Math 110: Homework 11

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Contents

1 Riesz's Theorem

Problem 1.1. Find a polynomial $p \in \mathscr{P}_3(\mathbb{R})$ such that

$$q'(1) = \int_0^1 p(t)q(t)\mathrm{d}t \qquad \forall q \in \mathscr{P}_3(\mathbb{R}).$$

Solution. To begin with, let us define the following inner product for our space:

$$\langle p, q \rangle := \int_0^1 p(t)q(t)\mathrm{d}t.$$

Then, let us orthonormalise the standard basis $1, x, x^2, x^3$ of our space using Gram-Schmidt as follows:

$$\begin{aligned}
e_1 &= \frac{1}{\|1\|} = \frac{1}{\sqrt{\langle 1, 1\rangle}} = \frac{1}{\sqrt{\int_0^1 1\mathrm{d}x}} = \frac{1}{\sqrt{1}} = 1 \\
e_2 &= \frac{x - \langle x, 1\rangle (1)}{\|x - \langle x, 1\rangle (1)\|} = \frac{x - \int_0^1 x\mathrm{d}x}{\left\|x - \int_0^1 x\mathrm{d}x\right\|} = \frac{x - \frac{1}{2}}{\|x - \frac{1}{2}\|} = \frac{x - \frac{1}{2}}{\sqrt{\langle x - \frac{1}{2}, x - \frac{1}{2}\rangle}} \\
&= \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 (x - \frac{1}{2})^2\mathrm{d}x}} = \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 x^2 - x + \frac{1}{4}}} = \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}}} = \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}} = \sqrt{12}\left(x - \frac{1}{2}\right) \\
e_3 &= \frac{x^2 - \langle x^2, e_2\rangle e_2 - \langle x^2, e_1\rangle e_1}{\|x^2 - \langle x^2, e_2\rangle e_2 - \langle x^2, e_1\rangle e_1\|} = \frac{x^2 - \sqrt{12}\left(x - \frac{1}{2}\right)\int_0^1 \sqrt{12}x^2\left(x - \frac{1}{2}\right)\mathrm{d}x - \int_0^1 x^2\mathrm{d}x}{\left\|x^2 - \sqrt{12}\left(x - \frac{1}{2}\right)\int_0^1 \sqrt{12}x^2\left(x - \frac{1}{2}\right)\mathrm{d}x - \int_0^1 x^2\mathrm{d}x\right\|} \\
&= \frac{x^2 - \sqrt{12}(x - \frac{1}{2})\int_0^1 \sqrt{12}x^3 - \frac{\sqrt{12}}{2}x^2\mathrm{d}x - \int_0^1 x^2\mathrm{d}x}{\left\|x^2 - \sqrt{12}(x - \frac{1}{2})\int_0^1 \sqrt{12}x^3 - \frac{\sqrt{12}}{2}x^2\mathrm{d}x - \int_0^1 x^2\mathrm{d}x\right\|} = \frac{x^2 - \sqrt{12}(x - \frac{1}{2})(\frac{\sqrt{12}}{4} - \frac{\sqrt{12}}{6}) - \frac{1}{3}}{\left\|x^2 - \sqrt{12}(x - \frac{1}{2})(\frac{\sqrt{12}}{4} - \frac{\sqrt{12}}{6}) - \frac{1}{3}\right\|} \\
&= \frac{x^2 - x + \frac{1}{6}}{\|x^2 - x + \frac{1}{6}\|} = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\langle x^2 - x + \frac{1}{6}, x^2 - x + \frac{1}{6}\rangle}} = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\int_0^1 (x^2 - x + \frac{1}{6})^2\mathrm{d}x}} \\
&= \frac{x^2 - x + \frac{1}{6}}{\sqrt{\int_0^1 x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\mathrm{d}x}} = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\frac{1}{5} - \frac{2}{4} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}}} = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\frac{1}{180}}} = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right) \\
e_4 &= \frac{x^3 - \langle x^3, e_3\rangle e_3 - \langle x^3, e_2\rangle e_2 - \langle x^3, e_1\rangle e_1}{\|x^3 - \langle x^3, e_3\rangle e_3 - \langle x^3, e_2\rangle e_2 - \langle x^3, e_1\rangle e_1\|} = \cdots = \sqrt{2800}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right)
\end{aligned}$$

Thus, e_1, e_2, e_3, e_4 is an orthonormal basis for our vector space. From here, let us define $\varphi(p) = p'(1)$.

Then, using Riesz's Representation Theorem, we know that there exists a unique p such that $\varphi(q) = \langle q, p\rangle = \langle p, q\rangle$.

