$$c(t,n) \geq \frac{t\, n^{1+1/t}}{e} - nt$$

Basis $\quad c(1,n)$ ✓

$\quad\quad\quad c(t,1)$ ✓

$c(T,N)$ $\quad \forall n,t \quad n \leq N \ \&\ t < T$

or $n < N \ \&\ t \leq T$

the claim held

$$c(T,N) \geq c(T, N-x) + (N-x) + c(T-1, x)$$

$$\geq \frac{T(N-x)^{1+\frac{1}{T}}}{e} - (N-x)T + (N-x) + \frac{(T-1)\, x^{1+\frac{1}{T-1}}}{e} - x(T-1)$$

$$= \frac{T}{e} N^{1+\frac{1}{T}} \left[\left(1 - \frac{x}{n}\right)^{1+\frac{1}{T}} + \left(1 - \frac{1}{T}\right) \frac{x^{1+1/(T-1)}}{N^{1+1/T}} \right]$$

$$+ \cancel{Tx} + N - \cancel{x} - \cancel{Tx} + \cancel{x} - NT$$

$$= \frac{TN^{1+1/T}}{e} \underbrace{\left[\left(1 - \frac{x}{N}\right)^{1+\frac{1}{T}} + \left(1 - \frac{1}{T}\right) \frac{x^{1+\frac{1}{(T-1)}}}{N^{1+\frac{1}{T}}} + \frac{e}{TN^{1/T}} \right]} - NT$$

if $[-] \geq 1$ then we are done

Geometric Arithmetic Mean Inequality

$$\alpha a + \beta b \geq a^{\alpha} \; b^{\beta}$$

where $\alpha + \beta = 1$

$\alpha, \beta, a, b \geq 0$

$\alpha = 1 - \frac{1}{T} \quad \beta = \frac{1}{T}$

$a = \frac{x^{1 + 1/(T-1)}}{N^{1 + 1/T}} \quad b = \frac{e}{N^{1/T}}$

$2nd + 3rd \geq \left(\frac{x^{T/(T-1)}}{N^{1+1/T}}\right)^{1-\frac{1}{T}} \left(\frac{e}{N^{1/T}}\right)^{1/T}$

$= \frac{x}{N^{1 - 1/T^2}} \cdot \frac{e^{1/T}}{N^{1/T^2}} = \frac{x e^{1/T}}{N}$

$\frac{x(1+\frac{1}{T})}{N}$

the increasing sequence $\left(1+\frac{1}{t}\right)^t$

tends to e as $t \to \infty$

$\left(1+\frac{1}{T}\right)^T < e \Rightarrow 1 + \frac{1}{T} < e^{1/T}$

$$[\ldots] \geq \left(1-\frac{x}{N}\right)^{1+\frac{1}{T}} + \frac{x}{N}\left(1+\frac{1}{T}\right)$$

Bernoulli's inequality

$$(1-\alpha)^t \geq 1-\alpha t \quad \forall t \geq 1$$
$$\forall \alpha \leq 1$$

$$[\ldots] \geq 1-\left(1+\frac{1}{T}\right)\frac{x}{N} + \left(1+\frac{1}{T}\right)\frac{x}{N} = 1$$

$$[\ldots] \geq 1$$

$$c(T,N) \geq \frac{T N^{1+\frac{1}{T}}}{e} - NT$$

If a sorting algorithm uses p processors & runs in t time, then

$$pt \geq c(t,n) \geq \frac{t n^{1+\frac{1}{t}}}{e} - nt$$

$$pt \geq \frac{t n^{1+\frac{1}{t}}}{e} - nt$$

$$p \geq \frac{n^{1+1/t}}{e} - n$$

$$p/n + 1 \geq \frac{n^{1/t}}{e}$$

$$n^{1/t} \leq e(p/n+1)$$

$$\frac{1}{t}\log n \le \log e(p/n+1)$$

$$\frac{\log n}{\log(e(p/n+1))} \le t$$

$$t = \Omega\left(\frac{\log n}{\log(p/n+1)}\right)$$

$$t = \Omega\left(\log_{(p/n+1)} n\right)$$

A lower bound for sorting n items using p/n processors

$p = n$ $\quad p/n = 1$ $\quad p/n+1 = 2$ $\quad t = \Omega(\log_2 n)$

$p = n\log n$ $\quad p/n = \log n$ $\quad t = \Omega\left(\frac{\log n}{\log\log n}\right)$

