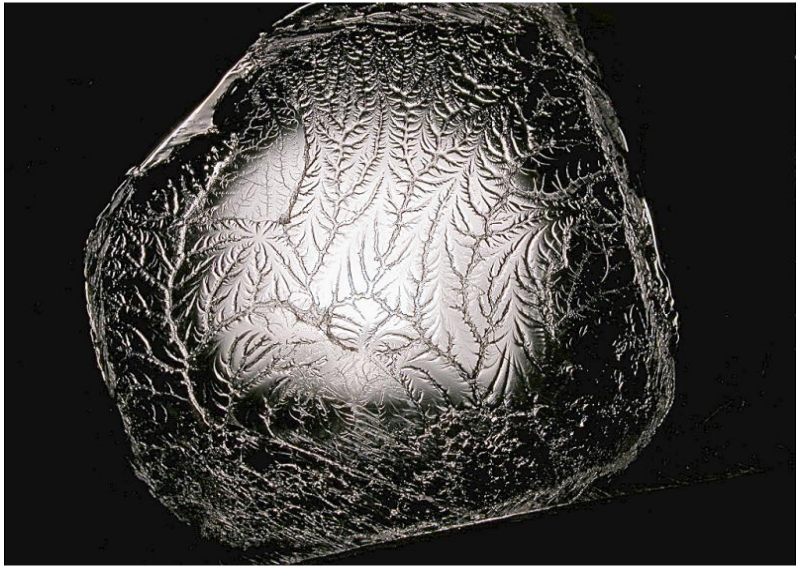
chaotic: Fractals are infinitely complex.

Amazingly, these beautiful objects of breath-taking complexity are generated by relatively simple mathematical processes.

Roughly speaking, ***fractals*** are complex geometric shapes with fine structure at arbitrarily small scales. Usually they have some degree of self-similarity. In other words, if we magnify a tiny part of a fractal, we will see features reminiscent of the whole. Sometimes the similarity is exact; more often it is only approximate or statistical.

Fractals

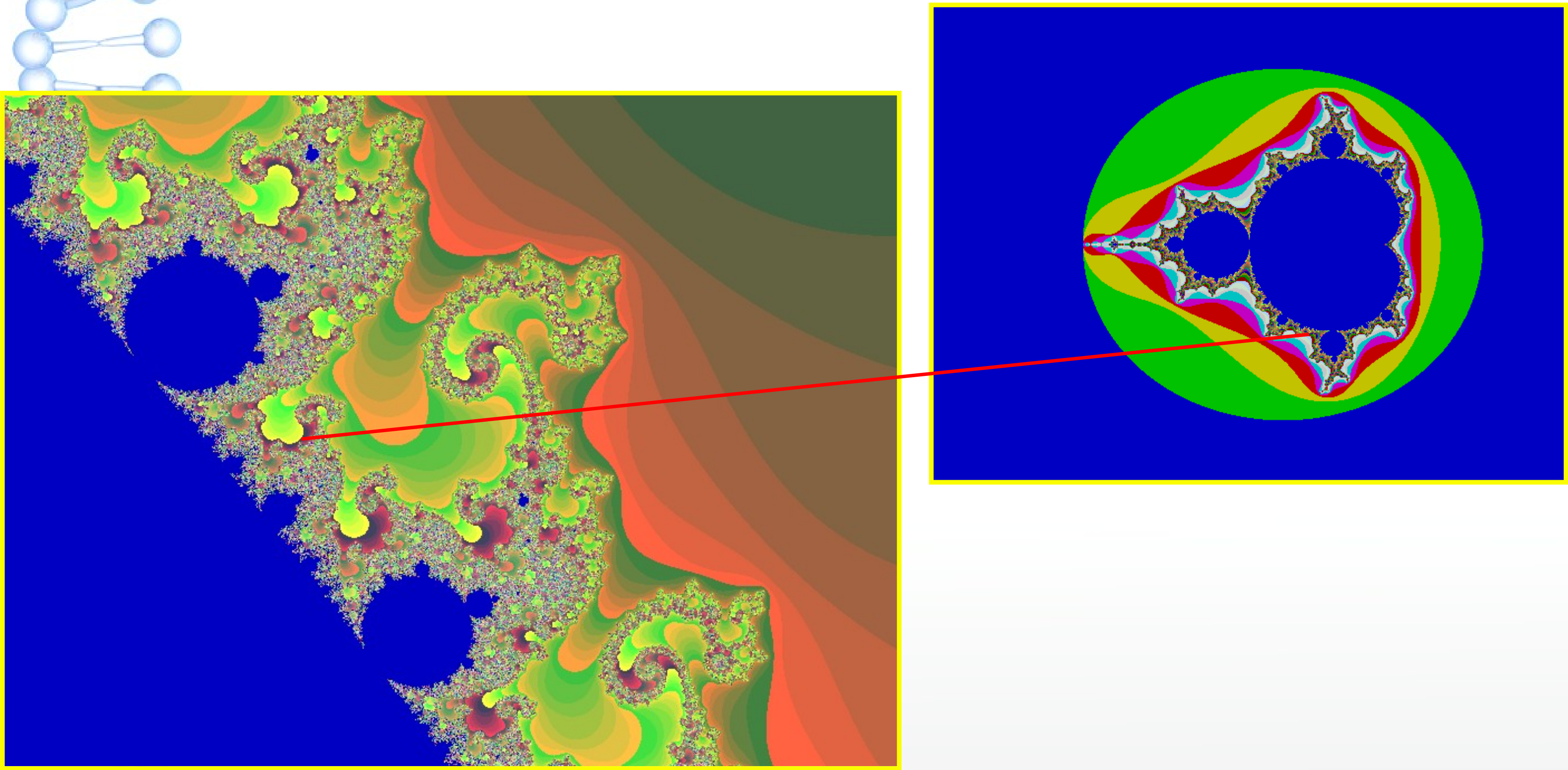
Frost crystals occurring naturally on cold glass form fractal patterns



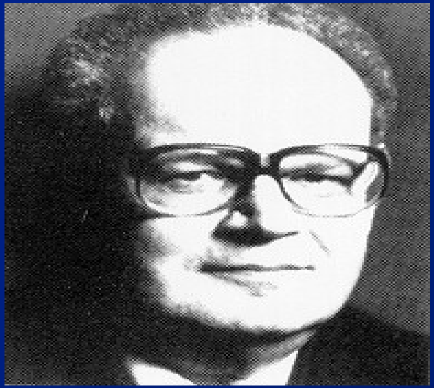
A fractal is formed when pulling apart two glue-covered acrylic sheets

Fractals

The first fractals were discovered by a french Mathematician named Gaston Julia who discovered them decades before the advent of computer graphics.



The Fractal Geometry of Nature



Why is geometry often described as cold and dry? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line...

...Nature exhibits not simply a higher degree but an altogether different level of complexity. The number of distinct scales of length of patterns is for all purposes infinite.

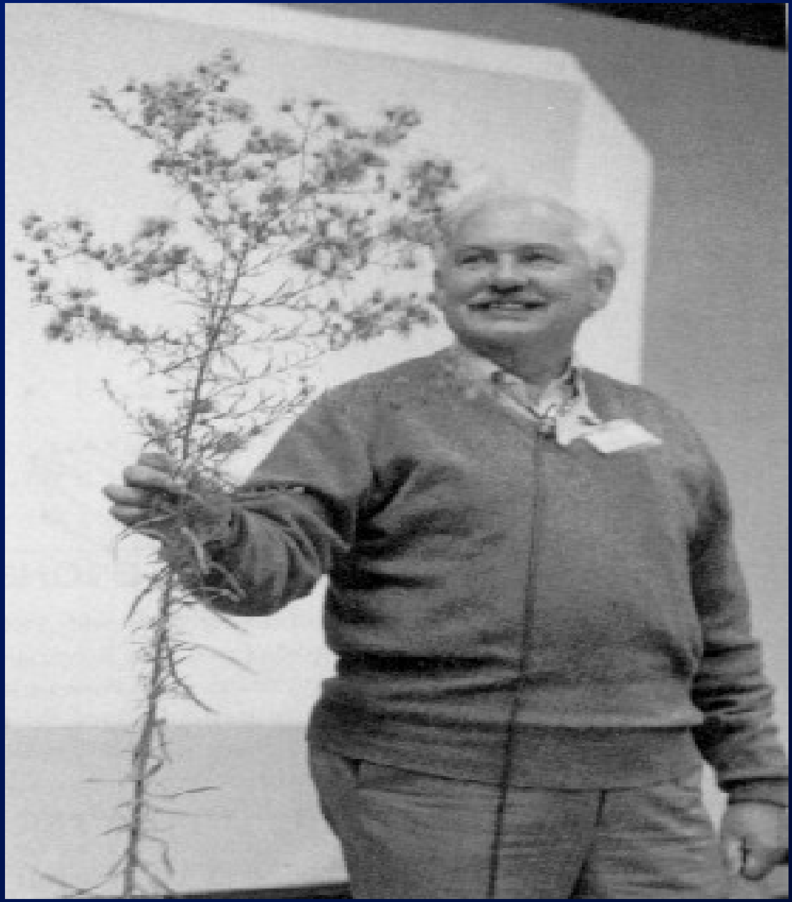
The existence of these patterns challenges us to study those forms that Euclid leaves aside as being formless, to investigate the morphology of the amorphous. Mathematicians have disdained this challenge, however, and have increasingly chosen to flee from nature by devising theories unrelated to anything we can see or feel.

---Benoit Mandelbrot (1984)

Aristid Lindenmeyer

Aristid Lindenmeyer invented L-Systems to model plant growth.

See the “fractal” he is holding?



Assignment 5

Consider the map $x_{n+1} = 3x_n - x_n^3$.

a) Find all the fixed points and classify their stability.

b) Draw a cobweb starting at $x_0 = 1.9$.

c) Draw a cobweb starting at $x_0 = 2.1$.

(Quadratic map) Consider the ***quadratic map*** $x_{n+1} = x_n^2 + c$.

a) Find and classify all the fixed points as a function of c.

b) Find the values of c at which the fixed points bifurcate, and classify those bifurcations.

c) For which values of c is there a stable 2-cycle? When is it superstable?

Assignment 5

(Cubic map) Consider the cubic map $x_{n+1} = f(x_n)$, where $f(x_n) = rx_n - x_n^3$.

a) Find the fixed points. For which values of r do they exist? For which values are they stable?
b) To find the 2-cycles of the map, suppose that $f(p) = q$ and $f(q) = p$. Show that p, q are roots of the equation $x(x^2 - r + 1)(x^2 - r - 1)(x^4 - rx^2 + 1) = 0$ and use this to find all the 2-cycles.
c) Determine the stability of the 2-cycles as a function of r.
d) Plot a partial bifurcation diagram, based on the information obtained.

