Quantum Fourier Transformation

$$\tilde{F}(k) = \int_{-\infty}^{\infty} f(x)\, \underbrace{e^{ikx}}_{\text{Kernel}}\, dx$$

$\boxed{x \to k}$

$x \leftarrow k$

Discrete integral transform:

$n \in \mathbb{N}$, set of natural numbers.

consider 2^n integers:

$$S_n = \{0, 1, 2, \ldots 2^n - 1\}$$

Now consider two variables $x, y \in S_n$

Define a kernel: $K(x, y)$

Then the discrete integral transformation of $f(x)$ is given by:

$$\boxed{\underset{\text{column vector}}{\tilde{f}(y)} = \sum_{x=0}^{N-1} \underbrace{K(y, x)}_{N \times N \text{ matrix}}\ \underset{\text{column vector}}{f(x)}}$$

$N = 2^n$

If K is unitary, then

$$\boxed{f(x) = \sum_{y=0}^{N-1} K^{+}(x, y)\, \tilde{f}(y)}$$

extend this concept to a Hilber space of n qubits:

say, $|x\rangle = |x_{n-1}\, x_{n-2} \ldots x_0\rangle$

x_{n-1}: 0 or 1, x_{n-2}: 0 or 1, x_0: 0 or 1

2 qubit

$|x\rangle = |00\rangle$

Now apply unitary operator U on $|x\rangle$

$$U|x\rangle = \sum_{y=0}^{N-1} |y\rangle \langle y | U | x\rangle$$

$U|x\rangle = I\, U|x\rangle$

$I = \sum_{y} |y\rangle\langle y|$

$$\boxed{U|x\rangle = \sum_{y=0}^{N-1} U(x,y)\,|y\rangle}$$

$$\tilde{f}(y) = \sum_{x=0}^{N-1} K(y,x)\, f(x)$$

If we can identify

$$U|x\rangle = \sum_{y=0}^{N-1} K(x,y)\,|y\rangle$$

then the unitary operator U computes the 'discrete integral transform' of the basis states.

$$U \sum_x f(x)\,|x\rangle$$

$$= \sum_x f(x)\, U|x\rangle = \sum_x f(x) \sum_y K(x,y)\,|y\rangle$$

$$= \sum_y \underbrace{\left(\sum_x K(y,x)\, f(x) \right)}\,|y\rangle$$

$$= \sum_y \tilde{f}(y)\,|y\rangle$$

$$\boxed{U \sum_x f(x)\,|x\rangle = \sum_y \tilde{f}(y)\,|y\rangle}$$

For Quantum Fourier transform:

$N = 2^n$

$$\boxed{K(x,y) = \frac{1}{\sqrt{N}}\, e^{2\pi i x y/N}}$$

$$= \frac{1}{\sqrt{N}}\, \omega_n^{xy}$$

$\omega_n = e^{2\pi i/N}$

$(101) = 5$

$1\times 2^2 + 0\times 2^1 + 1\times 2^0 = 5$

Single qubit: $n = 1$, $N = 2$

$$K(x,y) = \frac{1}{\sqrt{2}} e^{2\pi i xy/2}$$

$$= \frac{1}{\sqrt{2}} (-1)^{xy}$$

$$\omega = e^{2\pi i/2}$$
$$= e^{\pi i}$$
$$= -1$$

$x \to 0, 1$
$y \to 0, 1$

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

$\Rightarrow$ Hadamard Gate is a QFT in $\mathbb{C}^2$

$$\alpha |0\rangle + \beta |1\rangle \xrightarrow[H]{QFT} \alpha \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Case 2

$n = 2$, $N = 4$ $\Rightarrow$ $x, y \in \{0, 1, 2, 3\}$

$$\omega_2 = e^{2\pi i/4} = e^{\pi i/2} = i$$

$$K(x,y) = (i)^{xy}$$

$$K = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Prove that
K is unitary

$$\tilde{f}(x) = \frac{1}{\sqrt{N}} \sum_y e^{2\pi i xy/N} f(y)$$

$$f(y) = \frac{1}{\sqrt{N}} \sum_x e^{-2\pi i xy/N} \tilde{f}(x)$$

$$\langle x | KK^{\dagger} | y \rangle = \sum_{z=0}^{N-1} \langle x | K | z \rangle \langle z | K^{\dagger} | y \rangle$$

