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MIDSEM
KARTIKEYA SAXENA

Ans-2- (i)- $\ddot{x} = ax - x^2$

$\dot{x} = v = f(x, v)$

$\dot{v} = ax - x^2 = g(x, v)$

Fixed pts.

$v = 0 \quad ax - x^2 = 0$

$v = 0 \quad (a-x)x = 0$

$x = 0$ or $x = a$

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a-2x & 0 \end{pmatrix}$$

$\tau = 0, \quad \Delta = -(a-2x) = 2x - a$

$(0,0) \rightarrow \tau = 0, \Delta = -a$

$\lambda^2 - a = 0$

$a > 0$: $\lambda_1 = \sqrt{a}, \lambda_2 = -\sqrt{a}$

Saddle node.

$(a, 0)$ $\tau = 0, \Delta = a$

$\lambda^2 + a = 0$

$\lambda_1 = -i\sqrt{a} \quad \lambda_2 = i\sqrt{-a}$

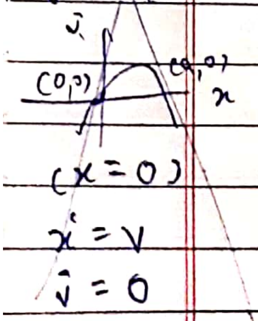
center (Border line) case

Nullclines

$(v = 0)$

$\dot{x} = 0$

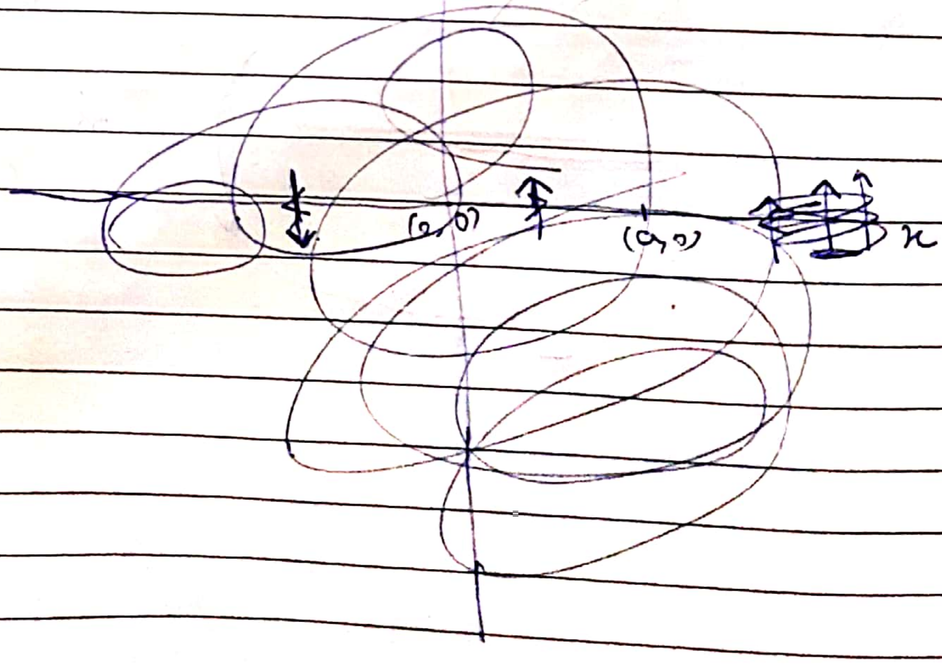
$\dot{v} = x(a-x)$



$(x = 0)$

$\dot{x} = v$

$\dot{v} = 0$



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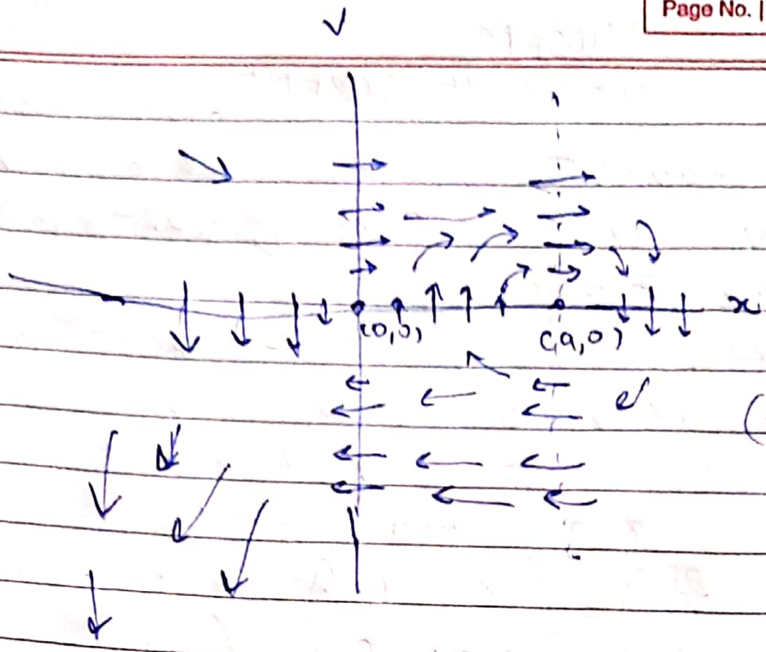
$a > 0$



