Homework 8

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1 Conjugates

Problem 1.1. Let $p \in \mathscr{P}(\mathbb{C})$. Define $q : \mathbb{C} \to \mathbb{C}$ by the formula

$$q(z) := p(z)\overline{p(\overline{z})}.$$

Prove that $q \in \mathscr{P}(\mathbb{R})$. If $\deg p = n$, then what is $\deg q$? Explain.

Solution. To begin with, we observe that we can rewrite any $p \in \mathscr{P}(\mathbb{C})$ by splitting it into a real and imaginary part as follows:

$$\begin{aligned}
p(z) &= c_0 + c_1 z + \ldots + c_n z^n \\
&= (a_0 + b_0 i) + (a_1 + b_1 i)z + \ldots + (a_n + b_n i)z^n \\
&= (a_0 + a_1 z + \ldots + a_n z^n) + (b_0 i + b_1 i z + \ldots + b_n i z^n) \\
&= (a_0 + a_1 z + \ldots + a_n z^n) + i(b_0 + b_1 z + \ldots + b_n z^n) \\
&= p_a(z) + i p_b(z),
\end{aligned}$$

where each $c_i \in \mathbb{C}$, but $a_i, b_i \in \mathbb{R}$.

Now, with this in mind, we observe the following:

$$\begin{aligned}
p(\overline{z}) &= p_a(\overline{z}) + i p_b(\overline{z}) \\
&= (a_0 + a_1\overline{z} + \ldots + a_n\overline{z}^n) + i(b_0 + b_1\overline{z} + \ldots + b_n\overline{z}^n) \\
\overline{p(\overline{z})} &= \overline{(a_0 + a_1\overline{z} + \ldots + a_n\overline{z}^n) + i(b_0 + b_1\overline{z} + \ldots + b_n\overline{z}^n)} \\
&= \overline{(a_0 + a_1\overline{z} + \ldots + a_n\overline{z}^n)} - i\overline{(b_0 + b_1\overline{z} + \ldots + b_n\overline{z}^n)} \\
&= (\overline{a_0} + \overline{a_1\overline{z}} + \ldots + \overline{a_n\overline{z}^n}) - i(\overline{b_0} + \overline{b_1\overline{z}} + \ldots + \overline{b_n\overline{z}^n}) \\
&= (a_0 + a_1 z + \ldots + a_n z^n) - i(b_0 + b_1 z + \ldots + b_n z^n) \\
&= p_a(z) - i p_b(z)
\end{aligned}$$

This means then that we get the following:

$$\begin{aligned}
q(z) &= p(z)\overline{p(\overline{z})} \\
&= (p_a(z) + i p_b(z))(p_a(z) - i p_b(z)) \\
&= (p_a(z))^2 + (p_b(z))^2
\end{aligned}$$

We note that since p_a and p_b consists of only real coefficients, it follows then that we can conclude that $q(z) \in \mathscr{P}(\mathbb{R})$.

Now, from this, we observe that if $\deg p = n$, then we see that $\deg q = 2n$. ■

2 Invariant Subspaces

Problem 2.1. Let $V = \mathscr{P}_3(\mathbb{R})$ and let D denote the differentiation operator on V. Determine, with proof, all subspaces of V invariant under the action of D.

Solution. To begin with, we note that the zero vector space $\{0\}$ is invariant under D. To prove this claim, we note that if $u \in \{0\}$, then $u = 0$.

$$D(u) = 0 \in \{0\}.$$

So, $\{0\}$ is indeed invariant under D.

From here, we claim that any invariant subspace U of V under D, other than $\{0\}$, must be $P_k(\mathbb{R})$ for $k = 0, 1, 2, 3$.

To show this claim, let us suppose that we have a subspace U of V which is invariant under D. Furthermore, let the highest degree of any polynomial $p \in U$ be n, where $n \geq 0$.

