Quantum Error Correction

Sender | Noisy channel | Receiver
Alice | | Bob

1 0 0 1 1 0 1 0 — Bits is sent ✓

1 0 0 1 0 0 1 0 — single bit error ✓

1 0 1 1 0 0 1 0 — Multiple bit error ✓

1 0 0 1 0 1 1 0

Consecutive bits are corrupted — Burst error

Parity bit

1 0 1 1 0 0 1 0 — no. of 1 = 4

1 0 1 0 0 0 1 0 → not acceptable

Logical bits

$$\begin{cases} 0_L = 000 \\ 1_L = 111 \end{cases}$$

Suppose you get : 1 0 1 → then correct it to 111

Errors in quantum communication

Decoherence

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \longrightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Bit flips

continuous evolution of quantum states
also introduce phase errors

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \; ; \quad |\psi'\rangle = e^{i\alpha}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$$

Say, a phase perturbation converts it to

$$\boxed{|\psi'\rangle = \frac{|0\rangle + e^{i\alpha}|1\rangle}{\sqrt{2}}}$$

A msmt of σ_z will still give you $|0\rangle$ or $|1\rangle$ with probability $\frac{1}{2}$

But if you make a msmt of σ_x it will project $+1$ with probability

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|\psi'\rangle = \frac{|0\rangle + e^{i\alpha}|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\alpha}\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} e^{i\alpha/2}\begin{pmatrix}e^{-i\alpha/2}\\e^{i\alpha/2}\end{pmatrix}$$

$$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad x = \pm 1$$

$$x = +1, \quad |x=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$P_{+1} = |+1\rangle\langle +1| = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$$

$$P_{+1}|\psi'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\alpha/2\\\cos\alpha/2\end{pmatrix} e^{i\alpha/2} = \cos\alpha/2 \, e^{i\alpha/2} \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}$$

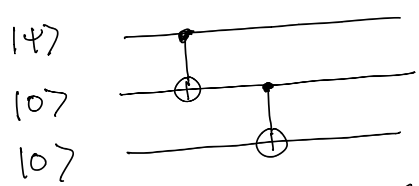
$$\boxed{\cos^2\frac{\alpha}{2}}$$

$0_L = 000$

Sender

Alice

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



Logical $|\psi_L\rangle$

$$|\psi\rangle_L = (\underset{\text{1st bit}}{\alpha|0\rangle + \beta|1\rangle})\ \underset{\text{2nd}}{|0\rangle}\ \underset{\text{3rd}}{|0\rangle}$$

$$= (\alpha|00\rangle + \beta|11\rangle)\ |0\rangle$$

$$\boxed{|\psi_L\rangle = \alpha|000\rangle + \beta|111\rangle}$$

Alice is sending ψ_L

Say, noise in the channel flips a bit with probability p and leaving it unchanged is $1-p$

The state received by Bob via the noisy channel is $|\psi_2\rangle_L$

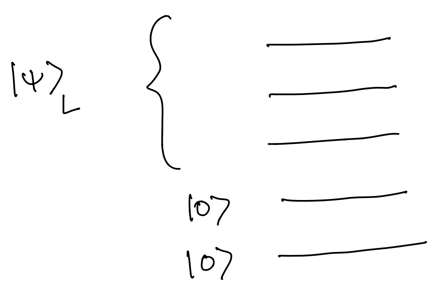
We will have the following possibilities

