Math 135: Homework 11

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Problems

7 Orderings and Ordinals

Problem 7.14. Assume that $\langle A, < \rangle$ is a partially ordered structure. Define the function F on A by the equation

$$F(a) = \{x \in A : x \leq a\}.$$

and let $S = \operatorname{ran} F$. Show that F is an isomorphism from $\langle A, < \rangle$ onto $\langle S, \subset_S \rangle$.

Solution. First, we will show that F is indeed a bijection. To do this, we first note that since $S = \operatorname{ran} F$, then by definition, F maps A onto S.

To show that it is injective, let us suppose for the sake of contradiction that for $a_1, a_2 \in A$ such that $F(a_1) = F(a_2)$, we have that $a_1 \neq a_2$. But if $a_1 \neq a_2$, then we note that $a_1 \in F(a_1)$ and $a_2 \in F(a_2)$, but $a_1 \neq a_2 \implies F(a_1) \neq F(a_2)$ as these two sets contain different elements. But this is a contradiction. Thus, F must be injective.

Since F is both one-to-one and onto, we note then that it is indeed a bijection.

To show that if $a_1 < a_2$ then $F(a_1) \subset_S F(a_2)$, we first note that by definition, $a_2 \in F(a_2)$. However, because $a_1 < a_2 \implies a_2 \not\leq a_1$, we have that $a_2 \notin F(a_1)$.

Next, we note that for all $a \in F(a_1)$, by definition we have that $a \leq a_1$. However, since $a \leq a_1 < a_2$, it follows that $a < a_2$; in other words, for all $a \in F(a_1)$, we have that $a \in F(a_2)$. And since $a_2 \in F(a_2)$ but $a_2 \notin F(a_1)$, we observe then that, indeed, $F(a_1) \subset_S F(a_2)$ as desired.

For the other direction, note that if $F(a_1) \subset F(a_2)$, it follows then that there exists some $a \in F(a_2)$ which isn't in $F(a_1)$. More specifically, by definition of F, we note that $F(a_1)$ contains all elements $a \leq a_1$, and $F(a_2)$ contains elements $a \leq a_2$. Since $F(a_1) \subset F(a_2)$, we see that all $a \in F(a_1)$ is also in $F(a_2)$. But $F(a_1) \subset F(a_2)$, meaning that $a_1 < a_2$ for there to be elements in $F(a_2)$ that isn't in $F(a_1)$.

Thus, we see that indeed, F is an isomorphism from $\langle A, < \rangle$ onto $\langle S, \subset_S \rangle$. ■

Problem 7.15a. Assume that $<$ is a well-ordering on A and that $t \in A$. Show that $\langle A, < \rangle$ is never isomorphic to $\langle \operatorname{seg} t, <^\circ \rangle$.

Solution. Suppose for the sake of contradiction that $\langle A, < \rangle$ is isomorphic to $\langle \operatorname{seg} t, <^\circ \rangle$. That is, for some $t \in A$, there exists some isomorphism $F : A \to \operatorname{seg} t$.

Now, we note that since F is isomorphic, it follows that for all $a_1 < a_2$ in A, we have that $F(a_1) < F(a_2)$. Furthermore, by a previous exercise, we note then that this means that $a \leq f(a)$ for all $a \in A$.

With this in mind, we observe then that $f(t) \in \operatorname{seg} t$, and thus by definition we have that $f(t) < t$. However, we note that this contradicts with the fact that $t \leq f(t)$. Thus, we conclude that no such isomorphism can exist. ■

Problem 7.15b. Show that in Theorem 7K, at most one of the three alternatives holds.

Solution. Suppose for the sake of contradiction that more than one of the three alternatives can hold.

Then, we consider the case where we're comparing A and B. Now, by the theorem, we have three cases:

1. $\langle A, <_A \rangle \simeq \langle B, <^\circ \rangle$
2. $\langle A, <_A \rangle \simeq \langle \operatorname{seg} b, <^\circ \rangle$, for some $b \in B$
3. $\langle \operatorname{seg} a, <_A \rangle \simeq \langle B, <^\circ \rangle$, for some $a \in A$

Suppose Case 1 holds and Case 2 holds as well. We have then that $\langle A, <_A\rangle \simeq \langle B, <^\circ\rangle$ and $\langle A, <_A\rangle \simeq \langle \operatorname{seg} b, <^\circ\rangle$.

However, since we can construct an isomorphism between $\langle A, <_A\rangle$ and $\langle B, <^\circ\rangle$, and between $\langle A, <_A\rangle$ and $\langle \operatorname{seg} b, <^\circ\rangle$, we note then that we can thus construct an isomorphism between $\langle \operatorname{seg} b, <^\circ\rangle$ and $\langle B, <^\circ\rangle$.

This contradicts with the previous subpart.

We can proceed similarly for if Case 1 and Case 3 holds at the same time.

For Case 2 and Case 3 both holding, we note that we have $\langle A, <_A\rangle \simeq \langle \operatorname{seg} b, <^\circ\rangle$ and $\langle \operatorname{seg} a, <_A\rangle \simeq \langle B, <^\circ\rangle$. But if this is the case, we can then construct an isomorphism between each set and their own initial segment.

First, we create an injection between $\operatorname{seg} a$ and A. Then, from A we know there's an isomorphism with $\operatorname{seg} b$, from which we can construct another injection to B. And B is isomorphic to $\operatorname{seg} a$.

