Counting Compositions

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1 Introduction

In elementary schools (and in Pre-K Montessori schools) children use Cuisenaire rods[4] to learn how numbers fit together. Figure 1 shows rod trains of lengths up to 14 with red (length 2) and green (length 3) rods. They are examples of *compositions*: ways to write an integer n as an ordered sum of positive integers. We write $\mathbb{P}$ for the set of positive integers.

For each subset R of $\mathbb{P}$ let $F(n, R)$ be the number of compositions of n that use summands only in the *rod set* R. We call those compositions *trains* — a term a second grader might use when working with Cuisenaire rods. When we want to list the elements of R we will write them in square brackets, usually in increasing order.

- $F(n, [1, 2]) = F_{n+1}$, the $(n+1)$st Fibonacci number. That's why we chose "F" for the general case.
- When you can use rods of any integral length, $F(n, \mathbb{P}) = 2^{n-1}$.
- $F(n, [1]) = 1$ for all n.

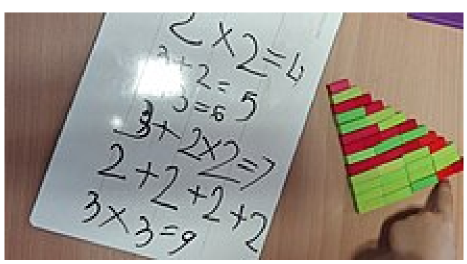

Figure 1: A young child using a 'staircase' of red and green rods to investigate ways of composing the counting numbers

- $F(n,[s,t])$ is the number of (nonnegative) integral solutions to the diophantine equation $sx+ty=n$ when the order of the $s+t$ summands is taken into account.
- $F(n,[2,3])$ counts the *Padovan numbers*[5][6].
- $F(n,[1,3])$ counts the *Narayana cow numbers*[7].

These sequences and others like them are well studied. Our contribution is an attempt to look at them all at once.

2 Recursion

Each of these sequences satisfies a simple linear recursion that counts the number of trains of total length n by looking at the possibilities for the first rod and the rest of the train. For rod set R

$$F(n,R)=\sum_{k\in R}F(n-k,R) \tag{2.1}$$

with initial conditions

$$F(n,R)=\begin{cases}1 \text{ if } n=0\\ 0 \text{ if } n<0\end{cases}$$

The first case counts the train that uses a single rod of length n when $n\in R$.

These linear recursions specify $F(n,R)$ with a linear recursion in which all the coefficients are 1. As our study progressed we realized that it made more sense to allow several kinds of rods of the same length. If, for example, there are red and white rods of length 2 and just white rods of length 3 then we could write $R=[2,2,3]$ and the number of compositions respecting rod colors would satisfy the recursion

$$F(n,[2,2,3])=2F(n-2,[2,2,3])+F(n-3,[2,2,3]).$$

In this generalization a rod set is a multiset of positive integers. For infinite rod sets we may sometimes assume a uniform bound on the number of times each positive integer can occur. Equation 2.1 remains unchanged when we interpret the summation over R in the obvious way.

3 Growth rates

We want to think about these counts for large n.

Theorem 1. *When* $\gcd(R)=1$ *the sequence* $F(n,R)$ *grows asymptotically at an exponential rate. That is*

$$\lim_{n\to\infty}\frac{F(n+1,R)}{F(n,R)}$$

exists, and is finite.

Proof. Note: *EB: We can prove this using generating functions, and will (perhaps later) if we must, but I think there should be a qualitative argument, or at least a convincing heuristic, that does not depend on any such machinery.* ☐

Definition 2. *The* growth rate $g(R)$ *for the rod set* R *is the limit in the previous theorem.*

For example

- $g(\mathbb{P}) = 2$.
- $g([1]) = 1$.
- $g([1,2]) = \varphi$, the golden mean.
- $g([2,3]) = 1.3247179572\ldots$, the *plastic number.*

When $\gcd(R) = d > 1$ the only nonzero train counts are for multiples of d, and

$$F(dn, R) = F(n, R/d).$$

We can no longer compute the ratio of successive counts. It's natural in that case to take the ratio of counts d steps apart, and define

$$g(R) = g(R/d)^{1/d}.$$

Different rod sets R can have the same growth rate. For example, calculations show counts for $[1,3,4]$, $[1,3,5,6]$, $[2,2,3]$ and $[1,3,5,\ldots]$ all grow at the Fibonacci rate φ, though they (necessarily) generate different sequences.

One way to explain the coincidence is to expand a term in the defining recursion:

$$\begin{aligned} F(n) &= F(n-1) + F(n-2) \\ &= F(n-1) + F(n-2-1) + F(n-2-2) \\ &= F(n-1) + F(n-3) + F(n-4) \end{aligned} \tag{3.1}$$

so the rod sets $[1,2]$ and $[1,3,4]$ define the same recursion with different initial conditions. They grow at the same rate.

Definition 3. *The rod sets* R *and* S *are* g-equivalent *(or just* equivalent*) when* $g(R) = g(S)$*. This equivalence relation partitions the power set of* $\mathbb{P}$ *into equivalence classes we call* families *of recursions.*

Our aim is to understand that partition. The first step is formalizing the construction in Equation!3.1.

4 Extensions and expansions

Note: *EB: Not sure about the naming convention here. I've also stated several theorems as such and not as conjectures even though I don't see just how to prove them without more work or using later machinery.*

I don't think we want to introduce antirods yet, but we should keep them in mind in this section so we can modify easily later.

Definition 4. *Suppose R and S are rod sets. The* extension *of $r \in R$ by S is the rod set T formed by replacing r by the sums $r + s$ for all $s \in S$. If $R = S$, we call the extension an* expansion*.*

Lemma 5. *A rod set T is an extension of rod set R by S if and only if $T = (R - \{a\}) \cup (a + S)$ for some $r \in R$.*

We turn these nouns into the verbs extend *and* expand *with the obvious meanings. We may sometimes use them for the results of a sequence of successive extensions rather than just one.*

For example, the expansion of $2 \in R = [1, 2]$ by $[1, 2]$ itself is

$$T = [1, 2 + 1, 2 + 2] = [1, 3, 4].$$

Expanding 1 by $[1, 2]$ yields $[1 + 1, 1 + 2, 3] = [2, 2, 3]$. Extending $1 \in [1, 3, 4]$ by $[2, 2, 3]$ yields $[3, 3, 3, 4, 4]$. Noting that $[3, 3, 3, 4, 4]$ contains $1+[2, 2, 3] = [3, 3, 4]$ recovers $[1, 3, 4]$ by contraction.

The following theorem shows this is in fact an expansion.

Theorem 6. *If R and S are equivalent rod sets then extending an element of R by S produces an equivalent rod set T.*

Proof. Note: *EB: When $R = S$ so the expansion is an extension we should be able to do this directly from the definition of the growth rate and the recursions, even for infinite rod sets.*

The more general case may need tree machinery.

□

Corollary 7. *Let*

$$R = [a, b, \ldots, z, D]$$

be a finite rod set of with maximum D. Let

$$T = [a, b, \ldots, z, a + D, b + D, \ldots, z + D, 2D].$$

Then $R \equiv S$.

Proof. T it the expansion of $\max(R)$ in R by R. □

Corollary 8. *Families are infinite.*

Question 1. *What does extension say about growth rates when R and S are not equivalent? Perhaps the growth rate of the extension is between $g(R)$ and $g(S)$.*

Theorem 9. *Rod sets R and S are equivalent if and only if there is a rod set T that is a (multistep) expansion of each.*

Proof. If each of R and S expands to the same T then the transitivity of equivalence shows R and S are equivalent.

To prove "only if", Let

$$T = [r + s \mid r \in R, s \in S]. \tag{4.1}$$

You can think of T as the expansion of the elements in R one at a time by elements of S, or the reverse. □

Question 2. *The sum construction in the previous theorem makes each family into a semigroup in a natural way. How can we exploit that?*

The growth rate function defines a linear order on the set of families. That linear order is related to two interesting partial orders on the set of rod multisets. The most natural is inclusion, which we write with $\subset$ for strict inclusion and $\subseteq$ when we want to allow equality.

