Math 135: Homework 5

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Problems

4 The Natural Numbers

Problem 4.2. Show that if a is a transitive, then a^+ is also a transitive set.

Solution. Let us suppose that a is a transitive set. Then, it follows that for all a_1, a_2, if $a_1 \in a_2$ and $a_2 \in a$, then $a_1 \in a$ as well. But we see that this means then that for all $a_2 \in a$, we have that $a_2 \subseteq a$.

So, let us take some $x \in a^+$. Then, either $x \in a$ or $x \in \{a\}$. In the first case, we observe that since a is a transitive set, then if $x \in a$ it follows that $x \subseteq a \subseteq a^+$. So, we see that $x \subseteq a^+$ as desired.

On the other hand, if $x \in \{a\}$, we observe that $x = a$, and thus $x = a \subseteq a^+$ as desired.

Therefore, we see that, indeed, a^+ is a transitive set. ■

Problem 4.4. Show that if a is a transitive set, then $\bigcup a$ is also a transitive set.

Solution. Let us suppose that a is a transitive set. Then, we observe that for all a_1, a_2, if $a_1 \in a_2$ and $a_2 \in a$, it follows that $a_1 \in a$.

Now, with this in mind, we see that $\bigcup a$ consists of the elements of the elements of a. Then, as a is transitive, the elements of the elements of a are also elements of a; we have that $\bigcup a \subseteq a$.

From here, let us take some $a_2 \in \bigcup a$. Since $\bigcup a \subseteq a$, it follows that $a_2 \in a$ as well.

Then, let us take some $a_1 \in a_2 \in a$. By definition, we note that $\bigcup a$ consists of the elements of the elements of a, so we see that $a_1 \in \bigcup a$ as well, and thus we see that $\bigcup a$ is transitive as desired. ■

Problem 4.6. Prove that if $\bigcup(a^+) = a$, then a is a transitive set.

Solution. Suppose that $\bigcup(a^+) = a$. Then, we observe the following:

$$\begin{aligned} a &= \bigcup(a^+) \\ &= \bigcup(a \cup \{a\}) \\ &= \bigcup a \cup \bigcup \{a\} \\ &= \bigcup a \cup a \end{aligned}$$

Then, from this, we see that $\bigcup a \subseteq a$. In other words, the elements of the elements of a are also elements of a, meaning that a is a transitive set as desired. ■

Problem 4.8. Let f be a one-to-one function from A into A, and assume that $c \in A - \operatorname{ran} f$. Define $h : \omega \to A$ by recursion:

$$\begin{aligned} h(0) &= c \\ h(n^+) &= f(h(n)). \end{aligned}$$

Show then that h is also one-to-one.

Solution. We shall show that h is one-to-one by induction.

First, let us define the following set:

$$S := \{n \in \omega : (\forall m, n)\, h(m) = h(n) \implies m = n\}.$$

We note that the condition for our set can also be written as $\forall m, n\, (m \neq n \implies h(m) \neq h(n))$.

Now, let us suppose that $m \neq 0$. Then, it follows that m is the successor of some natural number $p \in \omega$. That is, $m = p^+$. Then, we observe that

$$h(m) = h(p^+) = f(h(p)).$$

And we note that $c \in A - \operatorname{ran} f$, meaning that it isn't in $\operatorname{ran} f$. So, we know that $f(h(p)) \neq c$. This, we see that $h(0) = c \neq h(m)$ for all $m \neq 0$. Thus, we see that, indeed, $0 \in S$.

Next, let us suppose that $k \in S$. Then, we look at $m = k^+$. We observe the following:

$$f(h(k)) = h(k^+) = h(m) = h(p^+) = f(h(p)).$$

Now, because f is one-to-one, it follows that $h(k) = h(p)$. And by our induction hypothesis, we have that $k = p$. Then, because $k = p$, we see then that $k^+ = p^+ = m$. So, we see that indeed, $h(k^+) = h(m) \implies k^+ = m$, meaning that $k^+ \in S$, and thus S is inductive.

