# Plug and Play

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## 1 Introduction

Note: *Describe the source of the problem and the group makeup*

## 2 Plugs, Strips and Puzzles

Our mathematical model for a plug is a bit string, for example 1011 . Our current convention is that there are no leading or trailing 0 bits. (We may want to relax that convention at some time.)

The *length* of a plug is the length of the string; this one has length 4. The *number of prongs* is the number of 1 bits — 3 in this example.

Each plug has a *plugnumber* when we interpret its bit string as a binary integer. So 1011 is number $1 + 4 + 8 = 13$.[1]

Since plugs begin and end with 1 bits, the plugs of length $n$ have plugnumbers the odd integers between $2^{n-1} + 1$ and $2^n - 1$. [2]

We will want ways to talk about plugs other than specifying their bit strings or plugnumbers. In any context we can give a plug any name we like, or have variables whose values can be plugs.

The Stackexchange question that triggered our project asked about the 2 prong plugs. We'll call those the *classic plugs* and give them their own names: $T_k$ for the two prong plug of length $k$, It has $k - 2$ 0 bits between its 2 end prongs. $T_k$ has plugnumber $2^{k-1} + 1$. The stackexchange question uses the single prong plug 1 instead of the mathematically more natural length 2 two prong plug $T_2$ with no gap between the prongs.

A *power strip* (or just a *strip*) models a place to plug in plugs. Think of it as a finite sequence of slots some of which are occupied by the prongs of plugs. So

[1]This convention reads increasing bit significance from left to right, so the first bit is the units bit. We can change our minds and set the plugnumber for 1011 to $1 + 2 + 8 = 11$ if we wish, but let's decide soon and stick to our decision.

[2]This equivalence will need revision if we allow leading and trailing 0 s in plug bitstrings.

a power strip is a pair consisting of an array of slots and a set of (plug, offset) pairs such that all the plugs can be inserted simultaneously at the specified offsets. It is a *solved* if every slot in the strip is filled.

A *plug puzzle*, or, for us, just a puzzle, specifies the types amd numbers of plugs you are allowed to use to fill a strip — something like

$$\mathbb{E}(\text{list of allowable plug types}, \text{restrictions on numbers of plugs of each type})$$

A *solution* of length $n$ is a strip with $n$ slots filled with plugs that are consistent with the restrictions in the puzzle specification.

Given a puzzle we try to understand the number and shape of solutions. often as a function of the size of the strip.

Note: *Remember to say somewhere (perhaps not here) that plugs and solutions have a left to right direction. You can't turn them around. Perhaps add an electric system not quite analogy.*

The stackexchange post posed a sequence of puzzles:

$$\mathbb{T}_k = \mathbb{E}((\text{the one prong plug}, T_3, T_4, \ldots .T_k), \text{exactly 1 of each }).$$

The only possible solutions are strips of length $2k + 1$. The question asks for an efficient algorithm to count them.

# 3 Anything goes

That puzzle spurred our investigations,bput it is much too hard to start out with. In particular, we discovered that restricting the number of plugs of each type is a stumbling block. So for a while we will study just puzzles that specify the allowed plug types, with no limit on the number of each that can appear in a solution.

In the *anything goes* puzzle $\mathbb{A}$ you may use any plugs as oftem as you wish:

$$\mathbb{A} = \mathbb{E}(\text{all plug types })$$

Counting solutions begins to put us in touch with some classical notions in combinatorics.

**Theorem 1.** *Length $n$ solutions for the plug puzzle that allows arbitrary many instances of any plug correspond to the partitions of the $n$-element set $[n] = \{1, 2, \ldots , n\}$.*

Note: *Example here — wait until the good LaTeX plug representation is done.*

The Bell numbers $\{1, 2, 3, 5, 15, 52, \ldots$ count these.

# 4 Factoring

There is more information in a a solved strip than the shape of the partition that defines it. For example, two copies of 101 solve a strip of length 4. So do two copies of 11 . Both solutions correspond to a partition the four element set of slots into two sets of two slots each, but their geometry is different.

