Counting Compositions

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1 Introduction

In elementary schools (and in Pre-K Montessori schools) children use Cuisenaire rods[4] to learn how numbers fit together. Figure 1 shows ways to build rods of lengths up to 14 with red (length 2) and green (length 3) rods.

A *composition* of an integer n is a way to write n as an ordered sum of positive integers. We write $\mathbb{P}$ for the set of positive integers.

For each subset R of $\mathbb{P}$ let $F(n, R)$ be the number of compositions of n that use summands only in the *rod set* R. We call those compositions *trains* — a term a second grader might use when working with Cuisenaire rods. When we want to list the elements of R we will write them in square brackets, usually in increasing order.

- $F(n, [1, 2]) = F_{n+1}$, the $(n+1)$st Fibonacci number. That's why we chose "F" for the general case.
- When you can use rods of any integral length, $F(n, \mathbb{P}) = 2^{n-1}$.
- $F(n, [1]) = 1$ for all n.

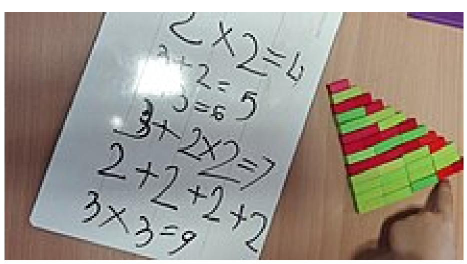

Figure 1: A young child using a 'staircase' of red and green rods to investigate ways of composing the counting numbers

- $F(n,[s,t])$ is the number of (nonnegative) integral solutions to the diophantine equation $sx+ty=n$ when the order of the $s+t$ summands is taken into account.
- $F(n,[2,3])$ counts the *Padovan numbers*[5][6].
- $F(n,[1,3])$ counts the *Narayana cow numbers*[7].

These sequences and others like them are well studied. Our contribution is an attempt to look at them all at once.

2 Recursion

Each of these sequences satisfies a simple linear recursion that counts the number of trains of total length n by looking at the possibilities for the first rod and the rest of the train. Let R be a set of rod lengths. Then

$$F(n,R)=\sum\{F(n-k,R)\mid k\leq n\}\ . \tag{2.1}$$

The initial conditions for this recursion come from the convention that $F(0,R)=1$ and the weak inequality on k. That captures the fact that when $n\in R$ there is at least one way to write n as a composition: use a single rod of length n.

These linear recursions specify $F(n,R)$ with a linear recursion in which all the coefficients are 1. As our study progressed we realized that it made more sense to allow several kinds of rods of the same length. If, for example, there are red and white rods of length 2 and just white rods of length 3 then we could write $R=[2,2,3]$ and the number of compositions respecting rod colors would satisfy the recursion

$$F(n,[2,2,3])=2F(n-2,[2,2,3])+F(n-3,[2,2,3]).$$

In this generalization a rod set is a multiset of positive integers. For infinite rod sets we may sometimes assume a uniform bound on the number of times each postive integer can occur.

When a rod multiset is actually a set we call it *classic*.

3 Growth rates

We want to think about these counts for large n.

Theorem 1. *When* $\gcd(R)=1$ *the sequence* $F(n,R)$ *grows asymptotically at an exponental rate. That is*

$$\lim_{n\to\infty}\frac{F(n+1,R)}{F(n,R)}$$

exists, and is finite.

Proof. Note: *EB: We can prove this using generating functions, and will (perhaps later) if we must, but I think there should be a qualitative argument, or at least a convincing heuristic, that does not depend on any such machinery.* □

Definition 2. *The* growth rate $g(R)$ *for the rod set* R *is the limit in the previous theorem.*

For example

- $g(\mathbb{P}) = 2$.
- $g([1]) = 1$.
- $g([1,2]) = \varphi$, the golden mean.
- $g([2,3]) = 1.3247179572\ldots$, the *plastic number*.

When $\gcd(R) = d > 1$ the only nonzero train counts are for multiples of d, and

$$F(dn, R) = F(n, R/d).$$

We can no longer compute the ratio of successive counts. It's natural in that case to take the ratio of counts d steps apart, and define

$$g(R) = g(R/d)^{1/d}.$$

Different rod sets R can have the same growth rate. For example, calculations show counts for $[1,3,4]$, $[1,3,5,6]$, $[2,2,3]$ and $[1,3,5,\ldots]$ all grow at the Fibonacci rate φ, though they (necessarily) generate different sequences.

One way to explain the coincidence is to expand a term in the defining recursion:

$$\begin{aligned} F(n) &= F(n-1) + F(n-2) \\ &= F(n-1) + F(n-2-1) + F(n-2-2) \\ &= F(n-1) + F(n-3) + F(n-4) \end{aligned} \tag{3.1}$$

so the rod sets $[1,2]$ and $[1,3,4]$ define the same recursion with different initial conditions. They grow at the same rate.

Definition 3. *The rod sets* R *and* S *are* g-equivalent *(or just* equivalent*) when* $g(R) = g(S)$. *This equivalence relation partitions the power set of* $\mathbb{P}$ *into equivalence classes we call* families *of recursions.*

Our aim is to understand that partition. The first step is formalizing the construction in Equation!3.1.

