Counting Compositions

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1 Introduction

In elementary schools (and in Pre-K Montessori schools) children use Cuisenaire rods[5] to learn how numbers fit together. Figure 1 shows rod trains of lengths up to 14 with red (length 2) and green (length 3) rods. They are examples of *compositions*: ways to write an integer n as an ordered sum of positive integers. We write $\mathbb{Z}_+$ for the set of positive integers.

Note: *DB: There are lots of other ways to use Cuisinaire rods (eg for fractions) and they are also a rich source of problems for preservice teachers and others (I use them in discrete math at BU, for ex). Not critical now, but when we get closer I would like to rewrite this sentence. Also we should discuss the new title... I think it's too general – e.g. sounds like counting the number of compositions of n. Maybe something with "train" and "family" in the title? Something like "Counting Compositions with Train Families?"*

Note: *DB: Need to decide what we're using for positive and also for non-negative integers, which we need a lot. I like $\mathbb{P}$ and $\mathbb{N}$, but $\mathbb{Z}^+$ is fine as long as we also have $\mathbb{N}$ or something else. I know that it's confusing because $\mathbb{N}$ sometimes includes and sometimes excludes 0. I've never seen $\mathbb{Z}_+$.*

For each subset R of $\mathbb{Z}_+$ let $F(n, R)$ be the number of compositions of n that use summands only in the *rod set* R. We call those compositions *trains* —

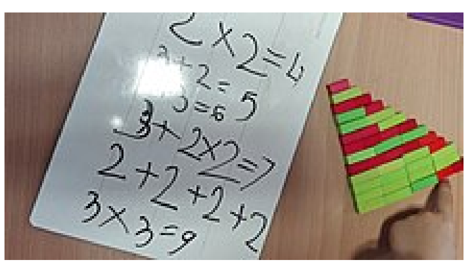

Figure 1: A young child using a 'staircase' of red and green rods to investigate ways of composing the counting numbers

a term a second grader might use when working with Cuisenaire rods. When we want to list the elements of R we will write them in square brackets, usually in nondecreasing order.

- $F(n, [1, 2]) = F_n$, the nth Fibonacci number when you start with initial conditions $F_1 = 1, F_2 = 2$. That's why we chose "F" for the general case.

  Note: *I like redefining fibonacci numbers for the whole paper, but I don't like using "F" for them because F is standard w/different initial conditions.... maybe we should be consistent w/what Art Benjamin does in his book, which is to use $f_n = F_{n+1}$. That would change all the notation, which I'm OK with....*

- When you can use rods of any integral length, $F(n, \mathbb{Z}_+) = 2^{n-1}$.
- $F(n, [1]) = 1$ for all n.
- $F(n, [2, 3])$ counts the *Padovan numbers*[6][7].
- $F(n, [1, 3])$ counts the *Narayana cow numbers*[8].

These sequences and others like them are well studied. Our contribution is an attempt to look at them in families determined by the rate at which they grow as a function of n.

Each of these sequences satisfies a simple linear recursion that counts the number of trains of total length n by looking at the possibilities for the first rod and the rest of the train. For rod set R

$$F(n, R) = \sum_{k \in R} F(n-k, R) \tag{1.1}$$

with initial conditions

$$F(n, R) = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n < 0 \end{cases}$$

Setting $F(0, R) = 1$ counts the train of length n formed by a single rod of length $n \in R$.

We allow several kinds of rods of the same length, so rod sets are multisets. If, for example, there are red and white rods of length 2 and just green rods of length 3 then we write $R = [2, 2, 3]$ and the number of compositions respecting rod colors satisfies the recursion

$$F(n, [2, 2, 3]) = 2F(n-2, [2, 2, 3]) + F(n-3, [2, 2, 3]).$$

In this generalization a rod set is a multiset of positive integers. For infinite rod sets we may sometimes assume a uniform bound on the number of times each positive integer can occur. Equation 1.1 remains unchanged when we interpret the summation over R in the obvious way.

2 Growth rates, generating functions and Cuisenaire polynomials

We want to think about these counts for large n.

Theorem 1. *When* $\gcd(R) = 1$ *the sequence* $F(n, R)$ *grows asymptotically at an exponential rate. That is*

$$\lim_{n\to\infty} \frac{F(n+1, R)}{F(n, R)}$$

exists, and is finite.

Proof. Note: *EB: This is well known. There is a standard proof using linear algebra. We prove it later using generating functions. I would like to offer here a qualitative argument or a convincing heuristic that does not depend on any such machinery. Here's one possibility: shortening rods increases the growth rate. The rod set* R *with* k *rods all of length* 1 *has recursion*

$$F(n, R) = kF(n-1, R)$$

so grows at rate k *so all finite rod sets grow at most exponentially fast.* □

Definition 2. *The* growth rate $g(R)$ *for the rod set* R *is the limit in the previous theorem.*

For example

- $g(\mathbb{Z}_+) = 2$.
- $g([1]) = 1$.
- $g([1, 2]) = \varphi$, the golden mean.
- $g([2, 3]) = 1.3247179572\ldots$, the *plastic number*.

When $\gcd(R) = d > 1$ the only nonzero train counts are for multiples of d, and

$$F(dn, R) = F(n, R/d).$$

We can no longer compute the ratio of successive counts. It's natural in that case to take the ratio of counts d steps apart, and define

$$g(R) = g(R/d)^{1/d}.$$

One way to show that the growth rate for the Fibonacci numbers is the golden mean is to construct their *generating function*. Imagine using them as the coefficients of a formal power series to define

$$\mathcal{F}(x) = 1 + 2x + 3x^2 + 5x^3 + 8x^4 + 13x^5 + \cdots .$$

Then the Fibonacci recursion tells us that

$$\mathcal{F}(x) = x\mathcal{F}(x) + x^2\mathcal{F}(x)$$

so

$$\mathcal{F}(x) = \frac{1}{1 - x - x^2}$$

For a reason that will become clear soon, it's useful to rewrite the denominator in that equation as

$$x^2 \left(\left(\frac{1}{x} \right)^2 - \frac{1}{x} - 1 \right).$$

That construction generalizes.

Definition 3. *If R is finite rod set define its* Cuisenaire polynomial

$$p_R(x) = x^{\max(R)} - \sum_{k \in R} x^{\max(R)-k} \ .$$

These monic integral polynomial in which the coefficient of each power less than the highest is nonpositive and the constant term is nonzero correspond bijectively to finite recursions defined by Equation 1.1. The generating function for the counts $F(n, R)$ is

$$\mathcal{F}_R(x) = \frac{1}{x^{\max(R)} p_R\left(\frac{1}{x}\right)}. \tag{2.1}$$

For example,

$$p_{[1,2]}(x) = x^2 - x - 1,$$

$$p_{[2,3]}(x) = x^3 - x - 1.$$

and

$$p_{[2,2,3]}(x) = x^3 - 2x - 1.$$

Theorem 4. *The growth rate of a finite rod set R is the a unique positive root of its Cuisenaire polynomial.*

Proof. Since the coefficients of the Cuisenaire polynomial change sign just once Descartes' Rule of Signs implies that it has a single positive real root.

Since the generating function $\mathcal{F}_R(x)$ has nonnegative coefficients, Pringsheim's theorem ([3], Theorem IV.6 p 240, [4]) says it has a pole on the real line at its radius of convergence. Equation 2.1 says that pole is the unique positive real root of the Cuisenaire polynomial.

That is the smallest pole of the generating function, which governs the growth rate [3], Theorem (find it).

Alternatively, you can prove the theorem using elementary linear algebra: the roots of the Cuisenaire polynomial are the eigenvalues of a matrix you are computing high powers of. □

With this information we can show that the growth rate of a linear recursion from a rod set that contains no repeated rods is between 1 and 2, The value of the Cuisenaire polynomial $p_R(x)$ at $x = 1$ can't be positive , since the first and last terms cancel and the other terms are negative. (It's strictly less than 0 except for the Cuisenaire polynomial $p_{[1]}(x) = x - 1$.) The value $p_R(2) > 0$ since it is the difference between $2^{\max(R)}$ and the smaller integer whose binary representation is given by the lower order terms. The polynomial is clearly increasing for $x > 2$ so the largest real root is between 1 and 2.

Definition 5. *The rod sets R and S are g-*equivalent *(or just* equivalent*) when $g(R) = g(S)$. This equivalence relation partitions the set of multisets of $\mathbb{Z}_+$ into equivalence classes we call* families *of recursions.*

3 Expansions

Playing with the recursion for the Fibonacci numbers you soon stumble on

$$\begin{aligned} F(n) &= F(n-1) + F(n-2) \\ &= F(n-1) + F(n-2-1) + F(n-2-2) \\ &= F(n-1) + F(n-3) + F(n-4) \end{aligned} \tag{3.1}$$

so the rod sets $[1, 2]$ and $[1, 3, 4]$ define essentially the same recursion.

Question 1. *Does this "essentially" mean just the same recursion but with different initial conditions? Note that the tree theorem below shows that the ratio of [1,3,4] train counts to [1,2] train counts has a known limiting value calculated with the shift polynomial. Perhaps that depends on the smaller eigenvalue.*

Our aim is to understand that play. The first step is formalizing the construction in Equation!3.1.

