PH551: Nonlinear Dynamics and Chaos

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Class Time Table:

Monday: 12:00-1:00PM

Tuesday: 12:00-1:00PM

Friday: 11:00-12:00PM

Two Quizes: 20

Assignments: 20

Project: 30

End Sem:30

Syllabus

Historical Development of Chaos: Newton [1642-1727], Laplace [1749-1827]-Determinism, Poincare [1854-1912]-Chaos in Three-Body Problem, Fluid Motion-Weather Prediction [1950] , Lorenz - Reincarnation of Chaos (1961), Robert May - Chaos in Population Dynamics, Universality of chaos and later developments, Deterministic Chaos- Main ingradients, Current Problems of Interest.

Dynamical systems: Importance of concepts of chaos, Fractals, and nonlinear dynamics in different natural and engineering processes. Introduction to dynamical systems, state space: continuous state with discrete time or continuous time variable, discrete state with discrete or continous time variable.

One-dimensional system: Fixed points and their local and global stability analysis, converting the dynamical problem into equivalent problem of potentials. Two-dimensional system: Fixed points and linear stability analysis. Nonlinear analysis with examples of pendulum. Dissipation and the divergence theorem, Poincare-Bendixon's Theorem, weakely nonlinear oscillators.

Syllabus

Three-dimensional system: Linear and nonlinear stability analysis with examples of Lorentz system, forced nonlinear oscillator, Poincare section and maps. Bifurcation theory: Bifurcations in 1D and 2D flows with examples of sadle-node, transcritical, pitchfork bifurcations in different physical systems. Hopf-bifurcations. Homoclinic and hetroclinic bifurcations.

One dimensional Maps and Chaos: Stability of fixed point and periodic orbits, quadratic maps, bifurcation in maps, characterization of chaos using Lyapunov exponents and Fourier spectrum.

Different Routes to Chaos: Quasiperiodic, intermittency, period doubling, etc.

Fractals and attractors: Introduction to countable and non-countable sets, Cantor set, Dimension of self-similar Fractals. Henon map, Rossler systems, Chemical chaos, forced-double well oscillators.

A brief phenomenology of turbulent flow: Phenomenology of Turbulent flow in classical (Kolgmogorov phenomenology for energy cascade) and quantum system (especially generation and phenomenology of turbulence in Bose-Einstein condensation and superfluid Helium).

Books

1. **Strogatz, S. Nonlinear Dynamics and Chaos. Reading, MA: Addison-Wesley, 2007.**

2. Lakshmanan, M and R. Rajasekar, Nonlinear Dynamics: Integrability, Chaos and Patterns, Springer, 2003.

3. Hilborn, Robert C. Chaos and Nonlinear Dynamics. Oxford University Press, Second edition, 2000.

4. Guckenheimer, J., and P. Holmes. Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. New York, NY: Springer-Verlag, 2002.

5. Drazin, P. G. Nonlinear systems. Cambridge, UK: Cambridge University Press, 1992.

6. Berge, P., Y. Pomeau, and C. Vidal. Order Within Chaos. New York, NY: Wiley 1987.

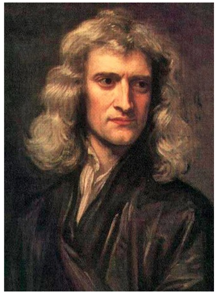
Possible project topics

- Nonlinear Dynamics in Physiological and Biological modelling.
- Studying spread of disease through mathematical models using linear stability analysis.
- Nonlinear Chemical Dynamics Oscillations and Patterns (belousov zhabotinsky reaction).
- Nonlinear dynamics of weather prediction model.
- Time series data for the Bitcoin to USD market for the presence of Chaotic attractors and its implications.
- Evolutionary Game Theory.
- Stochastic resonance in Lorenz model.
- Quantum Chaos.
- Kolmogorov-Arnold-Moser Theory.
- Poincare work on Chaos.

Historical Developments of Nonlinear Dynamics

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytical studies of planetary motion
1890s	Poincaré	Geometric approach, nightmares of chaos
1920–1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920–1960	Birkhoff	Complex behavior in Hamiltonian mechanics
	Kolmogorov	
	Arnol'd	
	Moser	
1963	Lorenz	Strange attractor in simple model of convection
1970s	Ruelle & Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators, and their applications

Historical Developments of Nonlinear Dynamics



Newton [1642-1727]

Newton's Laws: The equation of motion for a particle of mass m under a force field $\mathbf{F}(\mathbf{x}, t)$ is given by

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}, t)$$

Given initial condition $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, we can determine $\mathbf{x}(t)$ *in principle.*

Given initial condition $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, we can determine $\mathbf{x}(t)$ *in principle.*

Using Newton's laws we can understand dynamics of many complex dynamical systems, and predict their future quantitively. For example, the equation of a simple oscillator is

$$m\ddot{x} = -kx$$

Historical Developments of Nonlinea Dynamics

whose solution is

$$x(t) = A\cos(\sqrt{k/m}t) + B\sin(\sqrt{k/m}t),$$

with A and B to be determined using initial condition. The solution is simple oscillation.

Planetary motion (2 body problem)

$$\mu\ddot{\mathbf{r}} = -(\alpha/r^2)\hat{\mathbf{r}},$$

the solution is elliptical orbit for the planets. In fact the astronomical data matched quite well with the predictions. Newton's laws could explain dynamics of large number of systems, e.g., motion of moon, tides, motion of planets, etc.

He was unable to solve the three body problem!

Historical Developments of Nonlinear Dynamics

Laplace [1749-1827]- Determinism

Newton's law was so successful that the scientists thought that the world is deterministic. In words of Laplace

"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at any given moment knew all of the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit the data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain and the future just like the past would be present before its eyes."

Historical Developments of Nonlinear Dynamics

Poincare [1854-1912]-Chaos in Three-Body Problem

One of the first glitch to the dynamics came from three-body problem. The question posed was whether the planetary motion is stable or not. It was first tackled by Poincare towards the end of ninteenth century. He showed that we cannot write the trajectory of a particle using simple function. In fact, the motion of a planet could become random or disorderly (unlike ellipse). This motion was called chaotic motion later. In Poincare's words itself.

Historical Developments of Nonlinear Dynamics

"If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. but even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon. - in a 1903 essay "Science and Method"."

Clearly determinism does not hold in nature in the classical sense...

Historical Developments of Nonlinear Dynamics

Fluid Motion- Weather Prediction [1950]

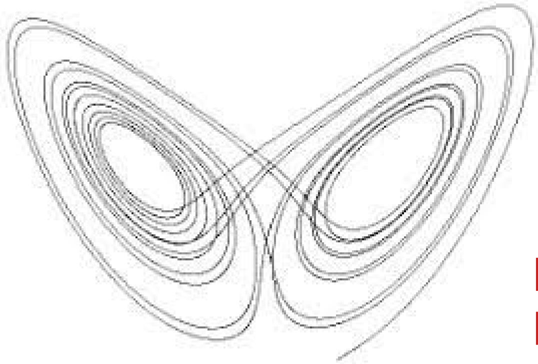
Motion of fluid parcel is given by

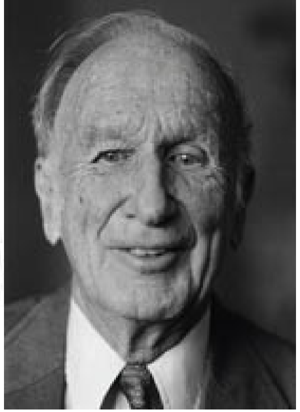
$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \nu \nabla^2 \mathbf{u}.$$

where ρ, $\mathbf{u}$, and p are the density, velocity, and pressure of the fluid, and ν is the kinetic viscosity of the fluid. The above equation is Newton's equation for fluids. There are some more equations for the pressure and density. These complex set of equations are typically solved using computers. The first computer solution was attempted by a group consisting of great mathematician named Von Neumann. Von Neumann thought that using computer program we could predict weather of next year, and possibly plan out vacation accordingly. However his hope was quickly dashed by Lorenz in 1963.

Historical Developments of Nonlinear Dynamics

Lorenz - Reincarnation of Chaos





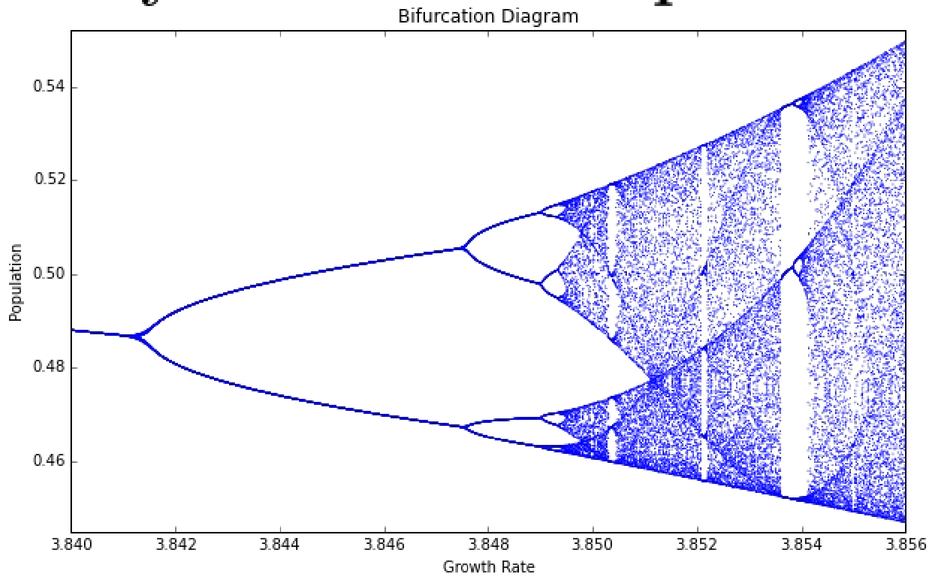
Edward N. Lorenz

In 1961, Edward Lorentz discovered the butterfly effect while trying to forecast the weather. He was essentially solving the convection equation. After one run, he started another run whose initial condition was a truncated one. When he looked over the printout, he found an entirely new set of results. The results was expected to be same as before.

Lorenz believed his result, and argued that the system is sensitive to the initial condition. This accidental discovery generated a new wave in science after a while. Note that the equations used by Lorenz do not conserve energy unlike three-body problem. These two kinds of systems are called dissipative and conservative systems, and both of them show chaos.

Historical Developments of Nonlinear Dynamics

Robert May - Chaos in Population Dynamics



Robert May

In 1976, May was studying population dynamics using simple equation

$$P_{n+1} = aP_n(1 - P_n)$$

where P_n is the population on the nth year. May observed that the time series of P_n shows constant, periodic, and chaotic solution.

Historical Developments of Nonlinear Dynamics

Universality of chaos and later developments

In 1979, Feigenbaum showed that the behaviour of May's model for population dynamics is shared by a class of systems. Later scientists discovered that these features are also seen in many experiments. After this discovery, scientists started taking chaos very seriously. Some of the pioneering experiments were done by Gollub, Libchaber, Swinney, and Moon.

Deterministic Chaos- Main ingradients

- Nonlinearity: Response not proportional to input forcing (somewhat more rigourous definition a bit later
- Sensitivity to initial conditions.
- Deterministic systems too show randomness (deterministic chaos). Even though noisy systems too show many interesting stochastic or chaotic behaviour, we will focus on deterministic chaos in these notes.

Dynamical System

A dynamical system is specified by a set of variables called state variables and evolution rules. The state variables and the time in the evolution rules could be discrete or continuous. Also the evolution rules could be either deterministic or stochastic. Given initial condition, the system evolves as

$$\mathbf{x}(0) \to \mathbf{x}(t).$$

The evolution rules for dyanamical systems are quite precise. Contrast this with psychological laws where the rules are not precise. In the present course we will focus on dynamical systems whose evolution is deterministic.

Dynamical System

The most generic way to characterize such systems is through differential equations. Some of the examples are

1. **One dimensional Simple Oscillator**: The evolution is given by

   $$m\ddot{x} = -kx,$$

   We can reduce the above equation to two first-order ODE. The ODEs are

   $$\begin{aligned} \dot{x} &= p/m, \\ \dot{p} &= -kx. \end{aligned}$$

   The state variables are x and p.

Dynamical System

2. **LRC Circuit:** The equation for a LRC circuit in series is given by

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = V_{applied}.$$

The above equation can be reduced to

$$\begin{aligned} \dot{Q} &= I, \\ L\dot{I} &= V_{applied} - RI - \frac{Q}{C}. \end{aligned}$$

The state variables are Q and I.

3. **Population Dynamics:** One of the simplest model for the volution of population P over time is given by

$$\dot{P} = \alpha P - P^2,$$

where α is a costant.

Dynamical System

A general dynamical system is given by $|x(t)\rangle = (x_1, x_2, ...x_n)^T$. Its evolution is given by

$$\frac{d}{dt}|x(t)\rangle = |f(|x(t)\rangle, t)\rangle$$

where **f** is a continuous and differentiable function. In terms of components the equations are

$$\begin{aligned}
\dot{x_1} &= f_1(x_1, x_2, ..., x_n, t), \\
\dot{x_2} &= f_2(x_1, x_2, ..., x_n, t) \\
. &\quad . \\
\dot{x_n} &= f_n(x_1, x_2, ..., x_n, t),
\end{aligned}$$

where f_i are continuous and differentiable functions. When the functions f_i are independent of time, the system is called *autonomous* system. However, when f_i are explicit function of time, the system is called *nonautonomous*. The three examples given above are autonomous sysetms.

Dynamical System

A nonautonomous system can be converted to an autonomous oney by renaming $t = x_{n+1}$ and

$$\dot{x}_{n+1} = 1.$$

An example of nonautonomous system is

$$\begin{aligned}\dot{x} &= p \\ \dot{p} &= -x + F(t).\end{aligned}$$

The above system can be converted to an autonomous system using the following procedure.

$$\begin{aligned}\dot{x} &= p \\ \dot{p} &= -x + F(t) \\ \dot{t} &= 1.\end{aligned}$$

In the above examples, the system variables evolve with time, and the evolution is described using ordinary differential equation. There are however many situations when the system variables are fields in which case the evolution is described using partial differential equation. We illustrate these kinds of systems using examples.

Dynamical System

1. **Diffusion Equation**

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$

Here the state variable is field $T(x)$. We can also describe $T(x)$ in Fourier space using Fourier coefficients. Since there are infinite number of Fourier modes, the above system is an infinite-dimensional. In many situations, finite number of modes are sufficient to describe the system, and we can apply the tools of nonlinear dynamics to such set of equations. Such systems are called low-dimensional models.

2. **Navier-Stokes Equation**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}.$$

Here the state variables are $\mathbf{u}(\mathbf{x})$ and $p(\mathbf{x})$.

Dynamical System

Number of variables →

Nonlinearity ↓

	$n=1$	$n=2$	$n\geq 3$	$n>>1$	Continuum
Linear	*Growth, decay, or equilibrium* Exponential growth RC circuit Radioactive decay	*Oscillations* Linear oscillator Mass and spring RLC circuit 2-body problem (Kepler, Newton)	Civil engineering, structures Electrical engineering	*Collective phenomena* Coupled harmonic oscillators Solid-state physics Molecular dynamics Equilibrium statistical mechanics	*Waves and patterns* Elasticity Wave equations Electromagnetism (Maxwell) Quantum mechanics (Schrödinger, Heisenberg, Dirac) Heat and diffusion Acoustics Viscous fluids
Nonlinear	Fixed points Bifurcations Overdamped systems, relaxational dynamics Logistic equation for single species	Pendulum Anharmonic oscillators Limit cycles Biological oscillators (neurons, heart cells) Predator-prey cycles Nonlinear electronics (van der Pol, Josephson)	*Chaos* Strange attractors (Lorenz) 3-body problem (Poincaré) Chemical kinetics Iterated maps (Feigenbaum) Fractals (Mandelbrot) Forced nonlinear oscillators (Levinson, Smale) Practical uses of chaos Quantum chaos ?	*The frontier* Coupled nonlinear oscillators Lasers, nonlinear optics Nonequilibrium statistical mechanics Nonlinear solid-state physics (semiconductors) Josephson arrays Heart cell synchronization Neural networks Immune system Ecosystems Economics	*Spatio-temporal complexity* Nonlinear waves (shocks, solitons) Plasmas Earthquakes General relativity (Einstein) Quantum field theory Reaction-diffusion, biological and chemical waves Fibrillation Epilepsy Turbulent fluids (Navier-Stokes) Life

Continuous state variables and discrete time

Many systems are described by discrete time. For example, hourly flow Q_n through a pipe could be described by

$$Q_{n+1} = f(Q_n).$$

where f is a continuous and single-valued function, and n is the index for hour. Another example is evolution of the normalized population P_n is the population at nth year then

$$P_{n+1} = aP_n(1 - P_n).$$

Here the population is normalized with respect to the maximum population to make P_n as a continuous variable. Physically the first term represents growth, while the second term represents saturation.

The above equations are called **difference equations**.

Note that if the time gap between two observations become very small, then the description will be closer to continuous time case.

Discrete state variables and discrete time

For some ynamical systems the system variables are discrete, and they evolve in discrete time. A popular example is game of life where each site has a living cell or a dead cell. The cell at a given site can change from live to dead or vise versa depending on certain rules. For example, a dead cell becomes alive if number of live neighbours are between 3 to 5 (neither under-populated or over-populated). These class of dynamical systems show very rich patterns and behaviour. Unfortunately we will not cover them in this course.

Discrete state variables and continuous time

The values of system variables of logic gates are discrete (0 or 1). However they depend on the external input that can exceed the threshold value in continuous time. Again, these class of systems are beyond the scope of the course.

In the present course we will focus on ordinary differential equations and difference equations that deal with continuous state variables but continuous and discrtet time respectively.

One Dimensional Systems

The evolution equation of an autonomous one-dimensional dynamical system (DS) is given by

$$\dot{x} = f(x).$$

The points x^* at which $f(x^*) = 0$ are called "*fixed points*" (FP); at these points

$$\dot{x} = 0.$$

Hence, if at $t = 0$ the system at x^*, then the system will remain at x^* at all time in future. Noet that a system can have any number of fixed points (0, 1, 2, ...).

Now let us explore the behaviour of the DS near the fixed points:

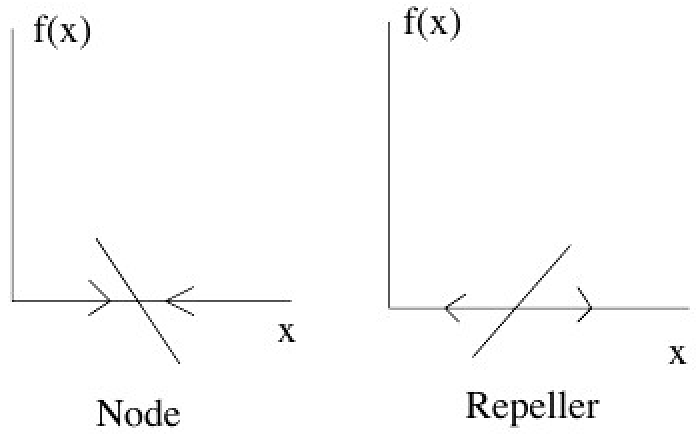
$$\dot{x} \approx f'(x^*)(x - x^*),$$

whose solution is

$$x(t) - x^* = (x(0) - x^*) \exp\left(f'(x^*)t\right).$$

One Dimensional Systems

Clearly, if $f'(x^*) < 0$, the system will approach x^*. This kind of fixed point is called a *node*. If $f'(x^*) > 0$, the system will go away from x^*, and the fixed point is called a *repeller*. These two cases are shown in the first two figures of the following diagram:

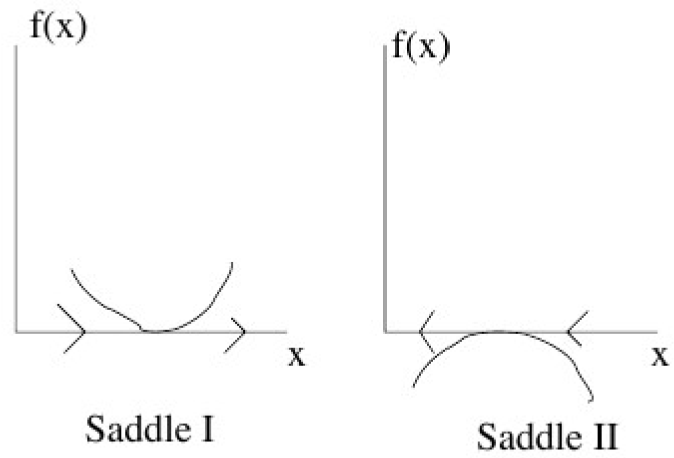


Note that the motion in only taking place along the x-axis.

One Dimensional Systems

On rare ocassions $f(x^*) = f'(x^*) = 0$. In these cases, the evolution near the fixed point will be determined by the second derivative of f, i.e.,

$$\dot{x} \approx \frac{f''(x^*)}{2}(x - x^*)^2.$$



If $f''(x^*) > 0$ then $\dot{x} > 0$ for both sides of x^*. If the system is to the left of x^*, it will tend towards x^*. On the other hand if the system is to the right of x^*, then it will go further away from x^*. This point is called *saddle point of Type I*. The reverse happens for $f''(x^*) < 0$, and the fixed point is called *saddle point of Type II*.

One Dimensional Systems

Examples

1. $\dot{x} = 2x$. The fixed point is $x^* = 0$. It is a repeller because $f'(0) = 2$ (positive).

