Math 110: Linear Algebra

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Contents

WEEK 1

INTRODUCTION

1.1 Lecture - 8/24/2023

1.1.1 Complex Numbers

Definition 1.1 (Complex Numbers)**.** We can represent complex numbers in three different ways, for $a, b \in \mathbb{R}$:

1. We can represent them as pairs: (a, b).
2. We can write them as $a + bi$.
3. We can also represent them geometrically; we can imagine a plane where the x-axis is the real axis, and the y-axis is the imaginary axis.

1.1.2 Addition of Complex Numbers

For complex numbers $\alpha = a + bi$ and $\beta = c + di$, we have that $\alpha + \beta = (a + bi) + (c + di) := (a + c) + (b + d)i$.

Geometrically, we can think of the diagonal of the parallelogram constructed by the two vectors corresponding to α and β.

In terms of pairs, for $\alpha = (a, b)$ and $\beta = (c, d)$, we have $\alpha + \beta = (a + c, b + d)$.

We proceed similarly for subtraction.

Remark 1.2. We note here that addition for complex numbers is commutative; that is, $\alpha + \beta = \beta + \alpha$.

We also note here that in the complex world, we have the additive identity as well: $0 + 0i$.

The additive inverse of α is simply $-\alpha$.

1.1.3 Multiplication of Complex Numbers

We observe that for $\alpha = a + bi$ and $\beta = c + di$, we can also define multiplication as: $\alpha\beta = (ac - bd) + (ad + bc)i$.

Similarly to addition, multiplication of complex numbers is also commutative.

For the multiplicative inverse, we can proceed as follow:

$$\begin{aligned}\frac{1}{a+bi} &= \frac{a-bi}{(a+bi)(a-bi)} \\ &= \frac{a-bi}{a^2+b^2}\end{aligned}$$

And since we are working with some $\alpha \neq 0$, then we know that at least one of $a, b \neq 0$. Thus, we have that $a^2 + b^2 \neq 0$ as well.

Thus, we have the multiplicative inverse of $\alpha = a + bi$ as $\frac{a}{a^2+b^2} + \frac{b}{a^2+b^2}i$.

To divide complex numbers α and β, then we simply have to do $(a+bi) \cdot \frac{1}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$.

We note here that both $\mathbb{R}$ and $\mathbb{C}$ are fields, with $\mathbb{R} \subsetneq \mathbb{C}$.

1.1.4 Complex Numbers, Polar Form, and Exponentials

We can picture a vector from the origin, at some angle θ from the real axis. We observe that $a = r\cos(\theta)$ and $b = r\sin(\theta)$, where r is the length of the vector (or the “modulus” of the complex number $a + bi$).

1.2 Discussion - 8/25/2023

Problem 1.1. Let $a = 1 + i$, $b = 2 - i$. Find the following:

1. $a + b$
2. $2a - 3b$
3. ab
4. b/a

Solution.

1. $a + b = (1 + 2) + (1 + (-1))i = 3 + 0i = 3$.
2. $2a - 3b = 2(1 + i) - 3(2 - i) = (2 + 2i) - (6 + 3i) = -4 + 5i$.
3. $ab = (1(2) - (1)(-1)) + (1(-1) + 2)i = 3 + i$
4. To find b/a, can find the conjugate of the denominator, and proceed with $\frac{(2-i)(1-i)}{(1+i)(1-i)} = \frac{1-3i}{2}$.

■

Problem 1.2. Solve the equation $x^3 = 1$ in $\mathbb{C}$.

Solution. We can tackle this problem by working with the polar form of complex numbers instead. Define $\omega := r\cos(\theta) + ir\sin(\theta)$.

Now, we observe that since we want to find $\omega^3 = 1$, then we have that $r = 1$. Next, we can think of a circle (with radius $r = 1$), and we rotate by $120°$, yielding us three rotations before returning to the original position. Then solution then to our problem $\omega^3 = 1$ are the three points for these rotations.