Cole's Merge Sort

is Optimal

Finding Connected Components of a graph

$G = (V, E)$

undirected graph

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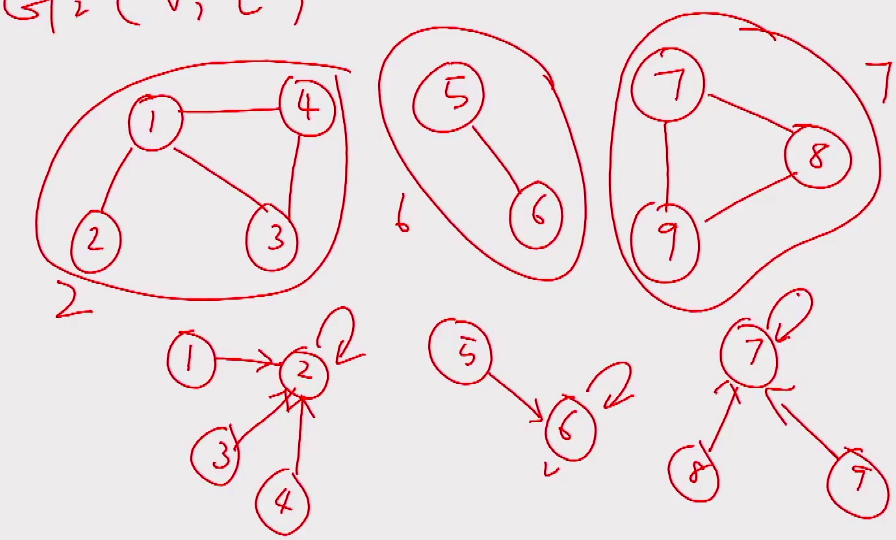
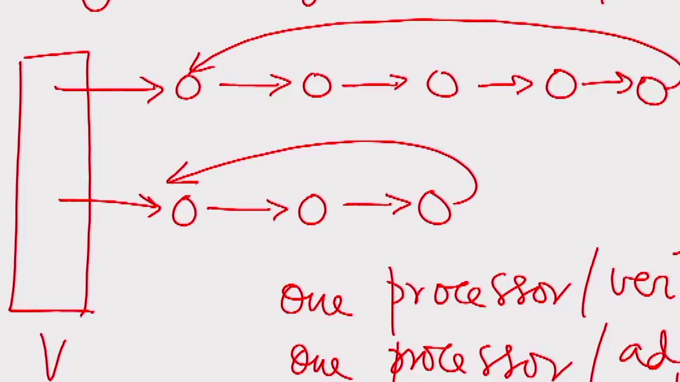
$G = (V, E)$

Define $p(v)$, $\forall v$



G in adjacency list representation



one processor / vertex

one processor / adj list entry

$n + 2m \longrightarrow n + m$ processor.

Step 0

$\forall v \in V$, define $p(v) = v$

Step 1

repeat

1.1 for $(u, v) \in E$ pardo

if (u belongs to star)

if $(p(u) < p(v))$

$p(p(u)) = p(v)$

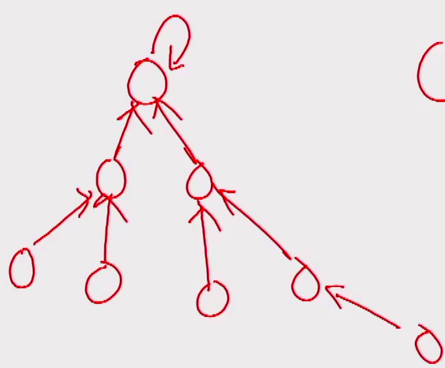
1.2 $\forall u \in V$

$$p(u) = p(p(u))$$

until (for every $(u,v) \in E$, $p(u) == p(v)$)

