Homework 4

Michael Pham

Fall 2023

Contents

1 When Does Linearity Occur

Problem 1.1. Let $a, b \in \mathbb{R}$. Define $T : \mathscr{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$Tp := (2p(1) + 5p'(2) + ap(-1)p(3), \int_{-1}^{1} x^3 p(x) \mathrm{d}x + b \sin p(0)).$$

Under what conditions on a, b is the map T linear?

Solution. We observe that for a map to be linear, it must be that for all $p, q \in \mathscr{P}(\mathbb{R})$, and $\alpha, \beta \in \mathbb{R}$, we have:

$$T(\alpha p + \beta q) = \alpha T(p) + \beta T(q).$$

Now, suppose we had some polynomial p such that $p(1) = 1, p'(2) = 1, p(-1) = -1, p(3) = 3$.

Similarly, suppose we had a polynomial q such that $q(1) = 2, p'(2) = 1, p(-1) = 0, p(3) = 4$.

Then, we observe the following for the first component of $T(p+q)$:

$$\begin{aligned} T(p+q) &= 2((p+q)(1)) + 5((p+q)'(2)) + a((p+q)(-1))((p+q)(3)) \\ &= 2(1+2) + 5(1+1) + a(-1+0)(3+4) \\ &= 6 + 10 - 7a \\ &= 16 - 7a. \end{aligned}$$

However, we observe that $T(p) + T(q)$ we have:

$$\begin{aligned} T(p) + T(q) &= 2p(1) + 5p'(2) + ap(-1)p(3) + 2q(1) + 5q'(2) + aq(-1)q(3) \\ &= 2(1) + 5(1) + a(-1)(3) + 2(2) + 5(1) + a(0)(4) \\ &= 2 + 5 - 3a + 4 + 5 + 0a \\ &= 16 - 3a \end{aligned}$$

However, we see that for $16 - 7a = 16 - 3a$, we must have that $4a = 0 \implies a = 0$.

Now, we consider some polynomial p such that $p(0) = \frac{\pi}{2}$.

Similarly, let us have some polynomial q such that $q(0) = \frac{\pi}{2}$.

For the second component of Tp, we know that $\int_{-1}^{1} x^3 p(x) \mathrm{d}x$ isn't multiplied by b, so we can instead just look at the $b \sin p(0)$ portion of it. Then, we observe the following for part of the second component of $T(p+q)$, we have that:

$$\begin{aligned} b \sin((p+q)(0)) &= \sin(\pi) \\ &= 0b \end{aligned}$$

However, for $T(p) + T(q)$, we see that we get:

$$\begin{aligned} b \sin(p(0)) + b \sin(q(0)) &= b \sin(\frac{\pi}{2}) + b \sin(\frac{\pi}{2}) \\ &= b + b \\ &= 2b \end{aligned}$$

Then, we see that for $2b = 0$, we must have that $b = 0$.

So, we know that if we want T to possibly be linear, we need $a = b = 0$. Then, we have the following for Tp

$$Tp \coloneqq (2p(1) + 5p'(2), \int_{-1}^{1} x^3 p(x) \mathrm{d}x).$$

Now, we will confirm that this is, indeed, linear. To do so, we observe the following:

$$\begin{aligned}
T(\alpha p + \beta q) &= \left(2((\alpha p + \beta q)(1)) + 5((\alpha p + \beta q)'(2)), \int_{-1}^{1} x^3((\alpha p + \beta q)(x))\mathrm{d}x\right) \\
&= \left(2(\alpha p(1) + \beta q(1)) + 5(\alpha p'(2) + \beta q'(2)), \int_{-1}^{1} x^3(\alpha p(x) + \beta q(x))\mathrm{d}x\right) \\
&= \left(2\alpha p(1) + 2\beta q(1) + 5\alpha p'(2) + 5\beta q'(2), \int_{-1}^{1} x^3 \alpha p(x) + x^3 \beta q(x)\mathrm{d}x\right) \\
&= \left((2\alpha p(1) + 5\alpha p'(2)) + (2\beta q(1) + 5\beta q'(2))\,, \int_{-1}^{1} x^3 \alpha p(x)\mathrm{d}x + \int_{-1}^{1} x^3 \beta q(x)\mathrm{d}x\right) \\
&= \left(2\alpha p(1) + 5\alpha p'(2), \int_{-1}^{1} x^3 \alpha p(x)\mathrm{d}x\right) + \left(2\beta q(1) + 5\beta q'(2), \int_{-1}^{1} x^3 \beta q(x)\mathrm{d}x\right) \\
&= \left(\alpha(2p(1) + 5p'(2)), \alpha \int_{-1}^{1} x^3 p(x)\mathrm{d}x\right) + \left(\beta(2q(1) + 5q'(2)), \beta \int_{-1}^{1} x^3 q(x)\mathrm{d}x\right) \\
&= \alpha \left(2p(1) + 5p'(2), \int_{-1}^{1} x^3 p(x)\mathrm{d}x\right) + \beta \left(2q(1) + 5q'(2), \int_{-1}^{1} x^3 q(x)\mathrm{d}x\right) \\
&= \alpha Tp + \beta Tq.
\end{aligned}$$

Thus, we see that since $T(\alpha p + \beta q) = \alpha Tp + \beta Tq$ for $a = b = 0$, then it follows that T is a linear map.