2. $\dot{x} = -x+1$. The fixed point is $x^* = 1$. It is node because $f'(1) = -1$ (negative).

3. $\dot{x} = (x-2)^2$. The fixed point is $x^* = 2$. This point is a saddle point of Type I because $f''(2) = 2$ (positive).

The above analysis provides us information about the behaviour near the fixed points. So they are called *local behaviour of the system.*

One Dimensional Systems

Consider the following nonlinear differential equation:

$$\dot{x} = \sin x.$$

$$dt = \frac{dx}{\sin x},$$

which implies

$$\begin{aligned} t &= \int \csc x \; dx \\ &= -\ln\left|\csc x + \cot x\right| + C. \end{aligned}$$

To evaluate the constant C, suppose that $x = x_0$ at $t = 0$. Then $C = \ln|\csc x_0 + \cot x_0|$. Hence the solution is

One Dimensional Systems

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|.$$

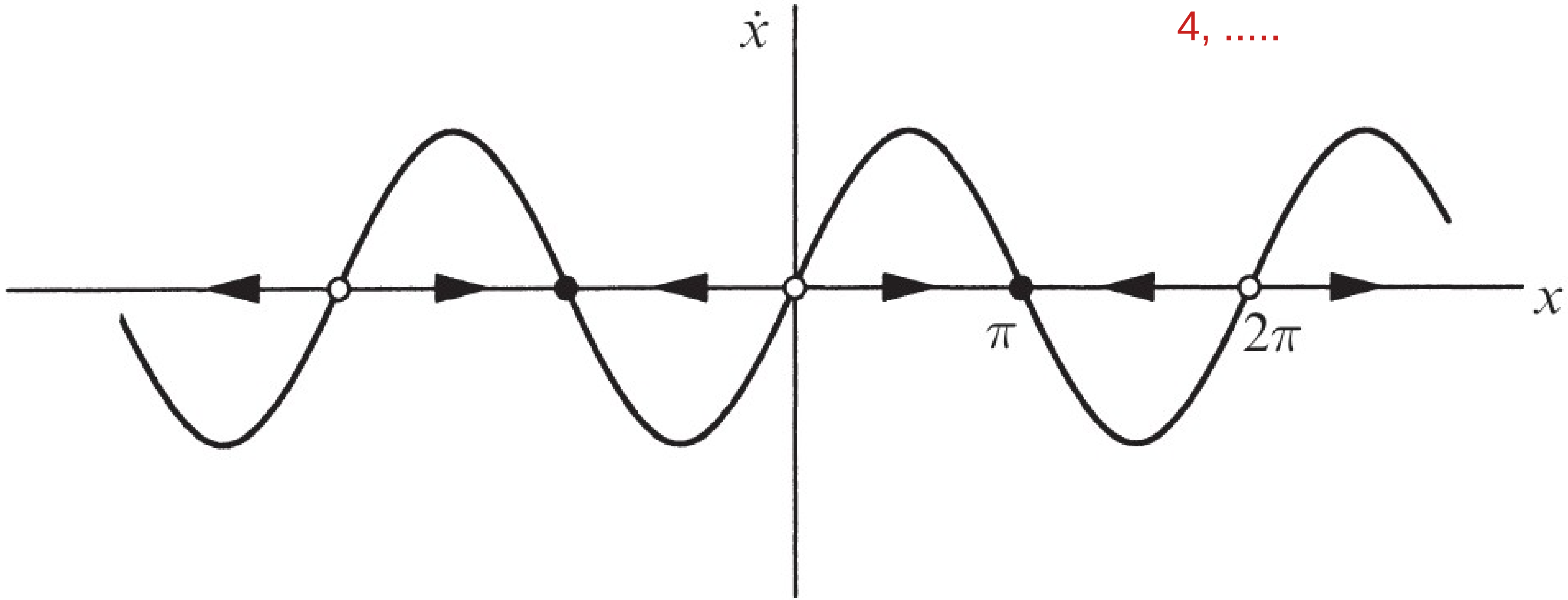
1. Suppose $x_0 = \pi/4$; describe the qualitative features of the solution $x(t)$ for all $t > 0$. In particular, what happens as $t \to \infty$?
2. For an *arbitrary* initial condition x_0, what is the behavior of $x(t)$ as $t \to \infty$?

One Dimensional Systems

Geometrical way of thinking the same problem....

Fixed Points: n π

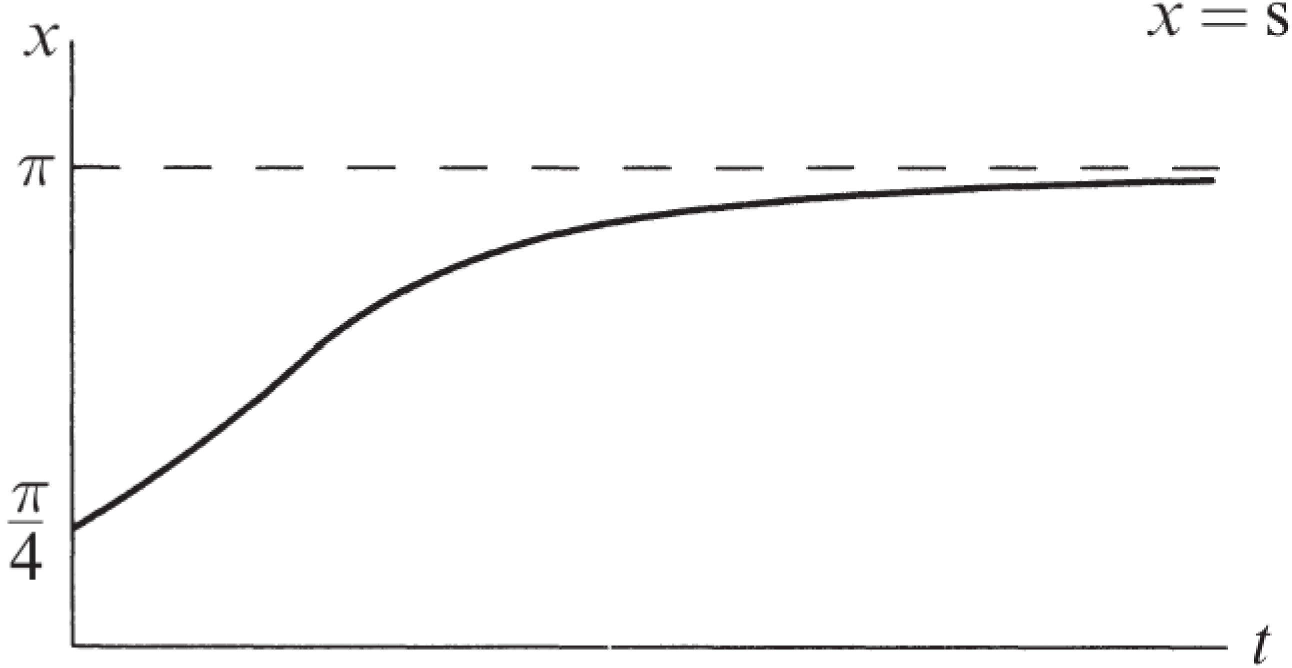
n=0, 1, 2, 3, 4, ...



the ***flow*** is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$.

One Dimensional Systems

$\dot{x} = \sin x$

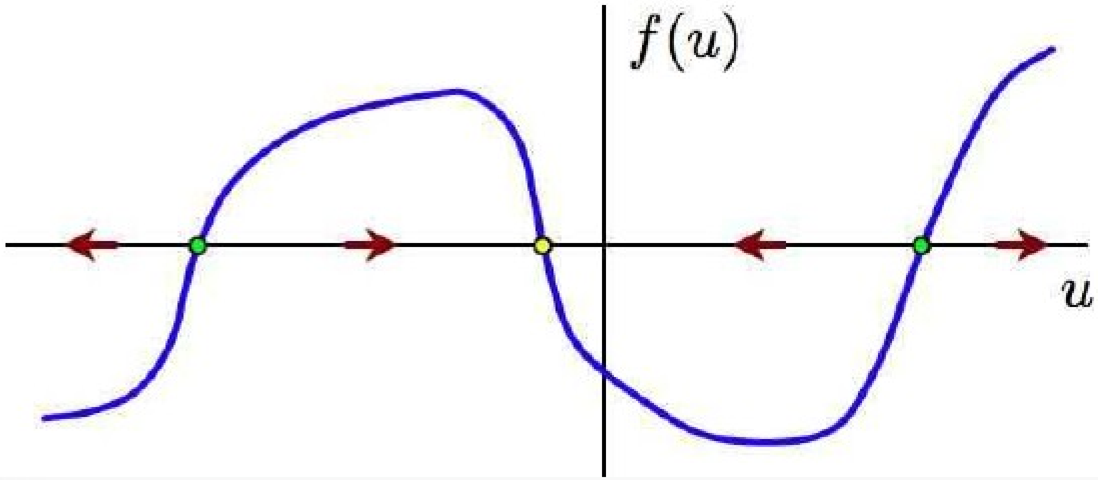


If particle starts from $x_0 = \pi/4$ it moves faster and faster

until it crosses $x = \pi/2$ (the point sinx reaches its maxumum.) Then it starts slowing down and eventually reaches to stable fixed point $x = \pi$

Recap to draw to phase flow for N=1 system

$$\dot{u} = f(u) \quad \Longrightarrow \quad \begin{cases} f(u) > 0 & \dot{u} > 0 \quad \Rightarrow \quad \text{move to right} \\ f(u) < 0 & \dot{u} < 0 \quad \Rightarrow \quad \text{move to left} \\ f(u) = 0 & \dot{u} = 0 \quad \Rightarrow \quad \text{fixed point} \end{cases}$$



Exercise

Sketch the phase flows of following examples. In each case identify all the fixed points and access their stability using linear stability analysis. Assume all constants A, a, b,c, γ, etc. are positive.

$$\dot{v} = -g$$
$$m\dot{v} = -mg - \gamma v$$
$$m\dot{v} = -mg - cv^2 \operatorname{sgn}(v)$$

$$\dot{u} = A\sin(u)$$
$$\dot{u} = A\,(u-a)(u-b)(u-c)$$
$$\dot{u} = au^2 - bu^3 \;.$$

$$\dot{x} = 4x^2 - 16$$
$$\dot{x} = x - x^3$$
$$\dot{x} = 1 + \tfrac{1}{2}\cos x$$

$$\dot{x} = e^{-x}\sin x$$
$$\dot{x} = 1 - 2\cos x$$

Exercise

Sketch the phase flows of following examples. In each case identify all the fixed points and access their stability using linear stability analysis.

$$\dot{x} = x(1-x)$$

$$\dot{x} = \tan x$$

$$\dot{x} = 1 - e^{-x^2}$$

$$\dot{x} = x(1-x)(2-x)$$

$$\dot{x} = x^2(6-x)$$

$$\dot{x} = \ln x$$

Solving the Dyanmical system using Potential

The problem of first order dynmaical system can be visulized using the Physical idea of potential energy also. Let us consider a situation in which particle is sliding down the wall of a potential well, where the **potential V(x)** is defined as

$$f(x) = -\frac{dV}{dx}.$$

The negative sign in the definition of V follows the standard convention in physics; it implies that the particle always moves "downhill" as the motion proceeds. To see this, we think of x as a function of t, and then calculate the time-derivative of $V(x(t))$. Using the chain rule, we obtain

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}.$$

Solving the Dyanmical system using Potential

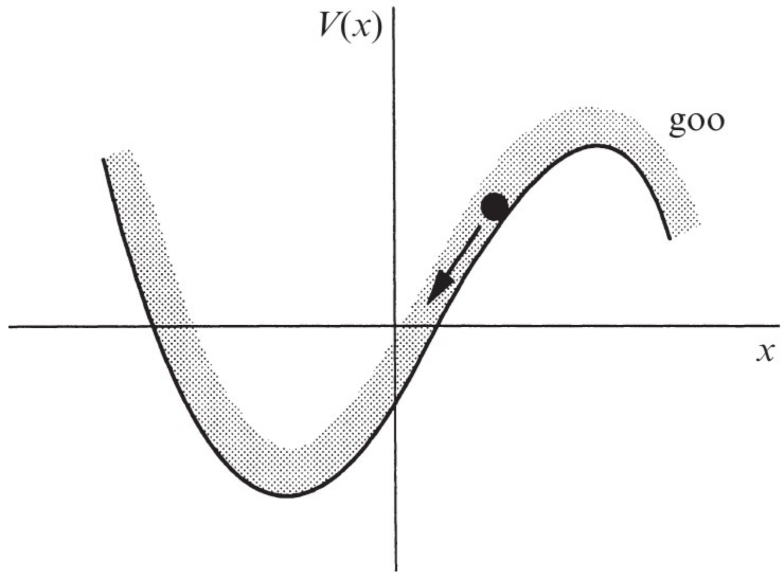
Now for a first-order system,

$$\frac{dx}{dt} = -\frac{dV}{dx},$$

since $\dot{x} = f(x) = -dV/dx$, by the definition of the potential. Hence,

$$\frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0.$$

Solving the Dyanmical system using Potential



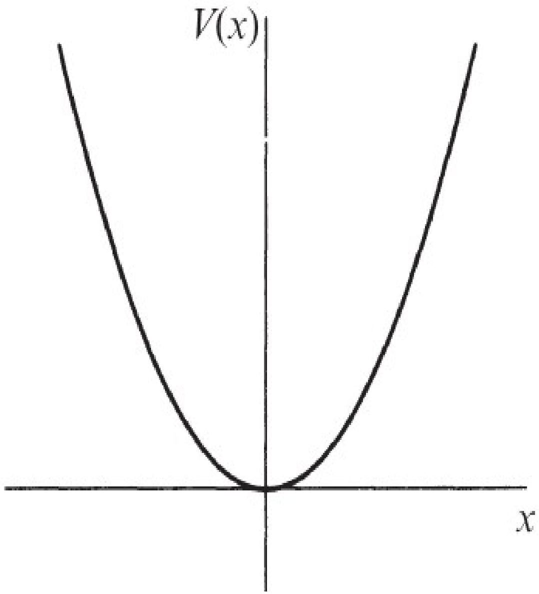
Thus $V(t)$ *decreases along trajectories*, and so the particle always moves toward lower potential. Of course, if the particle happens to be at an ***equilibrium*** point where $dV/dx = 0$, then V remains constant. This is to be expected, since $dV/dx = 0$ implies $\dot{x} = 0$; equilibria occur at the fixed points of the vector field. Note that local minima of $V(x)$ correspond to *stable* fixed points, as we'd expect intuitively, and local maxima correspond to unstable fixed points.

Solving the Dyanmical system using Potential

Graph the potential for the system $\dot{x} = -x$, and identify all the equilibrium points.

We need to find $V(x)$ such that

$-dV/dx = -x$. The general solution is $V(x) = \frac{1}{2}x^2 + C$, where C is an arbitrary constant. (It always happens that the potential is only defined up to an additive constant. For convenience, we usually choose $C = 0$.)



The only equilibrium point here is at x=0 and it's stable.

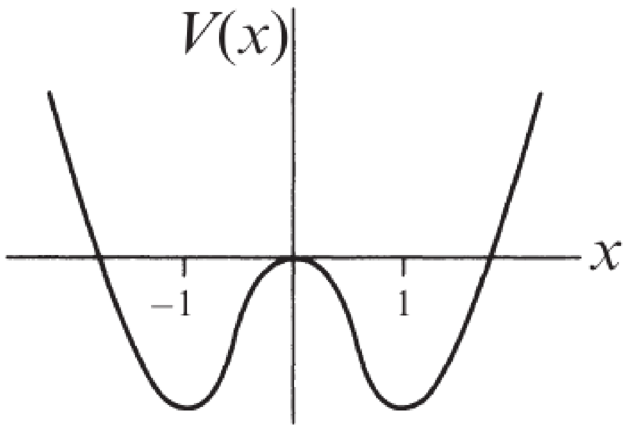
Solving the Dyanmical system using Potential

Graph the potential for the system $\dot{x} = x - x^3$, and identify all equilibrium points.

Solving $-dV/dx = x - x^3$ yields

$$V = -\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 + C.$$

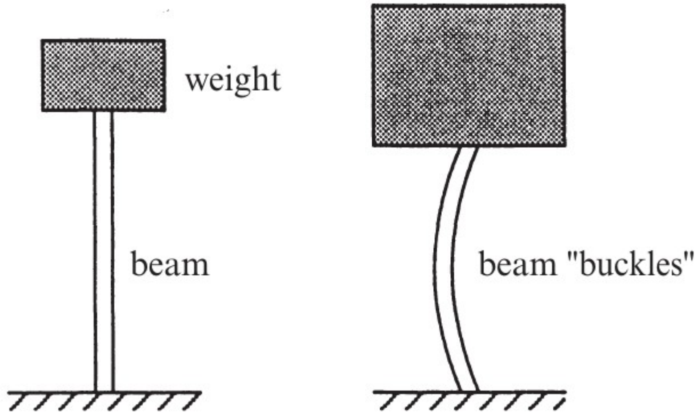
Consider C=0.



The fixed points will be corresponding to the local minima of the double well potential which are situated at x=-1 and x=1.

Bifurcation in 1d

The flow in 1d is very limited. We find that all the solutions either settle down to the equilibrium or move towards the infinity. One of the interesting thing to notice here is how nature of the fixed points changes upon varying the parameters. In particular we will see that the fixed points can be created or destroyed or their stability can be changed. The qualitative change in the dynamics is termed as **bifurcation**. Bifurcation is important as it provides us a model to understand the transition to the instabilities as the parameter is varied.



The beam gets bend as the load exceeds at certain critical value

Bifurcation in 1d

Saddle-Node Bifurcation

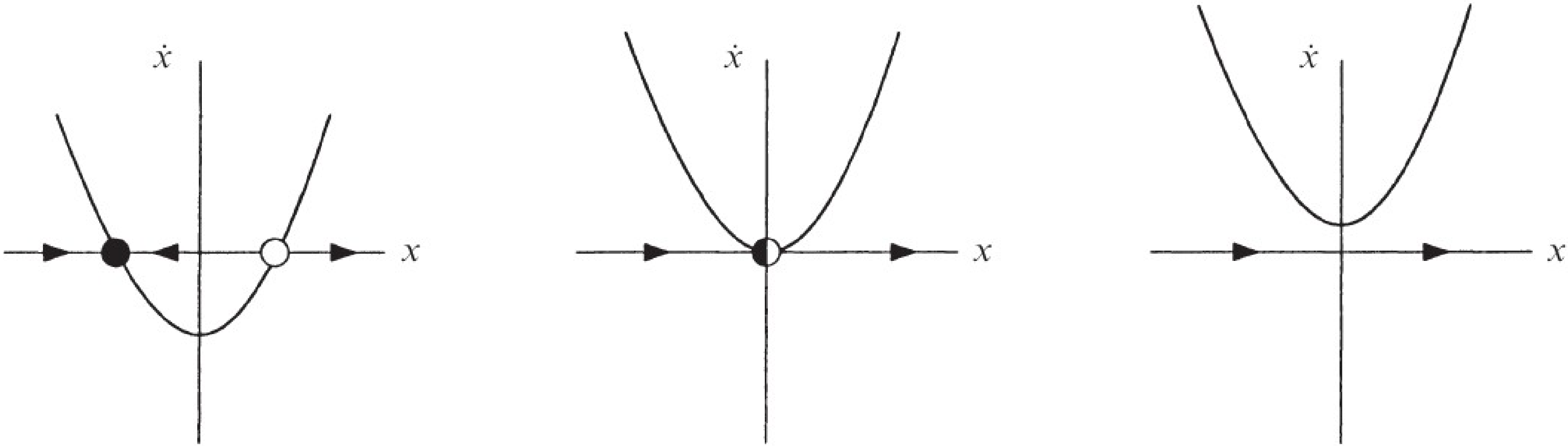
The saddle-node bifurcation is the basic mechanism by which fixed points are *created and destroyed*. As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

$$\dot{x} = r + x^2$$

where r is a parameter, which may be positive, negative, or zero

Bifurcation in 1d

Saddle-Node Bifurcation



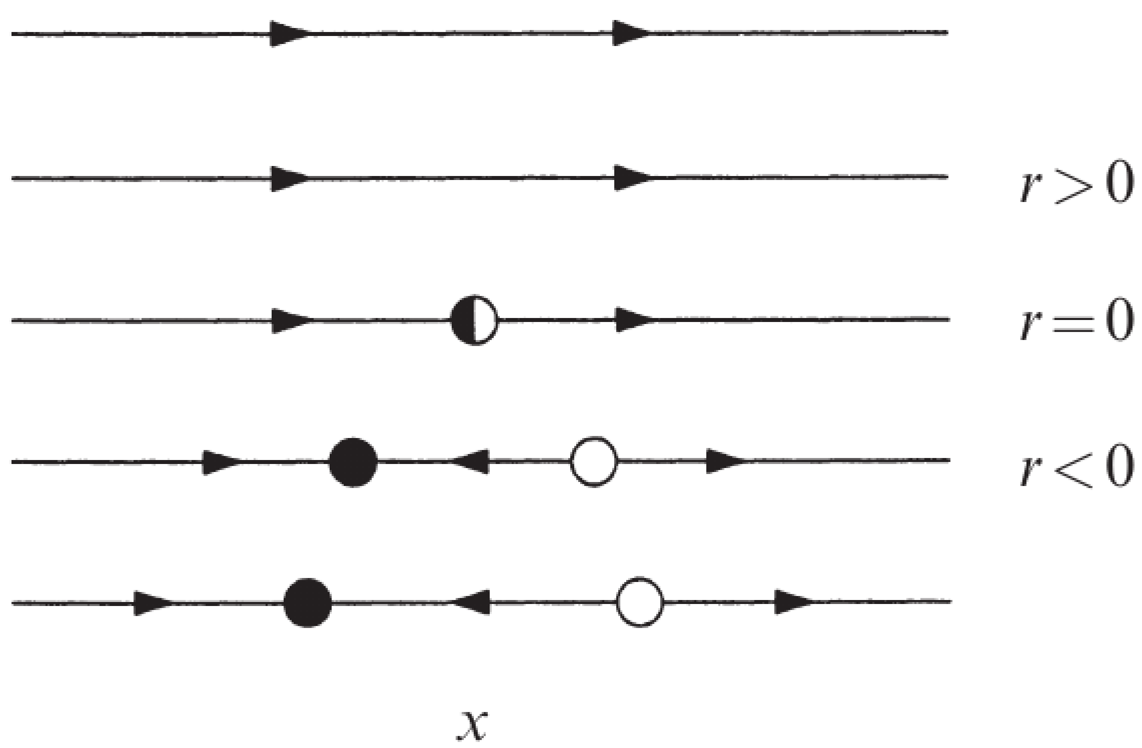
(a) $r<0$ (b) $r=0$ (c) $r>0$

In this example, we say that a *bifurcation* occurred at $r = 0$, since the vector fields for $r < 0$ and $r > 0$ are qualitatively different.

