Linear Algebra Done Right 3e Exercises

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Contents

Chapter 1

Vector Spaces

1.1 $\mathbb{R}^n$ and $\mathbb{C}^n$

Problem 1.1. Suppose a, b are real numbers, not both zero. Find real numbers c, d such that

$$1/(a+bi) = c + di.$$

Solution. To do this, we proceed as follows:

$$\begin{aligned} \frac{1}{a+bi} &= \frac{a-bi}{a^2-b^2} \\ \frac{a-bi}{a^2-b^2} &= c+di \\ \frac{a}{a^2-b^2} - \frac{b}{a^2-b^2}i &= c+di \end{aligned}$$

Thus, we see that

$$\begin{aligned} c &= \frac{a}{a^2-b^2} \\ d &= -\frac{b}{a^2-b^2} \end{aligned}$$

■

Problem 1.2. Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1.

Solution. We proceed as follows:

$$\begin{aligned}
\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \left(\frac{-1+\sqrt{3}i}{2}\right)^2 \cdot \frac{-1+\sqrt{3}i}{2} \\
&= \frac{1-2\sqrt{3}i-3}{4} \cdot \frac{-1+\sqrt{3}i}{2} \\
&= \frac{-2-2\sqrt{3}i}{4} \cdot \frac{-1+\sqrt{3}i}{2} \\
&= \frac{2-2\sqrt{3}i+2\sqrt{3}i+6}{8} \\
&= \frac{8}{8} \\
&= 1
\end{aligned}$$

■

Problem 1.3. Show that for all $\alpha, \beta \in \mathbb{C}$, we have:

$$\alpha + \beta = \beta + \alpha$$

Solution. We note that $\alpha = a + bi$ and $\beta = c + di$ for $a, b, c, d \in \mathbb{R}$. So, we observe the following:

$$\begin{aligned}
\alpha + \beta &= (a+bi) + (c+di) \\
&= (a+c) + (b+d)i \\
&= (c+a) + (d+b)i \\
&= (c+di) + (a+bi) \\
&= \beta + \alpha
\end{aligned}$$

■

Problem 1.4. Show that, for all $\alpha, \beta, \lambda \in \mathbb{C}$, we have:

$$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda).$$

Solution. We first observe that, by definition, we have for $a, b, c, d, e, f \in \mathbb{R}$:

$$\begin{aligned}
\alpha &= a + bi \\
\beta &= c + di \\
\lambda &= e + fi
\end{aligned}$$

Then, we observe the following:

$$\begin{aligned}
(\alpha + \beta) + \lambda &= ((a+bi) + (c+di)) + (e+fi) \\
&= ((a+c) + (b+d)i) + (e+fi) \\
&= (a+c+e) + (b+d+f)i \\
&= a + (c+e) + bi + (d+f)i \\
&= (a+bi) + ((c+e) + (d+f)i) \\
&= (a+bi) + ((c+di) + (e+fi)) \\
&= \alpha + (\beta + \lambda)
\end{aligned}$$

■

Problem 1.5. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$, for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Solution. We observe the following:

$$\begin{aligned}
(\alpha\beta)\lambda &= ((a+bi)(c+di))(e+fi) \\
&= (ac+adi+cbi-bd)(e+fi) \\
&= ((ac-bd)+(ad+cb)i)(e+fi) \\
&= (e(ac-bd)+e(ad+cb)i+f(ac-bd)i-f(ad+cb)) \\
&= eac-ebd+eadi+ecbi+faci-fbdi-fad-fcb
\end{aligned}$$

$$\begin{aligned}
\alpha(\beta\lambda) &- (a+bi)((c+di)(e+fi)) \\
&= (a+bi)(ce+cfi+edi-df) \\
&= ace+acfi+aedi-adf+cebi-bcf-bed-bdfi \\
&= eac-ebd+eadi+ecbi+faci-fbdi-fad-fcb
\end{aligned}$$

Thus, we see that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$. ■

Problem 1.6. Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that

$$\alpha + \beta = 0.$$

Solution. We observe that if $\alpha + \beta = 0$, then we have

$$\begin{aligned}
0 &= \alpha + \beta \\
&= (a+bi)+(c+di) \\
&= (a+c)+(b+d)i
\end{aligned}$$

So, we have $a+c=0$ and $b+d=0$. In other words, we have $c=-a$, and $d=-b$. We note here that this in fact implies uniqueness.

So, if we let $\beta = -a-bi$, then

$$\begin{aligned}
\alpha+\beta &= (a+bi)+(-a-bi) \\
&= (a-a)+(b-b)i \\
&= 0
\end{aligned}$$

■

Problem 1.7. Show that for every $\alpha \in \mathbb{C}$, where $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Solution. First, we want to check for existence.