Definition 10. *We say rod set R* dominates *S and write $R \succ S$ and $S \prec R$ when its elements are* smaller *in the following sense: you can convert S to R by a (possibly infinite) sequence of replacements of an element $s \in S$ by a* smaller *positive integer r.*

For example, $[2, 3, 5]$ dominates $[2, 4, 5]$ and $[2, 3, 6]$. It neither dominates nor is dominated by $[1, 3, 6]$.

Theorem 11. *Let R and S be rod sets*

1. *If $R \subset S$ then $g(R) < g(S)$.*
2. *If $R \succ S$ then $g(R) > g(S)$.*

Proof. Here are convincing heuristic arguments. We'll provide formal proofs later if we can't do it now using just the recursions.

1. Allowing more rods clearly allows more trains.
2. Shorter rods can only make it easier to compose trains of each length.

□

Note that if T is an expansion of R by an equivalent rod set S then T has more elements than R but they are larger. The two tendencies influence the growth rate in opposite directions and the magnitudes exacltly cancel.

Corollary 12. *For all rod sets R, $g(R) \geq 1$. For each classic rod set R, $g(R) \leq 2$.*

Proof.

$$1 = g([1]) \leq g(R).$$

If R is classic then

$$g(R) \leq g(\mathbb{P}) = 2.$$

□

5 Polynomials and generating functions

Imagine using the Fibonacci numbers as the coefficients of a formal power series to define the *generating function*

$$\mathcal{F}(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots .$$

Then the Fibonacci recursion tells us that

$$\mathcal{F}(x) = x\mathcal{F}(x) + x^2\mathcal{F}(x)$$

so

$$\mathcal{F}(x) = \frac{1}{1 - x - x^2}$$

For a reason that will become clear soon, it's useful to rewrite the denominator in that equation as

$$x^2 \left(\left(\frac{1}{x} \right)^2 - \frac{1}{x} - 1 \right)$$

That construction generalizes.

Definition 13. *If R is finite rod set define its* Cuiseneire polynomial

$$p_R(x) = x^{\max(R)} - \sum_{k \in R} x^{\max(R)-k} .$$

These monic integral polynomial in which the coefficient of each power less than the highest is nonpositive and the constant term is nonzero correspond bijectively to finite recursions defined by Equation 2.1. The generating function for the counts $F(n, R)$ is

$$\mathcal{F}_R(x) = \frac{1}{x^{\max(R)} p_R\left(\frac{1}{x}\right)}. \tag{5.1}$$

The classic Cuisenaire polynomials are those corresponding to classic rod sets: the coefficients other than the highest are all 0 or -1.

For example,

$$p_{[1,2]}(x) = x^2 - x - 1,$$

$$p_{[2,3]}(x) = x^3 - x - 1.$$

and

$$p_{[2,2,3]}(x) = x^3 - 2x - 1.$$

Lemma 14. *The root of a Cuisenaire polynomial with largest absolute value is real and at least* 1.

Proof. Pringsheim's theorem says the largest (in absolute value) root of the Cuisenaire polynomial is real. [3], Theorem IV.6 p 240. □

Theorem 15. *For a finite rod set R the growth rate $g(R)$ is the largest real root of its Cuisenaire polynomial.*

Proof. The partial fraction expansion of the Cuisenaire polynomial shows that the generating function is a sum of several geometric series. The series corresponding to the root whose ratio has the largest absolute value with the largest ratio dominates the ratio $F(n+1,R)/F(n,R)$ for large n.

Alternatively, you can prove the theorem using elementary linear algebra: the roots of the Cuisenaire polynomial are the eigenvalues of a matrix you are computing high powers of.

Alternatively, the largest root of the Cuisenaire polynomial is the smallest pole of the generating function, which governs the growth rate [3], Theorem (find it). □

As a corollary, we provide a second proof that the growth rate of a classic linear recursion is between 1 and 2, The value of the classic Cuisenaire polynomial $p_R(x)$ at $x = 1$ can't be positive , since the first and last terms cancel and the other terms are negative. (It's strictly less than 0 except for the Cuisenaire polynomial $p_{[1]}(x) = x - 1$.) The value $p_R(2) > 0$ since it is the difference between $2^{\max(R)}$ and the smaller integer whose binary representation is given by the lower order terms. The polynomial is clearly increasing for $x > 2$ so the largest real root is between 1 and 2.

With Cuisenaire polynomials at hand we can prove Theorem 11 for finite rod sets.

Theorem 16. *Let R and S be finite rod sets*

1. *If $R \subset S$ then $g(R) < g(S)$.*
2. *If $R \succ S$ then $g(R) > g(S)$.*

Proof. 1. Suppose $R \subset S$. It suffices to prove $g(R) < g(S)$ when $S = R \cup \{k\}$ for a single new rod k. If $k \leq \max(R)$ then

$$p_S(x) = p_R(x) - x^{\max(R)-k}$$

while if $k > \max(R)$ then

$$p_S(x) = x^{k-\max(R)} p_R(x) - 1.$$

In either case, $p_S(x) < p_R(x)$ at every positive root of $p_R(x)$. Therefore its largest positive root is larger than the largest positive root of $p_R(x)$. [1]

2. Suppose $R \succ S$. It suffices to prove $g(R) > g(S)$ when R and S differ in only one place, and the corresponding elements $r \in R$ and $s \in S$ satisfy $r < s$. If those elements are not the unique largest in their rod sets then R and S have the same maximum d and

$$p_R(x) = p_S(x) - x^{d-r} + x^{d-s}.$$

The two extra terms combine to a negative number when $x > 1$ is a root of $p_S(x)$, so the largest positive root of $p_R(x)$ is greater than that of $p_S(x)$.

A similar argument works when the replacement changes the maximum of either set.

□

The growth rate $g(R)$ is the root of $p_R(x)$ with the largest absolute value, but that polynomial need not be the minimal polynomial for $g(R)$: the monic polynomial of least degree with that root. For example,

$$\begin{aligned} p_{[3,8,9]}(x) &= x^9 - x^6 - x - 1 \\ &= (x^2+1)(x^7 - x^5 - x^4 + x^3 + x^2 - x - 1). \end{aligned} \tag{5.2}$$

Here $g([3,8,9]) = 1.20114\ldots$ is a root of the irreducible second factor.

That calls for

Definition 17. *For each rod set R let m_R be the minimal polynomial for the growth rate $g(R)$,*

So

$$m_{[3,8,9]}(x) = x^7 - x^5 - x^4 + x^3 + x^2 - x - 1 \neq p_{[3,8,9]}(x).$$

We showed earlier that $g([1,3,4])$ is equivalent to $g([1,2])$ by expanding the Fibonacci recursion. We can show that another way now by looking at the Cuisenaire polynomials:

$$\begin{aligned} p_{[1,3,4]}(x) &= x^4 - x^3 - x - 1 \\ &= (x^2+1)(x^2 - x - 1) \\ &= (x^2+1)p_{[1,2]}(x). \end{aligned}$$

Therefore $p_{[1,3,4]}$ and $p_{[1,2]}$ have the same real roots, so the same largest real root.

[1] In fact Descartes' rule of signs says there is exactly one positive root.

Theorem 18. *Rod sets R and S are equivalent if and only if $m_R(x) = m_S(x)$, so the minimal polynomial for the growth rate determines which rod sets are in a family. That minimal polynomial divides all the Cuisenaire polynomials for rod sets in the family.*

Proof. If $m(x)$ is the minimal polynomial for an algebraic number ξ then $m(x)$ divides any integral polynomial p(x) which has ξ as a root.

To see why, write

$$p(x) = q(x)m(x) + r(x)$$

where the remainder $r(x)$ has degree less than that of $m(x)$. Substitute ξ to conclude that $r(x)$ is the 0 polynomial.

The growth rate $g(R)$ is the largest positive root of the Cuisenaire polynomial $p_R(x)$, so its minimal polynomial must then be a factor. $\square$

The following conjecture is false. Irreducibility matters. It leads to the need for "negative rods".

