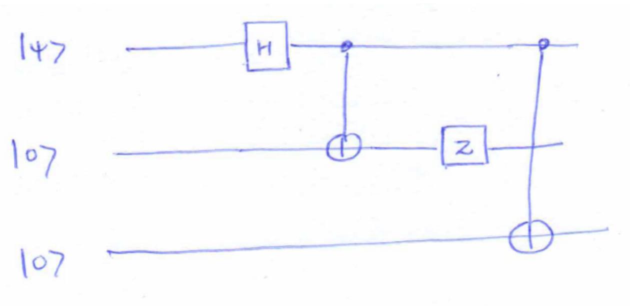
PH 441
End-Sem. Part I
24/11/2020
Marks: 20

1. Find the out put state of the following circuit, for:



Output state?

(a) $|\psi\rangle = |0\rangle$ and $\frac{|000\rangle - |111\rangle}{\sqrt{2}}$
(b) $|\psi\rangle = |1\rangle$ $\frac{|000\rangle + |111\rangle}{\sqrt{2}}$

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2. (a) A system is in a statistical mixture of states in which 25% are in the state $|0\rangle$ and the remaining in state $|1\rangle$. If the measurement is made in $\{|+\rangle, |-\rangle\}$, basis, what is the probability of finding the system in the $|+\rangle$ state? $1/2$
(b) Work out the reduced density matrix corresponding to the first qubit, for following state: $|\psi\rangle = \frac{1}{\sqrt{3}}[|00\rangle + |01\rangle + |10\rangle]$

1+2=3

3. Consider the following two-qubit state:

$$\frac{1}{\sqrt{11}}\left(|00\rangle + \sqrt{5}|01\rangle + \sqrt{2}\,|10\rangle + \sqrt{3}|11\rangle\right)$$

(a) If we measure the first qubit and get $|0\rangle$, then find out the state to which the second qubit collapses. $1/\sqrt{6}\,(|0\rangle + \sqrt{5}\,|1\rangle)$
(b) What is the probability that a projective measurement of the first qubit gives $|0\rangle$? $6/11$

2+2=4

4. What is the fundamental difference between ***Super-dense coding*** and ***quantum teleportation***. Show how the quantum teleportation protocol could be realized using the Bell state: $|\Phi^-\rangle = \frac{1}{\sqrt{2}}[|00\rangle - |11\rangle]$.

1+4=5

5. Consider the following table giving values of a 2 to 1 function $f(x)$ corresponding to three qubit inputs x. There exists a string ξ such that $f(x) = f(y)$ iff $x \oplus \xi = y$.

x	f(x)	x	f(x)
000	101	100	000
001	010	101	110
010	000	110	101
011	110	111	010

Table 1

The string in this case is $|110\rangle$. A Simon algorithm is executed and the second register is measured which gives: $|000\rangle$. If the first register is passed through Hadamard gates, what will be the result of measurement on the first register?

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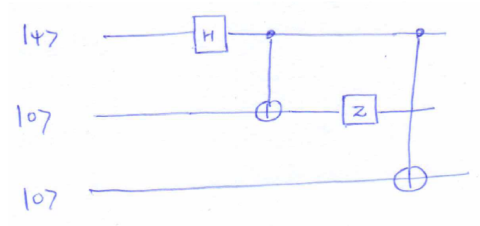
Solutions

1.



(a) $|\psi\rangle = |0\rangle$

$$|000\rangle \xrightarrow{H} \frac{|0\rangle+|1\rangle}{\sqrt{2}}|00\rangle \xrightarrow{CNOT} \frac{|000\rangle+|110\rangle}{\sqrt{2}} \xrightarrow{Z} \frac{|000\rangle-|110\rangle}{\sqrt{2}}$$

$$\Big\downarrow \text{CNOT}$$

$$\frac{|000\rangle-|111\rangle}{\sqrt{2}}$$

$$\boxed{\text{Thus output is } \frac{|000\rangle-|111\rangle}{\sqrt{2}} \equiv |-\rangle}$$

(b) Similarly for $|\psi\rangle = |1\rangle$

$$\boxed{\text{The output state is } \frac{|000\rangle+|111\rangle}{\sqrt{2}} \equiv |+\rangle}$$

2.