We observe that if $\alpha\beta = 1$, then

$$\begin{aligned}
1 &= \alpha\beta \\
&= (a+bi)\cdot(c+di)
\end{aligned}$$

Then, we want $c + di = \frac{1}{a+bi}$ for this to be true. We recall then, from 1.1, that we have:

$$c + di = \frac{a}{a^2 - b^2} - \frac{b}{a^2 - b^2} i.$$

From here, we will check for uniqueness. To do this, we proceed as such:

$$\begin{aligned}\beta &= 1 \cdot \beta \\ &= \left(\frac{1}{\alpha}\right) \alpha \cdot \beta \\ &= \left(\frac{1}{\alpha}\right) (\alpha \cdot \beta) \\ &= \frac{1}{\alpha} \cdot 1 \\ &= \frac{1}{\alpha} \\ &= \frac{1}{a + bi}\end{aligned}$$

■

Problem 1.8. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$.

Solution. We observe the following:

$$\begin{aligned}\lambda(\alpha + \beta) &= (e + fi)((a + bi) + (c + di)) \\ &= (e + fi)((a + c) + (b + d)i) \\ &= ea + ec + ebi + edi + fai + fci - fb - fd \\ &= (ea + ebi + fai - fb) + (ec + edi + fci - fd)\end{aligned}$$

We observe next that

$$\begin{aligned}\lambda\alpha &= (e + fi)(a + bi) \\ &= ea + ebi + fai - fb \\ \lambda\beta &= (e + fi)(c + di) \\ &= ec + edi + fci - ed\end{aligned}$$

Thus, we see that

$$(ea + ebi + fai - fb) + (ec + edi + fci - fd) = \lambda\alpha + \lambda\beta.$$

Therefore, we have that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$. ■

Problem 1.9. Show that $(x + y) + z = x + (y + z)$, for all $x, y, z \in \mathbb{F}^n$.

Solution. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and $z = (z_1, \ldots, z_n)$. We observe the following:

$$\begin{aligned}(x + y) + z &= (x_1 + y_1, \ldots, x_n + y_n) + (z_1, \ldots, z_n) \\ &= ((x_1 + y_1) + z_1, \ldots, (x_n + y_n) + z_n) \\ &= (x_1 + (y_1 + z_1), \ldots, x_n + (y_n + z_n)) \\ &= (x_1, \ldots, x_n) + (y_1 + z_1, \ldots, y_n + z_n) \\ &= x + (y + z)\end{aligned}$$

■

Problem 1.10. Show that $(ab)x = a(bx)$ for $x \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$.

Solution. We observe the following:

$$\begin{aligned}(ab)x &= (ab)(x_1, \ldots, x_n)\\ &= ((ab)x_1, \ldots, (ab)x_n)\\ &= (a(bx_1), \ldots, a(bx_n))\\ &= a(bx_1, \ldots, bx_n)\\ &= a(bx)\end{aligned}$$

■

Problem 1.11. Show that $1x = x$ for all $x \in \mathbb{F}^n$.

Solution. We observe the following:

$$\begin{aligned}1x &= 1 \cdot (x_1, \ldots, x_n)\\ &= (1x_1, \ldots, 1x_n)\\ &= (x_1, \ldots, x_n)\\ &= x\end{aligned}$$

■

1.2 Definition of Vector Spaces

Problem 1.12. Prove that $-(-v) = v$ for every $v \in V$.

Solution. We observe that

$$(-(-v)) + (-v) = 0$$

So, we see that $-v$ and $-(-v)$ are additive inverses. Now, we recall that each element $v \in V$ has a unique additive inverse.

We note then that

$$v + (-v) = 0$$

So, we know that v is also an additive inverse of $-v$. Thus, since additive inverses are unique, we can conclude that, in fact, we have

$$(-(-v)) = v.$$

■

Problem 1.13. Suppose that $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution. We shall proceed by cases.

We observe that if $a = 0$, then we have that $av = 0$. Thus, we are done.

On the other hand, if $a \neq 0$, then we observe the following:

$$\begin{aligned} v &= 1 \cdot v \\ &= (a^{-1}a)v \\ &= a^{-1}(av) \\ &= a^{-1}(0) \\ &= 0 \end{aligned}$$

Thus, we see that $v = 0$. ■

Problem 1.14. Suppose $v, w \in V$. Explain why exists a unique $x \in V$ such that $v + 3x = w$.