Fractals

Why the dimensions of the terms in both sides of an equation should be same?

Ans: Because the quantities should not depend upon the unit in which we measure them.

Suppose if you express a physical quantity in terms of the fundamental dimensions what should be the power

$$\dim Q = \mathsf{T}^a \mathsf{L}^b \mathsf{M}^c \mathsf{I}^d \Theta^e \mathsf{N}^f \mathsf{J}^g$$

The values of a, b, c ... are in general integer numbers.

We will see for the object with fractal geometry have the non-integer exponents, i.e., a, b, c ... are non-integers.

Fractals

Lewis Fry Richardson was a scientist interested in the origins of war.[1] Frequently, wars are associated with border disputes, so Richardson was studying various borders in Europe. He happend to notice that the length of the border between Spain and Portugal was given as 987 km in a Spanish encyclopedia and 1214 km (23% longer!) in a Portuguese encyclopedia. Why? Richardson traced the discrepancy to differences in the length of the "scale" that was used to measure the border in the two cases. Since a border can be very irregular (*e.g.*, if it is defined by a river or a coastline), the length we measure will depend on how much detail we account for, and how much detail we just skip over. For example, if you were to measure the length of an

irregular border on a map, you could set a pair of dividers to some suitable small interval ϵ, and step with it along the border. Suppose you took N steps. Then the length of the border is $N\epsilon$. But if you choose a shorter step

Fractals

size ϵ, you will now include some features in the border that were previously bridged by the larger step size. Your estimate of the total length will be larger than the estimate obtained with the bigger step size. In the case of the border between Spain and Portugal, we might expect that the smaller country (Portugal) will measure the length of its border in more detail than the larger country (Spain). If so, we would expect Portugal to estimate a longer border, which indeed they do. In Figure 1, we see a plot of empirical data on the lengths of borders and coastlines from Richardson's paper. The horizontal axis corresponds to the logarithm of the step size ϵ (in km). The step size ranges from about $10^{1.5}$ km $= 32$ km to 10^3 km $= 1000$ km. The vertical axis corresponds to the logarithm of the resulting estimate of the border length. Note that for each of the five borders the data lie on straight lines with negative slopes on this log-log plot. However, each line has a different slope.

Figure 1 also shows data for a smooth circle. Although the estimated circumference of a circle also depends on step size, it eventually levels off at the true circumference as the step size gets very small. This is not true for the coastlines. As the step size gets smaller, more features of the coastline are included. The lengths of the coastline show no sign of leveling off as the step size gets smaller. Richardson concluded that irregular lines such as borders and coastlines do not have an identifiable length. But could there be a way to express the length of such irregular lines in a unique way?

Fractals

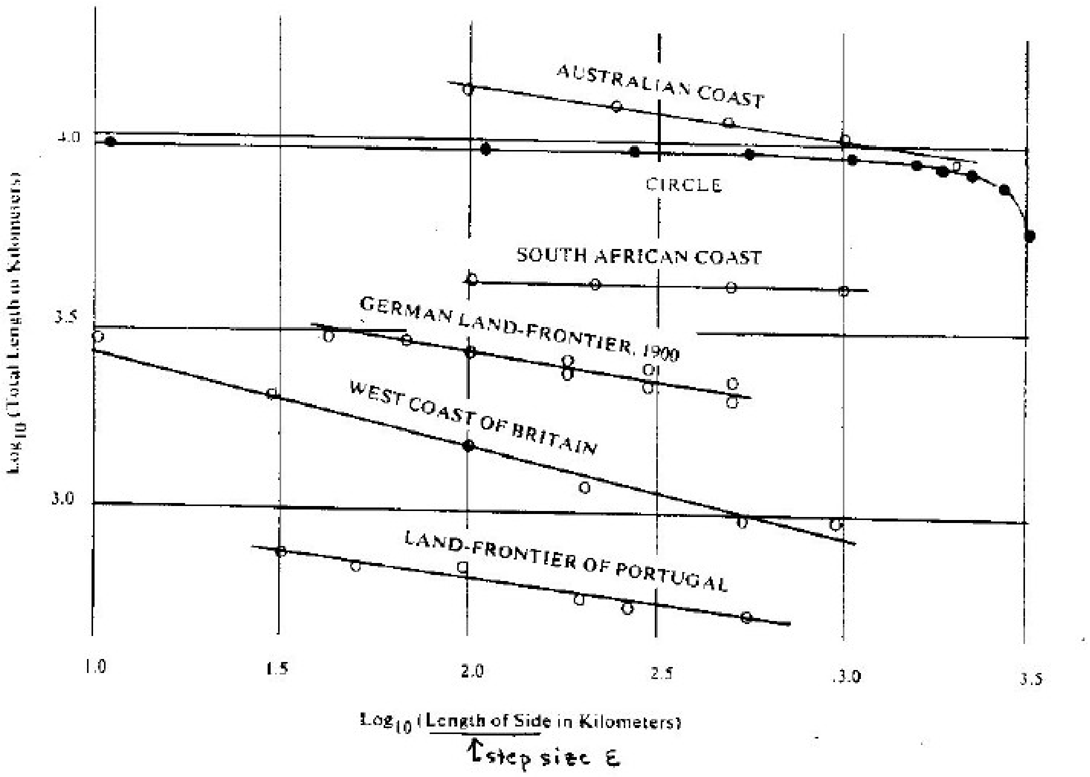


Figure 1

Plate 33 ☐ RICHARDSON'S EMPIRICAL DATA CONCERNING THE RATE OF INCREASE OF COASTLINES' LENGTHS

Fractals

The problem of the dependence of length on measuring scale was taken up by Benoit B. Mandelbrot who then basically founded a new branch of mathematics around 1975. The field is called fractal geometry for reasons that we will see in a moment. Mandelbrot is hailed as the "father of fractals". Mandelbrot addressed the question: if a coastline does not have length, then what property does it have? He sought a single number that expressed how much coastline there is, independent of the step size.

What went wrong in the measurement of the coastal size??

$$\ell = N\epsilon$$

where N is the number of steps. This equation is dimensionally correct since each side has dimensions of $\mathbf{L}^1$. Because it is dimensionally correct, it is independent of the units we use for ϵ. Also, the length of a straight line is independent of the step size ϵ. If we plot ℓ versus ϵ on a log-log plot, what

do we get? Since ℓ does not depend on ϵ, we get a line with zero slope

Fractals

In Figure 1, we saw that what we define as length ($N\epsilon$) is *not* independent of ϵ for coastlines. That is, our equation $\ell = N\epsilon$ does not appear to describe a fundamental characteristic of irregular lines such as coastlines. Perhaps this should give us a clue (from our discussion of dimensional analysis) that something in our problem is not *dimensionally* correct. We have assumed that we can ascribe to these irregular curves a physical quantity with dimensions of length **L**. But we found that the length ℓ depends on our step size ϵ in such a way that the dependence is a straight line (with a negative slope) on a log-log plot, indicating a power-law dependence; *i.e.*, ℓ depends on ϵ to some power. So, this gives us a clue that perhaps we can find a measure of an irregular line that is independent of the step size if we consider the step size ϵ raised to some power. Then the dimensionality of our measure of the irregular line will be length raised to some power.

Fractals

Mandelbrot proposed that the appropriate measure is something that we will call an "extent", given by

$$E = N\epsilon^D,$$

where, as before, N is the number of steps and ϵ is the step size. The dimensionality of the extent E will be $\mathbf{L}^D$. D is not necessarily an integer!

How do we find D? Let's again consider the data in Figure 1. We want to find a measure of length or "extent" that does not depend on the step size ϵ.

Since the length ℓ plotted versus the step size ϵ gives a straight line on a log-log plot, the equation relating ℓ and ϵ can be written

$$\ln \ell = \ln C + k \ln \epsilon$$

Fractals

where C and k are constants. Since the slope is negative, $k < 0$. Now let's rearrange the above equation so that only the constant term $\ln C$ is on the right-hand side. Then we will have a formula involving ℓ and ϵ that is constant — just what we want!

$$\begin{aligned} \ln \ell - k \ln \epsilon &= \ln C \\ \ln \ell \epsilon^{-k} &= \ln C \\ \ell \epsilon^{-k} &= C. \end{aligned}$$

But $\ell = N\epsilon$. Therefore,

$$N\epsilon\epsilon^{-k} = C \quad \text{or} \quad N\epsilon^{1-k} = C.$$

This looks just like Mandelbrot's equation $E = N\epsilon^{D}$ with $D = 1 - k$ and the constant C equal to the extent.

Fractals

What distinguishes a fractal curve (with $1 < D < 2$) from a smooth curve (with $D = 1$)?

One feature is the self-similarity that often appears in fractal objects.

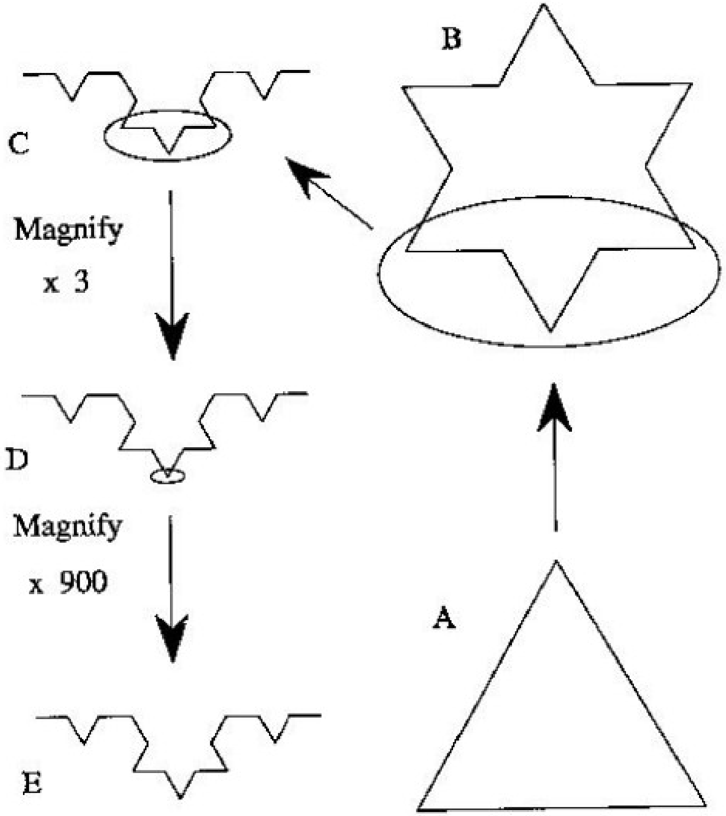
Self-similarity examples:

Koch curve or Koch Island

Start with an equilateral triangle (A). Then cut out the middle one-third of each side, replacing it with two sides of a smaller equilateral triangle, whose sides are one-third as long as the original sides, giving the six-pointed (12-sided) star

The middle one-third of each of the 12 sides is then cut out and replaced by two sides of a still smaller equilateral triangle

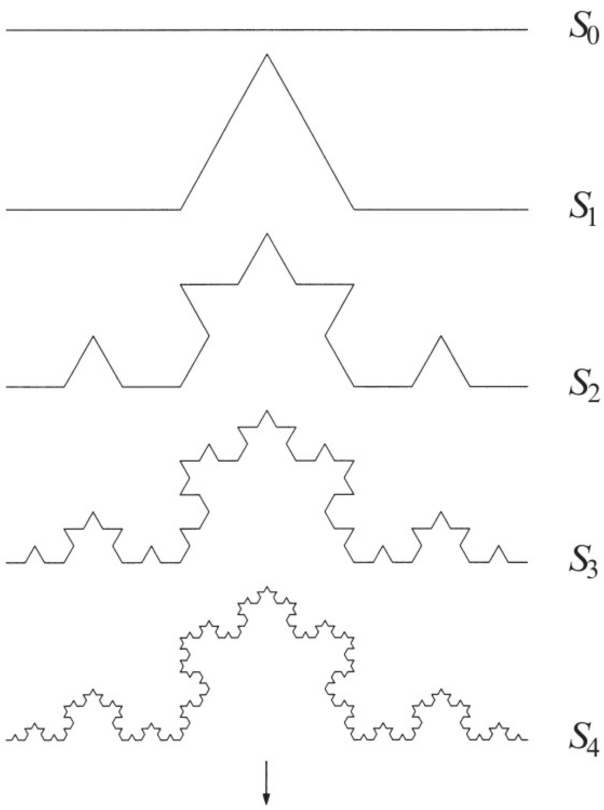
Fractals



Fractal dimension of a Koch island D is

$$\ln 4/\ln 3 \approx 1.26.$$

Fractals



von Koch curve K

What is the dimension of the von Koch curve? Since it's a curve, you might be tempted to say it's one-dimensional. But the trouble is that K has *infinite arc length*!

To see this, observe that if the length of S_0 is L_0, then the length of S_1 is $L_1 = \frac{4}{3}L_0$, because S_1 contains four segments, each of length $\frac{1}{3}L_0$. The length increases by a factor of $\frac{4}{3}$ at each stage of the construction, so $L_n = (\frac{4}{3})^n L_0 \to \infty$ as $n \to \infty$.

Moreover, the arc length between *any* two points on K is infinite, by similar reasoning. Hence points on K aren't determined by their arc length from a particular point, because every point is infinitely far from every other!

Fractals

This suggests that K is more than one-dimensional. But would we really want to say that K is two-dimensional? It certainly doesn't seem to have any "area." So the dimension should be *between* 1 and 2, whatever that means.

Countable and Uncountable Sets

Are some infinities larger than others? Surprisingly, the answer is yes. In the late 1800s, Georg Cantor invented a clever way to compare different infinite sets. Two sets X and Y are said to have the same *cardinality* (or number of elements) if there is an invertible mapping that pairs each element $x \in X$ with precisely one $y \in Y$. Such a mapping is called a ***one-to-one correspondence***; it's like a buddy system, where every x has a buddy y, and no one in either set is left out or counted twice.

A familiar infinite set is the set of natural numbers $\mathbf{N} = \{1,2,3,4, \ldots\}$. This set provides a basis for comparison—if another set X can be put into one-to-one correspondence with the natural numbers, then X is said to be ***countable***. Otherwise X is ***uncountable***.

Countable and Uncountable Sets

Show that the set of even natural numbers $E = \{2,4,6, \ldots\}$ is countable.

Solution: We need to find a one-to-one correspondence between E and $\mathbf{N}$. Such a correspondence is given by the invertible mapping that pairs each natural number n with the even number $2n$; thus $1 \leftrightarrow 2$, $2 \leftrightarrow 4$, $3 \leftrightarrow 6$, and so on.

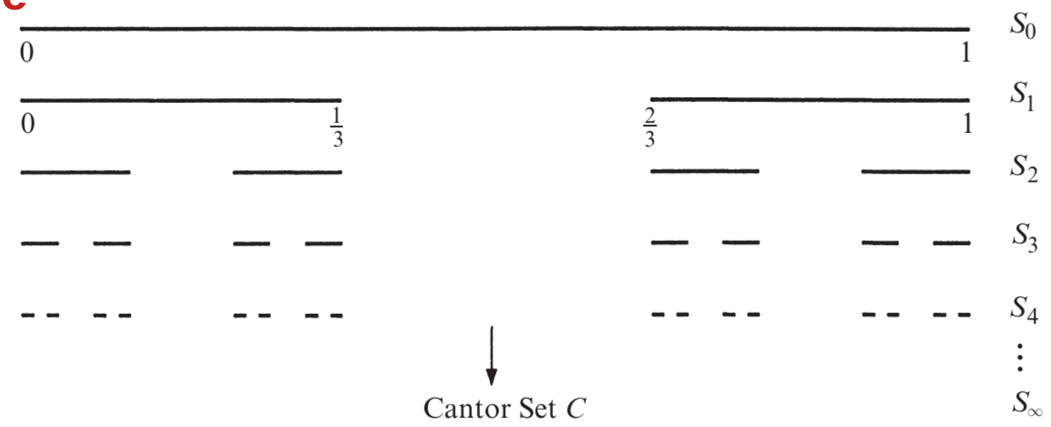
Cantor Set

Now we turn to another of Cantor's creations, a fractal known as the Cantor set.

Fractals

We start with the closed interval $S_0 = [0,1]$ and remove its open middle third, i.e., we delete the interval $(\frac{1}{3}, \frac{2}{3})$ and leave the endpoints behind. This produces the pair of closed intervals shown as S_1. Then we remove the open middle thirds of *those* two intervals to produce S_2, and so on. The limiting set $C = S_\infty$ is the ***Cantor set***.

It contains infinite number of infinitesimal pieces, separated by gaps of various size



$L_0 = 1,\ L_1 = \frac{2}{3},\ L_2 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2$, and in general, $L_n = \left(\frac{2}{3}\right)^n$

$L_n \to 0$ $n \to \infty$, the Cantor set has a total length of zero.

The Cantor set C has several properties that are typical of fractals more generally:

1. *C has structure at arbitrarily small scales.* If we enlarge part of C repeatedly, we continue to see a complex pattern of points separated by gaps of various sizes. This structure is neverending, like worlds within worlds. In contrast, when we look at a smooth curve or surface under repeated magnification, the picture becomes more and more featureless.
2. *C is self-similar.* It contains smaller copies of itself at all scales. For instance, if we take the left part of C (the part contained in the interval $\left[0, \frac{1}{3}\right]$) and enlarge it by a factor of three, we get C back again. Similarly, the parts of C in each of the four intervals of S_2 are geometrically similar to C, except scaled down by a factor of nine.

Fractals

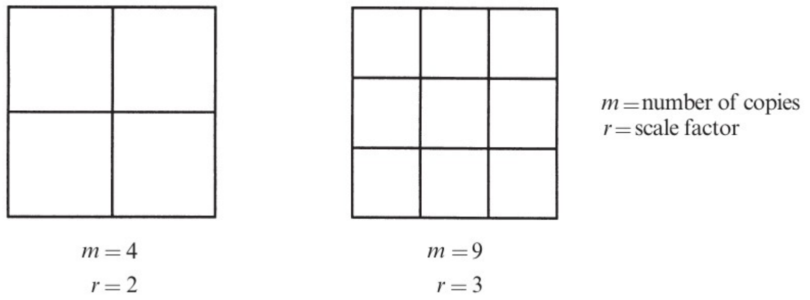
3. *The dimension of C is not an integer.*

We will compute the dimension and find that the dimension would be 0.67 for the cantor set points.

Similarity Dimension

The simplest fractals are self-similar, i.e., they are made of scaled-down copies of themselves, all the way down to arbitrarily small scales. The dimension of such fractals can be defined by extending an elementary observation about *classical* self-similar sets like line segments, squares, or cubes.

Fractals



If we shrink the square by a factor of 2 in each direction, it takes four of the small squares to equal the whole. Or if we scale the original square down by a factor of 3, then nine small squares are required. In general, if we reduce the linear dimensions of the square region by a factor of r, it takes r^2 of the smaller squares to equal the original.

Now suppose we play the same game with a solid cube. The results are different: if we scale the cube down by a factor of 2, it takes eight of the smaller cubes to make up the original. In general, if the cube is scaled down by r, we need r^3 of the smaller cubes to make up the larger one.

Fractals

The exponents 2 and 3 are no accident; they reflect the two-dimensionality of the square and the three-dimensionality of the cube. This connection between dimensions and exponents suggests the following definition. Suppose that a self-similar set is composed of m copies of itself scaled down by a factor of r. Then the ***similarity dimension*** d is the exponent defined by $m = r^d$, or equivalently,

$$d = \frac{\ln m}{\ln r}.$$

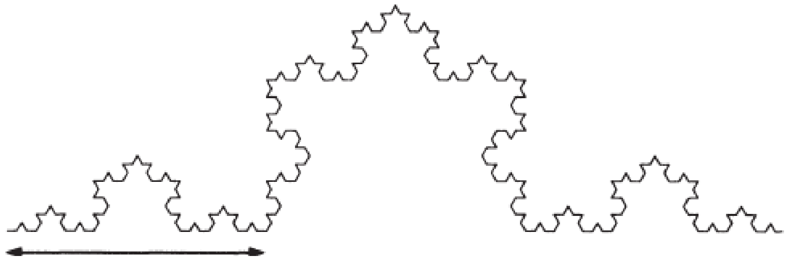
Find the similarity dimension of the Cantor set C.

The left half of the Cantor set is the original Cantor set, scaled down by a factor of 3

So $m = 2$ when $r = 3$. Therefore $d = \ln 2 / \ln 3 \approx 0.63$.

Fractals

Dimension of Von Koch curve:



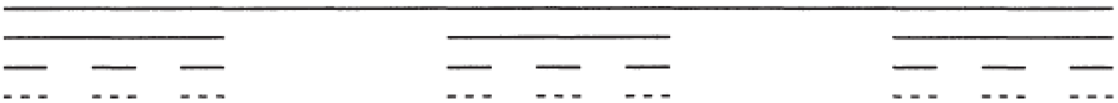
The curve is made up of four equal pieces, each of which is similar to the original curve but is scaled down by a factor of 3 in both directions.

