1. Find the normalized state $|\psi\rangle$ corresponding to the density matrix

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\left[I_2 + \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)\right]$$

Solution

$$|\psi\rangle\langle\psi| = \frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right]$$

$|\psi_1\rangle \neq |\psi_2\rangle$

$\rho_1 = \rho_2$

$$= \frac{1}{2}\left[\begin{pmatrix}1+\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}}\end{pmatrix}\right]$$

$$|\psi\rangle = \begin{pmatrix}a\\ b\end{pmatrix}$$

$$|\psi\rangle\langle\psi| = \begin{pmatrix}a\\ b\end{pmatrix}\begin{pmatrix}a^* & b^*\end{pmatrix}$$

$$= \begin{pmatrix}a^2 & ab\\ ba & b^2\end{pmatrix}$$

$$a^2 = \frac{\sqrt{2}+1}{2\sqrt{2}}, \quad b^2 = \frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$|\psi\rangle = \begin{pmatrix}\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}\\ \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}\end{pmatrix}$$

2. Given

$$\rho = \begin{pmatrix} \underline{\underline{3/4}} & \sqrt{2}e^{-i\phi}/4 \\ \sqrt{2}e^{i\phi}/4 & \underline{\underline{2/4}} \end{pmatrix}$$

(a) Is the matrix a <u>density matrix</u>?

(b) Is it a pure or mixed state?

(c) Find eigenvalues of ρ

(d) Find $tr(\sigma_x \rho)$

- $tr(\rho) = 1$
- $\rho = \rho^{\dagger}$
- Eigenvalues are non-negative

$$\underline{\rho^2 = \rho} \quad \text{(pure state)}$$

$$\rho^2 \neq \rho \quad \text{(mixed state)}$$

$$Tr(\rho^2) < 0 \quad (\quad '' \quad)$$

$$\rho^2 = \begin{pmatrix} 3/4 & \frac{\sqrt{2}e^{-i\phi}}{4} \\ \frac{\sqrt{2}e^{i\phi}}{4} & 1/4 \end{pmatrix} \begin{pmatrix} 3/4 & \frac{\sqrt{2}e^{-i\phi}}{4} \\ \frac{\sqrt{2}e^{i\phi}}{4} & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{11}{16} & \frac{\sqrt{2}}{4} e^{-i\phi} \\ \frac{\sqrt{2}}{4} e^{i\phi} & \frac{3}{16} \end{pmatrix}$$

$$\neq \rho$$

ρ represents a mixed state

$$\mathrm{Tr}(\rho^2) = \frac{14}{16} < 1$$

(c) $$\begin{vmatrix} 3/4 - \lambda & \frac{\sqrt{2}}{4} e^{-i\phi} \\ \frac{\sqrt{2}}{4} e^{i\phi} & 1/4 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - \lambda + \frac{1}{16} = 0$$

$$\Rightarrow \quad \lambda = \frac{2 \pm \sqrt{3}}{4}$$

$$\lambda_1 = \frac{2+\sqrt{3}}{4}, \quad \lambda_2 = \frac{2-\sqrt{3}}{4}$$

(d) $$\mathrm{Tr}(\sigma_x \rho) = \mathrm{Tr}(\rho \sigma_x)$$

$$= \mathrm{tr}\left[\begin{pmatrix} 3/4 & \frac{\sqrt{2} e^{-i\phi}}{4} \\ \frac{\sqrt{2} e^{i\phi}}{4} & 1/4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{2}} \cos\phi$$

3. Consider the following mixed states in the computational basis:

(i) $\frac{1}{2}\left\{\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle\right\} + \frac{1}{2}\left\{\frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle\right\}$

(ii) $\frac{9}{25}|0\rangle + \frac{16}{25}|1\rangle$ $\qquad \cancel{c_1|0\rangle + c_2|1\rangle}$

$\left|\frac{9}{25}\right|^2 + \left|\frac{16}{25}\right|^2 \neq 1$

Find the density operator corresponding to (i) and (ii)

Solution

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$$

(i) $|\psi\rangle = \frac{1}{2}|\psi_1\rangle + \frac{1}{2}|\psi_2\rangle$

$|\psi_1\rangle = \frac{3}{5}|0\rangle + 4/5\,|1\rangle$

$|\psi_2\rangle = \frac{3}{5}|0\rangle - 4/5\,|1\rangle$

$$\rho = \frac{1}{2}\begin{pmatrix} 9/25 & \frac{12}{25} \\ 12/25 & 16/25 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{pmatrix}$$

$\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$

$$= \begin{pmatrix} 9/25 & 0 \\ 0 & 16/25 \end{pmatrix}$$

$|\psi_1\rangle\langle\psi_1| = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}\begin{pmatrix} 3/5 & 4/5 \end{pmatrix}$

