Steps in Grover's algorithm

1. Generate the standard state

$$|s\rangle = \frac{1}{\sqrt{N}} \sum |x\rangle$$

2. The $(n+1)$th qubit is initialized to $\boxed{\frac{|0\rangle - |1\rangle}{\sqrt{2}}}$

3. Loop is applied m-times
(i.e. apply Grover operation m-times)

U_w (a) Apply the oracle (T-operation)

$$|s\rangle|y\rangle \longrightarrow \sum_{x=0}^{N-1} a_x |x\rangle \, \underline{(-1)^{f(x)}} \, |y\rangle$$

U_s (b) Apply Diffusion operators

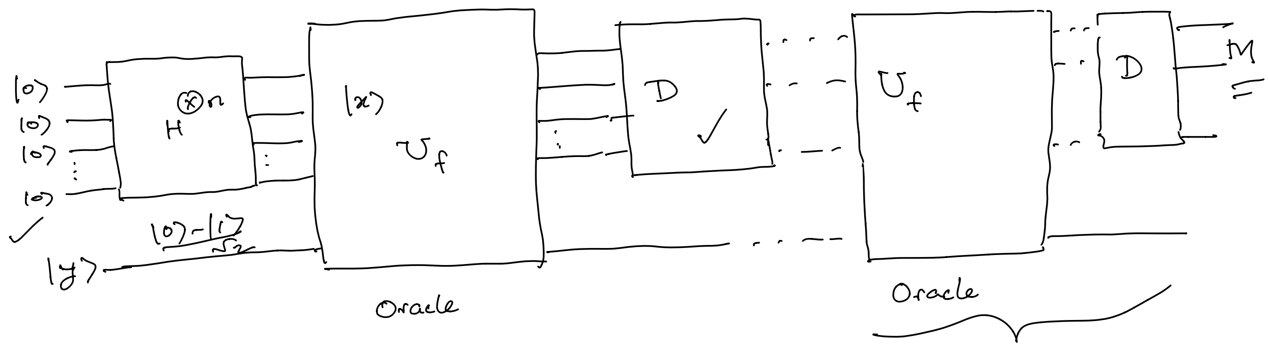
$$D = WRW$$

(c) Apply steps (a) and (b) $O(\sqrt{N})$ number of times

4. Measure the first register. With a very high probability, we can identify the marked state w.

5. The algorithm may fail to yield result, the probability of that is $O\left(\frac{1}{N}\right)$

Then go to step number 1



$$\left(D = -I + \frac{2J}{N} \right)$$

U_s

$$D|\psi\rangle = \sum \left(2\bar{a} - a_x \right) |x\rangle$$

$$|s\rangle \rightarrow |u\rangle, \; |w\rangle$$

Say, $|u\rangle$ denote the linear combination of all states for which $f(x) = 0$

$$|u\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq w} |x\rangle \qquad |s\rangle = \frac{1}{\sqrt{N}} \sum |x\rangle$$

$$\boxed{|s\rangle = \frac{1}{\sqrt{N}} |w\rangle + \sqrt{\frac{N-1}{N}} |u\rangle}$$

The marked state $|w\rangle$ is going to be inverted by operation T

Suppose, after j-th iterations, the amplitude of each unmarked state is u_j while that of the marked state is w_j

After (j+1) th iteration:

$$D = -I + \frac{2J}{N}$$

$$\begin{pmatrix} u_{j+1} \\ u_{j+1} \\ \vdots \end{pmatrix} = -\begin{pmatrix} u_j \\ u_j \\ \vdots \end{pmatrix} + \frac{2}{N} \begin{pmatrix} 1 & 1 & \cdots \\ 1 & 1 & \cdots \end{pmatrix} \begin{pmatrix} u_j \\ u_j \\ \vdots \\ w_j \end{pmatrix}$$

$$\begin{pmatrix} \vdots \\ w_{j+1} \\ \vdots \\ u_{j+1} \end{pmatrix} \qquad \begin{pmatrix} w_j \\ \vdots \\ u_j \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ u_j \end{pmatrix}$$

$$u_{j+1} = -u_j + \frac{2}{N}\left[(N-1)u_j + w_j\right]$$

$$= \frac{2}{N} w_j + \frac{1}{N}\left[2(N-1) - N\right] u_j$$

$$\boxed{u_{j+1} = \frac{2}{N} w_j + \frac{N-2}{N} u_j}$$

$$w_{j+1} = -w_j + \frac{2}{N}\left[(N-1)u_j + w_j\right]$$

$$\Rightarrow \boxed{w_{j+1} = \left(\frac{2}{N} - 1\right) w_j + \frac{2}{N}(N-1) u_j}$$

Before we calculated $(j+1)$ th item we have inverted the j-th value of w_j

$$\boxed{\begin{aligned} w_{j+1} &= \left(1 - \frac{2}{N}\right) w_j + \frac{2}{N}(N-1) u_j \\ u_{j+1} &= -\frac{2}{N} w_j + \frac{N-2}{N} u_j \end{aligned}}$$