$\dot{x} = v$

$\dot{v} = ax - x^2$

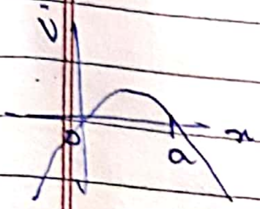
(center)

Null clience ~~then~~

$(v = 0)$

$\dot{x} = 0$

$\dot{v} = ax - x^2$



$(x = 0)$

$\dot{x} = v$

$\dot{v} = 0$

In IIIrd quadrant

$\dot{x} < 0$

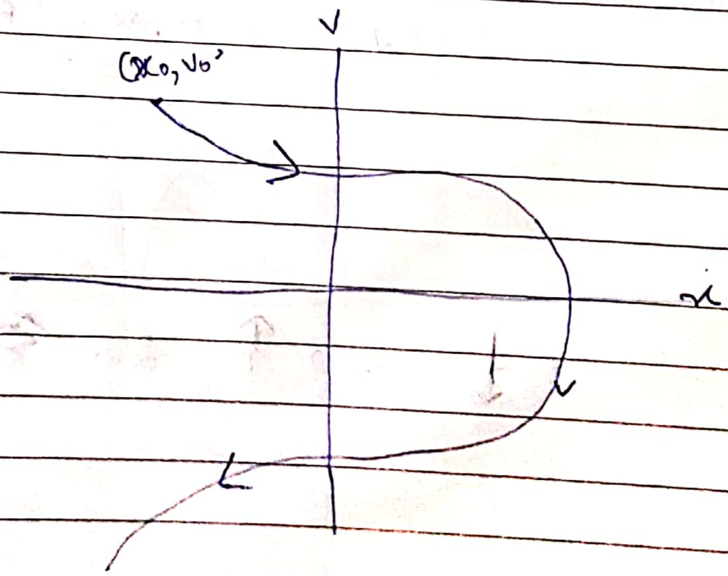
$\dot{v} < 0$ and eventually

$\dot{v}$ becomes dominant due to quadratic terms

$(x = a)$

$\dot{x} = v$

$\dot{v} = 0$



Example point.

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(iii) $\dot{r} = r(1-r^2) = r - r^3$

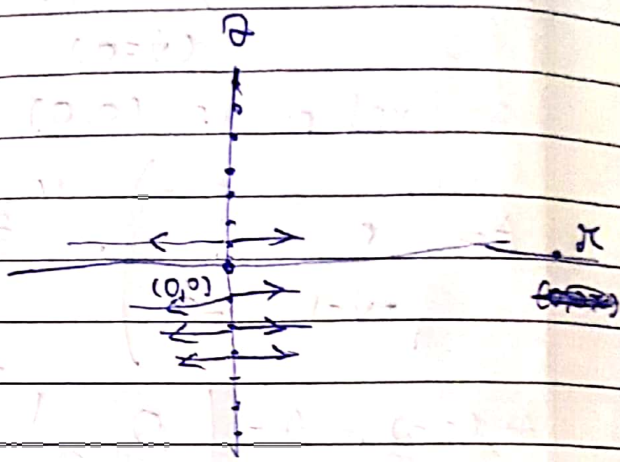
$\dot{\theta} = 1 - \cos\theta$

Fixed pts

$r = 0, \pm 1$ and $\theta = 2n\pi, n \in \mathbb{Z}$

$$A = \begin{pmatrix} 1-3r^2 & 0 \\ 0 & \sin\theta \end{pmatrix}$$

$$(0, 2n\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



$\tau = 1 \qquad \Delta = 0$

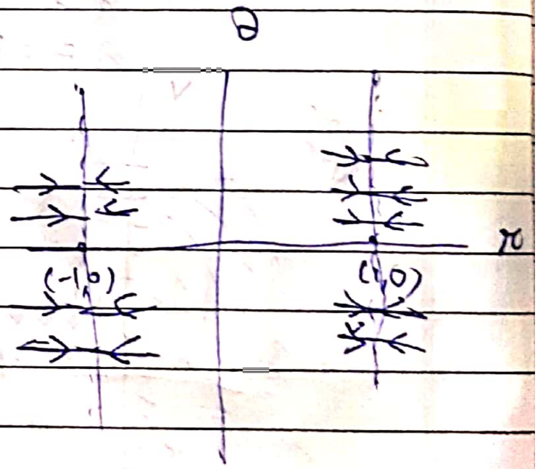
$\lambda^2 - \lambda = 0$

$\lambda(\lambda - 1) = 0$

$\lambda_1 = 0 \qquad \lambda_2 = 1$

$V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$(\pm 1, 2n\pi) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$



