Math 135: Homework 7

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Problems

6 Cardinal Numbers and the Axiom of Choice

Problem 6.1. Show that the equation

$$f(m,n) = 2^m(2n+1) - 1$$

defines a one-to-one correspondence between $\omega \times \omega$ and ω.

Solution. We will want to show that $f(m,n)$ is both one-to-one and onto.

First, we will show that it is one-to-one. Suppose that we have $f(m,n) = f(m',n')$. We want to show then that $(m,n) = (m',n')$.

To do this, we observe the following:

$$\begin{aligned} f(m,n) &= 2^m(2n+1) - 1 \\ f(m',n') &= 2^{m'}(2n'+1) - 1 \end{aligned}$$

Then, we have:

$$\begin{aligned} f(m,n) &= f(m',n') \\ 2^m(2n+1) - 1 &= 2^{m'}(2n'+1) - 1 \\ 2^{m-m'}(2n+1) &= 2n'+1 \end{aligned}$$

From here, we note that for any $n \in \omega$, $2n+1$ and $2n'+1$ are both odd numbers; they can't have 2 as one of their factors. In other words, we require for $2^{m-m'} = 2^0 = 1$ for the equality above to be true.

Then, this yields us:

$$\begin{aligned} m - m' &= 0 \\ m &= m' \end{aligned}$$

Furthermore, since this is the case, we have:

$$\begin{aligned} 2n+1 &= 2n'+1 \\ 2n - 2n' &= 0 \\ 2(n-n') &= 0 \\ n - n' &= 0 \\ n &= n' \end{aligned}$$

Thus, we have $m = m'$ and $n = n'$; in other words, we have that if $f(m,n) = f(m',n')$, then $(m,n) = (m',n')$ as desired. Thus, f is indeed one-to-one.

Next, to show onto, we want to show that for any $k \in \omega$, there exists $(m,n) \in \omega \times \omega$ such that $f(m,n) = k$.

For the case of $k = 0$, we observe that if we let $m = n = 0$, then we have:

$$\begin{aligned} 2^0(2(0)+1) - 1 &= 1(0+1) - 1 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

So, there exists m, n such that $f(m,n) = k = 0$.

And for $k = 1$, we observe that if we let $m = 1$ and $n = 0$, then we have:

$$\begin{aligned} 2^1(2(0)+1) - 1 &= 2(0+1) - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

So, there exists m, n such that $f(m, n) = k = 1$.

Now, for $k > 1$, we note that by the Fundamental Theorem of Arithmetic, k has some unique prime factorisation:

$$k = \prod_{i=1}^{j} p_i^{n_i},$$

where $p_1 < p_2 < \cdots < p_n$, and the n_i are positive integers.

Note that this prime factorisation will contain a 2^m term, where m is non-negative (with $m = 0$ if k is odd). Then, the product of the remaining primes in the unique factorisation of k will be an odd number; i.e. there exists some $n \in \omega$ such that $2n + 1 = \prod_{i=2}^{j} p_i^{n_i}$.

Now with this in mind, we first note that all $k' \in \omega$ must have some unique prime factorisation which we can rewrite as $k' = 2^m(2n+1)$, for some $m, n \in \omega$.

And if this is the case, then we have that for all $k \in \omega$, we have $k = k' - 1 = 2^m(2n+1) - 1$. Thus, we have shown that for all $k \in \omega$, there exists $(m, n) \in \omega \times \omega$ such that $f(m, n) = k$. Thus, f is indeed onto.

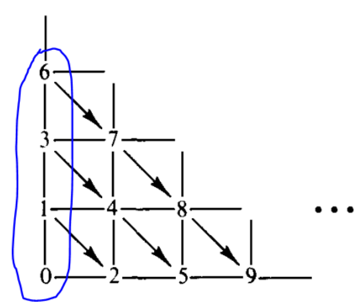
Therefore, we can conclude that f is a one-to-one correspondence between $\omega \times \omega$ and ω. ■

Problem 6.2. Show that in Fig. 32 we have:

$$\begin{aligned} J(m, n) &= [1 + 2 + \cdots + (m+n)] + m \\ &= \frac{1}{2}\left[(m+n)^2 + 3m + n\right] \end{aligned}$$

Solution. We will proceed by showing that J is both one-to-one and onto.