Bifurcation in 1d

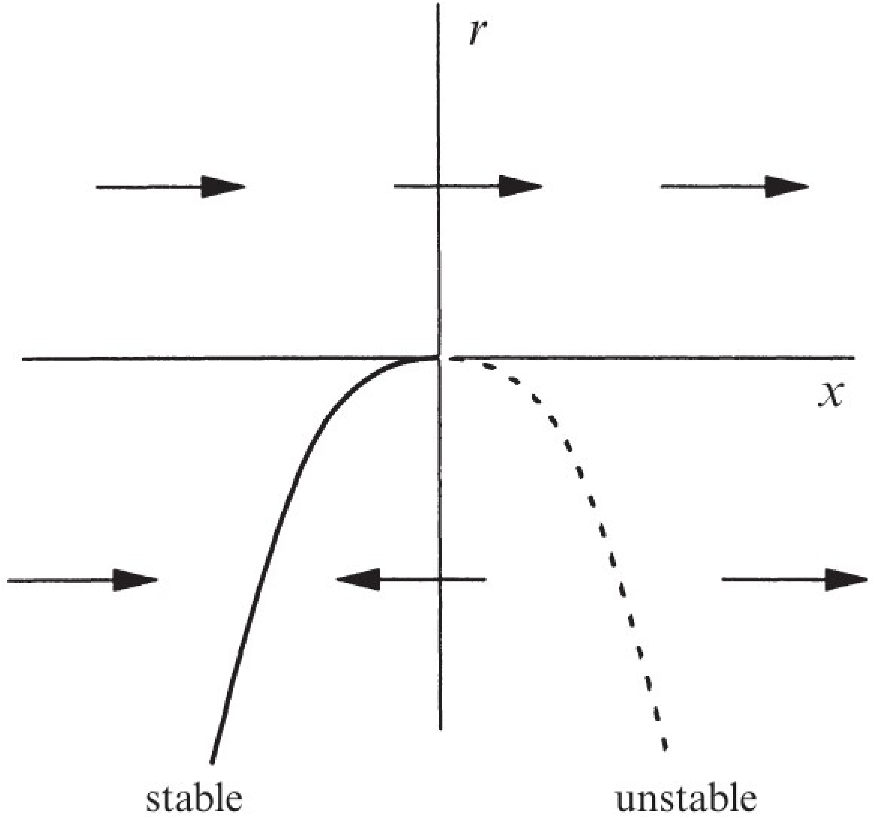
Saddle-Node Bifurcation

Graphical representation



Bifurcation in 1d

Saddle-Node Bifurcation



$$r = -x^2, \text{ i.e., } \dot{x} = 0$$

Bifurcation in 1D flow

Example:

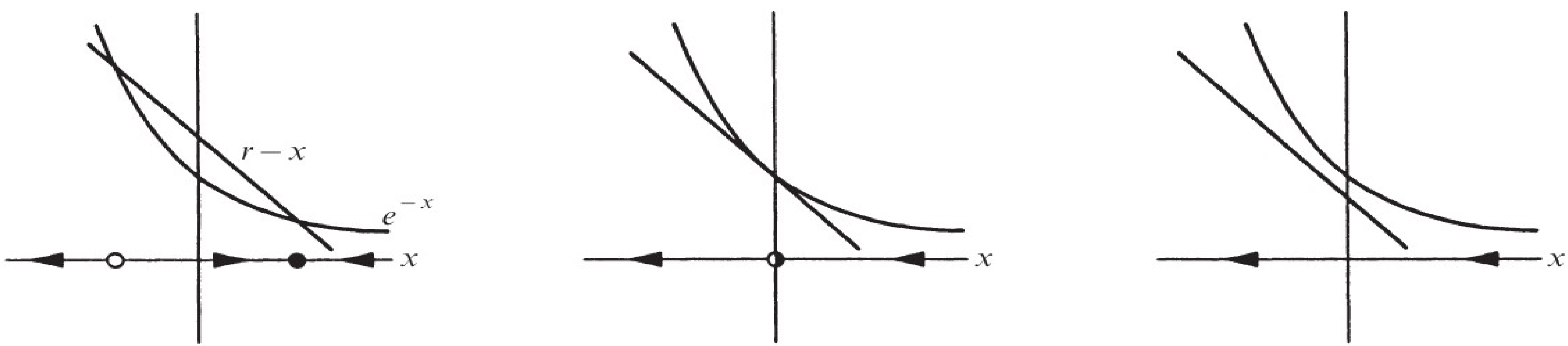
Saddle node bifurcation

$$\dot{x} = r - x - e^{-x}$$

The fixed points satisfy $f(x) = r - x - e^{-x} = 0$.

Fixed point can't by find quite easily due to its dependence on the complex function.

We will find the solution using the graphical method.



(a) (b) (c)

Bifurcation in 1D flow

Example:

$$\dot{x} = r - x - e^{-x}$$

To find the bifurcation point r_c, we impose the condition that the graphs of $r - x$ and e^{-x} intersect *tangentially*. Thus we demand equality of the functions *and* their derivatives:

$$e^{-x} = r - x$$

and

$$\frac{d}{dx} e^{-x} = \frac{d}{dx}(r - x).$$

The second equation implies $-e^{-x} = -1$, so $x = 0$. Then the first equation yields $r = 1$. Hence the bifurcation point is $r_c = 1$, and the bifurcation occurs at $x = 0$. ■

Bifurcation in 1D flow

Example:

$$\dot{x} = r - x - e^{-x}$$

In a certain sense, the examples $\dot{x} = r - x^2$ or $\dot{x} = r + x^2$ are representative of *all* saddle-node bifurcations; that's why we called them "prototypical." The idea is that, close to a saddle-node bifurcation, the dynamics typically look like $\dot{x} = r - x^2$ or $\dot{x} = r + x^2$.

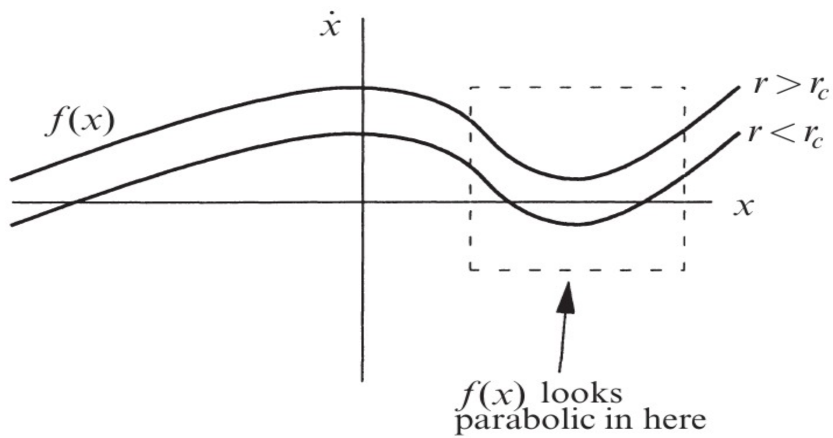
Using the Taylor expansion for e^{-x} about $x = 0$, we find

$$\begin{aligned}\dot{x} &= r - x - e^{-x} \\ &= r - x - \left[1 - x + \frac{x^2}{2!} + \cdots\right] \\ &= (r-1) - \frac{x^2}{2} + \cdots\end{aligned}$$

For appropriate scalins of r and x it can be shown that it has the typical form as

$$\dot{x} = r - x^2$$

Bifurcation in 1D flow



$$\dot{x} = f(x, r)$$

$$= f(x^*, r_c) + (x - x^*) \left.\frac{\partial f}{\partial x}\right|_{(x^*, r_c)} + (r - r_c) \left.\frac{\partial f}{\partial r}\right|_{(x^*, r_c)} + \tfrac{1}{2}(x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r_c)} + \cdots$$

Bifurcation in 1D flow

where we have neglected quadratic terms in $(r - r_c)$ and cubic terms in $(x - x^*)$. Two of the terms in this equation vanish: $f(x^*, r_c) = 0$ since x^* is a fixed point, and $\partial f / \partial x|_{(x^*, r_c)} = 0$ by the tangency condition of a saddle-node bifurcation. Thus

$$\dot{x} = a(r - r_c) + b(x - x^*)^2 + \cdots$$

where $a = \partial f / \partial x|_{(x^*, r_c)}$ and $b = \frac{1}{2} \partial^2 f / \partial x^2|_{(x^*, r_c)}$

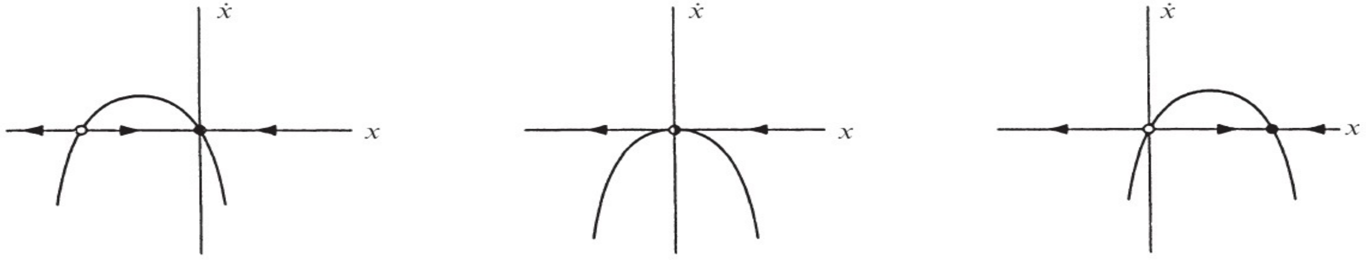
Bifurcation in 1D flow

Transcritical Bifurcation

There are certain cases in which the fixed point will remain for all the parameter range and may change its stability as the parameter is varied. The transcritical bifurcation describes mechanism of such change in the stability.

The normal form for a transcritical bifurcation is

$$\dot{x} = rx - x^2.$$



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

Bifurcation in 1D flow

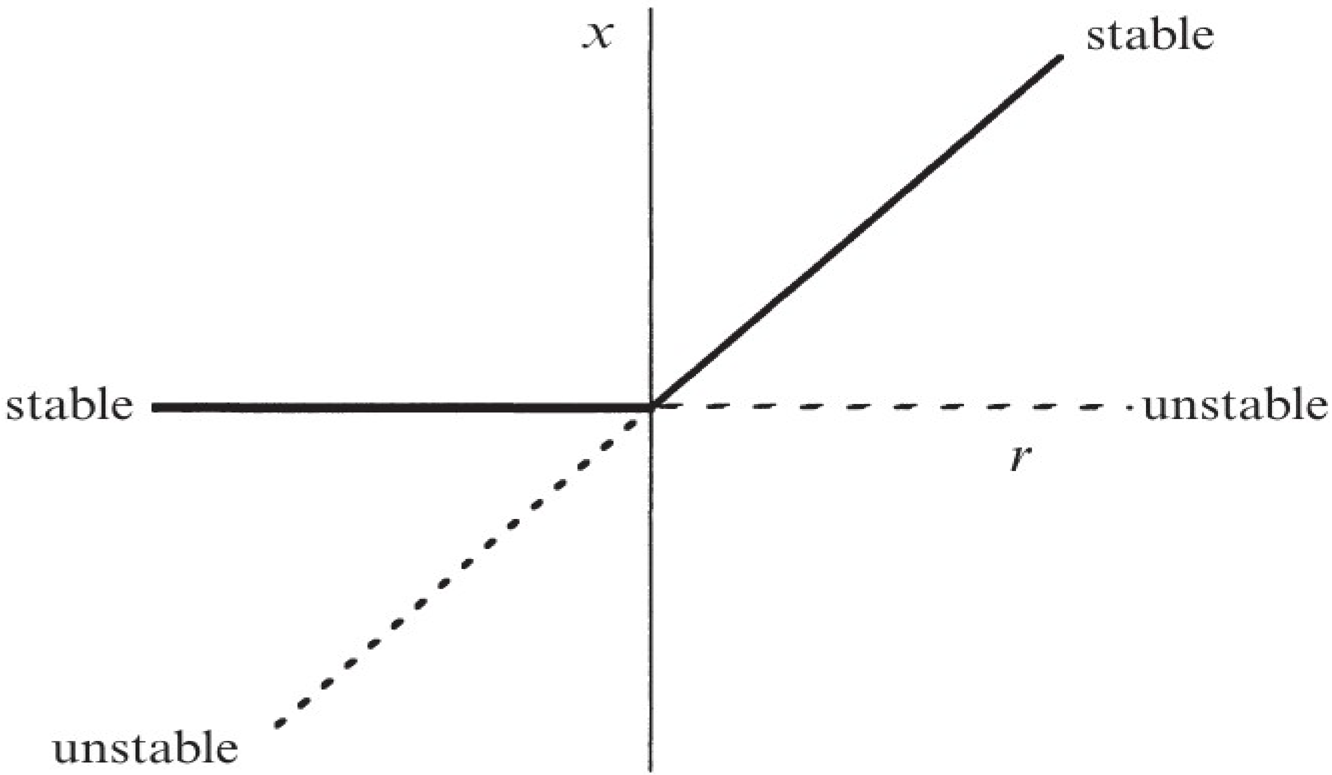
Transcritical Bifurcation

For $r < 0$, there is an unstable fixed point at $x^* = r$ and a stable fixed point at $x^* = 0$. As r increases, the unstable fixed point approaches the origin, and coalesces with it when $r = 0$. Finally, when $r > 0$, the origin has become unstable, and $x^* = r$ is now stable. Some people say that an ***exchange of stabilities*** has taken place between the two fixed points.

Please note the important difference between the saddle-node and transcritical bifurcations: in the transcritical case, the two fixed points don't disappear after the bifurcation—instead they just switch their stability.

Bifurcation in 1D flow

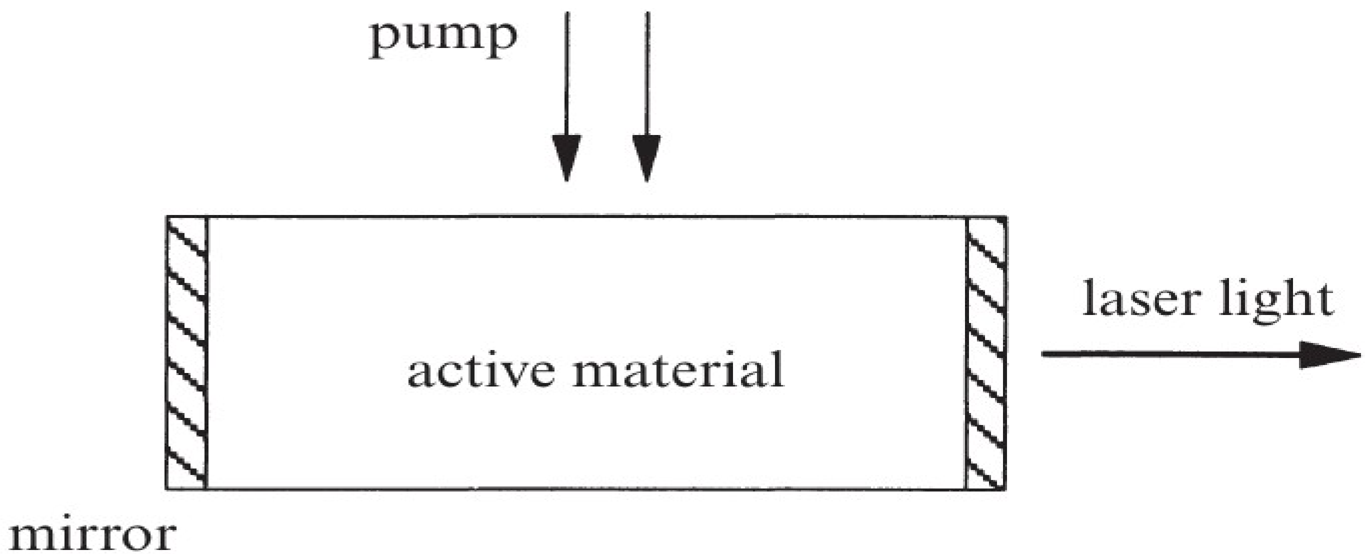
Transcritical Bifurcation



Bifurcation in 1D flow

Transcritical Bifurcation

Physical Example



Typical setup for the laser.

Bifurcation in 1D flow

Transcritical Bifurcation

$$\begin{aligned}\dot{n} &= \text{gain} - \text{loss} \\ &= GnN - kn.\end{aligned}$$

n(t) number of photon emmited

The gain term comes from the process of *stimulated emission*, in which photons stimulate excited atoms to emit additional photons. Because this process occurs via random encounters between photons and excited atoms, it occurs at a rate proportional to n and to the number of excited atoms, denoted by $N(t)$. The parameter $G > 0$ is known as the gain coefficient. The loss term models the escape of photons through the endfaces of the laser. The parameter $k > 0$ is a rate constant; its reciprocal $\tau = 1/k$ represents the typical lifetime of a photon in the laser.

Bifurcation in 1D flow

Transcritical Bifurcation

Now comes the key physical idea: after an excited atom emits a photon, it drops down to a lower energy level and is no longer excited. Thus N decreases by the emission of photons. To capture this effect, we need to write an equation relating N to n. Suppose that in the absence of laser action, the pump keeps the number of excited atoms fixed at N_0. Then the *actual* number of excited atoms will be reduced by the laser process. Specifically, we assume

$$N(t) = N_0 - \alpha n,$$

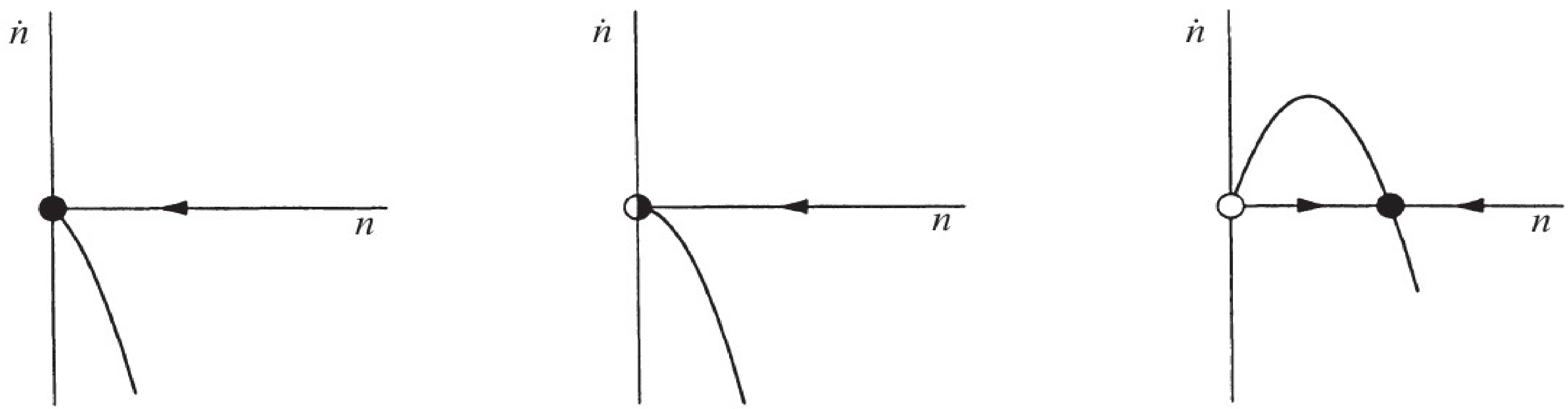
where $\alpha > 0$ is the rate at which atoms drop back to their ground states. Then

$$\begin{aligned} \dot{n} &= Gn(N_0 - \alpha n) - kn \\ &= (GN_0 - k)n - (\alpha G)n^2. \end{aligned}$$

Bifurcation in 1D flow

Transcritical Bifurcation

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$
$$= (GN_0 - k)n - (\alpha G)n^2.$$



$N_0 < k/G$ $\quad$ $N_0 = k/G$ $\quad$ $N_0 > k/G$

Bifurcation in 1D flow

Transcritical Bifurcation

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$
$$= (GN_0 - k)n - (\alpha G)n^2.$$

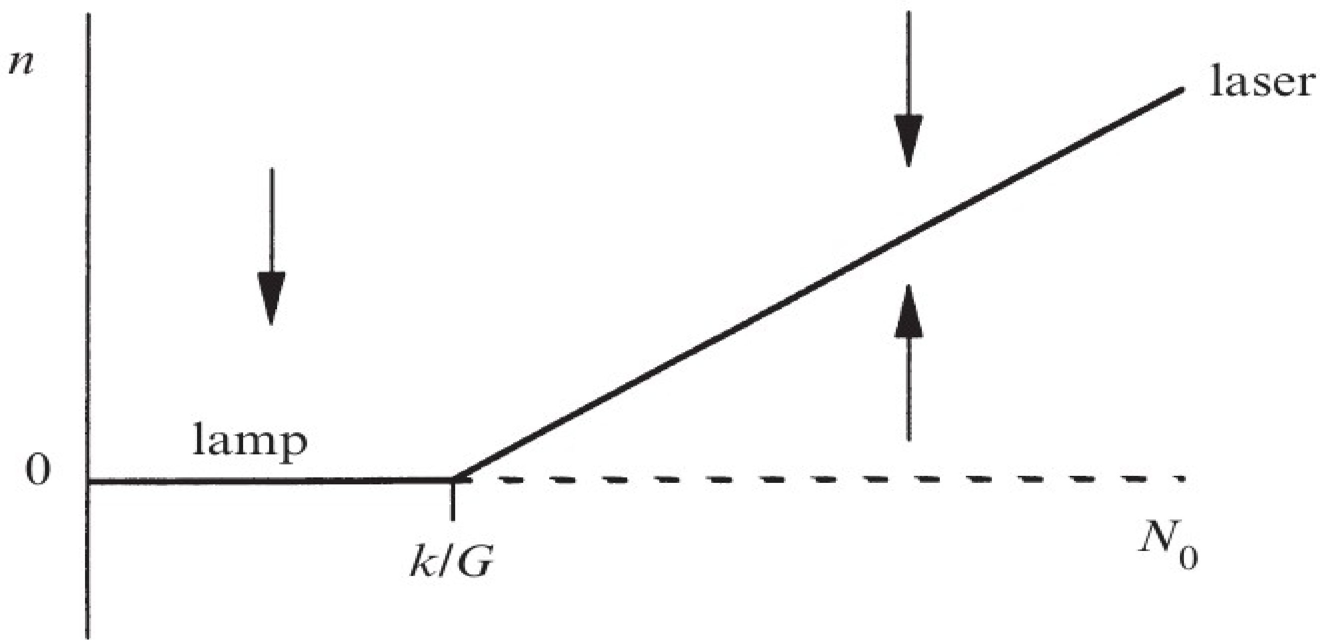
When $N_0 < k/G$, the fixed point at $n^* = 0$ is stable. This means that there is no stimulated emission and the laser acts like a lamp. As the pump strength N_0 is increased, the system undergoes a transcritical bifurcation when $N_0 = k/G$. For $N_0 > k/G$, the origin loses stability and a stable fixed point appears at $n^* = (GN_0 - k)/\alpha G > 0$, corresponding to spontaneous laser action. Thus

$N_0 = k/G$ can be interpreted as the ***laser threshold*** in this model.