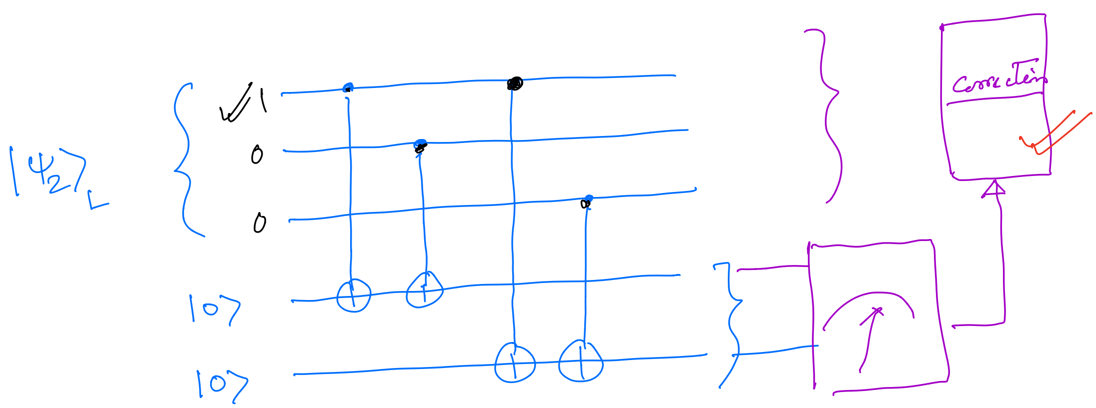
Bob receives	Probability of occurrence
$\alpha|000\rangle + \beta|111\rangle$	$(1-p)^3$
$\alpha|100\rangle + \beta|011\rangle$	$p(1-p)^2$
$\alpha|010\rangle + \beta|101\rangle$	$p(1-p)^2$
$\beta|110\rangle$	$p(1-p)^2$

$\alpha\,|001\rangle + \beta|110\rangle$ $\qquad$ Total $= 3p(1-p)^2$

$\left.\begin{array}{l} \alpha\,|110\rangle + \beta\,|001\rangle \\ \alpha\,|101\rangle + \beta\,|010\rangle \\ \alpha\,|011\rangle + \beta\,|100\rangle \end{array}\right\} \rightarrow p^2(1-p)$ $\qquad$ Total $= \underline{3p^2(1-p)}$

$\alpha\,|111\rangle + \beta\,|000\rangle \;\}$ $\qquad p^3$





$$(\alpha\,|000\rangle + \beta\,|111\rangle)\,|00\rangle = (\alpha\,|000\rangle + \beta\,|111\rangle)\,|00\rangle$$

$\Rightarrow$ ancilla bits are not affected at all.

$$|\psi_2\rangle_L = (\underline{\alpha\,|100\rangle + \beta\,|011\rangle})\,\underline{|00\rangle}$$

$=$

Bob's action results in $|\psi_3\rangle_L$

State received $|\psi_2\rangle_L$	After Bob's gate operation $|\psi_3\rangle_L$	Probability
$\alpha|000\rangle+\beta|111\rangle$	$(\alpha|000\rangle+\beta|111\rangle)|00\rangle$	$(1-p)^3$
$\alpha|100\rangle+\beta|011\rangle$	$(\alpha|100\rangle+\beta|011\rangle)|11\rangle$	$p(1-p)^2$
$\alpha|010\rangle+\beta|101\rangle$	$(\alpha|010\rangle+\beta|101\rangle)|10\rangle$	$p(1-p)^2$
$\alpha|001\rangle+\beta|110\rangle$	$(\alpha|001\rangle+\beta|110\rangle)|01\rangle$	$p(1-p)^2$
$\alpha(|110\rangle+\beta|001\rangle)$	$(\alpha|110\rangle+\beta|011\rangle)|01\rangle$	$p^2(1-p)$
$\alpha|101\rangle+\beta|010\rangle$	—— $|10\rangle$	$p^2(1-p)$
$\alpha|011\rangle+\beta|100\rangle$	—— $|11\rangle$	$p^2(1-p)$
$\alpha|111\rangle+\beta|000\rangle$	—— $|00\rangle$	p^3

msmt of ancilla may yield

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

Say Bob get $|00\rangle$

$\alpha|000\rangle+\beta|111\rangle$ with prob. $(1-p)^3$ (without errors)

$\alpha|111\rangle+\beta|000\rangle$ with prob. p^3 (errors)

$\boxed{\text{Bob take's no action}}$

Now, Say Bob gets $|01\rangle$

$\alpha|001\rangle+\beta|110\rangle$ with prob. $p(1-p)^2$ (with one error)

$\alpha|110\rangle+\beta|001\rangle$ with prob. $p^2(1-p)$ (with two errors)

H.O.

Then Bob applies a σ_x on the third qubit

$$\sigma_x \xrightarrow{3rd} (\alpha|001\rangle + \beta|110\rangle) \longrightarrow \alpha|000\rangle + \beta|111\rangle$$

$$\sigma_x \xrightarrow{3rd} (\alpha|110\rangle + \beta|001\rangle) \longrightarrow \alpha|111\rangle + \beta|000\rangle$$

✓ One is getting corrected fully while the other one is fully getting wrong!