Hence $m = 4$ when $r = 3$, and therefore $d = \ln 4 / \ln 3$.

Fractals

Even fifth Cantor sets:

Divide an interval into five equal pieces, delete the second and fourth subintervals and then repeat the process indefinitely.



$m = 3$ when $r = 5$

$$d = \ln 3 / \ln 5.$$

Fractals

Pointwise and Correlation Dimensions

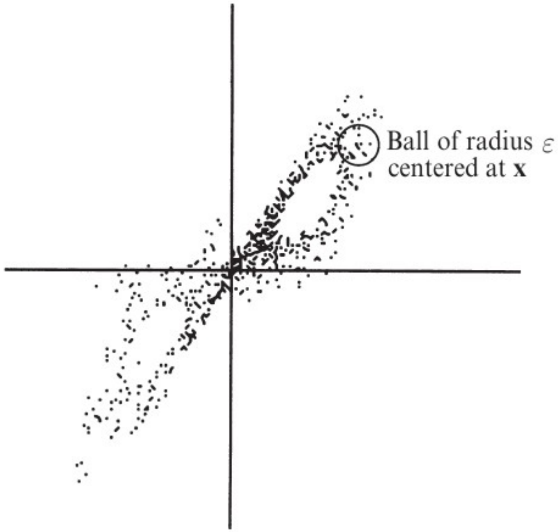
Suppose we are studying a chaotic system that settles down to a strange attractor in the phase space. Given that the strange attractors typically have fractal microstructure, how could we estimate the fractal dimension?

First we generate a set of very many points $\{\mathbf{x}_i, i = 1, \ldots, n\}$ on the attractor by letting the system evolve for a long time (after taking care to discard the initial transient, as usual). To get better statistics, we could repeat this procedure for several different trajectories. In practice, however, almost all trajectories on a strange attractor have the same long-term statistics so it's sufficient to run one trajectory for an extremely long time.

Fractals

There are several method to compute the dimesnion here we would propose one of the efficient method prposed by **Grassberger and Procaccia.**

Fix a point $\mathbf{x}$ on the attractor A. Let $N_{\mathbf{x}}(\varepsilon)$ denote the number of points on A inside a ball of radius ε about $\mathbf{x}$



Fractals

Most of the points in the ball are unrelated to the immediate portion of the trajectory through $\mathbf{x}$; instead they come from later parts that just happen to pass close to $\mathbf{x}$. Thus $N_{\mathbf{x}}(\varepsilon)$ measures how frequently a typical trajectory visits an ε neighborhood of $\mathbf{x}$.

Now vary ε. As ε increases, the number of points in the ball typically grows as a power law:

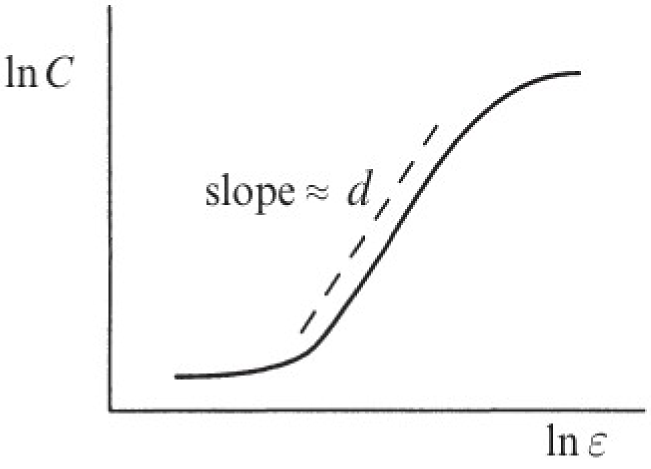
$$N_{\mathbf{x}}(\varepsilon) \propto \varepsilon^{d},$$

where d is called the ***pointwise dimension*** at $\mathbf{x}$. The pointwise dimension can depend significantly on $\mathbf{x}$; it will be smaller in rarefied regions of the attractor. To get an overall dimension of A, one averages $N_{\mathbf{x}}(\varepsilon)$ over many $\mathbf{x}$. The resulting quantity $C(\varepsilon)$ is found empirically to scale as

$$C(\varepsilon) \propto \varepsilon^{d},$$

where d is called the ***correlation dimension***.

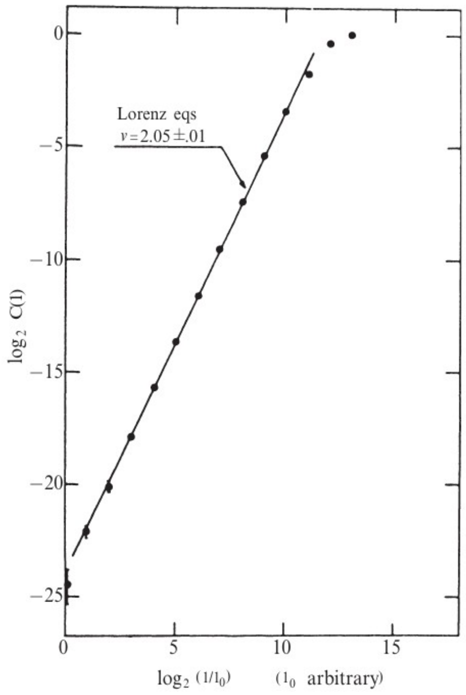
Fractals



The curve saturates at large ε because the ε-balls engulf the whole attractor and so $N_{\mathbf{x}}(\varepsilon)$ can grow no further. On the other hand, at extremely small ε, the only point in each ε-ball is $\mathbf{x}$ itself. So the power law is expected to hold only in the *scaling region* where

Fractals

Estimate the correlation dimension of the Lorenz attractor, for the standard parameter values $r = 28$, $\sigma = 10$, $b = \frac{8}{3}$.



Grassberger and Procaccia (1983)

Exercises

1.

$$x_{n+1} = \sinh x_n$$

The fixed points occur when

$$x = \sinh(x)$$

So the fixed point is $x^* = 0$.

As for stability,

$$f(x) = \sinh(x)$$

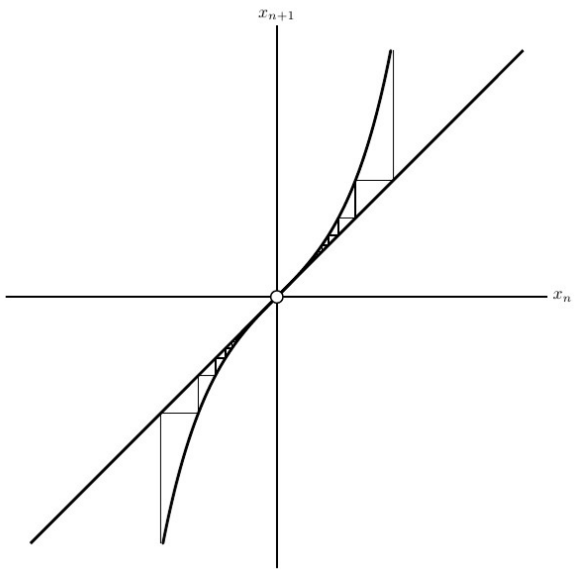
$$f'(x) = \cosh(x)$$

$$f'(0) = 1 \Rightarrow \text{ inconclusive, but the graph shows it's unstable}$$

Exercises

We can see this by taking some values

x_0	-0.50000	0.50000
x_1	-0.52110	0.52110
x_2	-0.54500	0.54500
x_3	-0.57238	0.57238
x_4	-0.60415	0.60415
x_5	-0.64158	0.64158
x_6	-0.68651	0.68651
x_7	-0.74173	0.74173
x_8	-0.81163	0.81163
x_9	-0.90372	0.90372
x_{10}	-1.03186	1.03186
x_{11}	-1.22497	1.22497
x_{12}	-1.55515	1.55515
x_{13}	-2.26231	2.26231
x_{14}	-4.75059	4.75059
x_{15}	-57.82220	57.82220



Exercises

2.

Find the value of r at which the logistic map has a superstable fixed point.

$$x = f(x) = rx(1-x) = rx - rx^2$$

$$rx^2 + (1-r)x = x(rx+1-r) = 0 \Rightarrow x^* = 0, \frac{r-1}{r}$$

$$f'(x) = r(1-2x)$$

$$f'(0) = r = 0$$

$x^* = 0$ is a superstable fixed point when $r = 0$

$$f'\left(\frac{r-1}{r}\right) = r\left(1 - 2\frac{r-1}{r}\right) = 0 \Rightarrow r = 0, 2$$

$x^* = \frac{r-1}{r}$ is a superstable fixed point when $r = 2$

Exercises

3. Analyze the long-term behavior of the map $x_{n+1} = rx_n/(1+x_n^2)$, where $r > 0$. Find and classify all fixed points as a function of r. Can there be periodic solutions? Chaos?

$$x_{n+1} = \frac{rx_n}{1+x_n^2}$$

The fixed points occur when

$$x = \frac{rx}{1+x^2}$$

$$x(1+x^2) = rx$$

$$x(1+x^2) - rx = x(1-r+x^2) = 0$$

$$x = 0, \pm\sqrt{1-r}$$

So there are three fixed points if $r \leq 1$ and one fixed point if $1 < r$.

$$f(x) = \frac{rx}{1+x^2}$$

$$f'(x) = \frac{r(1+x^2) - 2rx^2}{(1+x^2)^2} = \frac{r(1-x^2)}{(1+x^2)^2}$$

$r < 1 \Rightarrow x^* = 0$ is stable, and $x^* = \pm\sqrt{1-r}$ are unstable

$$f'(0) = r \qquad f'(\pm\sqrt{1-r}) = \frac{2}{r} - 1$$

Exercises

$r = 1 \Rightarrow f'(0) = 1$ is inconclusive

$$|x_{n+1}| = \left|\frac{x_n}{1+x_n^2}\right| < \left|\frac{x_n}{1}\right| = |x_n| \Rightarrow x^* = 0 \text{ is stable}$$

$1 < r \Rightarrow x^* = 0$ is stable

$|x_n|$ is monotonically decreasing when $r \leq 1$ and $x_n \neq 0$, which rules out any periodic orbits.

For r >1:

$$|x| < \left|\frac{rx}{1+x^2}\right| = \frac{r|x|}{1+x^2}$$

$$|x|(1+x^2) < r|x|$$

$$|x|(1-r+x^2) < 0 \Rightarrow x^2 < r-1 \Rightarrow |x| < \sqrt{r-1}$$

and

$$\left|\frac{rx}{1+x^2}\right| = \frac{r|x|}{1+x^2} < |x|$$

$$r|x| < |x|(1+x^2)$$

$$0 < |x|(1-r+x^2) \Rightarrow r-1 < x^2 \Rightarrow \sqrt{r-1} < |x|$$

Exercises

4.