First, we note that in the 0^{th} column of the diagram, each of the entry corresponds to the sum $\sum_{i=0}^{k} i$, where k is the k^{th} row of the entry, starting at $k = 0$. This column is circled in blue below:



Now, we will prove injectivity. To do this, we will want to show that if $\langle m, n\rangle \neq \langle m', n'\rangle$, then it follows that $J(m, n) \neq J(m', n')$.

Now, we note that for $\langle a, b\rangle$, we have $a + b = k$. With this in mind, we observe that if $\langle m, n\rangle \neq \langle m', n'\rangle$, then this means that $m + n = k \neq k' = m' + n'$.

With this in mind then, without loss of generality we will assume that $m + n < m' + n'$. In other words, $k < k'$.

Now, we observe that if $k < k'$, then it follows that $k + 1 \leq k'$.

Next, referring back to the diagram, we note that $m + n = k$ represents each diagonal. For example, if $m + n = 2$, then it will be the second diagonal (which has values $3, 4, 5$).

From here, we observe that for $m + n = k$, the minimum value that $J(m, n)$ can be will be the first value of the k^{th} diagonal. In other words, it'll be $\frac{1}{2}k(k + 1)$. Note that this is $J(0, k)$

On the other hand, we observe that the maximum value that $J(m, n)$ can be will be at the bottom of the diagonal; in other words, it'll be $\frac{1}{2}k(k + 1) + k$. This will be equal to $J(k, 0)$

Now, for injectivity, we want to show that for $k < k'$, we have that $J(m, n) < J(m', n')$. In other words, the maximum value of $J(m, n)$ will be less than the minimum value of $J(m', n')$.

To do this, we note then that we have for $m + n = k$ and $m' + n' = k'$ where $k < k'$ (i.e. $k + 1 \leq k'$):

$$\begin{aligned}
J(m', n') &\geq J(0, k') \\
&= \frac{1}{2}\left[k'(k' + 1)\right] \\
&\geq \frac{1}{2}\left[(k + 1)(k + 2)\right] \\
&= \frac{1}{2}k^2 + \frac{3}{2}k + 1 \\
&= \frac{1}{2}k^2 + \frac{1}{2}k + \frac{2}{2}k + 1 \\
&= \frac{1}{2}k(k + 1) + k + 1 \\
&> \frac{1}{2}k(k + 1) + k \\
&= J(k, 0) \\
&\geq J(m, n)
\end{aligned}$$

In other words, we see that, indeed, $J(m', n') > J(m, n)$. Following through with similar steps, we can then also show that if $m + n = k > m' + n' = k'$, then $J(m, n) > J(m', n')$. In other words, we have shown that if $\langle m, n\rangle \neq \langle m', n'\rangle$, then $J(m, n) \neq J(m', n')$; J is injective as desired.

Next, to show surjectivity, we observe that for every $y = J(m, n)$, we note that y will be on some k^{th} diagonal of the diagram. Then, we can do the following:

1. We let $m = y - \frac{1}{2}k(k + 1)$.
2. We let $n = k - m$.

Thus, we observe then that:

$$\begin{aligned}
J(m, n) &= \frac{1}{2}\left[(m + n)^2 + 3m + n\right] \\
&= \frac{1}{2}\left[(m + k - m)^2 + 3m + (k - m)\right] \\
&= \frac{1}{2}\left[k^2 + 2m + k\right] \\
&= \frac{1}{2}\left[k^2 + 2\left(y - \frac{1}{2}k(k + 1)\right) + k\right] \\
&= \frac{1}{2}\left[k^2 + 2y - k^2 - k + k\right] \\
&= \frac{1}{2}\left[2y\right] \\
&= y
\end{aligned}$$

Thus, we see that, indeed, for every $y \in \omega$, there exists some $\langle m, n \rangle \in \omega \times \omega$ such that $J(m, n) = y$.