Solution. We begin by noting that if we have $v + 3x = w$, then we see the following:

$$\begin{aligned} 3x &= w - v \\ x &= \frac{1}{3}(w - v) \end{aligned}$$

So, let us set $x = \frac{1}{3}(w - v)$. We observe the following then:

$$\begin{aligned} v + 3(\frac{1}{3}(w - v)) &= v + (w - v) \\ &= v + w + (-v) \\ &= w \end{aligned}$$

Now, we suppose that there exists some other x' such that $v + 3x' = w$. Then, we observe the following:

$$\begin{aligned} (v + 3x) - (v - 3x') &= w - w \\ v - v + 3x - 3x' &= 0 \\ 3(x - x') &= 0 \\ x - x' &= 0 \\ x &= x' \end{aligned}$$

Thus, we see that it must be that $x = x'$. ■

Problem 1.15. The empty set is not a vector space. Which requirement does it fail to satisfy?

Solution. We observe that the empty set $\emptyset = \{\}$ does not contain the element 0; it does not have the additive inverse. This also means then that any vector space V cannot be empty. ■

Problem 1.16. Show that the requirement of having an additive inverse for vector spaces can in fact be replaced by the following:

$$0v = 0, \forall v \in V.$$

Solution. We essentially want to show that $v + (-v) = 0 \iff 0v = 0$.

First, we shall proceed with the forward direction. Let us suppose that we have some element $v' = -v$ such that $v + (-v) = 0$.

Now, we observe the following:

$$
\begin{aligned}
0v &= (0+0)v \\
&= 0v + 0v0v + (-0v) & &= 0v + 0v + (-0v) \\
0 &= 0v
\end{aligned}
$$

Now, let us proceed with the backwards direction. Suppose that $0v = 0$. Then, we see the following:

$$
\begin{aligned}
v + (-v) &= (1 + (-1))v \\
&= 0v \\
&= 0
\end{aligned}
$$

Thus, we see that $v + (-v) = 0$; we have an additive inverse. ∎

Problem 1.17. Let ∞ and $-\infty$ denote two distinct objects, neither of which are in $\mathbb{R}$.

We define addition and scalar multiplication on $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$:

1. The sum and product of real numbers is as usual.
2. For $t \in \mathbb{R}$, we define:
   (a) Multiplication as:

$$
t(\infty) = \begin{cases} -\infty & t < 0 \\ 0 & t = 0 \\ \infty & t > 0 \end{cases}
$$

$$
t(-\infty) = \begin{cases} \infty & t < 0 \\ 0 & t = 0 \\ -\infty & t > 0 \end{cases}
$$

   (b) Addition as:

$$
\begin{aligned}
t + \infty &= \infty + t \\
&= \infty \\
\infty + \infty &= \infty \\
t + (-\infty) &= (-\infty) + t \\
&= -\infty \\
(-\infty) + (-\infty) &= -\infty \\
\infty + (-\infty) &= 0
\end{aligned}
$$

Is $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ a vector space over $\mathbb{R}$?

Solution. Suppose for the sake of contradiction that $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is indeed a vector space over $\mathbb{R}$. Then, we observe the following:

$$\begin{aligned}\infty &= (2 + (-1))\infty \\ &= 2\infty + (-1)\infty \\ &= \infty + (-\infty) \\ &= 0\end{aligned}$$

We note that this means that we don't have a unique additive identity element, and thus $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ cannot be a vector space. ■

1.3 Subspaces

For each of the following subsets of $\mathbb{F}^3$, determine if it is a subspace of $\mathbb{F}^3$.

Problem 1.18.

$$S_1 = \left\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\right\}.$$

Solution. First, we see that $(0, 0, 0)$ is contained in S:

$$0 + 2(0) + 3(0) = 0.$$

Next, we check for closure. First, consider some $v = (x_1, x_2, x_3)$ and $w = (y_1, y_2, y_3)$, where $v, w \in S$. So, we know that $x_1 + 2x_2 + 3x_2 = 0$, and $y_1 + 2y_2 + 3y_3 = 0$.

Now, we observe the following for closure under addition:

$$\begin{aligned}v + w &= (x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) &= (x + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

And for scalar multiplication, we observe that for some $\alpha \in \mathbb{F}$, we have:

$$\begin{aligned}\alpha v &= \alpha(x_1, x_2, x_3) \\ &= (\alpha x_1, \alpha x_2, \alpha x_3) \\ &= \alpha x_1 + 2(\alpha x_2) + 3(\alpha x_3) \\ &= \alpha(x_1 + 2x_2 + 3x_3) \\ &= \alpha(0) \\ &= 0\end{aligned}$$

Thus, we see that S_1 is indeed a subspace of $\mathbb{F}^3$. ■

Problem 1.19.

$$S_2 = \left\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\right\}.$$

Solution. We observe that S_2 is not a subspace of $\mathbb{F}^3$, as it does not contain the zero vector $(0, 0, 0)$:

$$\begin{aligned}0 + 2(0) + 3(0) &= 0 \\ &\neq 4\end{aligned}$$

■

Problem 1.20.

$$S_3 = \left\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\right\}.$$

Solution. We observe that S_3 is not a subspace of $\mathbb{F}^3$ as it isn't closed under addition.