Definition 6. *A rod set S is an* expansion *of a rod set R if it can be built from R by a sequence of transformations each of which replaces an integer $r \in R$ by the set of integers $r + R$.*

Note: *Overall, I like this way of defining, but I don't think you've stated it correctly. Every time we do the transformation, we're adding the seed, but you're using R to represent both the seed and the intermediate step. I think we need some subscripts and a sequence, each of which is in $\mathcal{F}_R$.*

For example, Equation 3.1 shows that $[1, 3, 4]$ is an expansion of $[1, 2]$. Expanding the 4 there tells us $[1, 3, 5, 6]$ is also an expansion of $[1, 2]$. Expanding the 1 in $[1, 2]$ leads to $[2, 2, 3]$.

The key to proving that expansions produce equivalent rod sets to factor their Cuisenaire polynomials.

Theorem 7. *If R and S are finite rod sets for which*

$$p_S(x) = q(x)p_R(x)$$

for some polynomial $q(x)$ that is positive when $x > 1$ then $g(R) = g(S)$.

Proof. The Cuisenaire polynomials have the same largest real root. □

Corollary 8. *The single step expansion S of $r \in R$ produces an equivalent rod set because*

$$p_S(x) = (x^r + 1)p_R(x)$$

Multistep expansions are a little more subtle, since after the first expansion the rod being expanded need not be in the rod set. For example, $[1, 3, 5, 6]$ is in the Fibonacci family because

$$\begin{aligned} p_{[1,3,5,6]}(x) &= x^6 - x^5 - x^3 - 1 \\ &= (x^4 + x^2 + 1)(x^2 - x - 1) \\ &= (x^4 + x^2 + 1)p_{[1,2]}(x). \end{aligned}$$

We study that subtlety systematically in the next section.

4 Trees

In this section we explore a tree structure that captures the rod sets generated by expansions.

Let $\mathcal{T}(n, R)$ be the set of trains of length n that are counted by $F(n, R)$. There is a natural way to organize the trains in a tree.[1]

Note: *EB: I think we should replace this Narayana example by the Fibonacci tree, since that's been our go to example all along.*

The trains for the two element Narayana rod set $R = [1, 3]$ live in a binary tree. Start by labeling the two children of the root 1 and 3. Then build the tree out recursively by creating two children below each node, appending either a 1 or a 3 to the label. In this infinite tree there will be one node labelled by each finite train of rods built from R. Define the sum of a node to be the sum of the entries in its label — the physical length of the train built from actual Cuisenaire rods.

Figure 4 shows the first five levels of that tree. Each node is tagged with its train (the path from the root) and the sum of the train. Level k contains all the trains with k rods.

Definition 9. *Let R be a rod set with k rods. Then* Tree(R) *is the complete k-ary tree with children at each node labelled by the elements of R. Thus the nodes of* Tree(R) *corresponds to the finite rod trains built from R. Label each node with its train and with the sum of the lengths of the rods in that train. There are then $F(n, R)$ nodes with sum n, divided among the subtrees at the root with $n - j$ of them in the subtree that starts with $j \in R$.*

[1] What an awkward mixed metaphor.

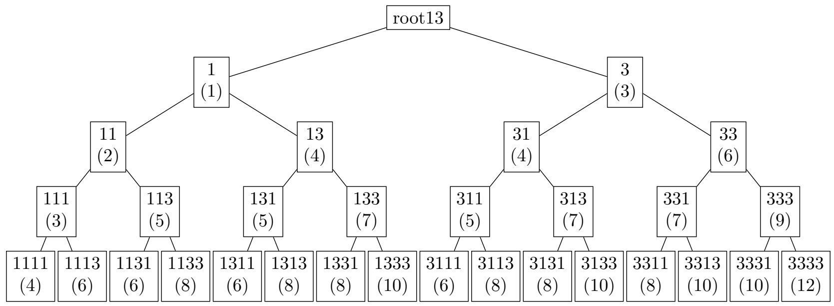

Figure 2: The tree for the rod set $[1, 3]$

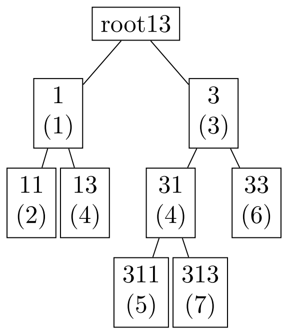

Figure 3: Tree($[1, 3]$) pruned to $[2, 4, 5, 6, 7]$ from subtree $[1, 3, 4]$

Note: *EB: Should the assertions in the previous definition be stated separately in a lemma or theorem with "proof obvious"?*

Suppose we prune Tree(R) by choosing a finite subtree of nodes and then adding all the children of the chosen nodes as leaves

Note: *DB Does this mean that we keep the root? So if want to get [1,4,6] the subtree is root and 3? We have to be careful as when we add rod/antirod pairs, we will often be keeping something attached to the root for the rod set. I think in theorem 11 we should be clearer about what is the "tree," i.e. that it's rooted.*

EB: I think we're OK if all subtrees are rooted and we count the root as an internal node. But let's talk this through to be sure.

Figure 4 shows such a pruning of Tree($[1, 3]$), starting with the subtree with sums $[1, 3, 4]$ and adding the leaves with sums $S = [2, 4, 5, 6, 7]$.

We wrote those leaf sums as a rod set S. In fact, R and S are equivalent. That follows from Theorem 7 and the Cuisenaire polynomial factorization

$$\begin{aligned} p_{[2,4,5,6,7]}(x) &= x^7 - x^5 - x^3 - x^2 - x^1 - 1 \\ &= (x^4 + x^3 + x + 1)(x^3 - x^2 - 1) \\ &= (x^4 + x^3 + x + 1)p_{[1,3]}(x) \end{aligned}$$

We can read off the quotient polynomial from the internal node sums $[1, 3, 4]$ of the pruned tree using the same algebra as that for calculating the Cuisenaire polynomial from the rod set, with $+$ signs instead of $-$ signs:

$$x^4 + x^3 + x + 1 = x^4 + x^{4-1} + x^{4-3} + x^{4-4}.$$

There is a direct way to understand the equivalence that works with actual rod trains. For example, consider

$$\tau = 1331131131331 \in \mathcal{T}(25, R).$$

Parse that into a train of rods from the pruned tree by following the tree until you reach a leaf, then starting again at the root to find

$$\begin{aligned} \tau &= 1331131131331 \\ &= (13)(311)(311)(313)(31) \\ &= 4557(4). \end{aligned}$$

The first four sums correspond to leaves, and so to a train in $\mathcal{T}(21, S)$. The leftover (31) corresponds to an internal node with sum 4. In the theorem that follows we will study this train parsing systematically in order to prove R and S are equivalent.

If there are multiple internal nodes with the same sum this construction will produce a polynomial factor with some coefficients greater than 1. For example, consider the full third level, with sums $[3, 5, 5, 5, 7, 7, 7, 9]$. Then

$$\begin{aligned} p_{[3,5,5,5,7,7,7,9]}(x) &= x^9 - x^6 - 3x^4 - 3x^2 - 1 \\ &= (x^6 + x^5 + x^4 + x^3 + 2x^2 + 1)(x^3 - x^2 - 1) \\ &= (x^6 + x^5 + x^4 + x^3 + 2x^2 + 1)p_{[1,3]}(x). \end{aligned}$$

Definition 10. *A* pruning *of the tree for finite rod set R is a pair $(\mathcal{Q}, \mathcal{S})$ where $\mathcal{Q}$ is a finite subtree and $\mathcal{S}$ is the set of children of nodes in $\mathcal{Q}$ that are not themselves in $\mathcal{Q}$. We write Q and S for the sums at the nodes of $\mathcal{Q}$ and $\mathcal{S}$ when we want to think of them as rod sets.*

Theorem 11. *The rod sets that result from pruning* $\mathrm{Tree}(R)$ *are precisely those generated by expanding R.*

Proof. Use tree induction. The theorem is clearly true for the seed R itself. Suppose it has been proved for some rod set S, which thus corresponds both to an expansion and a pruning of the tree. Then elements of S are the sums in the leaves of the pruning. Expanding one of those sums s using R creates leaf children that correspond precisely to the new elements in the corresponding expansion of $s \in S$ by R. $\square$

Theorem 12. *Let $(\mathcal{Q}, \mathcal{S})$ be a pruning of the tree for rod set R. Then*

$$F(n, R) = F(n, S) + \sum_{i \in Q} F(m - i, S). \tag{4.1}$$

Let $m = \max(Q)$ *and*

$$q(x) = x^m + \sum_{i \in Q} x^{m-i}. \tag{4.2}$$

Then

$$p_S(x) = q(x)p_R(x)$$

so R *and* S *are equivalent.*

Moreover, if γ *is the common growth rate for* R *and* S *then in the limit the train counts for* R *and* S *are proportional:*

$$\lim_{n\to\infty} \frac{F(n,R)}{F(n,S)} = q(1/\gamma). \tag{4.3}$$

Proof. Let $\tau = abc \cdots z$ be a finite rod train built from rods in R. Then τ corresponds to a path from the root in the tree for R. Let σ be the longest initial segment of τ that splits into subsegments ending in $\mathcal{S}$ when you follow the tree from the root, starting again when you reach a node in $\mathcal{S}$. Then leftover rods will end at a node in $\mathcal{Q}$.

Counting the trains in $\mathcal{T}(n,R)$ by grouping them according to which node is left over in this parsing leads to Equation 4.1.