In other words, J is surjective.

Thus, we conclude that, indeed, $J(m, n)$ is a one-to-one correspondence between $\omega \times \omega$ and ω as desired. ■

> **Problem 6.3.** Find a one-to-one correspondence between the open unit interval $(0, 1)$ and $\mathbb{R}$ that takes rationals to rationals and irrationals to irrationals.

Solution. We can construct a function as follows:

$$f(x) = \begin{cases} \frac{1}{x} - 2 & 0 < x \leq \frac{1}{2} \\ \frac{1}{x-1} + 2 & \frac{1}{2} < x < 1 \end{cases}$$

We note then that every rational will get mapped to a rational, whereas every irrational will get mapped to an irrational, by this function.

To check that it's a bijection, we see that if $f(x) = f(y)$, then:

$$\begin{aligned} \frac{1}{x} - 2 &= \frac{1}{y} - 2 \\ y - 2xy &= x - 2xy \\ y &= x \end{aligned}$$

Or, we have:

$$\begin{aligned} \frac{1}{x-1} + 2 &= \frac{1}{y-1} + 2 \\ (y-1) + 2(x-1)(y-1) &= (x-1) + 2(x-1)(y-1) \\ y &= x \end{aligned}$$

In either cases, we have $x = y$, so f is indeed injective.

And to prove surjectivity, we see that if $y \geq 0$, then:

$$\begin{aligned} y &= \frac{1}{x} - 2 \\ y + 2 &= \frac{1}{x} \\ x &= \frac{1}{y+2} \end{aligned}$$

And we see that

$$\begin{aligned} f(x) &= \frac{1}{\frac{1}{y+2}} - 2 \\ &= y + 2 - 2 \\ &= y \end{aligned}$$

And we see that if $y < 0$, then we have:

$$\begin{aligned} y &= \frac{1}{x-1} + 2 \\ y - 2 &= \frac{1}{x-1} \\ x &= \frac{1}{y-2} + 1 \end{aligned}$$

And so we see:

$$\begin{aligned} f(x) &= \frac{1}{\frac{1}{y-2} + 1 - 1} + 2 \\ &= \frac{1}{\frac{1}{y-2}} + 2 \\ &= y - 2 + 2 \\ &= y \end{aligned}$$

So, we see that for every $y \in \mathbb{R}$, there exists some $x \in (0, 1)$ such that $f(x) = y$.

Therefore, we see that f is bijective as desired. ■

Problem 6.4. Construct a one-to-one correspondence between the closed unit interval

$$[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\},$$

and the open unit interval $(0, 1)$.

Solution. We can construct the following function, where $n \in \omega$:

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{1}{2^{n+2}} & x = \frac{1}{2^n} \\ x & \text{otherwise} \end{cases}$$

To check injectivity, we observe that if $f(x) = f(y)$, then, either both $f(x) = \frac{1}{2}$ and $f(y) = \frac{1}{2}$, in which case $x = y = 0$.

Or:

$$\begin{aligned} \frac{1}{2^{n+2}} &= \frac{1}{2^{m+2}} \\ 2^{m+2} &= 2^{n+2} \\ m + 2 &= n + 2 \\ m &= n \end{aligned}$$

meaning that $x = \frac{1}{2^m} = \frac{1}{2^n} = y$.

Or, $x = f(x) = f(y) = y$.

In all cases, we see that $x = y$.

Then, to show surjectivity, we observe that if $y \in (0,1)$, if $y = \frac{1}{2}$, then we let $x = 0$ for $f(x) = y$.

If y is a negative power of two which is not 2^{-1}, then we can simply let $x = 4y$.

And if y is otherwise, we let $x = y$.