We see then that the solutions are:

1. $\omega_1 = 1 + 0i = 1$
2. $\omega_2 = \cos(120) + i\sin(120) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
3. $\omega_3 = \cos(240) + i\sin(240) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

■

Problem 1.3. Find $x \in \mathbb{R}^4$ such that $(4, -3, 1, 7) + 2x = (5, 9, -6, 8)$.

Solution. Let $x = (0.5, 6, -3.5, 0.5)$ ■

Problem 1.4. Let $V = \mathbb{R}_+ = \{a \in \mathbb{R} : a > 0\}$. Define addition and scalar multiplication by $v \bigoplus w = vw$, $c \bigotimes v = v^c$. Convince yourself that $(V, \bigoplus, \bigotimes)$ is a vector space.

Now, what is the zero vector of this vector space? What is the additive inverse of a vector?

Solution. We observe that the zero vector v is 1. We see that $v \bigoplus w = (1)w = w$. We also see that $v \bigotimes w = w^1 = w$.

The additive inverse of any vector $w \in V$ is $w' = \frac{1}{w}$, as $w \bigoplus w' = w(w') = \frac{w}{w} = 1$. ■

Problem 1.5. Prove that for all positive integers n, we have that $n(n+1)$ is even.

Solution. We proceed by cases.

If we have a positive integer which is even, we see then that we can rewrite $n = 2m$ for some $m \in \mathbb{Z}$. Then, we observe that $n(n+1) = 2m(2m+1) = 2(2m^2+m)$. By closure of integers, we have that $2m^2+m = j \in \mathbb{Z}$. Then, we have $n(n+1) = 2j$, which is even by definition.

If the integer n is odd, then we see that $n = 2m + 1$. Now, we have $n(n+1) = (2m+1)(2m+2) = (2m+1)(2(m+1)) = 2(m+1)(m+1)$. Again, since $\mathbb{Z}$ is closed under multiplication and addition, we have then that $(m+1)^2 = j \in \mathbb{Z}$, so $2j$ is also even. ■

Problem 1.6. Show that $\sqrt{2}$ is not a ratio of integers.

Solution. Suppose for the sake of contradiction that $\sqrt{2}$ can be expressed as a ratio of integers. Then, there exist $m, n \in \mathbb{Z}$ such that $\sqrt{2} = \frac{m}{n}$, where $\gcd(m, n) = 1$.

Now, observe that $2 = \frac{m^2}{n^2}$. Then, we see that $2n^2 = m^2$. But, we see then that this contradicts with the fact that $\gcd(m, n) = 1$, as this implies that m must have also been even. ■

Problem 1.7. Show that the opposite of a vector is unique.

Solution. We suppose for the sake of contradiction that for a vector v, there exists two different vectors which is the opposite of it.

Then, we see that $v + w = 0$ and $v + u = 0$. Then, we have that $v + w = v + u \implies w + u$. This thus contradicts with the fact that the two vectors are different; the opposite of a vector is unique. ■

WEEK 2

VECTOR SPACES

2.1 Lecture - 8/29/2023

2.1.1 Fields

Definition 2.1 (Fields)**.** We define a field to be a set $\mathbb{F}$ with two operations $(+, \cdot)$. We denote this by $\langle \mathbb{F}, +, \cdot \rangle$. These operations satisfying the following rules:

1. Closed under $+$ and $\cdot$
   - If we add or multiply two elements in $\mathbb{F}$, the result will also be in $\mathbb{F}$. In other words, we have $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$. The same for $\cdot$.
2. Commutativity
   - In other words, for all $a, b \in \mathbb{F}$, we have $a + b = b + a$. similarly, $a \cdot b = b \cdot a$.
3. Associativity
   - For all $a, b, c \in \mathbb{F}$, we have $a + (b + c) = (a + b) + c$, and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
4. Identities
   - There exist elements $0 \neq 1 \in \mathbb{F}$, where we have $0 + a = a$, and $1 \cdot a = a$.
5. Inverses
   - For any $a \in \mathbb{F}$, there exists $-a \in \mathbb{F}$ such that $a + (-a) = 0$. Furthermore, for any $a \in \mathbb{F}$, where $a \neq 0$, we have $a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1$.
6. Distributivity
   - For $a, b, c \in \mathbb{F}$, we have $c \cdot (a + b) = c \cdot a + c \cdot b$.