Each term in the sum on the right corresponds to a shift by i in the train counts for S. That multiplies the generating function by x^i. Therefore

$$\begin{aligned}\mathcal{F}_R(x) &= \mathcal{F}_S(x) + \sum_{i\in Q} x^i \mathcal{F}_S(x) \\ &= \left(1 + \sum_{i\in Q} x^i\right)\mathcal{F}_S(x)\end{aligned}$$

Equation 2.1 implies

$$\frac{1}{x^{\max(R)} p_R\left(\frac{1}{x}\right)} = \left(1 + \sum_{i\in Q} x^i\right) \frac{1}{x^{\max(S)} p_S\left(\frac{1}{x}\right)}$$

so

$$p_S\left(\frac{1}{x}\right) = x^{\max(R)-\max(S)} \left(1 + \sum_{i\in Q} x^i\right) p_R\left(\frac{1}{x}\right)$$

Substituting x for $1/x$ and noting that $m = \max(S) - \max(R)$ produces the desired

$$p_S(x) = q(x)p_R(x).$$

Finally, since

$$\lim_{n\to\infty} \frac{F(n-i,S)}{F(n,S)} = 1/\gamma^i,$$

dividing Equation 4.1 by $F(n,S)$ leads to Equation 4.3 □

For example, $[1,2]$ and $[1,3,4]$ are in the Fibonacci family, so

$$\lim_{n\to\infty} \frac{F(n+1,[1,2])}{F(n,[1,2])} = \lim_{n\to\infty} \frac{F(n+1,[1,3,4])}{F(n,[1,3,4])} = \varphi$$

but the fractions approach that limit at a different rate:

$$\lim_{n\to\infty} \frac{F(n+1,[1,2])}{F(n,[1,3,4])} = (1/\varphi)^2 + 1 \approx 1.38.$$

Note: *EB: The next conjecture is a converse to Theorem 12. I think it's true. If it's false we might be able to rescue it with antirods.*

Conjecture 13. *Let $p(x)$ and $s(x)$ be Cuisenaire polynomials such that*

$$s(x) = q(x)p(x)$$

for a polynomial $q(x)$ all of whose coefficients are nonnegative. Then the recursion corresponding to $s(x)$ is a proper pruning of the tree seeded by the recursion corresponding to $p(x)$.

Question 1. *Can we characterize the "rod sets" Q that determine the shift polynomials? For example, for the Fibonacci family, find all the shift polynomials.*

5 Families

Our goal is to characterize the families of recursions. Since expansion increases both the size of a rod set and its maximal element, we might hope that expanding some "small" rod set generates everything.

In the Fibonacci family, $[1,2]$ has the only quadratic Cuisenaire polynomial and is the only rod set with two elements. Unfortunately, expanding it does not find the whole family. [2] The rod set $[1,5,5,5,5,8]$ is in the Fibonacci family because

$$x^8 - x^7 - 4x^3 - 1 = (x^6 + x^4 + x^3 + 2x^2 - x + 1)(x^2 - x - 1).$$

The first factor on the right is positive for $x > 1$ so Theorem 7 applies. But $[1,5,5,5,5,8]$ is not an expansion of $[1,2]$ because every such expansion contains a pair of consective rods.

We can show, however that $x^2 - x - 1$ divides the Cuisenaire polynomial for every recursion in the family as a consequence of the following more general discussion.

Definition 14. *A family $\mathfrak{F}$ is determined by the growth rate $g(\mathfrak{F})$ shared by all the recursions in it. Let $m_{\mathfrak{F}}(x)$ be the minimal polynomial for that algebraic number.*

[2] Or perhaps fortunately, since the consequence makes interesting mathematics.

Theorem 15. *Rod set R is in family $\mathfrak{F}$ if and only if its Cuisenaire polynomial is a multiple of $m_{\mathfrak{F}}(x)$.*

Proof. $m_{\mathfrak{F}}(x)$ divides $p_R(x)$ then $g(\mathfrak{F})$ is a positive root of $p_R(x)$ Theorem 4 says the only such root is $g(R)$.

The converse follows from the fact that if $m(x)$ is the minimal polynomial for an algebraic number ξ then $m(x)$ divides any integral polynomial p(x) which has ξ as a root.

To see why, write

$$p(x) = q(x)m(x) + r(x)$$

where the remainder $r(x)$ has degree less than that of $m(x)$. Substitute ξ to conclude that $r(x)$ is the 0 polynomial.

The growth rate $g(R)$ is the unique positive root of the Cuisenaire polynomial $p_R(x)$, so its minimal polynomial must then be a factor. □

Corollary 16. *If R and S each expand to the same rod set T they and T are all in the same family.*

Question 2. *Is the converse to that corollary true? Do we already have a proof or a counterexample somewhere?*

Corollary 17. *When the minimal polynomial for a family is a Cuisenaire polynomial it is the Cuisenaire polynomial for the unique rod set in that family with minimum maximum rod.*

But the minimal polynomial for a family need not be one of the Cuisenaire polynomials for a rod set in the family. For example,

$$\begin{aligned} p_{[2,2,5]}(x) &= x^5 - 2x^3 - 1 \\ &= (x+1)(x^4 - x^3 - x^2 + x - 1) \\ &= (x+1)p_{[1,2,-3,4]} \end{aligned}$$

so the irreducible polynomial

$$x^4 - x^3 - x^2 + x - 1) = m_{\mathfrak{F}([2,2,5])}(x).$$

That would be the Cuisenaire polynomial for the currently disallowed rod set $[1, 2, -3, 4]$ corresponding to the recursion

$$F(n) = F(n-1) + F(n-2) - F(n-3) + F(n-4).$$

for which the train counts do indeed grow at the same rate as the counts for $[2, 2, 5]$. family. We will see later how to incorporate that rod set in our analysis

Statistically, the Cuisenaire polynomials seem to be irreducible most of the time. We randomly chose 100 rod sets of length 10 with rods between 1 and 100 . In three runs the proportion of irreducible Cuisenaire polynomials 0.90. 0.85, 0.89. The the same experiment with rods between 51 and 100 led to proportions were 0.9, 0.86, 0.91 so the irreducibility seems independent of the size of the rods.

With rod sets of length 20 rather than 10 the proportions were 0.98, 0.99, 1.00. The probability that a Cuisenaire polynomial is irreducible seems to increase with increasing rod set length. We suspect that in the limit almost all Cuisenaire polynomials are irreducible.

Question 3. *Can a family have two rod sets with the same minimum maximum rod? That would require a minimal polynomial $m(x)$ for the growth rate and two polynomials $q_1(x)$ and $q_2(x)$ of the same degree such that both $q_1(x)m(x)$ and $q_2(x)m(x)$ were Cuisenaire polynomials of minimum degree for the family.*

For $[7, 11]$ we have

$$x^{11} - x^4 - 1 = (x^2 - x + 1)(x^9 + x^8 - x^6 - x^5 + x^3 - x - 1).$$

The second factor is irreducible, so it is the minimal polynomial for the family. The first factor shows that the other factor of the Cuisenaire polynomial for the smallest rod in the family can have a negative coefficient.

Note: *EB. Do the Padovan's belong here? We already see that in the Fibonacci family the Cuisenaire polynomial which is the minimal polynomial does not expand to everything. The only extra fact about the Padovan's that's interesting is that there is a second rod set with just two rods.*

6 Antirods

Note: *EB: I think this is the next piece - introduce antirods starting with the recursion and train counts as the difference between numbers of trains with even and odd numbers of antirods. What I do see is how they work in the tree prunings.*

Our work so far seems to have convinced us that we want to use antirods only for intermediate steps in expansions. They should all be zapped by the end of an argument, so that there are no minus signs in our recursions.

None of the rest of the document has been touched yet.

Several strands led us reluctantly to allow negative numbers in rod sets.

We struggled to find a meaning for the negative rod. We first thought using a rod of length -3 might shorten a rod train by 3 units. That didn't work. We eventually discovered that the best way to think of -3 is as an *antirod* of length 3, which we write as $\bar{3}$. (We considered $\bar{3}$, which is less dramatic but better in black and white.)

We can make formal sense of antirods when pruning trees. The one in Figure 12 rooted at $[1, 2, \bar{3}, 4]$ has seven leaves with train lengths $[2, 3, \bar{4}, 5, 2, \bar{3}, 4]$. If we allow [rod, antirod] pairs of the same length to cancel, the leaves specify just the rod set $[2, 2, 5]$. The tree has one internal node with train length 1 that corresponds to the shift polynomial $x + 1$.

Antirods solve another problem. Recall that $[1, 5]$ is in the Padovan family seeded by $[2, 3]$ but not a pruning of the tree seeded by $[2, 3]$. Figure 12 shows

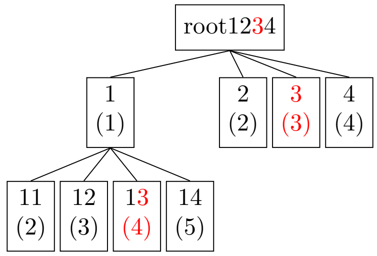

Figure 4: Pruning with antirods.