We observe that $(1,1,0)$ and $(0,0,1)$ are in S_3. However, we see that

$$\begin{aligned}(1,1,0) + (0,0,1) &= (1,1,1)\\ 1(1)(1) &= 1\\ &\neq 0\\ \therefore (1,1,1) &\notin S_3\end{aligned}$$

■

Problem 1.21.

$$S_4 = \left\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\right\}.$$

Solution. We first observe that $(0,0,0) \in S_4$.

Next, we test for closure under addition. Suppose we have $v = (x_1, x_2, x_3)$ and $w = (y_1, y_2, y_3)$, both in S_4. Then, we have that $x_1 = 5x_3$ and $y_1 = 5y_3$. Then, we observe the following:

$$\begin{aligned}v + w &= (x_1 + y_1, x_2 + y_2, x_3 + y_3)\\ x_1 + y_1 &= 5x_3 + 5y_3\\ &= 5(x_3 + y_3)\end{aligned}$$

And for scalar multiplication, we see that for some $\alpha \in \mathbb{F}$, we have

$$\begin{aligned}\alpha v &= \alpha(x_1, x_2, x_3)\\ \alpha x_1 &= \alpha(5x_3)\\ &= 5(\alpha x_3)\end{aligned}$$

Thus, we see that S_4 is indeed a subspace. ■

Problem 1.22. Show that the set of differentiable real-valued functions f on the interval $(-4,4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

Solution. First, let us denote this set as S. Now, we observe that the zero-function $f(x) = 0, \forall x \in (-4,4)$ is indeed in S:

$$\begin{aligned}f'(-1) &= 0\\ &= 3f(2)\end{aligned}$$

Now, for closure of addition, we observe that for $f, g \in S$, we have that $f + g$ is also differentiable. Now, we see that:

$$\begin{aligned}(f+g)'(-1) &= f'(-1) + g'(-1)\\ &= 3f(2) + 3g(2)\\ &= 3(f(2) + g(2))\\ &= 3(f+g)(2)\end{aligned}$$

And for scalar multiplication, we observe that for some $\alpha \in \mathbb{R}$, we have that αf is similarly a differentiable function. Now, we have:

$$\begin{aligned}(\alpha f)'(-1) &= \alpha f'(-1) \\ &= \alpha(3f(2)) \\ &= 3(\alpha f(2)) \\ &= 3(\alpha f)(2)\end{aligned}$$

Thus, it follows that S is indeed a subspace of $\mathbb{R}^{(-4,4)}$. ■

Problem 1.23. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Solution. Let us denote this set by S.

First, we will proceed with the forward direction. Let us suppose that S is indeed a subspace of $\mathbb{R}^{[0,1]}$. Then, we know that the zero function $f(x) = 0, \forall x \in [0,1]$ must be in S.

Then, we observe the following:

$$\begin{aligned}\int_0^1 f(x)\mathrm{d}x &= F(1) - F(0) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

Thus, we see that we must have that $b = 0$ for the zero function to be contained in S.

Now, let us proceed with the backwards direction. Suppose that $b = 0$. Then, we observe that the zero function is indeed contained in S as shown earlier.

Now, we will check for closure under addition. Suppose we have $f, g \in S$. Then, we observe that

$$\begin{aligned}\int_0^1 (f+g)(x)\mathrm{d}x &= \int_0^1 f(x) + g(x)\mathrm{d}x \\ &= \int_0^1 f(x)\mathrm{d}x + \int_0^1 g(x)\mathrm{d}x \\ &= 0 + 0 \\ &= 0\end{aligned}$$

For scalar multiplication, we see that for some $\alpha \in \mathbb{R}$, we have:

$$\begin{aligned}int_0^1(\alpha f)(x)\mathrm{d}x &= \int_0^1 \alpha f(x)\mathrm{d}x \\ &= \alpha \int_0^1 f(x)\mathrm{d}x \\ &- \alpha(0) \\ &= 0\end{aligned}$$

Thus, we see that for $b = 0$, we have a subspace of $\mathbb{R}^{[0,1]}$. ■

Problem 1.24. Is $\mathbb{R}^2$ a subspace of $\mathbb{C}^2$?

Solution. No; we observe that if $\mathbb{R}^2$ is indeed a subspace of $\mathbb{C}^2$, then we must have that:

$$i(1,1) = (i,i) \in \mathbb{R}^2.$$

However, of course, we see that $(i,i) \notin \mathbb{R}^2$; it isn't closed under scalar multiplication. Thus, we see that we have a contradiction, and thus $\mathbb{R}^2$ can't be a subspace of $\mathbb{C}^2$. ■

Problem 1.25. Is $\left\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\right\}$ a subspace of $\mathbb{R}^3$?