Then, using our orthonormal basis for $\mathscr{P}_3(\mathbb{R})$, we have

$$\begin{aligned}
p &= \varphi(e_1)e_1 + \varphi(e_2)e_2 + \varphi(e_3)e_3 + \varphi(e_4)e_4 \\
&= e_1'(1)e_1 + e_2'(1)e_2 + e_3'(1)e_3 + e_4'(1)e_4 \\
&= 12\left(x - \frac{1}{2}\right) + 180\left(x^2 - x + \frac{1}{6}\right) + 1680\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right) \\
&= 1680x^3 - 2340x^2 + 840x - 60
\end{aligned}$$

Thus, we have that $p(x) = 1680x^3 - 2340x^2 + 840x - 60$. ■

2 Finding the Orthonormal Projection

Problem (Setup). For the next questions in this section, let $V = C[-\pi, \pi]$ with the inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(t)\overline{g(t)}\mathrm{d}t$$

We will want to determine the orthogonal projection of the function $h(x) = \exp(2ix)$ on the given subspaces.

Before we proceed, we will first make note of the fact that all of the subspaces given will be of the form $\operatorname{span}(1, \cos x, \sin x, \ldots, \cos nx, \sin nx)$.

Next, by Euler's formula, we have that $e^{2ix} = \cos(2x) + i\sin(2x)$.

Next, we make the following claim:

Lemma 2.1. The list $1, \sin(x), \cos(x), \ldots, \sin(nx), \cos(nx)$ is orthogonal under our given inner product $\int_{-\pi}^{\pi} f(t)\overline{g(t)}\mathrm{d}t$.

Proof. To begin with, we note that since our bounds are $[-\pi, \pi]$, then $\overline{\sin x} = \sin x$ and $\overline{\cos x} = \cos x$. Furthermore, $\overline{1} = 1$. So, we can disregard the conjugation sign and instead see that our inner product is simply

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\mathrm{d}t.$$

Then, first we note that $\int_{-\pi}^{\pi}\cos(ax)\mathrm{d}x = \int_{-\pi}^{\pi}\sin(ax)\mathrm{d}x = 0$. Thus, we see that 1 is orthogonal to the rest of our list.

From here, we observe that $\sin(a(-x))\cos(b(-x)) = \sin(-ax)\cos(-bx) = -\sin(ax)\cos(bx)$. Thus, we see that $\sin(ax)\cos(bx)$ is an odd function. It follows then that

$$\langle \sin(ax), \cos(bx)\rangle = \int_{-\pi}^{\pi}\sin(ax)\cos(bx)\mathrm{d}x = 0.$$

We see then that $\sin(ax)$ and $\cos(bx)$ are thus orthogonal to each other.

Next, we note that for $a \neq b$, we have that $\sin(ax)\sin(bx) = \frac{1}{2}\left[\cos\left((a-b)x\right) - \cos\left((a+b)x\right)\right]$. Then, we see that

$$\begin{aligned}\langle \sin(ax), \sin(bx)\rangle &= \int_{-\pi}^{\pi}\sin(ax)\sin(bx)\mathrm{d}x = \int_{-\pi}^{\pi}\frac{1}{2}\left[\cos\left((a-b)x\right) - \cos\left((a+b)x\right)\right]\mathrm{d}x\\ &= \frac{1}{2}\int_{-\pi}^{\pi}\cos\left((a-b)x\right)\mathrm{d}x - \frac{1}{2}\int_{-\pi}^{\pi}\cos\left((a+b)x\right)\mathrm{d}x = 0\end{aligned}$$

Then, it follows that $\sin(ax)$ and $\sin(bx)$ are orthogonal to each other as well, where $a \neq b$.

Similarly, we see that $\cos(ax)\cos(bx) = \frac{1}{2}\left[\cos\left((a-b)x\right) + \cos\left((a+b)x\right)\right]$. Then, we observe that

$$\begin{aligned}\langle \cos(ax), \cos(bx)\rangle &= \int_{-\pi}^{\pi}\cos(ax)\cos(bx)\mathrm{d}x = \int_{-\pi}^{\pi}\frac{1}{2}\left[\cos\left((a-b)x\right) + \cos\left((a+b)x\right)\right]\mathrm{d}x\\ &= \frac{1}{2}\int_{-\pi}^{\pi}\cos\left((a-b)x\right)\mathrm{d}x + \frac{1}{2}\int_{-\pi}^{\pi}\cos\left((a+b)x\right)\mathrm{d}x = 0\end{aligned}$$

Thus, we see that $\cos(ax), \cos(bx)$ are orthogonal to each other too. Therefore, we can conclude then that indeed, our list is orthogonal under the given inner product. □

We will now keep all of this in mind as we proceed with answering the following questions.

Problem 2.1. Determine the orthogonal projection of $h(x)$ on the subspace $\operatorname{span}(1, \cos x, \sin x)$.