Calculate the Liapunov exponent for the linear map $x_{n+1} = rx_n$.

$$x_n = r^n x_0$$

$$\lambda = \frac{1}{n}\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n}\ln\left|\frac{r^n\delta_0}{\delta_0}\right| = \ln|r|$$

So the Liapunov exponent $\lambda = \ln|r|$.

5.

Exercises

Find and classify all bifurcations for the system $\dot{x} = y - ax$, $\dot{y} = -by + x/(1+x)$.

$$y = ax \qquad y = \frac{x}{b(1+x)} \Rightarrow (x, y) = (0, 0) \text{ and } \left(\frac{1}{ab} - 1, \frac{1}{b} - a\right)$$

Next find the linearization at the fixed points.

$$A = \begin{pmatrix} -a & 1 \\ \frac{1}{(1+x)^2} & -b \end{pmatrix}$$

$$(x, y) = (0, 0) \Rightarrow \Delta = ab - 1 \quad \tau = -(a + b)$$

$$(x, y) = \left(\tfrac{1}{ab} - 1, \tfrac{1}{b} - a\right) \Rightarrow \Delta = ab - (ab)^2 \quad \tau = -(a + b)$$

Exercises

Looking more closely at this system, we see that the fixed point goes to infinity when either a or b is zero. Also notice that there is only one fixed point when $ab = 1$, and there are two fixed points when $ab \neq 1$, $a \neq 0$, and $b \neq 0$. There are several cases for the stability of the two fixed points.

$ab > 1$ and $a + b > 0 \Rightarrow$ The origin is a stable node and the other fixed point is a saddle point.

$0 < ab < 1$ and $a + b > 0 \Rightarrow$ The origin is a saddle point and the other fixed point is a stable node.

$ab > 1$ and $a + b < 0 \Rightarrow$ The origin is an unstable node and the other fixed point is a saddle point.

$0 < ab < 1$ and $a + b < 0 \Rightarrow$ The origin is a saddle point and the other fixed point is an unstable node.

$ab < 0 \Rightarrow$ The origin and the other fixed point are saddle points. This last case completes all possible cases.

From this we can conclude that transcritical bifurcations occur along the boundary $a = \dfrac{1}{b}$ in parameter space.

6.

Exercises

Plot the stability diagram for the system $\ddot{x} + b\dot{x} - kx + x^3 = 0$, where b and k can be positive, negative, or zero. Label the bifurcation curves in the (b, k) plane.

$$\dot{x} = y \qquad \dot{y} = kx - by - x^3$$

Next we find the fixed points, which are the intersections of the nullclines.

$$y = 0 \qquad y = \frac{kx - x^3}{b} \Rightarrow (x, y) = (0, 0) \text{ and } (\pm\sqrt{k}, 0)$$

Now we find the linearization at the fixed points.

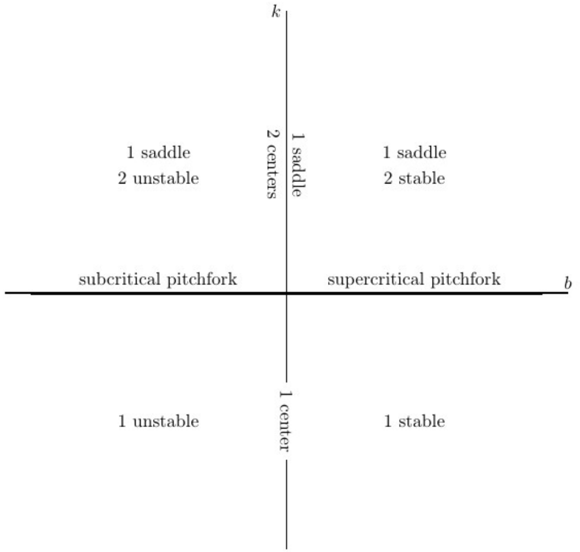
$$A = \begin{pmatrix} 0 & 1 \\ k - 3x^2 & -b \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} 0 & 1 \\ k & -b \end{pmatrix} \qquad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 + 4k}}{2}$$

$$A_{(\pm\sqrt{k},0)} = \begin{pmatrix} 0 & 1 \\ -2k & -b \end{pmatrix} \qquad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 8k}}{2}$$

Exercises

The second pair of fixed points comes into existence and splits off from the origin when $k > 0$, and the stability of all three fixed points is determined by the sign of b. If $b < 0$ and k become positive, then there is a subcritical pitchfork bifurcation. If $b > 0$ and k become positive, there is a supercritical pitchfork bifurcation. When $b = 0$ there is a bifurcation when k becomes positive too. The origin changes from a center to a saddle point, and the two new fixed points are centers.



Exercises

Let $x_{n+1} = f(x_n)$, where $f(x) = -(1+r)x - x^2 - 2x^3$.

a) Classify the linear stability of the fixed point $x^* = 0$.

b) Show that a flip bifurcation occurs at $x^* = 0$ when $r = 0$.

Exercises

$$x_{n+1} = f(x_n) = -(1+r)x_n - x_n^2 - 2x_n^3$$

a)

$$f'(x) = -(1+r) - 2x - 6x^2$$

$$f'(0) = -(1+r)$$

$$-1 < -(1+r) < 1 \Rightarrow 0 < r < 2$$

$$0 < r < 2 \Rightarrow \text{stable} \qquad r < 0 \text{ or } 2 < r \Rightarrow \text{unstable}$$

b)

A flip bifurcation will occur at $x = 0$ and $r = 0$ if the stability changes on either side of the r value.

$$f(0) = 0$$

$$r < 0 \Rightarrow f'(0) = -(1+r) < -1 \Rightarrow \text{unstable}$$

$$r = 0 \Rightarrow f'(0) = -(1+r) = -1 \Rightarrow \text{inconclusive}$$

$$0 < r \Rightarrow f'(0) = -(1+r) > -1 \Rightarrow \text{stable}$$

Hence at $x = 0$ and $r = 0$ a flip bifurcation occurs.

Strange attractors

So far we have discussed in detail about the fractal behaviour in the chaotic attractors, however we could not discuss few of the issues related to the genesis of the fractal nature in the chaotic attaractors. We neither know so far what causes the sensitive dependence on initial condition nor how a differential equation can generate a fractal attractor. The same issues were of great concern infront of the scientific comminity in the mid-1970s. At that time the only known examples of strange attractors were the Lorenz attractor (1963). Thus it was felt that there should be availability of other system where the things can be understood in more better and transparent way.

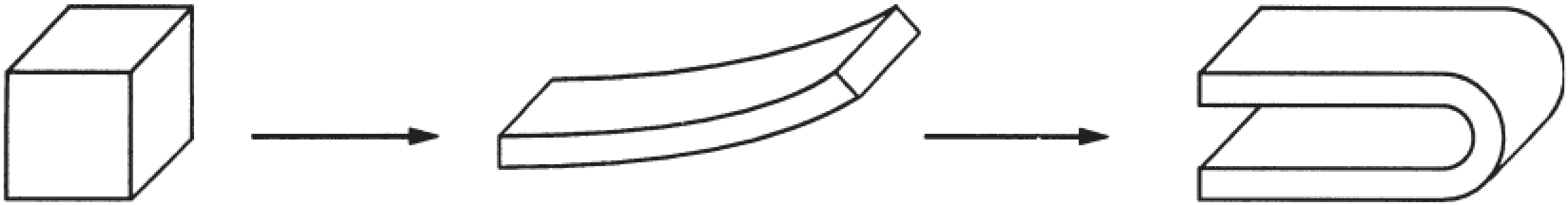
The answers of the many of the questions raised in this context was addressed by Henon (1976) and Rossler (1976) using the intuitive concepts of the stretching and floding that results in the formation of the strange attractors.

Strange attractors

Strange attractors have two properties that seem hard to reconcile. Trajectories on the attractor remain confined to a bounded region of phase space, yet they separate from their neighbors exponentially fast (at least initially). How can trajectories diverge endlessly and yet stay bounded?

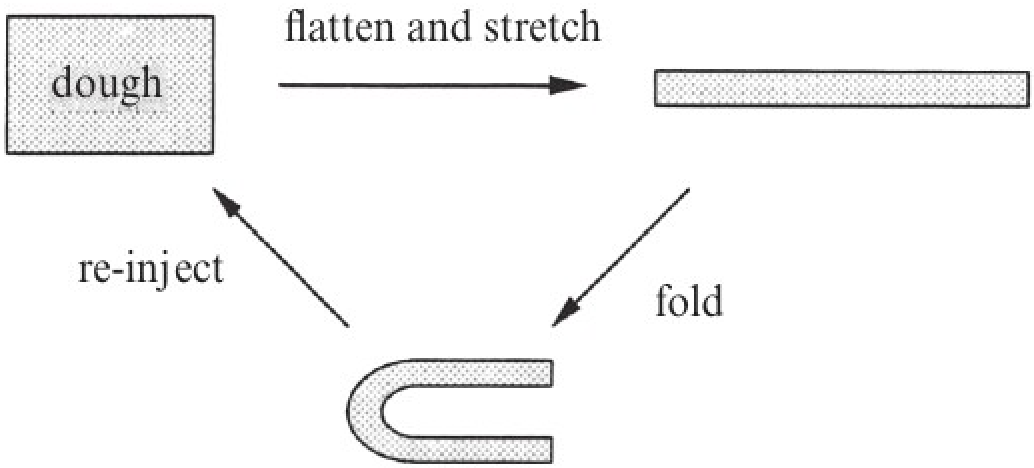
The basic mechanism involves repeated ***stretching and folding***.

Consider a small blob in the phase space. A strange attractor typically arises when the flow contracts (due to dissipations) the blob in some direction and stretches (due to sensitive on initial condition) it in others.



Strange attractors

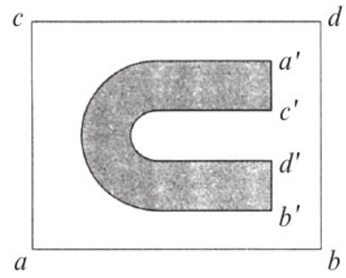
Example:



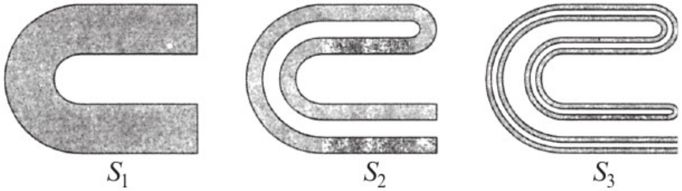
The dough is rolled out and flattened, then folded over, then rolled out again, and so on. After many repetitions, the end product is a flaky, layered structure—the culinary analog of a fractal attractor.