■

2 Linear Maps and Span

Problem 2.1. Suppose $T \in \mathcal{L}(V, W), v_1, \ldots, v_w \in V$ and the list $Tv_1, \ldots Tv_m$ spans W. Prove or disprove that the list $v_1, \ldots, v_m$ spans V.

Solution. We shall disprove this statement.

Let us consider the vector spaces $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$ over the field $\mathbb{F} = \mathbb{R}$.

Then, let us consider the following linear map T which sends any vector (a, b, c) in V to the vector (a, b) in W.

To verify that T is indeed linear, we observe that for vectors $v_1 := (a, b, c), v_2 := (d, e, f) \in V$, along with scalars $\alpha, \beta \in \mathbb{R}$, we have:

$$\begin{aligned}
T(\alpha v_1 + \beta v_2) &= T(\alpha(a, b, c) + \beta(d, e, f)) \\
&= T((\alpha a, \alpha b, \alpha c) + (\beta d, \beta e, \beta f)) \\
&= T((\alpha a + \beta d, \alpha b + \beta e, \alpha c + \beta f)) \\
&= (\alpha a + \beta d, \alpha b + \beta e) \\
\alpha T(v_1) + \beta T(v_2) &= \alpha T((a, b, c)) + \beta T((d, e, f)) \\
&= \alpha(a, b) + \beta(d, e) \\
&= (\alpha a, \alpha b) + (\beta d, \beta e) \\
&= (\alpha a + \beta d, \alpha b + \beta e)
\end{aligned}$$

Therefore, since $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$, we observe that T is indeed a linear map.

Next, let us consider the following vectors in V:

$$(1, 0, 0), (0, 1, 0).$$

Then, applying T on these vectors, we get:

$$(1, 0), (0, 1).$$

We observe that since these two vectors Tv_1, Tv_2 are the canonical basis for $\mathbb{R}^2$, it follows that they span W. However, we observe that the list v_1, v_2 does not span V, as a spanning list for V must be at least length 3. ∎

3 Maps and Linear Independence

Problem 3.1. Let $V = \mathscr{P}_2(\mathbb{R}), W = \mathbb{R}$. Are the maps

$$T_1 : f \mapsto f(0), \quad T_2 : f \mapsto f'(1), \quad T_3 : f \mapsto \int_0^1 f(x)\mathrm{d}x$$

in $\mathcal{L}(V, W)$? Are they linearly independent?

Solution. First, we will test whether $T_1, T_2, T_3 \in \mathcal{L}(V, W)$. Suppose we have $p, q \in \mathscr{P}_2(\mathbb{R})$, and $\alpha, \beta \in \mathbb{R}$. Then, we observe the following:

$$\begin{aligned} T_1(\alpha p + \beta q) &= (\alpha p + \beta q)(0) \\ &= \alpha p(0) + \beta q(0) \\ &= \alpha T_1 p + \beta T_1 q \end{aligned}$$

$$\begin{aligned} T_2(\alpha p + \beta q) &= (\alpha p + \beta q)'(1) \\ &= \alpha p'(1) + \beta q'(1) \\ &= \alpha T_2 p + \beta T_2 q \end{aligned}$$

$$\begin{aligned} T_3(\alpha p + \beta q) &= \int_0^1 (\alpha p + \beta q)(x)\mathrm{d}x \\ &= \int_0^1 \alpha p(x) + \beta q(x)\mathrm{d}x \\ &= \int_0^1 \alpha p(x)\mathrm{d}x + \int_0^1 \beta q(x)\mathrm{d}x \\ &= \alpha \int_0^1 p(x)\mathrm{d}x + \beta \int_0^1 q(x)\mathrm{d}x \\ &= \alpha T_3 p + \beta T_3 q \end{aligned}$$

Now, we want to prove linear independence. We observe that for linear independence to hold, only the trivial solution satisfies the following equation:

$$\begin{aligned} a_1 T_1 + a_2 T_2 + a_3 T_3 &= T_0 \\ (a_1 T_1 + a_2 T_2 + a_3 T_3)(p) &= T_0(p) \\ &= 0, \end{aligned}$$

where T_0 is the linear map such that for all $p \in V$, we have $T_0(p) = 0$.

To do this, let us first consider $p_1 = 1$. Applying the different transformations, we get:

$$\begin{aligned} T_1(p_1) &= 1 \\ T_2(p_1) &= 0 \\ T_3(p_1) &= 1 \end{aligned}$$

Then, putting p_1 into our equation we get $a_1 + a_3 = 0$.