Conjecture 19. *The fact that the minimal polynomial may be a proper factor of the Cuisenaire polynomial will turn out to be just a nuisance in the analysis that follows. It won't affect the conclusions of any of the theorems but may complicate the proofs.*

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I randomly chose 100 rod sets of length 10 with rods between 1 and 100 .

In three runs the proportion of irreducible cuisenaire polynomials was 0.90. 0.85, 0.89.
I ran the same experiment with rods between 51 and 100.The proportions were 0.9, 0.86, 0.91.

So the irreducibility seems independent of the size of the rods.

With rod sets of length 20 rather than 10 the proportions were 0.98, 0.99, 1.00

That means the probability of irreducibility increases with increasing length. I suspect tha
That says that although families are infinite, almost all  rod sets actually seed their own
The  reducible cuisenaire polynomials may or may not be the ones that seed their families. T

{'spots': [1, 5, 6, 8, 8], 'growthrate': 1.4791859598, 'cpoly':
'-2-x^2-x^3-x^7+ x^8',
'factors': '(x + 1)(x^7 - 2x^6 + 2x^5 - 2x^4 + 2x^3 - 3x^2 + 2x - 2)'}

{'spots': [2, 2, 3, 8, 9], 'growthrate': 1.6407279391, 'cpoly':
'-1-x-x^6-2x^7+ x^9',
'factors': '(x + 1)^2(x^7 - 2x^6 + x^5 - x^4 + x^3 - x^2 + x - 1)'}

{'spots': [1, 1, 4, 7, 7], 'growthrate': 2.1257731227, 'cpoly':
'-2-x^3-2x^6+ x^7',
'factors': '(x^2 + 1)(x^5 - 2x^4 - x^3 + 2x^2 - 2)'}
```

```
No repeated rods:
{'spots': [1, 2, 4, 5, 8], 'growthrate': 1.8208656546, 'cpoly':
'-1-x^3-x^4-x^6-x^7+ x^8',
'factors': '(x + 1)(x^7 - 2x^6 + x^5 - x^4 - x^2 + x - 1)'}

{'spots': [1, 1, 2, 6, 8], 'growthrate': 2.4261025199, 'cpoly':
'-1-x^2-x^6-2x^7+ x^8',
'factors': '(x + 1)(x^7 - 3x^6 + 2x^5 - 2x^4 + 2x^3 - 2x^2 + x - 1)'}

{'spots': [5, 6, 6, 7, 8], 'growthrate': 1.292620822, 'cpoly':
'-1-x-2x^2-x^3+ x^8',
'factors': '(x + 1)(x^7 - x^6 + x^5 - x^4 + x^3 - 2x^2 - 1)'}

