① (a)

$$x_{n+1} = 2x_n \qquad 0 \le x_n \le 1/2$$
$$= 2(1-x_n) \qquad 1/2 \le x_n \le 1$$

For $0 \le x_n \le 1/2$

$$x^* = 2x^* \implies x^* = 0$$

And for $1/2 \le x_n \le 1$

$$x^* = 2(1-x^*) \Rightarrow x^* = 2/3$$

For $0 \le x \le 1/2$ we have

$$f'(x^*) = 2 > 1$$

$x^* = 0$ is unstable.

for $x_n \in [1/2, 1]$

$$|f'(x^*)| = 2 > 1$$

$x^* = \frac{2}{3}$ is also unstable.

(b) $x_{n+1} = f(f(x_n))$

$$x_{n+1} = 2(2x_n) \qquad x \in [0, 1/4]$$
$$= 2[2(1-x_n)] \qquad x \in [3/4, 1]$$
$$= 2[1-2x_n] \qquad x \in [1/4, 1/2]$$
$$= 2[1-\{2(1-x_n)\}] \qquad x \in [1/2, 3/4]$$

$$x_{n+1} = 4x_n \qquad x \in [0, 1/4]$$

$$= 2(1-2x_n) \qquad x \in [1/4, 1/2]$$

$$= 2[1-(2-2x_n)] \qquad x \in [1/2, 3/4]$$

$$= 2[2(1-x_n)] \qquad x \in [3/4, 1]$$

$$x^* = 0 \qquad [0, 1/4]$$

$$x^* = 2/5 \qquad [1/4, 1/2]$$

$$x^* = 2/3 \qquad [1/2, 3/4]$$

$$x^* = \frac{4}{5} \qquad [3/4, 1]$$

$$\left.\begin{aligned} f'(x^*) &= 4 && [0, 1/4] \\ &= -4 && [1/4, 1/2] \\ &= 4 && [1/2, 3/4] \\ &= -4 && [3/4, 1] \end{aligned}\right\}$$

unstable as $\boxed{|f'(x^*)| > 1}$

(c) Lyapunov exponent

$$\lambda = \lim_{N \to \infty} \left[\frac{1}{N} \sum_{i=0}^{N-1} \ln \left(|f'(x_i)| \right) \right]$$

$$= \lim_{N \to \infty} \left[\frac{1}{N} \times N \ln 2 \right]$$

$$\Rightarrow \lambda = \ln 2 > 0$$

so the state will be chaotic.

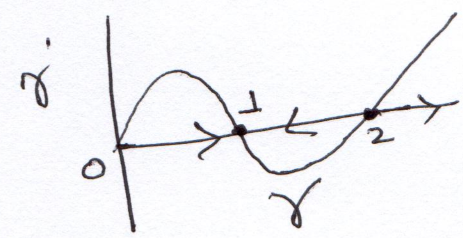
② (a) $\dot{r} = r(1-r^2)(4-r^2)$

$\dot{\theta} = 2 - r^2$

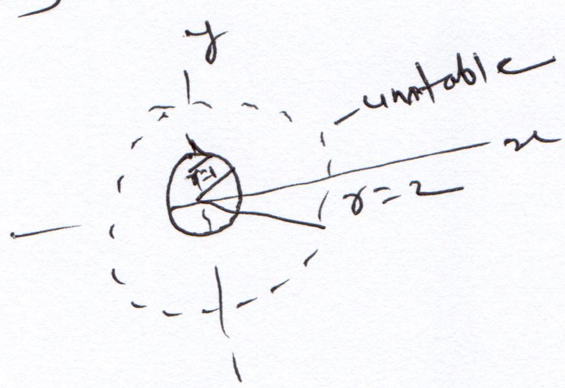
$\dot{\theta} > 0 \qquad r \in [0, \sqrt{2})$

