Date: / /
Page No.

SOLUTIONS:

Karthikeya Saxena, 180101034

Ans-1 $f(x_n) = x_{n+1} = \begin{cases} 2x_n & 0 \le x_n \le 1/2 \\ 2 - 2x_n & 1/2 \le x_n \le 1 \end{cases}$

i) - for fixed pts. $f(x) = x$

$x = 2x \quad 0 \le x \le 1/2$

$x = 0$

and $x = 2 - 2x \quad 1/2 \le x \le 1$

$3x = 2$

$x = 2/3$

Now for stability, we calculate $|f'(x)|$

$f'(x) = \begin{cases} 2 & 0 \le x \le 1/2 \\ -2 & 1/2 \le x \le 1 \end{cases}$

$|f'(x)| = 2 > 1 \quad \forall \quad 0 \le x \le 1$

Therefore $x = 0, 2/3$ are both unstable

ii) - for period two cycle : $f^2(x) = x$

$f(f(x)) = x$

Case:1- $0 \le x \le 1/2 \quad 0 \le f(x) \le 1/2$

$f(2x) = x$

$4x = x$

$x = 0 \rightarrow$ fixed pt.

Case:2- $0 \le x \le 1/2 \quad 1/2 \le f(x) \le 1$

$f(2x) = x$

$2 - 4x = x$

$\left.\begin{array}{l} x = 2/5 \\ f(x) = 4/5 \end{array}\right\}$ 2-cycle

Case:3- $1/2 \le x \le 1 \quad 0 \le f(x) \le 1/2$

$f(2 - 2x) = x$

$4 - 4x = x$

$\left.\begin{array}{l} x = 4/5 \\ f(x) = 2/5 \end{array}\right\}$ 2-cycle

Date : / /
Page No.

Case:4 $\frac{1}{2} \le x < 1$ $\frac{1}{2} < f(x) \le 1$

$f(2-2x) = x$

$2 - 2(2-2x) = x$

$2 - 4 + 4x = x$

$3x = 2$

$x = 2/3$ — fixed pt.

For stability: we need to calculate

$|f'(p)||f'(q)| = 4 > 1$ unstable. where p & q are pts in 2 cycle but $|f'(x)| = 2$ $\forall x \in [0,1]$

c) $\delta_0 e^{\lambda n} = \delta_n$ λ = lyapunov exponent

$n \to \infty$

$$e^{\lambda n} = \frac{\delta_n}{\delta_0}$$

$$\lambda = \frac{1}{n} \ln\left|\frac{\delta_n}{\delta_0}\right|$$

$$= \frac{1}{n} \ln\left|\frac{f^n(x_0+\delta_0) - f^n(x_0)}{x_0+\delta_0 - x_0}\right|$$

$$= \frac{1}{n} \ln|f^n(x_0)'|$$ $n \to \infty$

$$\lambda = \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)|$$ $x_i = f(x_{i-1})$

$$= \frac{1}{n} \times n \ln 2$$

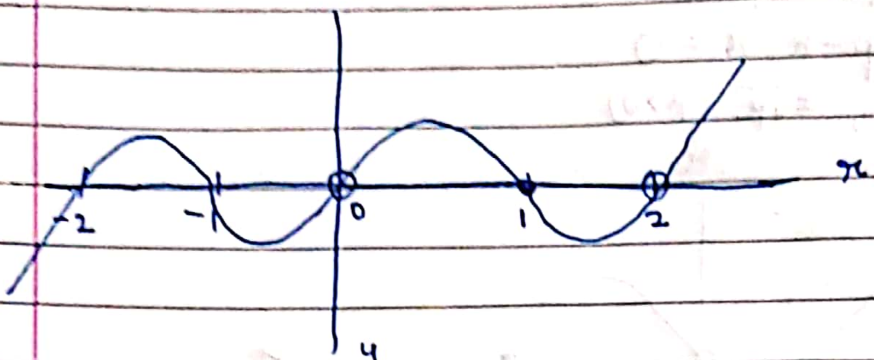
$$\lambda = \ln 2 = 0.693 > 0$$

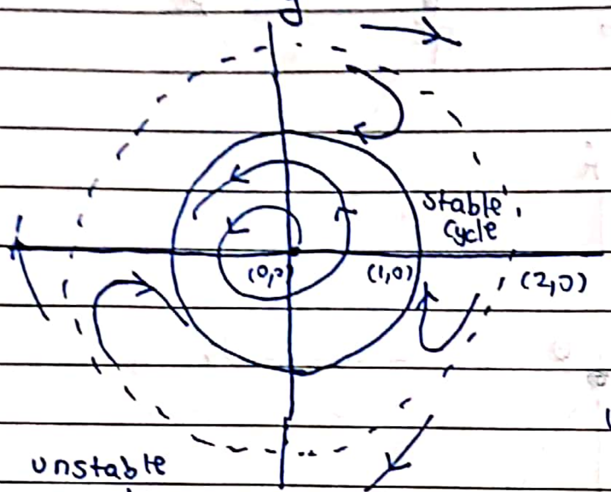
Thus the system exhibit chaotic behaviour due to sensitive dependence to initial condition plus it resembles Lorentz map which has aperiodic behavior.