{'spots': [2, 2, 7, 8, 9], 'growthrate': 1.5069090112, 'cpoly':
'-1-x-x^2-2x^7+ x^9',
'factors': '(x + 1)(x^8 - x^7 - x^6 + x^5 - x^4 + x^3 - x^2 - 1)'}
```

Question 3. *How will this extend to infinite rod sets? They still have growth rates and live in families.*

Theorem 20. *If R and S are finite and*

$$p_R(x) = v(x)p_S(x)$$

for some polynomial $v(x)$ that is positive when $x > 1$ then $g(R) = g(S)$.

Proof. The Cuisenaire polynomials have the same largest real root. □

Definition 21. *A* shift polynomial *is a monic integer polynomial all of whose coefficients are nonnegative and whose constant term is positive.*

Corollary 22. *If R and S are finite and*

$$p_S(x) = v(x)p_R(x)$$

for some shift polynomial $v(x)$ then $g(R) = g(S)$.

For example, $[1, 3, 5, 6]$ is in the Fibonacci family because

$$\begin{aligned} p_{[1,3,5,6]}(x) &= x^6 - x^5 - x^3 - 1 \\ &= (x^4 + x^2 + 1)(x^2 - x - 1) \\ &= (x^4 + x^2 + 1)p_{[1,2]}(x). \end{aligned}$$

Multiplication by the shift polynomial $x^{\max(R)} + 1$ turns the Cuisenaire polynomial for the rod set R into that for the equivalent rod set that extends $\max(R)$ by R.

Question 4. *How often do the Cuisenaire and minimal polynomials differ? What are the consequences when they do?*

Equation 5.2 suggests that when they do, the Cuisenaire polynomial is a shift polynomial times the minimal polynomial.

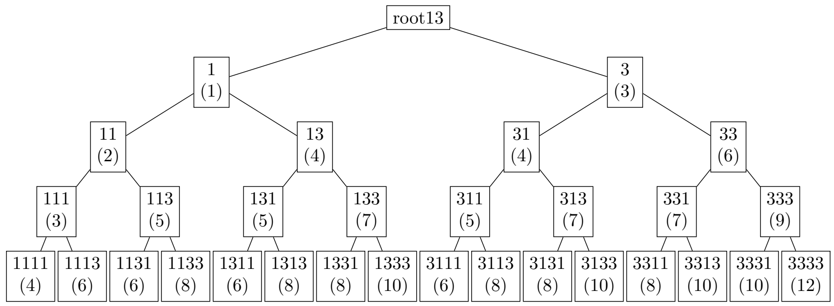


Figure 2: The tree for the rod set $[1, 3]$

6 Trees

In this section we explore a tree structure that captures the equivalent rod sets that result when we repeatedly expand a root rod set using itself. With the tree we can calculate the shift polynomial that relates the Cuisenaire polynomials.

Let $\mathcal{T}(n, R)$ be the set of compositions of n that are counted by $F(n, R)$. There is a natural way to organize the trains in a tree.[2] To build the tree for Narayana rod set $R = [1, 3]$ start with two children of the root, labeled 1 and 3. Build the tree out recursively by creating two children below each node, appending either a 1 or a 3 to the label. In this infinite tree there will be one node for each finite train of rods built from R. The label at the node describes the train. Define the sum of a node to be the sum of the entries in its label — the physical length of the train built from actual Cuisenaire rods.

Figure 6 shows the first five levels of that tree. Each node is tagged with its label (the path from the root) and the sum of the correponding train. Note that level k contains all the trains with k rods. Also note that the total number of rod trains of length m in the tree is $F(m, R)$, and if $R = [r_1, \ldots, r_j]$ then the number of those trains that appear as a left branch of the tree is $F(m - r_1, R)$, the number that appear as the second branch to the left is $F(m - r_2, R)$, ... ,and the number that appear as a right branch are $F(m - r_j, R)$.

Suppose we prune that tree so that the result is finite and contains all the siblings of any leaf. Figure **??** shows such a pruning, with leaf sums $S = [2, 4, 5, 6, 7]$.

We wrote those leaf sums as a rod set S because, in fact, R and S are equivalent. That follows from the following observation about the Cuisenaire polynomials.

$$
\begin{aligned}
p_{[2,4,5,6,7]}(x) &= x^7 - x^5 - x^3 - x^2 - x^1 - 1 \\
&= (x^4 + x^3 + x + 1)(x^3 - x^2 - 1) \\
&= (x^4 + x^3 + x + 1)p_{[1,3]}(x)
\end{aligned}
$$

[2] What an awkward mixed metaphor.

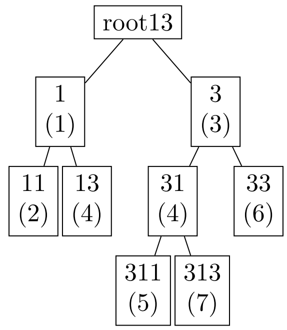


Figure 3: The tree for the rod set $[1, 3]$ pruned to $[2, 4, 5, 6, 7]$

We can read off the quotient polynomial from the internal node sums $[1, 3, 4]$ of the pruned tree using the same algebra as that for calculating the Cuisenaire polynomial from the rod set, with $+$ signs instead of $-$ signs:

$$x^4 + x^3 + x + 1 = x^4 + x^{4-1} + x^{4-3} + x^{4-4}.$$

There is a direct way to understand the equivalence that works with actual rod trains. For example, consider

$$\tau = 1331131131331 \in \mathcal{T}(25, R).$$

Parse that into a train of rods from the pruned tree by following the tree until you reach a leaf, then starting again at the root to find

$$\begin{aligned}\tau &= 1331131131331 \\ &= (13)(311)(311)(313)(31) \\ &= 4557(4).\end{aligned}$$

The first four sums correspond to leaves, and so to a train in $\mathcal{T}(21, S)$. The leftover (31) corresponds to an internal node with sum 4. In the theorem that follows we will study this train parsing systematically in order to prove R and S are equivalent.

If there are multiple internal nodes with the same sum this construction will produce a shift polynomial with some coefficients greater than 1. For example, consider the full third level, with sums $[3, 5, 5, 5, 7, 7, 7, 9]$. Then

$$\begin{aligned}p_{[3,5,5,5,7,7,7,9]}(x) &= x^9 - x^6 - 3x^4 - 3x^2 - 1 \\ &= (x^6 + x^5 + x^4 + x^3 + 2x^2 + 1)(x^3 - x^2 - 1) \\ &= (x^6 + x^5 + x^4 + x^3 + 2x^2 + 1)p_{[1,3]}(x).\end{aligned}$$

Theorem 23. *In the tree for finite rod set R let $\mathcal{Q}$ be a (finite?) set of nodes and Q the multiset of sums of nodes in $\mathcal{Q}$. Let $\mathcal{S}$ be the set of children of nodes* *in $\mathcal{Q}$ that are not themselves in $\mathcal{Q}$ and S the multiset of those nodes. Then*

$$F(n,R) = F(n,S) + \sum_{i \in Q} F(m-i, S). \tag{6.1}$$

Let $m = \max(Q)$ *and*

$$q(x) = x^m + \sum_{i \in Q} x^{m-i}. \tag{6.2}$$

Then

$$p_S(x) = q(x) p_R(x)$$

so R and S are equivalent.

Moreover, if γ is the common growth rate for R and S then in the limit the train counts for R and S are proportional:

$$\lim_{n \to \infty} \frac{F(n,R)}{F(n,S)} = q(1/\gamma). \tag{6.3}$$

Proof. Let $\tau = abc \cdots z$ be a finite rod train built from rods in R. Then τ corresponds to a path from the root in the tree for R. Let σ be the longest initial segment of τ that splits into subsegments ending in $\mathcal{S}$ when you follow the tree from the root, starting again when you reach a node in $\mathcal{S}$. Then leftover rods will end at a node in $\mathcal{Q}$.

Counting the trains in $\mathcal{T}(n,R)$ by grouping them according to which node is left over in this parsing leads to Equation 6.1.

Each term in the sum on the right corresponds to a shift by i in the train counts for S. That multiplies the generating function by x^i. Therefore

$$\begin{aligned}
\mathcal{F}_R(x) &= \mathcal{F}_S(x) + \sum_{i \in Q} x^i \mathcal{F}_S(x) \\
