1. Given two qubits in the state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, what are the probabilities of measuring $|++\rangle, |+-\rangle, |-+\rangle$ and $|--\rangle$?

1.

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Thus,

$$\underbrace{|01\rangle} = \frac{1}{2}\left[|++\rangle - |+-\rangle + |-+\rangle - |--\rangle \right]$$

$$\underbrace{|10\rangle} = \frac{1}{2}\left[|++\rangle + |+-\rangle - |-+\rangle - |--\rangle \right]$$

$$\therefore \; |\psi_{AB}\rangle = \frac{1}{\sqrt{2}} \frac{1}{2}\left[2|++\rangle - 2|--\rangle \right]$$

$$= \frac{1}{\sqrt{2}}\left[\underline{|++\rangle - |--\rangle} \right]$$

$+ -$
$- +$

Hence,

$$\boxed{\begin{aligned} P(|++\rangle) &= \tfrac{1}{2} \\ P(|--\rangle) &= \tfrac{1}{2} \\ P(|+-\rangle) &= 0 = P(|-+\rangle) \end{aligned}}$$

Find the expectation value of $\sigma_x \otimes \sigma_z$ measured in each of the Bell states.

2.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underline{\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$\sigma_x \otimes \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$|00\rangle, |01\rangle,$
$|10\rangle, |11\rangle$

$$\Rightarrow \sigma_x \otimes \sigma_z = |00\rangle\langle 10| - |01\rangle\langle 11| + |10\rangle\langle 00| - |11\rangle\langle 01|$$

Now, the Bell states are:

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \; ; \; |\phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \; ; \; |\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Thus,

$$|\phi^+\rangle = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array} \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array}, \quad |\phi^-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad |\psi^-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Hence

(i) $\langle \phi^+ | \sigma_x \otimes \sigma_z | \phi^+ \rangle$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}(0+0+0+0)$$
$$= 0$$

(ii) $\langle \Phi^- | \sigma_x \otimes \sigma_z | \Phi^- \rangle$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = 0$$ ✓

(iii) $\langle \psi^+ | \sigma_x \otimes \sigma_z | \psi^+ \rangle$

$$= \frac{1}{2}\begin{pmatrix}0 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & -1\\1 & 0 & 0 & 0\\0 & -1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}0 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = 0$$ ✓

(iv) $\langle \psi^- | \sigma_x \otimes \sigma_z | \psi^- \rangle$

$$= \frac{1}{2}\begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & -1\\1 & 0 & 0 & 0\\0 & -1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix}\begin{pmatrix}-1\\0\\0\\-1\end{pmatrix}$$

$$= 0$$ ✓

$\Rightarrow$ Expectation value of $\sigma_x \otimes \sigma_z$ measured in each of the Bell states yields zero!

3. Express U_{CCNOT} in terms of I and X

3.
$$U_{CCNOT} = |000\rangle\langle 000| + |001\rangle\langle 001|$$
$$+ |010\rangle\langle 010| + |011\rangle\langle 011|$$
$$+ |100\rangle\langle 100| + |101\rangle\langle 101|$$

$$+ |111\rangle\langle 110| + |110\rangle\langle 111|$$

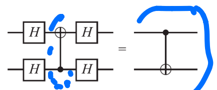
$$= |00\rangle\langle 00| \left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$$
$$+ |01\rangle\langle 01| \left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$$
$$+ |10\rangle\langle 10| \left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$$
$$+ |11\rangle\langle 11| \left(|1\rangle\langle 0| + |0\rangle\langle 1| \right)$$

$$= \left(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| \right) \otimes \underbrace{\left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)}_{I}$$
$$+ |11\rangle\langle 11| \otimes \underbrace{\left(|1\rangle\langle 0| + |0\rangle\langle 1| \right)}_{X}$$