Bifurcation in 1D flow

Transcritical Bifurcation

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$
$$= (GN_0 - k)n - (\alpha G)n^2.$$



Although this model perdicts the threshold value of pump beyond which the laser action begins it has lots of limitations (not consider the spontaneous emission, quantum mechanical effect, etc..)

Bifurcation in 1D flow

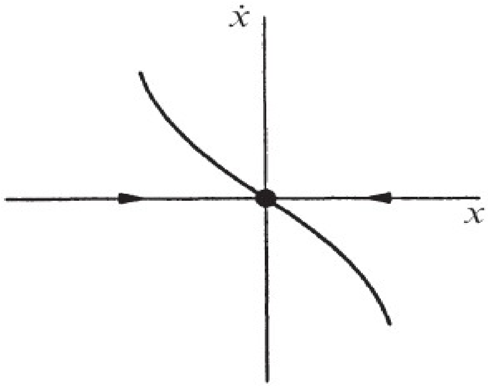
Pitchfork Bifurcation

This bifurcation is quite common where we have symmetry. As for an example if we have left right spatial symmetry that would be broken once bifurcation takes place.

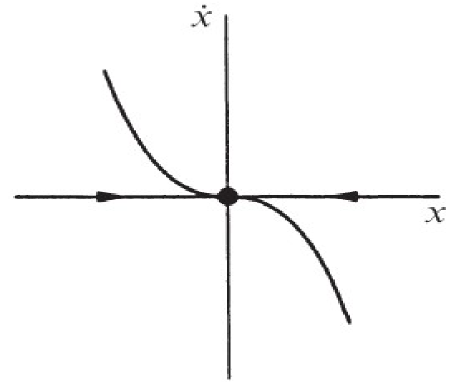
Supercritical Pitchfork Bifurcation

$$\dot{x} = rx - x^3 .$$

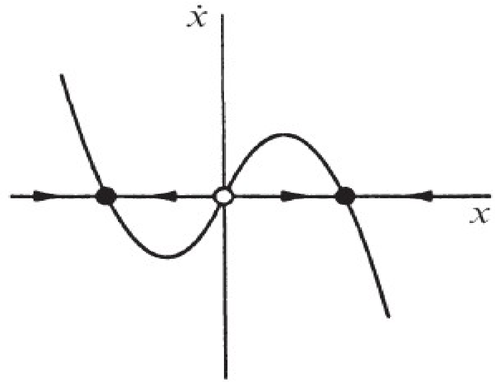
Note that this equation is ***invariant*** under the change of variables $x \to -x$.



(a) $r<0$



(b) $r=0$



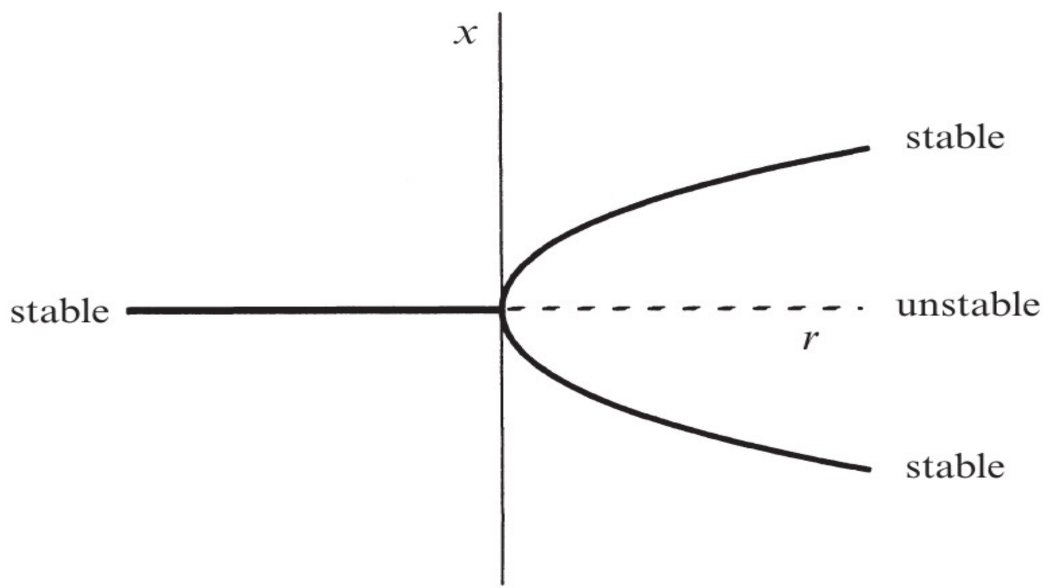
(c) $r>0$

Bifurcation in 1D flow

Pitchfork Bifurcation

When r<0 orgin is the only fixed point and its stable. When r=0, the origin is still stable but much weakly than those for r<0. Here the solution does not decay exponentially near the fixed point, but much slower than this. In Physics it is also known as critical slowing down. For r>0 the system becomes unstable at the origin and two new stable points appear.

$$x^* = \pm\sqrt{r}$$

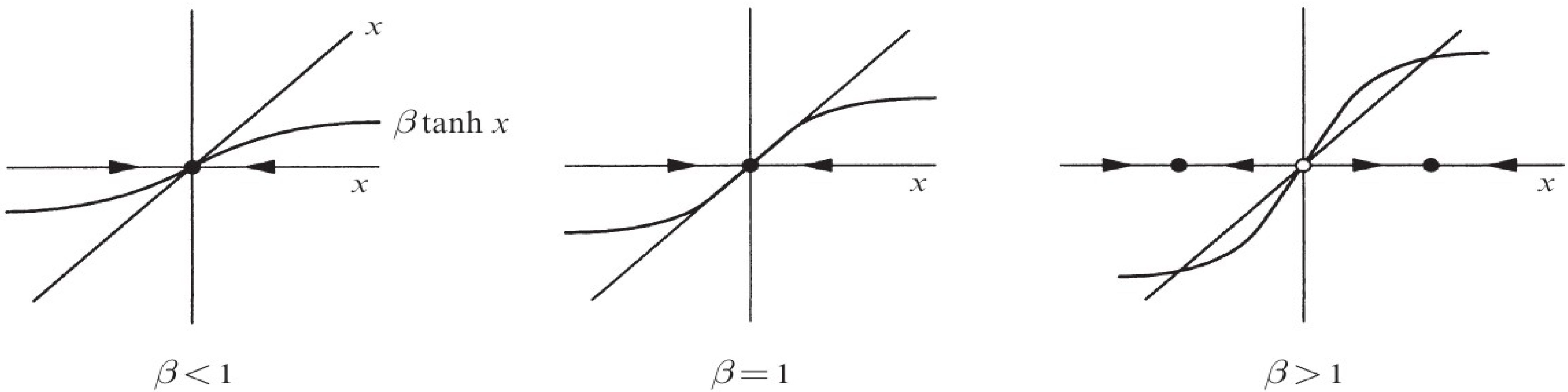


Bifurcation diagram

Bifurcation in 1D flow

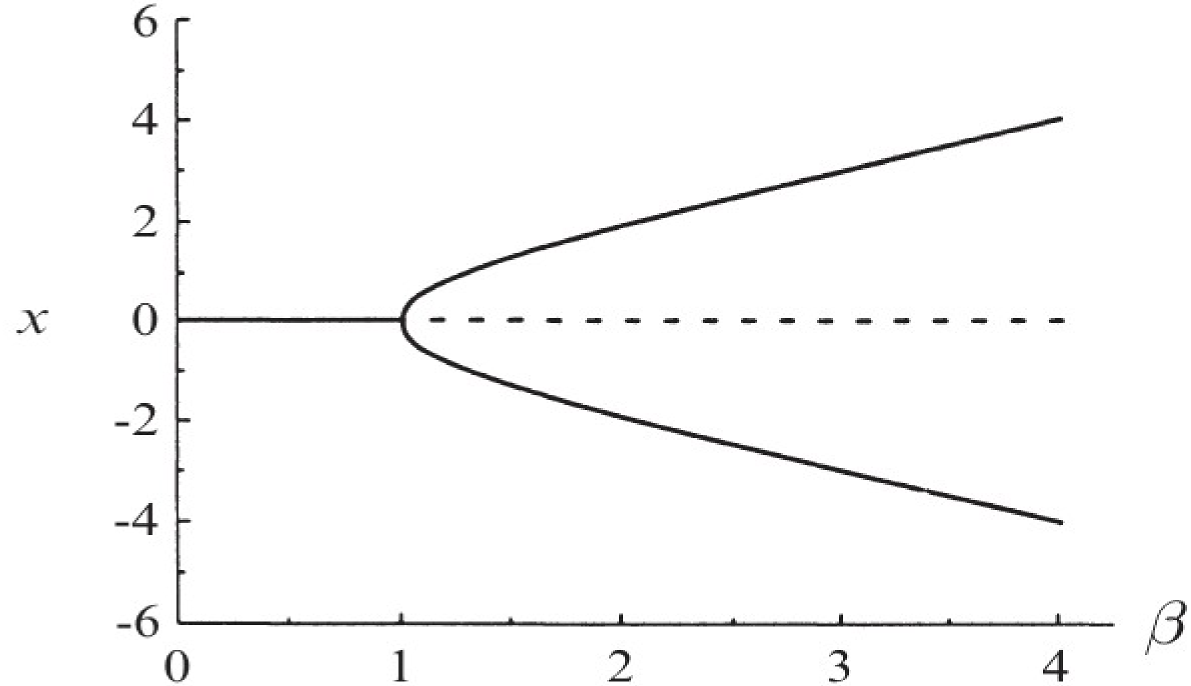
Pitchfork Bifurcation

$$\dot{x} = -x + \beta \tanh x$$



Bifurcation in 1D flow

Pitchfork Bifurcation



Bifurcation in 1D flow

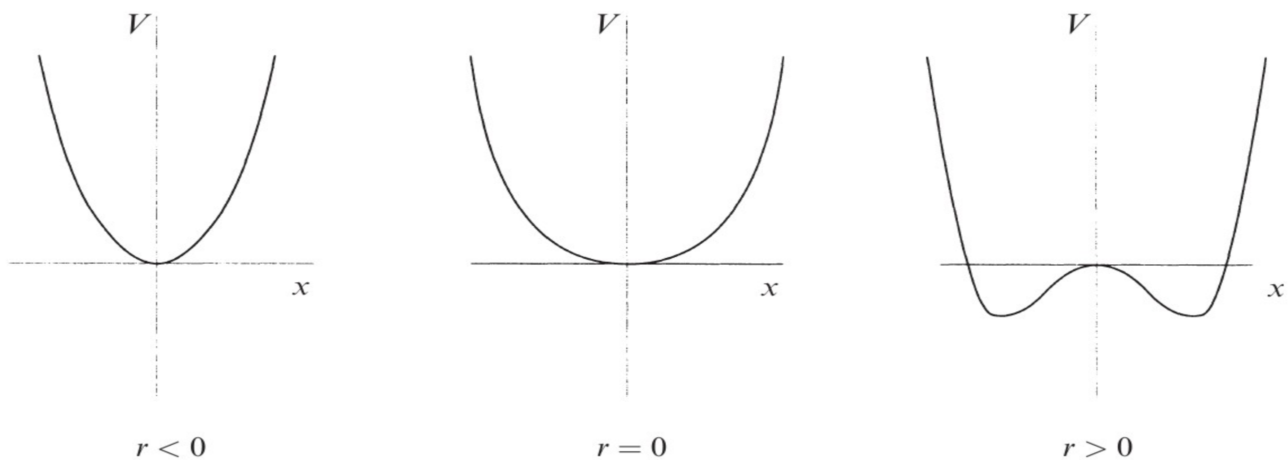
Pitchfork Bifurcation

Plot the potential $V(x)$ for the system $\dot{x} = rx - x^3$, for the cases $r < 0$, $r = 0$, and $r > 0$.

$$\dot{x} = f(x)$$

$$f(x) = -dV/dx.$$

$$-dV/dx = rx - x^3.$$



Bifurcation in 1D flow

Subcritical Pitchfork Bifurcation

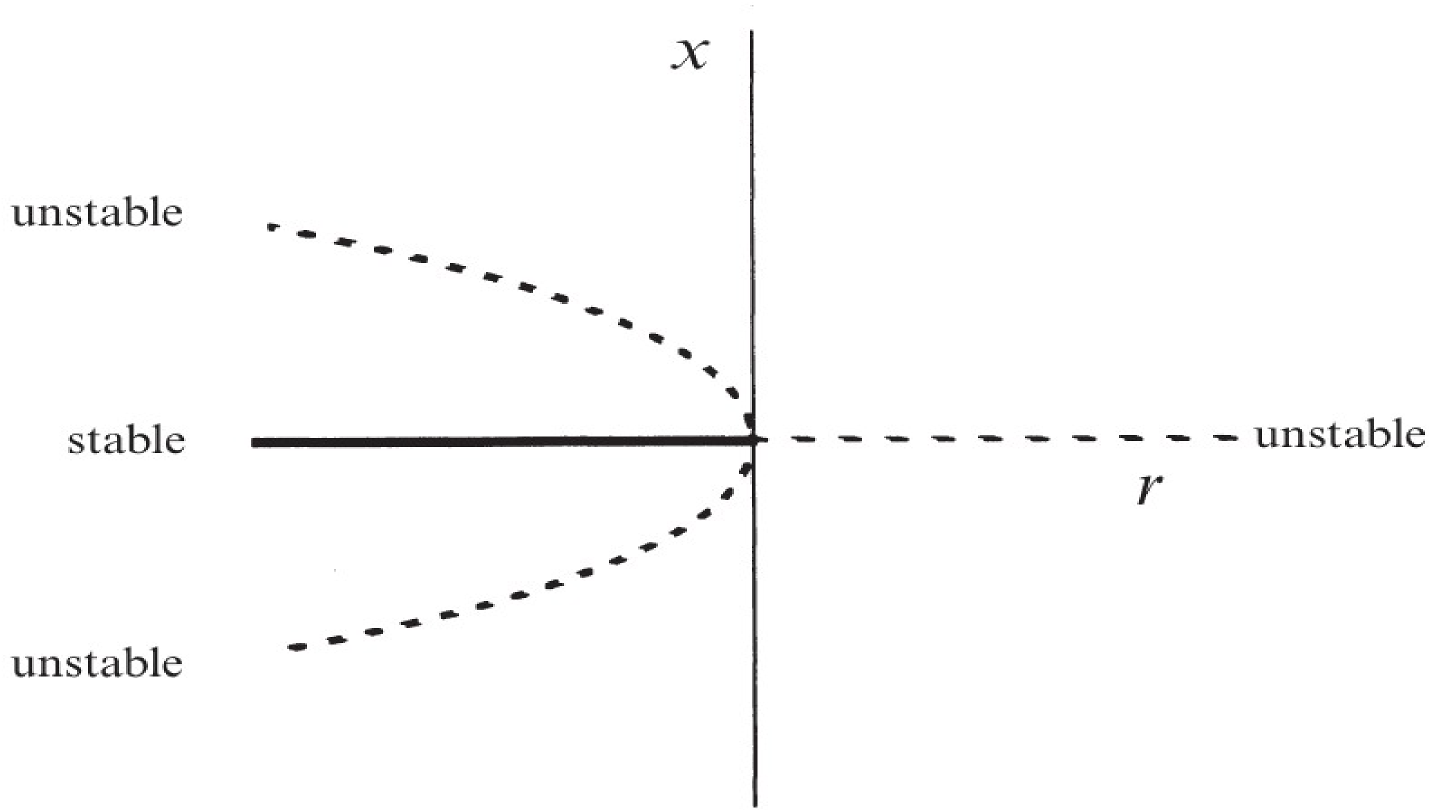
In the supercritical case $\dot{x} = rx - x^3$ discussed above, the cubic term is *stabilizing*: it acts as a restoring force that pulls $x(t)$ back toward $x = 0$. If instead the cubic term were *destabilizing*, as in

$$\dot{x} = rx + x^3,$$

then we'd have a ***subcritical*** pitchfork bifurcation.

Bifurcation in 1D flow

Subcritical Pitchfork Bifurcation



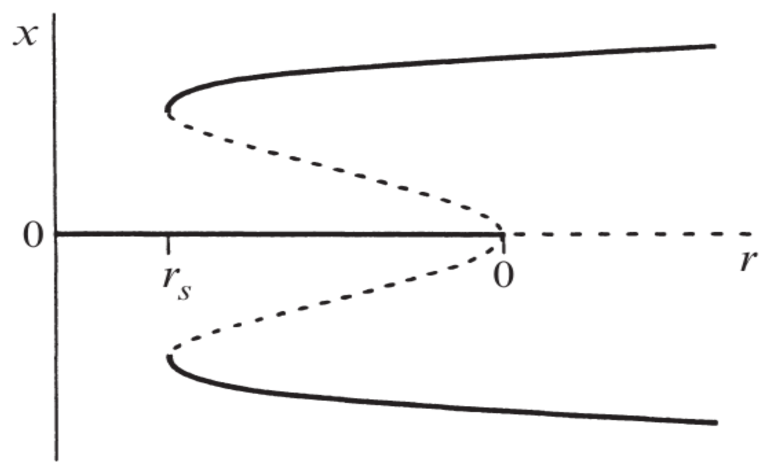
Bifurcation in 1D flow

Subcritical Pitchfork Bifurcation

In real physical systems, such an explosive instability is usually opposed by the stabilizing influence of higher-order terms. Assuming that the system is still symmetric under $x \to -x$, the first stabilizing term must be x^5. Thus the canonical example of a system with a subcritical pitchfork bifurcation is

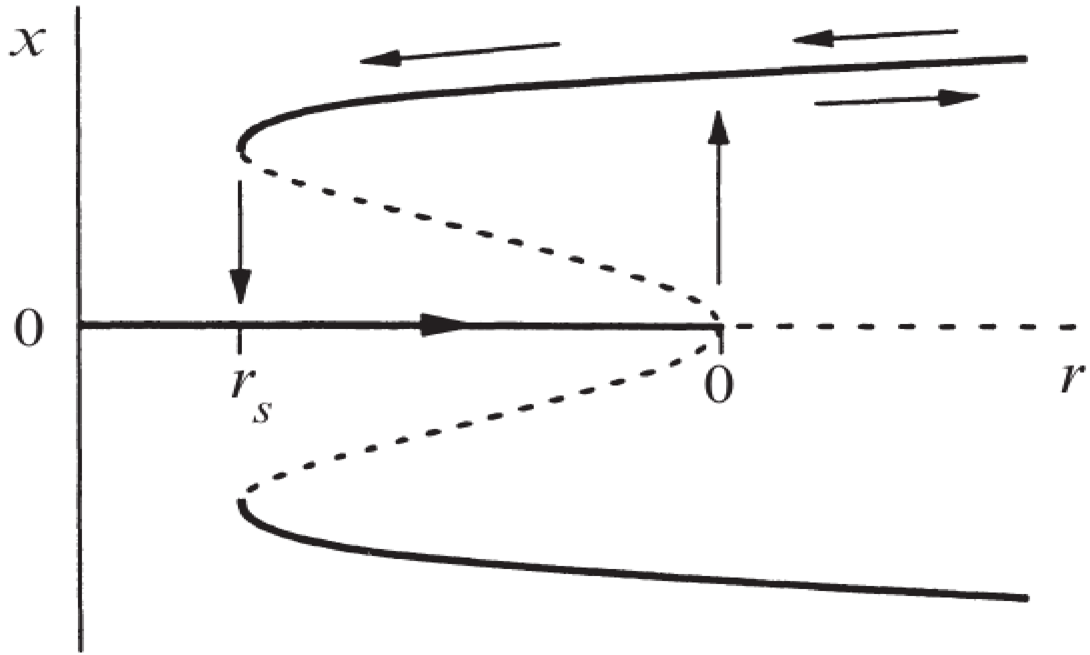
$$\dot{x} = rx + x^3 - x^5.$$

Exercise



Bifurcation in 1D flow

Subcritical Pitchfork Bifurcation



Presence of Hysteresis!!!