Definition 4. *Suppose* R *and* S *are rod sets. The* extension *of* $r \in R$ *by* S *is the rod set* T *formed by replacing* r *by the sums* $r + s$ *for all* $s \in S$. *If* R *and* S *are equivalent, we call the extension an* expansion.

We turn these nouns into the verbs extend *and* expand *with the obvious meanings.*

For example, the expansion of $2 \in R = [1, 2]$ by $[1, 2]$ itself is

$$T = [1, 2 + 1, 2 + 2] = [1, 3, 4].$$

Expanding 1 by $[1, 2]$ yields $[1 + 1, 1 + 2, 3] = [2, 2, 3]$. Extending $1 \in [1, 3, 4]$ by $[2, 2, 3]$ yields $[3, 3, 3, 4, 4]$. The following theorem shows this is in fact an expansion.

Theorem 5. *If R and S are equivalent rod sets then expanding an element of R by S produces an equivalent rod set T.*

Proof. We should be able to do this directly from the definition of the growth rate and the recursions, even for infinite rod sets.

(I've looked a little at what expansion does to the Cuisenaire polynomial and it doesn't seem helpful in this context. I suspect that is useful when expanding in the tree seeded by S,) □

Corollary 6. *Let*

$$R = [a, b, \ldots, z, D]$$

be a finite rod set of with maximum D. Let

$$T = [a, b, \ldots, z, a + D, b + D, \ldots, z + D, 2D].$$

Then $R \equiv S$.

Proof. T it the expansion of $\max(R)$ in R by R. □

Corollary 7. *Families are infinite.*

Question 1. *What does extension say about growth rates when R and S are not equivalent? Perhaps the growth rate of the extension is between $g(R)$ and $g(S)$.*

Theorem 8. *Rod sets R and S are equivalent if and only if there is a rod set T that is an expansion of each.*

Proof. If each of R and S expands to the same T then the transitivity of equivalence shows R and S are equivalent.

To prove "only if", Let

$$T = [r + s \mid r \in R, s \in S]. \tag{3.2}$$

You can think of T as the expansion of the elements in R one at a time by elements of S, or the reverse. □

Question 2. *The sum construction in the previous theorem makes each family into a semigroup in a natural way. How can we exploit that?*

The growth rate function defines a linear order on the set of families. That linear order is related to two interesting partial orders on the set of rod multisets. The most natural is inclusion, which we write with $\subset$ for strict inclusion and $\subseteq$ when we want to allow equality.

Definition 9. *We say rod set R* dominates *S and write $R \succ S$ and $S \prec R$ when its elements are* smaller *in the following sense: you can convert S to R by a (possibly infinite) sequence of replacements of an element $s \in S$ by a* smaller *positive integer r.*

For example, $[2, 3, 5]$ dominates $[2, 4, 5]$ and $[2, 3, 6]$. It neither dominates nor is dominated by $[1, 3, 6]$.

Theorem 10. *Let R and S be rod sets*

1. *If $R \subset S$ then $g(R) < g(S)$.*
2. *If $R \succ S$ then $g(R) > g(S)$.*

Proof. Here are convincing heuristic arguments. We'll provide formal proofs later if we can't do it now using just the recursions.

1. Allowing more rods clearly allows more trains.
2. Shorter rods can only make it easier to compose trains of each length.

□

Note that if T is an expansion of R by an equivalent rod set S then T has more elements than R but they are larger. The two tendencies influence the growth rate in opposite directions and the magnitudes exacltly cancel.

Corollary 11. *For all rod sets R, $g(R) \geq 1$. For each classic rod set R, $g(R) \leq 2$.*

Proof.

$$1 = g([1]) \leq g(R).$$

If R is classic then

$$g(R) \leq g(\mathbb{P}) = 2.$$

□

4 Polynomials and generating functions

Imagine using the Fibonacci numbers as the coefficients of a formal power series to define the *generating function*

$$\mathcal{F}(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots .$$

Then the Fibonacci recursion tells us that

$$\mathcal{F}(x) = x\mathcal{F}(x) + x^2\mathcal{F}(x)$$

so

$$\mathcal{F}(x) = \frac{1}{1 - x - x^2}$$

For a reason that will become clear soon, it's useful to rewrite the denominator in that equation as

$$x^2 \left(\left(\frac{1}{x} \right)^2 - \frac{1}{x} - 1 \right)$$

That construction generalizes.

Definition 12. *If R is finite rod set define its* Cuiseneire polynomial

$$p_R(x) = x^{\max(R)} - \sum_{k \in R} x^{\max(R)-k} .$$

These monic integral polynomial in which the coefficient of each power less than the highest is nonpositive and the constant term is nonzero correspond bijectively to finite recursions defined by Equation 2.1. The generating function for the counts $F(n,R)$ is

$$\mathcal{F}_R(x) = \frac{1}{x^{\max(R)} p_R\left(\frac{1}{x}\right)}. \tag{4.1}$$

The classic Cuisenaire polynomials are those corresponding to classic rod sets: the coefficients other than the highest are all 0 or -1.