Thus,

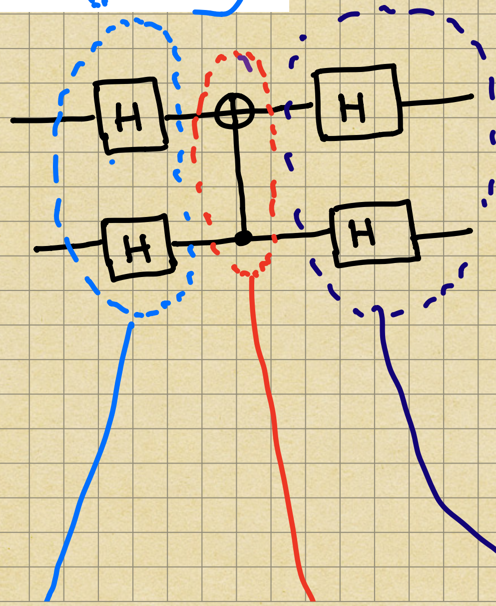
$$\boxed{U_{CCNOT} = \left(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| \right) \otimes I \checkmark + |11\rangle\langle 11| \otimes X \checkmark}$$

4. Show that the two circuits below are equivalent



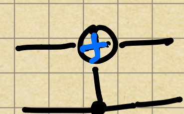
✓ ✓

4.



Remember

$$U_{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$



$$I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|$$

$$\left(U_H \otimes U_H\right)\left(I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|\right)\left(U_H \otimes U_H\right)$$

$$= U_H I U_H \otimes U_H\left(|0\rangle\langle 0|\right)U_H$$

$$+ U_H X U_H \otimes U_H\left(|1\rangle\langle 1|\right)U_H$$

$$U_H I U_H = U_H^2 = I$$

$$\boxed{\begin{aligned} U_H^2 &= I \checkmark \\ U_H I U_H &= I \checkmark \\ U_H X U_H &= Z \end{aligned}}$$

$$= I \otimes \tfrac{1}{2}\left(|0\rangle + |1\rangle\right)\left(\langle 0| + \langle 1|\right)$$

$$+ Z \otimes \tfrac{1}{2}\left(|0\rangle - |1\rangle\right)\left(\langle 0| - \langle 1|\right)$$

$$(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) = \left[|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 1| + |1\rangle\langle 0|\right.$$

$$= \tfrac{1}{2}\, I \otimes \left(I + X\right)$$

$$+ \tfrac{1}{2}\, Z \otimes \left(I - X\right)$$

$$= \tfrac{1}{2}\left[I \otimes (I + X) + Z \otimes (I - X)\right]$$

$$= \tfrac{1}{2}\left[(I + Z) \otimes I + (I - Z) \otimes X\right]$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes X$$

$$= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

$$I + Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= U_{CNOT}$$

5. Show that, $U_{AND} = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|)\otimes I + |11\rangle\langle 11| \otimes X$

5.

The logical AND gate is defined by

$$AND(x,y) \equiv x \wedge y \equiv \begin{cases} 1 & x=y=1 \\ 0 & \text{otherwis} \end{cases}$$

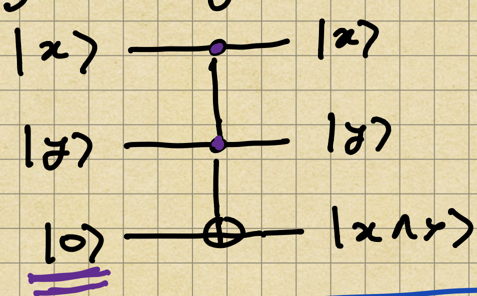
The logic function for AND is

$$f(x, y, 0) \equiv (x, y, x \wedge y)$$

$$U_{AND} |x, y, 0\rangle = |x, y, x\wedge y\rangle, \quad x, y \in \{0, 1\}$$

Note that the third qubit in the RHS is 1 if and only if $x=y=1$, and 0 otherwise.

Thus, clearly AND gate could be obtained by using a CCNOT gate.



Hence,

$$\boxed{U_{AND} = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I + |11\rangle\langle 11| \otimes X}$$

//

Measurement in Quantum computation

superposition of states → superpos.

↑

Input

Output

Measurement

↓

random — [one of the states

Von-Neumann measurements

↓ Projective measurements

Physically observables ⟶ Quantum operators

Let us make some msmts

$\{ \underline{\underline{M_m}} \}$ m : m no. of possibles results

O   ⊘   θ

$\underline{\underline{m}}$

say n possiblities are there

m : 1, 2, . . n

The probability of an outcome m is given by

$$\boxed{p(m) = \langle \psi | M_m^{+} M_m | \psi \rangle} \rightarrow \text{(i)}$$

$$\boxed{\sum_{m=1}^{n} M_m^{+} M_m = I} \rightarrow \text{(ii)}$$

The list of msmts must exhaust all possibilities

$$|\psi\rangle \longrightarrow ?$$

The post msmt state is given by

$$\boxed{\frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^{+} M_m | \psi \rangle}}} \rightarrow \text{(iii)}$$