Say Bob gets $|10\rangle$ as a msmt. of the ancilla bit :

$$(\alpha|010\rangle + \beta|101\rangle)|10\rangle$$

Now, Bob applies σ_x on the 2nd qubit

$$\sigma_x \xrightarrow{2nd} (\alpha|010\rangle + \beta|101\rangle)|10\rangle$$

$$\longrightarrow (\alpha|000\rangle + \beta|111\rangle)|10\rangle \checkmark$$

with prob. $p(1-p)^2$

$$\checkmark\ \alpha|101\rangle + \beta|010\rangle \xrightarrow{\sigma_x} \alpha|111\rangle + \beta|000\rangle$$

$p^2(1-p)$

Finally

Bob applies σ_x on the 1st qubit

$$(\alpha|100\rangle + \beta|011\rangle) \xrightarrow{\sigma_x} \alpha|000\rangle + \beta|111\rangle$$

$|11\rangle$

with prob. $p(1-p)^2$

$$\alpha|011\rangle + \beta|100\rangle \xrightarrow{\sigma_x} \alpha|111\rangle + \beta|000\rangle$$

with prob. $p^2(1-p)$ ✓

Q. How many of these states will continue to have errors?

A. Probability of that happening is

$$3p^2(1-p) + p^3$$

$$= 3p^2 - 2p^3 < p$$

Suppose $p = 0.01$

$$3 \times 10^{-4}$$

error is reduced by a factor of 300!

Shor's 9 qubit error correction code

Bit flip $a|0\rangle + b|1\rangle \longrightarrow a|1\rangle + b|0\rangle$

Phase flip $a|0\rangle + b|1\rangle \longrightarrow a|0\rangle - b|1\rangle$

$$|0\rangle + |1\rangle \longrightarrow |0\rangle - |1\rangle$$

$\sqrt{2}$ $\quad\quad\quad\quad\quad\quad \sqrt{2}$

$$|+\rangle \longrightarrow |-\rangle$$

Phase-flip errors

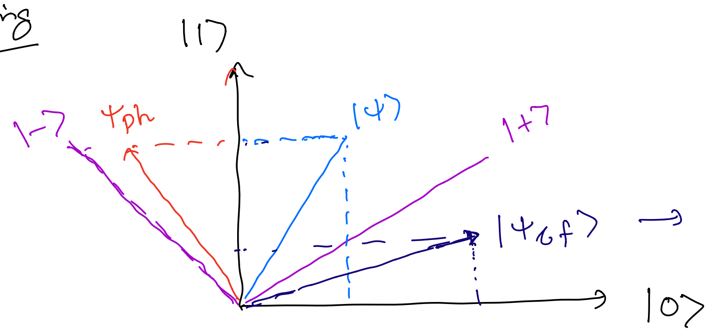
$$|0\rangle_L \equiv |000\rangle$$
$$|1\rangle_L \equiv |111\rangle$$

One can encode the triplet in a diagonal basis

— digression —

Geometrical meaning

$|\psi\rangle = a|0\rangle + b|1\rangle$, with $a|0\rangle \downarrow |1\rangle$, $b|1\rangle \downarrow |0\rangle$



$|\psi_{bf}\rangle \rightarrow$ This is the reflection of $|\psi\rangle$ about the diagonal basis $|+\rangle$

$a|0\rangle + b|1\rangle$

$\longrightarrow a|0\rangle - b|1\rangle$

$= -\left[-a|0\rangle + b|1\rangle\right]$

$a|000\rangle + b|111\rangle$, i.e. $a|0\rangle_L + b|1\rangle_L$

$$|\psi\rangle = a|+++\rangle + b|---\rangle$$

In the computational basis the state is:

$$|\psi\rangle = \frac{a+b}{2\sqrt{2}}\left[|000\rangle + |011\rangle + |101\rangle + |110\rangle\right]$$
$$+ \frac{a-b}{2\sqrt{2}}\left[|001\rangle + |010\rangle + |100\rangle + |111\rangle\right]$$