For example,

$$p_{[1,2]}(x) = x^2 - x - 1,$$

$$p_{[2,3]}(x) = x^3 - x - 1.$$

and

$$p_{[2,2,3]}(x) = x^3 - 2x - 1.$$

Lemma 13. *The root of a Cuisenaire polynomial with largest absolute value is real and at least* 1.

Proof. Pringsheim's theorem says the largest (in absolute value) root of the Cuisenaire polynomial is real. [3], Theorem IV.6 p 240. □

Theorem 14. *For a finite rod set R the growth rate $g(R)$ is the largest real root of its Cuisenaire polynomial.*

Proof. The partial fraction expansion of the Cuisenaire polynomial shows that the generating function is a sum of several geometric series. The series corresponding to the root whose ratio has the largest absolute value with the largest ratio dominates the ratio $F(n+1,R)/F(n,R)$ for large n.

Alternatively, you can prove the theorem using elementary linear algebra: the roots of the Cuisenaire polynomial are the eigenvalues of a matrix you are computing high powers of.

Alternatively, the largest root of the Cuisenaire polynomial is the smallest pole of the generating fumction, which governs the growth rate [3], Theorem (find it). □

As a corollary, we provide a second proof that the growth rate of a classic linear recursion is between 1 and 2, The value of the classic Cuisenaire polynomial $p_R(x)$ at $x = 1$ can't be positive , since the first and last terms cancel and the other terms are negative. (It's strictly less than 0 except for the Cuisenaire polynomial $p_{[1]}(x) = x - 1$.) The value $p_R(2) > 0$ since it is the difference between $2^{\max(R)}$ and the smaller integer whose binary representation is given by the lower order terms. The polynomial is clearly increasing for $x > 2$ so the largest real root is between 1 and 2.

With Cuisenaire polynomials at hand we can prove Theorem 10 for finite rod sets.

Theorem 15. *Let R and S be finite rod sets*

1. *If $R \subset S$ then $g(R) < g(S)$.*
2. *If $R \succ S$ then $g(R) > g(S)$.*

Proof.

1. Suppose $R \subset S$. It suffices to prove $g(R) < g(S)$ when $S = R \cup \{k\}$ for a single new rod k. If $k \leq \max(R)$ then
$$p_S(x) = p_R(x) - x^{\max(R)-k}$$
while if $k > \max(R)$ then
$$p_S(x) = x^{k-\max(R)} p_R(x) - 1.$$
In either case, $p_S(x) < p_R(x)$ at every positive root of $p_R(x)$. Therefore its largest positive root is larger than the largest positive root of $p_R(x)$. [1]

2. Suppose $R \succ S$. It suffices to prove $g(R) > g(S)$ when R and S differ in only one place, and the corresponding elements $r \in R$ and $s \in S$ satisfy $r < s$. If those elements are not the unique largest in their rod sets then R and S have the same maximum d and
$$p_R(x) = p_S(x) - x^{d-r} + x^{d-s}.$$
The two extra terms combine to a negative number when $x > 1$ is a root of $p_S(x)$, so the largest positive root of $p_R(x)$ is greater than that of $p_S(x)$.

   A similar argument works when the replacement changes the maximum of either set.

□

The growth rate $g(R)$ is the root of $p_R(x)$ with the largest absolute value, but that polynomial need not be the minimal polynomial for $g(R)$: the monic polynomial of least degree with that root. For example,

$$\begin{aligned} p_{[3,8,9]}(x) &= x^9 - x^6 - x - 1 \\ &= (x^2+1)(x^7 - x^5 - x^4 + x^3 + x^2 - x - 1). \end{aligned} \tag{4.2}$$

Here $g([3, 8, 9]) = 1.20114\ldots$ is a root of the irreducible second factor.

That calls for

[1] In fact Descartes' rule of signs says there is exactly one positive root.

Definition 16. *For each rod set R let m_R be the minimal polynomial for the growth rate $g(R)$,*

So
$$m_{[3,8,9]}(x) = x^7 - x^5 - x^4 + x^3 + x^2 - x - 1 \neq p_{[3,8,9]}(x).$$

We showed earlier that $g([1,3,4])$ is equivalent to $g([1,2])$ by expanding the Fibonacci recursion. We can show that another way now by looking at the Cuisenaire polynomials:

$$\begin{aligned} p_{[1,3,4]}(x) &= x^4 - x^3 - x - 1 \\ &= (x^2+1)(x^2 - x - 1) \\ &= (x^2+1)p_{[1,2]}(x). \end{aligned}$$

Therefore $p_{[1,3,4]}$ and $p_{[1,2]}$ have the same real roots, so the same largest real root.

Theorem 17. *Rod sets R and S are equivalent if and only if $m_R(x) = m_S(x)$, so the minimal polynomial for the growth rate determines which rod sets are in a family. That minimal polynomial divides all the Cuisenaire polynomials for rod sets in the family.*

Proof. If $m(x)$ is the minimal polynomial for an algebraic number ξ then $m(x)$ divides any integral polynomial p(x) which has ξ as a root.

To see why, write
$$p(x) = q(x)m(x) + r(x)$$
where the remainder $r(x)$ has degree less than that of $m(x)$. Substitute ξ to conclude that $r(x)$ is the 0 polynomial.