Strange attractors

More detailed picture of Pastry map:



The rectangle *abcd* is flattened, stretched, and folded into the ***horseshoe*** *a'b'c'd'*, also shown as S_1. In the same way, S_1 is itself flattened, stretched, and folded into S_2, and so on. As we go from one stage to the next, the layers become thinner and there are twice as many of them.



Now try to picture the limiting set S_∞. It consists of infinitely many smooth layers, separated by gaps of various sizes. In fact, a vertical cross section through the middle of S_∞ would resemble a *Cantor set*! Thus S_∞ is (locally) the product of a smooth curve with a Cantor set. The fractal structure of the attractor is a consequence of the stretching and folding that created S_∞ in the first place.

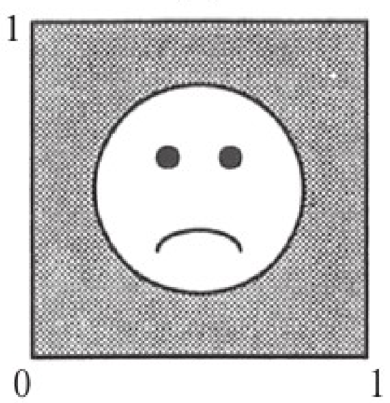
Strange attractors

Example:

The ***baker's map*** B of the square $0 \le x \le 1$, $0 \le y \le 1$ to itself is given by

$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n,\ ay_n) & \text{for } 0 \le x_n \le \frac{1}{2} \\ (2x_n - 1,\ ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \le x_n \le 1 \end{cases}$$

where a is a parameter in the range $0 < a \le \frac{1}{2}$. Illustrate the geometric action of B by showing its effect on a face drawn in the unit square.



Strange attractors

Example:

We will find that the map will be equivalent to the transformation which may be regarded as a product of two simpler transformation:

The square is stretched and flattened into 2Xa rectangle and then the rectangle is cut in half, yeilding two 1Xa rectangles, and the right half is stacked on top of the left half such that its base is at the level y=1/2.

Why is the procedure is same to the formula?

Strange attractors

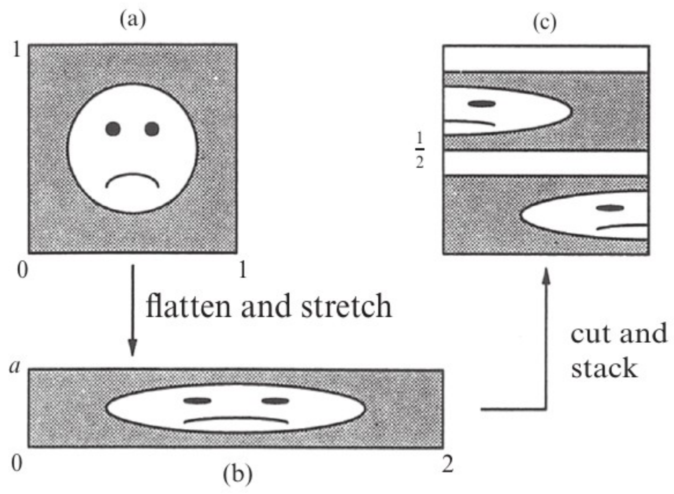
$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n,\ ay_n) & \text{for } 0 \le x_n \le \frac{1}{2} \\ (2x_n - 1,\ ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \le x_n \le 1 \end{cases}$$

Left half we have

$$(x_{n+1}, y_{n+1}) = (2x_n,\ ay_n).$$

So the horizontal direction is stretched by 2 and the vertical direction is contracted by a. The same is true for the right half of the rectangle, except the image is shifted left by 1 and up by 1/2 as

$$(x_{n+1}, y_{n+1}) = (2x_n,\ ay_n) + (-1, \tfrac{1}{2}).$$



Strange attractors

Hénon Map

Michel Hénon (1976)

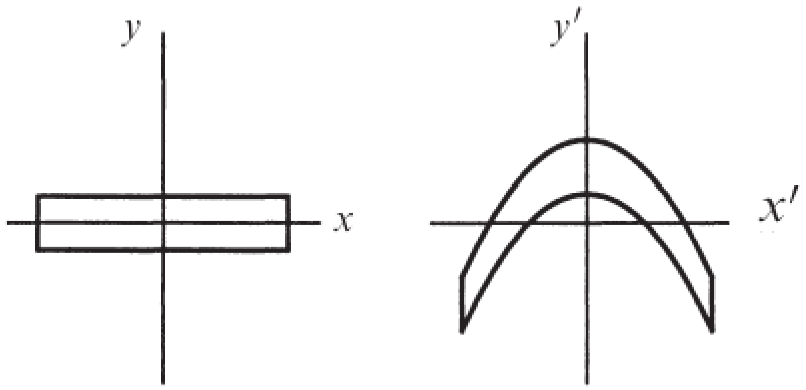
Two-dimensional map with a strange attractor.

The ***Hénon map*** is given by

$$x_{n+1} = y_n + 1 - a x_n^2, \qquad y_{n+1} = b x_n,$$

where a and b are adjustable parameters. Hénon (1976) arrived at this map by an elegant line of reasoning. To simulate the stretching and folding that occurs in the Lorenz system, he considered the following chain of transformations

Strange attractors



Start with a rectangular region elongated along the x-axis

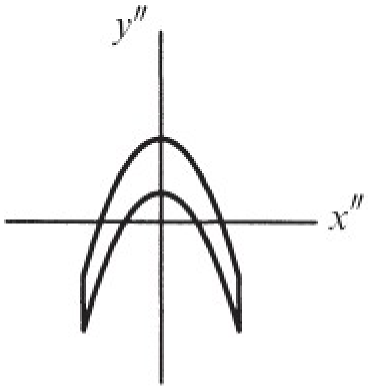
Then stretch and fold the retcangle by applying the transformation:

$$T': \quad x' = x, \quad y' = 1 + y - ax^2.$$

The primes denote iteration, not differentiation.

Strange attractors

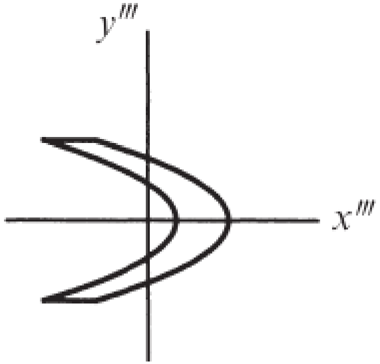
Now fold the region even more by contracting Figure



$$T'': \quad x'' = bx', \qquad y'' = y'$$

where $-1 < b < 1$.

$$T''': \quad x''' = y'', \qquad y''' = x''.$$



Strange attractors

Then the composite transformation $T = T'''T''T'$ yields the Hénon mapping

where we use the notation (x_n, y_n) for (x, y) and (x_{n+1}, y_{n+1}) for (x''', y''').

$$x_{n+1} = y_n + 1 - ax_n^2, \qquad y_{n+1} = bx_n,$$

Some trajectories of the Hénon map escape to infinity.

Show that the Hénon map T is invertible if $b \neq 0$, and find the inverse T^{-1}.

$$x_n = b^{-1}y_{n+1}, \; y_n = x_{n+1} - 1 + ab^{-2}(y_{n+1})^2.$$

Strange attractors

Show that the Hénon map contracts areas if $-1 < b < 1$.

$$x_{n+1} = f(x_n, y_n) \qquad\qquad y_{n+1} = g(x_n, y_n)$$

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}.$$

If $|\det \mathbf{J}(x, y)| < 1$ for all (x, y), the map is area-contracting.

For the Hénon map, we have $f(x, y) = 1 - ax^2 + y$ and $g(x, y) = bx$. Therefore

$$\mathbf{J} = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

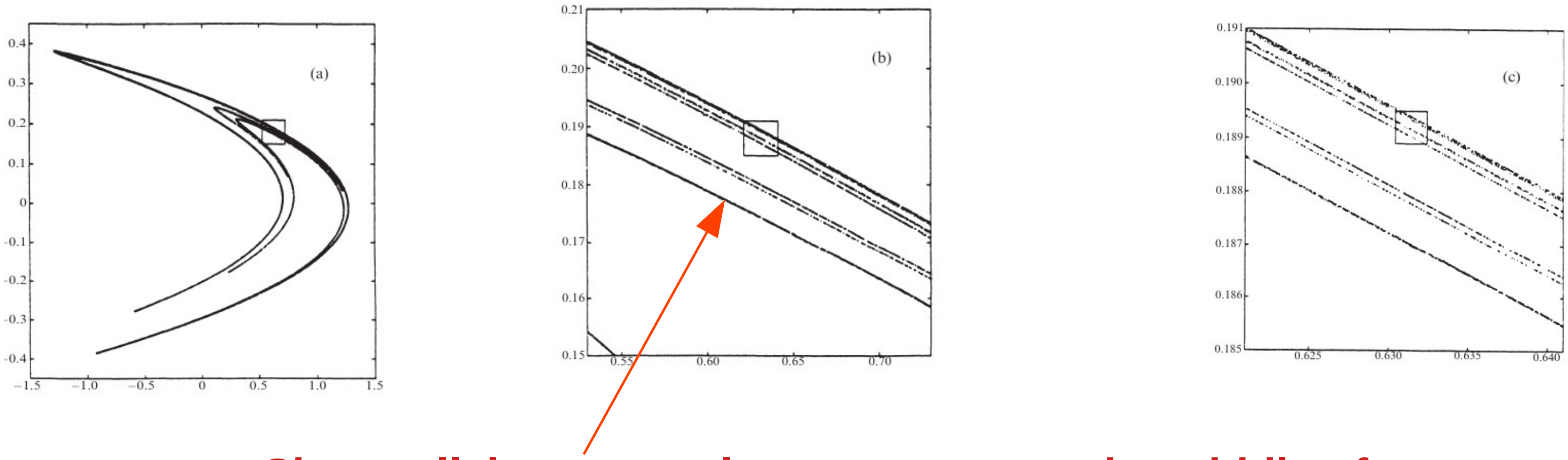
Strange attractors

and $\det \mathbf{J}(x,y) = -b$ for all (x,y). Hence the map is area-contracting for $-1 < b < 1$, as claimed. In particular, the area of any region is reduced by a *constant* factor of $|b|$ with each iteration.

Strange attractors

Zooming In on a Strange Attractor

$a = 1.4,\ b = 0.3$



Six parallel curves: a lone cureve near the middle of the frame, then two closely spaced curve above it, and then three more.

Poincare Map

Dynamical System describes the flow is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = F(\vec{x}, t)$$

Systems of such equations describe a *flow* in phase space.

The solution is often studied by considering the trajectories of such flows.

But the phase trajectory is itself often difficult to determine, if for no other reason than that the dimensionality of the phase space is too large.