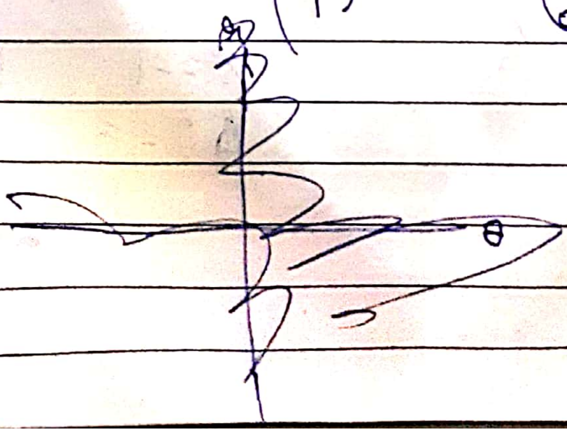
$\tau = -2 \qquad \Delta = 0$

$\lambda^2 + 2\lambda = 0$

$\lambda(\lambda + 2) = 0$

$\lambda_1 = 0 , \lambda_1 = -2$

$V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



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iii) → $\ddot{x} + x\dot{x} + x = 0$

$\dot{x} = v = f(x,v)$

$\dot{v} = -xv - x = g(x,v)$

Fixed pts.

$v = 0$, $-xv - x = 0$

$(\dot{x} = 0)$ $\quad -x(v+1) = 0$

$x = 0,\ v = -1$

$(\dot{v} = 0)$

so fixed pt. is $(0,0)$

$$A = \begin{pmatrix} 0 & 1 \\ -v-1 & -x \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

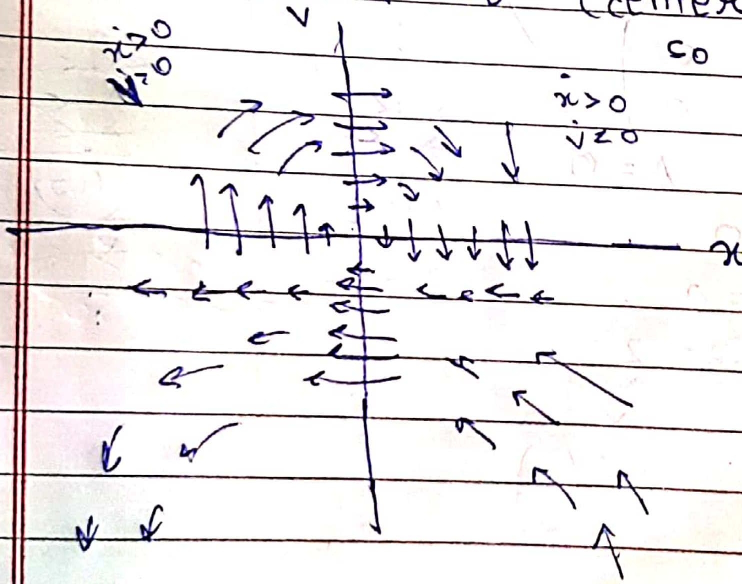
At $(0,0)$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\tau = 0$, $\Delta = 1$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$ (center) Borderline case

so not sure



(center)