The growth rate $g(R)$ is the largest positive root of the Cuisenaire polynomial $p_R(x)$, so its minimal polynomial must then be a factor. □

Conjecture 18. *The fact that the minimal polynomial may be a proper factor of the Cuisenaire polynomial will turn out to be just a nuisance in the analysis that follows. It won't affect the conclusions of any of the theorems but may complicate the proofs.*

Question 3. *How will this extend to infinite rod sets? They still have growth rates and live in families.*

Theorem 19. *If R and S are finite and*

$$p_R(x) = v(x)p_S(x)$$

for some polynomial $v(x)$ that is positive when $x > 1$ then $g(R) = g(S)$.

Proof. The Cuisenaire polynomials have the same largest real root. □

Definition 20. *A* shift polynomial *is a monic integer polynomial all of whose coefficients are nonnegative and whose constant term is positive.*

Corollary 21. *If R and S are finite and*

$$p_S(x) = v(x)p_R(x)$$

for some shift polynomial $v(x)$ then $g(R) = g(S)$.

For example, $[1, 3, 5, 6]$ is in the Fibonacci family because

$$\begin{aligned} p_{[1,3,5,6]}(x) &= x^6 - x^5 - x^3 - 1 \\ &= (x^4 + x^2 + 1)(x^2 - x - 1) \\ &= (x^4 + x^2 + 1)p_{[1,2]}(x). \end{aligned}$$

Multiplication by the shift polynomial $x^{\max(R)}+1$ turns the Cuisenaire polynomial for the rod set R into that for the equivalent rod set that extends $\max(R)$ by R.

Question 4. *How often do the Cuisenaire and minimal polynomials differ? What are the consequences when they do?*

Equation 4.2 suggests that when they do, the Cuisenaire polynomial is a shift polynomial times the minimal polynomial.

5 Trees

In this section we explore a tree structure with which to construct multiple rod sets equivalent to a given rod set. The tree provides both an algebraic proof by describing the shift polynomial that relates the Cuisenaire polynomials and proof based on a map of rod trains.

Let $\mathcal{T}(n, R)$ be the set of compositions of n that are counted by $F(n, R)$. There is a natural way to organize the trains in a tree.[2] To build the tree for Narayana rod set $R = [1, 3]$ start with two children of the root, labeled 1 and 3. Build the tree out recursively by creating two children below each node, appending either a 1 or a 3 to the label. In this infinite tree there will be one node for each finite train of rods built from R. The label at the node describes the train. Define the sum of a node to be the sum of the entries in its label — the physical length of the train built from actual Cuisenaire rods.

Figure 5 shows the first five levels of that tree. Each node is tagged with its label (the path from the root) and the sum of the correponding train. Level k contains all the trains with k rods.

Suppose we prune that tree so that the result is finite and contains all the siblings of any leaf. Figure **??** shows such a pruning, with leaf sums $S = [2, 4, 5, 6, 7]$.

We wrote those leaf sums as a rod set S because, in fact, R and S are equivalent. That follows from the following observation about the Cuisenaire

[2]What an awkward mixed metaphor.

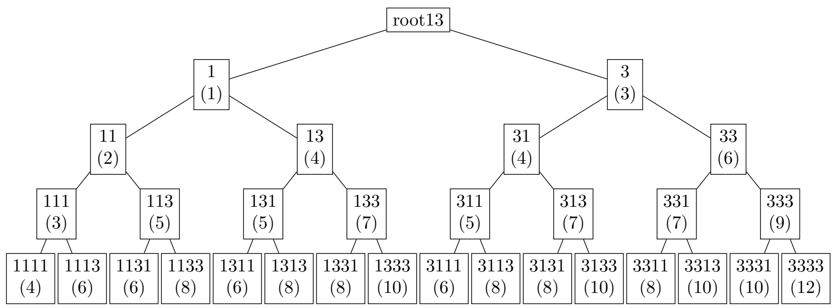

Figure 2: The tree for the rod set $[1, 3]$

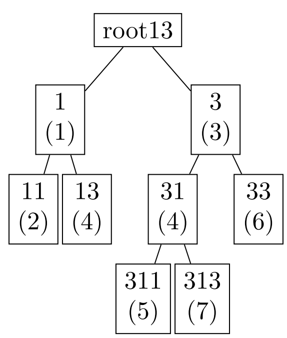

Figure 3: The tree for the rod set $[1, 3]$ pruned to $[2, 4, 5, 6, 7]$

polynomials.

$$\begin{aligned} p_{[2,4,5,6,7]}(x) &= x^7 - x^5 - x^3 - x^2 - x^1 - 1 \\ &= (x^4 + x^3 + x + 1)(x^3 - x^2 - 1) \\ &= (x^4 + x^3 + x + 1)p_{[1,3]}(x) \end{aligned}$$

We can read off the quotient polynomial from the internal node sums $[1, 3, 4]$ of the pruned tree using the same algebra as that for calculating the Cuisenaire polynomial from the rod set, with $+$ signs instead of $-$ signs:

$$x^4 + x^3 + x + 1 = x^4 + x^{4-1} + x^{4-3} + x^{4-4}.$$

There is a direct way to understand the equivalence that works with actual rod trains. For example, consider

$$\tau = 1331131131331 \in \mathcal{T}(25, R).$$

Parse that into a train of rods from the pruned tree by following the tree until you reach a leaf, then starting again at the root to find

$$\begin{aligned} \tau &= 1331131131331 \\ &= (13)(311)(311)(313)(31) \\ &= 4557(4). \end{aligned}$$

The first four sums correspond to leaves, and so to a train in $\mathcal{T}(21, S)$. The leftover (31) corresponds to an internal node with sum 4. In the theorem that follows we will study this train parsing systematically in order to prove R and S are equivalent.