$\dot{\theta} = 0 \qquad r = \sqrt{2}$

$\dot{\theta} < 0 \qquad r > \sqrt{2}$



$\dot{r} = 0 \Rightarrow r = 0, 1, 2$

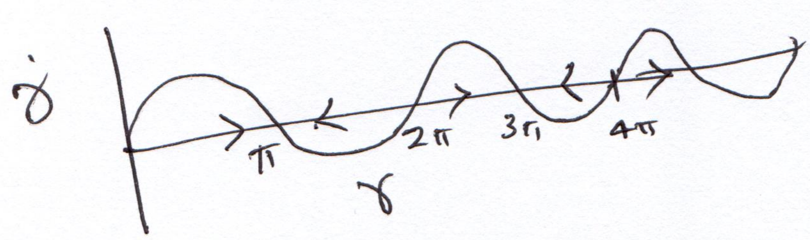


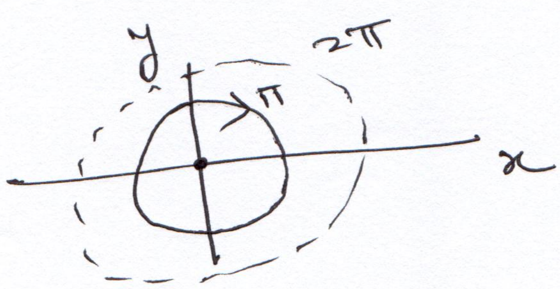
(b) $\dot{r} = r \sin r$

$\dot{\theta} = 1$

$\dot{r} = 0 \Rightarrow r \sin r = 0 \qquad [0, 1, 2, \dots]$

F. Pts. $r^* = n\pi$





(3)

(a) $\dot{x} = \nu x + 2y$ — (1)

$\dot{y} = \nu y + (z-a)x$ — (2)

$\dot{z} = 1 - xy$ — (3)

$$\Delta \cdot f = \frac{\partial}{\partial x}\dot{x} + \frac{\partial}{\partial y}\dot{y} + \frac{\partial}{\partial z}\dot{z}$$

$$= \nu + \nu + 0$$

$$= 2\nu$$

Therefore phase space volume is increasing with time.

(b) $\dot{x} = \dot{y} = \dot{z} = 0$

(1) $\Rightarrow \nu x + 2y = 0 \Rightarrow x = -\frac{1}{\nu} 2y$ — (A)

(2) $\Rightarrow \nu y + (z-a)x = 0 \Rightarrow \nu y = -x(z-a)$ — (B)

(3) $\Rightarrow 1 - xy = 0 \Rightarrow x = 1/y$ — (C)

from (A) & (B) we have

$$z = -\nu / y^2$$

Let $x = K \Rightarrow y = 1/k$, $z = -\nu k^2$

$$\nu y + (z-a)x = 0$$

$$\nu/k + (-\nu k^2 - a)k = 0$$

$$\Rightarrow \nu k^2 + a = \nu k^{-2} \Rightarrow \boxed{\nu[-k^2 + k^{-2}] = a}$$

F.P. $(k, \frac{1}{k}, -\nu k^2)$

for $\nu(k^{-2} - k^2) = a$

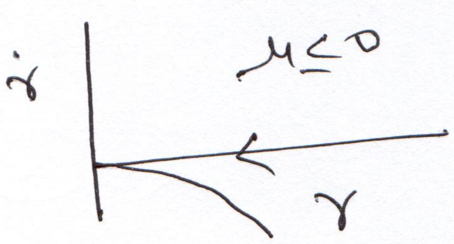
(4) $\dot{r} = \mu r - r^3$

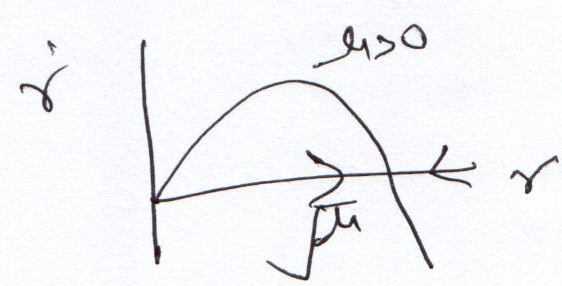
$\dot{\theta} = \omega + b r^2$

$\dot{\theta} > 0$ for $\omega, b > 0$

$\dot{r} = \mu r - r^3$

F. Pts. $r^* = 0, \sqrt{\mu}$





Supercritical Hopf bifurcation.

(5) (a) $x_{n+1} = 3x_n - x_n^3$

$x^* = 3x^* - x^{*3}$

$x^* = 0 \pm \sqrt{2}$

$$f'(x) = \frac{d}{dx}(3x - x^3) = 3 - 3x^2$$

For $x^* = 0$ $\quad f'(x^*) = 3 > 1 \Rightarrow$ unstable

$x^* = \pm\sqrt{2}$ $\quad f'(x^*) = -3$

$|f'(x)| > 1 \Rightarrow$ unstable.

