Measurement theory

Pure states

$$p(m) = \langle \psi | M_m^{\dagger} M_m | \psi \rangle \rightarrow (a)$$

$$\sum_{m=1}^{n} M_m^{\dagger} M_m = I \rightarrow (b)$$

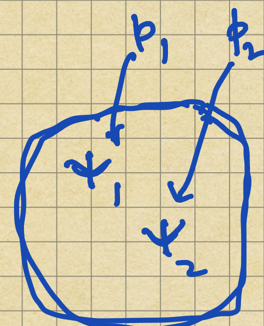
post msmt state : $\dfrac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^{\dagger} M_m | \psi \rangle}}$

$M_m = P_m$

$$\langle M \rangle = E(M) = \sum m\, p(m) = \langle \psi | M | \psi \rangle \rightarrow (c)$$

Mixed state

density operator



$$\rho = \sum_i \underline{\underline{p_i}}\, |\psi_i\rangle\langle\psi_i|$$

$$|\psi_i\rangle = \sum_j c_{ij}\, \underline{\underline{|j\rangle}}$$

what is the probability of finding $|\psi_i\rangle$ in $|j\rangle$

$\longrightarrow |c_{ij}|^2$

The probability to find the mixed state $|\psi\rangle$ in the basis state $|j\rangle$?

$$\sum_i \underline{\underline{p_i}} \, |\underline{\underline{\langle j | \psi_i \rangle}}|^2$$

$$= \sum_i p_i \, \langle \psi_i | j \rangle \, \underline{\langle j | \psi_i \rangle}$$

$$= \langle j | \Big(\sum_i p_i \, \underbrace{|\psi_i\rangle\langle\psi_i|}_{} \Big) | j \rangle$$

$$= \underline{\underline{\langle j | \rho | j \rangle}}$$

$$\underline{\underline{\rho_{jj}}}$$

$$\langle \psi | M_m^{\dagger} M_m | \psi \rangle$$

General msmt

For a mixed state the probability of outcome m is

- $$\underline{\underline{tr\left(M_m^{\dagger} M_m \rho\right)}}$$ ✓✓

- post msmt density operator

$$\checkmark \quad \frac{M_m \rho M_m^\dagger}{tr(M_m^\dagger M_m \rho)}$$

$$tr(M_m^\dagger M_m \rho)$$

$$= tr\left[M_m^\dagger M_m \sum_i p_i |\psi_i\rangle\langle\psi_i| \right]$$

$$= \sum_i p_i \, tr\left(\underbrace{M_m^\dagger M_m |\psi_i\rangle}_{} \langle\psi_i| \right)$$

$$\underline{|\phi\rangle} \; \underline{\langle\psi_i|}$$

$$= \sum_i p_i \langle\psi_i| M_m^\dagger M_m |\psi_i\rangle$$

consider a
One-qubit system

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

Let us measure z-component of spin

$$\sigma_z = \underline{|0\rangle\langle 0|} - \underline{|1\rangle\langle 1|}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M_0 = |0\rangle\langle 0|$$

$$\rho = |\psi\rangle\langle\psi|$$
$$= (a|0\rangle + b|1\rangle)(a^*\langle 0| + b^*\langle 1|)$$
$$= |a|^2\ |0\rangle\langle 0| + ab^*\ |0\rangle\langle 1|$$
$$+ ba^*\ |1\rangle\langle 0| + |b|^2\ |1\rangle\langle 1|$$

$$\mathrm{Tr}(M_0^\dagger M_0 \rho) = \mathrm{Tr}\left[|0\rangle\langle 0|0\rangle\langle 0|(\ \_\_\_\ )\right]$$
$$= |a|^2$$

post msmt $\rho = \dfrac{M_0 \rho M_0^\dagger}{\mathrm{Tr}(M_0^\dagger M_0 \rho)}$

$$= \frac{|0\rangle\langle 0|\ |a|^2}{|a|^2}$$
$$= |0\rangle\langle 0|$$

<u>Repeated measurements</u>

Say we make a msmt on the state $|\psi\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$