Q. What happens if we measure any of the qubits in a computational basis?

Say, you measure qubit no. 1

The state collapses either to :

$$\frac{a+b}{\sqrt{2}}\left(|000\rangle + |011\rangle\right) + \frac{a-b}{\sqrt{2}}\left(|001\rangle + |010\rangle\right) \rightarrow (i)$$

or

$$\frac{a+b}{\sqrt{2}}\left(|101\rangle + |110\rangle\right) + \frac{a-b}{\sqrt{2}}\left(|100\rangle + |111\rangle\right) \rightarrow (ii)$$

Say, the state collapsed to (i)

$$\frac{a+b}{\sqrt{2}}\left(|000\rangle + |011\rangle\right) + \frac{a-b}{\sqrt{2}}\left(|001\rangle + |010\rangle\right)$$

Introduce two ancilla and measure the ancilla after CNOT operations

$$|0\rangle \longrightarrow \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|1\rangle \longrightarrow \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

In the diagonal basis the collapsed state is

$$\left(a|+++\rangle + b|---\rangle\right) + \left(a|-++\rangle + b|+--\rangle\right)$$

$\Rightarrow$ The state is either without error or with a single bit flip error

Shor's 5 qubit error code

bit flip $\quad X\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} b \\ a \end{pmatrix}$

$\begin{pmatrix} a \end{pmatrix} \longrightarrow \begin{pmatrix} a \end{pmatrix}$

phase flip $Z\begin{pmatrix} \\ b \end{pmatrix} \quad \begin{pmatrix} \\ -b \end{pmatrix}$

phase and bit flip errors : $Y\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} -b \\ a \end{pmatrix}$

Shor's code

We will encode

$|0\rangle$ as $|0\rangle_L = |+++\rangle$

$|1\rangle$ as $|1\rangle_L = |---\rangle$

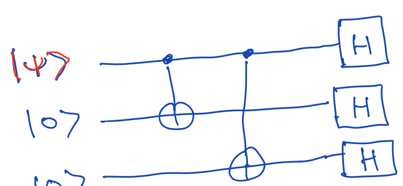
Our goal is to send $\boxed{a|0\rangle + b|1\rangle}$ through a quantum channel

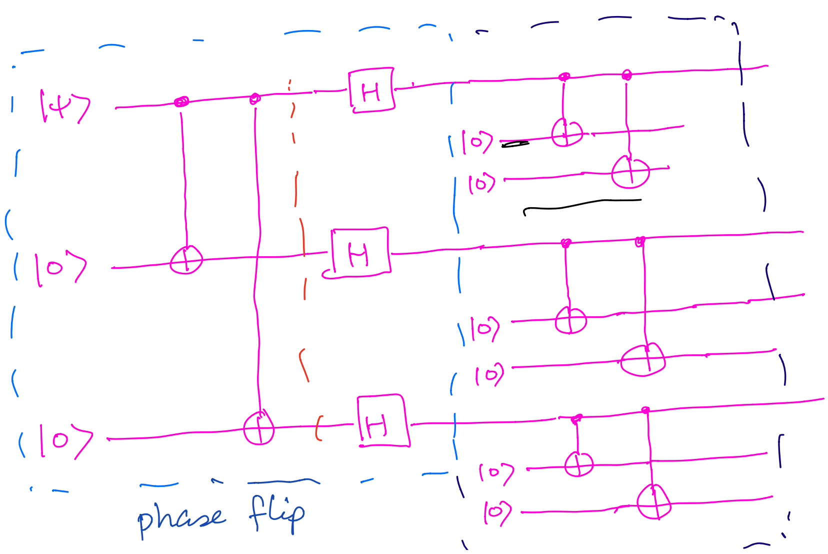
phase-flip encoding

$$(a|0\rangle + b|1\rangle)|0\rangle|0\rangle \xrightarrow{\text{CNOT}} a|000\rangle + b|111\rangle$$

$\downarrow$ Apply Hadamard gates on all 3 qubits

$$\frac{a}{2\sqrt{2}}\Big[(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)\Big] + \frac{b}{2\sqrt{2}}\Big[(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)\Big]$$





coding

Bit-flip coding

$$\frac{a}{2\sqrt{2}}\left[\underline{(|0\rangle+|1\rangle)|00\rangle}\ (|0\rangle+|1\rangle)|00\rangle\ (|0\rangle+|1\rangle)|00\rangle\right]$$

$$+\ \frac{b}{2\sqrt{2}}\left[\underline{(|0\rangle-|1\rangle)|00\rangle}\ (|0\rangle-|1\rangle)|00\rangle\ (|0\rangle-|1\rangle)|00\rangle\right]$$