Ans-2- a) $\dot{r} = r(1-r^2)(4-r^2) = r(r-2)(r-1)(r+1)(r+2)$

$\dot{\theta} = 2 - r^2$

Assuming $r>0$ radial system.





$\dot{\theta} > 0$ (counter clock wise) if $r < \sqrt{2}$

$\dot{\theta} < 0$ if $r > \sqrt{2}$

unstable fixed pt. at origin

b) $\dot{r} = r\sin r$

$\dot{\theta} = 1$

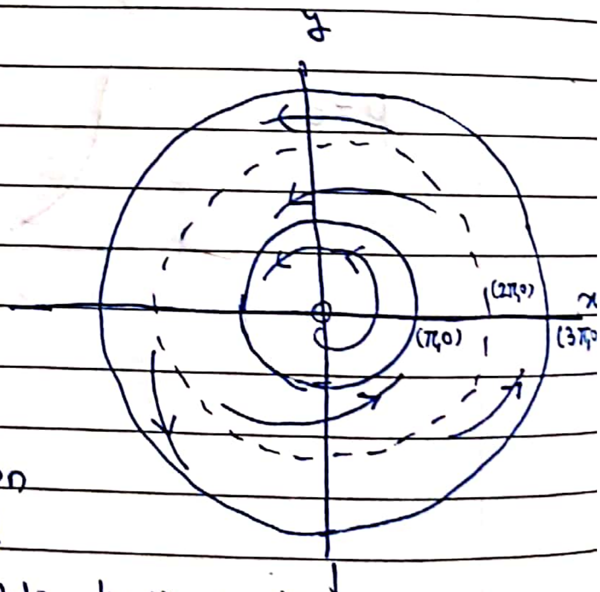
$\dot{r} = r\sin r = 0 = f(r)$

$r = 0,\ n\pi \quad n \in \mathbb{Z}$

$f'(r) = \sin r + r\cos r$

$f'(n\pi) = r\cos n\pi$

$$f'(n\pi) = \begin{cases} r > 0 & n \text{ is even} \\ -r < 0 & n \text{ is odd} \end{cases}$$



Alternating stable & unstable limit cycle.

Ans 4:- $\dot{r} = \mu r - r^3$ $\omega > 0, b > 0$

$\dot{\theta} = \omega + b r^2$

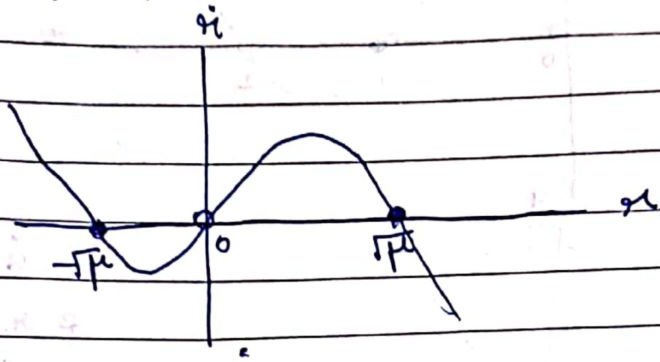
For fixed ~~pts~~ cycles:

$\dot{r} = 0$ $(\mu - r^2) r = 0$

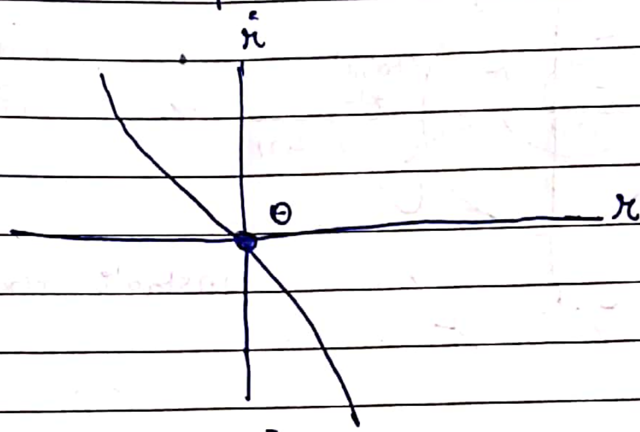
$r = 0, \pm\sqrt{\mu}$ $(\mu > 0)$

$\dot{\theta} > 0$

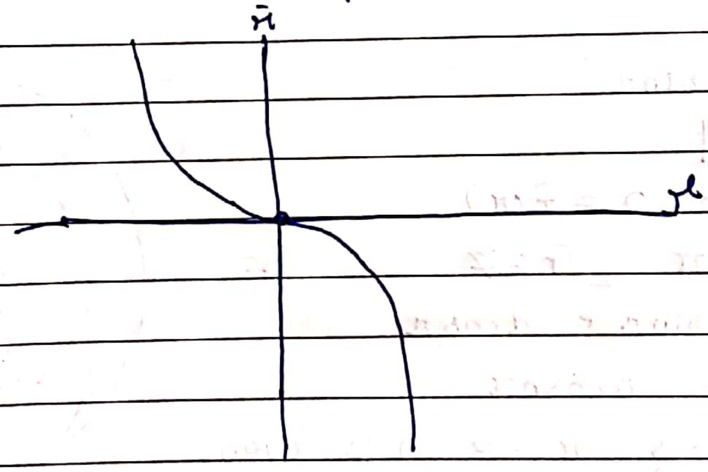
$\mu > 0$



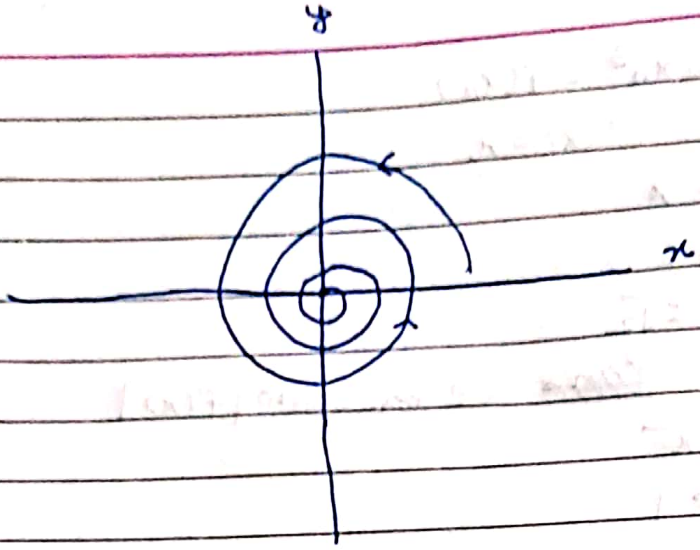
$\mu < 0$



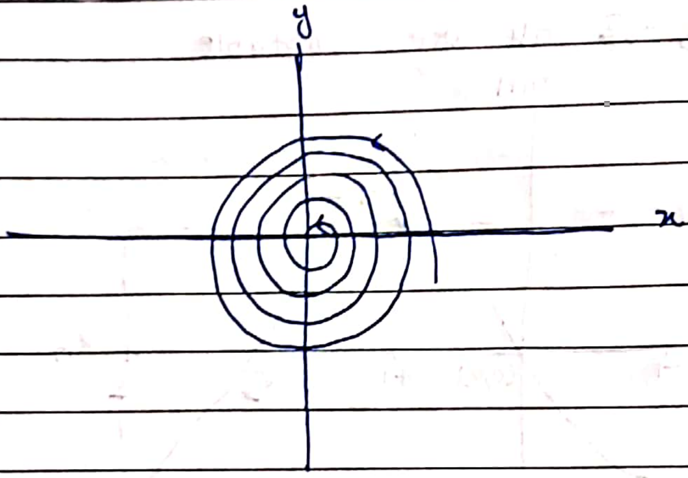
$\mu = 0$



$\mu < 0$

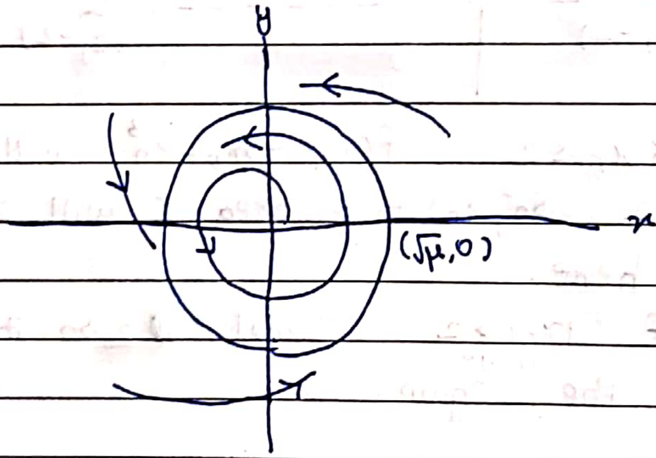


$\mu = 0$



The radial rate is slow

$\mu > 0$



As μ is varied supercritical hopf bifurcation occur at $\mu = 0$.