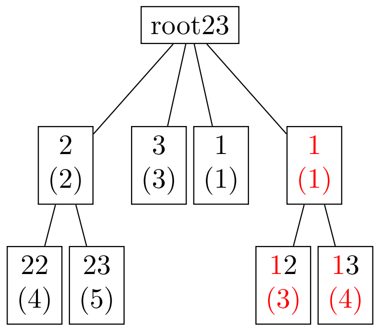

Figure 5: Using a rod/antirod pair.

how to put it in that tree by appending the [rod,antirod] pair $[1, \bar{1}]$ at the root. In the pruning the leaves are $[4, 5, 3, 1, \bar{3}, \bar{4}]$ and the internal nodes are $[\bar{1}, 2]$, confirming the factorization

$$\begin{aligned} p_{[1,5]}(x) &= x^5 - x^4 - 1 \\ &= (x^2 - x + 1)(x^3 - x - 1) \\ &= (x^2 - x + 1)p_{[2,3]}. \end{aligned}$$

The internal antirod node leads to the negative coefficient in the factor x^2-x+1, which is thus not a shift polynomial as we have defined it way above. But I think we should call it one anyway. The new definition is "what you multiply the minimal polynomial by to get the Cuisenaire polynomial."

The rod set $[1, 5, 5, 5, 5, 8]$ is the Fibonacci family because

$$x^8 - x^7 - 4x^3 - 1 = (x^2 - x - 1)(x^6 + x^4 + x^3 + 2x^2 - x + 1)$$

but it is not a pruning of the tree rooted at $[1, 2]$ because every such

7 Polynomials and generating functions

The following conjecture is false. Irreducibility matters. It leads to the need for "negative rods".

Conjecture 18. *The fact that the minimal polynomial may be a proper factor of the Cuisenaire polynomial will turn out to be just a nuisance in the analysis that follows. It won't affect the conclusions of any of the theorems but may complicate the proofs.*

```
That says that although families are infinite, almost all  rod sets
actually seed their own families!
The reducible cuisenaire polynomials may or may not be the
ones that seed their families.
They may be multiples of the seed.
That said, I can use the program to find examples where
they are the seeds - check whether factors have minus signs.
Here are a few (by hand from random sets of length 5 with rods from 1 to 10).

{'spots': [1, 5, 6, 8, 8], 'growthrate': 1.4791859598, 'cpoly':
'-2-x^2-x^3-x^7+ x^8',
'factors': '(x + 1)(x^7 - 2x^6 + 2x^5 - 2x^4 + 2x^3 - 3x^2 + 2x - 2)'}

{'spots': [2, 2, 3, 8, 9], 'growthrate': 1.6407279391, 'cpoly':
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{'spots': [1, 1, 4, 7, 7], 'growthrate': 2.1257731227, 'cpoly':
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Question 4. *How will this extend to infinite rod sets? They still have growth rates and live in families.*

Definition 19. *A* shift polynomial *is a monic integer polynomial all of whose*

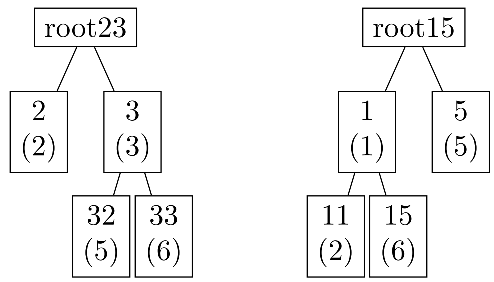

Figure 6: Two pruned Padovan trees that join.

coefficients are nonnegative and whose constant term is positive.

Question 5. *How often do the Cuisenaire and minimal polynomials differ? What are the consequences when they do?*

Equation **??** *suggests that when they do, the Cuisenaire polynomial is a shift polynomial times the minimal polynomial.*

Note: *EB: There are several examples that show that Theorem 12 cannot find all the rod sets starting from the one with Cuisenaire polynomial of least degree. They seem to come in three categories. Sometimes there is no translate by R to unexpand by. Sometimes the Cuisenaire polynomial is not the minimal polynomial. Sometimes the Padovan phenomenon gets in the way. Sometimes the shift polynomial has a negative term.*

We deal with those by getting the rod/antirod stuff right.

I haven't modified the rest of this section.

The Padovan family shows that this theorem cannot find all the rod sets starting from one with the Cuisenaire polynomial of least degree since $[1, 5]$ does not come from a pruning of the tree seeded by $[2, 3]$.

The Padovan family suggests a conjecture. Figure 11 shows that the trees seeded by $[2, 3]$ and $[1, 5]$ can each be pruned to generate the equivalent rod set $[2, 5, 6]$. The Cuisenaire polynomial for that rod set factors in two interesting ways:

$$\begin{aligned} x^6 - x^4 - x^1 - 1 &= (x+1)(x^2 - x + 1)(x^3 - x - 1) \\ &= (x+1)(x^5 - x^4 - 1) = (x+1)p_{[1,5]}(x) \\ &= (x^3 + x + 1)(x^3 - x - 1) = (x^3 + x + 1)p_{[2,3]}(x) \end{aligned}$$

so $[1, 5]$ and $[2, 3]$ are each equivalent to $[2, 5, 6]$ and hence to one another, but neither Cuisenaire polynomial is a multiple of the other.

Conjecture 20. *S results from a sequence of expansions of R using R to expand if and only if it comes from a proper pruning of the tree seeded by R. an expansion of R*

This next conjecture is a corollary.

Conjecture 21. *Rod sets R and S are equivalent if and only if there is a rod set T that comes from a proper pruning of the trees seeded by R and S. In that case there are shift polynomials $a(x)$ and $b(x)$ such that*

$$p_T(x) = a(x)p_R(x) = b(x)p_S(x)$$

Proof. This should follow once we really understand how expansions and prunings are the same. □

Note: *Here is an example to explore. Three seeds of length* 6 *for* $3/4$ *of Debbie's example (not the last one since my calculations stopped at rods of length 12). This is like the Padovans since the c-poly for* $[2, 3, 4, 5, 6, 7]$ *is irreducible and hence the m-poly for the family since it the one of minimal degree.*

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Question 6. *Given $R \equiv S$, what is the smallest T? (fewest number of rods)? Which has the least maximum? We might have to answer these questions to prove the conjecture.*

Note: *It would be interesting to find all the families with just one rod set with a minimal number of rods.*

The Padovans are the first example, maybe the only counterexample with two rod sets R and S each with two rods. I know there is no other example where the quotient of the Cuisenaire polynomials is a trinomial, and no other example with rods of length less than 50.

Conjecture: For any n there are only finitely many families with multiple seeds of size n.

Conjecture 22. *Some of the previous arguments work for infinite rod sets when appropriately modified.*

Question 7. *Are families with multiple seeds rare or common? Can we detect them? Can a family have multiple roots with different cardinalities?*

The rod sets [2.3] and $[1, 5]$ are the only ones of size 2 in the Padovan family. There can't be another set of the form $[2, b]$ since g([a,b]) is strictly monotone in the second entry. Similarly, there can't be a two rod set starting with anything greater than 2. The same argument shows there is at most one two rod set starting with 1. Then $[1, 5]$ happens to work.

In general, in any family there are only finitely many rod sets of any particular finite size, and you can search for them systematically once you know one of them.

Conjecture 23. *In the Fibonacci family the only rod sets with no repeated rods are* $[1, 3, 5, \ldots, 2k+1, 2k]$.

Question 8. *Can we tell when the minimal polynomial for a family is the Cuisenaire polynomial for the family? Can we characterize the minimal polynomials* $m_R(x)$*? Can we characterize those algebraic numbers that occur as growth rates?*

8 Antirods

Note: *EB Starting to think about antirods. When we see the whole picture we can decide whether to incorporate them from the start.*

What I do see is how they work in the tree prunings.

Our work so far seems to have convinced us that we want to use antirods only for intermediate steps in expansions. They should all be zapped by the end of an argument, so that there are no minus signs in our recursions.

Several strands led us reluctantly to allow negative numbers in rod sets.

When the Cuisenaire polynomial is not the minimal polynomial the minimal polynomial can (must?) have positive terms other than the leading term. For example,

$$\begin{aligned} p_{[2,2,5]}(x) &= x^5 - 2x^3 - 1 \\ &= (x+1)(x^4 - x^3 - x^2 + x - 1) \\ &= (x+1)p_{[1,2,-3,4]}. \end{aligned}$$

The rod set $[1, 2, -3, 4]$ corresponds the the recursion

$$F(n) = F(n-1) + F(n-2) - F(n-3) + F(n-4).$$

The train counts for that recursion grow at the same rate as the counts for $[2, 2, 5]$. It is the natural seed for the family.

We struggled to find a meaning for the negative rod. We first thought using a rod of length -3 might shorten a rod train by 3 units. That didn't work. We eventually discovered that the best way to think of -3 is as an *antirod* of length 3, which we write as $\bar{3}$. (We considered $\bar{3}$, which is less dramatic but better in black and white.)

We can make formal sense of antirods when pruning trees. The one in Figure 12 rooted at $[1, 2, \bar{3}, 4]$ has seven leaves with train lengths $[2, 3, \bar{4}, 5, 2, \bar{3}, 4]$. If we allow [rod, antirod] pairs of the same length to cancel, the leaves specify just the rod set $[2, 2, 5]$. The tree has one internal node with train length 1 that corresponds to the shift polynomial $x + 1$.

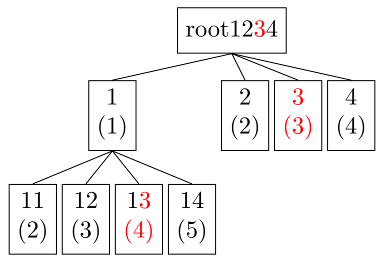

Figure 7: Pruning with antirods.

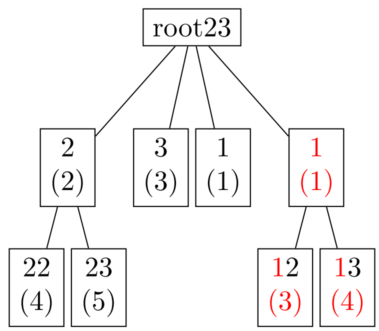

Figure 8: Using a rod/antirod pair.

Antirods solve another problem. Recall that $[1, 5]$ is in the Padovan family seeded by $[2, 3]$ but not a pruning of the tree seeded by $[2, 3]$. Figure 12 shows how to put it in that tree by appending the [rod,antirod] pair $[1, \bar{1}]$ at the root. In the pruning the leaves are $[4, 5, 3, 1, \bar{3}, \bar{4}]$ and the internal nodes are $[\bar{1}, 2]$, confirming the factorization

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9 Polynomials and generating functions

The following conjecture is false. Irreducibility matters. It leads to the need for “negative rods”.

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I randomly chose 100 rod sets of length 10 with rods between 1 and 100 .

In three runs the proportion of irreducible cuisenaire polynomials was 0.90. 0.85, 0.89.
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Question 9. *How will this extend to infinite rod sets? They still have growth rates and live in families.*

Definition 25. *A* shift polynomial *is a monic integer polynomial all of whose coefficients are nonnegative and whose constant term is positive.*

Question 10. *How often do the Cuisenaire and minimal polynomials differ? What are the consequences when they do?*

Equation **??** *suggests that when they do, the Cuisenaire polynomial is a shift polynomial times the minimal polynomial.*

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We deal with those by getting the rod/antirod stuff right.

I haven't modified the rest of this section.

The Padovan family shows that this theorem cannot find all the rod sets starting from one with the Cuisenaire polynomial of least degree since $[1, 5]$ does not come from a pruning of the tree seeded by $[2, 3]$. In fact, the theorem does not even capture the entire Fibonacci family. Let $S = [2, 4, 4, 4, 4, 7]$. Then

$$\begin{aligned}
p_S(x) &= x^7 - x^5 - 4x^3 - 1 \\
&= (x^5 + x^4 + x^3 + 2x^2 - x + 1)(x^2 - x - 1) \\
&= (x^5 + x^4 + x^3 + 2x^2 - x + 1)p_{[1,2]}(x)
\end{aligned}$$

The quotient polynomial is irreducible and positive for positive x, so S is in the Fibonacci family. It does not come from a pruning of the tree seeded by $[1, 2]$ for several reasons. That tree has only four nodes with sum 4 and their children do not make a proper pruning. If there were a proper pruning then the factor in Equation **??** would be a shift polynomial.

The Padovan family suggests a conjecture. Figure 11 shows that the trees seeded by $[2, 3]$ and $[1, 5]$ can each be pruned to generate the equivalent rod set $[2, 5, 6]$. The Cuisenaire polynomial for that rod set factors in two interesting ways:

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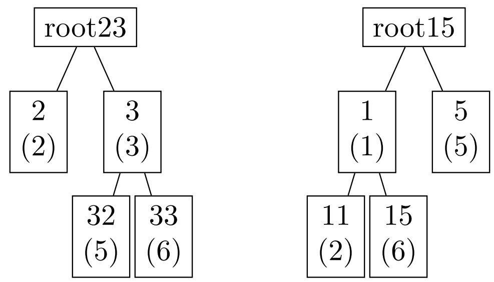
This next conjecture is a corollary.



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Proof. This should follow once we really understand how expansions and prunings are the same. $\square$

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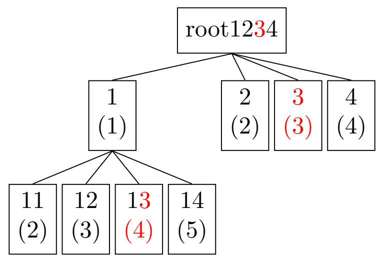

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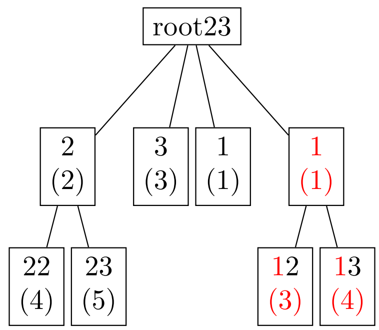

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'factors': '(x + 1)(x^7 - 3x^6 + 2x^5 - 2x^4 + 2x^3 - 2x^2 + x - 1)'}

{'spots': [5, 6, 6, 7, 8], 'growthrate': 1.292620822, 'cpoly':
'-1-x-2x^2-x^3+ x^8',
'factors': '(x + 1)(x^7 - x^6 + x^5 - x^4 + x^3 - 2x^2 - 1)'}
```