Now, we note that since U is invariant under D, it follows then that $Dp \in U$. However, observe that since p is of degree n, then Dp must be of degree $n-1$. And since $Dp \in U$, then we note that $D^2p = D(Dp) \in U$ as well, which has degree of $n-2$. And we can keep on iterating this n times, yielding us the list:

$$p, Dp, D^2p, \ldots, D^np.$$

We note from here that since each $p, Dp, \ldots, D^np$ are polynomials of different degrees $n, n-1, \ldots, 0$ respectively, it follows that they are linearly independent. Furthermore, since there are $n+1$ of these linearly independent polynomials, they in fact span – and are a basis of – $\mathscr{P}_n(\mathbb{R})$. Then, we note that for any $p \in \mathscr{P}_n(\mathbb{R})$, $p \in U$ as well. So, we have $\mathscr{P}_n(\mathbb{R}) \subseteq U$.

Now, to show the other inclusion, let us suppose that we have some $p \notin \mathscr{P}_n(\mathbb{R})$. This means then that $\deg p \geq n+1$. However, by the maximality of n, we can't have any polynomial of degree greater than n in U; in other words, any $p \in U$ must also be in $\mathscr{P}_n(\mathbb{R})$. So, we observe that $U \subseteq \mathscr{P}_n(\mathbb{R})$.

Thus, we can conclude that, indeed, $U = \mathscr{P}_n(\mathbb{R})$.

Therefore, we can conclude then that the remaining invariant subspaces of $V = \mathscr{P}_3(\mathbb{R})$ under D must be:

1. $\mathscr{P}_0(\mathbb{R})$,
2. $\mathscr{P}_1(\mathbb{R})$,
3. $\mathscr{P}_2(\mathbb{R})$,
4. $\mathscr{P}_3(\mathbb{R})$.

■

3 Limits? In an Algebra Class??

Problem 3.1. Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$ satisfies the condition: For any $\varphi \in V'$, and any $v \in V$, $\lim_{n\to\infty} \varphi(T^n v) = 0$. What does this imply about the eigenvalues of T?

Solution. To begin with, we claim that the following is true:

Lemma 3.1. If for any $\varphi \in V'$ and $v \in V$, $\lim_{n\to\infty} \varphi(T^n v) = 0$, then it follows that $\lim_{n\to\infty} T^n v = 0$.

Proof. Let us suppose for the sake of contradiction that $\lim_{n\to\infty} T^n v \neq 0$. Now, we note that since V is a finite-dimensional vector space, we can let $v_1, \dots, v_n$ be a basis for V. Furthermore, let $\varphi_1, \dots, \varphi_n$ be the dual basis of $v_1, \dots, v_n$.

Now, we note that we can rewrite $T^n v$ as a vector as such:

$$T^n v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Now, since we claim that $T^n v \neq 0$, then we know that at least one of $a_1, \dots, a_n \neq 0$. Denote this non-zero entry as a_i. Then, we can pick a φ_i that returns this a_i. However, this contradicts with our assumption that $\lim_{n\to\infty} \varphi(T^n v) = 0$ for all $\varphi \in V'$ and $v \in V$.

Thus, we can conclude that, in fact, we have $\lim_{n\to\infty} T^n v = 0$. $\square$

Now, we note that since V is some finite-dimensional vector space over $\mathbb{C}$, it follows then that we can always find at least one eigenvalue-eigenvector pair for V. We note then that for each eigenvalue-eigenvector pair, we have the following:

$$Tv = \lambda v.$$

From here, let us introduce the following lemma:

Lemma 3.2. Suppose V is a finite-dimensional complex vector space, and let $T \in \mathcal{L}(V)$. Now, for an eigenvalue λ and eigenvector v, we have:

$$T^n v = \lambda^n v.$$

Proof. We will prove this by induction.

Base Case: Suppose that $n = 1$. Then, by definition, we observe that $Tv = \lambda v$.

Induction Hypothesis: Suppose that our claim holds for $n = k$, for $1 \leq k$. That is, $T^n v = \lambda^n v$.