Date: / /
Page No.

Ans-5 — $x_{n+1} = 3x_n - x_n^3 = f(x_n)$

a) For fixed pts. $f(x) = x$.

$$3x - x^3 = x$$
$$x^3 = 2x$$

$x = 0$, $x = \pm\sqrt{2}$

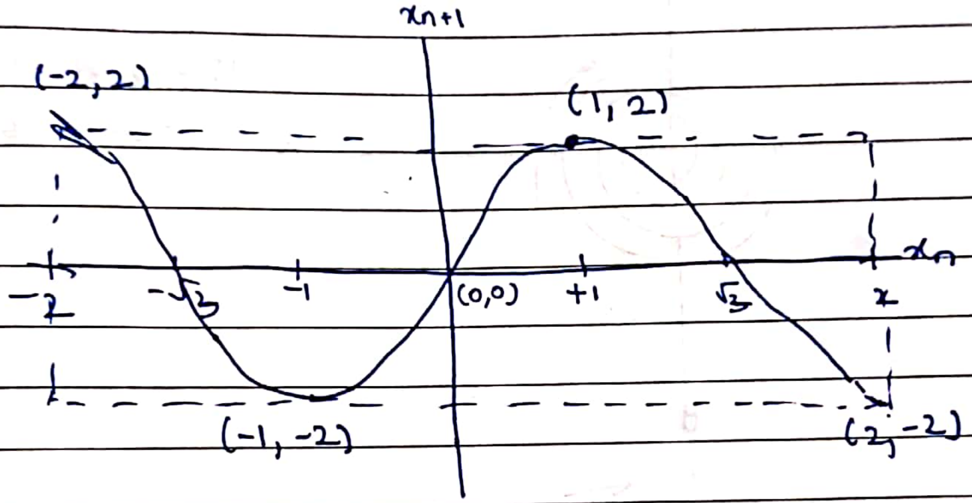
For stability we calculate $|f'(x)|$

$f'(x) = 3 - 3x^2$

$|f'(0)| = 3 > 1$

$|f'(\pm\sqrt{2})| = 3 > 1$

So $x = 0, \sqrt{2}, -\sqrt{2}$ all are unstable



b) So, if $-2 \le x_n \le 2$, $f(x_n) = 3x_n - x_n^3$ will also be $-2 \le f(x_n) \le 2$, So, in this area I will stay locked if I start here.

Similarly if $|x_n| > 2$ $|f(x_n)| > 2$ so it will never enter the above region.

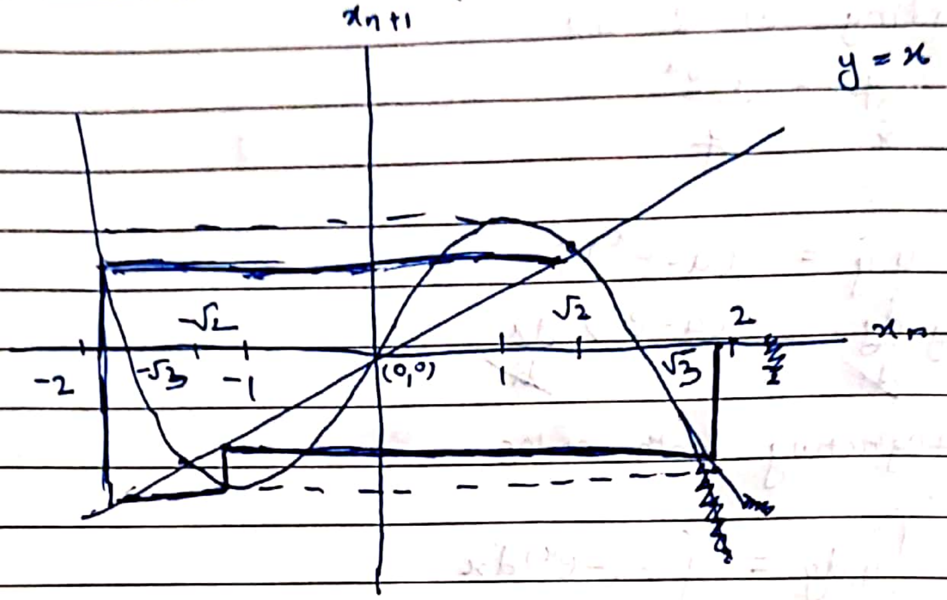
Date : / /
Page No.