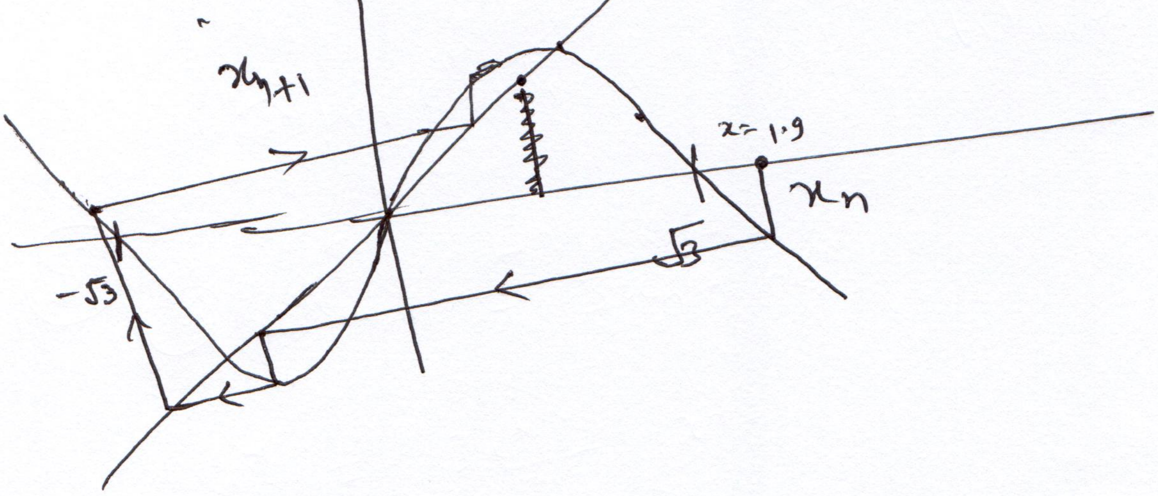
(b) $x_0 = 1.9$

$x_1 = 3 \times 1.9 - 1.9^3 = -1.159$

$x_2 = 3 \times (-1.159) - (-1.159)^3 = -1.92$

$x_3 = 1.319$; $x_4 = 1.622$

$x_5 = 0.3937$ - - - - -



$x_0 = 2.1$

$x_1 = -2.96$, $x_2 = 17.0776$ $x_3 = -4929.37$

6

$\ddot{x} = a - e^x$

$\dot{x}\ddot{x} = a\dot{x} - e^x\dot{x}$

$\ddot{x}\dot{x} - a\dot{x} + e^x\dot{x} = 0$

$\frac{d}{dt}\left[\frac{\dot{x}^2}{2} - ax + e^x\right] = 0$

$E = \frac{\dot{x}^2}{2} - ax + e^x$ is the conserved quantity.

$\dot{x} = y$

$\dot{y} = a - e^x$

$J = \begin{bmatrix} 0 & 1 \\ e^{-x} & 0 \end{bmatrix}$

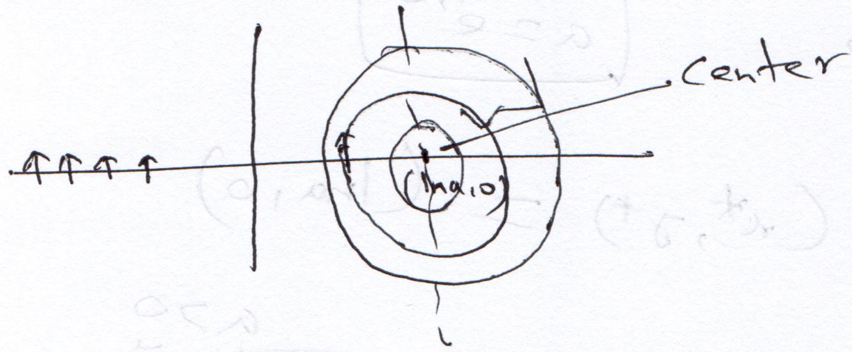
$Tr = 0$; $\Delta = e^x$.