Bifurcation in 1D flow

Assignments

For each of the following plot the phase space portrait, find the fixed points and their stability, draw the bifurcation diagram and identify the nature of bifurcations

$$\dot{x} = 1 + rx + x^2$$

$$\dot{x} = r + x - \ln(1 + x)$$

$$\dot{x} = r - \cosh x$$

$$\dot{x} = r + \tfrac{1}{2}x - x/(1 + x)$$

$$\dot{x} = rx + x^2$$

$$\dot{x} = x - rx(1 - x)$$

$$\dot{x} = rx - \ln(1 + x)$$

$$\dot{x} = x(r - e^x)$$

$$\dot{x} = rx + 4x^3$$

$$\dot{x} = rx - 4x^3$$

$$\dot{x} = rx - \sinh x$$

$$\dot{x} = x + \frac{rx}{1 + x^2}$$

Bifurcation in 1D flow

Assignments

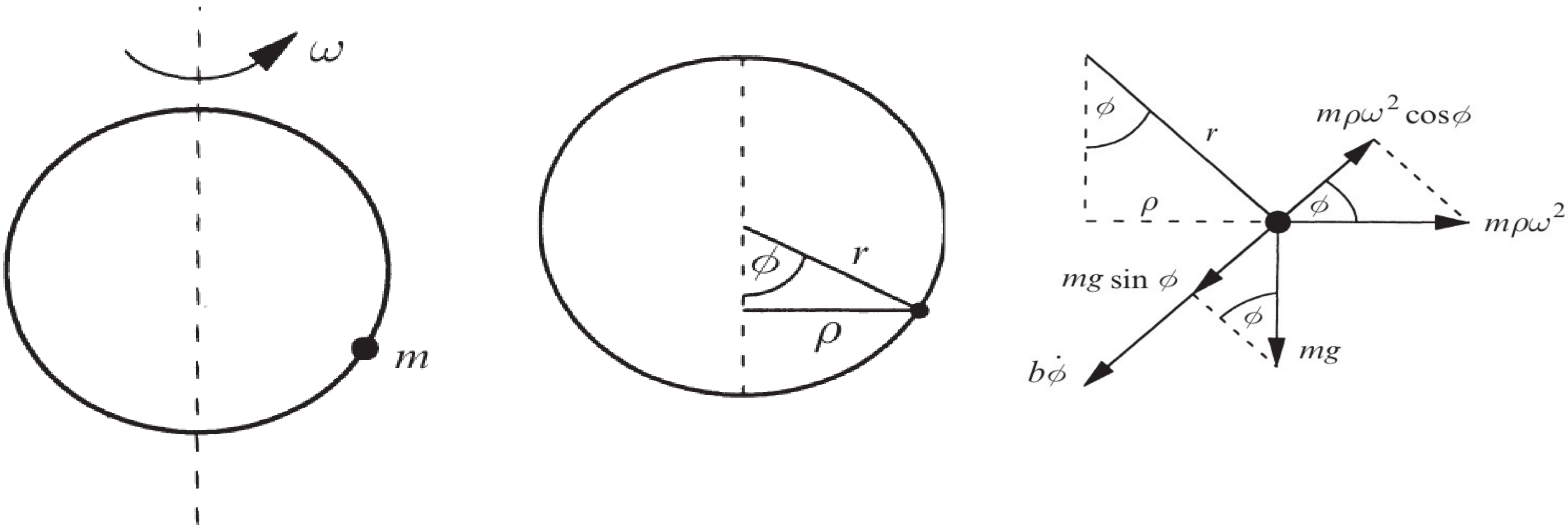
Consider the system $\dot{x} = rx - \sin x$.

a) For the case $r = 0$, find and classify all the fixed points, and sketch the vector field.
b) Show that when $r > 1$, there is only one fixed point. What kind of fixed point is it?
c) As r decreases from ∞ to 0, classify *all* the bifurcations that occur.
d) For $0 < r << 1$, find an approximate formula for values of r at which bifurcations occur.
e) Now classify all the bifurcations that occur as r decreases from 0 to $-\infty$.
f) Plot the bifurcation diagram for $-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

Bifurcation in 1D flow

Example of pitchfork bifurcation:

Overdamped Bead on a Rotating Hoop



Bifurcation in 1D flow

Example of pitchfork bifurcation:

Overdamped Bead on a Rotating Hoop

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi.$$

Under the overdamped situtaion we can ignore the inertia term or the second derivative of the angle or the angular acceleration term!!
Then the equation erduces to

$$\begin{aligned} b\dot{\phi} &= -mg\sin\phi + mr\omega^2 \sin\phi\cos\phi \\ &= mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right). \end{aligned}$$

fixed points where $\sin\,\phi = 0$,

$\phi^* = 0$ (the bottom of the hoop) and $\phi^* = \pi$ (the top).

Bifurcation in 1D flow

Overdamped Bead on a Rotating Hoop

The additional fixed points will be

$$\frac{r\omega^2}{g} > 1,$$

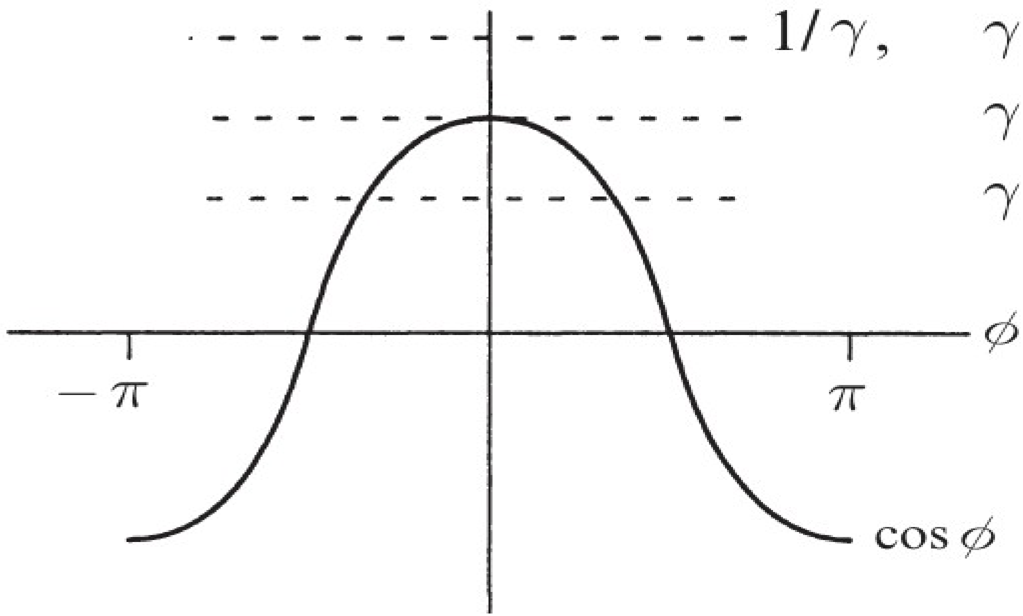
that is, *if the hoop is spinning fast enough.* These fixed points satisfy $\phi^* = \pm\cos^{-1}(g/r\omega^2)$. To visualize them, we introduce a parameter

$$\gamma = \frac{r\omega^2}{g}$$

and solve $\cos\phi^* = 1/\gamma$ graphically.

Bifurcation in 1D flow

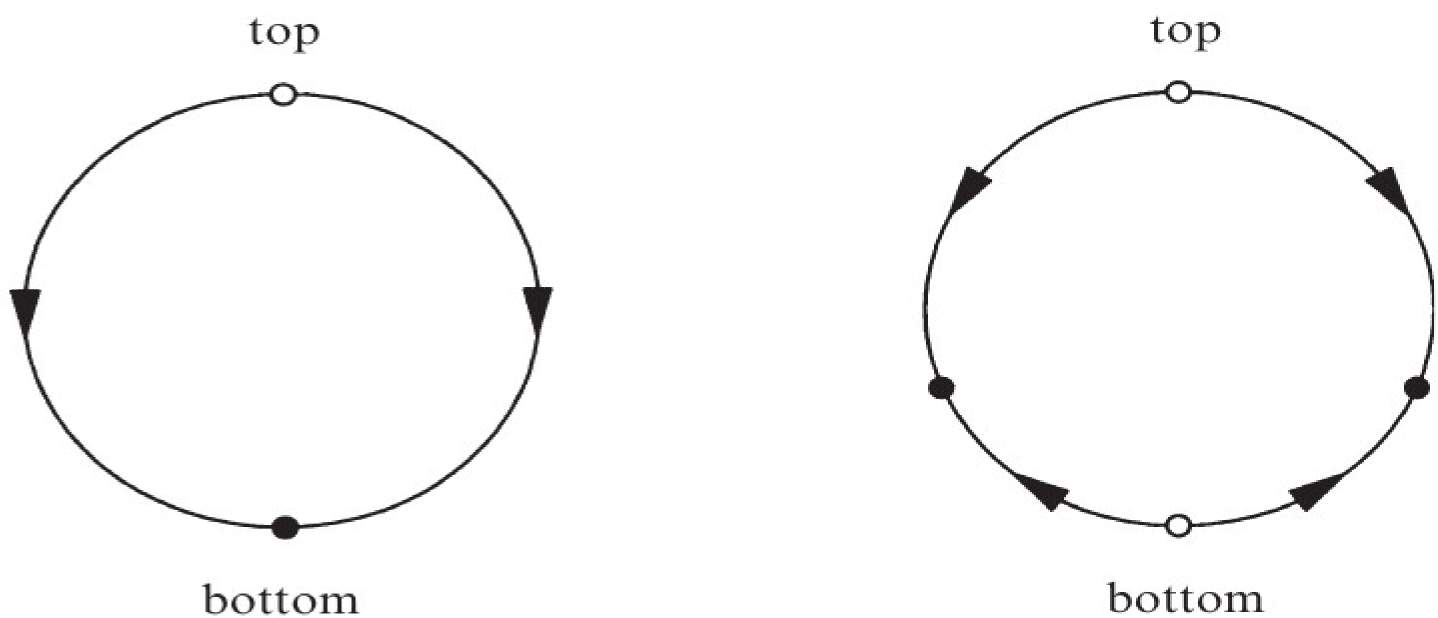
Overdamped Bead on a Rotating Hoop



As $\gamma \to \infty$, these intersections approach $\pm\pi/2$.

Bifurcation in 1D flow

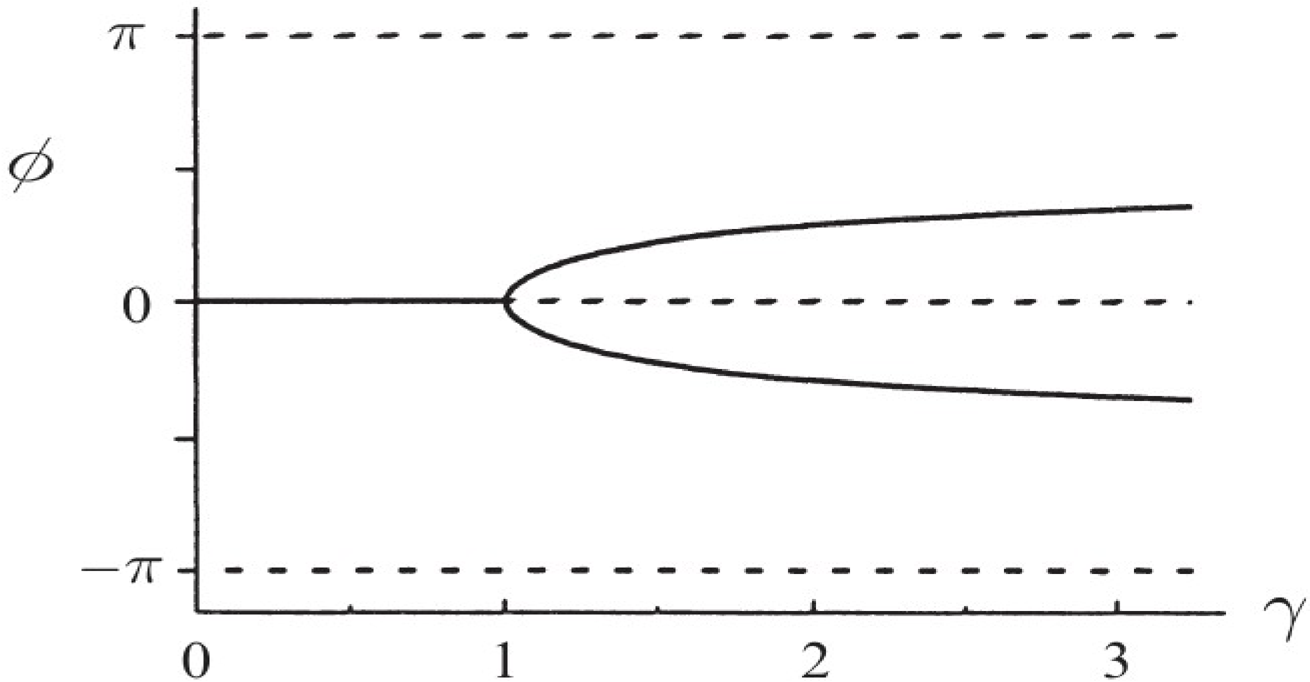
Overdamped Bead on a Rotating Hoop



$\gamma < 1$ $\gamma > 1$

Bifurcation in 1D flow

Overdamped Bead on a Rotating Hoop



Bifurcation in 1D flow

Imperfect Bifurcations and Catastrophes

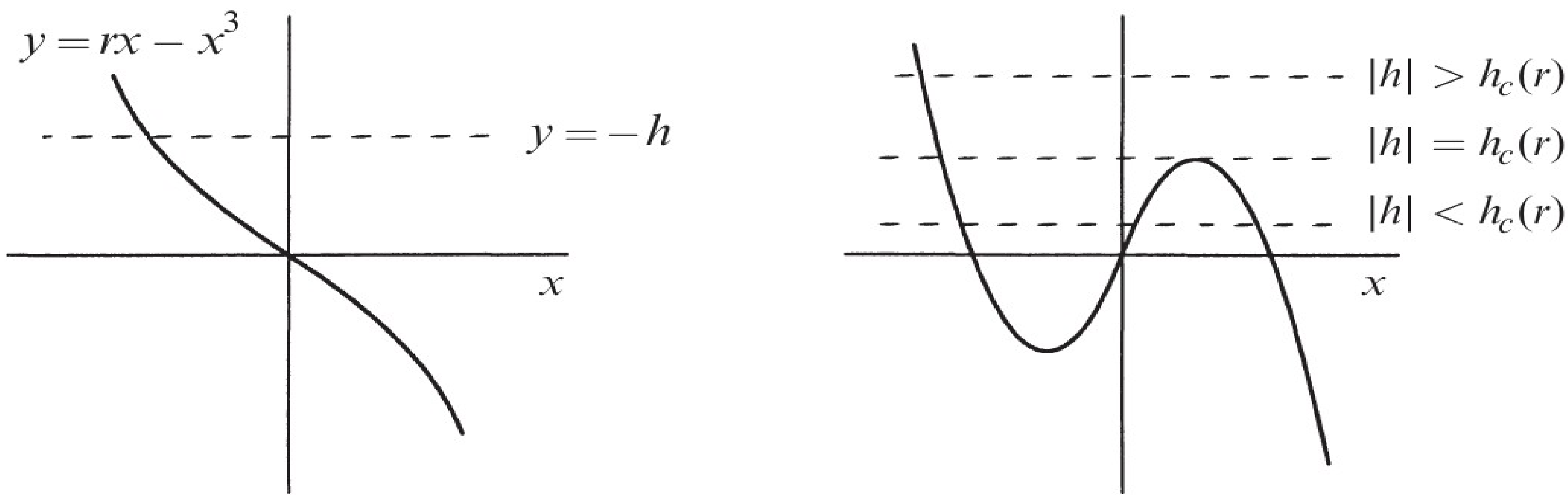
For example, consider the system

$$\dot{x} = h + rx - x^3.$$

If $h = 0$, we have the normal form for a supercritical pitchfork bifurcation, and there's a perfect symmetry between x and $-x$. But this symmetry is broken when $h \neq 0$; for this reason we refer to h as an ***imperfection parameter***.

Bifurcation in 1D flow

Imperfect Bifurcations and Catastrophes



(a) $r \leq 0$ (b) $r > 0$

Bifurcation in 1D flow

Imperfect Bifurcations and Catastrophes

To find the values of h at which this bifurcation occurs, note that the cubic has a local maximum when $\frac{d}{dx}(rx - x^3) = r - 3x^2 = 0$. Hence

and the value of the cubic at the local maximum is

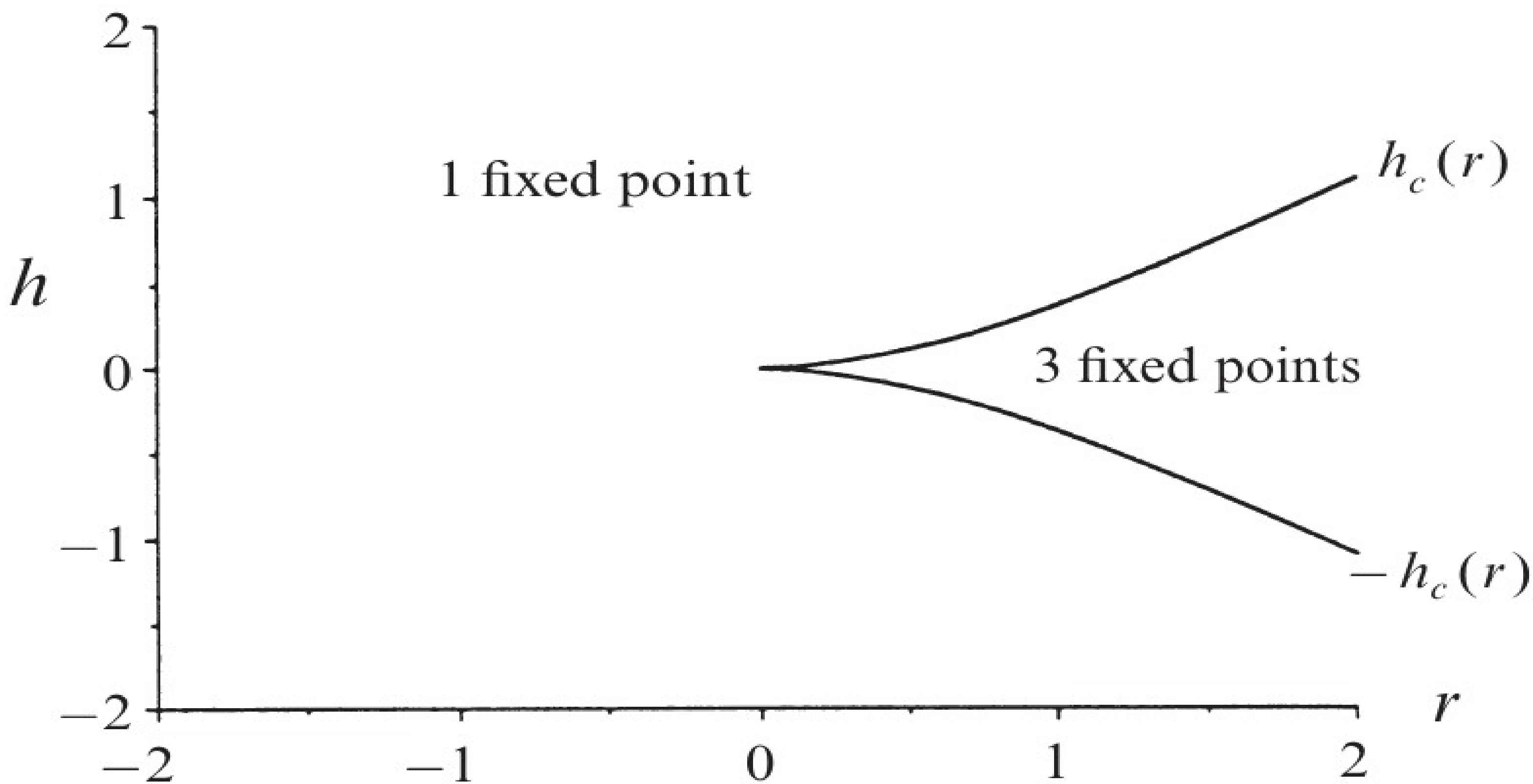
$$rx_{\max} - (x_{\max})^3 = \frac{2r}{3}\sqrt{\frac{r}{3}}.$$