b) $x_0 = 1.9$

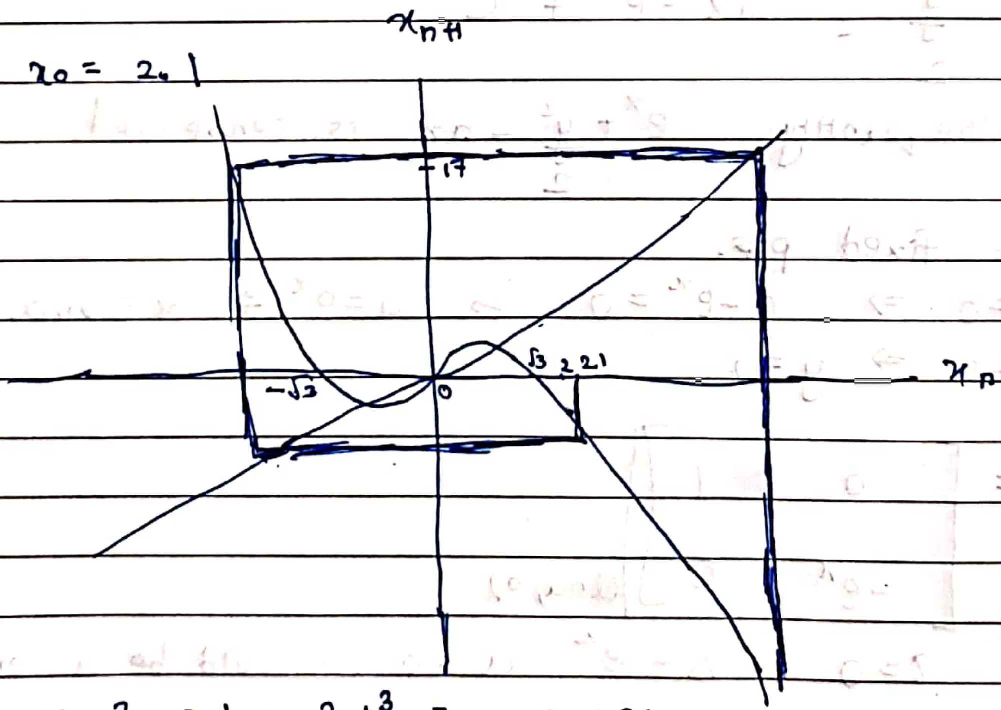
$x_1 = 3 \times 1.9 - 1.9^3 = -1.159$

$x_2 = 3 \times (-1.159) - 1.159^3 = -1.9201$

$x_3 = 3 \times (-1.9201) - (1.9201)^3 = 1.3186$



$x_0 = 2.1$



$x_1 = 3 \times 2.1 - 2.1^3 = -2.961$

$x_2 = 3 \times (-2.961) - (-2.961)^3 = 17.0776$

$x_3 = 3 \times (17.0776) - (17.0776)^3 = -4929.3589$

Date : / /
Page No.

Ans-6-
$$\ddot{x} = a - e^{x}$$

let $y = \dot{x}$, then

$$\dot{y} = a - e^{x} \quad -(i)$$
$$\dot{x} = y \quad -(ii)$$

dividing (i) & (ii)

$$\frac{\dot{y}}{\dot{x}} = \frac{a-e^{x}}{y}$$

$$y\dot{y} = (a-e^{x})\dot{x}$$
$$y\frac{dy}{\cancel{dt}} = (a-e^{x})\frac{dx}{\cancel{dt}}$$

Integrating both sides

$$\int y\,dy = \int (a-e^{x})\,dx$$

$$\frac{y^2}{2} = ax - e^{x} + C$$

So the quantity $e^{x} + \frac{y^2}{2} - ax$ is conserved.

For fixed pts.

$\dot{y} = 0 \Rightarrow a - e^{x} = 0 \Rightarrow a = e^{x} \Rightarrow x = \ln a \quad (a>0)$

$\dot{x} = 0 \Rightarrow y = 0$

$$J = \begin{bmatrix} 0 & 1 \\ -e^{x} & 0 \end{bmatrix}_{(\ln a, 0)}$$

$\tau = 0$, $\Delta = e^{x} = a > 0$. Could be a center.

$$V = e^{x} + \frac{y^2}{2} - ax$$

Scanned with CamScanner

$$\left.\frac{\partial V}{\partial x}\right|_{(\ln a,0)} = e^x - a = 0$$

$$\left.\frac{\partial V}{\partial y}\right|_{(\ln a,0)} = y = 0$$

Local maxima or minima or saddle
~~hence, it is a centre~~
In this case it is minima on slight changes we see that the function increases hence centre