(a) Here, $\rho = \begin{pmatrix} 1/4 & 0 \\ 0 & 3/4 \end{pmatrix}$

$$P(|+\rangle) = \langle + | \rho | + \rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1/4 & 0 \\ 0 & 3/4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \frac{1}{2}$$

(b) $$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle\right)$$

$$\rho_{12} = |\psi\rangle\langle\psi|$$
$$\rho_1 = \mathrm{Tr}_2(\rho_{12}) = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

3.

$$|\psi\rangle = \frac{1}{\sqrt{11}}\left(|00\rangle + \sqrt{5}\,|01\rangle + \sqrt{2}\,|10\rangle + \sqrt{3}\,|11\rangle\right)$$

If we get the first qubit as $|0\rangle$, it can be form the first two terms

$$P_1 = \frac{1}{11}, \quad P_2 = \frac{5}{11}$$

$\Rightarrow$ Probability of obtaining $|0\rangle$ in first qubit

$$P = P_1 + P_2 = \frac{6}{11}$$

Post measurement state

$$|\psi'\rangle = \frac{1/\sqrt{11}\left[\,|0\rangle + \sqrt{5}\,|1\rangle\right]}{\sqrt{P}}$$

$$= \frac{1}{\sqrt{6}}\left[\,|0\rangle + \sqrt{5}\,|1\rangle\right]$$

(a) Thus, the second qubit collapses to : $\boxed{\frac{1}{\sqrt{6}}\left[\,|0\rangle + \sqrt{5}\,|1\rangle\right]}$

(b) $\boxed{P = \frac{6}{11}}$

4. In superdense coding two classical bits are sent from one party to another by using a quantum channel.

On the other hand, in quantum teleportation, quantum bits are sent from one party to another by using a classical channel.

(Now discuss quantum teleportation as we discussed in the class for the case $|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$)

Steps

(i) $|\phi\rangle = a|0\rangle + b|1\rangle$, the qubit to be transmitted

(ii) Alice and Bob share:

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

(iii)
$$|\phi\rangle \otimes |\Phi^-\rangle = \frac{1}{\sqrt{2}}\left[a|000\rangle - a|011\rangle + b|100\rangle - b|111\rangle\right]$$

Alice has the first two qubits while Bob has the third.

(iv) Alice apply $U_{CNOT} \otimes I$ followed by $U_H \otimes I \otimes I$

$$(U_H \otimes I \otimes I)(U_{CNOT} \otimes I)\frac{1}{\sqrt{2}}\left[a|000\rangle - a|011\rangle + b|100\rangle - b|111\rangle\right]$$

$$= (U_H \otimes I \otimes I)\frac{1}{\sqrt{2}}\left[a|000\rangle - a|011\rangle + b|110\rangle - b|101\rangle\right]$$

$$= \frac{1}{2}\left[a(|0\rangle+|1\rangle)|00\rangle - a(|0\rangle+|1\rangle)|11\rangle + b(|0\rangle-|1\rangle)|10\rangle - b(|0\rangle-|1\rangle)|01\rangle\right]$$

$$= \frac{1}{2}\left[|00\rangle(a|0\rangle - b|1\rangle) + |10\rangle(a|0\rangle + b|1\rangle) + |01\rangle(-a|1\rangle + b|0\rangle) + |11\rangle(-a|1\rangle - b|0\rangle)\right]$$

Thus, Alice is going to obtain one of the following ~~qubits~~ states: $|00\rangle$, $|10\rangle$, $|01\rangle$, $|11\rangle$ with equal probability while Bob's ~~state~~ qubit collapses to $a|0\rangle - b|1\rangle$, $a|0\rangle + b|1\rangle$, $-a|1\rangle + b|0\rangle$ and $-a|1\rangle - b|0\rangle$ respectively.

(v) Now, after receiving the two classical bits, Bob knows the state of the qubit in his stand by apply the following ~~Decon~~ Decoding :

Received qubit	Bob's state	Decoding
00	$a\|0\rangle - b\|1\rangle$	Z
01	$-a\|1\rangle + b\|0\rangle$	$-Y$
10	$a\|0\rangle + b\|1\rangle$	I
11	$-a\|1\rangle - b\|0\rangle$	$-X$

5.

The second register measures : $|000\rangle$

Therefore the first register ~~contain~~ contains :

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[|010\rangle + |100\rangle\right]$$

Apply the Hadamard gates :

$$|\psi_1\rangle \xrightarrow{H^{\otimes 3}} \frac{1}{\sqrt{2}}\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}} + \frac{|0\rangle-|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{4}\left[\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)\left(|0\rangle + |1\rangle\right) + \left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right)\left(|0\rangle + |1\rangle\right)\right]$$

$$= \frac{1}{4}\left[\left(2|00\rangle - 2|11\rangle\right)\left(|0\rangle + |1\rangle\right)\right]$$

$$= \frac{1}{2}\left[|000\rangle + |001\rangle - |110\rangle - |111\rangle\right]$$