Math 135: Homework 9

Michael Pham

Spring 2024

Problems

6 Cardinal Numbers and the Axiom of Choice

Problem 6.18. Prove that the following statement is equivalent to the axiom of choice: For any set $\mathscr{A}$ whose members are nonempty sets, there is a function f with domain $\mathscr{A}$ such that $f(X) \in X$ for all $X \in \mathscr{A}$.

Solution. First, we will show that the axiom of choice implies the given statement.

Proof. Let us suppose that the axiom of choice holds.

Then, let us consider some set $\mathscr{A}$ whose members are nonempty sets. We want to show that there exists a function f with the desired property given in the statement for this set.

To do this, let us first consider the set $\bigcup \mathscr{A}$ which we know exists by the union axiom. Then, by the axiom of choice, we know that there exists a function $f : \mathscr{P}(\bigcup \mathscr{A}) \setminus \{\emptyset\} \to \bigcup \mathscr{A}$ such that $f(X) \in X$ for every nonempty $X \subseteq \bigcup \mathscr{A}$.

Then, if we restrict f to $\mathscr{A}$, we see then that indeed we have a function with the desired property for $\mathscr{A}$. □

Now, we will show the other direction.

Proof. Let us consider some non-empty set X. Then, we observe that the set $\mathscr{P}(X) \setminus \{\emptyset\}$ is in fact a collection of non-empty sets. So, by our assumption, we know that there exists a function f whose domain is $\mathscr{P}(X) \setminus \{\emptyset\}$ such that $f(x) \in x$ for all $x \in \mathscr{P}(X) \setminus \{\emptyset\}$. But since $x \in \mathscr{P}(X) \setminus \{\emptyset\}$, this means that $x \subseteq X$.

In other words, $f(x) \in x$ for every nonempty $x \subseteq X$. Furthermore, $\operatorname{ran} f \subseteq X$. But this is precisely the axiom of choice. □

Thus, we conclude that the two are indeed equivalent. ■

Problem 6.21. Assume that $\mathscr{A}$ is a non-empty set such that for every set B,

$$B \in \mathscr{A} \iff \text{every finite subset of } B \text{ is a member of } \mathscr{A}.$$

Show that $\mathscr{A}$ has a maximal element; i.e., an element that is not a subset of any other element of $\mathscr{A}$.

Solution. We will prove this by using Zorn's Lemma.

First, let us consider some $\mathscr{C} \subseteq \mathscr{A}$, where $\mathscr{C}$ is a chain. We want to show then that $\bigcup \mathscr{C} \in \mathscr{A}$.

For $\bigcup \mathscr{C} \in \mathscr{A}$, it must be then that every finite subset C of $\bigcup \mathscr{C}$ must be a member of $\mathscr{A}$. So, let us denote F to be a finite subset of $\bigcup \mathscr{C}$. Now, by definition of union, for each $x \in F$, there exists some $C_x \in \mathscr{C}$ such that $x \in C_x \in \mathscr{C}$.

Then, using the fact that F is finite and $\mathscr{C}$ being a chain, we note then that the set $C' := \{C_x : x \in F\}$ must have a maximal element $C_{\max}$.

Then, because $\mathscr{C}$ is a chain, it follows then that for all $C_x \in C'$, we have that $C_x \subseteq C_{\max}$. This means then that for all $x \in F$, we have $x \in C_x \subseteq C_{\max}$, meaning that $F \subseteq C_{\max}$.

And we note that since $\mathscr{C} \subseteq \mathscr{A}$, and we have $F \subseteq C_{\max} \in \mathscr{C} \subseteq \mathscr{A}$, we have then that $F \in \mathscr{A}$. Thus, we observe that every finite subset F of $\bigcup \mathscr{C}$ is in $\mathscr{A}$; thus, we that $\bigcup \mathscr{C} \in \mathscr{A}$ as desired.

From here, we can simply apply Zorn's Lemma to thus conclude that, indeed, $\mathscr{A}$ has a maximal element. ■

Problem 6.25. Assume that S is a function with domain ω such that $S(n) \subseteq S(n^+)$ for each $n \in \omega$. Assume that B is a subset of the union $\bigcup_{n\in\omega} S(n)$ such that for every infinite subset B' of B, there is some n such that $B' \cap S(n)$ is finite.