&= \left(1 + \sum_{i \in Q} x^i\right) \mathcal{F}_S(x)
\end{aligned}$$

Equation 5.1 implies

$$\frac{1}{x^{\max(R)} p_R\left(\frac{1}{x}\right)} = \left(1 + \sum_{i \in Q} x^i\right) \frac{1}{x^{\max(S)} p_S\left(\frac{1}{x}\right)}$$

so

$$p_S\left(\frac{1}{x}\right) = x^{\max(R)-\max(S)} \left(1 + \sum_{i \in Q} x^i\right) p_R\left(\frac{1}{x}\right)$$

Substituting x for $1/x$ and noting that $m = \max(S) - \max(R)$ produces the desired

$$p_S(x) = q(x) p_R(x).$$

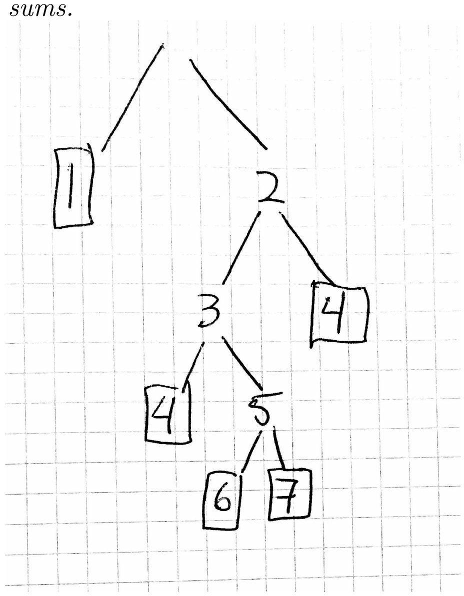
Finally, since

$$\lim_{n\to\infty} \frac{F(n-i,S)}{F(n,S)} = 1/\gamma^i,$$

dividing Equation 6.1 by $F(n,S)$ leads to Equation 6.3 $\square$

Note: *Ethan: provide an example verifying the ratio numerically — from Fibonacci. DB – see below, not sure if this is too much detail at first, and the second part has been a while and this may not be how we want to do it.... but something to go from...*

Example 24. *Consider the tree for the Fibonacci numbers, where we show only the sums.*



We have $R = [1,2], S = [1,4,4,6,7]$, *and* $Q = \{2,3,5\}$. *Consider the following rod trains that are counted by* $F(9,R)$: 121122 *and* 122121. *Following the tree as described above, we have* $121122 = (1)(211)(22) = 144$; *this rod set is counted in both* $F(9,R)$ *and* $F(9,S)$. *Now consider* $122121 = (1)(22)(1)(21) = 141(21)$. *This rod train consists of* 141, *which is something counted in* $F(9-3,S) = F(6,S)$ *followed by* 21; *the total number of rods of this form are* $F(6,S)$, *and any rod that ends in something that stops at the internal node* 3 *will be of this form. We also can have rods trains counted by* $F(n,R)$ *that consist of trains counted by* $F(n-2,S)$ *followed by a* 2 *or that consist of trains counted by* $F(n-5,S)$ *followed by* 212 . *We have the equation* $F(n,R) = F(n,S) + F(n-2,S) + F(n-3,S) + F(n-5,S)$.

Now divide the above equation by $F(n,S)$ *to obtain*

$$\frac{F(n,R)}{F(n,S)} = 1 + \frac{F(n-2,S)}{F(n,S)} + \frac{F(n-3,S)}{F(n,S)} + \frac{F(n-5,S)}{F(n,S)}$$

If γ *is the growth rate for* $F(n,S)$ *then in the limit we have*

$$\frac{F(n,R)}{F(n,S)} = 1 + \frac{1}{\gamma^2} + \frac{1}{\gamma^3} + \frac{1}{\gamma^5}$$

or $F(n,R) = F(n,S)(1 + \frac{1}{\gamma^2} + \frac{1}{\gamma^3} + \frac{1}{\gamma^5})$

Theorem 25. *The rod sets that result from pruning the tree seeded by R are precisely those generated by repeated expansion of R using R itself.*

Proof. Use tree induction. The theorem is clearly true for the seed R itself. Suppose it has been proved for some rod set S, which thus corresponds both to an expansion and a pruning of the tree. Then elements of S are the sums in the leaves of the pruning. Expanding one of those sums s using R creates leaf children that correspond precisely to the new elements in the corresponding expansion of $s \in S$ by R. □

Note: *EB: The next conjecture is a converse to Theorem 23. I think it's true. If it's false we might be able to rescue it with antirods.*

Conjecture 26. *Let $p(x)$ and $s(x)$ be Cuisenaire polynomials such that*

$$s(x) = q(x)p(x)$$

for a (shift) polynomial $q(x)$ all of whose coefficients are nonnegative. Then the recursion corresponding to $s(x)$ is a proper pruning of the tree seeded by the recursion corresponding to $p(x)$.

Corollary 27. *This is a new proof that expansion of R by R creates equivalent rod sets — something we proved a long time ago directly using the recursion.*

Note: *EB: There are several examples that show that Theorem 23 cannot find all the rod sets starting from the one with Cuisenaire polynomial of least degree. They seem to come in three categories. Sometimes there is no translate by R to unexpand by. Sometimes the Cuisenaire polynomial is not the minimal polynomial. Sometimes the Padovan phenomenon gets in the way. Sometimes the shift polynomial has a negative term.*

We deal with those by getting the rod/antirod stuff right.

I haven't modified the rest of this section.

The Padovan family shows that this theorem cannot find all the rod sets starting from one with the Cuisenaire polynomial of least degree since $[1,5]$ does not come from a pruning of the tree seeded by $[2,3]$. In fact, the theorem does not even capture the entire Fibonacci family. Let $S = [2,4,4,4,4,7]$. Then

$$\begin{aligned} p_S(x) &= x^7 - x^5 - 4x^3 - 1 \\ &= (x^5 + x^4 + x^3 + 2x^2 - x + 1)(x^2 - x - 1) \\ &= (x^5 + x^4 + x^3 + 2x^2 - x + 1)p_{[1,2]}(x) \end{aligned}$$

The quotient polynomial is irreducible and positive for positive x, so S is in the Fibonacci family. It does not come from a pruning of the tree seeded by $[1,2]$

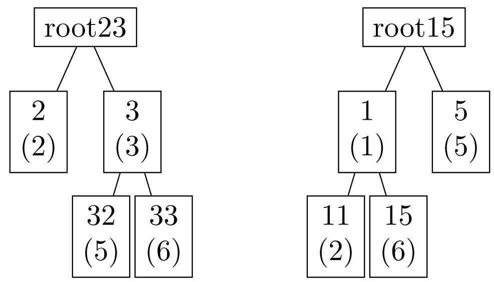


Figure 4: Two pruned Padovan trees that join.

for several reasons. That tree has only four nodes with sum 4 and their children do not make a proper pruning. If there were a proper pruning then the factor in Equation **??** would be a shift polynomial.

The Padovan family suggests a conjecture. Figure 6 shows that the trees seeded by $[2, 3]$ and $[1, 5]$ can each be pruned to generate the equivalent rod set $[2, 5, 6]$. The Cuisenaire polynomial for that rod set factors in two interesting ways:

$$\begin{aligned} x^6 - x^4 - x^1 - 1 &= (x + 1)(x^2 - x + 1)(x^3 - x - 1) \\ &= (x + 1)(x^5 - x^4 - 1) = (x + 1)p_{[1,5]}(x) \\ &= (x^3 + x + 1)(x^3 - x - 1) = (x^3 + x + 1)p_{[2,3]}(x) \end{aligned}$$

so $[1, 5]$ and $[2, 3]$ are each equivalent to $[2, 5, 6]$ and hence to one another, but neither Cuisenaire polynomial is a multiple of the other.

Conjecture 28. *S results from a sequence of expansions of R using R to expand if and only if it comes from a proper pruning of the tree seeded by R. an expansion of R*

This next conjecture is a corollary.

Conjecture 29. *Rod sets R and S are equivalent if and only if there is a rod set T that comes from a proper pruning of the trees seeded by R and S. In that case there are shift polynomials $a(x)$ and $b(x)$ such that*

$$p_T(x) = a(x)p_R(x) = b(x)p_S(x)$$

Proof. This should follow once we really understand how expansions and prunings are the same. □

Note: *Here is an example to explore. Three seeds of length 6 for 3/4 of Debbie's example (not the last one since my calculations stopped at rods of length 12). This is like the Padovans since the c-poly for $[2, 3, 4, 5, 6, 7]$ is irreducible and hence the m-poly for the family since it the one of minimal degree.*