Solution. Yes; we observe that if $a^3 = b^3$, then it follows that $a = b$. Then, from here, we can proceed as follows from previous problems (namely, 1.21). ■

Problem 1.26. Is $\left\{(a,b,c) \in \mathbb{C}^3 : a^3 = b^3\right\}$ a subspace of $\mathbb{C}^3$?

Solution. No; we recall from 1.2 that

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = 1.$$

In fact, the following vector is in our subset

$$\left(1, \frac{-1+\sqrt{3}i}{2}, 0\right)$$
$$\left(1, \frac{-1-\sqrt{3}i}{2}, 0\right)$$

However, we note that

$$\left(1, \frac{-1+\sqrt{3}i}{2}, 0\right) + \left(1, \frac{-1-\sqrt{3}i}{2}, 0\right) = (2,-1,0)$$

is not in our subset; and thus, it isn't closed under addition. Therefore, it isn't a subspace of $\mathbb{C}^3$. ■

Problem 1.27. Give an example of a nonempty subset U of $\mathbb{R}^2$ that is closed under addition and under taking additive inverses, but U is not a subspace of $\mathbb{R}^2$.

Solution. We can define U to be as follow:

$$U \coloneqq \left\{(x,y) \in \mathbb{R}^2 : x,y \in \mathbb{Z}\right\}.$$

We observe then that for any $x,y \in \mathbb{Z}$, we have $-x,-y \in \mathbb{Z}$ such that $(x,y) + (-x,-y) = (x-x, y-y) = (0,0) \in U$.

Furthermore, we see that it is closed under addition, as for u, u', we have:

$$\begin{aligned} u + u' &= (x,y) + (x',y') \\ &= (x+x', y+y') \end{aligned}$$

And by closure of integers, we see that $x + x' \in \mathbb{Z}$ and $y + y' \in \mathbb{Z}$.

However, we note that for $\alpha \in \mathbb{R}$, we have:

$$\begin{aligned} \alpha u &= \alpha(x, y) \\ &= (\alpha x, \alpha y) \end{aligned}$$

However, if $\alpha \notin \mathbb{Z}$, then it follows that $\alpha x, \alpha y \notin \mathbb{Z}$, and thus we have that $\alpha u = (\alpha x, \alpha y) \notin U$; thus we see that it isn't closed under scalar multiplication. ■

Problem 1.28. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that it is closed under scalar multiplication, but is not a subspace of $\mathbb{R}^2$.

Solution. Let us define U to be as follows

$$U := \left\{(x, y) \in \mathbb{R}^2 : x = 0 \vee y = 0 \in\right\}.$$

Then, we observe that for $\alpha \in \mathbb{R}$, we have three cases:

1. If $x = 0$, then $\alpha u = \alpha(0, y) = (\alpha(0)\alpha(y)) = (0, \alpha y)$. And we see that $x = 0$ still, so $\alpha u \in U$.
2. If $y = 0$, then we see that $\alpha u = \alpha(x, 0) = (\alpha(x), \alpha(0)) = (\alpha x, 0)$. Again, since $y = 0$, we see that $\alpha u \in U$.
3. If $x, y = 0$, then it follows that $\alpha u = \alpha(0, 0) = (0, 0)$. And we see that $\alpha u \in U$ still.

However, we note now that while $u = (x, 0), u' = (0, y) \in U$, for $x, y \neq 0$, we see that

$$\begin{aligned} u + u' &= (x, 0) + (0, y) \\ &= (x, y) \end{aligned}$$

However, since $x, y \neq 0$, then $u + u' \notin U$. Thus, it is not closed under addition and thus not a subspace of $\mathbb{R}^2$. ■

Problem 1.29. A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$?

Solution. Before we start, let us recall the following lemma:

Lemma 1.1. $\sqrt{2} \notin \mathbb{Q}$.

Now, first, let us define $f(x) = \sin(\sqrt{2}x)$ and $g(x) = \cos(x)$. Then, let us define $h(x) = f(x) + g(x)$.

Now, we observe that $h(0) = \sin(0) + \cos(0) = 1$. Then, we also know that since f, g are both periodic functions, then $h(x)$ is too. This means then that there exists some p such that $h(0) = h(p) = h(-p)$.

However, we note now that if $h(0) = 1$, then it follows that

$$1 = \sin(\sqrt{2}p) + \cos(p) = \sin(-\sqrt{2}p) + \cos(-p) = -\sin(\sqrt{2}p) + \cos(p).$$

However, we note here that since $\sin(\sqrt{2}p) + \cos(p) = -\sin(\sqrt{2}p) + \cos(p)$, then it follows that $2\sin(\sqrt{2}p) = 0$; thus, we have that $\sin(\sqrt{2}p) = 0$. Thus, we know that $\cos(p) = 1 \implies p = 2\pi k$, for some $k \in \mathbb{Z}$.