Similarly, the value at the minimum is the negative of this quantity. Hence saddle-node bifurcations occur when $h = \pm h_c(r)$, where

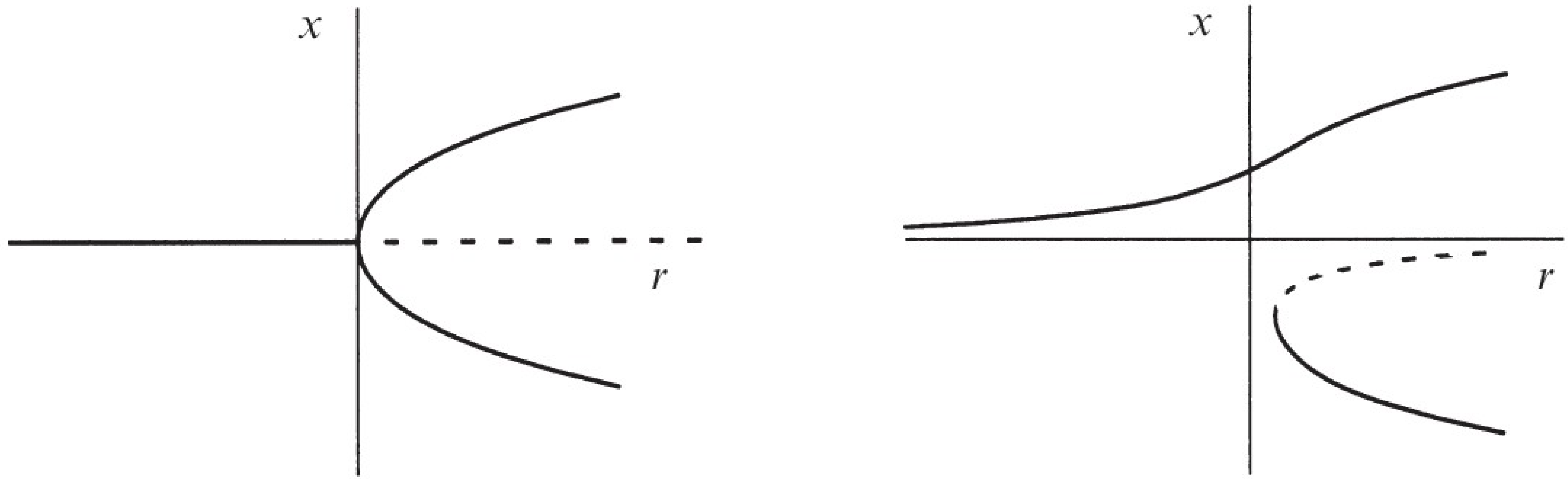
$$h_c(r) = \frac{2r}{2}\sqrt{\frac{r}{3}}.$$

Bifurcation in 1D flow

Imperfect Bifurcations and Catastrophes



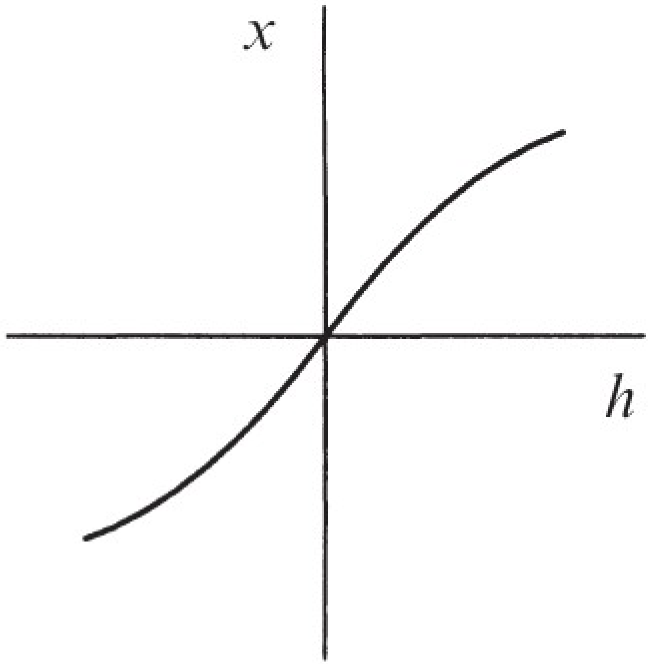
Bifurcation in 1d



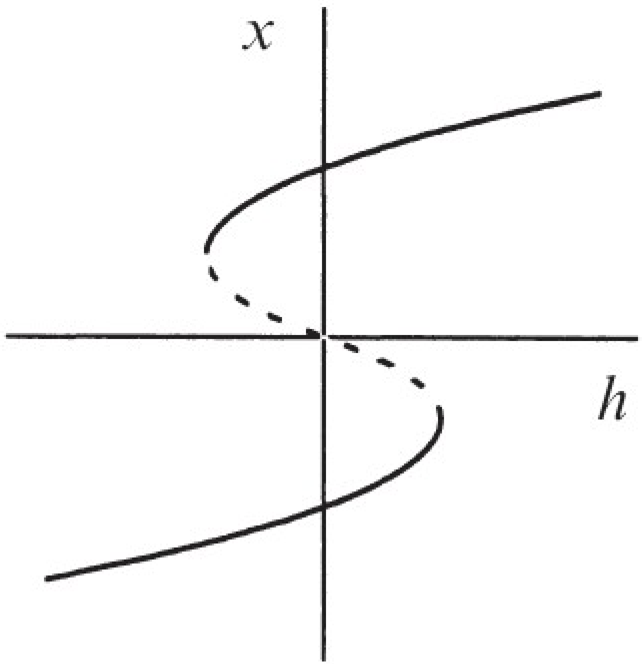
(a) $h = 0$ (b) $h \neq 0$

Bifurcation in 1d

Bifurcation diagram with h in various ‘r’ region



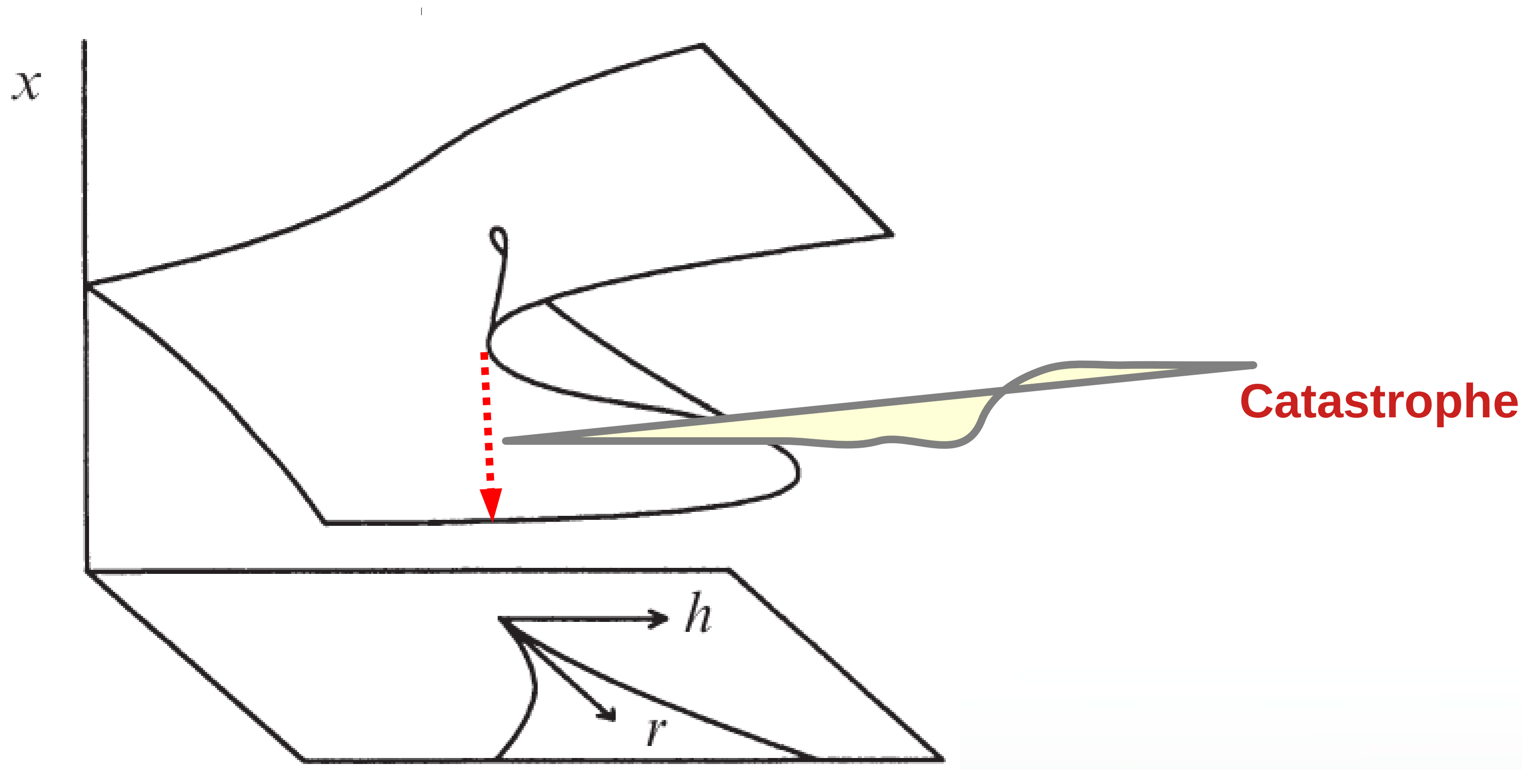
(a) $r \leq 0$



(b) $r > 0$

Bifurcation in 1d

Bifurcation diagram with h-r plane



Imperfect Bifurcations and Catastrophes

Example

Insect Outbreak

For a biological example of bifurcation and catastrophe, we turn now to a model for the sudden outbreak of an insect called the spruce budworm. This insect is a serious pest in eastern Canada, where it attacks the leaves of the balsam fir tree. When an outbreak occurs, the budworms can defoliate and kill most of the fir trees

Ludwig et al. (1978) proposed and analyzed an elegant model of the interaction between budworms and the forest. They simplified the problem by exploiting a separation of time scales: the budworm population evolves on a *fast* time scale (they can increase their density fivefold in a year, so they have a characteristic time scale of months), whereas the trees grow and die on a *slow* time scale (they can completely replace their foliage in about 7–10 years, and their life span in the absence of budworms is 100–150 years.) Thus, as far as the budworm dynamics are concerned, the forest variables may be treated as constants. At the end of the analysis, we will allow the forest variables to drift very slowly—this drift ultimately triggers an outbreak.

Imperfect Bifurcations and Catastrophes

Example

Insect Outbreak

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - p(N).$$

$$p(N) = \frac{BN^2}{A^2 + N^2}$$

p(N) represents the death rate of the insects due to the predators.

Imperfect Bifurcations and Catastrophes

Dimensionless Formulation

$$x = N / A,$$

which yields

$$\frac{A}{B}\frac{dx}{dt} = \frac{R}{B} Ax\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1+x^2}.$$

$$\tau = \frac{Bt}{A}, \qquad r = \frac{RA}{B}, \qquad k = \frac{K}{A}.$$

Imperfect Bifurcations and Catastrophes

Dimensionless Formulation

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2},$$

Analysis of Fixed Points

$$x^* = 0;$$

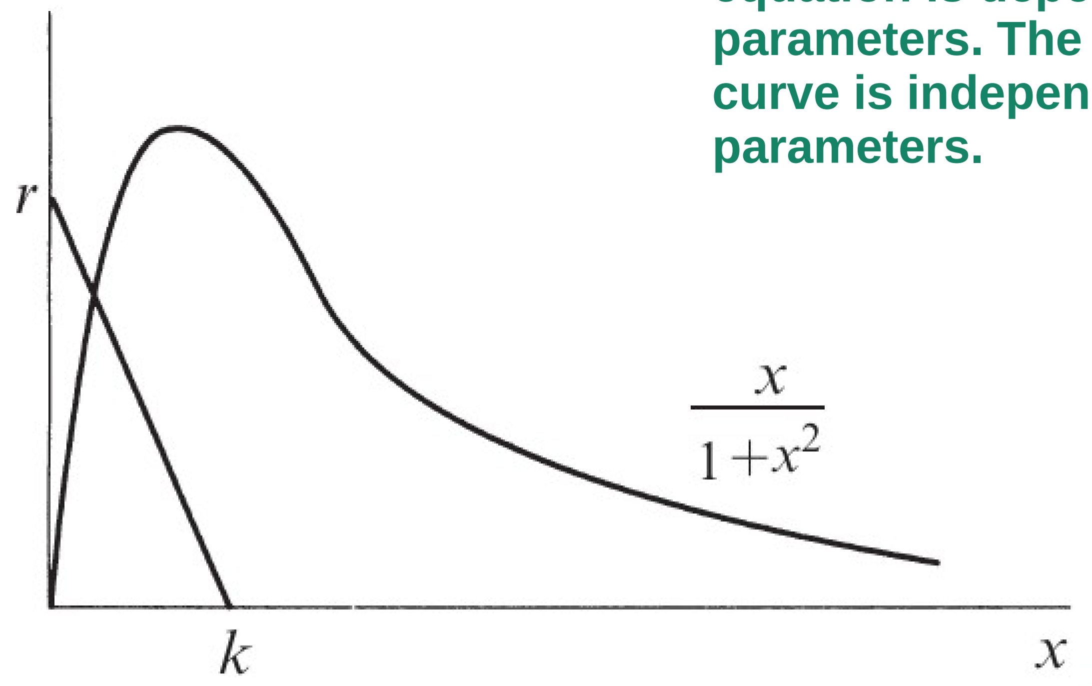
Always a fixed point and it is unstable!!

Other fixed points are given by

$$r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}.$$

We will find the solution graphically.

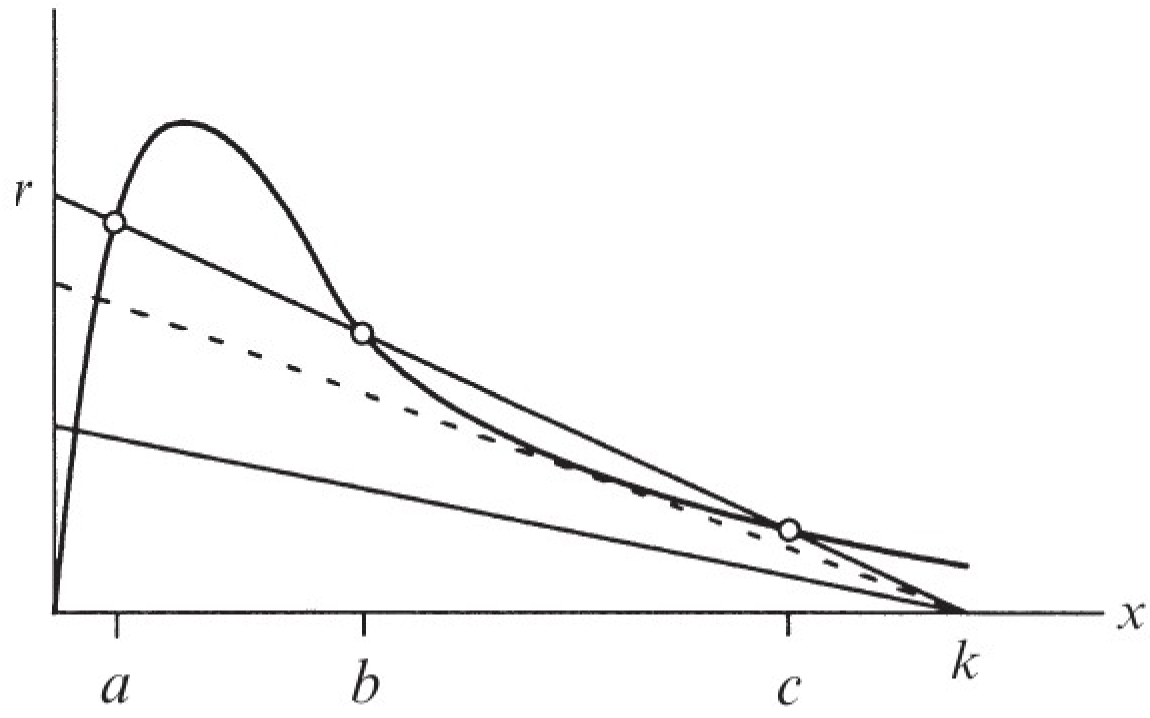
Imperfect Bifurcations and Catastrophes



Only the left hand side of the equation is dependent on the parameters. The right side curve is independent of the parameters.

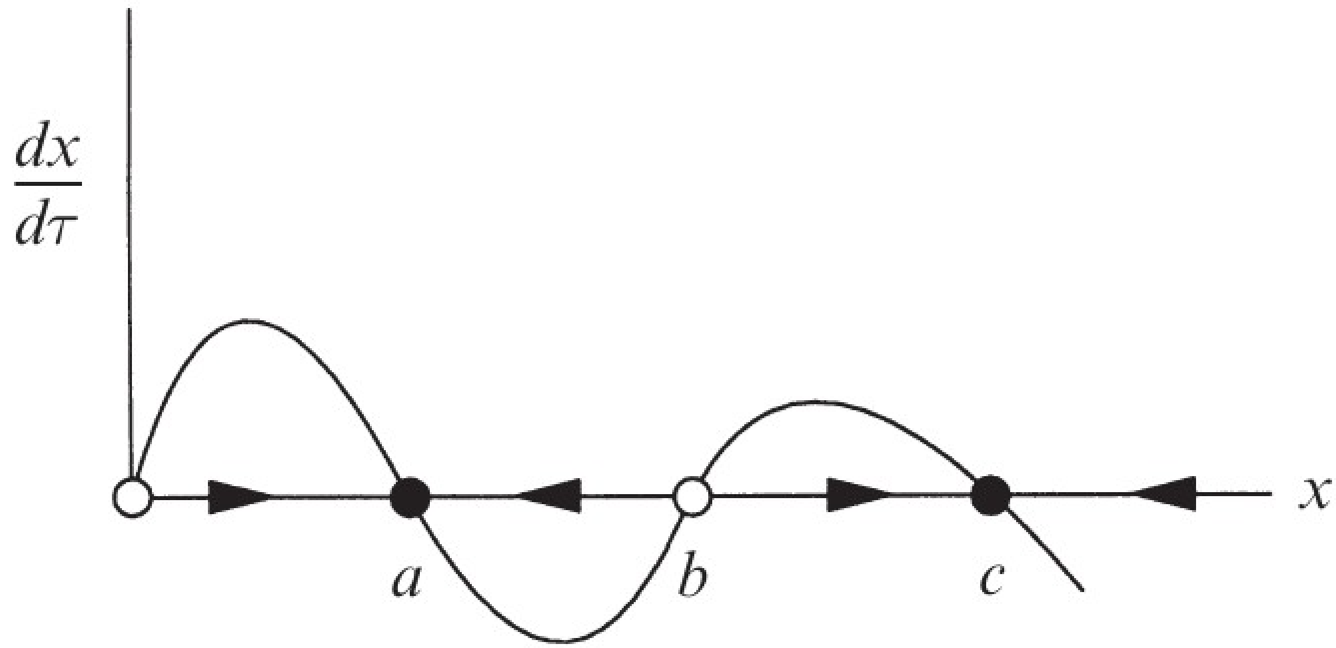
For small k there is only one intersection.

Imperfect Bifurcations and Catastrophes



For large k there may be three fixed point. However, as r is decreased for fixed k there is saddle node bifurcation takes place when line intersects tangentially.

Imperfect Bifurcations and Catastrophes



The smaller **fixed point 'a'** is called refuge level of the budworm population, while the **larger point 'c'** is the **outbreak** level. From the point of view of controling the pest budworm one would like to keep the population at **'a'** away from **'c'.**

an outbreak occurs if and only if $x_0 > b$.

An outbreak can also be triggered by a saddle-node bifurcation.

