Density matrix

$$\begin{cases} Tr(\rho^2) < 1 & \text{for mixed state} \\ Tr(\rho^2) = 1 & \text{for pure state} \end{cases}$$

Say we have,

50% $\longrightarrow$ $|0\rangle$ ✓

50% $\longrightarrow$ $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$$\rho = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

Sum of diagonal elements = 1

off-diagonal $\longrightarrow$ coherence

$\boxed{Tr(\rho) = 1}$ for all states

Say, ✓ 75% $|0\rangle$

25% $|1\rangle$

$$\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$$

(completely mixed state)

50% $|0\rangle$

50% $|1\rangle$

✓ $\rho_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

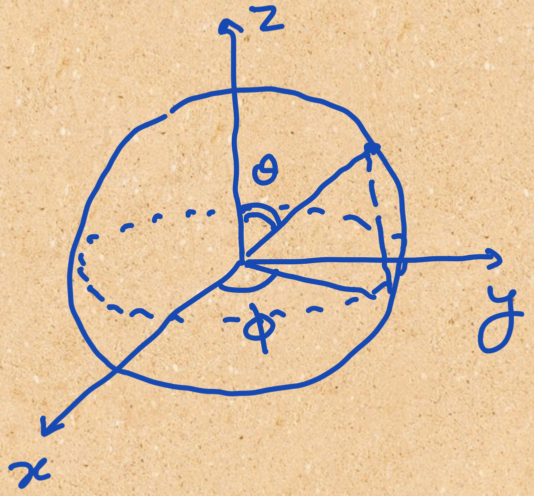
Maximally mixed state

50% $\quad \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

50% $\quad \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$ $\qquad \rho_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

$$\underline{\underline{\rho_1 = \rho_2}}$$

Bloch sphere



$$\sigma_n = \vec{\sigma} \cdot \hat{n}$$

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

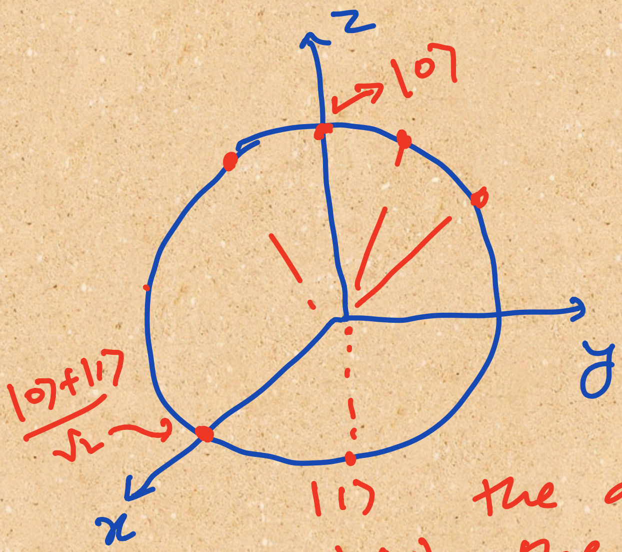
$$\sigma_n = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

Eigenvalues of σ_n : $\lambda = \pm 1$

Eigenstates associated with $\lambda = +1$

$$|\theta, \phi\rangle = |\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix}$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



Take the Pauli matrix along the direction $\hat{n}$, σ_n

The state corresponding to $|\psi\rangle$ is given by that ray in the direction $\hat{n}$ for which the eigenvalue of $\sigma_n = +1$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{array}{l} \theta = 0 \\ \phi = \pi \end{array}$$

Eigenvalues of $\sigma_z = \pm 1$

Eigenvector corresponding to $\underline{\underline{\lambda = +1}}$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |0\rangle$

$$\rho = |\psi\rangle\langle\psi|$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \underline{\cos^2\frac{\theta}{2}} & e^{-i\phi}\,\underline{\cos\frac{\theta}{2}\sin\frac{\theta}{2}} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \underline{\underline{\sin^2\frac{\theta}{2}}} \end{pmatrix}$$

Any 2×2 matrix can be expressed in terms of Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ and I

$$\rho = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix}$$

$$= \frac{I}{2} + \frac{1}{2}\underbrace{\sin\theta\cos\phi}_{n_x}\,\sigma_x + \frac{1}{2}\underbrace{\sin\theta\sin\phi}_{n_y}\,\sigma_y + \frac{1}{2}\underbrace{\cos\theta}_{n_z}\,\sigma_z$$

$$\boxed{\rho = \frac{1}{2}\left[I + \hat{n}\cdot\vec{\sigma}\right]}$$

✓ all points exactly lies on the surface of Bloch sphere.