Thus we seek a geometric depiction of the trajectories in a lower-dimensional space—in essence, a view of phase space without *all* the detail.

Poincare Map

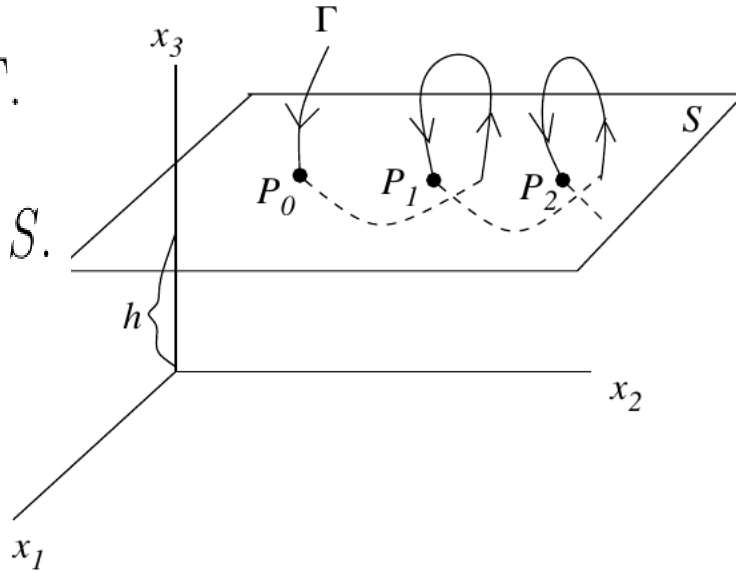
Construction of Poincaré sections

Suppose we have a 3-D flow Γ. Instead of directly studying the flow in 3-D, consider, e.g., its intersection with a plane ($x_3 = h$):

- Points of intersection correspond (*in this case*) to $\dot{x}_3 < 0$ on Γ.

- Height h of plane S is chosen so that Γ continually crosses S.

- The points P_0, P_1, P_2 form the 2-D **Poincaré section**.



Poincare Map

The Poincaré section is a continuous mapping T of the plane S into itself:

$$P_{k+1} = T(P_k) = T\left[T(P_{k-1})\right] = T^2(P_{k-1}) = \ldots$$

Since the flow is deterministic, P_0 determines P_1, P_1 determines P_2, etc.

The Poincaré section reduces a *continuous* flow to a **discrete-time mapping**. However the time interval from point to point is not necessarily constant.

Poincare Map

Some geometric properties of the flow and Poincare section are same:

- Dissipation $\Rightarrow$ areas in the Poincaré section *should* contract.
- If the flow has an attractor, we should see it in the Poincaré section.

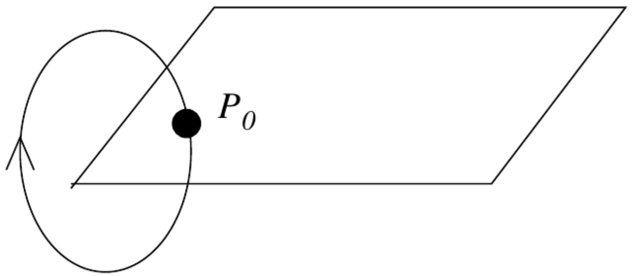
Types of Poincaré sections

1 Periodic

The flow is a closed orbit (e.g., a limit cycle):

P_0 is a fixed point of the Poincaré map:

$$P_0 = T(P_0) = T^2(P_0) = \ldots.$$



Poincare Map

We proceed to analyze the stability of the fixed point.

To first order, a Poincaré map T can be described by a matrix M defined in the neighborhood of P_0:

$$M_{ij} = \left.\frac{\partial T_i}{\partial x_j}\right|_{P_0} .$$

In this context, M is called a *Floquet matrix*. It describes how a point $P_0 + \delta$ moves after one intersection of the Poincaré map.

A Taylor expansion about the fixed point yields:

$$T_i(P_0 + \delta) \simeq T_i(P_0) + \left.\frac{\partial T_i}{\partial x_1}\right|_{P_0} \cdot \delta_1 + \left.\frac{\partial T_i}{\partial x_2}\right|_{P_0} \cdot \delta_2, \qquad i = 1, 2$$

Poincare Map

Since $T(P_0) = P_0$,

$$T(P_0 + \delta) \simeq P_0 + M\delta$$

Therefore

$$\begin{aligned} T\Big(T(P_0+\delta)\Big) &\simeq T(P_0 + M\delta) \\ &\simeq T(P_0) + M^2\delta \\ &\simeq P_0 + M^2\delta \end{aligned}$$

After m interations of the map,

$$T^m(P_0 + \delta) - P_0 \simeq M^m\delta.$$

Poincare Map

Stability therefore depends on the properties of M.

Assume that δ is an eigenvector of M. (There will always be a projection onto an eigenvector.) Then

$$M^m \delta = \lambda^m \delta,$$

where λ is the corresponding eigenvalue.

Therefore

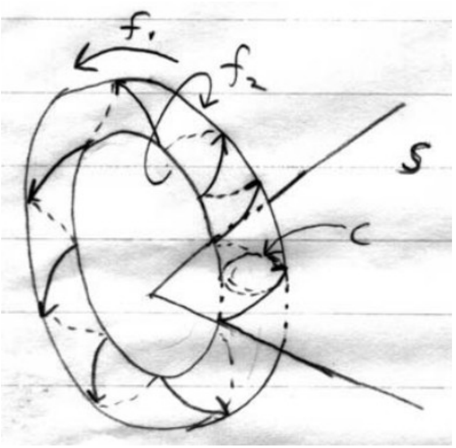
$$\begin{aligned} |\lambda| &< 1 \Rightarrow \text{ linearly stable} \\ |\lambda| &> 1 \Rightarrow \text{ linearly unstable} \end{aligned}$$

Conclusion: a periodic map is unstable if one of the eigenvalues of the Floquet matrix crosses the unit circle in the complex plane.

Poincare Map

.2 Quasiperiodic flows

Consider a 3-D flow with two fundamental frequencies f_1 and f_2. The flow is a torus T^2:



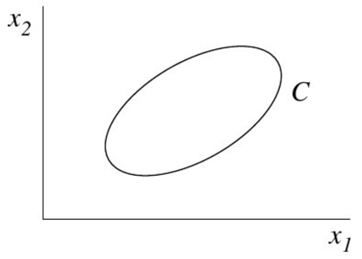
The points of intersection of the flow with the plane S appear on a closed curve C.

Poincare Map

.2 Quasiperiodic flows

As with power spectra, the form of the resulting Poincaré section depends on the ratio f_1/f_2:

- **Irrational** f_1/f_2. The frequencies are called *incommensurate*. The closed curve C appears continuous, e.g.



– The trajectory on the torus T^2 never repeats itself exactly.

Poincare Map

- **Rational** f_1/f_2.
  - f_1 and f_2 are *frequency locked.*
  - There are finite number of intersections (points) along the curve C.
  - Trajectory repeats itself after n_1 revolutions and n_2 rotations.
  - The Poincaré section is periodic with

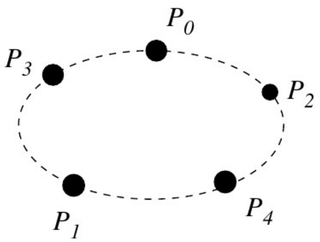
$$\text{period } = n_1/f_1 = n_2/f_2$$

  - The Poincaré section contains just n_1 points. Thus

$$P_i = T^{n_1}(P_i)$$

Poincare Map

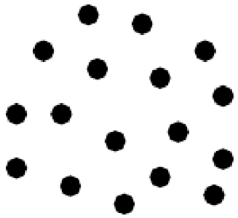
– Example, $n_1 = 5$:



.3 Aperiodic flows

Aperiodic flows may no longer lie on some reasonably simple curve.

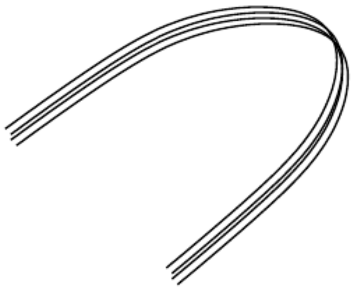
In an extreme case, one has just a point cloud:



Poincare Map

.3 Aperiodic flows

Deterministic aperiodic systems often display more order, however. In some cases they create mild departures from a simple curve, e.g.



Such cases arise from strong dissipation (and the resulting contraction of areas in phase space).

Turbulence motion

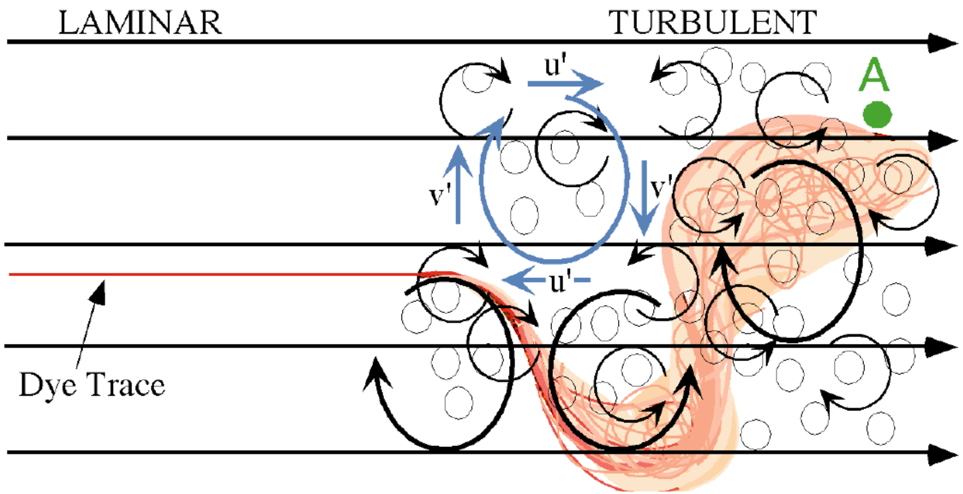
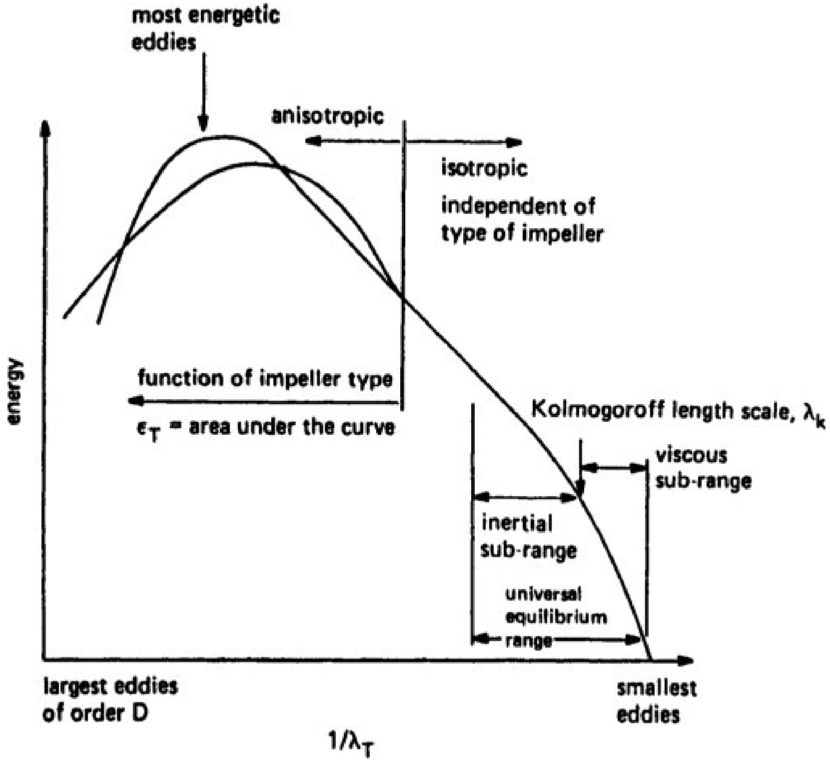
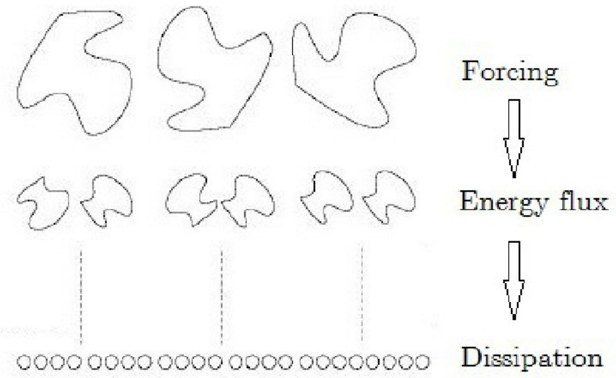