Next, consider $p_2 = 2x$. We see that

$$\begin{aligned} T_1(p_2) &= 0 \\ T_2(p_2) &= 2 \\ T_3(p_2) &= 1 \end{aligned}$$

Thus, putting p_2 into our equation yields us $2a_2 + a_3 = 0$.
Finally, for $p_3 = 3x^2$, we observe that

$$\begin{aligned} T_1(p_3) &= 0 \\ T_2(p_3) &= 6 \\ T_3(p_3) &= 1 \end{aligned}$$

So, putting p_3 into our equation yields us $6a_2 + a_3 = 0$.
Then from here, we can construct the following system of linear equations:

$$\begin{aligned} a_1 + a_3 &= 0 \\ 2a_2 + a_3 &= 0 \\ 6a_2 + a_3 &= 0 \end{aligned}$$

$$\begin{aligned} 4a_2 &= 0 \\ a_2 &= 0 \end{aligned}$$

$$\begin{aligned} 2a_2 + a_3 &= 0 \\ 2(0) + a_3 &= 0 \\ a_3 &= 0 \end{aligned}$$

$$\begin{aligned} a_1 + a_3 &= 0 \\ a_1 + 0 &= 0 \\ a_1 &= 0 \end{aligned}$$

Thus, we see that since $a_1 = a_2 = a_3 = 0$, it follows then that T_1, T_2, T_3 are linearly independent. ■

4 Testing for Commutativity

Problem 4.1. Suppose that V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\operatorname{range} S \subset \operatorname{null} T.$$

Prove or disprove that $ST = TS = 0$.

Solution. We first note that, by definition, we have that $\operatorname{null} T = \{v \in V : Tv = 0\}$, and $\operatorname{range} S = \{Sv : v \in V\}$.

Then, from these definitions, we observe that because $\operatorname{range} S \subset \operatorname{null} T$, then this implies that every vector in the form of Sv, for $v \in V$, gets mapped to 0 by T. In other words, we have that $TS(v) = T(Sv) = 0$.

Now, we shall show that it is possible to have $ST \neq TS = 0$. To do this, let us first consider the vector space $V = \mathbb{R}^2$. Now, we recall that a linear map between two finite-dimensional vector spaces can be represented with a matrix. So, let us consider some linear map T defined by:

$$T := \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Then, we want to find some linear map S such that $TS = 0$. In other words, we have:

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ a-c & b-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With this in mind, we see that we have the following system of equations:

$$a - c = 0$$
$$b - d = 0$$

Thus, $a = c$ and $b = d$. Then, we can let $a = c = 2$, and $b = d = 3$. This then yields us the following:

$$S := \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

We then observe:

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2-2 & 3-3 \\ 2-2 & 3-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

However, we see that

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+3 & -2+-3 \\ 2+3 & -2+-3 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 5 & -5 \end{bmatrix}$$

Therefore, we see that while $TS = 0$, we have that $ST \neq 0$. ■

5 Linear Maps and Dimensionality

Problem 5.1. Suppose V is a nonzero finite-dimensional vector space, and $\mathcal{L}(V, W)$ is finite-dimensional for some vector space W. Prove or disprove that W is finite-dimensional.

Solution. Let us suppose for the sake of contradiction that W is infinite-dimensional.

As W is infinite-dimensional, then we have infinitely many linearly independent vectors $w_1, w_2, \ldots, w_n, \ldots$.

Now, let us consider some vector space V, where $\dim(V) = n > 0$. We now consider some basis $\{v_1, \ldots, v_n\}$ of V.

Now, let us define some linear map T_i to be as follows:

$$T_i(v_j) = \begin{cases} w_i, & j = 1 \\ 0, & 2 \leq j \leq n \end{cases}$$

We observe then that each T_i maps v_1 to a vector w_i. Now, we will show that $T_1, T_2, \ldots, T_n, \ldots$ is also linearly independent. To do this, we observe the following:

$$\begin{aligned} a_1 T_1(v_1) + a_2 T_2(v_1) + \ldots + a_n T_n(v_1) + \ldots &= 0 \\ a_1 w_1 + a_2 w_2 + \ldots + a_n w_n + \ldots &= 0 \end{aligned}$$

And since $w_1, w_2, \ldots, w_n, \ldots$ is linearly independent, it follows then that $a_1 = a_2 = \cdots = a_n = \cdots = 0$.

So, we see that $T_1, \ldots, T_n, \ldots$ is linearly independent.

It follows then that since $T_1, T_2, \ldots, T_n, \ldots \subset \mathcal{L}(V, W)$, then $\mathcal{L}(V, W)$ must be infinite-dimensional. However, this is a contradiction as we said that $\mathcal{L}(V, W)$ is finite-dimensional+.

Therefore, we see that W must be finite-dimensional. ■