Math 135: Homework 10

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Problems

7 Orderings and Ordinals

Problem 7.1a. Assume that $<_A$ and $<_B$ are partial orderings on A and B respectively, and that f is a function from A into B satisfying

$$x <_A y \implies f(x) <_B f(y)$$

for all $x \in A$.

Can we conclude that f is one-to-one?

Solution. f isn't one-to-one.

We shall construct a counterexample. First, let us define $A = \{0, 1, 2\}$ and $\{0, 1\} = B$. Then, we have the linear ordering $<_A$ to be $1 <_A 2$. On other hand, let $<_B$ be the usual ordering on B.

Then, from here, let us define some function F as follows:

- $F(0) = 0$
- $F(1) = 0$
- $F(2) = 1$.

We note then that F indeed follows the constraints given in the problem statement.

Then, because of this, we see then that $0 \neq_A 1$ but $F(0) = F(1)$. Thus, we see that injectivity doesn't hold. ■

Problem 7.1b. Can we conclude that

$$x <_A y \iff f(x) <_B f(y)?$$

Solution. This is false.

We note that $<_A$ and $<_B$ are partial orderings, meaning that *at most* one of the three holds:

1. $x <_A y$, or
2. $x =_A y$, or
3. $y <_A x$.

However, it is possible for elements in A to not be related to one another. For example, let us consider the sets $A = \{0, 1, 2\} = B$, where $<_A$ is $1 <_A 2$, but 0 isn't related to the other elements. Also, let $<_B$ be the usual ordering.

Then, from here, let us define some function F as follows:

- $F(0) = 1$
- $F(1) = 1$
- $F(2) = 2$.

Then, we observe that while indeed $1 <_A 2$ and $F(1) <_B F(2)$, we note that while $F(0) <_B F(2)$, we don't have that $0 <_A 2$. Thus, we have found a counterexample. ■

Problem 7.5. Assume that $<$ is a well-ordering on A, and that $f : A \to A$ satisfies the following condition:

$$x < y \implies f(x) < f(y)$$

for all x and y in A. Show that $x \leq f(x)$ for all $x \in A$.

Solution. Suppose for the sake of contradiction that there exists some $x \in A$ such that $f(x) < x$. In other words, let us suppose that the following set is nonempty:

$$A' := \{a \in A : f(a) < a\}.$$

Now, we note that $A' \subseteq A$. Furthermore, we note by well-ordering of $<$ and the fact that A' is a non-empty subset of A, we know then that there exists some least element $x \in A'$.

Then, we observe that $f(x) < x \implies f(f(x)) < f(x) < x$. In other words, we observe that $f(x) \in A'$ and that $f(x) < x$. However, this is a contradiction with the minimality of x.

Thus, we conclude that for all $x \in A$, we have that $x \leq f(x)$ as desired. ■

Problem 7.6. Assume that S is a subset of the real numbers that is well-ordered. Show that S is countable.

Solution. Suppose that $S \subseteq \mathbb{R}$ is well-ordered. In order to show that S is countable, we will want to create an injection $f : S \to A$, where A is some countable set.

Then, let us consider the set $A := \{q_n : n \in \omega\}$. Next, for any $s \in S$, we define its successor s^+ to be the next element in s. Note that if s is a maximal element of S, we define $s^+ = s + 1$.

Now, with all of this in mind, note that since $s \neq s^+$, we observe that the set $\mathbb{Q} \cap (s, s^+)$ is non-empty, and thus there exists some least rational. So, for any $s \in S$, we let q_s be the least rational in A such that $s < q_s < s^+$.

Now, to show injectivity, let us suppose that for $q_a = q_b$, we have $a \neq b$. Without loss of generality, suppose then that $a < b$.

Then, we observe that we have $a < q_a = q_b < a^+$, and $b < q_b = q_a$. Then, we note that combining these inequalities together, we have:

$$a < b < q_a = q_b < a^+.$$

However, we note here that a^+ is defined to be the least element in S such that $a < a^+$. But if we have $a < b$ and $b \in S$ as well, this contradicts with the minimality of a^+. In other words, it can't be that $a < b$. A similar argument follows to show why $b < a$ can't happen.