If there are multiple internal nodes with the same sum this construction will produce a shift polynomial with some coefficients greater than 1. For example, consider the full third level, with sums $[3, 5, 5, 5, 7, 7, 7, 9]$. Then

$$\begin{aligned} p_{[3,5,5,5,7,7,7,9]}(x) &= x^9 - x^6 - 3x^4 - 3x^2 - 1 \\ &= (x^6 + x^5 + x^4 + x^3 + 2x^2 + 1)(x^3 - x^2 - 1) \\ &= (x^6 + x^5 + x^4 + x^3 + 2x^2 + 1)p_{[1,3]}(x). \end{aligned}$$

Theorem 22. *In the tree for finite rod set R let Q be the sums for a finite set of nodes and S the sums for the children of those nodes. Then*

$$F(n, R) = F(n, S) + \sum_{i \in Q} F(m - i, S). \tag{5.1}$$

Let $m = \max(Q)$ and

$$q(x) = x^m + \sum_{i \in Q} x^{m-i}. \tag{5.2}$$

Then

$$p_S(x) = q(x)p_R(x)$$

so R and S are equivalent.

Moreover, if γ is the common growth rate for R and S then in the limit the train counts for R and S are proportional:

$$\lim_{n\to\infty} \frac{F(n,R)}{F(n,S)} = q(1/\gamma). \tag{5.3}$$

Proof. Let $\tau = abc\cdots z$ be a finite rod train built from rods in R. Let σ be the longest initial segment of τ that splits into subsegments whose sums are in S when you follow the tree from the root, starting again when you reach a leaf. Then the sum of the leftover rods will correspond to a sum in Q.

Counting the trains in $\mathcal{T}(n,R)$ by grouping them according to which node is left over in this parsing leads to Equation 5.1.

Each term in the sum on the right corresponds to a shift by i in the train counts for S. That multiplies the generating function by x^i. Therefore

$$\begin{aligned}\mathcal{F}_R(x) &= \mathcal{F}_S(x) + \sum_{i\in Q} x^i \mathcal{F}_S(x)\\ &= \left(1 + \sum_{i\in Q} x^i\right)\mathcal{F}_S(x)\end{aligned}$$

Equation 4.1 implies

$$\frac{1}{x^{\max(R)} p_R\left(\frac{1}{x}\right)} = \left(1 + \sum_{i\in Q} x^i\right) \frac{1}{x^{\max(S)} p_S\left(\frac{1}{x}\right)}$$

so

$$p_S\left(\frac{1}{x}\right) = x^{\max(R)-\max(S)}\left(1 + \sum_{i\in Q} x^i\right) p_R\left(\frac{1}{x}\right)$$

Substituting x for $1/x$ and noting that $m = \max(S) - \max(R)$ produces the desired

$$p_S(x) = q(x)p_R(x).$$

Finally, since

$$\lim_{n\to\infty} \frac{F(n-i,S)}{F(n,S)} = 1/\gamma^i,$$

dividing Equation 5.1 by $F(n,S)$ leads to Equation 5.3 □

Note: *Ethan: provide an example verifying the ratio numerically — from Fibonacci.*

The Padovan family shows that this theorem cannot find all the rod sets starting from one with the Cuisenaire polynomial of least degree since $[1,5]$

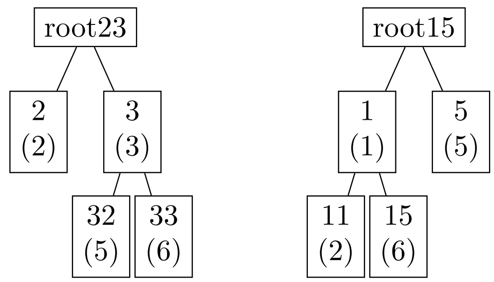

Figure 4: Two pruned Padovan trees that join.

does not come from a pruning of the tree seeded by $[2, 3]$. In fact, the theorem does not even capture the entire Fibonacci family. Let $S = [2, 4, 4, 4, 4, 7]$. Then

$$\begin{aligned}
p_S(x) &= x^7 - x^5 - 4x^3 - 1 \\
&= (x^5 + x^4 + x^3 + 2x^2 - x + 1)(x^2 - x - 1) \\
&= (x^5 + x^4 + x^3 + 2x^2 - x + 1)p_{[1,2]}(x)
\end{aligned}$$

The quotient polynomial is irreducible and positive for positive x, so S is in the Fibonacci family. It does not come from a pruning of the tree seeded by $[1, 2]$ for several reasons. That tree has only four nodes with sum 4 and their children do not make a proper pruning. If there were a proper pruning then the factor in Equation **??** would be a shift polynomial.