Null clines

$\dot{x} = 0 = v$

$\dot{v} = -x$

$\dot{x} = v$ $\quad (x=0)$

$\dot{v} = 0$

$\dot{x} = -1$ $\quad (v = -1)$

$\dot{v} = 0$

Ans-1- $U(x) = -x^2$

$-U(x) = x^2$

$-\dfrac{dU(x)}{dx} = 2x = \ddot{x}$

$\ddot{x} = 2x$

$\dot{x} = v$

$\dot{v} = 2x$

Fixed pt. $(0,0)$

$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

$\tau = 0, \ \Delta = -2$

$\lambda^2 - 2 = 0$

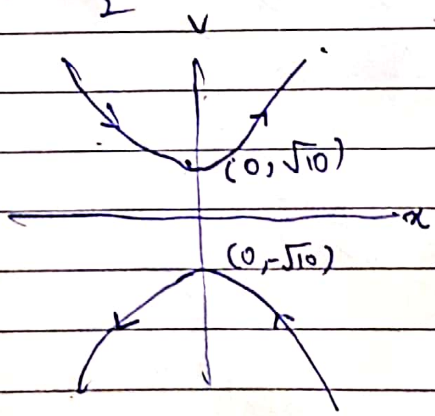
$\lambda = \pm\sqrt{2}$ saddle node

$E = \frac{1}{2}v^2 + (-x^2) = \frac{v^2}{2} - x^2$

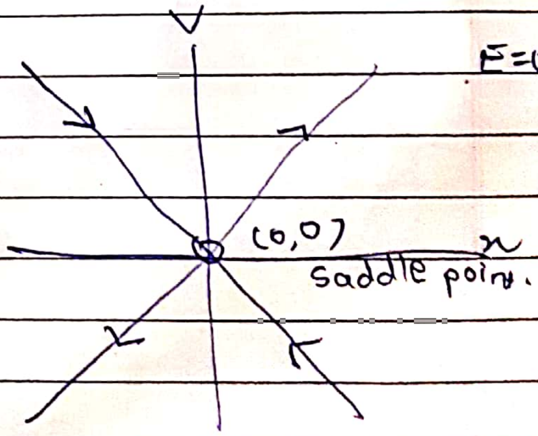
$E = 5, \quad \frac{v^2}{2} - x^2 = 5$ $\quad E = -5 \quad \frac{v^2}{2} - x^2 = -5$

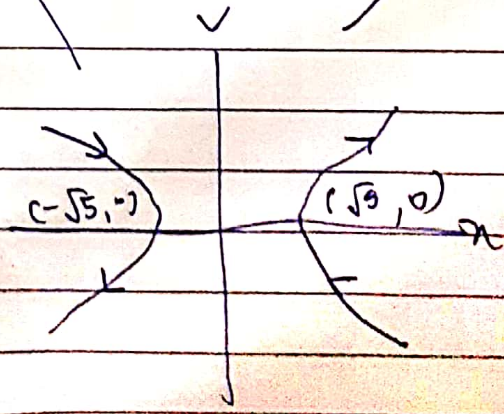
$E = 0, \quad \frac{v^2}{2} - x^2 = 0$

E=5



E=0





E=-5

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Ans-3: $\dot{N} = -aN\ln(bN) = f(N) \quad a,b > 0$

fixed pt. $f(N^*) = 0$

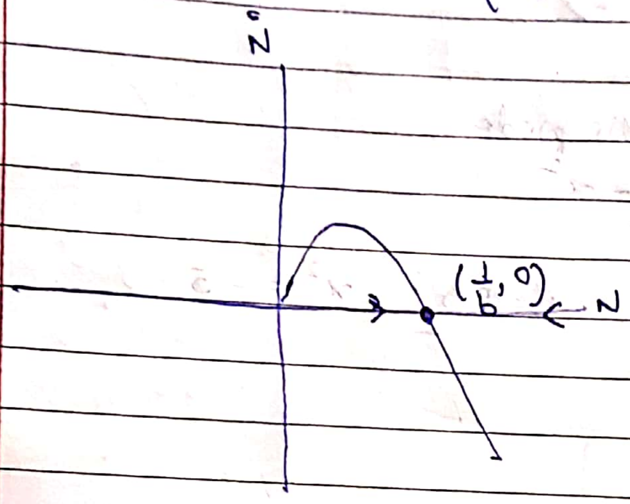
$-aN^*\ln(bN^*) = 0$

$N^* = \frac{1}{b}$

$$\frac{df(N)}{dN} = -a\left(\ln(bN) + \frac{\cancel{N}}{\cancel{b}\cancel{N}}\cdot\cancel{b}\right)$$

$$f'(N) = -a\left(\ln(bN) + 1\right)$$

$$f'\left(\frac{1}{b}\right) = -a(1) = -a < 0 \text{ stable}$$



a influence the slope at fixed pt. and b decides the fixed pt.

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~~Ans-4- $\dot{x} = \mu x - x^2$~~
~~$\dot{y} = -y$~~

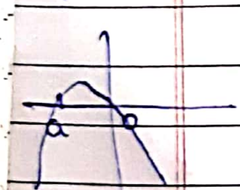
Ans-1- continued.

$a<0$ $(0,0) \to \tau = 0, \Delta = -a$
$\lambda^2 - a = 0$
$\lambda = \pm i\sqrt{-a}$
center
border line case
So not sure.