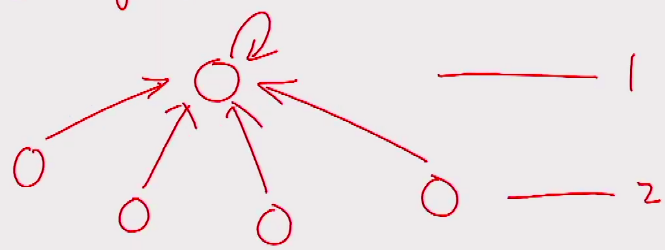
<ins>parent pointers</ins>

initially, $\forall v$, $p(v) = v$



Star graph

Tree of 2 levels



one processor / vertex

detect if it belongs to a star graph

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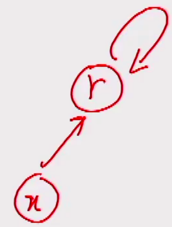
Star check

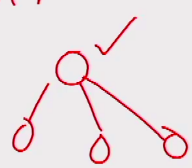


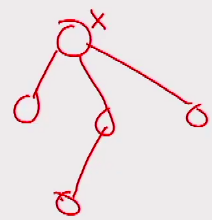
if $(p(v) \neq p(p(v)))$

v is not a star graph

if (v is not in a star)

inform $p(p(v))$ that it is not in a star





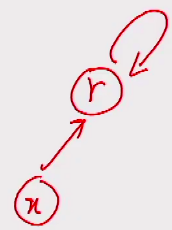
if (v is a child of a root)

check with the parent

Star check

if $(p(v) \neq p(p(v)))$

v is not in a star graph



Step 1

repeat

1.1 for $(u, v) \in E$ pardo

if (u belongs to star)

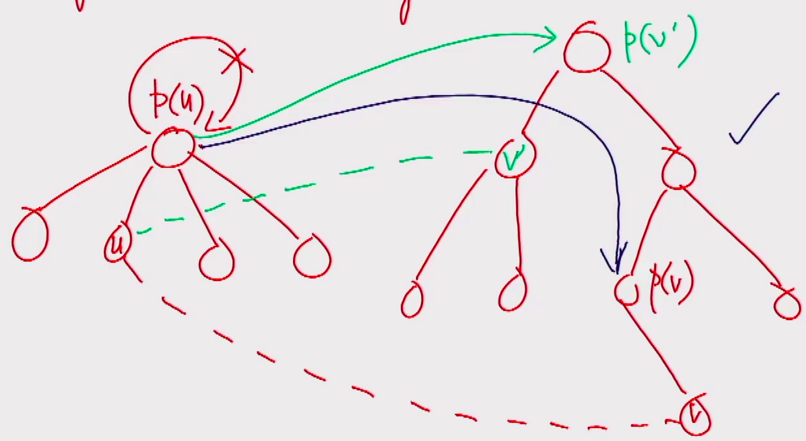
$$\left.\begin{array}{l} \text{if } (p(u) < p(v)) \\ \quad p(p(u)) = p(v) \end{array}\right\} \underline{\text{Hooking}}$$

1.2 $\forall u \in V$

$$\left.\begin{array}{l} \\ p(u) = p(p(u)) \end{array}\right\} \text{Contracting}$$

until (for every $(u,v) \in E$, $p(u) == p(v)$)

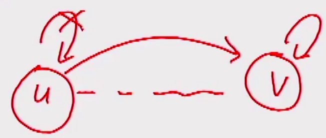
if u belongs to a star



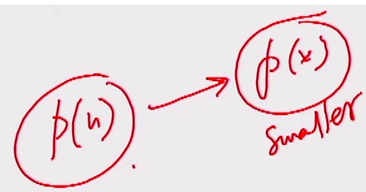
Correctness

No cycle is formed in the hooking step.