Inductive Step: Now, we observe that for $T^{n+1}v$, we have the following:

$$\begin{aligned} T^{n+1}v &= T(T^n v) \\ &= T(\lambda^n v) \\ &= \lambda^n T(v) \\ &= \lambda^n(\lambda v) \\ &= \lambda^{n+1} v. \end{aligned}$$

$\square$

Then, we note that for each eigenvalue-eigenvector pair, we have the following:

$$\begin{aligned}\lim_{n\to\infty} T^n v &= \lim_{n\to\infty} \lambda^n v \\ &= 0\end{aligned}$$

From here, we note that since v is an eigenvector, it follows that $v \neq 0$. Then, we note then that since λ is some scalar in $\mathbb{C}$, for $\lim\limits_{n\to\infty} \lambda^n v$ to be true, it must follow then that $|\lambda| < 1$. ■

4 Commutativity?

Problem 4.1. Suppose that V is finite-dimensional, $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, and let $S \in \mathcal{L}(V)$ having the same eigenvectors (but not necessarily the same eigenvalues) as T.

Prove that $TS = ST$.

Solution. Let $n = \dim V$.

Now, to begin with, we note that since T has $\dim V$ distinct eigenvalues, then the list of $\dim V$ eigenvectors $v_1, \ldots, v_n$ of T corresponding to these distinct eigenvalues will be linearly independent. Furthermore, we note that since $v_1, \ldots, v_n$ are linearly independent and there are n vectors, they in fact span V and are a basis of V.

Then, we note that any vector $v \in V$ can be expressed as a (unique) linear combination of $v_1, \ldots, v_n$. In particular, we can rewrite v as:

$$v = a_1 v_1 + \ldots + a_n v_n.$$

Now, we note that S, T have the same eigenvectors. Then, let us denote $Tv_i = \mu_i v_i$, and $Sv_i = \lambda_i v_i$. Then, we observe the following:

$$\begin{aligned}
TS(v) &= TS(a_1 v_1 + \ldots + a_n v_n) \\
&= TS(a_1 v_1) + \ldots + TS(a_n v_n) \\
&= a_1 T(S(v_1)) + \ldots + a_n T(S(v_n)) \\
&= a_1 T(\lambda_1 (v_1)) + \ldots + a_n T(\lambda_n (v_n)) \\
&= a_1 \lambda_1 T(v_1) + \ldots + a_n \lambda_n T(v_n) \\
&= a_1 \lambda_1 \mu_1 v_1 + \ldots + a_n \lambda_n \mu_n v_n \\
&= a_1 \mu_1 \lambda_1 v_1 + \ldots + a_n \mu_n \lambda_n v_n \\
&= a_1 \mu_1 S(v_1) + \ldots + a_n \mu_n S(v_n) \\
&= a_1 S(\mu_1 v_1) + \ldots + a_n S(\mu_n v_n) \\
&= a_1 S(T(v_1)) + \ldots + a_n S(T(v_n)) \\
&= ST(a_1 v_1) + \ldots + ST(a_n v_n) \\
&= ST(a_1 v_1 + \ldots + a_n v_n) \\
&= ST(v)
\end{aligned}$$

Thus, we have that $TS(v) = ST(v)$ for all $v \in V$. In other words, $TS = ST$, as desired. ■

Problem 4.2. Give an example of such opeartors T, S on $\mathbb{R}^2$, neither of which is a multiple of the identity operator.

Solution. We define S, T to have the following matrix representations:

$$\mathcal{M}(T) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \quad \mathcal{M}(S) = \begin{bmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Now, we observe that the eigenvalues of T are 2 and 1 by construction. Furthermore, we see that the eigenvectors are $(2,1)$ and $(1,0)$:

$$\begin{aligned}
\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} &= \begin{bmatrix}4\\2\end{bmatrix}\\
&= 2\begin{bmatrix}2\\1\end{bmatrix}\\
\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}1\\0\end{bmatrix}\\
&= 1\begin{bmatrix}1\\0\end{bmatrix}
\end{aligned}$$

Meanwhile, we note that for S, we eigenvalues are $1, \frac{1}{2}$, and the eigenvectors are $(1,0)$ and $(2,1)$:

$$\begin{aligned}
\begin{bmatrix}1 & -1\\0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}1\\0\end{bmatrix}\\
&= 1\begin{bmatrix}1\\0\end{bmatrix}\\
\begin{bmatrix}1 & -1\\0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} &= \begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}
\end{aligned}$$

And from here, we note that we have:

$$\begin{aligned}
TS &= \begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\0 & \frac{1}{2}\end{bmatrix}\\
&= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\\
&= \begin{bmatrix}1 & -1\\0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\\
&= ST
\end{aligned}$$

