Math 135: Homework 1

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Spring 2024

Contents

1 Axioms

Problem 1.1 (Problem 2.2). Give an example of sets A and B for which $\bigcup A = \bigcup B$, but $A \neq B$.

Solution. We can consider the following sets:

$$A = \{\{a\}, \{b\}\}$$
$$B = \{\{a, b\}\}$$

Since the sets A and B contain different elements, we see that they are in fact not equal. However, note here that:

$$\bigcup A = \{a, b\}$$
$$\bigcup B = \{a, b\}.$$

Since $\bigcup A$ and $\bigcup B$ have the same elements, we see that they are in fact equal by the Extension Axiom. Thus, we have sets A, B for which $\bigcup A = \bigcup B$, but $A \neq B$. ■

Problem 1.2 (Problem 2.6a). Show that for any set A, $\bigcup \mathscr{P} A = A$.

Solution. To do this, we shall the following inclusions: $\bigcup \mathscr{P} A \subseteq A$, and $A \subseteq \bigcup \mathscr{P} A$.

First, we will show that $A \subseteq \bigcup \mathscr{P} A$. To do this, we will first introduce the following lemma:

Lemma 1.1. Every member of a set A is a subset of $\bigcup A$.

Proof. Suppose that $x \in A$.

Then, for all $t \in x$, we observe that $t \in \bigcup A$ by definition. Thus, we observe that $x \subseteq \bigcup A$ by definition. □

Now, note that since $A \subseteq A$, it follows then that $A \in \mathscr{P} A$. Then, by our lemma, we observe that $A \subseteq \bigcup \mathscr{P} A$.

Next, we will show that $\bigcup \mathscr{P} A \subseteq A$. To do this, let us first consider some $a \in \bigcup \mathscr{P} A$. Then, there exists some $t \in \mathscr{P} A$ such that $a \in t$. This means then that there exists some $t \subseteq A$ to which a belongs to. Thus, we have $a \in t \subseteq A$, meaning that $a \in A$. Since the choice of a was arbitrary, we see then that, indeed, $\bigcup \mathscr{P} A \subseteq A$.

Thus, we can conclude that, in fact, $A = \bigcup U \mathscr{P} A$ as desired. ■

Problem 1.3 (Problem 2.6b). Show that $A \subseteq \mathscr{P} \bigcup A$. When does equality hold?

Solution. We will first show the inclusion. Suppose that $a \in A$. Then, by Lemma 1.1, we note that $a \subseteq \bigcup A$, and thus $a \in \mathscr{P} \bigcup A$ by definition of the power set. Thus, we see that $A \subseteq \mathscr{P} \bigcup A$.

In order for equality to hold, we first note that for any set X, the empty set will be an element of the power set $\mathscr{P} X$. Furthermore, we note that for elements $Y \in X$ which is a set itself, $Y \bigcup \emptyset = Y$ (this follows from our result in Problem 2.17, since $\emptyset \subseteq Y$). So, our set X must contain the empty set, and furthermore, we need $X = \{\emptyset\}$ for equality to hold:

$$\begin{aligned} X &= \{\emptyset\} \\ \mathscr{P} \bigcup X &= \mathscr{P}(\emptyset) \\ &= \{\emptyset\} \\ &= X \end{aligned}$$

■

Problem 1.4 (Problem 2.8). Show that there is no set to which every singleton belongs.

Solution. First, let us suppose for the sake of contradiction that there exists some set S to which every singleton belongs in.

From here, let us then consider the set $S' = \bigcup S$. We note here that, by definition of $\bigcup S$, S' will thus contain be a set to which every set belonged to. However, as discussed in class, we can't have a set containing all sets; if this is the case, then it would lead to Russel's Paradox, thus leading to a contradiction.

Therefore, we can conclude that there is no set to which every singleton belongs. ■

Problem 1.5 (Problem 2.9). Give an example of sets a and B for which $a \in B$ but $\mathscr{P}a \notin \mathscr{P}B$.

Solution. Let us consider the following sets:

$$a = \{\emptyset\}$$
$$B = \{\{\emptyset\}\}.$$

We see then that, indeed, $a \in B$. However, we note now that, by definition of power set, we have:

$$\mathscr{P}a = \{\emptyset, \{\emptyset\}\}$$
$$\mathscr{P}B = \{\emptyset, \{\{\emptyset\}\}\}.$$

Thus, we see that while $a \in B$, $\mathscr{P}a \notin \mathscr{P}B$. ■

Problem 1.6 (Problem 2.10). Show that if $a \in B$, then $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$.

Solution. To begin with, we note that if $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$, then by definition of the power set, it suffices to show that we have $\mathscr{P}a \subseteq \mathscr{P}\bigcup B$.

To do this, let us consider some element $x \in \mathscr{P}a$. Then, by definition of the power set, we have that x must be a subset of a. That is, $x \subseteq a \in B$.

Then, we note that for any $t \in x \subseteq a \in B$, we have $t \in a \in B$. Thus, $t \in \bigcup B$. This means then that $x \subseteq \bigcup B$, and thus $x \in \mathscr{P}\bigcup B$.