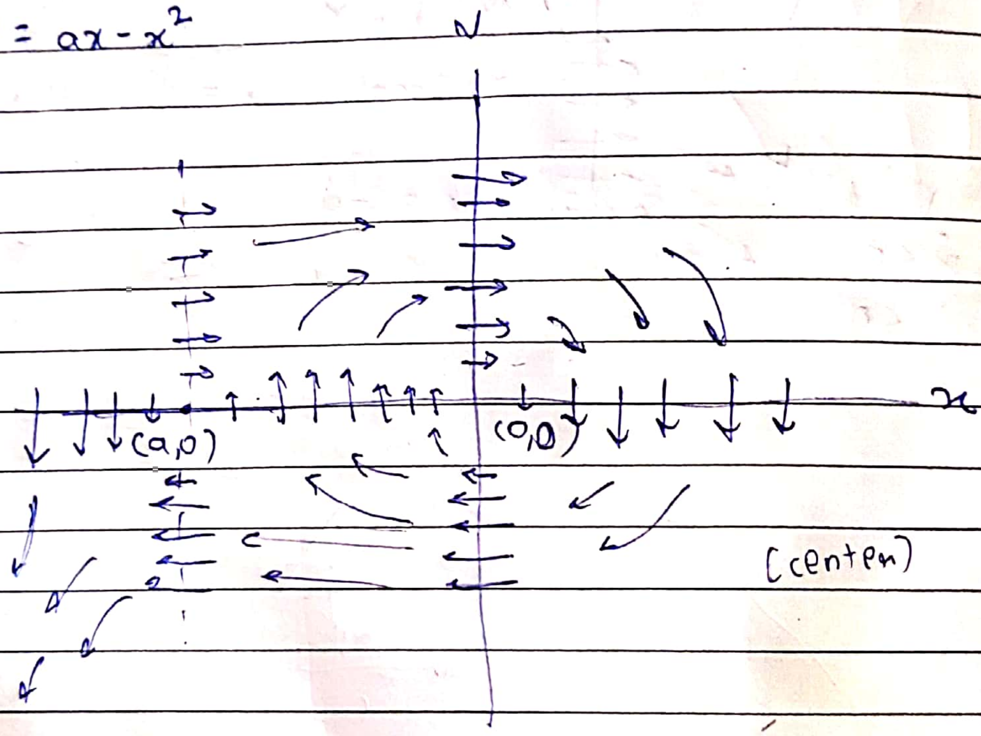
$(a,0)$ $\tau = 0, \Delta = a$
$\lambda^2 + a = 0$
$\lambda = \pm\sqrt{-a}$
Saddle point

$\dot{x} = v$
$\dot{v} = ax - x^2$

Null cline
$(v=0)$
$\dot{x} = 0$
$\dot{v} = x(a-x)$



$(x=0)$
$\dot{x} = v$
$\dot{v} = 0$
$(x=a)$
$\dot{x} = v$
$\dot{v} = 0$



Same graph for $a>0$
shifted and ~~node~~ fixed
Point stability swaps

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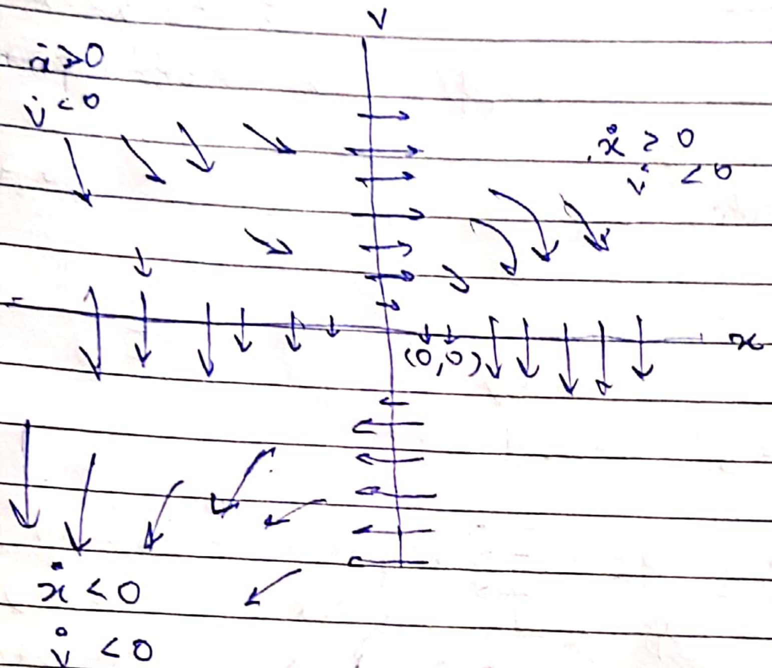
$\dot{x} = v$
$\dot{v} = ax - x^2$

$a=0$ $\quad \dot{x} = v$
$\dot{v} = -x^2$

Single fixed pt. $(0,0)$.