Q. (1) First measure $\underline{\{|0\rangle, |1\rangle\}}$ basis

then measure in $\underline{\{|+\rangle, |-\rangle\}}$

(2) First measure $\{|+\rangle, |-\rangle\}$

then measure in $\{|0\rangle, |1\rangle\}$

A. (1) and (2) will not be same.

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

msmt in basis $\{|0\rangle, |1\rangle\}$

get $\underline{\underline{|0\rangle}}$ with probability $\underline{\underline{|a|^2}}$

Now measure in $\{|+\rangle, |-\rangle\}$ basis

get $\underline{\underline{|+\rangle}}$ with probability $\underline{\underline{\frac{1}{2}}}$

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$P(0,+) = \frac{|a|^2}{2} \checkmark$$

$$P(+,0) = ? \quad \frac{|a+b|^2}{4}$$

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle$$

$$= a\,\frac{|+\rangle + |-\rangle}{\sqrt{2}} + b\,\frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$= \frac{a+b}{\sqrt{2}}\,|+\rangle + \frac{a-b}{\sqrt{2}}\,|-\rangle$$

$$p(+) = \frac{|a+b|^2}{2}$$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$: \frac{1}{2}$$

$$P(+,0) = \frac{|a+b|^2}{4} \quad //$$

Positive operator-valued measure

POVM

It is a <u>non-projective</u> measurement.

Unlike projective measurements

| POVM msmt need not commute.

$$\boxed{P_m P_{m'} = P_m \delta_{mm'}}$$

$$|m\rangle \underbrace{\langle m | m' \rangle}_{\delta_{mm'}} \langle m'| = P_m \delta_{mm'}$$

$$\sum_m M_m^\dagger M_m = I$$

$\langle \psi | A | \psi \rangle \geq 0$

POVM

A collection of positive operators $\{E_i\}$

$$\sum_i E_i = I$$

E_i must be Hermitian therefore one can always find out a representation of E_i such that

$$\boxed{E_i = M_i^\dagger M_i} \checkmark \qquad \underline{\underline{M_m}}$$

$$E_i^\dagger = E_i$$

$$p(i) = \langle \psi | E_i | \psi \rangle$$

$$= Tr(\rho E_i)$$

post msmt state

$$\rho \longrightarrow \rho' = \frac{M_i \rho M_i^\dagger}{Tr(\rho E_i)}$$

we get it if we read it

$$\rho \longrightarrow \sum_i M_i \rho M_i^\dagger$$

POVM

1. $E_1 = \frac{1}{2} |0\rangle\langle 0|$ ✓

$E_2 = \frac{1}{2} |1\rangle\langle 1|$ ✓

✓ $E_3 = I - E_1 - E_2$

$\sum E_i = I = E_1 + E_2 + E_3$

n dimen

n msmt operators

2.
$$\begin{cases} E_1 = \dfrac{\sqrt{2}}{\sqrt{2}+1} |1\rangle\langle 1| \\ E_2 = \dfrac{\sqrt{2}}{\sqrt{2}+1} \left[|0\rangle - |1\rangle\right]\left[\langle 0| - \langle 1|\right] \\ E_3 = I - E_2 - E_1 \end{cases}$$

Say, we are given

$$|\psi_1\rangle = |0\rangle \longrightarrow$$
$$|\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

M_0
M_1

projective msmts will not be able to distinguish $|\psi_1\rangle$ and $|\psi_2\rangle$.

POVM

Suppose we get the result

$$E_1 = \frac{\sqrt{2}}{\sqrt{2}+1} |1\rangle\langle 1|$$

clearly the state cannot be $|\psi_1\rangle = |0\rangle$

Because $|\psi_1\rangle$ is orthogonal

to E_1

$\Rightarrow$ if we get the result E_1, the state must be $|\psi_2\rangle$

$E_1 |\psi_1\rangle = 0$

If msmt give the result E_2 then the state cannot be $|\psi_2\rangle$

$$E_2 |\psi_2\rangle = 0$$

$$E_1 |\psi_2\rangle \neq 0$$

If you get $E_1 \longrightarrow \psi_2$

$E_2 \longrightarrow \psi_1$

what about E_3 ?

then we cannot come to a conclusion