$f(x) = V(x,y)\big|_{y=0} = e^x - ax$, $f''(x) = e^x > 0$ upward parabola

$f(y)\ V(x,y)\big|_{x=\ln a} = \frac{y^2}{2} + a - a\ln a$ upword parabola

$a=0$

$\dot{x} = y$

$\dot{y} = -e^x$

Nullclines:

$\dot{x} = 0$

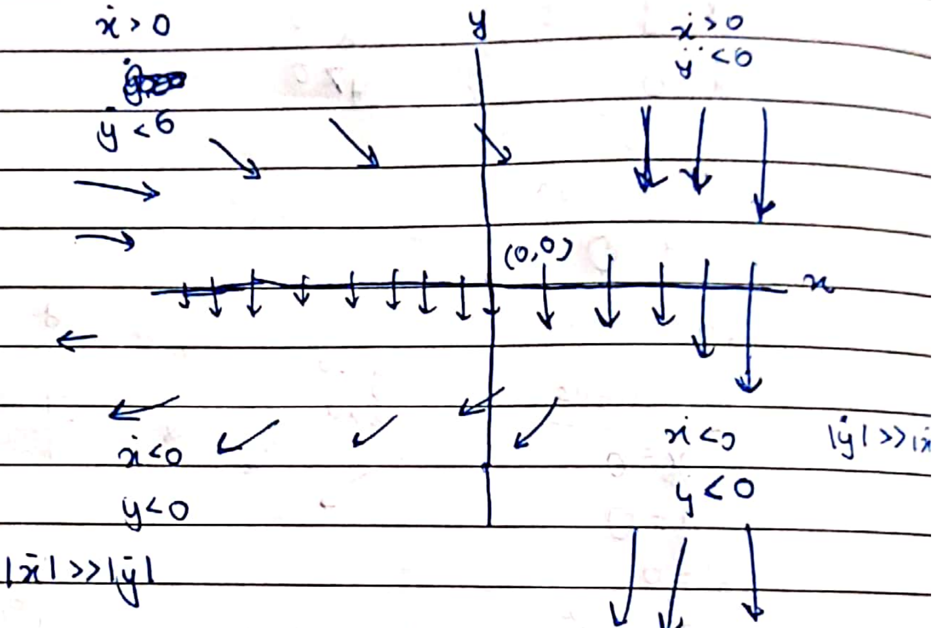
$y = 0$

$\dot{x} = 0$

$\dot{y} = -e^x$

$\dot{y} = 0$

no sol^n



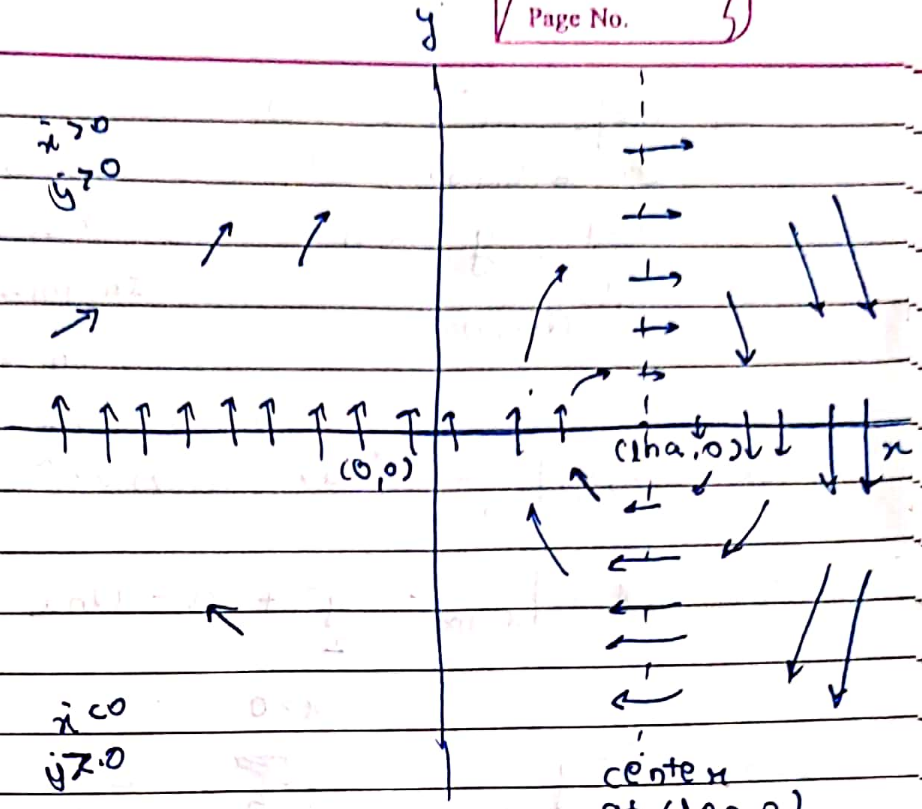
No fixed pts.