Figure 1. Tracer transport in laminar and turbulent flow. The straight, parallel black lines are streamlines, which are everywhere parallel to the mean flow. In laminar flow the fluid particles follow the streamlines exactly, as shown by the linear dye trace in the laminar region. In turbulent flow eddies of many sizes are superimposed onto the mean flow. When dye enters the turbulent region it traces a path dictated by both the mean flow (streamlines) and the eddies. Larger eddies carry the dye laterally across streamlines. Smaller eddies create smaller scale stirring that causes the dye filament to spread (diffuse).

Chaos in Fluids





Chaos in Fluids

Chaos can occur in fluids as well. If we take $\nabla \cdot \mathbf{v} = 0$ and ρ to be constant and uniform, the Navier-Stokes equation says:

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{\nabla \mathcal{P}}{\rho} + \nu \nabla^2 \mathbf{v}$$

Strong turbulence in a fluid falls in the category of being irregular in $\mathbf{x}$ with no characteristic size for features. This is certainly more advanced than our examples, and indeed a full formalism for turbulence remains to be invented. One thing we can do to characterize strong turbulence is apply dimensional analysis.

Chaos in Fluids

There are several scaling laws for turbulence in 3 dimensions. Recall that vortices (eddies) appear at all length scales λ and are efficient at transferring energy. Let us define L as the size of the fluid container, λ_0 as the scale where dissipation is important (for Reynolds number $R \approx 1$), ϵ as the mean energy transfer per unit time per unit mass, and v_λ as the velocity variation at length scale λ. Note that the dimensions $[\nu] = m^2/s$ and $[\epsilon] = (kgm^2/s^2)(1/skg) = m^2/s^3$. There are three scales to consider.

1. At $\lambda \approx L$, there can be no dependence on ν, so $\epsilon \propto \dfrac{v_L^3}{L}$. (This is the scale with the most kinetic energy and the largest energy.)

2. At $\lambda_0 \ll \lambda \ll L$, there can still be no ν, so here $\epsilon \propto \dfrac{v_\lambda^3}{\lambda}$. Note that this is independent of the properties ρ, ν and the scale L of the fluid!

Chaos in Fluids

3. At $\lambda \approx \lambda_0$, because $R = \frac{v_0 \lambda_0}{\nu} \approx 1$, then $v_0 \approx \frac{\nu}{\lambda_0}$. This is where the energy dissipation occurs. Here we only have ν and λ_0 present, so $\epsilon \propto \frac{\nu^3}{\lambda_0^4}$.

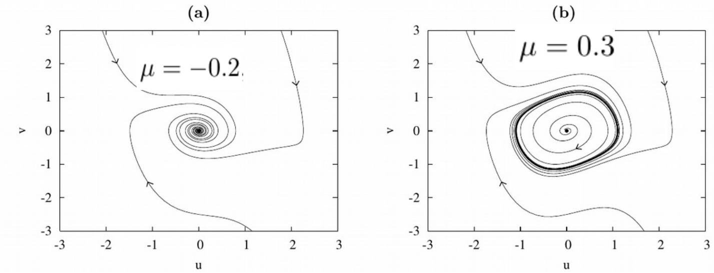
Rather than using λ and v_λ, the universal result for the case $\lambda_0 \ll \lambda \ll L$ is often written in terms of the wavenumber $k \propto \dfrac{1}{\lambda}$ and kinetic energy per unit mass per unit wave number, $E(k)$. The kinetic energy per unit mass can be written as $E(k)\,dk$. Here $E(k)$ behaves as a rescaled version of the energy with slightly different dimensions, $[E(k)] = m^3/s^2$. Analyzing its dimensions in relation to ϵ and k we note that $m^3/s^2 = (m^2/s^3)^{2/3}(1/m)^{-5/3}$ which yields

$$E(k) \sim \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

This is the famous Kolmogorov scaling law for strong turbulence. It provides a mechanism by which we can make measurements and probe a universal property of turbulence in many systems.

Hopf-Bifurcation

The term Hopf bifurcation (also sometimes called Poincaré-Andronov-Hopf bifurcation) refers to the local birth or death of a periodic solution (self-excited oscillation) from an equilibrium as a parameter crosses a critical value. It is the simplest bifurcation not just involving equilibria and therefore belongs to what is sometimes called *dynamic* (as opposed to *static*) *bifurcation theory*. In a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearised flow at a fixed point becomes purely imaginary. This implies that a Hopf bifurcation can only occur in systems of dimension two or higher.



The origin $(u, v) = (0, 0)$ is a fixed point for each μ, with eigenvalues $\lambda(\mu), \bar{\lambda}(\mu) = \frac{1}{2}\left(\mu \pm i\sqrt{4-\mu^2}\right)$. The system has a Hopf bifurcation at $\mu = 0$. We have $\omega = -1$, $d = \frac{1}{2}$ and $a = -\frac{1}{8}$, so the bifurcation is supercritical and there is a stable isolated periodic orbit (*limit cycle*) if $\mu > 0$

$$\dot{u} = v,$$
$$\dot{v} = -u + (\mu - u^2)v.$$

Hopf-Bifurcation in Lorenz Model

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - xz - y$$

$$\dot{z} = xy - bz$$

$$C^{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$$

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}_{C^{\pm}} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{pmatrix}$$

Hopf-Bifurcation in Lorenz Model

$$\det\begin{pmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b-\lambda \end{pmatrix}$$

$$= (-\sigma-\lambda)((-1-\lambda)(-b-\lambda)+b(r-1))-\sigma((-b-\lambda)+b(r-1))$$

$$= -(\sigma+\lambda)((1+\lambda)(b+\lambda)+b(r-1))+\sigma(b+\lambda-b(r-1))$$

$$= -(\sigma+\lambda)(b+(1+b)\lambda+\lambda^2+br-b)+\sigma(b-br+b)+\sigma\lambda$$

$$= -(\sigma+\lambda)(\lambda^2+(1+b)\lambda+br)+\sigma(2b-br)+\sigma\lambda$$

$$= -\sigma\lambda^2-\sigma(1+b)\lambda-\sigma br-\lambda^3-(1+b)\lambda^2-br\lambda+2\sigma b-b\sigma r+\sigma\lambda$$

$$= -\lambda^3-(\sigma+1+b)\lambda^2-b(r+\sigma)\lambda-2b\sigma(r-1)=0$$

$$= \lambda^3+(\sigma+1+b)\lambda^2+b(r+\sigma)\lambda+2b\sigma(r-1)=0$$

For Hopf bifurcation:

Assuming a root of the form $\lambda = iw$

Hopf-Bifurcation in Lorenz Model

$$\lambda^3 + (\sigma + 1 + b)\lambda^2 + b(r + \sigma)\lambda + 2b\sigma(r - 1) = 0$$

$$(iw)^3 + (\sigma + 1 + b)(iw)^2 + b(r + \sigma)(iw) + 2b\sigma(r - 1) = 0$$

$$-iw^3 - (\sigma + 1 + b)w^2 + ib(r + \sigma)w + 2b\sigma(r - 1) = 0$$

Both the real and imaginary parts have to separately equal 0.

$$-(\sigma + 1 + b)w^2 + 2b\sigma(r - 1) = 0 \Rightarrow w = \pm\sqrt{\frac{2b\sigma(r - 1)}{(\sigma + 1 + b)}}$$

$$-iw^3 + ib(r + \sigma)w = -iw(w^2 - b(r + \sigma)) = 0 \Rightarrow w = 0, \pm\sqrt{b(r + \sigma)}$$

Equating the two roots for w and solving for r

Hopf-Bifurcation in Lorenz Model

$$\pm\sqrt{\frac{2b\sigma(r-1)}{(\sigma+1+b)}} = \pm\sqrt{b(r+\sigma)}$$

$$\frac{2b\sigma(r-1)}{(\sigma+1+b)} = b(r+\sigma)$$

$$2b\sigma(r-1) = b(r+\sigma)(\sigma+1+b)$$

$$2b\sigma r - 2b\sigma = br(\sigma+1+b) + b\sigma(\sigma+1+b)$$

$$2b\sigma r - br(\sigma+1+b) = b\sigma(\sigma+1+b) + 2b\sigma$$

$$r(2b\sigma - b(\sigma+1+b)) = b\sigma(\sigma+1+b) + 2b\sigma$$

$$rb(\sigma-1-b) = b\sigma(\sigma+3+b)$$

$$r = \sigma\frac{(\sigma+3+b)}{(\sigma-1-b)}$$

$$\sigma = 10, b = 8/3, \qquad r_H \simeq 24.74$$