Example 2.2 (Finite Field of Five Elements). Suppose we have the integers mod 5. Now, we construct the following table for $+$:

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

We see that commutativity holds as the table is symmetric along the main diagonal; if we flip the columns and rows, we will still have the same table. We see that since there is a zero in every row and column, then an additive inverse exists for every number.

Now, for $\cdot$, we have the following table:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Again, we see that it is commutative by the same argument. Same applies for finding an inverse, though we have to make sure that we are only looking at rows/columns that are non-zero, and that we check if every row/column has a 1.

Remark 2.3 ($0 \cdot a = 0$). We will show that $0 \cdot a = 0$, for any $a \in \mathbb{F}$. We can proceed as follow:

$$\begin{aligned} 0 \cdot a &= (0+0) \cdot a \\ &= 0 \cdot a + 0 \cdot a \end{aligned}$$

So, we have that $0 \cdot a = 0 \cdot a + 0 \cdot a$.

From here, we know that there exists some $-a \in \mathbb{F}$ such that $0 \cdot a + (-a) = 0$.

So, we have $0 \cdot a + (-a) = 0 \cdot a + (0 \cdot a + (-a))$. So, we have $0 = 0 \cdot a + 0 \implies 0 \cdot a = 0$.

Remark 2.4 (Uniqueness of Additive Inverses). We will prove that additive inverses are unique.

Suppose for contradiction, we have two elements $b, c \in \mathbb{F}$ such that $a + b = 0, a + c = 0$.

We see then that since $a+b = 0 = a+c$. Then, we have $a+b = a+c$. Then, we have $(a+b)+b = (a+c)+b$. From here, we see that $(a+b)+b = (a+b)+c$. Thus, we have $b = c$.

2.1.2 Vector Spaces

Definition 2.5 (Vector Spaces)**.** A vector space V consists of two sets and two operations. We usually denote it by $\langle V, \mathbb{F}, +, \cdot \rangle$.

V is a set (of vectors).

$\mathbb{F}$ is a set (of numbers/scalars). We note here that $\mathbb{F}$ must be a field its own operations.

We define the operation $+ : V \times V \to V$, and $\cdot : \mathbb{F} \times V \to V$.

Example 2.6 (Canonical Example)**.** The following are canonical examples: $\mathbb{R}^n, \mathbb{C}^n$ or, generally, $\mathbb{F}^n$.

Given an $n \in \mathbb{N}$, we define $\mathbb{R}^n \coloneqq \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{R}, \forall j = 1, 2, \ldots, n\}$.

We also have $\mathbb{R}^\infty \coloneqq \{(x_1, x_2, \ldots) : x_j \in \mathbb{R}, \forall j \in \mathbb{N}\}$.

Definition 2.7 (Vector Addition)**.** Given $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{F}^n$, we can define addition by adding the corresponding components together as follow:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) \coloneqq (x_1 + y_1, \ldots, x_n + y_n).$$

We see that we have a neutral vector whose components are all equal to 0 in regards to addition. We also see $(-x_1, \ldots, x_n) + (x_1, \ldots, -x_n) = (0_1, \ldots, 0_n)$.

We see then that the additive structure of our vector space must follow the same conditions seen in fields as well; $\langle V, + \rangle$ must be an Abelian group.