```
[1, 3, 6, 7, 9, 11]
 x**11-x**10-x**8-x**5-x**4-x**2-1
 (x**4 - x**3 + x**2 - x + 1)*(x**7 - x**5 - x**4 - x**3 - x**2 - x - 1)

[1, 4, 5, 6, 7, 9]
x**9-x**8-x**5-x**4-x**3-x**2-1
(x**2 - x + 1)*(x**7 - x**5 - x**4 - x**3 - x**2 - x - 1)

[2, 3, 4, 5, 6, 7]
x**7-x**5-x**4-x**3-x**2-x**1-1
x**7 - x**5 - x**4 - x**3 - x**2 - x - 1
```

Question 5. *Given $R \equiv S$, what is the smallest T? (fewest number of rods)? Which has the least maximum? We might have to answer these questions to prove the conjecture.*

Note: *It would be interesting to find all the families with just one rod set with a minimal number of rods.*

The Padovans are the first example, maybe the only counterexample with two rod sets R and S each with two rods. I know there is no other example where the quotient of the Cuisenaire polynomials is a trinomial, and no other example with rods of length less than 50.

Conjecture: For any n there are only finitely many families with multiple seeds of size n.

Conjecture 30. *Some of the previous arguments work for infinite rod sets when appropriately modified.*

Question 6. *Are families with multiple seeds rare or common? Can we detect them? Can a family have multiple roots with different cardinalities?*

The rod sets [2.3] and $[1,5]$ are the only ones of size 2 in the Padovan family. There can't be another set of the form $[2,b]$ since g([a,b]) is strictly monotone in the second entry. Similarly, there can't be a two rod set starting with anything greater than 2. The same argument shows there is at most one two rod set starting with 1. Then $[1,5]$ happens to work.

In general, in any family there are only finitely many rod sets of any particular finite size, and you can search for them systematically once you know one of them.

Conjecture 31. *In the Fibonacci family the only rod sets with no repeated rods are $[1,3,5,\ldots,2k+1,2k]$.*

Question 7. *Can we tell when the minimal polynomial for a family is the Cuisenaire polynomial for the family? Can we characterize the minimal polynomials $m_R(x)$? Can we characterize those algebraic numbers that occur as growth rates?*

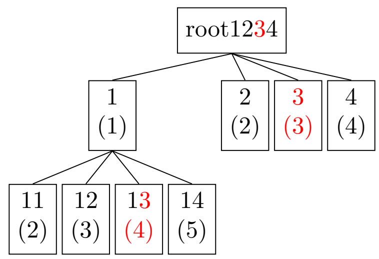


Figure 5: Pruning with antirods.

7 Antirods

Note: *EB Starting to think about antirods. When we see the whole picture we can decide whether to incorporate them from the start.*

What I do see is how they work in the tree prunings.

Our work so far seems to have convinced us that we want to use antirods only for intermediate steps in expansions. They should all be zapped by the end of an argument, so that there are no minus signs in our recursions.

Several strands led us reluctantly to allow negative numbers in rod sets.

When the Cuisenaire polynomial is not the minimal polynomial the minimal polynomial can (must?) have positive terms other than the leading term. For example,

$$\begin{aligned} p_{[2,2,5]}(x) &= x^5 - 2x^3 - 1 \\ &= (x+1)(x^4 - x^3 - x^2 + x - 1) \\ &= (x+1)p_{[1,2,-3,4]}. \end{aligned}$$

The rod set $[1, 2, -3, 4]$ corresponds the the recursion

$$F(n) = F(n-1) + F(n-2) - F(n-3) + F(n-4).$$

The train counts for that recursion grow at the same rate as the counts for $[2, 2, 5]$. It is the natural seed for the family.

We struggled to find a meaning for the negative rod. We first thought using a rod of length -3 might shorten a rod train by 3 units. That didn't work. We eventually discovered that the best way to think of -3 is as an *antirod* of length 3, which we write as 3. (We considered $\bar{3}$, which is less dramatic but better in black and white.)

We can make formal sense of antirods when pruning trees. The one in Figure 7 rooted at $[1, 2, 3, 4]$ has seven leaves with train lengths $[2, 3, 4, 5, 2, 3, 4]$. If we allow [rod, antirod] pairs of the same length to cancel, the leaves specify just the rod set $[2, 2, 5]$. The tree has one internal node with train length 1 that corresponds to the shift polynomial $x + 1$.

Antirods solve another problem. Recall that $[1, 5]$ is in the Padovan family seeded by $[2, 3]$ but not a pruning of the tree seeded by $[2, 3]$. Figure 7 shows

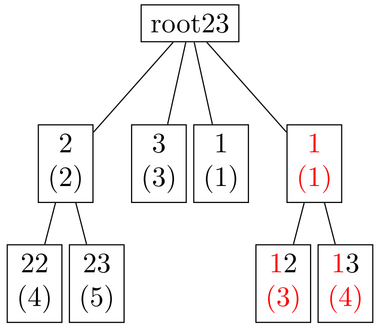


Figure 6: Using a rod/antirod pair.

how to put it in that tree by appending the [rod,antirod] pair $[1, 1]$ at the root. In the pruning the leaves are $[4, 5, 3, 1, 3, 4]$ and the internal nodes are $[1, 2]$, confirming the factorization

$$\begin{aligned} p_{[1,5]}(x) &= x^5 - x^4 - 1 \\ &= (x^2 - x + 1)(x^3 - x - 1) \\ &= (x^2 - x + 1)p_{[2,3]}. \end{aligned}$$

The internal antirod node leads to the negative coefficient in the factor $x^2 - x + 1$, which is thus not a shift polynomial as we have defined it way above. But I think we should call it one anyway. The new definition is “what you multiply the minimal polynomial by to get the Cuisenaire polynomial.”

The rod set $[1, 5, 5, 5, 5, 8]$ is the Fibonacci family because

$$x^8 - x^7 - 4x^3 - 1 = (x^2 - x - 1)(x^6 + x^4 + x^3 + 2x^2 - x + 1)$$

but it is not a pruning of the tree rooted at $[1, 2]$ because every such rod set will contain a pair of consecutive rods. But you can find a pruning if you follow the instructions implicit in the factorization: you add $[5, 5]$ and expand as in Figure 7.

I went looking for an example with two antirods. I found none in the family database I built, but figured out how to manufacture one, by using a cyclotomic polynomial (all roots of absolute value 1) as a shift polynomial.

For example, using the cyclotomic polynomial $x^4 - x^3 + x^2 - x + 2$ (which happens to be the polynomial whose roots are the primitive 10th roots of unity) as a factor in

$$(x^4 - x^3 + x^2 - x + 2)(x^2 - x - 1) = x^6 - 2x^5 + x^4 - x^3 + 2x^2 - x - 2$$

says that $[1, 1, -2, 3, -4, -4, 5, 6, 6]$ is in the Fibonacci family, and suggests that is a pruning of the tree rooted at $[1, 2]$ if you expand using rod/antirod pairs $[1, 1]$ and $[3, 3]$ as well as $[1, 2]$ at the root.

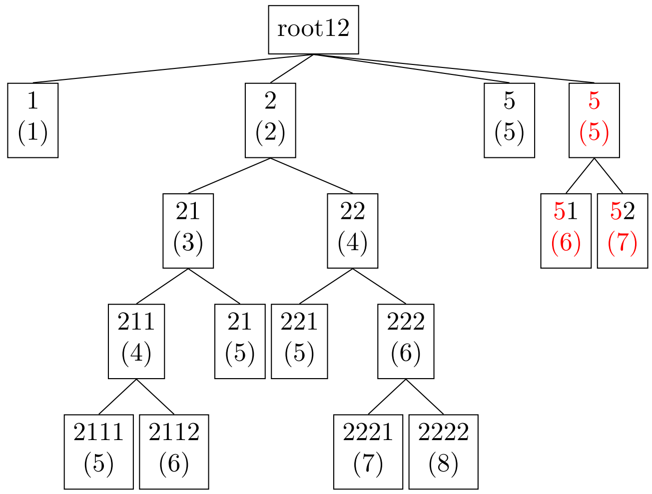


Figure 7: Using a rod/antirod pair in the Fibonacci family.

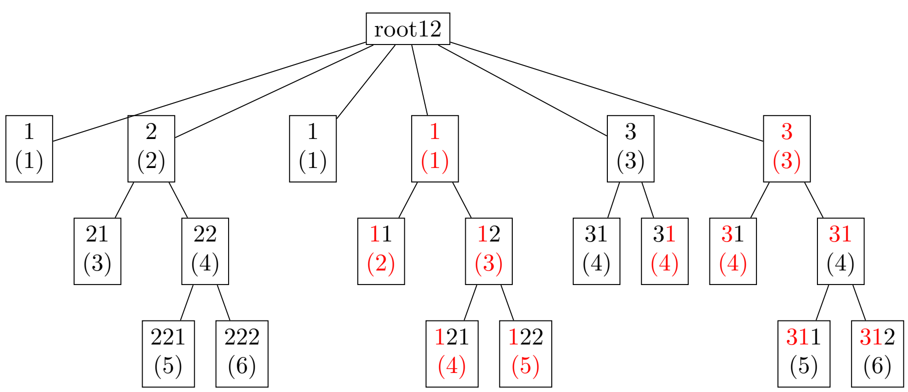


Figure 8: A complex member of the Fibonacci family.

Figure 7 shows one way that works. The leaves are

$$[1, 3, 5, 6, 1, 2, 4, 5, 4, 4, 4, 5, 6] = [1, 1, 2, 3, 4, 4, 5, 6].$$

The internal nodes are

$$[2, 4, 1, 3, 3, 3, 4] = [1, 2, 3, 4, 4],$$

It illustrates two new interesting features. First, there is rod/antirod cancellation both for the leaves and for the internal nodes. Second, in the rightmost part of the tree we see that the train 312 counts as an ordinary rod of length 6 since the number of antirods in the train is even.

This tree may not be optimal. I found it by trial and error. You will see one expansion of 3 by the pair $[1, 1]$ leading to leaves (31) and 31 of length 4 that then just cancel. I used that hack to convert the 3 from a leaf to an internal node. There's a third expansion by $[1, 1]$ for the last (3).

Note that $1, 2, 3, 4$ appear with opposite parity in the leaves and the internal nodes. Is this a coincidence?

The next step is to generalize Theorem 23. I'm sure it will work, since the proof comes directly from the recursion. When we think it through we should see just where the fact that you take the difference between the counts for the trains with an even and an odd number of antirods. That may call for an extra hypothesis.

When we understand the proof we might also see what happens when you use a shift polynomial with a root that's larger in absolute value than the growth factor from the Cuisenaire polynomial. (Debbie says she knows something about this.)

Definition 32. *A minimal/shift polynomial is* pure *if it has no positive/negative lower order terms. (Equivalent: the corresponding rod sets have no antirods.)*

Conjecture 33. *A rod set is in a family if and only if it is a pruning of the tree seeded by the rod set corresponding to the minimal polynomial for the growth rate of the family.*

We know that a rod set is in that family if and only if its Cuisenaire polynomial is a multiple of the minimal polynomial for the family by a factor (we call it a shift polynomial even though it may have negative coefficients) whose largest root is no larger than the growth factor.

The proof will need an algorithm with input the shift polynomial (that is, the factor by which you multiply the minimal polynomial for the family growth rate) that produces the pruning.

I think this is easy when everything is pure:

Conjecture 34. *If the shift polynomial and the minimal polynomial are both pure then start the tree with the seeding rod set at the root and recursively expand everything that appears in the shift rod set. You will be left with leaves that exactly correspond to the target rod set.*

Proof. Should be straightforward. It works on the random examples I tried — for example, $[3,4,6,7,9,10]$ is in the $[1,4]$ family because its Cuisenaire polynomial factors as

$$(x^4 - x^3 - 1)(x^6 + x^5 + x^4 + x + 1).$$

The shift polynomial corresponds to the rod set $[1,2,5,6]$. If you start the tree seeded by $[1,4]$ and expand the nodes in that rod set (to make them internal) you end up with just what you expect. ☐

How do we generalize these results when the shift or minimal polynomial is impure? This may help: you can always add rod/antirod pair.

Lemma 35. *For any rod set R and any n the rod sets $R \cup \{\pm n\} \cup R \mp n$ is in the R family.*

Proof. Expand one of the nodes $\pm n$. Then that node becomes the only internal node in the tree and the shift polynomial is $x^n \mp 1$, whose roots are roots of 1 and so have modulus 1. ☐

Here's an example with both impure:

$$(x^4 - x^3 + x^2 - x + 2)(x^4 - x^3 - x^2 + x - 1) = x^8 - 2x^7 + x^6 + x^3 - 4x^2 + 3x - 2$$

The root rod set is $[1,2,-3,4]$, the target is $[1,1,-2,-5,6,6,6,-7,-7,-7,8,8]$ and the internal nodes are $[-1,2,-3,4,4]$. The only internal node missing from the root rod set is -1. If you start at the root with edges to an extra rod/antirod pair, so with $[1,2,-3,4,1,-1]$, then expanding what you should finds the target in the leaves.

Conjecture 36. *Add any rod/antirod pairs you need in order to create nodes you can then expand to match the prescribed internal nodes.*

That works for the complex Fibonacci example in Figure 7, but only with a hack at the end introducing another rod/antirod pair.

It works too for Fibonacci target $[1,5,5,5,8]$ but the tree is less compact than the one in Figure 7. That one uses just a single $[5,5]$ pair rather than four and an $[8,8]$ pair.

How do we avoid dealing with trees like this which wants to produce something in the Fibonacci family?