$a > 0$

$\dot{y} = a - e^x$

$\dot{x} = y$

Fixed pt. $(\ln a, 0)$



Nullclines

$\dot{y} = 0$

$e^x = a$

$x = \ln a$

$\dot{y} = 0$

$\dot{x} = y$

$\dot{x} = 0$

$y = 0$

$\dot{y} = a - e^x$

$\dot{x} = 0$

$a < 0$

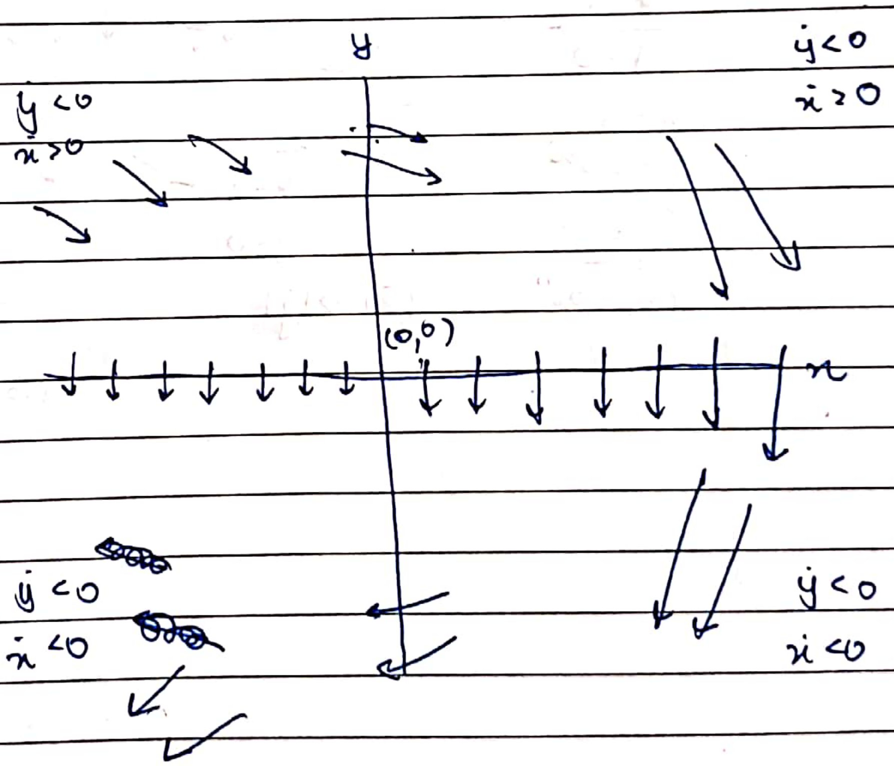
Nullclines

$\dot{x} = 0$

$y = 0$

$\dot{x} = 0$

$\dot{y} = a - e^x$



Date : / /
Page No.

Ans-7- $x_{n+1} = x_n^2 + c$

a) For fixed pts. $f(x) = x$

$$x^2 + c = x \Rightarrow x^2 - x + c = 0$$

$$x = \frac{1 \pm \sqrt{1-4c}}{2} \quad \& \; 1-4c \geq 0 \quad c \leq 1/4$$

Two fixed pts for $c \leq 1/4$

For stability we calculate $|f'(x)|$

$f'(x) = 2x$

Let $x_1 = \frac{1 + \sqrt{1-4c}}{2}$ and $x_2 = \frac{1-\sqrt{1-4c}}{2}$

$|f(x_1)| = |1+\sqrt{1-4c}|$

$|f(x_1)| > 1 \quad \forall \quad c < 1/4$

x_1 is unstable for $c < 1/4$

$|f(x_2)| = |1-\sqrt{1-4c}|$

$$|f(x_2)| < 1$$

$$|1-\sqrt{1-4c}| < 1$$

$$-1 < 1-\sqrt{1-4c} < 1$$

$$-2 < -\sqrt{1-4c} < 0$$

$$0 < \sqrt{1-4c} < 2$$

$$0 < 1-4c < 4$$

$$-1 < -4c < 3$$

$$-3 < 4c < 1$$

$$\frac{-3}{4} < c < 1/4$$

x_2 is stable for $-3/4 < c < 1/4$ and unstable for $c < -3/4$

Scanned with CamScanner

Date : / /
Page No.