$\rightarrow \sqrt{N-1}$

Define: $c_j = \sqrt{N-1}\, u_j$

$$\begin{pmatrix} w_{j+1} \\ c_{j+1} \end{pmatrix} = \begin{pmatrix} 1 - \frac{2}{N} & \frac{2}{N}\sqrt{(N-1)} \\ -\frac{2}{N}\sqrt{N-1} & 1 - \frac{2}{N} \end{pmatrix} \begin{pmatrix} w_j \\ c_j \end{pmatrix}$$

Recall, $\sin\theta = \frac{1}{\sqrt{N}}$, $\cos\theta = \sqrt{1 - \frac{1}{N}}$

$$\boxed{\begin{pmatrix} w_{j+1} \\ c_{j+1} \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} w_j \\ c_j \end{pmatrix}}$$

$$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}^{j} \begin{pmatrix} w_1 \\ c_1 \end{pmatrix}$$

$$= \begin{pmatrix} \sin(2j+1)\theta \\ \cos(2j+1)\theta \end{pmatrix} \qquad \begin{pmatrix} w_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

$\Rightarrow$ The measurement of the first register will give marked state with a probability $\sin^2(2m+1)\theta$

Example

$N = 8 \qquad D = -I + \frac{2J}{N}$

$$D = \begin{pmatrix} -0.75 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & -0.75 & 0.25 & & & & & \\ \vdots & & & & & & & \\ 0.25 & & & & & & & \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \vdots & & & & & \end{pmatrix}$$

$$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \cdots$$

$$DT = \begin{pmatrix} -0.75 & .25 & 0.25 & \cdots \\ 0.25 & -.75 & 0.25 & \\ 0.25 & & \ddots & \\ & & & +.75 \end{pmatrix}$$

$$DT \begin{pmatrix} u \\ u \\ u \\ w \\ u \\ u \\ u \\ u \end{pmatrix} = \begin{pmatrix} .75u - .25w \\ .75u - .25w \\ 0.75u - .25w \\ 1.75u + 0.75w \\ \\ \end{pmatrix}$$

$$u_1 = 0{\cdot}75\, u_0 - 0{\cdot}25\, w_0$$
$$w_1 = 1{\cdot}75\, u_0 + 0{\cdot}75\, w_0$$
$$7u_1^2 + w_1^2 = 7u_0^2 + w_0^2$$
$$\begin{pmatrix} u_n \\ w_n \end{pmatrix} = \begin{pmatrix} 0{\cdot}75 & -0{\cdot}25 \\ 1{\cdot}75 & 0{\cdot}75 \end{pmatrix}^k \begin{pmatrix} u_0 \\ w_0 \end{pmatrix}$$

How many iterations are required to find one marked item out of 64 items in an unstructured database using Grover's algorithm?

$$m = \frac{\pi}{4}\sqrt{N} = \frac{\pi}{4}\sqrt{64} = \frac{\pi \times 8}{4} \approx 6$$

What's the probability of failure of Grover's algorithm for finding a single marked state out of 8 items after two applications of Grover's rotations?

Solution

Here $N = 8$

Amplitude of each state $= \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$

Application U_w on $|s\rangle$ inverts components parallel to w:

$$a_w = -\frac{1}{2\sqrt{2}}$$

all other states $a_x = +\frac{1}{2\sqrt{2}}$

mean amplitude $\bar{a} = \frac{1}{8}\left[7\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right] = \frac{3}{8\sqrt{2}}$

Now apply U_s : $a_x \rightarrow 2\bar{a} - a_x$

$$a_w \rightarrow 2\left(\frac{3}{8\sqrt{2}}\right) - \left(-\frac{1}{2\sqrt{2}}\right) = \frac{5}{2\sqrt{2}}$$

other $a_x \rightarrow 2\left(\frac{3}{8\sqrt{2}}\right) - \left(\frac{1}{2\sqrt{2}}\right) = \frac{1}{4\sqrt{2}}$

$\Rightarrow$ after one iteration, amplitude of the marked state is 5 times that of unmarked state

2nd iteration

U_w :

$$a_w = -\frac{5}{4\sqrt{2}}$$

$$a_x = \frac{1}{4\sqrt{2}}$$

New mean

$$\bar{a} = \frac{1}{8}\left[7\,\frac{1}{4\sqrt{2}} - \frac{5}{4\sqrt{2}}\right] = \frac{1}{16\sqrt{2}}$$

Apply U_s :

$$a_x \rightarrow 2\bar{a} - a_x$$

$$a_w \rightarrow 2\left[\frac{1}{16\sqrt{2}}\right] - \left(-\frac{5}{4\sqrt{2}}\right) = \frac{11}{8\sqrt{2}}$$

Probability of success after two iterations

$$= |a_w|^2 = \frac{121}{128}$$

$$\Rightarrow \boxed{\text{Probability of failure} = 1 - \frac{121}{128} = \frac{7}{128}}$$