$$= \sum_{z=0}^{N-1} K(x,z)\, K^{\dagger}(z,y)$$

$$= \frac{1}{N} \sum_{z=0}^{N-1} e^{2\pi i xz/N}\, e^{-2\pi i zy/N}$$

$$= \frac{1}{N} \sum_{z=0}^{N-1} e^{2\pi i z(x-y)/N}$$

$\frac{r^n-1}{r-1}$

Say $x \neq y$

Then,

$$\frac{1}{N} \sum e^{2\pi i z(x-y)/N}$$

$$= \frac{1}{N} \frac{e^{2\pi i (x-y)} - 1}{e^{2\pi i (x-y)/N} - 1} \qquad x \neq y$$

$$= 0$$

$e^{2\pi i n} = 1$

If $x = y$

$$\frac{1}{N} \sum_{z=0}^{N-1} \frac{e^{2\pi i z(x-y)}}{N}$$

$$= 1$$

Thus,

$$\boxed{\langle x | KK^{\dagger} | y \rangle = \delta_{xy}}$$

Periodic function

$$f(x + P) = f(x)$$

$\uparrow$ period

We are going to have an oracle which

calculates the periodic function and outputs it to the target register which is initialized to $|0\rangle$.

Illustration with 3 qubits

- To find period P

Input register contain a uniform linear superposition of 3 qubit basis states

First register contain:

$$\frac{1}{\sqrt{8}}\Big[|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\Big]$$

$$= \frac{1}{\sqrt{8}}\Big[|0\rangle + |1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle\Big]$$

Target register: $|000\rangle \equiv |0\rangle$

Oracle calculates $f(x)$

The output:

$$|\psi\rangle = U_f \frac{1}{\sqrt{8}} \sum_{x=0}^{7} |x\rangle \; |0\rangle$$

$$= \frac{1}{\sqrt{8}} \sum_{x=0}^{7} |x\rangle \, |f(x)\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{8}}\Big[|0, f(0)\rangle + |1, f(1)\rangle + |2, f(2)\rangle + |3, f(3)\rangle + |4, f(4)\rangle + |5, f(5)\rangle + |6, f(6)\rangle + |7, f(7)\rangle\Big]$$

Now apply QFT on the first register:

$$|\psi'\rangle = QFT\,|\psi\rangle$$

$$= \frac{1}{\sqrt{8}}\frac{1}{\sqrt{8}} \sum_{x=0}^{7}\sum_{y=0}^{7} e^{2\pi i xy/8} \, |y, f(x)\rangle$$

Let us group 8 Terms corresponding to each value of y.

$$|\psi'\rangle = \frac{1}{8}\left[\underbrace{|0\rangle \sum_{x=0}^{7} |f(x)\rangle} + \underline{|1\rangle \sum_{x=0}^{7} e^{2\pi i x/8} |f(x)\rangle + \cdots} + |7\rangle \sum_{x=0}^{7} e^{14\pi i x/8} |f(x)\rangle\right]$$

$$= \frac{1}{8}\, |0\rangle \left[|f(0)\rangle + |f(1)\rangle + \cdots + |f(7)\rangle\right]$$
$$+ \frac{1}{8}\, |1\rangle \left[f(0) + e^{2\pi i/8} |f(1)\rangle + \cdots + e^{2\pi i 7/8} |f(7)\rangle\right]$$
$$+ \cdots + \frac{1}{8}\, |7\rangle \left[f(0) + e^{14\pi i/8} |f(1)\rangle + \cdots\right]$$

Assume $f(x+P) = f(x)$

$P = 2$ (period $= 2$)

Then, $f(0) = f(2) = f(4) = f(6) = a$

$f(1) = f(3) = f(5) = f(7) = b$

Look at first term: $|y\rangle = \underline{|0\rangle}$

$$\frac{1}{8}\, |0\rangle \left[|f(0)\rangle + |f(1)\rangle + \cdots + |f(7)\rangle\right]$$
$$= \frac{1}{8}\, |0\rangle \left[4|a\rangle + 4|b\rangle\right]$$
$$= \frac{1}{2} \left[|a\rangle + |b\rangle\right] |0\rangle$$

For state $|1\rangle$:

Co-efficient of $|1\rangle = 0$

$$\boxed{|\psi'\rangle = \underline{|0\rangle}\, \frac{1}{2}\left[|a\rangle + |b\rangle\right] + \underline{|4\rangle}\, \frac{1}{2}\left[|a\rangle - |b\rangle\right]}$$

If we measure the first register
and get $|0\rangle$ or $|4\rangle$
then we can conclude that the
Period is 2.