```
      root
     /    \
    3      -3
   / \
  4   5
```

which leads to leaf rod set $[-3,4,5]$. The Cuisenaire polynomial factors as $(x-1)(x+1)(x^3+x+1)$? The largest root is complex with absolute value 1.21.

Here's an example we have to understand. Start with the polynomial

$$x^2 - 3x + 2 = (x-1)(x-2)$$

corresponding to the rod set $[1, 1, 1, 2, 2]$.

There are two ways to see this as a pruned tree.

If you think of $x - 1$ as the Cuisenaire polynomial then [1] is the root and the rod set corresponding to the internal nodes for the shift polynomial $x - 2$ is $[1, 1]$. Then the pruning is

```
 1    1   -1    1   -1
           |         |
          -2        -2
```

If you think of $x - 2$ as the Cuisenaire polynomial then $[1, 1]$ is the root and the rod set corresponding to the internal nodes for the shift polynomial $x - 1$ is [1]. Then the pruning is

```
1   1    1   -1
             / \
           -2 -2
```

What is going on with the actual recursions?

If you think the problem comes from the antirods in the shift polynomial rod set you can ask the same kind of questions about

$$(x-3)(x-2)(x-1).$$

There each of the three possible shift polynomials has a positive constant term, so no antirod for the maximum internal node.

8 Periodicity

Note: *EB This section needs to be rewritten using expansions — I think they will explain all the arithmetic progression theorems.*

We can construct some infinite rod sets with a periodicity argument.

Theorem 37. *Let B be a set of distinct positive representatives of equivalence classes modulo m. Suppose $m \notin B$. Let R be the union of the arithmetic progressions of period m starting at the elements of B. Then Then $g(R) = g(B \cup \{m\})$.*

Note: EB Here's the old statement:

> *Let B be a finite rod set and $m \notin B$ such that for all $k \in \mathbb{P}$ the sets B and $B + mk$ are disjoint. Let*
>
> $$R = B \cup (B+m) \cup (B+2m) \cup \cdots .$$
>
> *Then $g(R) = g(B \cup \{m\})$.*

Proof. Note: *EB: Rewrite proof to match new statement.* Temporarily generalize our recursion notation so that for a subet $A \subset \mathbb{P}$

$$F(n, A, R)^* = \sum_{k \in A} F(n-k, R),$$

Then

$$\begin{aligned} F(n,R) &= F(n,B,R)^* + F(n,B+m,R)^* + F(n,B+2m,R)^* + \cdots \\ &= F(n,B,R)^* + F(n-m,B,R)^* + F(n-2m,B,R)^* + \cdots \\ &= F(n,B,R)^* + F(n-m,R) \\ &= \sum_{k\in B} F(n-k,R) + F(n-m,R) \\ &= \sum_{k \in B\cup\{m\}} F(n-k,R). \end{aligned}$$

That is the same recursion satisfied by $F(n, B \cup \{m\})$, so $g(R) = g(B \cup \{m\})$. $\square$

Corollary 38. *When $b > 1$ and $m \in \mathbb{P}$ then for $B = \{b\}$ the rod set R is the arithmetic progression $[b, b+m, b+2m, \ldots]$ and its growth rate is $g([b, m])$.*

So

$$g([2, 3, 4, \ldots]) = g([1, 2]) = \varphi$$

and

$$g([3, 4, \ldots]) = g([2, 3]),$$

the growth rate of the Padovan sequence.

Whenever $m > \max(B)$ the hypotheses are satisfied. For example, with $B = [1, 3]$ and $m = 4$ and then 5

$$g([1, 3, 5, 7, 9, 11, \ldots]) = g([1, 3, 4] = \varphi$$

and

$$g([3, 5, 7, 91, 3, 6, 8, 11, 13, 16, 18,]) = g([1, 3, 5]) = 1.570147\ldots\ .$$

More interesting is this example with $B = [1, 4]$ and $m = 2 < 4 = \max(B)$. Then $R = [1, 3, 4, 5, 6, 7, \ldots] = \mathbb{P}\backslash\{2\}$ and

$$g(\mathbb{P}\backslash\{2\}) = g([1, 2, 4]) = 1.754877\ldots.$$

That example suggests the following theorem.

Theorem 39. *For each $k \in \mathbb{P}$*

$$g(\mathbb{P}\backslash\{k\}) = g([1, 2, \ldots, k-1, k+1, k+2, \ldots]) = g([1, 2, \ldots, k, 2k])$$

Proof. Note: *EB Debbie asserts this and I've checked it numerically but I don't see a proof using the previous theorem. So provide one, or figure out what generalization is the one we need.* $\square$

Question 8. *Can we say anything in general about cofinite rod sets? They always end with the sequence of integers greater than m for some m. Maybe cofinite sets that omit a gap.*

For example,

```
g([1,5,6,7,8,...,79] = 1.5289463545197037
g([1, 4, 6, 7, 8]) =   1.528946354519709

g([1,6,7,8,...,79]) = 1.4655712318766887
g([1, 5, 6, 8, 10]) = 1.4655712318767704
(in the Narayana family so maybe there's a better example)
```

Note: *EB I don't know where this assertion belongs.*

The rod sequence R containing all rods of odd length except the rod of length 1 corresponds to $B = \{3\}$ and $m = 2$, so $g(R) = g(\{2, 3\})$, the growth rate of the Padovan sequence.

So far we have been discussing infinite rod sequences with an eventual periodic structure. Now we turn to more general infinite rod sequences.

Conjecture 40. *Let R be any infinite sequence of rod lengths and R_k the set of rods lengths in R no longer than k. Then $g(R_k)$ is an increasing sequence bounded by 2, so it has a limit. That limit is $g(R)$.*

Here's some evidence for that very likely conjecture. It shows a neat kind of interpolation between a Padovan recurrence and a Fibonacci recurrence. Below are the growth rates. The first line is the Padovan