$\lambda = \pm \frac{\sqrt{-4e^x}}{2} = \pm \frac{2ie^{x/2}}{2}$

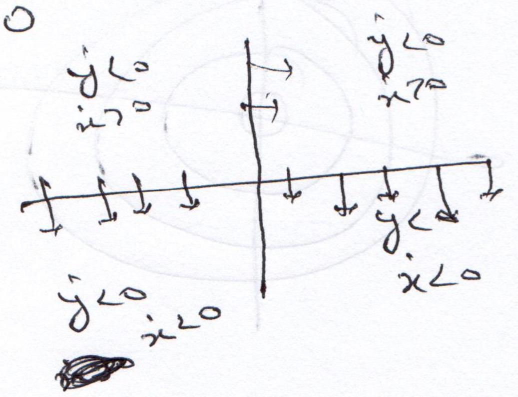
$= \pm ie^{x/2}$

For $a > 0$ $\quad \left.\begin{array}{l}\dot{x}=0 \Rightarrow y=0 \\ \dot{y}=0 \Rightarrow a=e^{x}\end{array}\right\}$

Fixed pts. are $(\ln a, 0)$

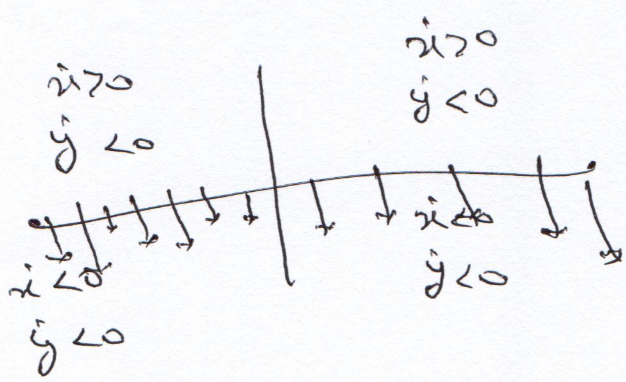


$a < 0$



No fixed pts.

$a = 0$



No fixed pt.

(7) $x_{n+1} = x_n^2 + c$

(a) F. pts. $\Rightarrow$ $x^*_{1,2} = x^*_\pm = \dfrac{1 \pm \sqrt{1-4c}}{2}$ $\quad c \le 1/4$

$f'(x)\big|_{x^*} = 1 \pm \sqrt{1-4c}$

$f'(x)\big|_{x^*_1} = \left|1+\sqrt{1-4c}\right| > 1$ for $c < 1/4$

Therefore ~~x~~ $x = x_1^*$ is unstable F.P.t.

$f'(x)\big|_{x^*_2} = \left|1-\sqrt{1-4c}\right| < 1$

$1 - \sqrt{1-4c} < 1$

$-1 \le 1 - \sqrt{1-4c}$

$-3/4 < c < 1/4$

x_2 is stable for

$$\boxed{-3/4 < c < 1/4}$$

(b) $c > 1/4$: No fixed point

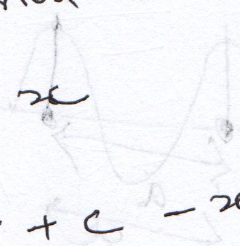
$c = 1/4$: one fixed pt. (Marginal stable)

$-\frac{3}{4} < c < 1/4$: Two fixed (one stable and another unstable).

6 (c) For period - 2 bifurcation, ~~we know~~

$f(f(x^*)) = x$ will give the points

corresponding to period 2.



$(x^2 + c)^2 + c = x$

$x^4 + 2x^2 c + c^2 + c - x = 0$

We know that $x^2 \cancel{+x} + c$ are already a soln. of the above eq.

So, $(x^2 + x + c + 1)(x^2 - x + c) = 0$

Two cycle pts. will be given by

$$x^* = \frac{-1 \pm \sqrt{1 - 4(c+1)}}{2} \qquad c \leq -3/4$$

These pts. will be stable

when $|f'(p)|\,|f'(q)| < 1$

where p, q are roots of $x^2 + x + c + 1 = 0$.

$|4(c+1)| < 1$

$\Rightarrow -\frac{5}{4} \leq c < -3/4$

So period two pts. will become unstable at

$c = -3/4$

For superstable 2-cycle we have

$4(c+1) = 0 \Rightarrow c = -1$.