b) at ~~[illegible]~~ $c = 1/4$ there is saddle node bifurcation as 2 fixed pts are born of opp. stability

For 2-cycle

$$ff(x) = x$$

$$(x^2+c)^2 + c = x$$

$$x^4 + 2x^2c + c^2 + c = x$$

$$x^4 + 2x^2c - x + c^2 + c = 0$$

Now, the sol^{n}s of $f(x) = x$, so we need to factor those out

$$\begin{array}{r|l} & x^2 + x + c + 1 \\ x^2 - x + c & x^4 + 2x^2c - x + c^2 + c \\ & -(x^4 - x^3 + cx^2) \\ & x^3 + x^2c \\ & -(x^3 - x^2 + cx) \\ & (c+1)x^2 - (c+1)x + c(c+1) \\ & (c+1)x^2 - (c+1)x + c(c+1) \\ & 0 \end{array}$$

$$(x^2 + x + (+1))(x^2 - x + c) = 0.$$

$$x = \frac{-1 \pm \sqrt{1 - 4(c+1)}}{2}$$

$$1 - 4(c+1) \geq 0$$
$$-3 \geq 4c$$
$$c \leq -3/4$$

For stability of 2cycle, with p and q where $q = f(p)$ & p and q are roots of $x^2 + x + c + 1 = 0$, we calculate

$$|f'(p)||f'(q)| = |4pq| = |4(c+1)|$$

$$|4(c+1)| < 1$$

$$-1 < 4(c+1) < 1$$

$$-1/4 < c+1 < 1/4$$

$$-5/4 < c < -3/4$$

So a flip bifurcation occur at $c = -3/4$ where fixed pt. x_2 becomes unstable &

Date : / /
Page No.

a stable period 2 cycle is born

c) For super stable 2-cycle

$4(c+1) = 0$

$c = -1$

For stable 2-cycle

$-5/4 < c < -3/4$

Ans-3 - $\dot{x} = vx + zy$ $\quad a, v > 0$

$\dot{y} = vy + (z-a)x$

$\dot{z} = 1 - xy$

b) For fixed pts.

$\dot{x} = 0 \qquad vx = -zy \quad -(i)$

$\dot{y} = 0 \qquad vy = -(z-a)x \quad -(ii)$

$\dot{z} = 0 \qquad xy = 1 \quad -(iii)$

$x = 1/y$ - from (iii)

$\frac{v}{y} = -zy$ - subs. x in (i)

$\Rightarrow v = -zy^2$

$\Rightarrow z = \frac{-v}{y^2}$

$vy = -\left(\frac{-v}{y^2} - a\right)\frac{1}{y}$ - subs. x & z in (ii)

$\Rightarrow vy^2 = \frac{v}{y^2} + ay^2$

$\Rightarrow vy^4 = v + ay^2$

$\Rightarrow y^4 - \frac{a}{v}y^2 - 1 = 0$

$\Rightarrow y^2 = \frac{\frac{a}{v} \pm \sqrt{\frac{a^2}{v^2} + 4}}{2}$

$\frac{a}{v} < \sqrt{\frac{a^2}{v^2} + 4}$

$\therefore \frac{a}{v} - \sqrt{\frac{a^2}{v^2} + 4} < 0$

So $y^2 = \frac{\frac{a}{v} - \sqrt{\frac{a^2}{v^2} + 4}}{2} < 0$

so this sol^n is not valid.

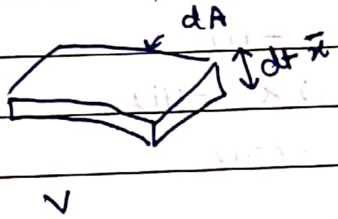
$$y = \sqrt{\frac{\frac{a}{v} + \sqrt{\frac{a^2}{v^2} + 4}}{2}}$$

$$x = \frac{1}{y} = \sqrt{\frac{2}{\frac{a}{v} + \sqrt{\frac{a^2}{v^2} + 4}}}$$

$$z = \frac{-v}{y^2} = \frac{-2v}{\frac{a}{v} + \sqrt{\frac{a^2}{v^2} + 4}}$$

Only one fixed pt.

4) For volume contraction



$$V + dV = V + \iint dA\, \bar{x}\, dt$$

$$dV = \iint \bar{x}\, dt\, dA$$

$$\dot{V} = \frac{dV}{dt} = \iint \bar{x}\, dA$$

$$\equiv \iiint \nabla \cdot \bar{x}\, dV$$

$$\nabla \cdot \bar{x} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$$

$$= v + v + 0$$

$$= 2v > 0$$

The system is not dissipative as volume is increasing