Projective measurement

M_m are orthogonal projectors P_m $\quad (M_m = P_m)$

$$P_m = P_m^{+}$$

$$P_m^2 = P_m \qquad P_m P_m |\psi\rangle \to \underline{\underline{|\psi'\rangle}}$$

$$P_m = |m\rangle\langle m|$$

$$P_m \underline{\underline{|\psi\rangle}} = \underline{\underline{|m\rangle}} \langle m|\psi\rangle$$

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle \quad \boxed{M_m = P_m}$$

$$\equiv \langle\psi| P_m^\dagger P_m |\psi\rangle$$

$$\boxed{p(m) = \langle\psi| P_m |\psi\rangle}$$

$$P_m^\dagger = P_m$$

$$P_m^2 = P_m$$

probability of obtaining a result m

say, we have an ensemble of states $|\psi\rangle$

we wish to measure an observable M

$$E(M) = \langle M\rangle = \sum_m \underline{\underline{m}}\, \underline{\underline{p(m)}}$$

$$= \sum_m m \langle\psi| P_m |\psi\rangle$$

$$= \langle \psi | \sum_m m P_m | \psi \rangle$$

$$\underline{\underline{M}} = \sum m P_m \qquad \left[\begin{aligned} A &= \sum_i \lambda_i \underline{\underline{|\lambda_i\rangle\langle\lambda_i|}} \\ &= \sum_i \lambda_i \cdot P_i \end{aligned} \right.$$

$$E(M) = \langle \psi | M | \psi \rangle$$

Let us consider a single qubit state in the basis
$\underline{\underline{|0\rangle}}, \underline{|1\rangle}$

$$\underline{\underline{M_0}} = |0\rangle\langle 0| \quad ; \quad \underline{\underline{M_1}} = |1\rangle\langle 1|$$

$$\underline{|\psi\rangle = a|0\rangle + b|1\rangle}$$

What is the prob. of picking up '1' out of msmt?

$$p(1) = \langle \psi | M_1 | \psi \rangle$$
$$p(0) = \langle \psi | M_0 | \psi \rangle$$

$$p(1) = \langle \psi | 1 \rangle \langle 1 | \psi \rangle$$
$$= (a^* \underline{\langle 0|} + b^* \langle 1|) |1\rangle\langle 1| \left(\begin{aligned} &a|0\rangle \\ &+ b|1\rangle \end{aligned} \right)$$
$$= |b|^2$$
$$p(0) = |a|^2$$

post msmt state

$$\text{post msmt} = \frac{M|\psi\rangle}{\sqrt{\langle\psi|M^\dagger M|\psi\rangle}}$$

$$= \frac{|1\rangle\langle 1|(a|0\rangle + b|1\rangle)}{\sqrt{}}$$

$$= \frac{b|1\rangle}{|b|} \qquad b = |b|e^{i\phi}$$

$$= e^{i\phi}|1\rangle$$

global phase

what about msmt in diagonal basis

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}; \quad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$\boxed{|\psi\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle}$$

$$p(+) = \langle\psi|M_+|\psi\rangle \qquad M_+ = |+\rangle\langle +|$$

$$= \langle\psi|+\rangle\langle +|\psi\rangle \qquad M_- = |-\rangle\langle -|$$

$$p(+) = \frac{|a+b|^2}{2}$$

$$p(-) = \frac{|a-b|^2}{2}$$

post msmt state $= \dfrac{M_+ |\psi\rangle}{\langle\psi| M_+^\dagger M_+ |\psi\rangle}$

$$= e^{i\theta} |+\rangle$$

Two qubit cases

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

Say we want to make msmt on the 1st qubit

as 0

$$M_0 = M_{00} + M_{01}$$

$$= |00\rangle\langle 00| + |01\rangle\langle 01|$$

$$= |0\rangle\langle 0| \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$M_0 = |0\rangle\langle 0| \otimes I$$

$$M_1 = M_{10} + M_{11}$$

$$= |1\rangle\langle 1| \otimes I$$

prob. of measuring 1st qubit as 0 :

$$\langle \psi | M_0 | \psi \rangle$$

$$= |a|^2 + |b|^2$$

$\boxed{\begin{array}{c}\text{15 mins} \\ 5\end{array}}$