Show that B is a subset of some $S(n)$.

Solution. Let us suppose for the sake of contradiction that B isn't a subset of every $S(n)$. Then, this means that for every $n \in \omega$, we have that $B \setminus S(n) \neq \emptyset$. Then, for every $n \in \omega$, there exists some $b_n \in B \setminus S(n)$.

From here, by the axiom of choice, we know that there exists some set $B' \coloneqq \{b_n : n \in \omega\}$. Furthermore, we note that $B' \subseteq B$.

Now, we claim that B' is infinite.

Proof. First, recall that $B \subseteq \bigcup_{n\in\omega} S(n)$.

Then, we note that if B' wasn't infinite, then there exists some m such that $B' \subseteq \bigcup_{n\leq m} S(n) = S(m)$. Then, we note that $B' \subseteq S(m)$. However, this means then that for any $b_n \in B'$, we have that $b_n \in S(m)$ as well.

However, this is a contradiction with the fact that $b_n \in B' \subseteq B$, and $B \not\subseteq S(n)$ for any $n \in \omega$. In other words, b_n shouldn't be in any $S(n)$ for all $n \in \omega$.

Thus, we conclude that B' must be infinite. □

Now, since B' is an infinite subset of B, we note then that there exists some m such that $B' \cap S(m)$ is infinite. Then, the set $\{k : b_k \in S(m)\}$ must be infinite as well. Then, this means that there exists some $k \in \omega$ where $k > m$ such that $b_k \in S(m)$. However, we note that we said that $b_k \notin S(k) \supseteq S(m)$. Thus, we have arrived at a contradiction.

Therefore, we can conclude that it must be that B is a subset of some $S(n)$. ■

Problem 6.27a. Let A be a collection of circular disks in the plane such that no two of which intersect. Show that A is countable.

Solution. To begin with, we note that every disk $d \in A$ will contain some rational point $p = (x, y) \in \mathbb{Q} \times \mathbb{Q}$. So, for each disk, we can pick said point p. Now, we note that because the disks can't overlap with one another, it follows then that from each disk we can pick out a unique p.

With this in mind, we can construct an injection $f : A \hookrightarrow \mathbb{Q} \times \mathbb{Q}$ where each disk $d \in A$ gets mapped to a unique point $p = (x, y) \in \mathbb{Q} \times \mathbb{Q}$; and we see that no two disks can map to the same point due to them not overlapping, and thus f is indeed an injection.

Therefore, we observe that $\operatorname{card}(D) \leq \operatorname{card}(\mathbb{Q} \times \mathbb{Q}) = \operatorname{card}(\mathbb{Q})$. In other words, we see that D is countable as desired. ■

Problem 6.27b. Let B be a collection of circles in the plane such that no two of which intersect. Need B be countable?

Solution. We observe that the set B is not countable.

To prove our claim, let us first define a circle as follows: for any (positive) $r \in \mathbb{R}$, we define:

$$b_r : \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\right\}.$$

Then, we observe that b_r is a circle and that for $r \neq s$, we note then that b_r and b_s don't intersect with one another.

Then, we can define B to be the collection of such circles:

$$B := \left\{b_r : r \in \mathbb{R}^+\right\}.$$

Thus, we observe that we can construct a bijection between B and $\mathbb{R}$ by mapping each $b_r \in B$ to $r \in \mathbb{R}$. In other words, we see that B is indeed uncountable. ■

Problem 6.27c. Let C be a collection of figure eights in the plane such that no two of which intersect. Need C be countable?

Solution. We shall prove that C is in fact countable.

Each figure eight $c \in C$ contains two loops which we shall denote by c_1 and c_2. Note that for each loop in c, it contains two different rational points $p \in \mathbb{Q}^2$ and $q \in \mathbb{Q}^2$ inside.

Now, we observe that for any two figure eights c_a and c_b, one of five scenarios must occur:

1. c_a is inside $c_{b,1}$ (the first loop of c_b).
2. c_a is inside $c_{b,2}$.
3. c_b is inside $c_{a,1}$.
4. c_b is inside $c_{a,2}$.
5. c_a and c_b are outside of one another.