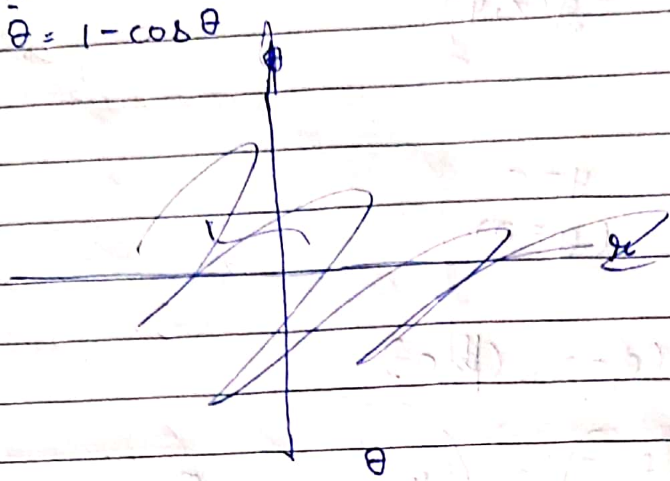


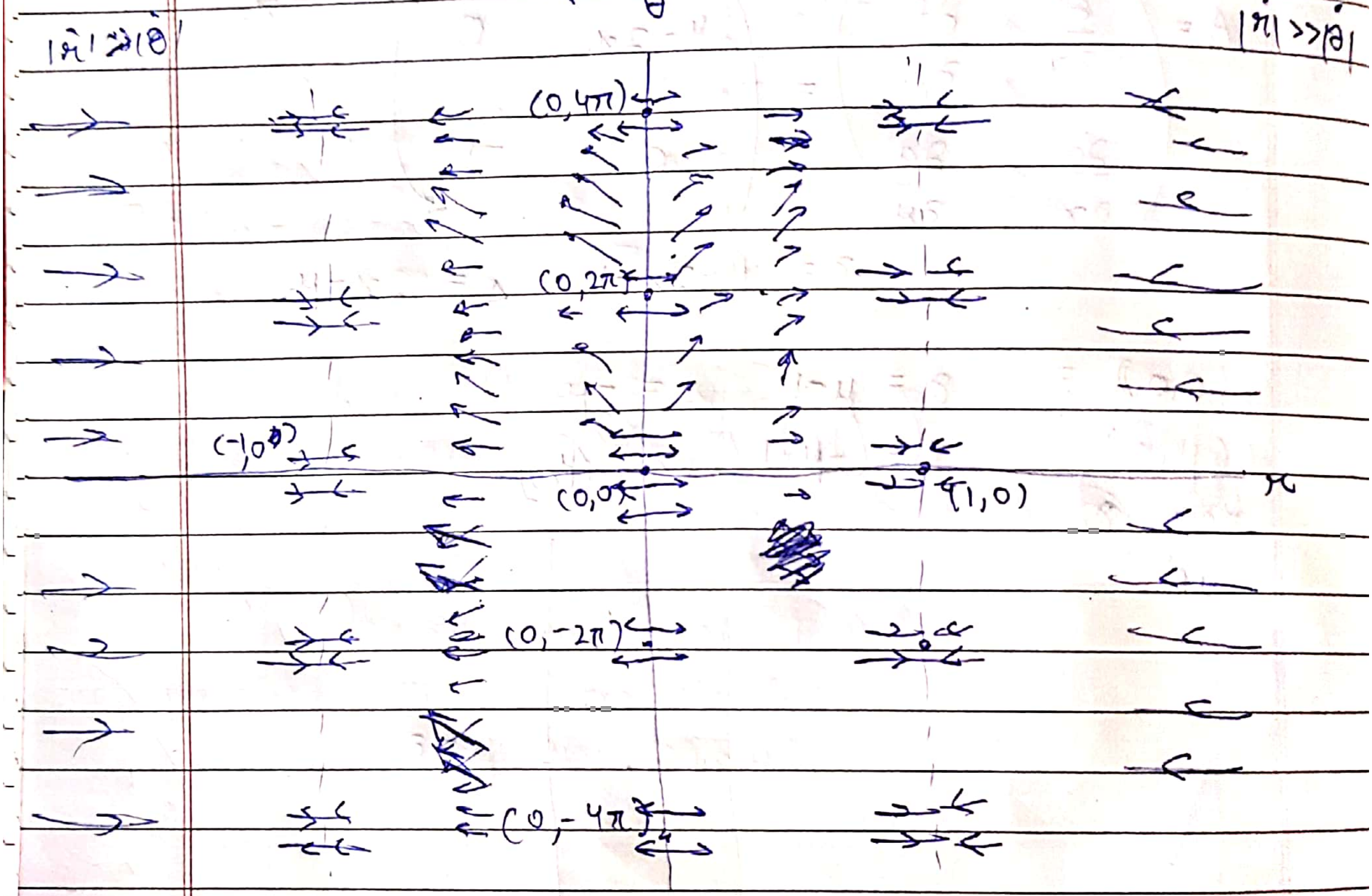
Null clines.
$(v=0)$
$\dot{x} = 0$
$\dot{v} = -x^2$
$(x=0)$
$\dot{x} = v$
$v = 0$