$\forall v \quad p(v) \leq v$ $\qquad p(v) = v$ initially



1.1 $\quad p(p(u)) = p(v)$



$\forall (u, v)$ s.t.

u is in a star graph

$p(u) > p(v)$

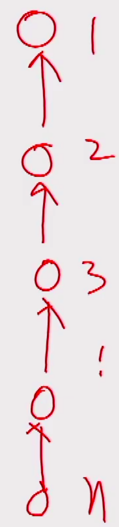
p-graph is a forest

pointer jumping is valid

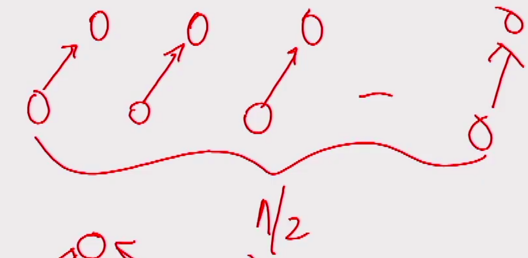
reduces the ht of the tree

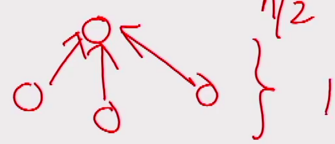
Look at

the total height of all the trees put together the total height $\leq n-1$



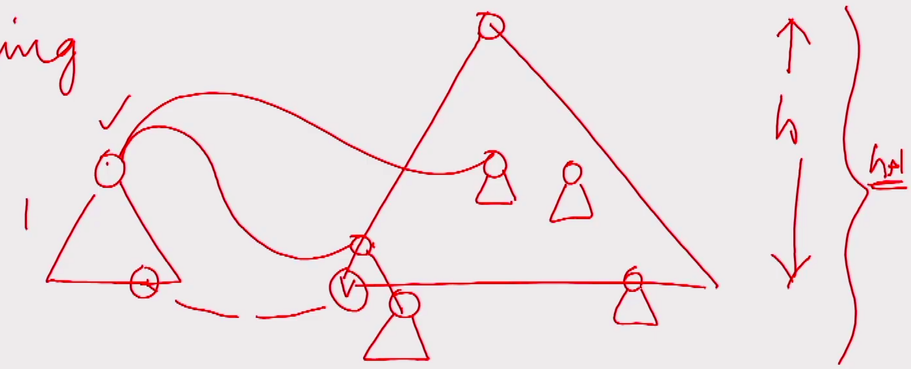
height $n-1$





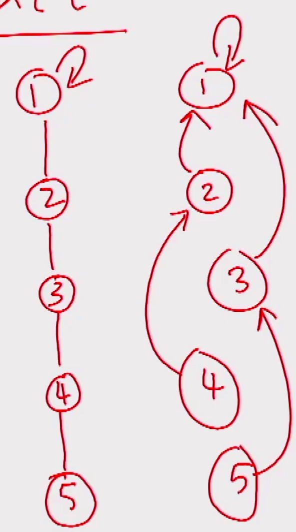
in each iteration
the total height
reduces by $\geq 3/2$

Hooking



h+1

Contract



ht : 4

ht' : 2

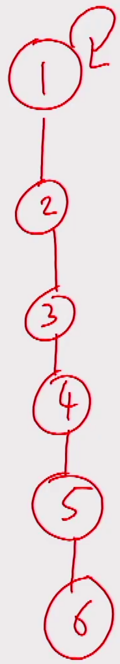
$$\frac{ht}{ht'} = 2$$

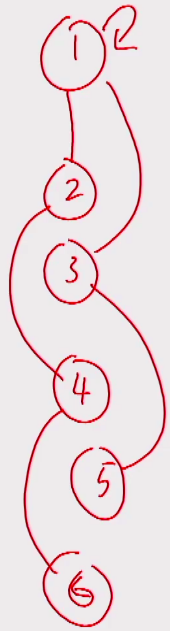
even height

factor 2

$ht = 5$

$\underline{ht = 2n-1}$





$ht' = 3$

$ht' = n$

$$\frac{ht}{ht'} = \frac{5}{3}$$

$$\frac{ht}{ht'} = \frac{2n-1}{n}$$

$n = 2 \rightarrow \underline{\frac{3}{2}}$

total ht reduces by 3/2
in each iter.

initially, ht $\leq n-1$

$\log_{3/2} n = O(\log n)$

CCs can ~~fe~~ be found in
$O(\log n)$ time
$n+m$ processors on
CRCW ARBITRARY
PRAM