Imperfect Bifurcations and Catastrophes

Calculating the Bifurcation Curves

Now we need to compute the curves in (k,r) space where the system undergoes the saddle node bifurcation. Unlike the model that we discussed earlier **here we will not be able to get ‘r’ explicitly in the function of ‘k’ .**

The condition for saddle node bifuctaion would be

$$r\left(1-\frac{x}{k}\right)=\frac{x}{1+x^2}$$

and

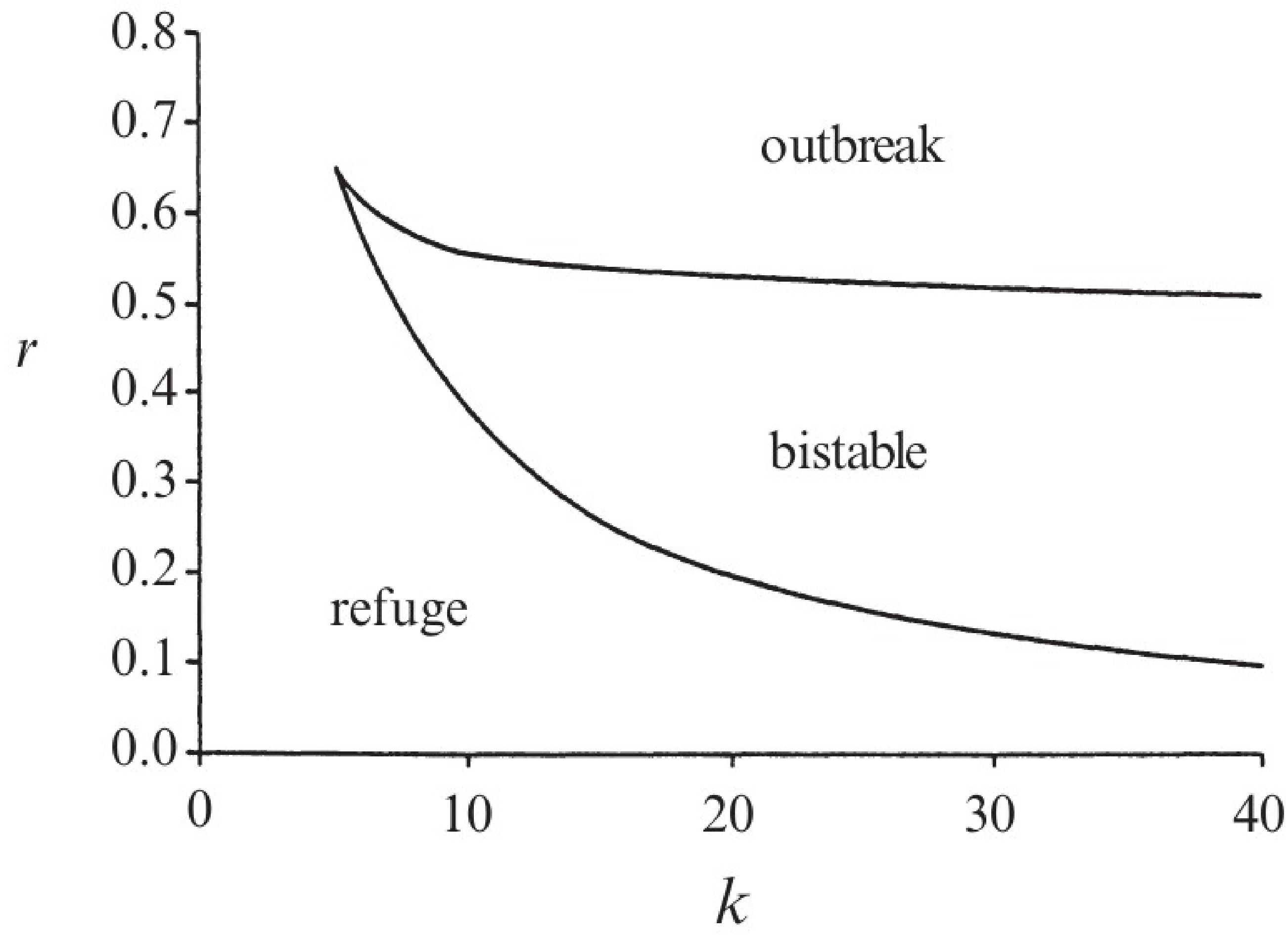
$$\frac{d}{dx}\left[r\left(1-\frac{x}{k}\right)\right]=\frac{d}{dx}\left[\frac{x}{1+x^2}\right]. \longrightarrow -\frac{r}{k}=\frac{1-x^2}{(1+x^2)^2}.$$

$$r=\frac{2x^3}{(1+x^2)^2}.$$

$$k=\frac{2x^3}{x^2-1}.$$

Imperfect Bifurcations and Catastrophes

Calculating the Bifurcation Curves



Imperfect Bifurcations and Catastrophes

Part of second assignment

(Imperfect transcritical bifurcation) Consider the system $\dot{x} = h + rx - x^2$.

When $h = 0$, this system undergoes a transcritical bifurcation at $r = 0$. Our goal is to see how the bifurcation diagram of x^* vs. r is affected by the imperfection parameter h.

a) Plot the bifurcation diagram for $\dot{x} = h + rx - x^2$, for $h < 0$, $h = 0$, and $h > 0$.

b) Sketch the regions in the (r, h) plane that correspond to qualitatively different vector fields, and identify the bifurcations that occur on the boundaries of those regions.

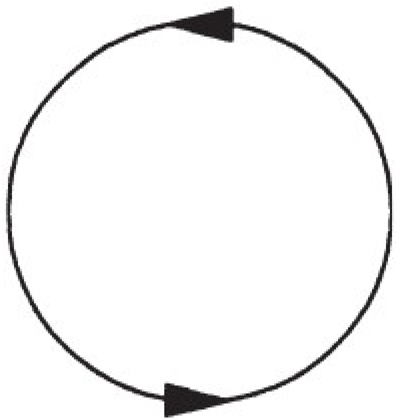
c) Plot the potential $V(x)$ corresponding to all the different regions in the (r, h) plane.

One dimensional flow

So far we've concentrated on the equation $\dot{x} = f(x)$, which we visualized as a vector field on the line. Now it's time to consider a new kind of differential equation and its corresponding phase space. This equation,

$$\dot{\theta} = f(\theta),$$

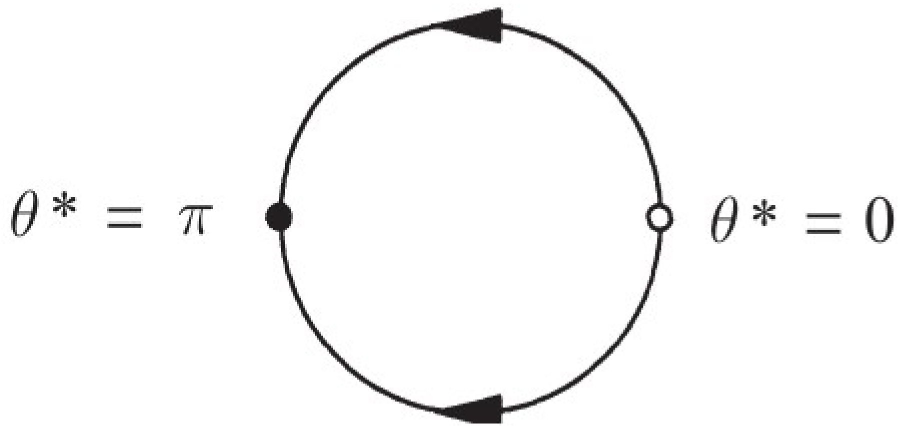
corresponds to a ***vector field on the circle.***



Mainly applicable for the periodic flow!

One dimensional flow

Sketch the vector field on the circle corresponding to $\dot{\theta} = \sin\theta$.



Convention if flow is positive the arrow is shown in the counterclockwise otherwise in clockwise direction.

One dimensional flow

Uniform Oscillator

A point on a circle is often called an *angle* or a ***phase***. Then the simplest oscillator of all is one in which the phase θ changes uniformly:

$$\dot{\theta} = \omega$$

where ω is a constant. The solution is

$$\theta(t) = \omega t + \theta_0,$$

which corresponds to uniform motion around the circle at an angular frequency ω. This solution *is* ***periodic***, in the sense that $\theta(t)$ changes by 2π, and therefore returns to the same point on the circle, after a time $T = 2\pi/\omega$. We call T the ***period*** of the oscillation.

One dimensional flow

Nonuniform Oscillator

$$\dot{\theta} = \omega - a \sin \theta$$

arises in many different branches of science and engineering. Here is a partial list:

Electronics (phase-locked loops)

Biology (oscillating neurons, firefly flashing rhythm, human sleep-wake cycle)

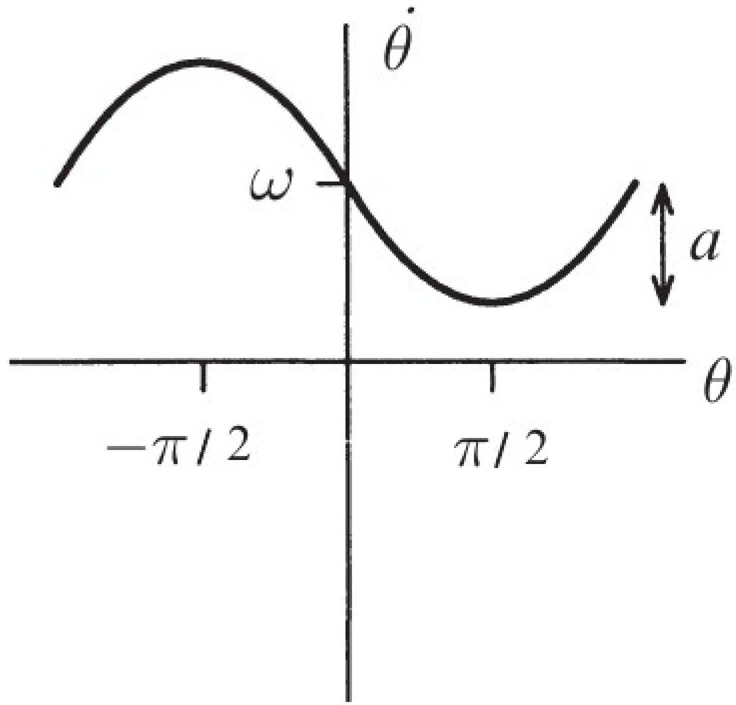
Condensed-matter physics (Josephson junction, charge-density waves)

Mechanics (Overdamped pendulum driven by a constant torque)

we assume that $\omega > 0$ and $a \geq 0$

One dimensional flow

Nonuniform Oscillator

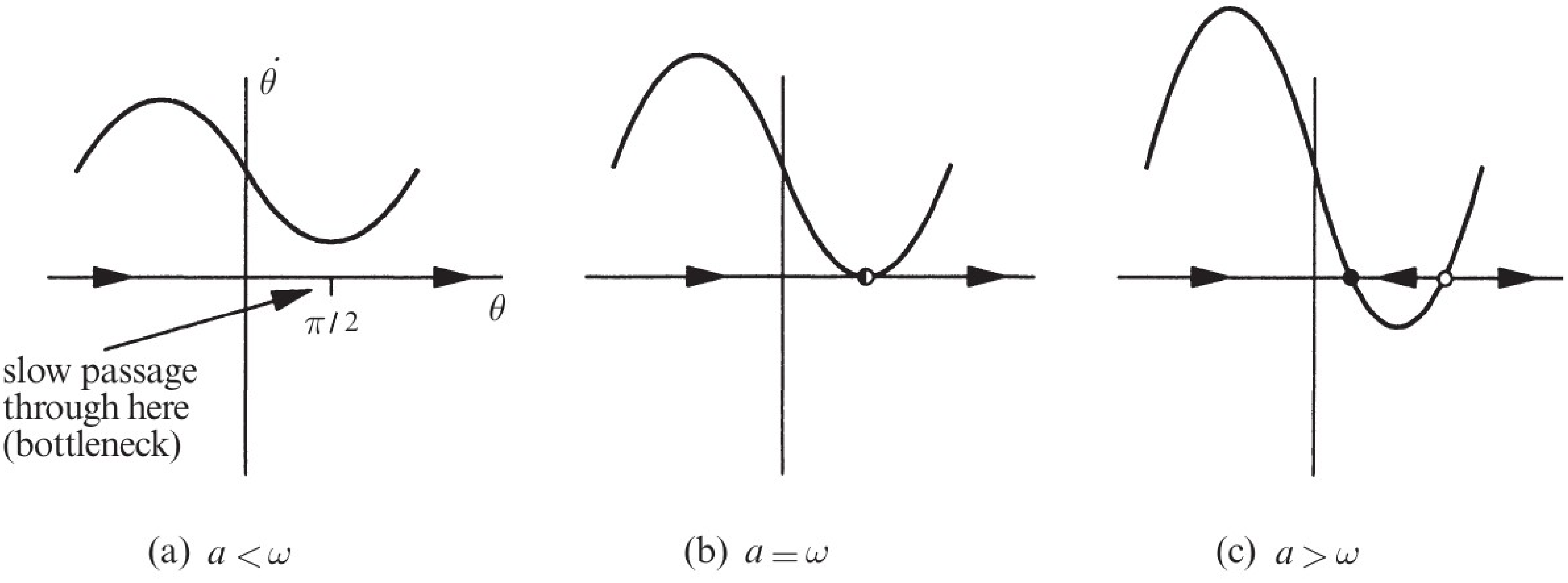


If a=0 it reduces to the uniform oscillator.

the flow is fastest at $\theta = -\pi/2$ and slowest at $\theta = \pi/2$

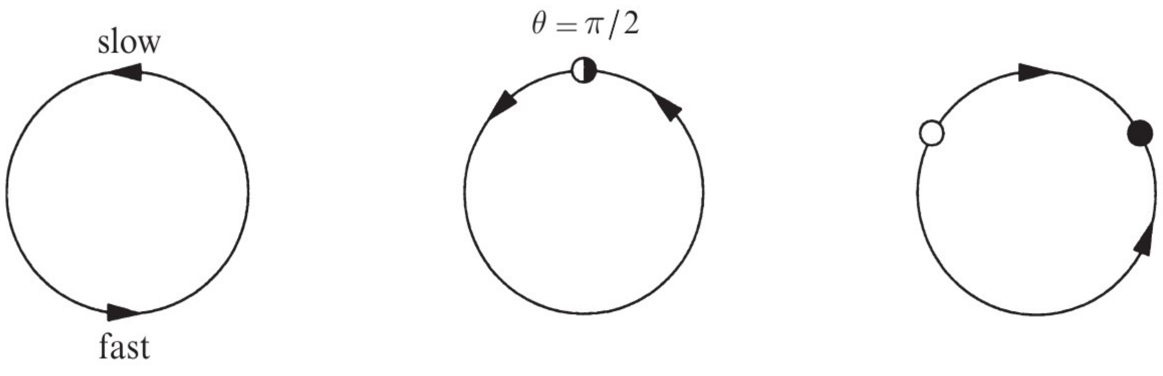
One dimensional flow

Nonuniform Oscillator



One dimensional flow

Nonuniform Oscillator



(a) $a < \omega$ (b) $a = \omega$ (c) $a > \omega$

$$\sin\theta^* = \omega/a, \qquad \cos\theta^* = \pm\sqrt{1-(\omega/a)^2}.$$

Their linear stability is determined by

$$f'(\theta^*) = -a\cos\theta^* = \mp a\sqrt{1-(\omega/a)^2}.$$

Thus the fixed point with $\cos\theta^* > 0$ is the stable one, since $f'(\theta^*) < 0$.

One dimensional flow

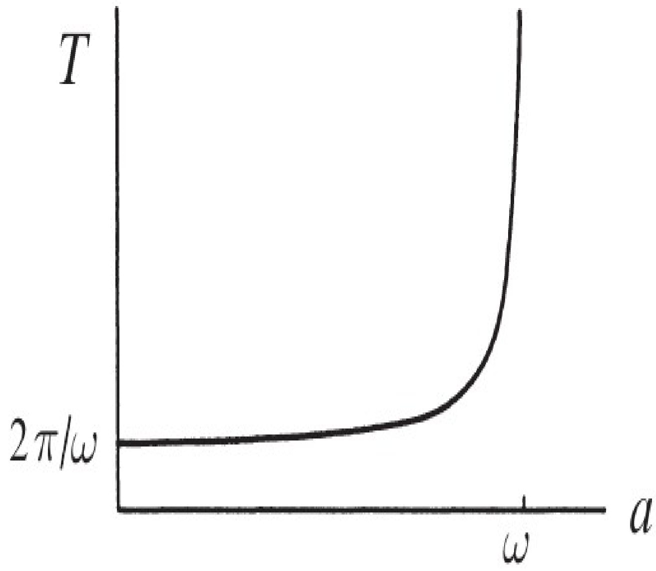
Nonuniform Oscillator

For $a < \omega$:

$$T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta$$

$$= \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta}$$

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}} \, .$$



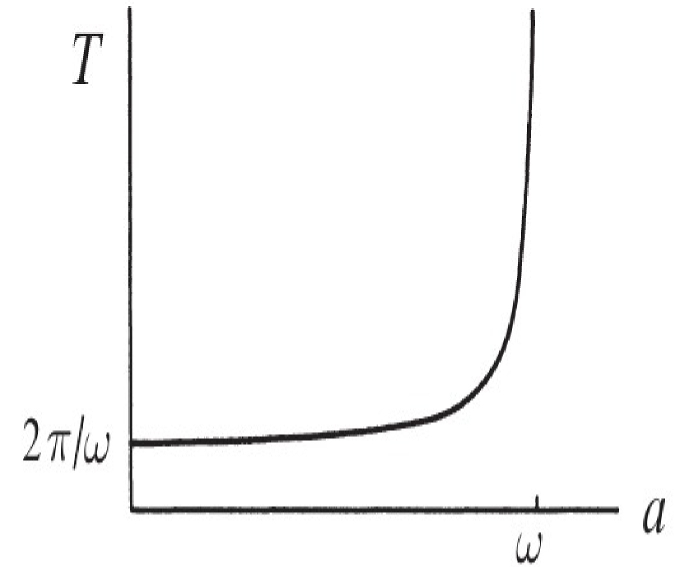
Nonuniform Oscillator

We can estimate the order of the divergence by noting that

$$\sqrt{\omega^2 - a^2} = \sqrt{\omega + a}\sqrt{\omega - a}$$
$$\approx \sqrt{2\omega}\sqrt{\omega - a}$$

as $a \to \omega^-$. Hence

$$T \approx \left(\frac{\pi\sqrt{2}}{\sqrt{\omega}}\right)\frac{1}{\sqrt{\omega - a}},$$



which shows that T blows up like $(a_c - a)^{-1/2}$, where $a_c = \omega$.

Two dimensional flow

In one dimensional flow so far we have seen the flow which approches towards a point or drifts away from the point. Due to this the corresponding trajectory either remains constant or changes monotonically in the confined space. In two dimension we will witness more varities of the flow.

A ***two-dimensional linear system*** is a system of the form

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Two dimensional flow

Such a system is ***linear*** in the sense that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then so is any linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Notice that $\dot{\mathbf{x}} = \mathbf{0}$ when $\mathbf{x} = \mathbf{0}$, so $\mathbf{x}^* = \mathbf{0}$ is always a fixed point for any choice of A.

The solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ can be visualized as trajectories moving on the (x, y) plane, in this context called the ***phase plane***. Our first example presents the phase plane analysis of a familiar system.

