1. The density matrix of a system is given as: $\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

2

(A) The density matrix represents a pure state.

(B) The density matrix represents a mixed state

(C) The probability that a measurement of the system will find it in state $|0\rangle$ is 1/5

(D) The expectation value of σ_x in this state is 1/5.

1.

$$\rho = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

✓ $\mathrm{Tr}(\rho) = 1$

$$\rho^2 = \frac{1}{25}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$= \frac{1}{25}\begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

✓ $= \rho$

$\Rightarrow$ ρ represents a pure state!

- The probability that a msmt of the system will find it in the state $|0\rangle$ is

$$\rho_{11} = \frac{1}{5} \quad //$$

- $\langle \sigma_x \rangle = \mathrm{Tr}(\rho\, \sigma_x)$

$$= \mathrm{Tr}\left[\frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \mathrm{Tr}\left[\frac{1}{5}\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\right]$$

$$= \frac{4}{5} \;//$$

correct options: (A), (C)

2. Let a density matrix be represented by $\rho = m|0\rangle\langle 0| + n|1\rangle\langle 1|$. It follows that

1

(A) m+n=1

(B) $m^2+n^2=1$

(C) $(m+n)^2=1$

(D) trace$(\rho^2) = m^2 + n^2$

2.

$$\rho = m|0\rangle\langle 0| + n|1\rangle\langle 1|$$

$$= m\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + n\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$= m\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + n\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}m & 0\\0 & n\end{pmatrix}$$

$$\rho^2 = \begin{pmatrix}m & 0\\0 & n\end{pmatrix}\begin{pmatrix}m & 0\\0 & n\end{pmatrix} = \begin{pmatrix}m^2 & 0\\0 & n^2\end{pmatrix}$$

Thus, $\mathrm{Tr}(\rho) = m+n = 1$

$$\Rightarrow (m+n)^2 = 1$$

$$\mathrm{Tr}(\rho^2) = m^2+n^2$$

Correct options are:

(A), (C), (D)

3. Consider a two qubit state $\frac{1}{\sqrt{7}}(|00\rangle + \sqrt{2}|01\rangle + \sqrt{3}|10\rangle + |11\rangle)$. If the first qubit is measured, and we obtain $|0\rangle$, then the second qubit collapses to:

2

(A) $\frac{1}{\sqrt{3}}[|0\rangle + \sqrt{2}|1\rangle]$

(B) $\frac{1}{\sqrt{3}}[|1\rangle + \sqrt{2}|0\rangle]$

(C) $\frac{1}{\sqrt{7}}[|0\rangle + \sqrt{2}|1\rangle]$

(D) $\frac{1}{\sqrt{7}}[2|0\rangle + \sqrt{3}|1\rangle]$

3.

$$|\psi\rangle = \frac{1}{\sqrt{7}}\left[|00\rangle + \sqrt{2}|01\rangle + \sqrt{3}|10\rangle + |11\rangle\right]$$

First qubit msmt results in $|0\rangle$

So, the second qubit collapses to:

$$\alpha\left[\frac{1}{\sqrt{7}}|0\rangle + \frac{\sqrt{2}}{\sqrt{7}}|1\rangle\right]$$

Now, $\left(\frac{\alpha}{\sqrt{7}}\right)^2 + \left(\frac{\sqrt{2}\alpha}{\sqrt{7}}\right)^2 = 1$

$$\Rightarrow \frac{\alpha^2}{7} + \frac{2\alpha^2}{7} = 1$$

$$\Rightarrow 3\alpha^2 = 7 \Rightarrow \alpha = \sqrt{\frac{7}{3}}$$

Thus, the second qubit collapses to:

$$\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

$$= \frac{1}{\sqrt{3}}\left[|0\rangle + \sqrt{2}|1\rangle\right]$$

correct option : (A)