However, we note as well that since $\sin(\sqrt{2}p) = 0$, then we know that $\sqrt{2}p = l\pi$ for some $l \in \mathbb{Z}$.

However, this means then that:

$$\frac{\sqrt{2}p}{p} = \frac{l\pi}{2k\pi}$$
$$\sqrt{2} = \frac{l}{2k}$$

However, this implies that $\sqrt{2} \in \mathbb{Q}$; however, this is a contradiction, and thus we see that the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ can't be a subspace of $\mathbb{R}^{\mathbb{R}}$. ■

Problem 1.30. Suppose that U_1, U_2 are subspaces of V. Prove that $U_1 \cap U_2$ is also a subspace of V.

Solution. First, we note that the zero vector 0 must be in both U_1 and in U_2 as they are both subspaces of V; thus, we see that $0 \in U_1 \cap U_2$.

Next, we observe that, by definition, we have that for any $x, y \in U_1 \cap U_2$, it must be that it is also in U_1 and also in U_2.

Now, since $x, y \in U_1$, we know $x + y \in U_1$ too, as U_1 is a subspace, and thus is closed under addition. Similarly, since $x, y \in U_2$, then $x + y \in U_2$ as well by the same argument. Thus, we see that $x + y \in U_1 \cap U_2$.

The same argument follows for scalar multiplication.

Thus, we see that $U_1 \cap U_2$ is indeed a subspace of V as well. ■

Remark 1.2. We note here that a similar argument can be used to prove that the intersection of every collection of subspaces of V is also a subspace of V:

Proof. We suppose that for $i \in I$, U_i is a subspace of V. Now, we want to show that $\bigcap_{i \in I} U_i$ is indeed a subspace as well.

To do this, we note that by definition of a subspace, each U_i for $i \in I$ must contain the zero vector 0; thus, we see that $0 \in \bigcap_{i \in I} U_i$.

Next, we will check for closure under addition. To do this, we note that if $x, y \in \bigcap_{i \in I} U_i$, then it follows that for each $i \in I$, $x, y \in U_i$. This means then that, since each U_i is a subspace, it follows that $x + y \in U_i$ for each $i \in I$. Thus, we can then conclude that $x + y \in \bigcap_{i \in I} U_i$ too. Thus, it is closed under addition.

For scalar multiplication, we take some $\alpha \in \mathbb{F}$. Then, we note that for $x \in \bigcap_{i \in I} U_i$, $x \in U_i$ for $i \in I$. And since each U_i is a subspace, it is closed under scalar multiplication, and thus $\alpha x \in U_i$ for $i \in I$; thus, we can conclude that $\alpha x \in \bigcap_{i \in I} U_i$ too.

Thus, we see that $\bigcap_{i \in I} U_i$ is indeed a subspace of V. ■

Problem 1.31. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution. We shall proceed by the backwards direction. Let us suppose that there are two subspaces U_1, U_2 of V, and without loss of generality, let us suppose that $U_1 \subseteq U_2$.

We note then that $U_1 \cup U_2$ is in fact U_2. And since U_2 is a subspace of V, then of course $U_1 \cup U_2 = U_2$ is as well.

Now, let us proceed with the forward direction. We will proceed by contradiction; that is, suppose we have two subspaces U_1, U_2 such that $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$, and that $U_1 \cup U_2$ is a subspace of V.

From here, let us consider two vectors $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$. We observe then that $u_1 + u_2 \in U_1 \cup U_2$. Then, by definition, we see that $u_1 + u_2$ is either in U_1 or U_2.

We now consider two cases.

First, if $u_1 + u_2$ is in U_1, then we see that by property of subspaces, we have that $u_1 + u_2 + (-u_1) = u_2 \in U_1$. However, this is a contradiction as u_2 can't be contained in U_1 by construction.

Similarly, if $u_1 + u_2 \in U_2$, then we see that $u_1 + u_2 + (-u_2) = u_1 \in U_2$. Again, this is a contradiction.

Therefore, we see that it must follow that either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. ■

Problem 1.32. Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two. Also note that we are not working in a field containing two elements.

Solution. Once again, we begin with the backwards direction. We consider subspaces U_1, U_2, U_3 of V. Without loss of generality, let us suppose that U_1 contains U_2 and U_2. It follows then that $U_1 \cup U_2 \cup U_3 = U_1$. And since U_1 is a subspace of V, it follows that $U_1 \cup U_2 \cup U_3 = U_1$ is also a subspace of V.

Now, for the forward direction. We first consider $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. We denote this union by W.

Then, we see that $U_3 \subseteq W$ becomes the two-union case which we have shown last problem; we see then that one of these subspaces must contain the other two.

