Math 135: Homework 2

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Problems

3 Relations and Functions

Problem 3.1. Suppose that we attempted to generalize the Kuratowski definitions of ordered pairs to ordered triples with the following definition:

$$\langle x, y, z\rangle^* = \{\{x\}, \{x, y\}, \{x, y, z\}\}.$$

Show that the definition is unsuccessful by giving examples of objects u, v, w, x, y, z with $\langle x, y, z\rangle^* = \langle u, v, w\rangle^*$ but with either $y \neq v$ or $z \neq w$ (or both)

Solution. We observe that if we let $x = x, y = y, z = x$ and $u = x, v = y, w = y$, we have the following:

$$\begin{aligned}
\langle x, y, z\rangle^* &= \{\{x\}, \{x, y\}, \{x, y, x\}\} \\
&= \{\{x\}, \{x, y\}, \{x, y\}\} \\
&= \{\{x\}, \{x, y\}\} \\
\langle u, v, w\rangle^* &= \{\{x\}, \{x, y\}, \{x, y, y\}\} \\
&= \{\{x\}, \{x, y\}, \{x, y\}\} \\
&= \{\{x\}, \{x, y\}\}
\end{aligned}$$

So, we see that $\langle x, y, z\rangle^* = \langle u, v, w\rangle^*$, but that $z \neq w$. Thus, the definition is unsuccessful. ■

Problem 3.2a. Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Solution. We shall proceed as follows:

$$\begin{aligned}
\langle x, y\rangle \in A \times (B \cup C) &\iff (x \in A) \wedge (y \in (B \cup C)) \\
&\iff (x \in A) \wedge ((y \in B) \vee (y \in C)) \\
&\iff ((x \in A) \wedge (y \in B)) \vee ((x \in A) \wedge (y \in C)) \\
&\iff (\langle x, y\rangle \in A \times B) \vee (\langle x, y\rangle \in A \times C) \\
&\iff \langle x, y\rangle \in (A \times B) \cup (A \times C)
\end{aligned}$$

■

Problem 3.2b. Show that if $A \times B = A \times C$, and $A \neq \emptyset$, then $B = C$.

Solution. We observe that $A \times B = A \times C$ means that we have the following:

$$\langle x, y\rangle \in A \times B \iff \langle x, y\rangle \in A \times C$$

Then, since $A \neq \emptyset$, we observe that for all $y \in B$, there exists some $x \in A$ such that:

$$x \in A \wedge y \in B \iff \langle x, y\rangle \in A \times B \iff \langle x, y\rangle \in A \times C \iff x \in A \wedge y \in C$$

Then, we see that $y \in C$ as well by going forward in the implications above. Similarly, if $y \in C$, then we can go backwards and thus get that $y \in B$.

Since this holds for all y, we see that, indeed, $B = C$ as desired. ■

Problem 3.4. Show that there exists no set to which every ordered pair belongs.

Solution. We first recall the definition of an ordered pair:

$$\langle x, y\rangle := \{\{x\}, \{x, y\}\}.$$

Now, let us suppose for the sake of contradiction that the set to which every ordered pair belongs does exist. We denote this set by S.

Then, with this in mind, we can have the following ordered pair:

$$\langle x, x\rangle = \{\{x\}, \{x, x\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}.$$

We note that this is a subset of S. From here, we note then that this means then that the set of all singletons S' is also be a subset of S.

However, recall from a previous homework problem that S' cannot exist.

Thus, we have a contradiction and conclude that such a set S cannot exist. ■

Problem 3.5a. Assume that A and B are given sets, and show that there exists a set C such that for any y,

$$y \in C \iff y = \{x\} \times B \text{ for some } x \text{ in } A.$$

In other words, show that $\{\{x\} \times B : x \in A\}$ is a set.

Solution. To begin with, we observe that $\{x\} \subseteq A \implies \{x\} \times B \subseteq A \times B$, for some $x \in A$.

Then, we see that $\{x\} \times A \in \mathscr{P}(A \times B)$. Now, we know that $A \times B$ is a set, and by the Power Set Axiom, so is $\mathscr{P}(A \times B)$. Furthermore, by definition of $A \times B$, we note then that any $t \in \mathscr{P}(A \times B)$ is in $\mathscr{P}\mathscr{P}\mathscr{P}(A \cup B)$.

Then, that means that by the Subset Axiom, we can construct the following set:

$$C := \{t \in \mathscr{P}\mathscr{P}\mathscr{P}(A \cup B) : \exists x(t = \{x\} \times B) \wedge (x \in A)\}$$

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Problem 3.5b. With A, B, C as above, show that $A \times B = \bigcup C$.

Solution. We will show that $A \times B = \bigcup C$ by first showing that $A \times B \subseteq \bigcup C$, then we will show that $\bigcup C \subseteq A \times B$.

To begin with, we will show that $A \times B \subseteq \bigcup C$. Let us denote $z = \langle x, y\rangle$. Then, we see that if $z = \langle x, y\rangle \in A \times B$, then by definition we have that $x \in A$ and $y \in B$.

More specifically, we note that $\langle x, y\rangle \in \{x\} \times B$. Then, by definition of the union, this means that $\langle x, y\rangle = z \in \bigcup\{\{x\} \times B : x \in A\} = \bigcup C$ as desired. Thus, we have shown that $A \times B \subseteq \bigcup C$.

On the other hand, let $z \in \bigcup C = \bigcup\{\{x\} \times B : x \in A\}$. Then, we note that, by definition of union, we have that $z \in \{x\} \times B$. Then, note that $\{x\} \times B \subseteq A \times B$, meaning that $z \in A \times B$. Thus, $\bigcup C \subseteq A \times B$ as desired.