And since x was arbitrary, we can thus conclude that we have $\mathscr{P}a \subseteq \mathscr{P}\bigcup B$; in other words, given that $a \in B$, we have that $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$ as desired. ■

2 Algebra of Sets

Problem 2.1 (Problem 2.17). Show that the following conditions are equivalent:

1. $A \subseteq B$,
2. $A \setminus B = \emptyset$,
3. $A \cup B = B$,
4. $A \cap B = A$.

Solution. To begin with, we will show that the first two conditions 1. and 2. are equivalent:

Proof. Let us suppose that $A \subseteq B$. Then,

$$\begin{aligned} A \subseteq B &\iff \forall x\,(x \in A \implies x \in B) \\ &\iff \forall x\,(x \notin A \vee x \in B) \\ &\iff \forall x \neg\,(x \in A \wedge x \in B) \\ &\iff \neg\,(\exists x\,(x \in A \wedge x \in B)) \\ &\iff A \setminus B = \emptyset \end{aligned}$$

□

Next, we will show that 1. and 3. are equivalent:

Proof. To do this, we proceed as follows:

Suppose that $A \subseteq B$. Then, we have:

$$\begin{aligned} A \subseteq B &\iff \forall x\,(x \in A \implies x \in B) \\ &\iff \forall x\,(x \in A \vee x \in B \iff x \in B) \end{aligned}$$

However, note here that the set $\{x : x \in A \vee x \in B\}$ is in fact $A \cup B$. Thus, we have:

$$\forall x\,(x \in A \vee x \in B \iff x \in B) \iff A \cup B = B.$$

□

Finally, we will show that 1. and 4. are equivalent:

Proof. First, suppose that $A \subseteq B$.

Then, we observe the following:

$$\begin{aligned} A \subseteq B &\iff \forall x(x \in A \implies x \in B) \\ &\iff \forall x(x \in A \wedge x \in B \iff x \in A) \\ &\iff A \cap B = A. \end{aligned}$$

□

Thus, we have shown that, indeed, the four conditions are equivalent as desired. ■

Problem 2.2 (Problem 2.19)**.** Is $\mathscr{P}(A \setminus B)$ always equal to $\mathscr{P}A - \mathscr{P}B$. Is it ever equal to $\mathscr{P}A \setminus \mathscr{P}B$.

Solution. No.

We observe that the empty set $\emptyset$ will always be an element of $\mathscr{P}X$, for any set X.

Then, we note that $\emptyset \in \mathscr{P}(A \setminus B)$.

Furthermore, we have $\emptyset \in \mathscr{P}A$ and $\emptyset \in \mathscr{P}B$.

However, we note that the set $\mathscr{P}A \setminus \mathscr{P}B$ consists of elements in $\mathscr{P}A$ that is not in $\mathscr{P}B$; since $\emptyset$ is in both $\mathscr{P}A$ and $\mathscr{P}B$, then $\emptyset \notin \mathscr{P}A \setminus \mathscr{P}B$.

However, we note that while $\emptyset \in \mathscr{P}(A \setminus B)$, we have that $\emptyset \notin \mathscr{P}A \setminus \mathscr{P}B$. Thus, we can conclude that these sets can never be equal. ■

Problem 2.3 (Problem 2.24a). Show that if A is nonempty, then $\mathscr{P}\bigcap A = \bigcap\{\mathscr{P}X : X \in A\}$.

Solution. We shall proceed as follows: suppose we have some $a \in \mathscr{P}\bigcap A$. Then, by definition, we have the following:

$$\begin{aligned} a \in \mathscr{P} \cap A &\iff a \subseteq \cap A \\ &\iff \forall X \in A, a \subseteq X \\ &\iff \forall X \in A, a \in \mathscr{P}X \\ &\iff a \in \bigcap\{\mathscr{P}X : X \in A\}. \end{aligned}$$

■

Problem 2.4 (Problem 2.24b). Show that

$$\bigcup\{\mathscr{P}X : X \in A\} \subseteq \mathscr{P}\bigcup A.$$

Under what condition does equality hold?

Solution. To begin with, let us consider some $x \in \bigcup\{\mathscr{P}X : X \in A\}$.

Then, we observe the following:

$$\begin{aligned} x \in \bigcup\{\mathscr{P}X : X \in A\} &\iff \exists X \in A, x \in \mathscr{P}X \\ &\iff x \subseteq X \end{aligned}$$

However, note here that $X \subseteq \bigcup A$ by definition, so we have:

$$\begin{aligned} x \subseteq X &\implies x \subseteq \bigcup A \\ &\implies x \in \mathscr{P}\bigcup A. \end{aligned}$$

Thus, we see that $\bigcup\{\mathscr{P}X : X \in A\} \subseteq \mathscr{P}\bigcup A$.

For equality to hold, we note that we want the following to be the case as well:

$$\mathscr{P}\bigcup A \subseteq \bigcup\{\mathscr{P}X : X \in A\}.$$

Then, let us take some $x \in \mathscr{P}\bigcup A$. Then, this means that $x \subseteq \bigcup A$. Now, in order for us to get to $x \subseteq X$ (so that $x \in \bigcup\{\mathscr{P}X : X \in A\}$), we need that $\bigcup A \subseteq X$, for some $X \in A$.

Thus, we see that, in fact, equality holds when $\bigcup A \subseteq X$. ■