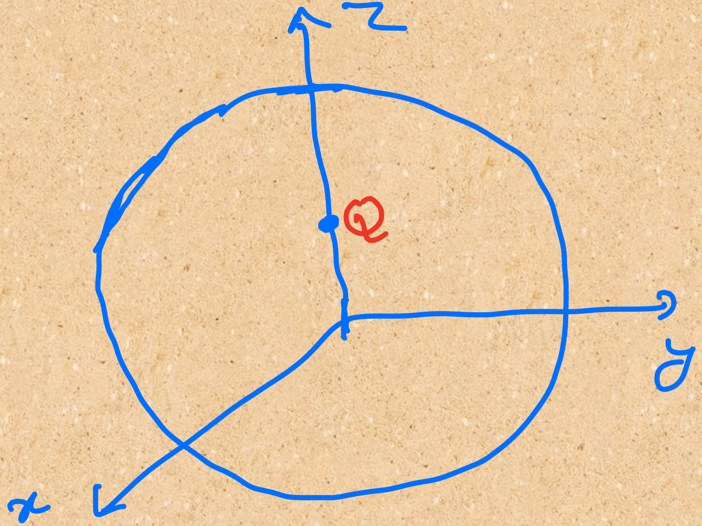
Mixed state

say, $\boxed{\rho = \frac{1}{2}\left[I + \vec{a}\cdot\vec{\sigma}\right]}$

with $|\vec{a}| < 1$

ρ will lie inside the Bloch sphere

For example Take $\hat{z}$ direction



$a_z = \frac{1}{2}$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\rho = \frac{1}{2}\left[I + \frac{1}{2}\sigma_z \right]$$

$$= \frac{1}{2}\begin{pmatrix} 1 + \frac{1}{2} & \underline{\underline{0}} \\ \underline{\underline{0}} & 1 - \frac{1}{2} \end{pmatrix}$$

$Tr(\rho) = 1$

$$\rho = \frac{1}{2}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$\boxed{\rho = \underline{\underline{\frac{3}{4}}}\, |0\rangle\langle 0| + \underline{\underline{\frac{1}{4}}}\, |1\rangle\langle 1|}$$

$$\underline{\underline{\langle \sigma_x \rangle}} = Tr(\rho\, \sigma_x)$$

$$= \frac{1}{2}\, Tr\left[\underbrace{(I + \vec{a}\cdot\vec{\sigma})}\, \underbrace{\sigma_x} \right]$$

$$= \frac{1}{2}\, Tr\left[\underline{\underline{\sigma_x (I}} + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z) \right]$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$$

$Tr(AB)$
$= Tr(BA)$

$$Tr(\sigma_x) = Tr(\sigma_y) = Tr(\sigma_z) = 0$$

$$\langle \sigma_x \rangle = \frac{1}{2}\, Tr(a_x)$$

$$= a_x$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle \sigma_y \rangle = a_y$$

$$Tr(\rho^2) = Tr\left[\frac{1}{4} (I + \vec{a}\cdot\vec{\sigma})^2 \right]$$

$$= \frac{1}{4} \mathrm{Tr}\left[I^2 + a_x^2 \sigma_x^2 + a_y^2 \sigma_y^2 + a_z^2 \sigma_z^2 \right]$$

$\sigma_x \sigma_y = i\sigma_z$, $\sigma_y \sigma_z = i\sigma_x$ ...

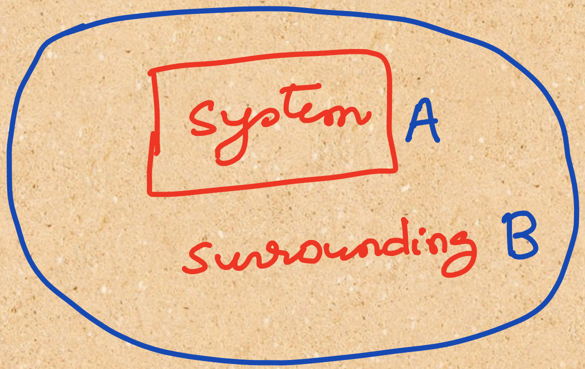
$\sigma_x \sigma_y + \sigma_y \sigma_x = 0$

$$\mathrm{Tr}(\rho^2) = \frac{1}{4} \mathrm{Tr}\left[I + |a|^2 \right]$$
$$= \frac{1}{2}\left(1 + |a|^2 \right)$$

Since $|a|^2 < 1$

$$\therefore \boxed{\mathrm{Tr}(\rho^2) < 1}$$

Reduced density matrix



$$\boxed{\rho_{A+B}}$$

$A+B$

We are interested in

studying $\underline{\underline{A}}$ only!