In a power strip of length $n$ there are $n-1$ *gaps* between slots. The *thickness* or *thickness array* is a the sequence of integers that counts the number of plugs that cover each of the gaps. A solution (that is, a filled strip) is *prime* if its thickness is never 0. Any solution is a concatenation of prime filled strips: the *factors*.

When a puzzle specifies no restrictions on the number of each kind of plug then factors and products (concatenations) of solutions are themselves solutions. When that happens we can relate the number of prime solutions and the number of solutions.

Let $\mathrm{cs}(n)$ be the number of solved strips of length $n$ and $\mathrm{csp}(n)$ the number of prime solutions of length $n$. Let $\mathrm{cp}(n)$ be the number of solved strips that use $n$ plugs and $\mathrm{cpp}(n)$ the number of prime solutions that use $n$ plugs.

Some of these counts may be 0. Consider the puzzle that uses just the classic plug 11 . Then $cs(n) = 1$ if $n$ is even, 0 if $n$ is odd. $csp(2) = 1$ and 0 otherwise.

We did a brute force calculation for the number of prime solutions to the anything goes puzzle, and found $\{1, 1, 2, 6, 22, \ldots\}$. That sequence seems to be A074664 in The On-Line Encyclopedia of Integer Sequences. The comments there about permutations and partitions strongly support

**Conjecture 2.** *The prime solutions to problem $\mathbb{A}$ are characterized by the equivalent conditions defining the OEIS sequence A074664.*

This easy theorem might help prove the conjecture:

**Theorem 3.** *Suppose that every factor of a solved strip solves its smaller strip and any concatenation of solved strips is a solution. Then*

$$\mathrm{cs}(n) = \mathrm{csp}(1)\,\mathrm{cs}(n-1) + \mathrm{csp}(2)\,\mathrm{cs}(n-2) + \cdots + \mathrm{csp}(n-2)\,\mathrm{cs}(2) + \mathrm{csp}(n)$$

*and the same recursion hold for the counts* cp) *and* cpp).

*Proof.* Each solved strip begins with a prime solution of length $k \leq n$ which can be chosen $\mathrm{csp}(k)$ ways and ends with one of the $\mathrm{cs}(n-k)$ solved strips of length $n-k$. Replace "length" by "nimber of plugs" to prove the second assertion. $\square$

Note: *So far we've used thickness only to find factors. But there is probably more information to exploit. What's the maximum thickness? What patterns are possible in the thickness vector?*

# 5 Disallowing the Plug 1

Note: *Also a section by Debbie. Also again, not really a theorem and notation should probably be changed. Table needs to be better labeled. I have similar work for when the plug 1 is allowed that I ran out of time before I put up.*

**Theorem 4.** *Let $S'(n,k)$ represent the number of set partitions with no singletons, i.e. the number of solutions to the plug problem with $k$ plugs, where the plug 1 is not allowed. Then $S'(n,k) = kS'(n-1,k) + (n-1)S'(n-2,k-1)$, with $S'(2,1) = 1$ and $S'(n,k) = 0$ when $k > \frac{n}{2}$*

*Proof.* Consider adding position $n-1$ in a strip of length $n$ (where position numbering starts from 0) to form a solution with $k$ plugs. We can add a prong at position $n-1$ to one of the existing plugs in a solution for a strip of length $n-1$ with $k$ plugs in $kS'(n-1,k)$, which counts all possibilities where position $n-1$ is part of a plug with 3 or more prongs. If position $n-1$ is part of a plug with 2 prongs, then we have $n-1$ choices for the other end of the plug, and we have a total of $(n-1)S'(n-2,k-1)$ solutions. □

The solutions form a triangle, with $n$ on the vertical axis and $k$ on the horizontal:

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 |
| 3 | 1 | 0 | 0 | 0 |
| 4 | 1 | 3 | 0 | 0 |
| 5 | 1 | 10 | 0 | 0 |
| 6 | 1 | 25 | 15 | 0 |
| 7 | 1 | 56 | 105 | 0 |

## 6 Puzzles with just one plug type

Note: *When we do enough of these we may see metapatterns.*

- 11...1 This puzzle has just one prime, the plug itself, of length $k$.
- 100...1 These are the classic plugs.

