Counting Compositions

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1 Introduction

In elementary schools (and in Pre-K Montessori schools) children use Cuisenaire rods[3] to learn how numbers fit together. Figure 1 shows ways to build rods of lengths up to 14 with red (length 2) and green (length 3) rods.

To save time and typing, we will often use the term "integer" for a nonnegative integer – the natural numbers $\mathbb{N}$ starting at 1.

A *composition* of an integer n is a way to write n as an ordered sum of integers. For each subset R of integral rod lengths let $C(n, R)$ be the number of compositions of n that use summands only in R. We will always write R in increasing order.

Many interesting combinatorial sequences come up in this context.

- When you can use rods of any integral length, $C(n, \mathbb{N})) = 2^{n-1}$.
- $C(n, \{1\}) = 1$ for all n.
- $C(n, \{1, 2\}) = F_n$, the nth Fibonacci number.

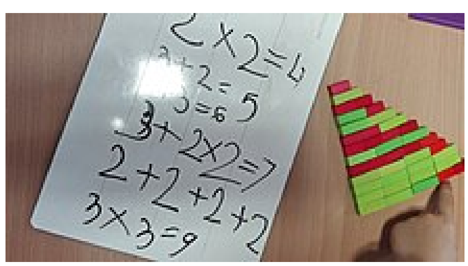

Figure 1: A young child using a 'staircase' of red and green rods to investigate ways of composing the counting numbers

- $C(n, \{s, t\})$ is the number of (nonnegative) integral solutions to the diophantine equation $sx + ty = n$ when the order of the $s + t$ summands is taken into account.

- $C(n, \{2, 3\})$ counts the *Padovan numbers*[**?**][5].

2 Recursion

These sequences and others like them are well studied. Our contribution (if there is one) is an attempt to look at them all at once.

Each of these sequences satisfies a simple linear recursion:

$$C(n, R) = \sum \{C(n - k, R) \mid k < n \text{ and } k \in R\} \tag{2.1}$$

Any such linear recursion grows exponentially at an asymptotically rate $g(R)$.[6]

- $g(\mathbb{N})) = 2$.
- $g(\{1\}) = 1$.
- $g(\{1, 2\}) = \varphi$, the golden mean.
- $g(\{2, 3\}) = 1.3247179572\ldots$, the *plastic number*.

If R is finite define its order $o(R)$ to be its largest element. Then information about $g(R)$ is encoded in the the polynomial

$$p_R(x) = x^{o(R)} - \sum_{k \in R} x^{o(R)-k}$$

For example

$$p_{\{1,2\}}(x) = x^2 - x - 1$$

and

$$p_{\{2,3\}}(x) = x^3 - x - 1.$$

A monic polynomial is a *Cuisenaire polynomial* if the coefficient of each power less than the highest is 0 or -1 and the constant term is -1. Cuisenaire polynomials correspond bijectively to finite recursions defined by Equation 2.

Theorem 1. *For each sequence R the sequence $C(n, R)$ grows exponentially with an asymptotic growth factor $g(R)$ that satisfies $1 \le g(R) \le 2$.*

Proof. $C(n, R)$ does have a finite exponential growth rate $g(R)$, which must be at least 1 since $C(n, \{1\}) = 1$ solutions "grow" at that rate. For any R, $C(n, R)$ grows no faster than $C(n, \mathbb{N})$ so has asymptotic growth rate at most 2.

There's a nice alternative proof for R that uses the Cuisenaire polynomial $p_R(x)$. Its value at 1 can't be positive , since the first term and the last term cancel and the other terms are negative. (It's strictly less than 0 except for the Cuisenaire polynomial $p_{\{1\}}(x) = x + 1$.) The value $p_R(2) < 2$ since it's the difference between $2^{o(R)}$ and an integer less than that (since we have its binary representation). The polynomial is clearly increasing for $x > 2$ so the largest real root is between 1 and 2. $\square$

3 Families

Different sets R can define for which $g(R)$ is the same. $g(\{1,2\}) = \varphi$ since that R generates the Fibonacci numbers. Calculation shows $R = \{1,3,4\}$ and $R = \{1,3,5,6\}$ have the same rate (to many decimal places), though they (necessarily) generate different sequences. We can understand this kind of coincidence several ways.

If you expand the Fibonacci sequence recursion one level deeper you see that

$$F(n) = F(n-1) + F(n-2) = F(n-1) + F(n-3) + F(n-4)$$

so the sequences $\{1,2\}$ and $R = \{1,3,4\}$ define the same recursion with different initial conditions. Asymptotically they grow at the same rate.

The Cuisenaire polynomials are related:

$$\begin{aligned} p_{\{1,3,4\}}(x) &= x^4 - x^3 - x - 1 \\ &= (x^2+1)(x^2-x-1) \\ &= (x^2+1)p_{\{1,2\}}(x). \end{aligned}$$

Therefore $p_{\{1,3,4\}}$ and $p_{\{1,1\}}$ have the same roots, so the same largest real root.

Definition 2. *The sets R and S are g-*equivalent *(or just* equivalent *when $g(R) = g(S)$. This equivalence relation partitions the power set of $\mathbb{N}$ into equivalence classes we call* families *of recursions.*

Our aim is to understand that partition.

Theorem 3. *If R and S are finite and*

$$p_R(x) = v(x)p_S(x)$$

for some polynomial $v(x)$ that is positive when $x > 1$ then $g(R) = g(S)$.