Solution. To begin with, by our lemma from earlier, we know that $1, \cos x, \sin x, \dots, \cos nx, \sin nx$ are all orthogonal to each other. Then, because $h(x) = e^{2ix} = \cos 2x + i \sin 2x$, we note then that it is in fact orthogonal to the given subpsace. Therefore, we see that the orthogonal projection of $h(x)$ onto U is simply 0. ■

Problem 2.2. Determine the orthogonal projection of $h(x)$ onto the subspace

$$U := \operatorname{span}(1, \cos x, \sin x, \cos 2x, \sin 2x).$$

Solution. We observe that since $h(x) = \cos 2x + i \sin 2x$, it in fact is an element of our given subspace. Thus, the orthogonal projection of $h(x)$ onto U is simply $h(x) = \cos 2x + i \sin 2x$. ■

Problem 2.3. Determine the orthogonal projection of $h(x)$ onto the subspace

$$U := \operatorname{span}(1, \cos x, \sin x, \dots, \cos nx, \sin nx) \qquad \text{(for } n > 2.\text{)}$$

Solution. Again, we note that since $h(x)$ is in fact in the span of U, then the orthogonal projector of $h(x)$ onto U will simply be $h(x)$ itself. ■

3 Minimisation I

Problem 3.1. Find $p \in \mathscr{P}_3(\mathbb{R})$ such that $p(-1) = 0$, $p'(-1) = 0$, and the following is minimised:

$$\int_0^1 (1 - 5x - p(x))^2 \mathrm{d}x.$$

Solution. To begin with, let us define the inner product of our space to be

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\mathrm{d}t.$$

From here, we want to minimise $\|1 - 5x - p(x)\|$. To do this, we first define U to be

$$U := \{p \in \mathscr{P}_3(\mathbb{R}) : p(-1) = 0, p'(-1) = 0\}.$$

Then, we can see that a basis for U must be $(x+1)^2, (x+1)^3$.

Now, using Gram-Schmidt, we can orthonormalise these vectors to get the following:

$$\begin{aligned} e_1 &= \frac{(x+1)^2}{\|(x+1)^2\|} = \sqrt{\frac{5}{31}}(x+1)^2 \\ e_2 &= \frac{(x+1)^3 - \langle (x+1)^3, e_1 \rangle e_1}{\|(x+1)^3 - \langle (x+1)^3, e_1 \rangle e_1\|} = 2\sqrt{\frac{217}{313}}\left((x+1)^3 - \frac{105}{62}(x+1)^2\right). \end{aligned}$$

From here, let $q(x) = 1 - 5x$. Then, the closest point $p \in U$ to q is:

$$p = \langle q, e_1 \rangle e_1 + \langle q, e_2 \rangle e_2.$$

Through computations, we get then that

$$p(x) = -\frac{95}{124}(x+1)^2 - \frac{791}{626}\left((x+1)^3 - \frac{105}{62}(x+1)^2\right).$$

∎

4 Minimisation II

Problem 4.1. Find $p \in \mathscr{P}_3(\mathbb{R})$ such that $p(-1) = 0$, $p'(-1) = 0$, and the following is minimised:

$$p(0)^2 + \int_0^1 (1 - 5x - p'(x))^2 \mathrm{d}x.$$

Solution. To begin with, we will define the following inner product for our space:

$$\langle f, g \rangle \coloneqq f(0)g(0) + \int_0^1 f'(t)g'(t)\mathrm{d}t.$$

From here, we note that once again, the basis for U will once again be $(x+1)^2, (x+1)^3$. Then, we orthonormalise it again with Gram-Schmidt to get:

$$\begin{aligned} e_1 &= \frac{(x+1)^2}{\|(x+1)^2\|} = \sqrt{\frac{3}{31}}(x+1)^2 \\ e_2 &= \frac{(x+1)^3 - \left\langle (x+1)^3, e_1 \right\rangle e_1}{\|(x+1)^3 - \langle (x+1)^3, e_1 \rangle e_1\|} = 2\sqrt{\frac{155}{2081}}\left((x+1)^3 - \frac{141}{62}(x+1)^2\right). \end{aligned}$$

From here, we let $q(x) = x - \frac{5}{2}x^2$ as before, and let

$$p = \langle q, e_1 \rangle e_1 + \langle q, e_2 \rangle e_2.$$

Through some computations, we will see then that we will get:

$$p(x) = -\frac{16}{31}(x+1)^2 - \frac{1315}{2081}\left((x+1)^3 - \frac{141}{62}(x+1)^2\right).$$

■

5 Orthogonal Projector

Problem 5.1. Let $V = \mathbb{R}^3$ equipped with the standard inner product. Prove or disprove: any linear operator $P \in \mathcal{L}(V)$ such that $P^2 = P$ is an orthogonal projector.

Solution. This is false. Let us consider the following example:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

First, note that

$$P^2 = P(P) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

However, to confirm that it is not an orthogonal projector, we can consider the vector $v = (0, 2, 2)$. We note here that $\|v\| = \sqrt{0(0) + 2(2) + 2(2)} = \sqrt{8}$.

However, note that $Pv = (0, 0, 4)$. Note then that $\|Pv\| = \sqrt{0^2 + 0^2 + 4^2} = \sqrt{16}$. Then, we see that $\|Pv\| > \|v\|$, thus violating one of the properties of an orthogonal projector (more concretely, it doesn't follow $\|Pv\| \leq \|v\|$ for all $v \in V$).

Therefore, we can conclude that P is not an orthogonal projector. ∎