Thus, for every y we see there exists an x such that $f(x) = y$. Therefore, f is surjective.

Thus, we see that this is indeed bijective. ■

Problem 6.6. Let κ be a nonzero cardinal number. Show that there does not exist a set to which every set of cardinality κ belongs.

Solution. We can let $\kappa = 1$. Then, suppose for the sake of contradiction that there exists a set to which every set of cardinality $\kappa = 1$ belongs to.

We note that a set with cardinality $\kappa = 1$ is a singleton. And we have proven in a previous homework that the set of all singletons cannot exist. In other words, a set to which every set of cardinality $\kappa = 1$ belongs to doesn't exist. ■

Problem 6.7. Assume that A is finite and $f : A \to A$. Show that f is one-to-one iff $\operatorname{ran} f = A$.

Solution. First, suppose that f is one-to-one. Then, let $B = f[\![A]\!]$. Then, we have $B \subseteq A$ and $B \approx A$.

However, note that if $B \subset A$, then $\operatorname{card} B < \operatorname{card} A$. But, we have that $B \approx A$, so it follows that $B = A$. Therefore, we see that f is indeed onto; i.e. $\operatorname{ran} f = A$.

On the other hand, suppose that $\operatorname{ran} f = A$. Now, since A is finite, it has a cardinality of n, for some $n \in \omega$.

From here, let's suppose for the sake of contradiction that there exists some $y \in A$ such that for $x \neq x'$, we have $f(x) = f(x') = y$.

Then, there's $n - 2$ elements left in the domain and $n - 1$ in the range that need to be paired with each other. However, since f is a function, an element in $\operatorname{dom} f$ can't be mapped to two elements in $\operatorname{ran} f$. By the Pigeonhole Principle then, there's at least one element in A which doesn't have a pre-image.

Thus, we have a contradiction. So, we conclude that f is indeed one-to-one. ■

Problem 6.13. Show that a finite union of finite sets is finite.

Solution. We can proceed by induction.

First, we observe that for a set A with cardinality 0, we have then that $A = \emptyset$. Then, $\bigcup A = \emptyset$, so, indeed, we have that $\bigcup A$ is finite as well.

Next, suppose that our claim holds for set A with cardinality n.

Now, we look at A whose cardinality is n^+. Observe then that because A is finite, it follows that there exists a bijection between A and n^+, and thus some bijective function $f : n^+ \to A$.

Then, we have:

$$\begin{aligned}\bigcup A &= \bigcup_{k \in n^+} f(k) \\ &= \bigcup_{k \in n} f(k) \cup f(n)\end{aligned}$$

By our induction hypothesis, we have that $\bigcup_{k\in n} f(k)$ is finite. And since $f(n)$ is also finite, we thus have that $\bigcup_{k\in n} f(k) \cup f(n)$ is finite too.

Therefore, by induction, we conclude that the claim holds as desired. ■

Problem 6.14. Define a permutation of K to be any one-to-one function from K onto K. We can then define the factorial operation on cardinal numbers by the equation

$$\kappa! = \operatorname{card}\{f : f \text{ is a permutation of } K\}$$

where K is any set of cardinality κ. Show that $\kappa!$ is well defined.

Solution. Suppose we have sets K_0 and K_1. Let $\operatorname{card} K_0 = \kappa = \operatorname{card} K_1$.

Then, because the cardinalities of K_0 and K_1 are the same, we can thus construct a bijection between them.

To show that $\kappa!$ is well-defined, we will have to show then that there exists a bijection between the set of permutations of K_0 (which we will denote at K_0') and permutations of K_1 (which we denote as K_1').

Then to do this, we recall that there exists a bijection g between K_0 and K_1. So, for each permutation f of K_1, we can first send this permutation to K_0 with our bijection g, which we then permute. After, we can send the permutation of K_0 back to K_1 using g^{-1}.

Thus, we observe then that there exists a bijection between the set of permutations f of K_0 and of K_1; i.e., we have shown that $\kappa!$ is well-defined. ■