■

5 Polynomials

Problem 5.1. Let $S, T \in \mathcal{L}(V)$ and suppose S in invertible. Prove that, for any polynomial $p \in \mathscr{P}(\mathbb{F})$, we have:

$$p(STS^{-1}) = Sp(T)S^{-1}.$$

Solution. First, we note that for some $p \in \mathscr{P}(\mathbb{F})$, we note then that it must be of degree n. Then, p is in fact in $\mathscr{P}_n(\mathbb{F})$. Let us define $p(x)$ to be

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n.$$

Then, we observe the following:

$$p(STS^{-1}) = a_0 I + a_1 STS^{-1} + a_2(STS^{-1})^2 + \ldots + a_n(STS^{-1})^n.$$

Now, we observe the following for some $(STS^{-1})^n$:

$$\begin{aligned} (STS^{-1})^n &= (STS^{-1})(STS^{1}) \cdots (STS^{-1}) && \text{(n times)} \\ &= ST^n S^{-1}. \end{aligned}$$

Now, it follows then that we in fact have:

$$\begin{aligned} p(STS^{-1}) &= a_0 I + a_1 STS^{-1} + a_2(STS^{-1})^2 + \ldots + a_n(STS^{-1})^n \\ &= a_0 I + a_1 STS^{-1} + a_2 ST^2 S^{-1} + \ldots + a_n ST^n S^{-1} \end{aligned}$$

From here, we note that $I = SIS^{-1}$. Furthermore, since $S, T \in \mathcal{L}(V)$, then $\lambda STS^{-1} = S\lambda TS^{-1}$, for some $\lambda \in \mathbb{F}$.

Then, with this in mind, we see that

$$\begin{aligned} a_0 I + a_1 STS^{-1} + a_2 ST^2 S^{-1} + \ldots + a_n ST^n S^{-1} &= Sa_0 IS^{-1} + Sa_1 TS^{-1} + Sa_2 T^2 S^{-1} + \ldots + Sa_n T^n S^{-1} \\ &= S(a_0 I + a_1 T + \ldots + a_n T^n)S^{-1} \\ &= Sp(T)S^{-1}. \end{aligned}$$

Thus, we have that $p(STS^{-1}) = Sp(T)S^{-1}$. ∎

Problem 5.2. How are the subspaces of V invariant under T related to the subspaces invariant under STS^{-1}.

Solution. We claim that there exists a bijection between the set of subspaces U which is invariant under T and subspaces W which is invariant under STS^{-1}.

Now, to begin with, we note by definition that if a subspace U is invariant under T, then $T(U) \subseteq U$. Similarly, if a subspace U is invariant under STS^{-1}, then $STS^{-1}(U) \subseteq U$.

Now, we note that if we have a subspace U which is invariant under T, then we have the following:

$$\begin{aligned} STS^{-1}(S(U)) &= ST(U) \\ S(T(U)) &\subseteq S(U) \\ STS^{-1}(S(U)) &\subseteq S(U) \end{aligned}$$

This tells us that if we have a T-invariant subspace U, then $S(U)$ will be invariant under STS^{-1}.

Now, we observe that for some subspace W invariant under STS^{-1}, we have the following:

$$\begin{aligned} STS^{-1}(W) &\subseteq W \\ S^{-1}(STS^{-1}(W)) &\subseteq S^{-1}(W) \\ TS^{-1}(W) &\subseteq S^{-1}(W) \\ T(S^{-1}(W)) &\subseteq S^{-1}(W) \end{aligned}$$

So, we note that for a subset W to be invariant under STS^{-1}, it follows that $S^{-1}(W)$ must also be invariant under T.

We observe then that since we have the following:

1. If U is T-invariant, then $S(U)$ is STS^{-1}-invariant, and
2. If W is STS^{-1}-invariant, then $S^{-1}(W)$ is T-invariant,

then in fact, there exists a bijection between the set of all T-invariant subspaces U and the set of all STS^{-1}-invariant subspaces W.

More concretely, let f be a function which maps from the set of all T-invariant subspaces to the set of all STS^{-1}-invariant subspaces. Then, let U be a subspace which is T-invariant. We have then that

$$f : U \mapsto S(U).$$

■