{'spots': [2, 2, 7, 8, 9], 'growthrate': 1.5069090112, 'cpoly':
'-1-x-x^2-2x^7+ x^9',
'factors': '(x + 1)(x^8 - x^7 - x^6 + x^5 - x^4 + x^3 - x^2 - 1)'}

Question 14. *How will this extend to infinite rod sets? They still have growth rates and live in families.*

Definition 30. *A* shift polynomial *is a monic integer polynomial all of whose coefficients are nonnegative and whose constant term is positive.*

Question 15. *How often do the Cuisenaire and minimal polynomials differ? What are the consequences when they do?*

Equation **??** *suggests that when they do, the Cuisenaire polynomial is a shift polynomial times the minimal polynomial.*

Note: *EB: There are several examples that show that Theorem 12 cannot find all the rod sets starting from the one with Cuisenaire polynomial of least degree. They seem to come in three categories. Sometimes there is no translate by R to unexpand by. Sometimes the Cuisenaire polynomial is not the minimal polynomial. Sometimes the Padovan phenomenon gets in the way. Sometimes the shift polynomial has a negative term.*

We deal with those by getting the rod/antirod stuff right.

I haven't modified the rest of this section.

The Padovan family shows that this theorem cannot find all the rod sets starting from one with the Cuisenaire polynomial of least degree since $[1, 5]$ does not come from a pruning of the tree seeded by $[2, 3]$. In fact, the theorem does not even capture the entire Fibonacci family. Let $S = [2, 4, 4, 4, 4, 7]$. Then

$$
\begin{aligned}
p_S(x) &= x^7 - x^5 - 4x^3 - 1 \\
&= (x^5 + x^4 + x^3 + 2x^2 - x + 1)(x^2 - x - 1) \\
&= (x^5 + x^4 + x^3 + 2x^2 - x + 1)p_{[1,2]}(x)
\end{aligned}
$$

The quotient polynomial is irreducible and positive for positive x, so S is in the Fibonacci family. It does not come from a pruning of the tree seeded by $[1, 2]$ for several reasons. That tree has only four nodes with sum 4 and their children do not make a proper pruning. If there were a proper pruning then the factor in Equation **??** would be a shift polynomial.

The Padovan family suggests a conjecture. Figure 11 shows that the trees seeded by $[2, 3]$ and $[1, 5]$ can each be pruned to generate the equivalent rod set $[2, 5, 6]$. The Cuisenaire polynomial for that rod set factors in two interesting ways:

$$
\begin{aligned}
x^6 - x^4 - x^1 - 1 &= (x + 1)(x^2 - x + 1)(x^3 - x - 1) \\
&= (x + 1)(x^5 - x^4 - 1) = (x + 1)p_{[1,5]}(x) \\
&= (x^3 + x + 1)(x^3 - x - 1) = (x^3 + x + 1)p_{[2,3]}(x)
\end{aligned}
$$

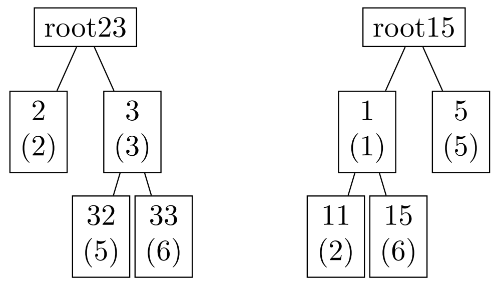

Figure 12: Two pruned Padovan trees that join.

so $[1, 5]$ and $[2, 3]$ are each equivalent to $[2, 5, 6]$ and hence to one another, but neither Cuisenaire polynomial is a multiple of the other.

Conjecture 31. *Rod sets R and S are equivalent if and only if there is a rod set T that comes from a proper pruning of the trees seeded by R and S. In that case there are shift polynomials $a(x)$ and $b(x)$ such that*

$$p_T(x) = a(x)p_R(x) = b(x)p_S(x)$$

Proof. This should follow once we really understand how expansions and prunings are the same. $\square$

Note: *Here is an example to explore. Three seeds of length 6 for 3/4 of Debbie's example (not the last one since my calculations stopped at rods of length 12). This is like the Padovans since the c-poly for $[2, 3, 4, 5, 6, 7]$ is irreducible and hence the m-poly for the family since it the one of minimal degree.*

```
[1, 3, 6, 7, 9, 11]
 x**11-x**10-x**8-x**5-x**4-x**2-1
 (x**4 - x**3 + x**2 - x + 1)*(x**7 - x**5 - x**4 - x**3 - x**2 - x - 1)

[1, 4, 5, 6, 7, 9]
x**9-x**8-x**5-x**4-x**3-x**2-1
(x**2 - x + 1)*(x**7 - x**5 - x**4 - x**3 - x**2 - x - 1)

[2, 3, 4, 5, 6, 7]
x**7-x**5-x**4-x**3-x**2-x**1-1
x**7 - x**5 - x**4 - x**3 - x**2 - x - 1
```

Question 16. *Given $R \equiv S$, what is the smallest T? (fewest number of rods)? Which has the least maximum? We might have to answer these questions to prove the conjecture.*

Note: *It would be interesting to find all the families with just one rod set with a minimal number of rods.*

The Padovans are the first example, maybe the only counterexample with two rod sets R and S each with two rods. I know there is no other example where the quotient of the Cuisenaire polynomials is a trinomial, and no other example with rods of length less than 50*.*

Conjecture: For any n there are only finitely many families with multiple seeds of size n.