The Padovan family suggests a conjecture. Figure 5 shows that the trees seeded by $[2, 3]$ and $[1, 5]$ can each be pruned to generate the equivalent rod set $[2, 5, 6]$. The Cuisenaire polynomial for that rod set factors in two interesting ways:

$$\begin{aligned}
x^6 - x^4 - x^1 - 1 &= (x + 1)(x^2 - x + 1)(x^3 - x - 1) \\
&= (x + 1)(x^5 - x^4 - 1) = (x + 1)p_{[1,5]}(x) \\
&= (x^3 + x + 1)(x^3 - x - 1) = (x^3 + x + 1)p_{[2,3]}(x)
\end{aligned}$$

so $[1, 5]$ and $[2, 3]$ are each equivalent to $[2, 5, 6]$ and hence to one another, but neither Cuisenaire polynomial is a multiple of the other.

Conjecture 23. *S results from a sequence of expansions of R using R to expand if and only if it comes from a proper pruning of the tree seeded by R. an expansion of R*

This next conjecture is a corollary.

Conjecture 24. *Rod sets R and S are equivalent if and only if there is a rod set T that comes from a proper pruning of the trees seeded by R and S. In that case there are shift polynomials $a(x)$ and $b(x)$ such that*

$$p_T(x) = a(x)p_R(x) = b(x)p_S(x)$$

Proof. This should follow once we really understand how expansions and prunings are the same. □

Note: *Here is an example to explore. Three seeds of length* 6 *for* 3/4 *of Debbie's example (not the last one since my calculations stopped at rods of length 12). This is like the Padovans since the c-poly for* $[2, 3, 4, 5, 6, 7]$ *is irreducible and hence the m-poly for the family since it the one of minimal degree.*

```
[1, 3, 6, 7, 9, 11]
 x**11-x**10-x**8-x**5-x**4-x**2-1
 (x**4 - x**3 + x**2 - x + 1)*(x**7 - x**5 - x**4 - x**3 - x**2 - x - 1)

[1, 4, 5, 6, 7, 9]
x**9-x**8-x**5-x**4-x**3-x**2-1
(x**2 - x + 1)*(x**7 - x**5 - x**4 - x**3 - x**2 - x - 1)

[2, 3, 4, 5, 6, 7]
x**7-x**5-x**4-x**3-x**2-x**1-1
x**7 - x**5 - x**4 - x**3 - x**2 - x - 1
```

Question 5. *Given* $R \equiv S$*, what is the smallest* T*? (fewest number of rods)? Which has the least maximum? We might have to answer these questions to prove the conjecture.*

Note: *It would be interesting to find all the families with just one rod set with a minimal number of rods.*

The Padovans are the first example, maybe the only counterexample with two rod sets R *and* S *each with two rods. I know there is no other example where the quotient of the Cuisenaire polynomials is a trinomial, and no other example with rods of length less than* 50*.*

Conjecture: For any n *there are only finitely many families with multiple seeds of size* n*.*

Conjecture 25. *Some of the previous arguments work for infinite rod sets when appropriately modified.*

Question 6. *Are families with multiple seeds rare or common? Can we detect them? Can a family have multiple roots with different cardinalities?*

The rod sets [2.3] and $[1, 5]$ are the only ones of size 2 in the Padovan family. There can't be another set of the form $[2, b]$ since g([a,b]) is strictly monotone in the second entry. Similarly, there can't be a two rod set starting with anything greater than 2. The same argument shows there is at most one two rod set starting with 1. Then $[1, 5]$ happens to work.

In general, in any family there are only finitely many rod sets of any particular finite size, and you can search for them systematically once you know one of them.

Conjecture 26. *In the Fibonacci family the only rod sets with no repeated rods are* $[1, 3, 5, \ldots, 2k+1, 2k]$.

Question 7. *Can we tell when the minimal polynomial for a family is the Cuisenaire polynomial for the family? Can we characterize the minimal polynomials* $m_R(x)$*? Can we characterize those algebraic numbers that occur as growth rates?*

6 Peroiodicity

Note: *EB This section needs to be rewritten using expansions — I think they will explain all the arithmetic progression theorems.*

We can construct some infinite rod sets with a periodicity argument.

Theorem 27. *Let* B *be a set of distinct positive representatives of equivalence classes modulo* m*. Suppose* $m \notin B$*. Let* R *be the union of the arithmetic progressions of period* m *starting at the elements of* B*. Then Then* $g(R) = g(B \cup \{m\})$*.*

Note: EB Here's the old statement:

> *Let* B *be a finite rod set and* $m \notin B$ *such that for all* $k \in \mathbb{P}$ *the sets* B *and* $B + mk$ *are disjoint. Let*
>
> $$R = B \cup (B+m) \cup (B+2m) \cup \cdots .$$
>
> *Then* $g(R) = g(B \cup \{m\})$*.*