continued

Ans-2- (ii) $\dot{r} = r(1-r^2)$

$\dot{\theta} = 1 - \cos\theta$





along the axis the ~~vec~~ osscillates and become horizontal at each 2π interval

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Ans-4 a) $\dot{x} = \mu x - x^2 = f(x,y)$

$\dot{y} = -y = g(x,y)$

Fixed pts

$\mu x - x^2 = 0$ $\qquad$ $y = 0$

$x = 0$ or μ $\qquad$ $(\dot{y} = 0)$

$(\dot{x} = 0)$

Fixed pts = $(0,0)$ $(\mu,0)$

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \mu - 2x & 0 \\ 0 & -1 \end{pmatrix}$$

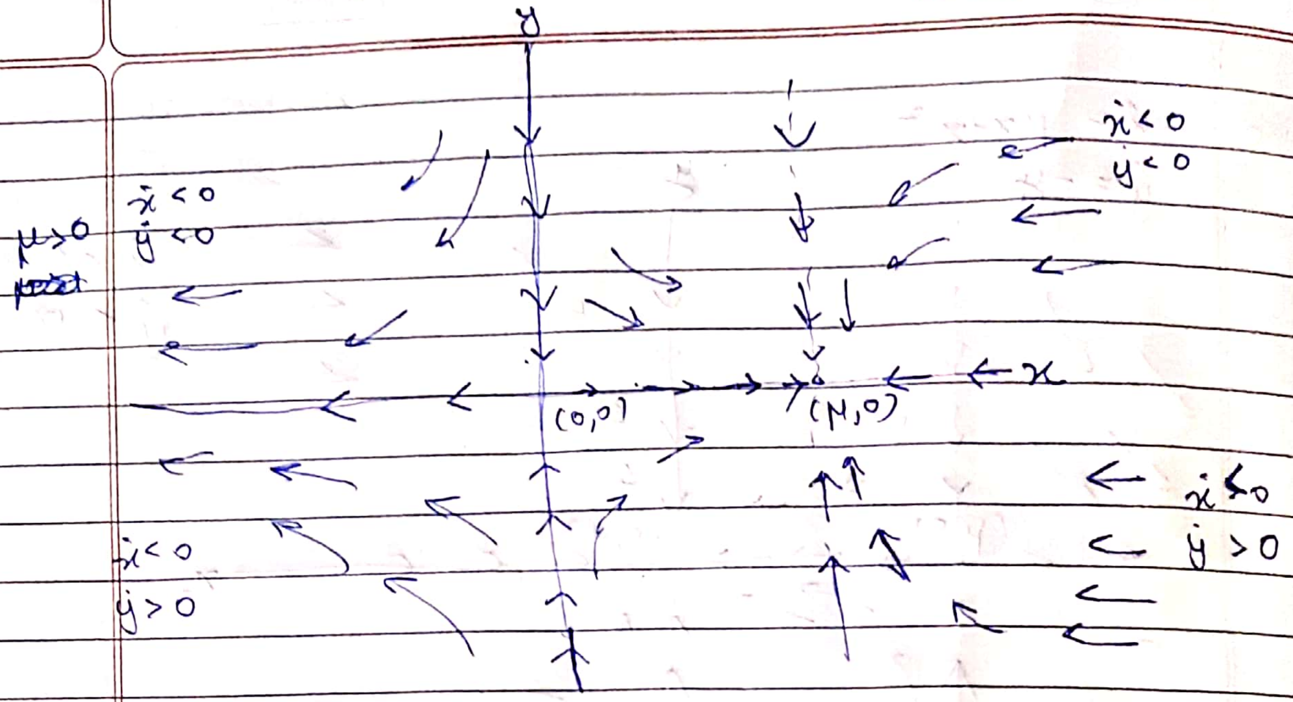
$\tau = \mu - 2x - 1$ $\qquad$ $\Delta = 2x - \mu$