Conjecture 32. *Some of the previous arguments work for infinite rod sets when appropriately modified.*

Question 17. *Are families with multiple seeds rare or common? Can we detect them? Can a family have multiple roots with different cardinalities?*

The rod sets [2.3] and $[1, 5]$ are the only ones of size 2 in the Padovan family. There can't be another set of the form $[2, b]$ since g([a,b]) is strictly monotone in the second entry. Similarly, there can't be a two rod set starting with anything greater than 2. The same argument shows there is at most one two rod set starting with 1. Then $[1, 5]$ happens to work.

In general, in any family there are only finitely many rod sets of any particular finite size, and you can search for them systematically once you know one of them.

Conjecture 33. *In the Fibonacci family the only rod sets with no repeated rods are* $[1, 3, 5, \ldots, 2k + 1, 2k]$.

Question 18. *Can we tell when the minimal polynomial for a family is the Cuisenaire polynomial for the family? Can we characterize the minimal polynomials $m_R(x)$? Can we characterize those algebraic numbers that occur as growth rates?*

12 Antirods

Note: *EB Starting to think about antirods. When we see the whole picture we can decide whether to incorporate them from the start.*

What I do see is how they work in the tree prunings.

Our work so far seems to have convinced us that we want to use antirods only for intermediate steps in expansions. They should all be zapped by the end of an argument, so that there are no minus signs in our recursions.

Several strands led us reluctantly to allow negative numbers in rod sets.

When the Cuisenaire polynomial is not the minimal polynomial the minimal polynomial can (must?) have positive terms other than the leading term. For example,

$$\begin{aligned} p_{[2,2,5]}(x) &= x^5 - 2x^3 - 1 \\ &= (x + 1)(x^4 - x^3 - x^2 + x - 1) \\ &= (x + 1)p_{[1,2,-3,4]}. \end{aligned}$$

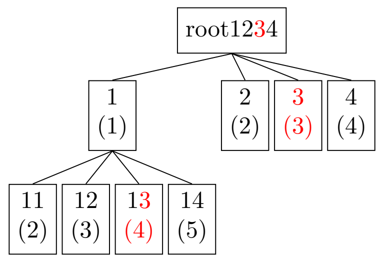

Figure 13: Pruning with antirods.

The rod set $[1, 2, -3, 4]$ corresponds the the recursion

$$F(n) = F(n-1) + F(n-2) - F(n-3) + F(n-4).$$

The train counts for that recursion grow at the same rate as the counts for $[2, 2, 5]$. It is the natural seed for the family.

We struggled to find a meaning for the negative rod. We first thought using a rod of length -3 might shorten a rod train by 3 units. That didn't work. We eventually discovered that the best way to think of -3 is as an *antirod* of length 3, which we write as $\bar{3}$. (We considered $\bar{3}$, which is less dramatic but better in black and white.)

We can make formal sense of antirods when pruning trees. The one in Figure 12 rooted at $[1, 2, \bar{3}, 4]$ has seven leaves with train lengths $[2, 3, \bar{4}, 5, 2, \bar{3}, 4]$. If we allow [rod, antirod] pairs of the same length to cancel, the leaves specify just the rod set $[2, 2, 5]$. The tree has one internal node with train length 1 that corresponds to the shift polynomial $x + 1$.

Antirods solve another problem. Recall that $[1, 5]$ is in the Padovan family seeded by $[2, 3]$ but not a pruning of the tree seeded by $[2, 3]$. Figure 12 shows how to put it in that tree by appending the [rod,antirod] pair $[1, \bar{1}]$ at the root. In the pruning the leaves are $[4, 5, 3, 1, \bar{3}, \bar{4}]$ and the internal nodes are $[\bar{1}, 2]$, confirming the factorization

$$\begin{aligned} p_{[1,5]}(x) &= x^5 - x^4 - 1 \\ &= (x^2 - x + 1)(x^3 - x - 1) \\ &= (x^2 - x + 1)p_{[2,3]}. \end{aligned}$$

The internal antirod node leads to the negative coefficient in the factor $x^2 - x + 1$, which is thus not a shift polynomial as we have defined it way above. But I think we should call it one anyway. The new definition is "what you multiply the minimal polynomial by to get the Cuisenaire polynomial."

The rod set $[1, 5, 5, 5, 5, 8]$ is the Fibonacci family because

$$x^8 - x^7 - 4x^3 - 1 = (x^2 - x - 1)(x^6 + x^4 + x^3 + 2x^2 - x + 1)$$

but it is not a pruning of the tree rooted at $[1, 2]$ because every such rod set will contain a pair of consecutive rods. But you can find a pruning if you follow

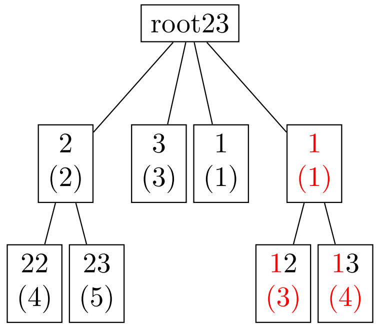

Figure 14: Using a rod/antirod pair.

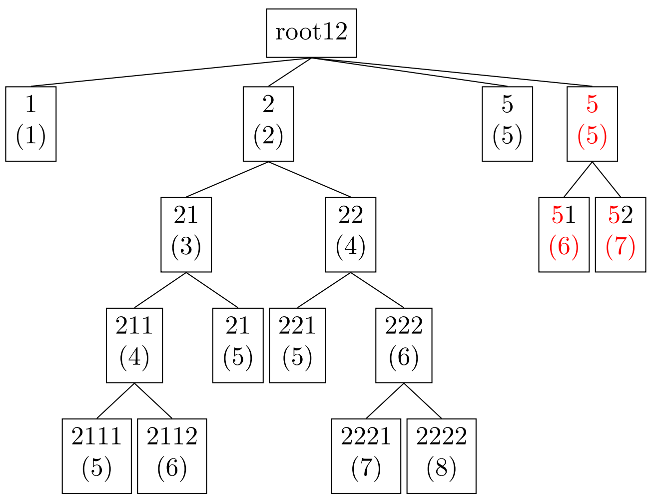

Figure 15: Using a rod/antirod pair in the Fibonacci family.

the instructions implicit in the factorization: you add $[5, \bar{5}]$ and expand as in Figure 12.

I went looking for an example with two antirods. I found none in the family database I built, but figured out how to manufacture one, by using a cyclotomic polynomial (all roots of absolute value 1) as a shift polynomial.

For example, using the cyclotomic polynomial $x^4 - x^3 + x^2 - x + 2$ (which happens to be the polynomial whose roots are the primitive 10th roots of unity) as a factor in

$$(x^4 - x^3 + x^2 - x + 2)(x^2 - x - 1) = x^6 - 2x^5 + x^4 - x^3 + 2x^2 - x - 2$$

says that $[1, 1, -2, 3, -4, -4, 5, 6, 6]$ is in the Fibonacci family, and suggests that is a pruning of the tree rooted at $[1, 2]$ if you expand using rod/antirod pairs $[1, \bar{1}]$ and $[3, \bar{3}]$ as well as $[1, 2]$ at the root.

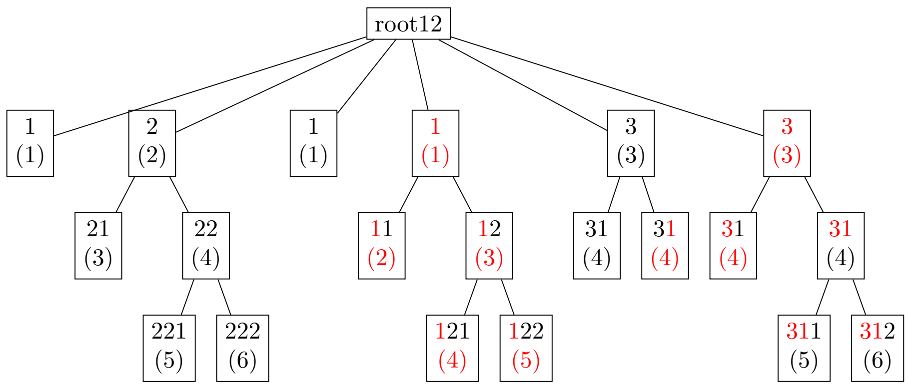

Figure 16: A complex member of the Fibonacci family.

Figure 12 shows one way that works. The leaves are

$$[1, 3, 5, 6, 1, \bar{2}, \bar{4}, \bar{5}, 4, \bar{4}, \bar{4}, 5, 6] = [1, 1, \bar{2}, 3, \bar{4}, \bar{4}, 5, 6].$$

The internal nodes are

$$[2, 4, \bar{1}, \bar{3}, 3, \bar{3}, 4] = [\bar{1}, 2, \bar{3}, 4, 4],$$

It illustrates two new interesting features. First, there is rod/antirod cancellation both for the leaves and for the internal nodes. Second, in the rightmost part of the tree we see that the train $\bar{3}\bar{1}2$ counts as an ordinary rod of length 6 since the number of antirods in the train is even.