Then, without loss of generalisation, let us suppose that Scenario 1 occurs; that is, c_a is inside $c_{b,1}$. Now, let us pick rational points $p_a \in c_{a,1}$ and $q_a \in c_{a,2}$, and let us pick $p_b \in c_{b,1}$ and $q_b \in c_{b,2}$. Note that these points are in $\mathbb{Q}^2$.

We observe that while p_b might be contained in one of the two loops of c_a as well, we note that since c_a is inside of $c_{b,1}$, it follows then that any point q_b we pick from $c_{b,2}$ is not contained in c_a.

A similar argument will show that we can always pick a point from the "outer" figure eight that isn't contained in the "inner" figure eight for Scenarios 2 to 4.

In the case of Scenario 5, it follows trivially from them being outside one another that any points p, q we pick from the loops of c_a and c_b aren't contained in the other figure eight.

With this in mind, we see that for any figure eight $c \in C$, we can map it to some pair of coordinates $(p, q) \in \mathbb{Q}^2 \times \mathbb{Q}^2$. Furthermore, we have shown that no two figure eights c can map to the same pair of coordinates (p, q); in other words, there exists an injection from C to $\mathbb{Q}^2 \times \mathbb{Q}^2$.

Then, we observe that $\operatorname{card}(C) \leq \operatorname{card}\left(\mathbb{Q}^2 \times \mathbb{Q}^2\right)$. Then, we note that $\mathbb{Q}^2$ is countable, and the cartesian product of countable sets is also countable. Thus, we conclude that, indeed, C is countable as well. ■

Problem 6.28. Find a set $\mathscr{A}$ of open intervals in $\mathbb{R}$ such that every rational number belongs to one of those intervals, but $\bigcup \mathscr{A} \neq \mathbb{R}$.

Solution. We can define $\mathscr{A}$ as follows:

$$\mathscr{A} := \left\{\left(\sqrt{2} + n - 1, \sqrt{2} + n\right) : n \in \mathbb{Z}\right\}.$$

We see then that every rational must belong to one of these intervals. However, since we're excluding irrationals of the form $\sqrt{2} + n$, we note that the union can't be equal to $\mathbb{R}$. Thus, $\mathscr{A}$ satisfies the desired requirements. ■

Problem 6.32. Let $\mathscr{F}A$ be the collection of all finite subsets of A. Show that if A is infinite, then $A \approx \mathscr{F}A$.

Solution. Let us denote A to be an infinite set, and $\mathscr{F}A$ to be a collection of all finite subsets of A.

Now, we note then that $\operatorname{card}(A) \leq \operatorname{card}(\mathscr{F}A)$ due to the fact that for each element $a \in A$, we have that $\{a\} \in \mathscr{F}A$ by definition.

Now, we want to show then that $\operatorname{card}(\mathscr{F}A) \leq \operatorname{card}(A)$ for them to be equal.

To do this, we shall denote F_n to be the subset of $\mathscr{F}A$ which contains all subsets of A whose cardinality is n. We note then that $\mathscr{F}A = \{\emptyset\} \sqcup \bigsqcup_{n=0}^{\infty} F_n$.

This means then that we have:

$$\begin{aligned}\operatorname{card}(\mathscr{F}A) &= \operatorname{card}\left(\bigsqcup_{n=0}^{\infty} F_n\right)\\ &= \sum_{n=0}^{\infty} \operatorname{card}(F_n)\end{aligned}$$

From here, we note then that the cardinality of each F_n is in fact at most equal to the cardinality of A^n. We note that we can construct a surjection between A^n and F_n; to do this, we consider a subset A'^n of A^n whose members are all of length n and have distinct elements within them. Then, we note that for each member in A'^n, we can map it to an element $\{a_1, \ldots, a_n\} \in F_n$. Thus, every element of F_n is mapped to by something in A'^n.

Now, noting that A is infinite, we see that $\operatorname{card}(F_n) \leq \operatorname{card}(A^n) = \operatorname{card}(A)^n = \operatorname{card}(A)$.