Example:

$$m\ddot{x} + kx = 0$$

$$\dot{x} = v$$

$$\dot{v} = -\frac{k}{m}x.$$

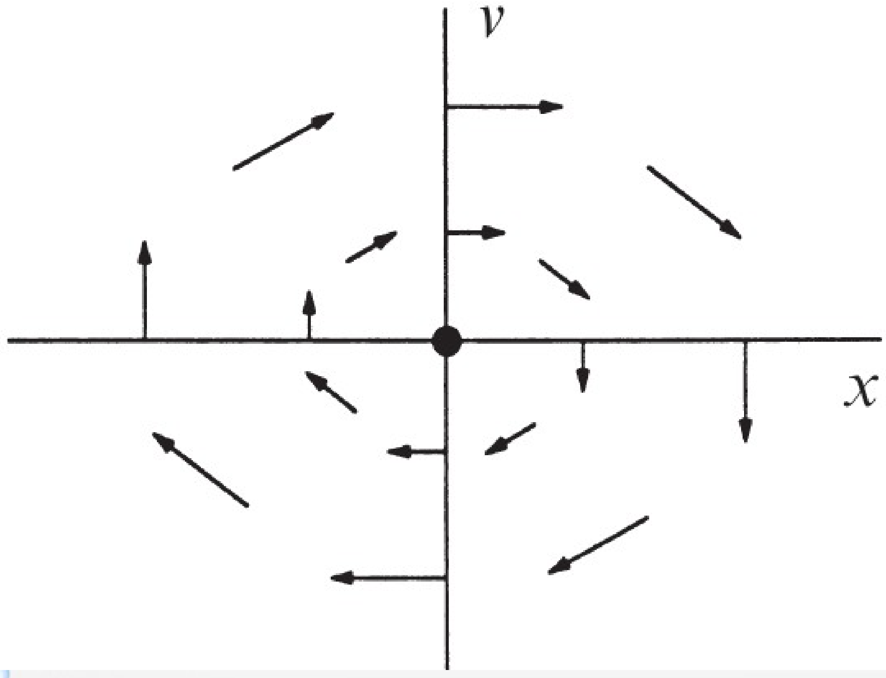
Two dimensional flow

$$\dot{x} = v$$

$$\omega^2 = k/m.$$

$$\dot{v} = -\omega^2 x.$$

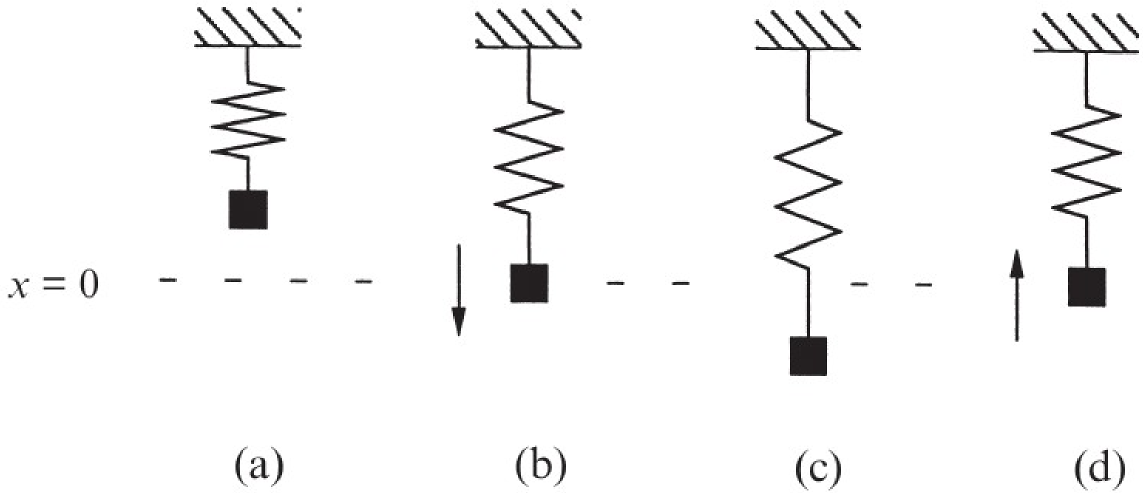
$(\dot{x}, \dot{v}) = (0,0)$ when $(x, v) = (0,0)$; hence the origin is a ***fixed point.***

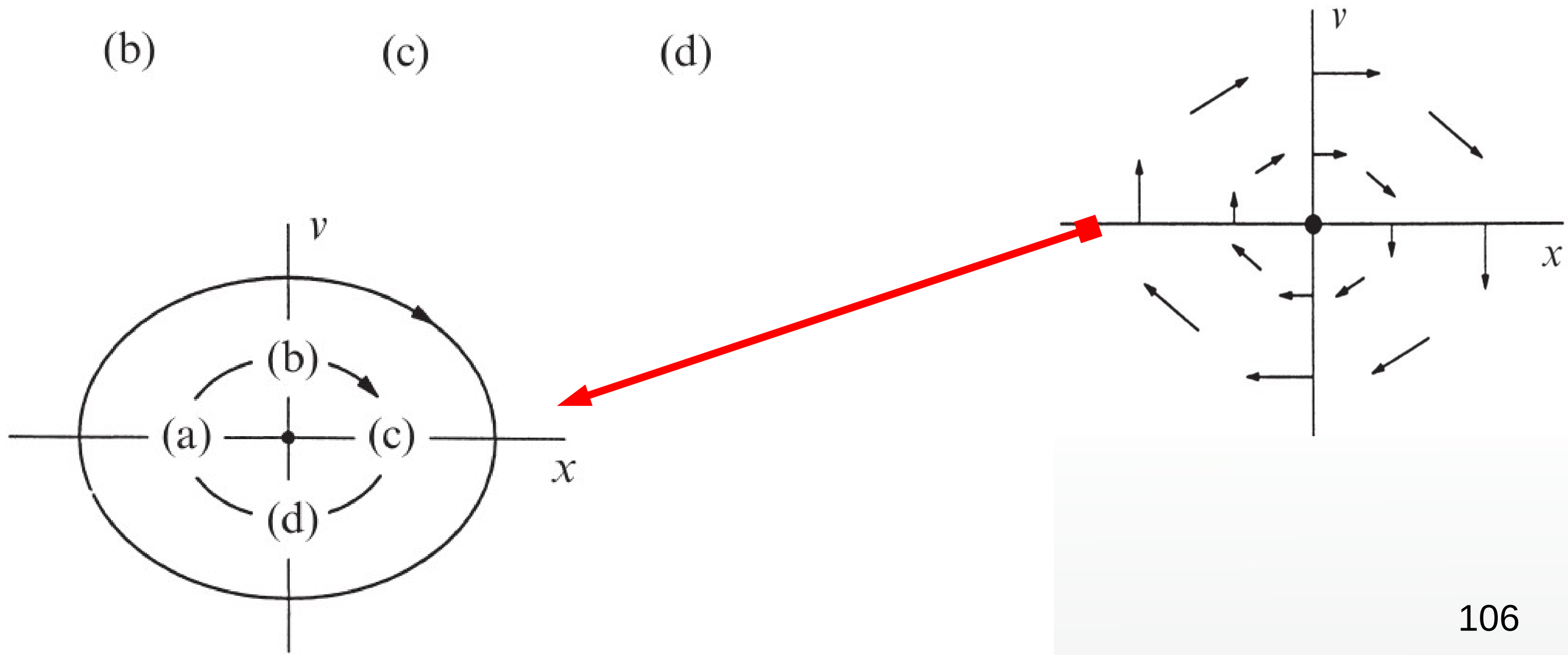


Then $v = 0$ and so $(\dot{x}, \dot{v}) = (0, -\omega^2 x)$.

Vector field will be vertically downward for positive x and vertically upward for negative x.

Two dimensional flow





Two dimensional flow

Example:

$$\dot{\mathbf{x}} = A\mathbf{x}, \text{ where } A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}.$$

a varies from $-\infty$ to $+\infty$.

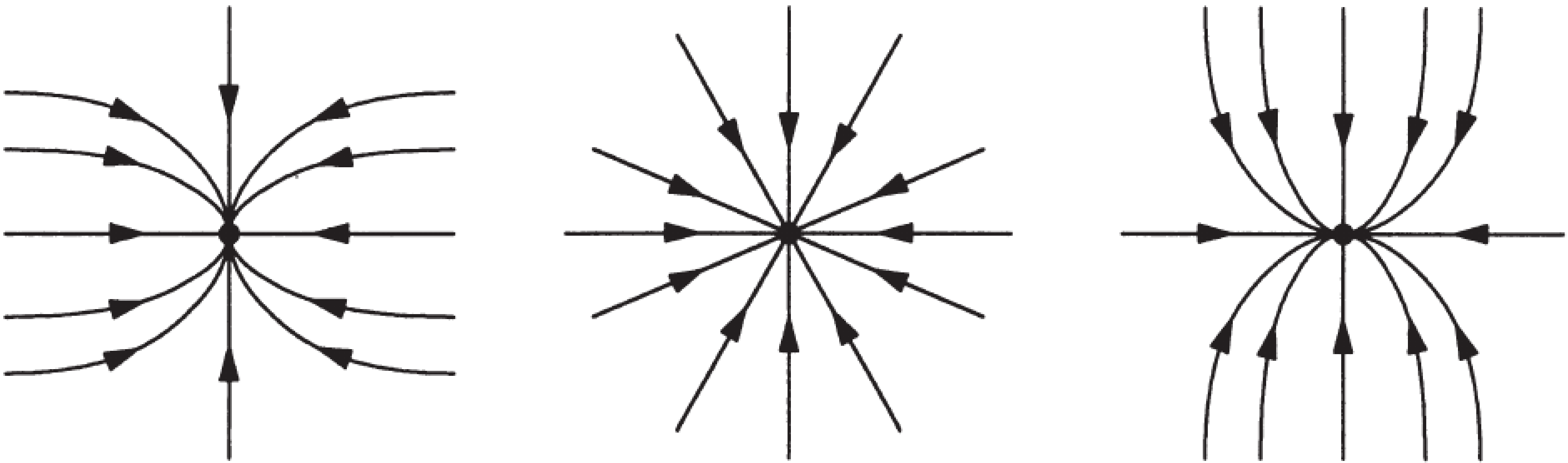
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\dot{x} = ax$$
$$\dot{y} = -y$$

System is uncoupled!

$$x(t) = x_0 e^{at}$$
$$y(t) = y_0 e^{-t}.$$

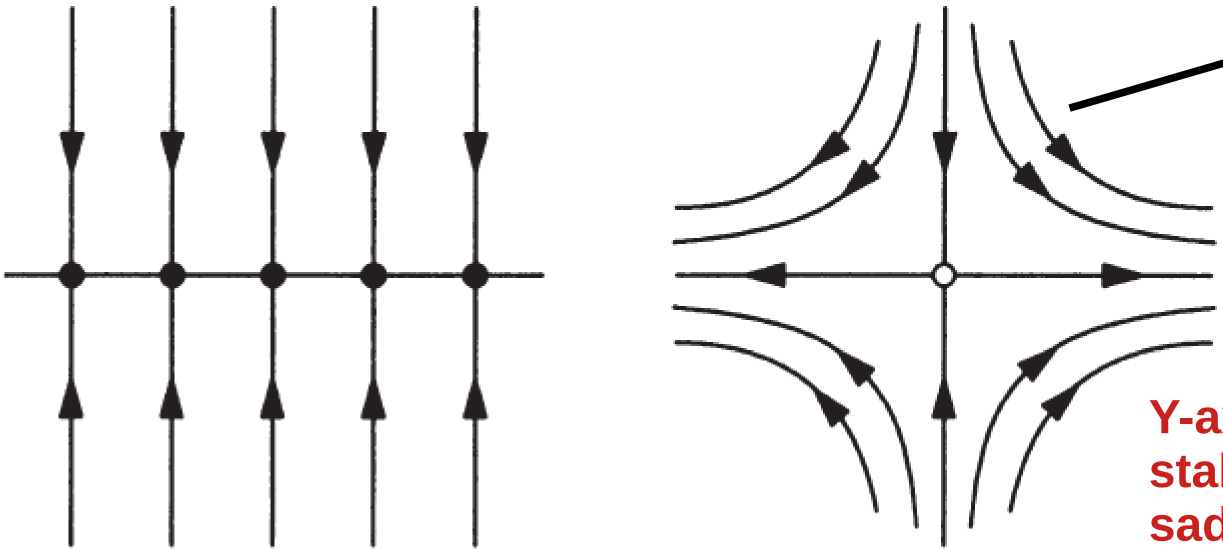
Two dimensional flow



(a) $a < -1$ (b) $a = -1$ (c) $=1 < a < 0$

$\mathbf{x}^* = \mathbf{0}$ is called a ***stable node.***

Two dimensional flow



(d) $a=0$ (e) $a>0$

Saddle node point at x=0.

Y-axis is called the stable manifold of the saddle point and x-axis is called the unstable manifold.

when $a=0$

$x(t) \equiv x_0$ and so there's an entire ***line of fixed points*** along the x-axis.

Two dimensional flow

So far we have noticed that the x and y axes play a crucial geometrical role. They determined the direction of the trajectories as $t \to \pm\infty$

For the general case, we would like to find the analog of these straight-line trajectories. That is, we seek trajectories of the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v},$$

where $\mathbf{v} \neq \mathbf{0}$ is some fixed vector to be determined, and λ is a growth rate, also to be determined. If such solutions exist, they correspond to exponential motion along the line spanned by the vector $\mathbf{v}$.

Two dimensional flow

To find the conditions on $\mathbf{v}$ and λ, we substitute $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ into $\dot{\mathbf{x}} = A\mathbf{x}$, and obtain $\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}$. Canceling the nonzero scalar factor $e^{\lambda t}$ yields

$$A\mathbf{v} = \lambda\mathbf{v},$$

which says that the desired straight line solutions exist if $\mathbf{v}$ is an ***eigenvector*** of A with corresponding ***eigenvalue*** λ.

Let's consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

Two dimensional flow

the characteristic equation becomes

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0.$$

Expanding the determinant yields

$$\lambda^2 - \tau\lambda + \Delta = 0$$

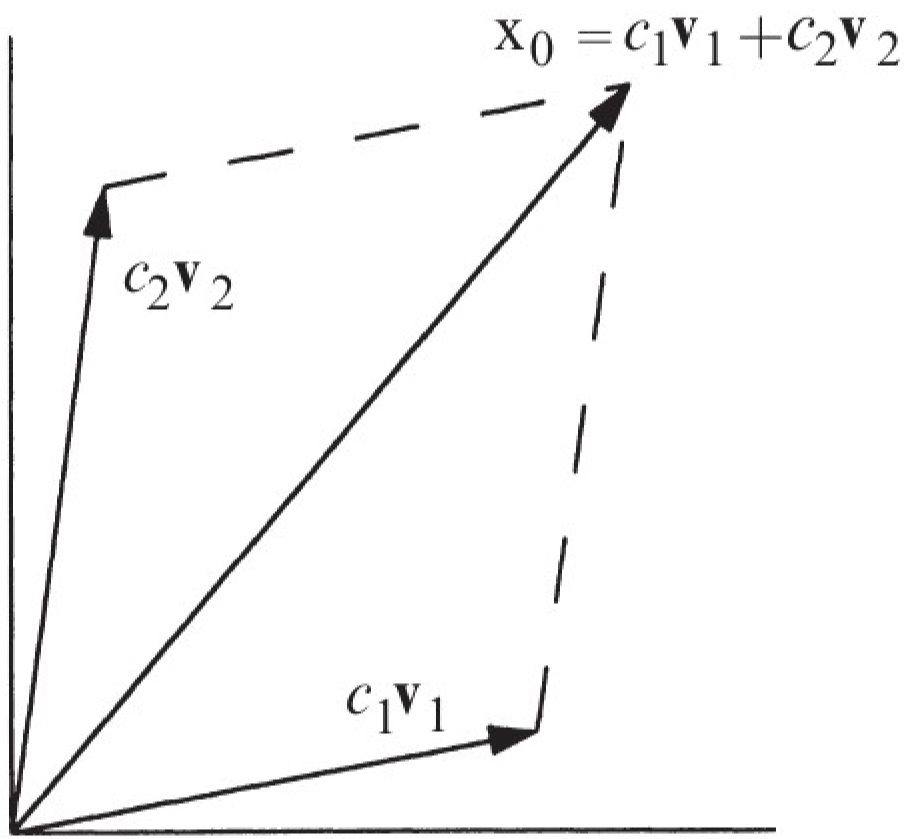
where

$$\tau = \text{trace}(A) = a + d,$$
$$\Delta = \det(A) = ad - bc.$$

Then

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \quad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

Two dimensional flow



The solution will have the form

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Two dimensional flow

Example:

Solve the initial value problem $\dot{x} = x + y, \quad \dot{y} = 4x - 2y,$ subject to the initial condition $(x_0, y_0) = (2, -3)$.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

First we find the eigenvalues of the matrix A. The matrix has $\tau = -1$ and $\Delta = -6$, so the characteristic equation is $\lambda^2 + \lambda - 6 = 0$. Hence

$$\lambda_1 = 2, \ \lambda_2 = -3.$$

Two dimensional flow

Eigenvectors would be

Next we find the eigenvectors. Given an eigenvalue λ, the corresponding eigenvector $\mathbf{v} = (v_1, v_2)$ satisfies

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

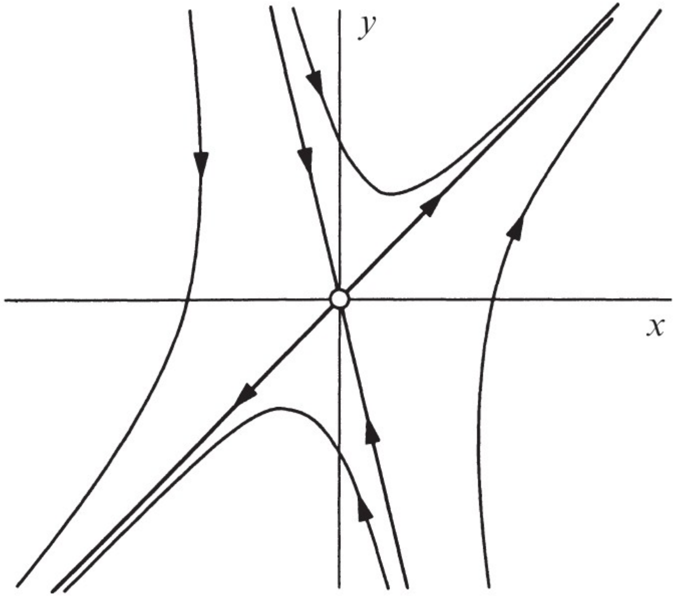
$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

Two dimensional flow

Finally using the initial condition $(x_0, y_0) = (2, -3)$ we have

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

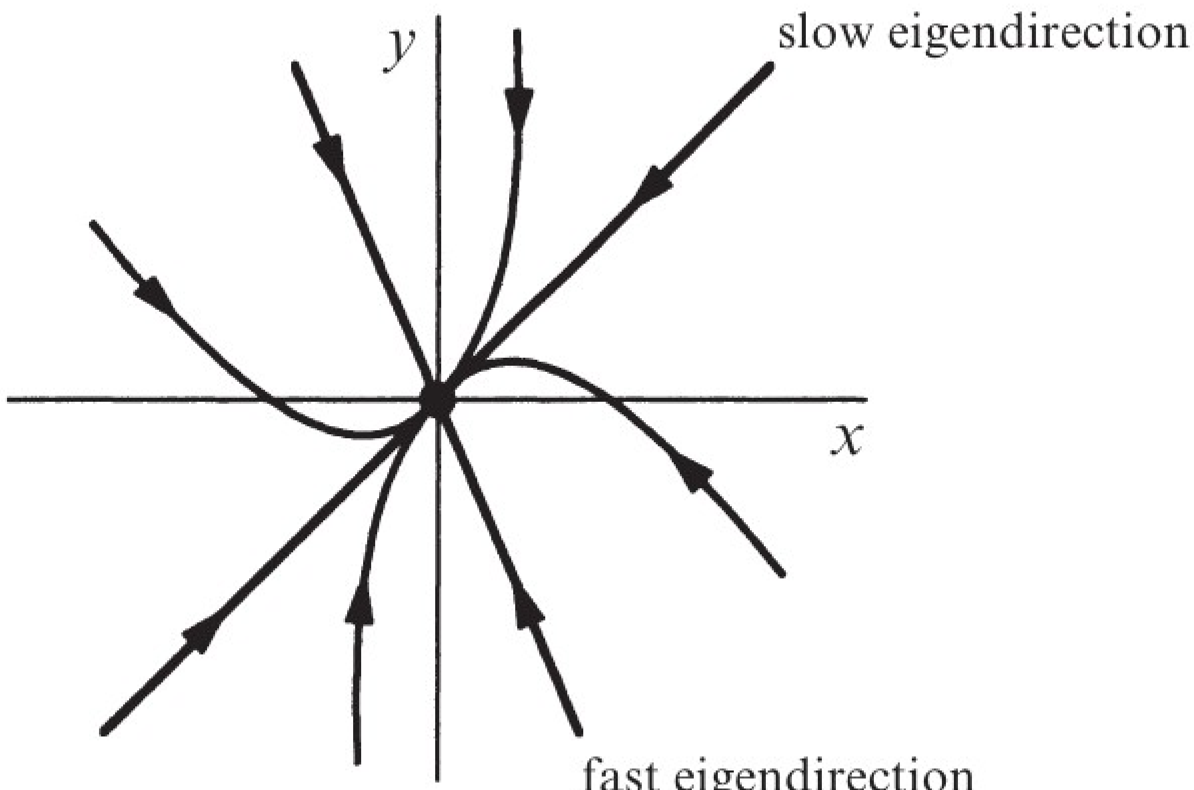
After getting an idea about the solution along the eigen direction now we will draw a full picture of the phase space.



Two dimensional flow

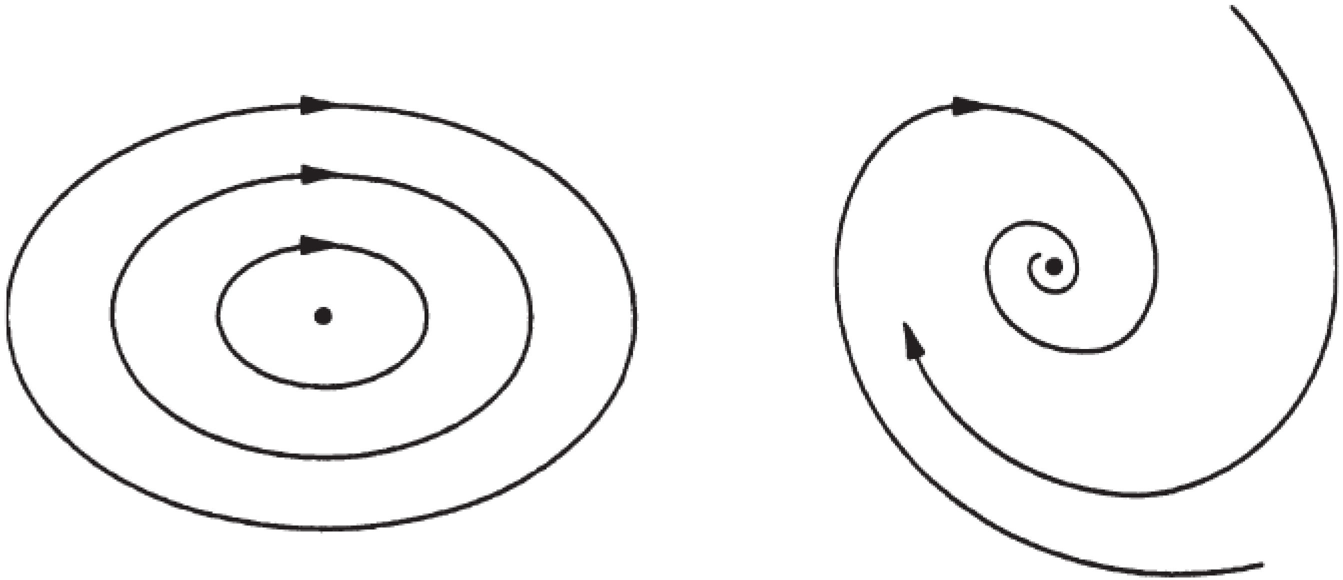
Example:

Sketch a typical phase portrait for the case $\lambda_2 < \lambda_1 < 0$.



Two dimensional flow

For complex eigen values the phase space diagram would be either circular or spiral.



(a) center (b) spiral