This tree may not be optimal. I found it by trial and error. You will see one expansion of 3 by the pair $[1, \bar{1}]$ leading to leaves (31) and $3\bar{1}$ of length 4 that then just cancel. I used that hack to convert the 3 from a leaf to an internal node. There's a third expansion by $[1, \bar{1}]$ for the last $\overline{(3)}$.

Note that $1, 2, 3, 4$ appear with opposite parity in the leaves and the internal nodes. Is this a coincidence?

The next step is to generalize Theorem 12. I'm sure it will work, since the proof comes directly from the recursion. When we think it through we should see just where the fact that you take the difference between the counts for the trains with an even and an odd number of antirods. That may call for an extra hypothesis.

When we understand the proof we might also see what happens when you use a shift polynomial with a root that's larger in absolute value than the growth factor from the Cuisenaire polynomial. (Debbie says she knows something about this.)

Definition 34. *A minimal/shift polynomial is* pure *if it has no positive/negative lower order terms. (Equivalent: the corresponding rod sets have no antirods.)*

Conjecture 35. *A rod set is in a family if and only if it is a pruning of the tree seeded by the rod set corresponding to the minimal polynomial for the growth rate of the family.*

We know that a rod set is in that family if and only if its Cuisenaire polynomial is a multiple of the minimal polynomial for the family by a factor (we call it a shift polynomial even though it may have negative coefficients) whose largest root is no larger than the growth factor.

The proof will need an algorithm with input the shift polynomial (that is, the factor by which you multiply the minimal polynomial for the family growth rate) that produces the pruning.

I think this is easy when everything is pure:

Conjecture 36. *If the shift polynomial and the minimal polynomial are both pure then start the tree with the seeding rod set at the root and recursively expand everything that appears in the shift rod set. You will be left with leaves that exactly correspond to the target rod set.*

Proof. Should be straightforward. It works on the random examples I tried — for example, $[3, 4, 6, 7, 9, 10]$ is in the $[1, 4]$ family because its Cuisenaire polynomial factors as

$$(x^4 - x^3 - 1)(x^6 + x^5 + x^4 + x + 1).$$

The shift polynomial corresponds to the rod set $[1, 2, 5, 6]$. If you start the tree seeded by $[1, 4]$ and expand the nodes in that rod set (to make them internal) you end up with just what you expect. □

How do we generalize these results when the shift or minimal polynomial is impure? This may help: you can always add rod/antirod pair.

Lemma 37. *For any rod set R and any n the rod sets $R \cup \{\pm n\} \cup R \mp n$ is in the R family.*

Proof. Expand one of the nodes $\pm n$. Then that node becomes the only internal node in the tree and the shift polynomial is $x^n \mp 1$, whose roots are roots of 1 and so have modulus 1. □

Here's an example with both impure:

$$(x^4 - x^3 + x^2 - x + 2)(x^4 - x^3 - x^2 + x - 1) = x^8 - 2x^7 + x^6 + x^3 - 4x^2 + 3x - 2$$

The root rod set is $[1, 2, -3, 4]$, the target is $[1, 1, -2, -5, 6, 6, 6, -7, -7, -7, 8, 8]$ and the internal nodes are $[-1, 2, -3, 4, 4]$. The only internal node missing from the root rod set is -1. If you start at the root with edges to an extra rod/antirod pair, so with $[1, 2, -3, 4, 1, -1]$, then expanding what you should finds the target in the leaves.

Conjecture 38. *Add any rod/antirod pairs you need in order to create nodes you can then expand to match the prescribed internal nodes.*

That works for the complex Fibonacci example in Figure 12, but only with a hack at the end introducing another rod/antirod pair.

It works too for Fibonacci target $[1, 5, 5, 5, 8]$ but the tree is less compact than the one in Figure 12. That one uses just a single $[5, \bar{5}]$ pair rather than four and an $[8, \bar{8}]$ pair.

How do we avoid dealing with trees like this which wants to produce something in the Fibonacci family?

```
      root
     /    \
    3     -3
   / \
  4   5
```

which leads to leaf rod set $[-3, 4, 5]$. The Cuisenaire polynomial factors as $(x-1)(x+1)(x^3+x+1)$? The largest root is complex with absolute value 1.21.

Here's an example we have to understand. Start with the polynomial

$$x^2 - 3x + 2 = (x-1)(x-2)$$

corresponding to the rod set $[1, 1, 1, \bar{2}, \bar{2}]$.

There are two ways to see this as a pruned tree.

If you think of $x - 1$ as the Cuisenaire polynomial then $[1]$ is the root and the rod set corresponding to the internal nodes for the shift polynomial $x - 2$ is $[\bar{1}, \bar{1}]$. Then the pruning is

```
 1    1   -1    1   -1
           |         |
          -2        -2
```

If you think of $x - 2$ as the Cuisenaire polynomial then $[1, 1]$ is the root and the rod set corresponding to the internal nodes for the shift polynomial $x - 1$ is $[\bar{1}]$. Then the pruning is

```
1   1    1   -1
            / \
          -2 -2
```

What is going on with the actual recursions?

If you think the problem comes from the antirods in the shift polynomial rod set you can ask the same kind of questions about

$$(x-3)(x-2)(x-1).$$

There each of the three possible shift polynomials has a positive constant term, so no antirod for the maximum internal node.

13 Periodicity

Note: *EB This section needs to be rewritten using expansions — I think they will explain all the arithmetic progression theorems.*

We can construct some infinite rod sets with a periodicity argument.

Theorem 39. *Let B be a set of distinct positive representatives of equivalence classes modulo m. Suppose $m \notin B$. Let R be the union of the arithmetic progressions of period m starting at the elements of B. Then Then $g(R) = g(B \cup \{m\})$.*

Note: EB Here's the old statement:

> *Let B be a finite rod set and $m \notin B$ such that for all $k \in \mathbb{Z}_+$ the sets B and $B + mk$ are disjoint. Let*
>
> $$R = B \cup (B + m) \cup (B + 2m) \cup \cdots .$$
>
> *Then $g(R) = g(B \cup \{m\})$.*

Proof. Note: *EB: Rewrite proof to match new statement.* Temporarily generalize our recursion notation so that for a subet $A \subset \mathbb{Z}_+$

$$F(n, A, R)^* = \sum_{k \in A} F(n - k, R),$$

Then

$$\begin{aligned} F(n, R) &= F(n, B, R)^* + F(n, B + m, R)^* + F(n, B + 2m, R)^* + \cdots \\ &= F(n, B, R)^* + F(n - m, B, R)^* + F(n - 2m, B, R)^* + \cdots \\ &= F(n, B, R)^* + F(n - m, R) \\ &= \sum_{k \in B} F(n - k, R) + F(n - m, R) \\ &= \sum_{k \in B \cup \{m\}} F(n - k, R). \end{aligned}$$

That is the same recursion satisfied by $F(n, B \cup \{m\})$, so $g(R) = g(B \cup \{m\})$. □

Corollary 40. *When $b > 1$ and $m \in \mathbb{Z}_+$ then for $B = \{b\}$ the rod set R is the arithmetic progression $[b, b + m, b + 2m, \ldots]$ and its growth rate is $g([b, m])$.*

So

$$g([2, 3, 4, \ldots]) = g([1, 2]) = \varphi$$

and

$$g([3, 4, \ldots]) = g([2, 3]),$$

the growth rate of the Padovan sequence.

Whenever $m > \max(B)$ the hypotheses are satisfied. For example, with $B = [1, 3]$ and $m = 4$ and then 5

$$g([1, 3, 5, 7, 9, 11, \ldots]) = g([1, 3, 4] = \varphi$$

and

$$g([3, 5, 7, 91, 3, 6, 8, 11, 13, 16, 18,]) = g([1, 3, 5]) = 1.570147\ldots .$$

More interesting is this example with $B = [1, 4]$ and $m = 2 < 4 = \max(B)$. Then $R = [1, 3, 4, 5, 6, 7, \ldots] = \mathbb{Z}_+ \backslash \{2\}$ and

$$g(\mathbb{Z}_+ \backslash \{2\}) = g([1, 2, 4]) = 1.754877 \ldots.$$

That example suggests the following theorem.

Theorem 41. *For each $k \in \mathbb{Z}_+$*

$$g(\mathbb{Z}_+ \backslash \{k\}) = g([1, 2, \ldots, k-1, k+1, k+2, \ldots]) = g([1, 2, \ldots, k, 2k])$$

Proof. Note: *EB Debbie asserts this and I've checked it numerically but I don't see a proof using the previous theorem. So provide one, or figure out what generalization is the one we need.* □

Question 19. *Can we say anything in general about cofinite rod sets? They always end with the sequence of integers greater than m for some m. Maybe cofinite sets that omit a gap.*

For example,

```
g([1,5,6,7,8,...,79] = 1.5289463545197037
g([1, 4, 6, 7, 8]) =   1.528946354519709

g([1,6,7,8,...,79]) = 1.4655712318766887
g([1, 5, 6, 8, 10]) = 1.4655712318767704
(in the Narayana family so maybe there's a better example)
```

Note: *EB I don't know where this assertion belongs.*

The rod sequence R containing all rods of odd length except the rod of length 1 corresponds to $B = \{3\}$ and $m = 2$, so $g(R) = g(\{2, 3\})$, the growth rate of the Padovan sequence.