```
[2, 3] 1.324717957244746
[2, 3, 4] 1.465571231876768
[2, 3, 4, 5] 1.534157744914267
[2, 3, 4, 5, 6] 1.5701473121960543
[2, 3, 4, 5, 6, 7] 1.590005373901364
[2, 3, 4, 5, 6, 7, 8] 1.6013473337876367
[2, 3, 4, 5, 6, 7, 8, 9] 1.607982727928201
[2, 3, 4, 5, 6, 7, 8, 9, 10] 1.6119303965641198
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11] 1.6143068232571485
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] 1.6157492027552105
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] 1.6166296843945727
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] 1.6171692963550925
```

Note: *EB You can probably do something similar starting with $[a, a + 1]$.*

9 Infinite rod sets

Our goal is to prove that every real number greater than or equal to 1 is the growth rate for some family.

Lemma 41. *Let*

$$p(x) = a_n x^n - a_m x^m - \cdots - a_0$$

be a polynomial of degree d whose only positive coefficient is the coefficient of x^n and whose second nontrivial term is of degree m. In particular, all Cuisenaire polynomials satisfy this condition . Then each of the derivatives $p^{(i)}(x)$ for $i \le m$ (the ones with at least two terms) has exactly one positive real root γ_i and

$$\gamma_0 \ge \gamma_1 \ge \cdots \ge \gamma_m.$$

Proof. The coefficients of $p(x)$ and all these derivatives have exactly one change of sign, so Descartes' Rule of Signs implies those polynomials all have exactly one positive root. Since all the derivatives are positive for large enough x, the zeroes of the derivatives must occur in decreasing order. □

Lemma 42. *Let R be a finite rod set with growth rate $g(R) = \gamma$, the positive real root of $p_R(x)$. For $n > \max(R)$ let*

$$S = R \cup \{n^{(k)}\},$$

R with k new rods of length n. Then

$$p_S(x) = x^{n-\max(R)} p_R(x) - k \tag{9.1}$$

and

$$\gamma < g(S) < \gamma + k/\sigma$$

where

$$\sigma = \gamma^{n-\max(R)} p_R'(\gamma).$$

Proof. Equation 9.1 follows from the definition of the Cuisenaire polynomial.

Then the slope of the tangent L to $p_S(x)$ at the point $A = (\gamma, p_S(\gamma)) = (\gamma, -k)$ is

$$\begin{aligned}\sigma &= p_S'(\gamma)\\ &= (n - \max(R) - 1)\gamma^{n-\max(R)} p_R(\gamma) + \gamma^{n-\max(R)} p_R'(\gamma)\\ &= \gamma^{n-\max(R)} p_R'(\gamma).\end{aligned}$$

Therefore L meets the x-axis at point $B = (0, \gamma + k/\sigma)$.

Since the derivative $\sigma > 0$, the unique inflection point of $p_S(x)$ is to the left of γ. That implies $p_S(x)$ is convex to the right of γ, so lies above its tangent L. Thus it meets the x-axis at $g(S)$ between γ and $\gamma + k/\sigma$).

Figure 9 illustrates this argument.

□

Lemma 43. *Let R be a finite rod set. Then for $n > \max(R)$,*

$$g(R \cup \{n\})$$

is a decreasing sequence with limit $g(R)$.

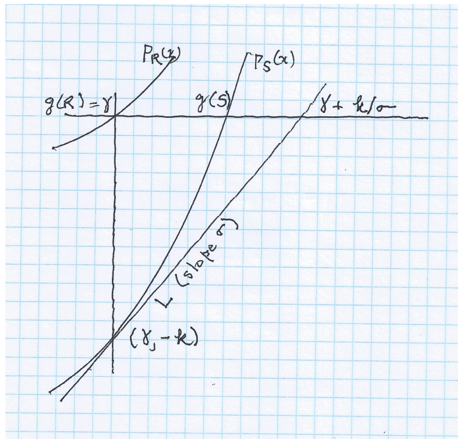


Figure 9: Illustrating Lemma 42.

Proof. Suppose $\epsilon > 0$. Since the slope in the preceding Lemma grows monotonically without bound as n increases we can find an N such that

$$g(R \cup \{n\}) - \gamma < \epsilon \text{ for all } n > N$$

Figure 9 illustrates this argument for $R = [1, 2]$ and $n = 5$.

□

Theorem 44. *Every real number greater than or equal to* 1 *is the growth rate for some family.*

Proof. Suppose $\gamma > 1$.

....

□

Question 9. $\mathbb{P}$ *is the only classic rod set its family. Does this family contain finite non classic rod sets (growth rate 2)? Is it infinite?*

Conjecture 45. *The greedy algorithm recovers a rod set from its growth rate if you start the algorithm with the minimal element of the rod set.*

Well this can't be true since there are multiple rod sets in a family that start at the same place. So conjecture that it find the lexicographically first.

This checks with lots of examples, with arbitrary starts that lead to finite rod sets.

Question 10. *A set of rod lengths is determined by its characteristic function: a function* $d : \mathbb{N} \to \{0, 1\}$ *and hence as a path in the infinite complete binary tree. Finite sets* R *index the internal nodes nodes of the tree — put the value of* $g(R)$ *there. Then* $g(R)$ *will be unchanged when you move to the right child and increases when you move left, Does this structure help us understand* g*? Understand families?*

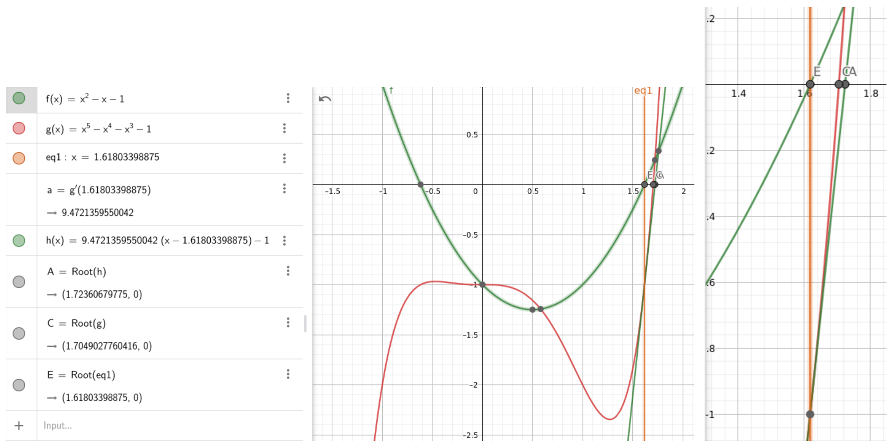


Figure 10: Illustrating Lemma 43 for $p_{[1,2]}(x)$ and $p_{[1,2,5]}(x)$.

10 Antirods

The material here is the beginning of work on rod sets that contain negative integer entries.

I think we should call those antirods (not negative rods), by analogy with antiparticles in physics. We may find that a rod — antirod pair annihilate each other.

When the cuisenaire polynomial factors as $p(x) = q(x)m(x)$, with $m(x)$ the irreducible minimal polynomial for the growth rate, we know that $m(x)$ has just one positive real root, that it is at least 1 and that it is the largest (in absolute value) root of $p(x)$. Thus $q(x)$ has no positive real roots and only small complex ones.

10.1 Examples

For $[7, 11]$ we have

$$x^{11} - x^4 - 1 = (x^2 - x + 1)(x^9 + x^8 - x^6 - x^5 + x^3 - x - 1)$$

so $q(x)$ need not be a shift polynomial. In this case its roots are roots of unity, but that might not always be true.

For $[2, 2, 5]$ the Cuisenaire polynomial factors as

$$x^5 - 2x^3 - 1 = (x + 1)(x^4 - x^3 - x^2 + x - 1)$$

The rod set for the minimal polynomial is $[1, 2, -3, 4]$. We can work out the values for the corresponding recursion

$$F(n) = F(n-1) + F(n-2) - F(n-3) + F(n-4)$$

starting with entries

$$F(1) = F(2) = F(-3) = F(4) = 1$$

to generate the sequence

$$F(1) = 1, F(2) = 2, F(3) = 2 + 1 - 1 = 2, 4, 5, 9, 12, 20, 28, \ldots$$

OEIS recognizes this as the expansion of $1/(1 - x - x^2 + x^3 - x^4)$

There must be a way to interpret these counts as counts of rod trains, and a way to organize them in a prunable tree ...

10.2 What minimal polynomials occur?

Suppose $m(x)$ is irreducible and has just one positive real root γ that's the largest in absolute value, so it's the minimal polynomial for γ.

Suppose it is not a Cuisenaire polynomial, so the corresponding rod set has some antirods.

Will the recursion defined by this rod set have nonegative values?

Will γ be the growth rate for a family that has a nonnegative rod set?

Is there an appropriate polynomial q to multiply m by to get an honest Cuisenaire polynomial?

If not, what other hypotheses on $m(x)$ might you need?

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