Therefore, we conclude that $A \times B = \bigcup C$. ■

Problem 3.6. Show that a set A is a relation iff $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$.

Solution. We will first show the forward direction.

Proof. ($\Longrightarrow$) Let us suppose that A is a relation. Then, by definition of a relation, any $a \in A$ is an ordered pair.

Then, let us take $a = \langle x, y \rangle \in A$. We note then that since $\langle x, y \rangle \in A$, then we know that:

1. There exists some y such that $\langle x, y \rangle \in A$. Thus, we see that $x \in \operatorname{dom} A$.
2. There exists some x such that $\langle x, y \rangle \in A$. Thus, we see that $y \in \operatorname{ran} A$.

Therefore, $a = \langle x, y \rangle \in \operatorname{dom} A \times \operatorname{ran} A$, and thus we see that $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$. □

For the backwards direction, we proceed as follows:

Proof. ($\Longleftarrow$) Let us suppose that $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$. Then, we note that every element in $\operatorname{dom} A \times \operatorname{ran} A$ is an ordered pair by definition, and thus, since $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$, every element in A must also be an ordered pair.

Thus, we see that A is a relation by definition. □

Thus, indeed, a set A is a relation iff $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$. ■

Problem 3.7. Show that if R is a relation, then $\operatorname{fld} R = \bigcup\bigcup R$.

Solution. Suppose that R is a relation.

Note that, by definition, $\operatorname{fld} R = \operatorname{dom} R \cup \operatorname{ran} R$.

Furthermore, we have:

$$\operatorname{dom} R := \left\{ t \in \bigcup\bigcup R : \exists b \left(\langle t, b \rangle \in R \right) \right\}$$
$$\operatorname{ran} R := \left\{ t \in \bigcup\bigcup R : \exists a \left(\langle a, t \rangle \in R \right) \right\}$$

Then, we observe the following:

$$\begin{aligned}
x \in \operatorname{fld} R &\iff x \in \operatorname{dom} R \cup \operatorname{ran} R \\
&\iff (x \in \operatorname{dom} R) \vee (x \in \operatorname{ran} R) \\
&\iff \left(x \in \left\{ t \in \bigcup\bigcup R : \exists b \left(\langle t, b \rangle \in R \right) \right\} \right) \vee \left(x \in \left\{ t \in \bigcup\bigcup R : \exists a \left(\langle a, t \rangle \in R \right) \right\} \right) \\
&\iff x \in \bigcup\bigcup R
\end{aligned}$$

Thus, we see that $\operatorname{fld} R = \bigcup\bigcup R$.

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Problem 3.8. Show that for any set A, we have:

$$\operatorname{dom} \bigcup A = \bigcup \{\operatorname{dom} R : R \in A\}$$
$$\operatorname{ran} \bigcup A = \bigcup \{\operatorname{ran} R : R \in A\}.$$

Solution. To begin with, we will prove the first statement.

Proof. Suppose that $x \in \operatorname{dom} \bigcup A$. Then, by definition of $\operatorname{dom} \bigcup A$, there exists some b such that $\langle x, b\rangle \in \bigcup A$.

Then, this means that there exists some $R \in A$ such that $\langle x, b\rangle \in R$. By definition then, we see that $x \in \operatorname{dom} R$, and thus $x \in \bigcup \{\operatorname{dom} R : R \in A\}$. Therefore, $\operatorname{dom} \bigcup A \subseteq \bigcup \{\operatorname{dom} R : R \in A\}$.

On the other hand, suppose that $x \in \bigcup \{\operatorname{dom} R : R \in A\}$. Then, this means that for some $R \in A$, we have that $x \in \operatorname{dom} R$. And by definition, this means that there exists some b such that $\langle x, b\rangle \in R$.

From here, we note then that $R \subseteq \bigcup A$, so we have that $\langle x, b\rangle \in \bigcup A$. This means then that there exists a b such that $\langle x, b\rangle \in \bigcup A$. However, this is precisely the definition of $x \in \operatorname{dom} \bigcup A$ as desired. Thus, we see that $\bigcup \{\operatorname{dom} R : R \in A\} \subseteq \operatorname{dom} \bigcup A$.

Thus, we can conclude that $\operatorname{dom} \bigcup A = \bigcup \{\operatorname{dom} R : R \in A\}$.

□

Next, we will prove the second statement.

Proof. Suppose that $y \in \operatorname{ran} \bigcup A$. Then, by definition of $\operatorname{ran} \bigcup A$, there exists some a such that $\langle a, y\rangle \in \bigcup A$.

Then, this means that there exists some $R \in A$ such that $\langle a, y\rangle \in R$. By definition then, we see that $y \in \operatorname{ran} R$, and thus $y \in \bigcup \{\operatorname{ran} R : R \in A\}$. Therefore, $\operatorname{ran} \bigcup A \subseteq \bigcup \{\operatorname{ran} R : R \in A\}$.

On the other hand, suppose that $y \in \bigcup \{\operatorname{ran} R : R \in A\}$. Then, this means that for some $R \in A$, we have that $y \in \operatorname{ran} R$. And by definition, this means that there exists some a such that $\langle a, y\rangle \in R$.

From here, we note then that $R \subseteq \bigcup A$, so we have that $\langle a, y\rangle \in \bigcup A$. This means then that there exists a a such that $\langle a, y\rangle \in \bigcup A$. However, this is precisely the definition of $y \in \operatorname{ran} \bigcup A$ as desired. Thus, we see that $\bigcup \{\operatorname{ran} R : R \in A\} \subseteq \operatorname{ran} \bigcup A$.

And thus, we see that $\operatorname{ran} \bigcup A = \bigcup \{\operatorname{ran} R : R \in A\}$ as desired. □

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