$|\psi_2\rangle\langle\psi_2| = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}\begin{pmatrix} 3/5 & -4/5 \end{pmatrix}$

(ii) $$|\psi\rangle = \underline{\underline{\frac{9}{25}}}\,|0\rangle + \underline{\underline{\frac{16}{25}}}\,|1\rangle$$

$$= \frac{9}{25}\,|\psi_1\rangle + \frac{16}{25}\,|\psi_2\rangle$$

$$\rho = \frac{9}{25}\,|0\rangle\langle 0| + \frac{16}{25}\,|1\rangle\langle 1|$$

$$= \underline{\underline{\begin{pmatrix} 9/25 & 0 \\ 0 & 16/25 \end{pmatrix}}}$$

$$\rho_{(ii)} = \underline{\rho_{(i)}}$$

mixed states in (i) and (ii) are <u>indistinguishable</u>

4. Consider the entangled state

$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle \right)$ with $\rho = |\psi\rangle\langle\psi|$

Find the reduced density matrix ρ_1.

Solution

$$\rho = |\psi\rangle\langle\psi|$$

$$= \frac{1}{2} \left[|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10| \right]$$

$$\rho_1 = tr_2(\rho)$$

$$= \frac{1}{2} \left[|0\rangle\langle 0| \, tr(|1\rangle\langle 1|) + |1\rangle\langle 1| \right]$$

$$= \frac{1}{2} \left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$$

5. Consider the entangled state in four-dimensional vector space

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{|0\rangle} \otimes \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{|1\rangle} - \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \right]$$

Find the density matrix $\rho = |\psi\rangle\langle\psi|$

Find $\rho_1 = tr_2(\rho)$, $\rho_2 = tr_1(\rho)$

Solution

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle \right]$$

$$\rho = \frac{1}{2}\left[|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10| \right]$$

$$\rho_1 = tr_2(\rho) = \frac{1}{2}\left[\underline{|0\rangle\langle 0| + |1\rangle\langle 1|} \right]$$

$$\rho_2 = tr_1(\rho) = \frac{1}{2}\left[|1\rangle\langle 1| + |0\rangle\langle 0| \right]$$

6. Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle$$

and the product state $|\phi\rangle = |11\rangle$

Find the probability of finding $|\psi\rangle$ in the state $|\phi\rangle$.

Solution

$$\left|\langle\phi|\psi\rangle\right|^2$$

$$= \frac{2}{3}$$

7. ~~Exercise 2.57~~ Problem (**Cascaded measurements are single measurements**) Suppose $\{L_l\}$ and $\{M_m\}$ are two sets of measurement operators. Show that a measurement defined by the measurement operators $\{L_l\}$ followed by a measurement defined by the measurement operators $\{M_m\}$ is physically equivalent to a single measurement defined by measurement operators $\{N_{lm}\}$ with the representation $N_{lm} \equiv M_m L_l$.

Solution

$$\{L_l\}$$
$$\{M_m\}$$

$$M\,L\,|\psi\rangle$$

$$N = ML$$

$$L\,|\psi\rangle$$

post msmt state after 'L':

$$|\phi\rangle = \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}$$

$$\langle\phi|M_m^\dagger M_m|\phi\rangle = \frac{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}{\langle\psi|L_l^\dagger L_l|\psi\rangle}$$

Post msmt state

$$\frac{M_m|\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}} = \frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}}$$

$$= N_{lm}|\psi\rangle \Big/ \sqrt{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}$$

8.

Exercise [redacted]8: Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M, with corresponding eigenvalue m. What is the average observed value of M, and the standard deviation?

Solution

$$\langle M \rangle = \langle \psi | M | \psi \rangle = m$$

$$(\Delta M)^2 = \langle M^2 \rangle - \langle M \rangle^2$$

$$\langle M^2 \rangle = \langle \psi | M^2 | \psi \rangle = m^2$$

$$(\Delta M)^2 = 0$$

9.

Exercise 2.09 Show that any measurement where the measurement operators and the POVM elements coincide is a projective measurement.

Solution

Let M_m be a measurement operator.

Now, given : measurement operator and the POVM element coincide

$\Rightarrow$ $E_m = M_m^\dagger M_m$

$= M_m$

POVM element

measurement operator

Recall, in the class we wrote $E_i = M_i^\dagger M_i$ which was due to the fact that E_i is hermitian. M_i is not necessarily a measurement operator

Now,

$\langle\psi|E_m|\psi\rangle \geq 0$

$\Rightarrow \langle\psi|M_m|\psi\rangle \geq 0 \,, \; \forall\, |\psi\rangle$

$\Rightarrow$ M_m is a positive operator

$\Downarrow$

M_m is Hermitian

Thus,

$E_m = M_m^\dagger M_m$

$= M_m M_m$

$= M_m^2$

$= M_m$

$\Rightarrow$ M_m is a projector

$\Rightarrow$ The measurement is a projective measurement.