Next, let us suppose that U_1, U_2 are not contained in each other. From here, let us consider $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$.

We will now show that $au_1 + u_2 \in U_3$. To do this, we observe that ■

Remark 1.3. We remark that if we had a field containing two elements, then if we let $V =$

Problem 1.33. Suppose U is a subspace of V. What is $U + U$?

Solution. We note that since U is a subspace of V, we then consider $u \in U$ and $u' \in U$. We see then that $u + u' \in U + U$. However, since $u, u' \in U$, then by closure of addition, we have that $u + u' \in U$ as well. Thus, we see that every vector in $U + U$ is also in U; $U + U \subseteq U$.

Similarly, for $u \in U$, we observe that $u = u + 0$, and thus every $u \in u$ is also in $U + U$. So, we have $U \subseteq U + U$.

Thus, we see that $U = U + U$. ■

Problem 1.34. If U, W are subspaces of V, is $U + W = W + U$?

Solution. We consider $x \in U$ and $y \in W$. Now, we note that addition is commutative in V.

So, it follows that for $x + y \in U + W$, we have $x + y = y + x \in W + U$. So, we see that $U + W \subseteq W + U$.

Similarly, we note that for $y + x \in W + U$, we have $y + x = x + y \in U + W$. So, $W + U \subseteq U + W$.

Therefore, we can conclude that, indeed, $U + W = W + U$. ■

Problem 1.35. If U_1, U_2, U_3 are subspaces of V, does it follow that

$$U_1 + (U_2 + U_3) = (U_1 + U_2) + U_3?$$

Solution. We consider x, y, z in U_1, U_2, U_3 respectively. Now, we note that addition is commutative in V.

We observe that $x + (y + z) \in U_1 + (U_2 + U_3)$. Now, $x + (y + z) = (x + y) + z \in (U_1 + U_2) + U_3$.

So, we see that $U_1 + (U_2 + U_3) \subseteq (U_1 + U_2) + U_3$.

A similar argument follows for the other direction.

Thus, we conclude that, indeed, we have $U_1 + (U_2 + U_3) = (U_1 + U_2) + U_3$. ∎

Problem 1.36. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution. We observe that for some subspace U of V, and for the subspace $W = \{0\}$, we have that $U + W = W + U = U$. Thus, the additive identity is simply $\{0\}$.

For an additive inverse, we observe that for a subspace U, there exists a subspace W such that $U + W = 0$. However, we note that since $U \subseteq U + W$ and $W \subseteq U + W$, it follows then that the only way for $U + W = 0$ to occur is if both $U, W = \{0\}$. ∎

Problem 1.37. Prove or disprove the following statement:

If U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Solution. We will provide a counterexample.

Suppose that $V = \mathbb{R}^2$, $W = \operatorname{Span}\{(1,1)\}$, $U_1 = \operatorname{Span}\{(1,0)\}$, and $U_2 = \operatorname{Span}\{(0,1)\}$.

Now, we observe that $U_1 + W = U_2 + W$. However, it is clear that $U_1 \neq U_2$. Thus, we have found a counterexample. ∎

Problem 1.38. Prove or disprove the following statement:

If U_1, U_2, W are subspaces of V such that

$$U_1 \oplus W = U_2 \oplus W,$$

then $U_1 = U_2$.

Solution. We will provide a counterexample.

Suppose that $V = \mathbb{R}^2$, $W = \operatorname{Span}\{(1,1)\}$, $U_1 = \operatorname{Span}\{(1,0)\}$, and $U_2 = \operatorname{Span}\{(0,1)\}$.

Now, we observe that $U_1 \oplus W = U_2 \oplus W$. However, it is clear that $U_1 \neq U_2$. Thus, we have found a counterexample. ∎

Problem 1.39. Suppose
$$U = \left\{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\right\}.$$
Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

Solution. We observe that we can simply let $W = \left\{(0, 0, a, b, c) \in \mathbb{F}^5 : a, b, c \in \mathbb{F}\right\}$.

First, we will show that $U \cap W = \{0\}$. To do this, we note that we can rewrite U to be

$$\begin{aligned}(x, y, x+y, x-y, 2x) &= (x, 0, x, x, 2x) + (0, y, y, -y, 0) \\ &= x(1, 0, 1, 1, 2) + y(0, 1, 1, -1, 0)\end{aligned}$$

So, we can in fact rewrite $U = \operatorname{Span}\{(1, 0, 1, 1, 2), (0, 1, 1, -1, 0)\}$. We denote the first and second vectors as u_1, u_2 respectively.

Similarly for W, we can rewrite it as $W = \operatorname{Span}\{(0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$. We denote these vectors as w_1, w_1, w_3 respectively.