With this in mind, we observe the following:

$$\begin{aligned}\sum_{n=0}^{\infty} \operatorname{card}(F_n) &\leq \sum_{n=0}^{\infty} \operatorname{card}(A)\\ &= \aleph_0 \cdot \operatorname{card}(A)\\ &= \operatorname{card}(A)\end{aligned}$$

Thus, putting this all together, we see that we have $\operatorname{card}(A) \leq \operatorname{card}(\mathscr{F}A) \leq \operatorname{card}(A)$; thus, we conclude that $\operatorname{card}(\mathscr{F}A) = \operatorname{card}(A)$. In other words, $\mathscr{F}A \approx A$ as desired. ■

Problem 6.35. Find a collection $\mathscr{A}$ of $2^{\aleph_0}$ sets of natural numbers such that any two distinct members of $\mathscr{A}$ have finite intersections.

Solution. To begin with, we note that the set of prime numbers P is countably infinite; in other words, $\operatorname{card}(P) = \aleph_0$.

From here, we note that the set of all subsets of P — that is, the set $\mathscr{P}(P)$ — has cardinality of $2^{\aleph_0}$ (in other words, it's uncountable). Then, since the set of finite subsets of a countable set is countable, the set of infinite subsets of P must thus be uncountable.

With this in mind, let us pick out some infinite subset P_1 of P with elements $\{p_0, p_1, \ldots\}$. We can then construct the set A_1 as follows:

$$A_1 := \{p_0, p_0 p_1, \ldots\},$$

where we order its elements in increasing order.

Then, we note that we can construct a bijection between P_1 and A_1. Furthermore, we note that every integer has a unique factorization (up to the order of the factors). With this fact in mind, for a different infinite subset P_2 of P, we note then that A_1 and A_2 will share only a finite number of elements with each other.

Thus, we observe that the collection $\mathscr{A}$ of these sets A will have $2^{\aleph_0}$ sets of natural numbers and each distinct members of $\mathscr{A}$ will have finite intersections. ■

Problem 6.36. Show that for an infinite cardinal κ, we have $\kappa! = 2^\kappa$, where $\kappa!$ is defined as in Exercise 14.

Solution. We first define K to be a set with cardinality $\operatorname{card}(K) = \kappa$.

Then, we can define the set $\mathbb{K} := \{\pi : K \to K : \pi \text{ is a permutation of } K\}$. We note then that $\kappa! = \operatorname{card}(\mathbb{K})$.

Now, we can show that $\kappa! = 2^\kappa$ by showing that $2^\kappa \leq \kappa! \leq 2^\kappa$.

To first show that $\kappa! \leq 2^\kappa$, we note that $\mathbb{K} \subseteq \mathscr{P}(K \times K)$. So, we have:

$$\begin{aligned}
\kappa! &= \operatorname{card}(\mathbb{K}) \\
&\leq \operatorname{card}(\mathscr{P}(K \times K)) \\
&= 2^{\kappa \cdot \kappa} \\
&= 2^\kappa
\end{aligned}$$

For the other inequality, we first consider the disjoint union of K with itself. Let us denote this by K'. Note then that $\operatorname{card}(K') = \kappa' = \kappa + \kappa = \kappa$.

Now, we consider the set of permutations of K', $\mathbb{K}'$, which has cardinality $\kappa'!$. Then, for any permutation $f \in \mathbb{K}'$, we define a function $g \in {}^2K'$ such that $g(k) = 0$ if $f(k)$ is in the first copy of K', and $g(k) = 1$ if $f(k)$ is in the second copy of K'.

We observe then that this is a surjection from $\mathbb{K}'$ onto ${}^2K$. Then, since surjections have an injective right-inverse, we note then that we can construct some injection from ${}^2K$ to $\mathbb{K}'$. Furthermore, since $\operatorname{card}(\mathbb{K}') = \operatorname{card}(\mathbb{K})$, we can construct a bijection between them. Thus, there exists an injection from ${}^2K$ to $\mathbb{K}$. In other words, we have $2^\kappa \leq \kappa!$.

Then, putting this all together, we see then that we have $2^\kappa \leq \kappa! \leq 2^\kappa$. Therefore, we conclude that $2^\kappa = \kappa!$ as desired. ■