$\xrightarrow{\text{CNOT operations}}$

$$\frac{a}{2\sqrt{2}}\left[(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)\right]$$

$$+\ \frac{b}{2\sqrt{2}}\left[(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\right]$$

Use short hand notation:

$$(+) \longrightarrow \frac{|000\rangle+|111\rangle}{\sqrt{2}}$$

$$(-) \longrightarrow \frac{|000\rangle-|111\rangle}{\sqrt{2}}$$

$$= \quad a\,|+++\rangle + b\,|---\rangle$$

Assume that only at most one qubit is affected after going through the noisy channel

Say p is the probability that a single qubit is affected.

Then the probability that no qubit is affected is $\underline{(1-p)^9}$

$p \lessapprox 0.01$

$$(1-p)^9 = \underline{1-9p+36p^2} + \cdots$$

The prob. that any 9 qubit is subjected to error and remaining 8 unaffected :

$$9p(1-p)^8 = 9p(1-8p+\cdots)$$
$$= 9p - 72p^2 + \cdots$$

Thus, probability of zero or 1 error

$$(1-9p+36p^2) + (9p-72p^2)$$
$$= 1 - 36p^2$$

$\Rightarrow$ Probability of having more than one error is $36p^2$

Because p is small $36p^2 \ll p$

Decoding

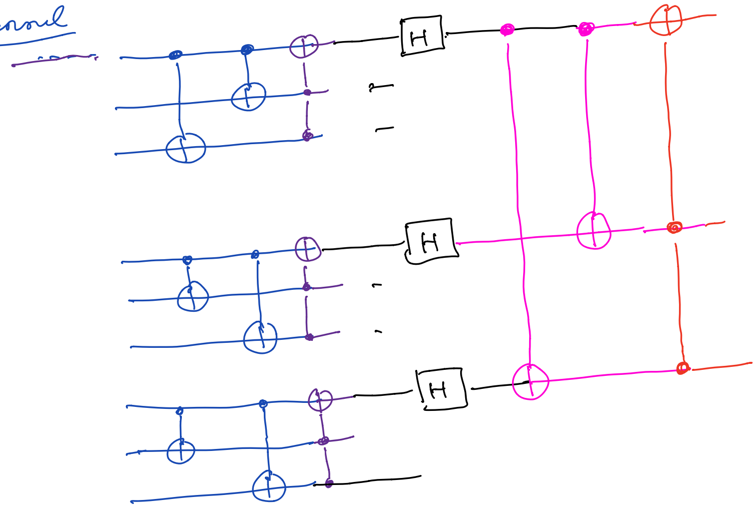
At the end of coding we obtain :

$$a\left[\frac{|000\rangle+|111\rangle}{\sqrt{2}} \cdot \frac{|000\rangle+|111\rangle}{\sqrt{2}} \cdot \frac{|000\rangle+|111\rangle}{\sqrt{2}}\right]$$
$$+b\left[\frac{|000\rangle-|111\rangle}{\sqrt{2}} \cdot \frac{|000\rangle-|111\rangle}{\sqrt{2}} \cdot \frac{|000\rangle-|111\rangle}{\sqrt{2}}\right]$$

Let us assume that the 1st qubit is flipped, by error

$$\Bigg\| \; a\left[\frac{|100\rangle-|011\rangle}{\sqrt{2}} \cdot \frac{|000\rangle+|111\rangle}{\sqrt{2}} \cdot \frac{|000\rangle+|111\rangle}{\sqrt{2}}\right]$$
$$+b\left[\frac{|100\rangle+|011\rangle}{\sqrt{2}} \cdot \frac{|000\rangle-|111\rangle}{\sqrt{2}} \cdot \frac{|000\rangle-|111\rangle}{\sqrt{2}}\right]$$

channel



CNOT operation

$$a\left[\frac{|111\rangle - |011\rangle}{\sqrt{2}} \cdot \frac{|000\rangle + |100\rangle}{\sqrt{2}} \cdot \frac{|000\rangle + |100\rangle}{\sqrt{2}}\right]$$

$$+\, b\left[\frac{|111\rangle + |011\rangle}{\sqrt{2}} \cdot \frac{|000\rangle - |100\rangle}{\sqrt{2}} \cdot \frac{|000\rangle - |100\rangle}{\sqrt{2}}\right]$$