Thus, we conclude that, indeed, $a = b$, and thus we have found an injection from S to some countable set A. Thus, we conclude that S is countable as well. ■

Problem 7.7a. Let C be some fixed set. Apply transfinite recursion to ω, using for $\gamma(x, y)$ the formula

$$y = C \cup \bigcup\bigcup \operatorname{ran} x.$$

Let F be the γ-constructed function on ω.

Calculate $F(0)$, $F(1)$, and $F(2)$. Make a good guess as to what $F(n)$ is.

Solution. We observe that $F(0) = C \cup \bigcup\bigcup \operatorname{ran}(F \restriction \operatorname{seg}(0)) = C \cup \emptyset = C$.

For $F(1) = C \cup \bigcup\bigcup \operatorname{ran}(F \restriction \operatorname{seg}(1)) = C \cup \bigcup\bigcup\{F(0)\} = C \cup \bigcup\bigcup\{C\} = C \cup \bigcup C$.

And $F(2) = C \cup \bigcup\bigcup \operatorname{ran}(F \restriction \operatorname{seg}(2)) = C \cup \bigcup\bigcup\{F(0), F(1)\} = C \cup \bigcup\bigcup\{C, C \cup \bigcup C\} = C \cup \bigcup(C \cup \bigcup C) = C \cup \bigcup C \cup \bigcup\bigcup C$.

And we have $F(n) = C \cup \bigcup C \cup \cdots \cup \bigcup_1 \bigcup_2 \cdots \bigcup_n C$.

And in fact, we note that this simply becomes $F(n) = C \cup \bigcup F(n-1)$.

For a concrete formula for this, we can define F as follows:

$$F(0) = C$$
$$F(n^+) = C \cup \bigcup F(n).$$

■

Problem 7.7b. Show that if $a \in F(n)$, then $a \subseteq F(n^+)$.

Solution. We see that since $a \in F(n)$, then by definition of F, we note that $F(n^+) = C \cup \bigcup F(n)$. Then, by definition of union, we have that $a \in F(n)$ means that $a \subseteq \bigcup F(n)$. In other words, $a \subseteq F(n^+)$ as desired. ■

Problem 7.7c. Let $\overline{C} = \bigcup \operatorname{ran} F$. Show that $\overline{C}$ is a transitive set, and that $C \subseteq \overline{C}$.

Solution. First, we will show that $C \subseteq \overline{C}$.

Proof. To do this, we note that $\operatorname{ran} F = \{F(n) : n \in \omega\}$. Then, we have that $\bigcup \operatorname{ran} F = \bigcup\{F(n) : n \in \omega\}$.

From here, we note that $F(0) = C$. Then, we see that $C \in \operatorname{ran} F$. Then, by definition, we see then that $F(0) = C \subseteq \bigcup \operatorname{ran} F = \overline{C}$. □

To prove transitivity of $\overline{C}$, we want to show that for some element $c \in \overline{C}$ and some $c' \in c$, we have that $c' \in \overline{C}$.

Proof. To do this, let us take some element $c \in \overline{C}$ and some element $c' \in c$. We note that for $c \in \overline{C}$, this means that $c \in \bigcup \operatorname{ran} F$. In other words, there exists some $n \in \omega$ such that $c \in F(n) \in \operatorname{ran} F$.

Now, we note that since $c \in F(n)$, then by definition of F, we have that $c \subseteq F(n^+)$. But since $c' \in c$, we note then that $c' \in F(n^+) \in \operatorname{ran} F$. And this means then that $c' \in \bigcup \operatorname{ran} F$. Thus, we have shown that, indeed, $\overline{C}$ is transitive as desired. □

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Problem 7.8. Show that the subset axioms are provable from the other axioms.

Solution. We want to show that for some set A and formula $\psi(x)$, $\{x \in A : \psi(x)\}$ is indeed a set.

To do this with replacement, we first define $\psi(x)$ be some formula, and then we define $\varphi(x, y)$ to be:

$$\varphi(x, y) := x = y \land \psi(x).$$

Then, we observe that the condition $x = y$ ensures that φ behaves like a function, and thus we can apply the replacement schema. So, we have that for any set A, there exists some set B such that $\forall y(y \in B \iff (\exists x \in A)\varphi(x, y))$.

In other words, we have:

$$\{y : (\exists x \in A)(x = y \land \psi(x))\}.$$

Or, we can rewrite this as:

$$\{x \in A : \psi(x)\}.$$

And by the replacement axioms, we see that this is indeed a set. Thus, we have proven the desired result. ■