However, this means then that they're all isomorphic with one another; i.e., $\langle A, <\rangle$ and $\langle B, <^\circ\rangle$ are isomorphic to their own initial segment. Thus, we have another contradiction.

So, we conclude that at most one of the scenarios can occur. ■

Problem 7.16. Assume that α and β are ordinal numbers with $\alpha \in \beta$. Show that $\alpha^+ \in \beta^+$. Conclude then that whenever $\alpha \neq \beta$, $\alpha^+ \neq \beta^+$.

Solution. We observe that $\alpha \in \beta$. Since α^+ is the least ordinal larger than α, then by definition we have $\alpha \in \alpha^+$, and $\alpha^+ \subseteq \beta$.

Similarly, by definition of β^+, we have that $\beta \in \beta^+ \iff \beta \subset \beta^+$.

Then, we have:

$$\alpha \subset \alpha^+ \subseteq \beta \subset \beta^+.$$

Thus, we have $\alpha^+ \subset \beta^+ \iff \alpha^+ \in \beta^+$.

Then, we note that if $\alpha \neq \beta$, then either $\alpha \in \beta$ or $\beta \in \alpha$. And thus we have then that either $\alpha^+ \in \beta^+$ or $\beta^+ \in \alpha^+$; in other words, we have $\alpha^+ \neq \beta^+$ as desired. ■

Problem 7.19. Assume that A is a finite set and that $<$ and $\prec$ are linear orderings on A. Show that $\langle A, <\rangle$ and $\langle A, \prec\rangle$ are isomorphic.

Solution. First, we note that since A is finite, then it follows that $<$ and $\prec$ are not only linear orderings on A, but are in fact well-orderings.

From here, we observe that every non-empty subset of A must thus have a minimal element with respect to $<$ and $\prec$. Furthermore, since A is a finite set, it follows then that there exists some $m \in \omega$ such that $A \approx m$.

With this in mind, we can construct an isomorphism $F : \langle A, <\rangle \to \langle A, \prec\rangle$ as follows:

- First, for $n = 0$, we denote a_0 to be the least element in A with respect to $<$. Also, we denote a_0' to be the least element in A with respect to $\prec$. Then, we let $a_0 \mapsto a_0'$.
- Then, for $n = k$, we map the minimal element $a_k \in \langle A, <\rangle \setminus \{a_0, \dots, a_{k-1}\}$ to the minimal element $a_k' \in \langle A, \prec\rangle \setminus \{a_0', \dots, a_{k-1}'\}$.

By construction, we see that if $a < b$, then we have that $F(a) \prec F(b)$.

For injectivity, let us suppse for contradiction that for $a, b \in A$, $F(a) =_\prec F(b)$ but $a \neq_< b$. Then, since $<$ is a linear ordering, we note that either $a < b$ or $b < a$.

Without loss of generality, let us suppose the first case holds. Then, we see that if $a < b$, it follows then that $F(a) \prec F(b)$; however, this means then that $F(a) \not\preceq F(b)$ which is a contradiction. So, indeed, F is injective.

For surjectivity, we want to show that for all $a' \in A$, there exists some $a \in A$ such that $F(a) = a'$. Then, we note that for all $n \in \omega$, we have that $a'_n = F(a_n)$. Thus, we see that indeed F is a surjection.

So, we conclude that F is an isomorphism as desired. ■

Problem 7.23i. Assume that A is a set and define α to be the set of ordinals dominated by A. Show that α is a cardinal number.

Solution. Suppose for the sake of contradiction that α is not a cardinal number. Then, there exists some ordinal $\beta < \alpha$ such that β is a cardinal number of α.

Now, we note that since $\beta < \alpha$, it follows then that $\beta \in \alpha$. This means then that β is dominated by A. However, by definition of α, we note that α isn't dominated by A else $\alpha \in \alpha$ which is a contradiction. Thus, β can't be equinumerous to α.

Therefore, we conclude that $\operatorname{card} \alpha = \alpha$; i.e., α is a cardinal number as desired. ■

Problem 7.23ii. Show that $\operatorname{card} A < \alpha$.

Solution. Suppose for the sake of contradiction that $\operatorname{card} A \geq \alpha$. Then, we note that $\operatorname{card} \alpha \leq \operatorname{card} A$. But if this was the case, we note then that α is thus dominated by A; this is a contradiction.

Therefore, we conclude that $\operatorname{card} A < \alpha$. ■

Problem 7.23iii. Show that α is the least cardinal greater than $\operatorname{card} A$.

Solution. Suppose for contradiction that there exists some β such that $\operatorname{card} A < \beta < \alpha$. Then, we have $\operatorname{card} A < \beta$; in other words, we see that β is not dominated by A.

Then, since α is the set of ordinals which is dominated by A, we have then that $\alpha \leq \beta$ by definition. But this is thus a contradiction.

Therefore, we conclude that α is indeed the least cardinal greater than $\operatorname{card} A$. ■

Problem 7.24. Show that for any ordinal number α, there exists a cardinal number κ that is (as an ordinal) larger than α.

Solution. We note that this follows from the previous question. Recall that any set A is equinumerous to some ordinal α.

Then, with this in mind, we observe that we can define κ to be a set of ordinals dominated by A as per the previous question. Then, we note that κ is itself a cardinal number and that $\operatorname{card} \alpha = \operatorname{card} A < \kappa$. It follows then that $\alpha < \kappa$.

Thus, we have proven the claim as desired. ■