Therefore, we see that h is one-to-one. ■

Problem 4.9. Let f be a function from B into B, and assume that $A \subseteq B$. We have two possible methods for constructing the "closure" C of A under f. First, define C^* to be the intersection of the closed supersets of A:

$$C^* = \bigcap \{X : A \subseteq X \subseteq B \land f[\![X]\!] \subseteq X\}$$

Alternatively, we could apply the recursion theorem to obtain the function h for which

$$h(0) = A$$
$$h(n^+) = h(n) \cup f[\![h(n)]\!]$$

Clearly $h(0) \subseteq h(1) \subseteq \cdots$, define C_* to be $\bigcup \operatorname{ran} h$; in other words

$$C_* = \bigcup_{i \in \omega} h(i)$$

Show that $C^* = C_*$.

Solution. First, we show that $C^* \subseteq C_*$ by showing that $f[\![C_*]\!] \subseteq C_*$.

Proof. We first observe that, by definition, we have that for all $n \in \omega$, we have that $f[\![h(n)]\!] \subseteq h(n^+)$. Then, we have the following:

$$f[\![C_*]\!] = f\left[\!\!\left[\bigcup_{i \in \omega} h(i)\right]\!\!\right] = \bigcup_{i \in \omega} f[\![h(i)]\!] \subseteq \bigcup_{i \in \omega} h(i^+) \subseteq \bigcup_{i \in \omega} h(i) = C_*.$$

Thus, we see that $f[\![C_*]\!] \subseteq C_*$, and thus we have that $C^* \subseteq C_*$. □

Next, we will show that $C_* \subseteq C^*$ by using induction to show that $h(n) \subseteq C^*$.

Proof. Let us construct a set S to be as follows:

$$S \coloneqq \{n \in \omega : h(n) \subseteq C^*\}.$$

We also define the set C' to be

$$C' \coloneqq \{X : A \subseteq X \subseteq B \wedge f[\![X]\!] \subseteq X\}.$$

We see then that $C^* = \bigcap C'$.

Now, we first see that $h(0) = A$. Then, we observe that by definition, for all $X \in C'$, $A \subseteq X$. Then, since $C^* = \bigcap C'$, it follows that $A \subseteq C^*$. So, we have that $h(0) = A \subseteq C^*$, meaning that $0 \in S$.

Next, suppose that $k \in S$. Then, we look at k^+.

We see that $h(k^+) = h(k) \cup f[\![h(k)]\!]$. Then, by our induction hypothesis, we have that $h(k) \subseteq C^*$, and by definition of C^* it must be that $h(k) \subseteq X$ for all $X \in C'$.

Then, we note that because $h(k) \subseteq X$ for all $X \in C'$, it follows then that $f[\![h(k)]\!] \subseteq f[\![X]\!] \subseteq X$, for all $X \in C'$. But this means then that $f[\![h(k)]\!] \subseteq C*$ by definition.

Therefore, we have that $k^+ \in S$ as desired, meaning that S is inductive. So we have that $h(n) \subseteq C^*$, and thus $C_* \subseteq C^*$. □

Then, since we have shown that $C^* \subseteq C_*$ and $C_* \subseteq C^*$, it follows that these two sets are indeed the same. ■

Problem 4.10. In Exercise 9, assume that B is the set of real numbers, $f(x) = x^2$, and A is the closed interval $\left[\frac{1}{2}, 1\right]$. What is the set called C^* and C_*.

Solution. Using our recursion definition, we observe that $h(0) = A = \left[\frac{1}{2}, 1\right]$. Then, we see that $h(1) = h(0) \cup f[\![h(0)]\!] = h(0) \cup \left[\frac{1}{4}, 1\right]$.

We observe then that we can keep on repeating this process and see that the lower bound will converges to 0.

Thus, we see that $C_* = [0, 1] = C^*$. ■