Now, we want to prove that these vectors are linearly independent. Then, we want the following to hold true only when $a_1 = \cdots = a_5 = 0$:

$$\begin{aligned}a_1u_1 + a_2u_2 + a_3w_1 + a_4w_2 + a_5w_3 &= 0 \\ a_1(1, 0, 1, 1, 2) + a_2(0, 1, 1, -1, 0) + a_3(0, 0, 1, 0, 0) + a_4(0, 0, 0, 1, 0) + a_5(0, 0, 0, 0, 1) &= 0\end{aligned}$$

So, with this in mind, we observe that we get the following:

$$\begin{aligned}a_1 &= 0 \\ a_2 &= 0 \\ a_1 + a_2 + a_3 &= 0 \\ a_1 - a_2 + a_4 &= 0 \\ 2a_1 + a_5 &= 0\end{aligned}$$

Then from here, we see that this system of equation shows us that $a_1 = a_2 = a_3 = a_4 = a_5 = 0$. Thus, they're linearly independent.

Furthermore, since we have a list of five linearly independent vectors, then it follows that $U \oplus W = \mathbb{F}^5$. ■

Problem 1.40. Suppose we have a function $f : \mathbb{R} \to \mathbb{R}$. We define an even function to be as follows:

$$f(-x) = f(x).$$

And we define an odd function to be
$$f(-x) = -f(x).$$

Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o be the set of real-valued odd functions of $\mathbb{R}$. Prove that $U_e \oplus U_o = \mathbb{R}^{\mathbb{R}}$.

Solution. First, we will begin by showing that $U_e + U_0 = \mathbb{R}^{\mathbb{R}}$. Let us consider $f_e(x) \in U_e$ and $f_o(x) \in U_o$.

Here, we note that we can rewrite f_e and f_o as follows:

$$\begin{aligned} f_e(x) &= \frac{f(x)+f(-x)}{2} \\ f_o(x) &= \frac{f(x)-f(-x)}{2} \end{aligned}$$

Then, we observe the following:

$$\begin{aligned} f_e(-x) &= \frac{f(-x)+f(-(-x))}{2} \\ &= \frac{f(x)+f(-x)}{2} \\ &= f_e(x) \\ f_o(-x) &= \frac{f(-x)-f(-(-x))}{2} \\ &= \frac{-f(x)+f(-x)}{2} \\ &= -\frac{f(x)-f(-x)}{2} \\ &= -f_o(x) \end{aligned}$$

Next, we see that

$$\begin{aligned} f_e(x)+f_o(x) &= \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} \\ &= \frac{2f(x)}{2} \\ &= f(x) \end{aligned}$$

Thus, we see that $U_e + U_o = \mathbb{R}^{\mathbb{R}}$.

Now, from here, we will show that $U_e \cap U_o = \{0\}$. To do this, we let $f \in U_e \cap U_o$. Since $f \in U_e \cap U_o$, we see that $f(-x) = f(x)$ and also $f(-x) = -f(x)$. Then, we observe that

$$\begin{aligned} f(-x) &= f(-x) \\ f(x) &= -f(x) \\ 2f(x) &= 0 \\ f(x) &= 0 \end{aligned}$$

Thus, we see that for all $x \in \mathbb{R}$, $f(x) = 0$; in other words, $f = 0$. Therefore, we see that $U_e \cap U_o = \{0\}$.

Thus, we have proven that $U_e \oplus U_o = \mathbb{R}^{\mathbb{R}}$. ■

Chapter 2
Finite-Dimensional Vector Spaces

2.1 Span and Linear Independence

Problem 2.1. Suppose that v_1, v_2, v_3, v_4 spans V. Show that

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Solution. We observe that we can express each of $v_1, \ldots, v_4$ in terms of the second list:

$$\begin{aligned}
v_1 &= (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4 \\
v_2 &= (v_2 - v_3) + (v_3 - v_4) + v_4 \\
v_3 &= (v_3 - v_4) + v_4 \\
v_4 &= v_4
\end{aligned}$$

Thus, since we can express each vector $v_1, \ldots, v_4$ as a linear combination of $v_1 - v_2, \ldots, v_4$, we see then that they must also span V. ■

Problem 2.2. Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list $(1 + i, 1 - i)$ is linearly independent.

Solution. We want to show that the following holds true only when $a_1 = a_2 = 0$:

$$a_1(1 + i) + a_2(1 - i) = 0.$$

We see then that we have the following:

$$\begin{aligned}
a_1 + a_2 &= 0 \\
a_1 i - a_2 i &= 0
\end{aligned}$$

$$()a_1 - a_2)i = 0$$

So, we see that $a_1 + a_2 = 0$ and $a_1 - a_2 = 0$. This means then that $a_1 = a_2 = 0$.
Thus, we see that they're linearly independent. ■

Problem 2.3. content...