Definition 2.8 ((Scalar) Multiplication)**.** We define multiplication to be that as follow:

Suppose we have some $c \in \mathbb{F}$, and some $(x_1, \ldots, x_n) \in \mathbb{F}^n$, we define multiplication as follow:

$$c\,(x_1, \ldots, x_n) \coloneqq (cx_1, \ldots, cx_n)$$

For rules, we observe that we have distributivity as follows, for scalars λ, μ and vectors u, v:

$$\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$$
$$(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$$

We also have associativity for the scalars λ, μ:

$$\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$$

We also have the following rules:

$$1 \cdot v = v$$

Remark 2.9. We can think of scalar multiplication geometrically as stretching/compressing(/flipping) the vector.

Remark 2.10. We also note here that, for the associativity of $\cdot$ for vector spaces, the $\cdot$ operator is different in $\lambda \cdot (\mu \cdot v)$ and $(\lambda \cdot \mu) \cdot v$; the latter occurs in the field between two scalars, whereas the former is between a scalar and a vector – the dot is being overloaded.

2.2 Lecture - 8/31/2023

Example 2.11. Let S be any given set which is non-empty, and $\mathbb{F}$ to be a field. We define $\mathbb{F}^S$ as follow:

$$\mathbb{F}^S := \{f : S \to \mathbb{F}\}.$$

Take $\mathbb{F}$ as our field. We define, for any functions $f, g \in \mathbb{F}^S$, addition to be $(f+g)(s) := f(s) + g(s)$, and for some $\lambda \in \mathbb{F}$, we define (scalar) multiplication to be $(\lambda \cdot f)(s) := \lambda \cdot f(s)$.

Remark 2.12. We note here that in $(f+g)(s) = f(s) + g(s)$, the first $+$ takes place in $\mathbb{F}^S$, while the second $+$ takes place in $\mathbb{F}$.

Similarly, the first $\cdot$ takes place in $\mathbb{F} \times \mathbb{F}^S$, whereas the second $\cdot$ takes place in $\mathbb{F}$.

Example 2.13. Define $\mathbb{F} := \mathbb{R}$, and $S := \{1, 2\}$. Any function $f \in \mathbb{R}^S$ is completely described by $(f(1), f(2))$.

We see that adding two functions will result in adding the pairs together:

$$(f(1), f(2)) + (g(1), g(2)) = ((f+g)(1), (f+g)(2))$$

Likewise, with scalar multiplication, we have:

$$\lambda \cdot (f(1), f(2)) = ((\lambda \cdot f)(1), (\lambda \cdot f)(2))$$

Remark 2.14. Notice that if $S = \mathbb{N}$ (or any other countable set), then $\mathbb{F}^s$ can be understood as $\mathbb{F}^\infty$, which is an infinite sequence.

2.2.1 Subspaces

The following are basic observations in V:

- The zero vector is unique.
- The additive inverses are unique.
- $0 \cdot v = 0$
- $(-1) \cdot v = -v$

Definition 2.15 (Subspaces)**.** Suppose we have a vector space V. Now, we have some $U \subseteq V$. We see that the following conditions must be met for U to be a subspace:

- Closure under addition and scalar multiplication.
- The zero vector is contained in U.

All the other rules for a vector space is inherited by virtue of the fact that $U \subseteq V$. We note here that U is also a vector space, contained within a vector space.

Subspaces can be added.

WEEK 3
IDK

3.1 Lecture - 9/5/2023

3.1.1 Recap

Recall that a subspace $U \subseteq V$, where V is a vector space, must satisfy the following two requirements:

1. Closure under vector addition and scalar multiplication.
2. The zero vector is contained in U.

We note that subspaces can be added. For example, suppose we have subspaces U, W of V. Then, we define addition of subspaces as:

$$U + W \coloneqq \{u + w : u \in U, w \in W\}.$$

Remark 3.1. We can add other things together as well. For example, we can add subsets together too.

Theorem 3.2. The sum $U + W$ is also a subspace of V.