$f(x) = 0, 1$

$f(0) = 0$
$f(1) = 0$

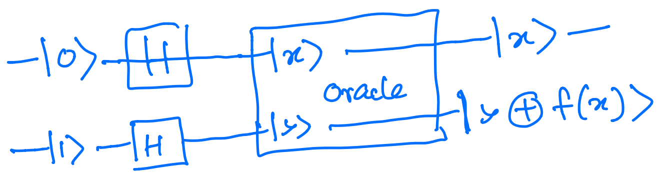
$f(0) = 1$
$f(1) = 1$

Deutsch Algorithm

$f(x)$ → constant
→ balanced

at one shot

Quantum computation



$f(0) = 1$
$f(1) = 0$

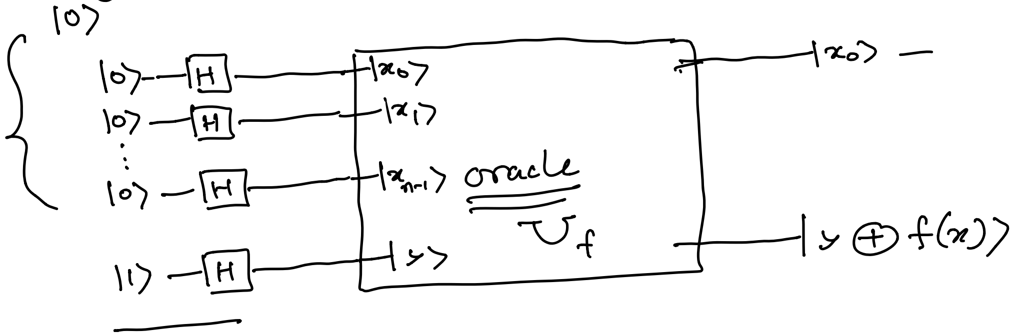
$f(0) = 0$
$f(1) = 1$

Input is single qubit

output is also a single qubit

Deutsch - Jozsa Algorithm

$|0\rangle^{\otimes n}$



Let us consider a 3 qubit system

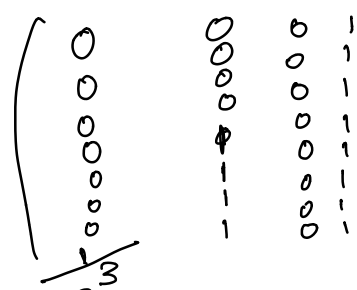
2^3 possibilities

000
001
010

0
1

$$\begin{cases} f(000) = 0 \\ f(001) = 1 \neq 0 \end{cases}$$

stop



$$\frac{2^3}{2} + 1$$

If

$f(000) = 0$

$f(001) = 0$

$f(010) = 0$

$f(011) = 0$

$f(100) = 0$

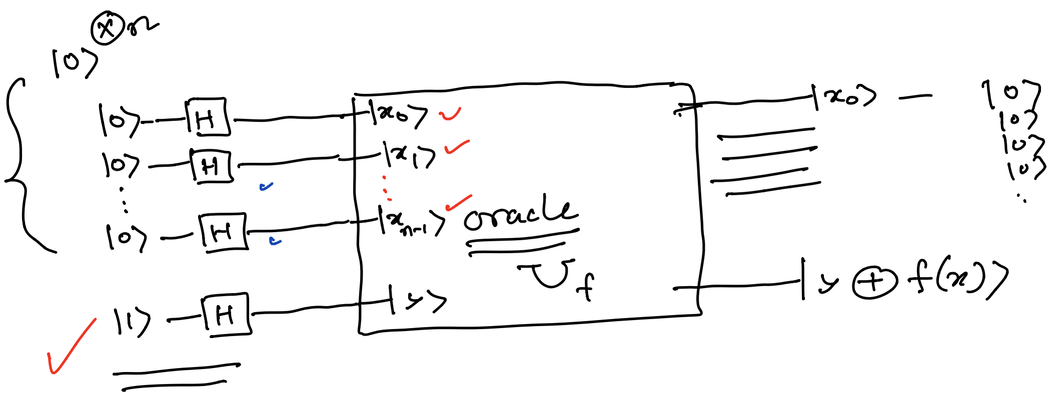
constant

stop

$$\boxed{\frac{2^n}{2} + 1}$$

$N = 2^n$

$$\frac{N}{2} + 1$$



Single qubit case

$$|0\rangle - \boxed{H} - \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1\rangle - \boxed{H} - \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|x\rangle - \boxed{H} - \frac{|0\rangle + (-1)^x |1\rangle}{\sqrt{2}}$$

$$\boxed{|x\rangle - \boxed{H} - \frac{|0\rangle + (-1)^x |1\rangle}{\sqrt{2}}}$$

$$|0\rangle^{\otimes n} \xrightarrow{H} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \cdots \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{2^{n/2}} \sum_x |x\rangle$$