$(0,0) =$ $\qquad$ $\tau = \mu - 1$, $\Delta = -\mu$

$(\mu,0)$ $\qquad$ $\tau = -\mu - 1$ $\quad$ $\Delta = \mu$

$\mu < 0$

$\mu > 0$



Null clines

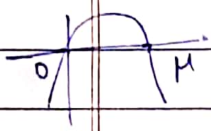
(0,0) - saddle

$(\mu, 0)$ - stable node

$(y=0)$

$\dot{x} = \mu x - x^2 = x(\mu - x)$

$\dot{y} = 0$



$(x=0)$

$\dot{x} = 0$

$\dot{y} = -y$

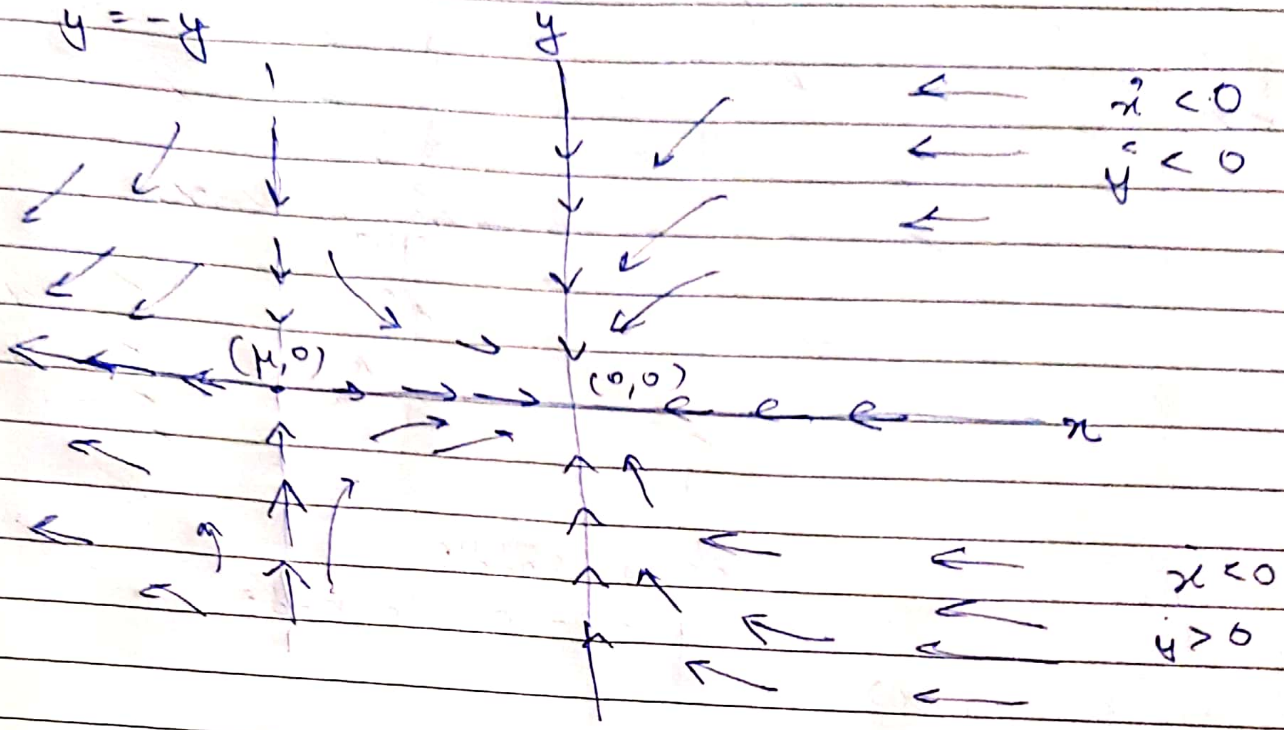
$(x=\mu)$

$\dot{x} = 0$

$\dot{y} = -y$

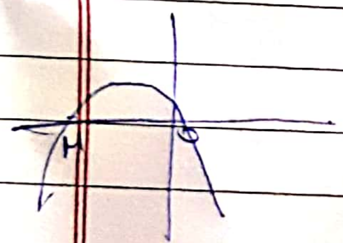
$\mu < 0$

$\dot{x} = \mu x - x^2$
$\dot{y} = -y$



Null clines
$(y = 0)$
$\dot{x} = (\mu - x)x$
$\dot{y} = 0$

$(\mu, 0)$ saddle node
$(0,0)$ stable node.



It is a transcritical bifurcation as the stability swaps at $\mu = 0$

$(x = \mu)$
$\dot{x} = 0$
$\dot{y} = -y$