Proof. We can verify this as follows:

1. Since U, W are subspaces, they are both closed under addition and scalar multiplication. Then we see that $\lambda(u + w) = \lambda u + \lambda w$; $\lambda u \in U, \lambda w \in W$. Similarly, $(u_1 + w_1) + (u_2 + w_2) = (u_1 + u_2) + (w_1 + w_2)$. And we see that $(u_1 + u_2) \in U$ and $(w_1 + w_2) \in W$. So, we see that $U + W$ is closed under addition and scalar multiplication.
2. For the zero vector, since both U, W contains the zero vector, then we can take $0 \in U$ and $0 \in W$. Then, we see that $0 + 0 = 0$, so $U + W$ contains the zero vector as well.

■

A Little Thought Experiment

Suppose we have the line $y = x$ in $\mathbb{R}^2$ and the point $(0, 2)$. We note that when adding them together, we have a line shifted up two units. However, while the line $y = x$ is a subspace of $\mathbb{R}^2$, the point $(0, 2)$ isn't; the result is no longer a subspace. We can verify this by the fact that we don't have the zero vector.

Now, suppose that we instead added the point $(2, 2)$; while the singleton isn't a subspace itself, the result of the addition yields a subspace still. This is because the point $(2, 2)$ is an element of the subspace formed by $y = x$.

Remark 3.3. The key takeaway from this experiment is that, while adding two subspaces together guarantees a subspace, we can't guarantee the result if the sets aren't known subspaces.

3.1.2 Direct Sums

Definition 3.4 (Direct Sum of Subspaces)**.** A sum $U + W$ of 2 subspaces U, W of V is called a direct sum if every vector in this sum has a unique representation as $u + w$, where $u \in U, w \in W$.

Example 3.5. Let us take two different lines going through the origin. We observe that the result is the entire plane $\mathbb{R}^2$.

We observe from here that every vector has a unique vector representation because if we have two different vectors, then we have the following:

$$v_1 + w_1 = v_2 + w_2$$
$$v_1 - v_2 = w_2 - w_1$$

Since $U \cap W = \{0\}$, we get $u_1 = u_2$ and $w_1 = w_2$.

Theorem 3.6. $U + W$ is a direct sum (of subspaces) if and only if:

- $U \cap W = \{0\}$, or
- 0 has a unique representation $0_{\in U} + 0_{\in W}$.

Proof. We shall prove the equivalences by cycling through them.

1. Suppose we have a direct sum as per the definition. Then, every vector $u + w$ must have a unique representation, including the zero vector 0. Since $0 = 0 + 0$, we know that this must be the only representation.
2. We now want to show that if 0 has a unique representation, then $U \cap W = \{0\}$. We shall proceed by contraposition.

   Let us suppose that $U \cap W \neq \{0\}$. Then, we see that there is a non-zero vector $v \in U \cap W$. That means then that $v + (-v) = 0$, where $v \in U$ and $-v \in W$. So, we see that 0 does not have a unique representation.
3. The final implication of $U \cap W = \{0\}$ implying direct sum's proof is the same as what we saw in our example.

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Remark 3.7. Suppose we take two subspaces: is the intersection always at the origin? We see that this isn't the case, as if we take the intersection between some line in $\mathbb{R}^2$ and $\mathbb{R}^2$ itself (which is its own subspace!), we see that the intersection is actually the line itself.

Remark 3.8. If U and W are subspaces, then their intersection will also be a subspace. First, we observe that since $0 \in U$ and $0 \in W$, then $0 \in U \cap W$.

For closure, we see that since U, W are both closed, then suppose we have $v_1, v_2 \in U \cap W$. Since v_1, v_2 are in both U and W, then $v_1 + v_2 \in U$ and $v_1 + v_2 \in W$. Thus, we have that $v_1 + v_2 \in U \cap W$ as well.

Similarly, suppose we have some vector $v \in U \cap W$. Then, $v \in U \wedge v \in W$. This means then that $\lambda v \in U \wedge \lambda v \in W$, where $\lambda \in \mathbb{F}$; then we see that $\lambda v \in U \cap W$ too.

Thus, we see that $U \cap W$ is also a subspace.