Proof. The Cuisenaire polynomials have the same largest real root. $\square$

Corollary 4. *Suppose*

$$R = \{a, b, \ldots, z, D\}$$

a finite set of order D. Let

$$S = \{a, b, \ldots, z, a+D, b+D, \ldots, z+D, 2D\}.$$

Then $R \equiv S$.

Proof.

$$p_S(x) = (x^D + 1)p_R(x),$$

□

Corollary 5. *The families of g-equivalent sets are infinite.*

These shifts are not the only only way to find equivalent sets of rods. The Padovan sequence offers an example:

$$\begin{aligned} p_{\}1,5\}}(x) &= x^5 - x^4 - 1 \\ &= (x^2 - x + 1)(x^3 - x - 1) \\ &= (x^2 - x + 1)p_{\{2,3\}}(x). \end{aligned}$$

The fact that in this case roots of the factor $v(x)$ are all roots of unity does provide a clue. Before we can take advantage of that fact have to look at Cuisenaire polynomials a little more closely.

The growth rate $g(R)$ is the root of $p_R(x)$ with the largest absolute value, but that polynomial need not be the primitive polynomial for $g(R)$. Equation **??** shows that for $R = \{1, 5\}$.

For a while we hoped that each family would contain one set for which the Cuisenaire polynomial would be the primitive polynomial for the family growth rate, but that fails for the family containg $R = \{7, 8, 9\}$ since

$$\begin{aligned} p_R(x) &= x^9 - x^2 - x - 1 \\ &= (x^2 + 1)(x^7 - x^5 + x^3 - x - 1). \end{aligned}$$

Neither of these irreducible factors is a Cuisenaire polynomial.

That calls for

Definition 6. *For each rod set R let q_R be the minimal polynomial for the growth rate $g(R)$,*

Theorem 7. *Rod sets R and S are equivalent if and only if $q_R(x) = q_S(x) = q(x)$, so the polynomial $q(x)$ really belongs to the family.*

Proof. Straightfowrward. □

Conjecture 8. *For each rod set R in the family $\mathcal{F}$*

$$p_R(x) = v(x)q_mathcalF(x)$$

where all the roots of v are roots of unity in the complex plane.

Moreover, none of the roots of $q_mathcalF(x)$ lie on the unit circle.

Proof. Since $g(R)$ is a root of p_r its minimal polynomial $q_mathcalF$ must be a factor of g_R. T

That the quotient v has only roots of unity for roots remains to be proved.

So does "moreover". □

Question 1. *Can we tell when $p_R = q_mathcalF$? Can we characterize the families for which no R has this property?*

Question 2. *Can we characterize the minimal polynomials $q_mathcalF$? Sometimes they are Cuisenaire polynomials. sometimes not. Can we characterize those algebraic numbers that occur as growth rates?*

4 Infinite rod sets

Infinite rod length sets R do not correspond to recursions of fixed length. But they do exhibit an asympotic growth rate $g(R) \leq 2$. To study those I think we need to move from Cuisenaire polynomials to Cuisenaire generating functions.

I started looking at everything but the rod of length 1. Then I found that for each k the sequence $C(n, \{2, 3, \ldots, k\})$ begins with a stretch of Fibonacci numbers that increases as k increases. This should not be hard to prove.

Below are the growth rates. The first line is the Padovan sequence. The table suggests that as k increases the growth rate increases to the golden ratio.

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[2] 1.0
[2, 3] 1.324717957244746
[2, 3, 4] 1.465571231876768
[2, 3, 4, 5] 1.534157744914267
[2, 3, 4, 5, 6] 1.5701473121960543
[2, 3, 4, 5, 6, 7] 1.590005373901364
[2, 3, 4, 5, 6, 7, 8] 1.6013473337876367
[2, 3, 4, 5, 6, 7, 8, 9] 1.607982727928201
[2, 3, 4, 5, 6, 7, 8, 9, 10] 1.6119303965641198
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11] 1.6143068232571485
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] 1.6157492027552105
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] 1.6166296843945727
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] 1.6171692963550925
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Conjecture 9. *Let R be any infinite sequence of rod lengths and R_k the set of rods lengths in R no longer than k. Then $g(R_k)$ is an increasing sequence bounded by 2, so it has a limit. That limit is $g(R)$.*

Conjecture 10. *Any real number between 1 and 3 is $g(R)$ for some set of rod lengths.*

Question 3. *A set of rod lengths is determined by its characteristic function: a function $d : \mathbb{N} \to \{0, 1\}$ and hence as a path in the infinite complete binary tree. Finite sets R index the internal nodes nodes of the tree — put the value of $g(R)$ there. Then $g(R)$ will be unchanged when you move to the right child and increases when you move left, Does this structure help us understand g?*

References

[1] Number of ways to arrange pairs of integers with distance constraint math.stackexchange.com/questions/4124452/number-of-ways-to-arrange-pairs-of-integers-with-distance-constraint

[2] Miklós Bóna, *A* Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory, 4th edition, World Scientific, 2017.

[3] en.wikipedia.org/wiki/Cuisenaire_rods

[4] en.wikipedia.org/wiki/Padovan_sequence

[5] oeis.org/A000931

[6] Standard reference for linear recursions. Some well known text? Wilf GeneratingFunctionology?