Proof. Note: *EB: Rewrite proof to match new statement.* Temporarily generalize our recursion notation so that for a subet $A \subset \mathbb{P}$

$$F(n, A, R)^* = \sum_{k \in A} F(n-k, R),$$

Then

$$\begin{aligned}
F(n, R) &= F(n, B, R)^* + F(n, B+m, R)^* + F(n, B+2m, R)^* + \cdots \\
&= F(n, B, R)^* + F(n-m, B, R)^* + F(n-2m, B, R)^* + \cdots \\
&= F(n, B, R)^* + F(n-m, R) \\
&= \sum_{k \in B} F(n-k, R) + F(n-m, R) \\
&= \sum_{k \in B \cup \{m\}} F(n-k, R).
\end{aligned}$$

That is the same recursion satisfied by $F(n, B \cup \{m\})$, so $g(R) = g(B \cup \{m\})$. $\square$

Corollary 28. *When* $b > 1$ *and* $m \in \mathbb{P}$ *then for* $B = \{b\}$ *the rod set* R *is the arithmetic progression* $[b, b+m, b+2m, \ldots]$ *and its growth rate is* $g([b, m])$*.*

So

$$g([2,3,4,\ldots]) = g([1,2]) = \varphi$$

and

$$g([3,4,\ldots]) = g([2,3]),$$

the growth rate of the Padovan sequence.

Whenever $m > \max(B)$ the hypotheses are satisfied. For example, with $B = [1,3]$ and $m = 4$ and then 5

$$g([1,3,5,7,9,11,\ldots]) = g([1,3,4] = \varphi$$

and

$$g([3,5,7,91,3,6,8,11,13,16,18,]) = g([1,3,5]) = 1.570147\ldots\ .$$

More interesting is this example with $B = [1,4]$ and $m = 2 < 4 = \max(B)$. Then $R = [1,3,4,5,6,7,\ldots] = \mathbb{P}\backslash\{2\}$ and

$$g(\mathbb{P}\backslash\{2\}) = g([1,2,4]) = 1.754877\ldots.$$

That example suggests the following theorem.

Theorem 29. *For each $k \in \mathbb{P}$*

$$g(\mathbb{P}\backslash\{k\}) = g([1,2,\ldots,k-1,k+1,k+2,\ldots]) = g([1,2,\ldots,k,2k])$$

Proof. Note: *EB Debbie asserts this and I've checked it numerically but I don't see a proof using the previous theorem. So provide one, or figure out what generalization is the one we need.* ☐

Question 8. *Can we say anything in general about cofinite rod sets? They always end with the sequence of integers greater than m for some m. Maybe cofinite sets that omit a gap.*

For example,

```
g([1,5,6,7,8,...,79] = 1.5289463545197037
g([1, 4, 6, 7, 8]) =   1.528946354519709

g([1,6,7,8,...,79]) = 1.4655712318766887
g([1, 5, 6, 8, 10]) = 1.4655712318767704
(in the Narayana family so maybe there's a better example)
```

Note: *EB I don't know where this assertion belongs.*

The rod sequence R containing all rods of odd length except the rod of length 1 corresponds to $B = \{3\}$ and $m = 2$, so $g(R) = g(\{2,3\})$, the growth rate of the Padovan sequence.

So far we have been discussing infinite rod sequences with an eventual periodic structure. Now we turn to more general infinite rod sequences.

Conjecture 30. *Let R be any infinite sequence of rod lengths and R_k the set of rods lengths in R no longer than k. Then $g(R_k)$ is an increasing sequence bounded by 2, so it has a limit. That limit is $g(R)$.*

Here's some evidence for that very likely conjecture. It shows a neat kind of interpolation between a Padovan recurrence and a Fibonacci recurrence. Below are the growth rates. The first line is the Padovan

```
[2, 3] 1.324717957244746
[2, 3, 4] 1.465571231876768
[2, 3, 4, 5] 1.534157744914267
[2, 3, 4, 5, 6] 1.5701473121960543
[2, 3, 4, 5, 6, 7] 1.590005373901364
[2, 3, 4, 5, 6, 7, 8] 1.6013473337876367
[2, 3, 4, 5, 6, 7, 8, 9] 1.607982727928201
[2, 3, 4, 5, 6, 7, 8, 9, 10] 1.6119303965641198
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11] 1.6143068232571485
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] 1.6157492027552105
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] 1.6166296843945727
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] 1.6171692963550925
```

Note: *EB You can probably do something similar starting with $[a, a+1]$.*

7 Infinite rod sets

Our goal is to prove that every real number greater than or equal to 1 is the growth rate for some family.