So far we have been discussing infinite rod sequences with an eventual periodic structure. Now we turn to more general infinite rod sequences.

Conjecture 42. *Let R be any infinite sequence of rod lengths and R_k the set of rods lengths in R no longer than k. Then $g(R_k)$ is an increasing sequence bounded by 2, so it has a limit. That limit is $g(R)$.*

Here's some evidence for that very likely conjecture. It shows a neat kind of interpolation between a Padovan recurrence and a Fibonacci recurrence. Below are the growth rates. The first line is the Padovan

```
[2, 3] 1.324717957244746
[2, 3, 4] 1.465571231876768
[2, 3, 4, 5] 1.534157744914267
[2, 3, 4, 5, 6] 1.5701473121960543
[2, 3, 4, 5, 6, 7] 1.590005373901364
[2, 3, 4, 5, 6, 7, 8] 1.6013473337876367
[2, 3, 4, 5, 6, 7, 8, 9] 1.607982727928201
```

```
[2, 3, 4, 5, 6, 7, 8, 9, 10] 1.6119303965641198
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11] 1.6143068232571485
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] 1.6157492027552105
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] 1.6166296843945727
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] 1.6171692963550925
```

Note: *EB You can probably do something similar starting with* $[a, a+1]$.

14 Infinite rod sets

Our goal is to prove that every real number greater than or equal to 1 is the growth rate for some family.

Lemma 43. *Let*

$$p(x) = a_n x^n - a_m x^m - \cdots - a_0$$

be a polynomial of degree d whose only positive coefficient is the coefficient of x^n *and whose second nontrivial term is of degree m. In particular, all Cuisenaire polynomials satisfy this condition . Then each of the derivatives* $p^{(i)}(x)$ *for* $i \leq m$ *(the ones with at least two terms) has exactly one positive real root* γ_i *and*

$$\gamma_0 \geq \gamma_1 \geq \cdots \geq \gamma_m.$$

Proof. The coefficients of $p(x)$ and all these derivatives have exactly one change of sign, so Descartes' Rule of Signs implies those polynomials all have exactly one positive root. Since all the derivatives are positive for large enough x, the zeroes of the derivatives must occur in decreasing order. □

Lemma 44. *Let R be a finite rod set with growth rate* $g(R) = \gamma$*, the positive real root of* $p_R(x)$*. For* $n > \max(R)$ *let*

$$S = R \cup \{n^{(k)}\},$$

R with k new rods of length n. Then

$$p_S(x) = x^{n-\max(R)} p_R(x) - k \tag{14.1}$$

and

$$\gamma < g(S) < \gamma + k/\sigma$$

where

$$\sigma = \gamma^{n-\max(R)} p_R'(\gamma).$$

Proof. Equation 14.1 follows from the definition of the Cuisenaire polynomial.

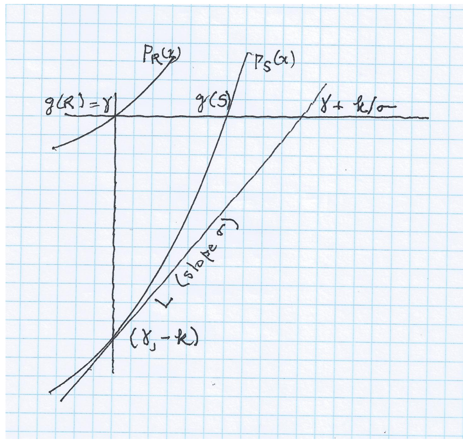

Figure 17: Illustrating Lemma 44.

Then the slope of the tangent L to $p_S(x)$ at the point $A = (\gamma, p_S(\gamma)) = (\gamma, -k)$ is

$$\begin{aligned}\sigma &= p_S'(\gamma)\\ &= (n - \max(R) - 1)\gamma^{n-\max(R)} p_R(\gamma) + \gamma^{n-\max(R)} p_R'(\gamma)\\ &= \gamma^{n-\max(R)} p_R'(\gamma).\end{aligned}$$

Therefore L meets the x-axis at point $B = (0, \gamma + k/\sigma)$.

Since the derivative $\sigma > 0$, the unique inflection point of $p_S(x)$ is to the left of γ. That implies $p_S(x)$ is convex to the right of γ, so lies above its tangent L. Thus it meets the x-axis at $g(S)$ between γ and $\gamma + k/\sigma$).

Figure 14 illustrates this argument.

□

Lemma 45. *Let R be a finite rod set. Then for $n > \max(R)$,*

$$g(R \cup \{n\})$$

is a decreasing sequence with limit $g(R)$.

Proof. Suppose $\epsilon > 0$. Since the slope in the preceding Lemma grows monotonically without bound as n increases we can find an N such that

$$g(R \cup \{n\}) - \gamma < \epsilon \text{ for all } n > N$$

Figure 14 illustrates this argument for $R = [1, 2]$ and $n = 5$.

□

Theorem 46. *Every real number greater than or equal to 1 is the growth rate for some family.*

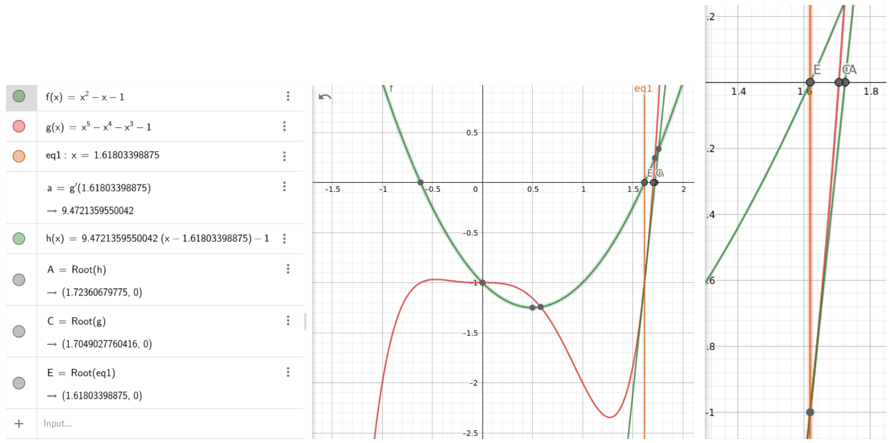

Figure 18: Illustrating Lemma 45 for $p_{[1,2]}(x)$ and $p_{[1,2,5]}(x)$.

Proof. Suppose $\gamma > 1$.

....

□

Question 20. *$\mathbb{Z}_+$ is the only classic rod set its family. Does this family contain finite non classic rod sets (growth rate 2)? Is it infinite?*

Conjecture 47. *The greedy algorithm recovers a rod set from its growth rate if you start the algorithm with the minimal element of the rod set.*

Well this can't be true since there are multiple rod sets in a family that start at the same place. So conjecture that it find the lexicographically first.

This checks with lots of examples, with arbitrary starts that lead to finite rod sets.

Question 21. *A set of rod lengths is determined by its characteristic function: a function $d : \mathbb{N} \to \{0, 1\}$ and hence as a path in the infinite complete binary tree. Finite sets R index the internal nodes nodes of the tree — put the value of $g(R)$ there. Then $g(R)$ will be unchanged when you move to the right child and increases when you move left, Does this structure help us understand g? Understand families?*

15 Antirods

The material here is the beginning of work on rod sets that contain negative integer entries.

I think we should call those antirods (not negative rods), by analogy with antiparticles in physics. We may find that a rod — antirod pair annihilate each other.

When the cuisenaire polynomial factors as $p(x) = q(x)m(x)$, with $m(x)$ the irreducible minimal polynomial for the growth rate, we know that $m(x)$ has just one positive real root, that it is at least 1 and that it is the largest (in absolute value) root of $p(x)$. Thus $q(x)$ has no positive real roots and only small complex ones.

15.1 Examples

For $[2, 2, 5]$ the Cuisenaire polynomial factors as

$$x^5 - 2x^3 - 1 = (x + 1)(x^4 - x^3 - x^2 + x - 1)$$

The rod set for the minimal polynomial is $[1, 2, -3, 4]$. We can work out the values for the corresponding recursion

$$F(n) = F(n - 1) + F(n - 2) - F(n - 3) + F(n - 4)$$

starting with entries

$$F(1) = F(2) = F(-3) = F(4) = 1$$

to generate the sequence

$$F(1) = 1, F(2) = 2, F(3) = 2 + 1 - 1 = 2, 4, 5, 9, 12, 20, 28, \ldots$$

OEIS recognizes this as the expansion of $1/(1 - x - x^2 + x^3 - x^4)$

There must be a way to interpret these counts as counts of rod trains, and a way to organize them in a prunable tree ...

15.2 What minimal polynomials occur?

Suppose $m(x)$ is irreducible and has just one positive real root γ that's the largest in absolute value, so it's the minimal polynomial for γ.

Suppose it is not a Cuisenaire polynomial, so the corresponding rod set has some antirods.

Will the recursion defined by this rod set have nonegative values?

Will γ be the growth rate for a family that has a nonnegative rod set?

Is there an appropriate polynomial q to multiply m by to get an honest Cuisenaire polynomial?

If not, what other hypotheses on $m(x)$ might you need?

References

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