Average out $\rightarrow$ Trace out B

$$\boxed{\rho_A = tr_B(\rho_{AB})}$$

Reduced density matrix

$(A + B)$

$$\rho_{AB} = \underbrace{|a_1\rangle\langle a_2|}_{\text{system}} \otimes \underbrace{|b_1\rangle\langle b_2|}_{\text{environment}}$$

$$\rho_A = tr_B(\rho_{AB})$$

$$= |a_1\rangle\langle a_2| \; tr\left[|b_1\rangle\langle b_2|\right]$$

basis states $\underline{H_B}$

Say $\longrightarrow |i_B\rangle$ are basis states in H_B

$$\rho_A = |a_1\rangle\langle a_2| \, tr\left(|b_1\rangle\langle b_2|\right)$$

$$tr_B\left(|b_1\rangle\langle b_2|\right) = \sum_{i_B} \underline{\langle i_B|b_1\rangle} \, \underline{\langle b_2|i_B\rangle}$$

$$= \sum_{i_B} \langle b_2|i_B\rangle\langle i_B|b_1\rangle$$

$$= \langle b_2 \left\{ \underline{\sum_{i_B} |i_B\rangle\langle i_B|} \right\} |b_1\rangle$$

$$tr_B\left(|b_1\rangle\langle b_2|\right) = \underline{\langle b_2|b_1\rangle}$$

$$\boxed{\rho_A = \underline{\langle b_2|b_1\rangle} \; \underline{|a_1\rangle\langle a_2|}}$$

<u>Partial trace</u> essentially averages out the effect of <u>environment</u>

Entangled Bell state

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$\rho = |\Phi^+\rangle\langle\Phi^+|$$

$$\overset{A}{(|0\rangle} \otimes \overset{B}{|0\rangle)} \overset{A}{(\langle 0|} \otimes \overset{B}{\langle 0|)}$$

$$= \frac{1}{2}\Big[|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\Big]$$

say we want to focus on the 1st qubit only.

$\rightarrow$ Trace out second qubit

$$\rho_1 = Tr_2\, \rho$$

$$Tr(|b_1\rangle\langle b_2|) \Rightarrow \langle b_2|b_1\rangle$$

$$= \frac{1}{2}\Big[|0\rangle\langle 0|\, Tr(|0\rangle\langle 0|) + |0\rangle\langle 1|\, Tr(|0\rangle\langle 1|) + |1\rangle\langle 0|\, Tr(|1\rangle\langle 0|) + |1\rangle\langle 1|\, Tr(|1\rangle\langle 1|)\Big]$$

$$= \frac{1}{2}\Big[|0\rangle\langle 0|\, \underbrace{\langle 0|0\rangle}_{=1} + |0\rangle\langle 1|\, \underbrace{\langle 1|0\rangle}_{=0} + |1\rangle\langle 0|\, \underbrace{\langle 0|1\rangle}_{=0} + |1\rangle\langle 1|\, \underbrace{\langle 1|1\rangle}_{=1}\Big]$$

$$= \frac{1}{2}\Big[|0\rangle\langle 0| + |1\rangle\langle 1|\Big]$$

Mixed state

Say we have a state:

$$|\psi\rangle = \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}}$$

Alice

Bob

What is the reduced density matrix for Alice?

$$\rho = |\psi\rangle\langle\psi|$$

$$\rho_A = \mathrm{tr}_B\, \rho = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\psi\rangle = |0\,1\rangle$$

A ← B

$$\rho_{AB} = |\psi\rangle\langle\psi|$$

$$= |0_A 1_B\rangle\langle 0_A 1_B|$$

$$= \left(|0\rangle_A \otimes |1\rangle_B\right)\left(\langle 0|_A \otimes \langle 1|_B\right)$$

$$= |0\rangle_A\langle 0|_A \otimes |1\rangle_B\langle 1|_B$$

$$\rho_A = Tr_B(\rho_{AB})$$

$$= \underline{\underline{|0\rangle\langle 0|}}\ tr(|1\rangle\langle 1|)$$

$$= \langle 1|1\rangle$$

$$= 1$$

$$= |0\rangle\langle 0|$$

$$|\psi\rangle = |\overset{A}{1}\overset{B}{1}\rangle$$

$$\rho_{AB} = |11\rangle\langle 11|$$