CCNOT operation

$$a\left[\frac{|011\rangle - |111\rangle}{\sqrt{2}} \cdot \frac{|000\rangle + |100\rangle}{\sqrt{2}} \cdot \frac{|000\rangle + |100\rangle}{\sqrt{2}}\right]$$

$$+\, b\left[\frac{|011\rangle + |111\rangle}{\sqrt{2}} \cdot \frac{|000\rangle - |100\rangle}{\sqrt{2}} \cdot \frac{|000\rangle - |100\rangle}{\sqrt{2}}\right]$$

$$= a\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\,|11\rangle \cdot \frac{|0\rangle + |1\rangle}{\sqrt{2}}\,|00\rangle \cdot \frac{|0\rangle + |1\rangle}{\sqrt{2}}\,|00\rangle\right]$$

$$+\, b\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\,|11\rangle \cdot \frac{|0\rangle - |1\rangle}{\sqrt{2}}\,|00\rangle \cdot \frac{|0\rangle - |1\rangle}{\sqrt{2}}\,|00\rangle\right]$$

1st, 4th, 7th Qubit undergo Hadamard operation

$$a\left[|1\rangle_1 \;|0\rangle_4 \;|0\rangle_7\right] + b\left[|0\rangle_1 \;|1\rangle_4 \;|1\rangle_7\right]$$

[we are not worried about 2nd, 3rd, 5th, 6th, 8th, 9th qubit]

$\xrightarrow{\text{CNOT}}$ $a\left[|1\rangle_1 |1\rangle_4 |1\rangle_7\right] + b\left[|0\rangle_1 |1\rangle_4 |1\rangle_7\right]$

$\xrightarrow{\text{CCNOT}}$ $\left[a|0\rangle_1 + b|1\rangle_1\right] |1\rangle_4 |1\rangle_7$

$a|0\rangle + b|1\rangle$

You got the original qubit

Problem

In a three qubit code, assume that the probability of a single bit flip error is 0.05. If the ancilla bit is measured to be $|00\rangle$ find the probability that the measured state is error free.

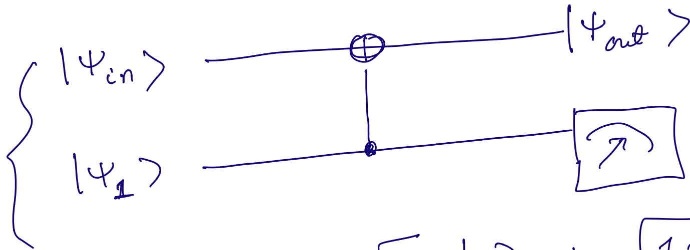
$\alpha|000\rangle + \beta|111\rangle$ with prob. $(1-p)^3$ (without error)

$\alpha|111\rangle + \beta|000\rangle$ with prob. p^3 (error)

$$\text{Required probability} = \frac{(1-p)^3}{(1-p)^3 + p^3}$$

$$= 0.99854$$

Problem



$$|\psi_1\rangle = \sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle$$

what kind of process (bit flip, phase flip or both) is depicted and with what probability?

$$|\psi_{in}\rangle = a|0\rangle + b|1\rangle$$

$$\text{Input} = \sqrt{p}\left(a|00\rangle + b|10\rangle\right) + \sqrt{1-p}\left(a|01\rangle + b|11\rangle\right)$$

$$\underline{\text{Output}} = \sqrt{p}\left(a|00\rangle + b|10\rangle\right) + \sqrt{1-p}\left(a|11\rangle + b|01\rangle\right)$$

✓

Say a msmt is made on the 2nd qubit.

If the 2nd qubit msmt turns out to be $|1\rangle$, then the input gets flipped in the output, otherwise input does not change.

Thus, prob. of bit flip = Prob. (second qubit $= |1\rangle$)

$$= \left(\sqrt{1-p}\,a\right)^2 + \left(\sqrt{1-p}\,b\right)^2$$

$$= (1-p)(a^2+b^2)$$

$$= \underline{\underline{1-p}}$$