Lemma 31. *Let*

$$p(x) = a_n x^n - a_m x^m - \cdots - a_0$$

be a polynomial of degree d whose only positive coefficient is the coefficient of x^n and whose second nontrivial term is of degree m. In particular, all Cuisenaire polynomials satisfy this condition . Then each of the derivatives $p^{(i)}(x)$ for $i \leq m$ (the ones with at least two terms) has exactly one positive real root γ_i and

$$\gamma_0 \geq \gamma_1 \geq \cdots \geq \gamma_m.$$

Proof. The coefficients of $p(x)$ and all these derivatives have exactly one change of sign, so Descartes' Rule of Signs implies those polynomials all have exactly one positive root. Since all the derivatives are positive for large enough x, the zeroes of the derivatives must occur in decreasing order. ☐

Lemma 32. *Let R be a finite rod set with growth rate $g(R) = \gamma$, the positive real root of $p_R(x)$. For $n > \max(R)$ let*

$$S = R \cup \{n^{(k)}\},$$

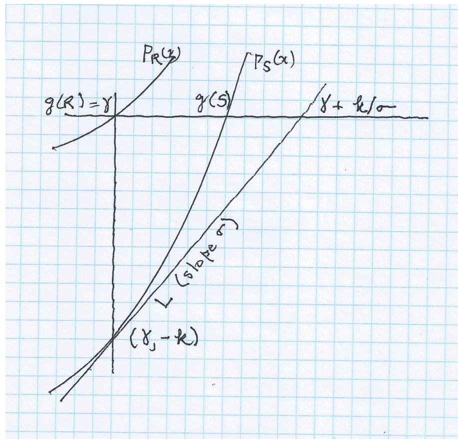

Figure 5: Illustrating Lemma 32.

R with k new rods of length n. Then

$$p_S(x) = x^{n-\max(R)} p_R(x) - k \tag{7.1}$$

and

$$\gamma < g(S) < \gamma + k/\sigma$$

where

$$\sigma = \gamma^{n-\max(R)} p_R'(\gamma).$$

Proof. Equation 7.1 follows from the definition of the Cuisenaire polynomial.

Then the slope of the tangent L to $p_S(x)$ at the point $A = (\gamma, p_S(\gamma)) = (\gamma, -k)$ is

$$\begin{aligned}
\sigma &= p_S'(\gamma) \\
&= (n - \max(R) - 1)\gamma^{n-\max(R)} p_R(\gamma) + \gamma^{n-\max(R)} p_R'(\gamma) \\
&= \gamma^{n-\max(R)} p_R'(\gamma).
\end{aligned}$$

Therefore L meets the x-axis at point $B = (0, \gamma + k/\sigma)$.

Since the derivative $\sigma > 0$, the unique inflection point of $p_S(x)$ is to the left of γ. That implies $p_S(x)$ is convex to the right of γ, so lies above its tangent L. Thus it meets the x-axis at $g(S)$ between γ and $\gamma + k/\sigma$).

Figure 7 illustrates this argument.

□

Lemma 33. *Let R be a finite rod set. Then for $n > \max(R)$,*

$$g(R \cup \{n\})$$

is a decreasing sequence with limit $g(R)$.

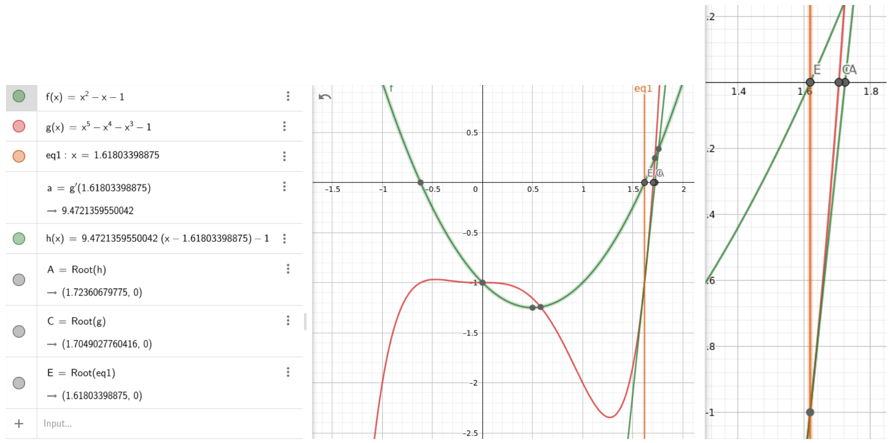

Figure 6: Illustrating Lemma 33 for $p_{[1,2]}(x)$ and $p_{[1,2,5]}(x)$.

Proof. Suppose $\epsilon > 0$. Since the slope in the preceding Lemma grows monotonically without bound as n increases we can find an N such that

$$g(R \cup \{n\}) - \gamma < \epsilon \text{ for all } n > N$$

Figure 7 illustrates this argument for $R = [1, 2]$ and $n = 5$.

□

Theorem 34. *Every real number greater than or equal to* 1 *is the growth rate for some family.*

Proof. Suppose $\gamma > 1$.

....

□

Question 9. $\mathbb{P}$ *is the only classic rod set its family. Does this family contain finite non classic rod sets (growth rate 2)? Is it infinite?*

Conjecture 35. *The greedy algorithm recovers a rod set from its growth rate if you start the algorithm with the minimal element of the rod set.*

Well this can't be true since there are multiple rod sets in a family that start at the same place. So conjecture that it find the lexicographically first.

This checks with lots of examples, with arbitrary starts that lead to finite rod sets.

Question 10. *A set of rod lengths is determined by its characteristic function: a function* $d : \mathbb{N} \to \{0, 1\}$ *and hence as a path in the infinite complete binary tree. Finite sets* R *index the internal nodes nodes of the tree — put the value of* $g(R)$ *there. Then* $g(R)$ *will be unchanged when you move to the right child and increases when you move left, Does this structure help us understand* g*? Understand families?*

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