$|x\rangle$ is n qubit basis state

$$|0\rangle^{\otimes 2} \xrightarrow{H} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{2} \left[|00\rangle + |01\rangle + |10\rangle + |11\rangle \right]$$

$$\boxed{|0\rangle^{\otimes n} \xrightarrow{H} \frac{1}{\underline{2^{n/2}}} \sum |x\rangle}$$

uniform linear superposition of n qubit basis states

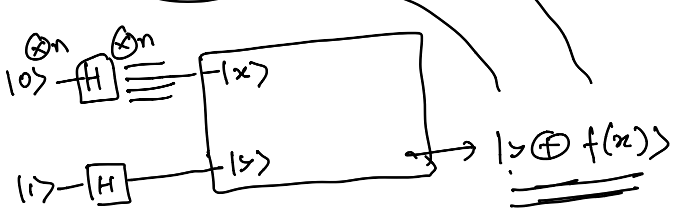
Input : $\underline{\underline{\dfrac{1}{2^{n/2}} \displaystyle\sum_{x=0}^{2^n - 1} |x\rangle}}$

2-qubit i.e. $n=2$

$$\frac{1}{2} \left(|\underline{x_0}\rangle + |\underline{x_1}\rangle + |\underline{x_2}\rangle + |\underline{x_3}\rangle \right)$$

$$= \frac{1}{2} \left(|00\rangle + |01\rangle + |10\rangle + |11\rangle \right)$$

Target : $\dfrac{|\underline{0}\rangle - |1\rangle}{\sqrt{2}}$



Output

$$\frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle \otimes \frac{1}{\sqrt{2}} \left[|f(x)\rangle - |\overline{f(x)}\rangle \right]$$

$f(x) = 0$ or 1

$\overline{f(x)} = 1$ or 0

$$\frac{1}{2^{\frac{n+1}{2}}} \sum_{x=0}^{2^n-1} |x\rangle \otimes (-1)^{f(x)} \left(|0\rangle - |1\rangle \right)$$

$$= \underbrace{\left[\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle \right]}_{\text{first register}} \otimes \underbrace{\frac{|0\rangle - |1\rangle}{\sqrt{2}}}_{\text{2nd register}}$$

$$\boxed{\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle \xrightarrow{H^{\otimes n}} ?}$$

$$|x_i\rangle \xrightarrow{H} \frac{|0\rangle + (-1)^{x_i} |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \sum_{k_i=0}^{1} (-1)^{x_i k_i} |k_i\rangle$$

$$|x\rangle \equiv |x_{n-1}, \ldots \ldots x_1 \cdots x_0\rangle$$

$$|00\rangle + |01\rangle + |10\rangle + |11\rangle$$

$$\underset{x_1\ x_2}{|00\rangle} \xrightarrow{H^{\otimes 2}} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + (-1)^{x_1}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + (-1)^{x_2}|1\rangle}{\sqrt{2}}$$

$$= \frac{1}{(2^2)^{1/2}} \sum_{k_1=0}^{1} (-1)^{x_1 k_1} |k_1\rangle \otimes \sum_{k_2=0}^{1} (-1)^{x_2 k_2} |k_2\rangle$$

$$\underset{x_1\ x_2}{|00\rangle} \quad = \frac{1}{(2^2)^{1/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} (-1)^{x_1 k_1 + x_2 k_2} |k_1 k_2\rangle$$

$$\boxed{|x\rangle \xrightarrow{H^{\otimes n}} \frac{1}{(2^n)^{1/2}} \sum_{k_{n-1}=0}^{1} \cdots \sum_{k_0=0}^{1} (-1)^{\sum x_i k_i} |k_{n-1} \cdots k_0\rangle}$$

Thus, the Output will be

$$\left[\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle\right] \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \left[\frac{1}{2^{n/2}} \sum_x (-1)^{f(x)} \frac{1}{2^{n/2}} \sum_k (-1)^{x \cdot k} |k\rangle\right] \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \left[\frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{k=0}^{2^n-1} (-1)^{f(x)+x.k} |k\rangle \right] \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$\Big(x.k \underset{\substack{\uparrow \\ \text{sum of bitwise product}}}{=} x_0k_0 + x_1k_1 + \cdots + x_{n-1}k_{n-1} \Big)$$

Let us now analyse the Output expression for the function $f(x)$, which can be either a constant function or balanced one.