  **Conjecture 5.** *Using just the classic plug of length $k$, the only prime is the one of length $2k-2$ constructed from $k-2$ copies in the obvious way.*

- 1011 This puzzle has no solutions.

**Question 1.** *Are there any single plug puzzles with more than one prime?*

**Question 2.** *Can we characterize the single plug puzzles with no solutions?*

## 7 Puzzles with two plug types

## 8 Puzzles using 2 Prong Plugs

Note: *There seem to be two attempts at this count. Perhaps one from Adam (moved here) or perhaps both from Deb. I'm not sure what is being counted. We should look at this with our codified definitions.*

*First attempt:*
Using only classic plugs of length $2^k$ where $k > 0$:

| length | csp | cs |
|---|---|---|
| 2 | 1 | 1 |
| 4 | 1 | 2 |
| 6 | 1 | 4 |
| 8 | 3 | 11 |
| 10 | 12 | 33 |
| 12 | 14 | 86 |
| 14 | – | – |

Note: *Deb's work:*

In this section we count strips that can be made eith just the two prong plugs:

$$\mathbb{T} = \mathbb{E}(\{T_k | k = 1, 2, \ldots\})$$

This is the stackexchange puzzle, using $T_0$ instead of the single prong plug ( 1), with no restrictions on the number of each kind of plug.

Note: *What is below doesn't really warrant a theorem... maybe a proposition or a lemma or just within the text, not sure. Also the $t(n)$ notation is temporary pending my comment in the Specifying Variations section. Also should we use length $n$ for the strip for consistency, which would make all odd lengths 0, or use $2n$ as I did below?*

**Theorem 6.** *For the puzzle $\mathbb{T}$ there are no solutions of odd length. For each $n$,*

$$\mathrm{cs}(2n) = (2n - 1) \cdot (2n - 3) \cdots 3 \cdot 1.$$

*Proof.* Solutions of length $2n$ correspond to partitions of the set of $2n$ slots into subsets of size two. There are $2n - 1$ ways to choose a partner for the plug whose first prong occupies the first slot , $2n - 3$ ways to choose a partner for the plug whose first prong occupies the next empty slot, and so on. □

We directly compute several values of $\mathrm{csp}(n)$, the number of prime strips of length $2n$ , using the "brute force" equation that subtracts composite plugs corresponding to all compositions of $n$ into $k$ parts, where $k \leq n$,

$$pt(n) = t(n) - \sum_{i_1 + i_2 + \ldots i_k = n} pt(i_1) pt(i_2) \cdots pt(i_k)$$

For example, we note that $pt(1) = 1$ and compute $\mathrm{csp}(2) = \mathrm{cs}(2) - pt(1)pt(1) = 3 - 1 \cdot 1 = 2$. Also we have $pt(3) = t(3) - pt(1)pt(2) - pt(2)pt(1) - pt(1)pt(1)pt(1) = 15 - 2 - 2 - 1 = 10$.

Note: *Ethan: is this essentially Theorem 3?* We obtain the following table with the first few values of $pt(n)$, along with the ratio $\dfrac{pt(n)}{t(n)}$.

| $n$ | $pt(n)$ | $\frac{pt(n)}{t(n)}$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 0.67 |
| 3 | 10 | 0.67 |
| 4 | 74 | 0.70 |
| 5 | 706 | 0.75 |
| 6 | 8162 | 0.79 |

# 9 Asymptotics

Note: *Placeholder for work on asymptotic estimates or bounds for puzzles where we can make some progress but can't find good closed form or known desriptions of the counting functions.*

# 10 Infinite strips

Note: *Placeholder for work on infinite prime strips.*