Case I

- $f(x)$ is constant

Then, the phase factor

$$(-1)^{f(x)+x.k} = \underbrace{\text{constant}}_{a} \cdot (-1)^{x.k}$$

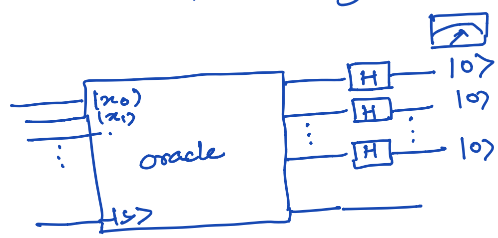
Thus, output

$$= \underbrace{\left[\frac{1}{2^n} \sum_{x} \sum_{k} a\, (-1)^{x.k} |k\rangle \right]} \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Because of $(-1)^{x.k}$ term we have equal number of positive and negative terms and the terms inside $\underbrace{[\quad\quad]}$ will cancel out EXCEPT for the term with $|k=0\rangle$

Hence, the first register, for $f(x) =$ constant, will be : $|k=0\rangle = |000\cdots 0\rangle = \underline{\underline{|0\rangle^{\otimes n}}}$

$\Rightarrow$ If we make a measurement and find that the first register is $|0\rangle^{\otimes n}$



Then the function is a constant function

Case 2 The function $f(x)$ is balanced.

In this case let us look at the co-efficient of $|k=0\rangle$ in the following term :

$$\left[\frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1}(-1)^{f(x)+x.k}\,|k\rangle\right]\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

It is : $$\sum_{x=0}^{2^n-1}(-1)^{f(x)}$$

Now, as $f(x)$ is balanced, half of $f(x) = 1$ and other half of $f(x) = 0$

So clearly, $$\sum_{x=0}^{2^n-1} (-1)^{f(x)} = 0 \quad \text{for } \underline{f(x) \text{ balanced}}$$

This implies that we will not get $|0\rangle$ in the first register if $f(x)$ is balanced.

$\Rightarrow$ If first register measurement gives anything other than $|0\rangle$, the function is balanced.

Example

- In Deutsch - Jozsa algorithm, if the input is a linear combination of two qubit basis states $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$, i.e. $|x\rangle = \frac{1}{2}\left(|00\rangle+|01\rangle+|10\rangle+|11\rangle\right)$ and if the function has the property that $f(00) = 0$, $f(01) = f(10) = f(11) = 1$, what will be the state of the first register before any measurement is made?

Solution

Here, input, $|x\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

Now, $f(00) = 0$, $f(01) = f(10) = f(11) = 1$

In D-S algorithm: $\underset{\text{Target}}{|y\rangle} = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

So,
$$|x\rangle \otimes |y\rangle = \frac{1}{2\sqrt{2}}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \otimes \left(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\Big[|000\rangle + |010\rangle + |100\rangle + |110\rangle - |001\rangle - |011\rangle - |101\rangle - |111\rangle\Big]$$

Output

$$|x\rangle \otimes |y \oplus f(x)\rangle$$

$$= \frac{1}{2\sqrt{2}}\Big[|000\rangle + |011\rangle + |101\rangle + |111\rangle - |001\rangle - |010\rangle - |100\rangle - |110\rangle\Big]$$

$0 \oplus f(00) = f(00) = 0$

$0 \oplus f(01) = f(01) = 1$

$1 \oplus f(00) = \overline{f(00)} = 1$

$$= \frac{1}{2\sqrt{2}}\Big[|00\rangle\left(|0\rangle - |1\rangle\right) + |01\rangle\left(|1\rangle - |0\rangle\right) + |10\rangle\left(|1\rangle - |0\rangle\right) + |11\rangle\left(|1\rangle - |0\rangle\right)\Big]$$

$$= \underbrace{\frac{1}{2}\Big[\left(|00\rangle - |01\rangle - |10\rangle - |11\rangle\right)\Big]}_{\text{First register}} \otimes \frac{\left(|0\rangle - |1\rangle\right)}{\sqrt{2}}$$

Now, Hadamard gate is applied on the 1st register, which is a two qubit register.

In this case

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

So, $\left[H \otimes H\right] \frac{1}{2}\left[\left(|00\rangle - |01\rangle - |10\rangle - |11\rangle\right)\right]$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\left[-|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]$$

Thus the state of the first register after the execution of Deutsch - Jozsa algorithm will be:

$$\frac{1}{2}\left[-|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]$$