Homework 5

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Contents

1 Direct Sums of Null and Range

Problem 1.1. Let $V := \mathbb{C}^3$. Give an example of a map $T \in \mathcal{L}(V, V)$ such that $V = \operatorname{null} T \oplus \operatorname{range} T$, with both $\operatorname{null} T, \operatorname{range} T$ non-zero. Or prove that none such map exists.

Solution. Let us consider the linear map from $V = \mathbb{C}^3 \to V$ represented by the following matrix:

$$T := \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Now, we observe the following for the basis vectors $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1)$:

$$\begin{aligned} Tv_1 &= (1, 1, 0) \\ Tv_2 &= (0, -1, 1) \\ Tv_3 &= (-1, 0, -1) \end{aligned}$$

From here, we note that Tv_1, Tv_2, Tv_3 forms a spanning set for $\operatorname{range} T$ but not a basis; we note that $-Tv_1 - Tv_2 = Tv_3$. Then, let us remove Tv_3 from this list, giving us Tv_1, Tv_2.

Since neither of these vectors are scalar multiples of each other, we note that it is a basis for $\operatorname{range} T$. Thus, we see that a basis for $\operatorname{range} T$ is:

$$\{(1, 1, 0), (0, -1, 1)\}.$$

Next, we note that for some vector $v = (a, b, c)$, applying T onto it yields us the vector:

$$Tv = (a - c, a - b, b - c).$$

In order for us to find vectors in $\operatorname{null} T$, we want some v such that $Tv = 0$. In this case, we note that only when $a = b = c$ do we get that $Tv = 0$; in other words, a basis for $\operatorname{null} T$ is

$$\{(1, 1, 1)\}.$$

From here, we observe that for $\operatorname{range} T + \operatorname{null} T = \{(1, 1, 0), (0, -1, 1), (1, 1, 1)\}$, we want to check for linear independence. In other words, we want to see whether the following holds true only when $a_1 = a_2 = a_3 = 0$:

$$a_1(1, 1, 0) + a_2(0, -1, 1) + a_3(1, 1, 1) = 0.$$

Then, we observe that we get the following system of equations:

$$\begin{aligned} a_1 + a_3 &= 0 \\ a_1 - a_2 + a_3 &= 0 \\ a_2 + a_3 &= 0 \end{aligned}$$

From here, we note that since $a_1 + a_3 = 0$, then:

$$\begin{aligned} a_1 - a_2 + a_3 &= 0 \\ a_1 + a_3 - a_2 &= 0 \\ 0 - a_2 &= 0 \\ a_2 &= 0 \end{aligned}$$

$$\begin{aligned} a_2 + a_3 &= 0 \\ 0 + a_3 &= 0 \\ a_3 &= 0 \end{aligned}$$

$$\begin{aligned} a_1 + a_3 &= 0 \\ a_1 + 0 &= 0 \\ a_1 &= 0 \end{aligned}$$

Thus, since $a_1 = a_2 = a_3 = 0$, it follows then that the vectors $(1, 1, 0), (0, -1, 1), (1, 1, 1)$ are all linearly independent. Thus, we have that

$$\operatorname{null} T \oplus \operatorname{range} T = \operatorname{span}\left((1, 1, 0), (0, -1, 1), (1, 1, 1)\right).$$

And we note that since it is the span of three linearly independent vectors, it must be then that it spans all of $\mathbb{C}^3$, and thus we can conclude that

$$\operatorname{null} T \oplus \operatorname{range} T = V.$$

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2 Nullity of T

Problem 2.1. Given an example of a map $T \in \mathcal{L}(\mathbb{R}^6, \mathbb{R}^2)$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4, x_5, x_6) : x_1 = -x_2, x_3 + x_5 = 0, x_1 - x_4 - x_5 = 0\},$$

or prove that none such map exists.

Solution. We first note that since T maps from $\mathbb{R}^6$ to $\mathbb{R}^2$, we know that $\dim V = 6$, and also that $\dim \operatorname{range} T \leq 2$.

Now, we observe the following:

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ \dim V &\leq \dim \operatorname{null} T + 2 \\ 4 &\leq \dim \operatorname{null} T \end{aligned}$$

In other words, by the Rank-Nullity Theorem, we see that $\dim \operatorname{null} T$ must be at least 4.

We now observe the following, for each vector $v \in \operatorname{null} T$:

$$\begin{aligned} x_1 &= -x_2 \\ x_3 + x_5 &= 0 \\ x_1 - x_4 - x_5 &= 0 \end{aligned}$$

$$\begin{aligned} x_2 &= -x_1 \\ x_5 &= -x_3 \\ x_4 &= x_1 - x_5 \\ &= x_1 + x_3 \end{aligned}$$

So, we can rewrite each vector $(x_1, x_2, x_3, x_4, x_5, x_6) \in \operatorname{null} T$ as:

$$(x_1, -x_1, x_3, x_1 + x_3, -x_3, x_6),$$

where $x_1, \ldots, x_6 \in \mathbb{R}$.

Now, we observe that this vector can in fact be rewritten as

$$x_1(1, -1, 0, 1, 0, 0) + x_3(0, 0, 1, 1, -1, 0) + x_6(0, 0, 0, 0, 0, 1).$$

Thus, we see that the following is a basis for $\operatorname{null} T$:

$$\{(1, -1, 0, 1, 0, 0), (0, 0, 1, 1, -1, 0), (0, 0, 0, 0, 0, 1)\}.$$

However, this means then that $\dim \operatorname{null} T = 3$, which contradicts with what the Rank-Nullity theorem states. Thus, such a map does not exists. ■

3 Basis for Null and Range

Problem 3.1. Suppose $T : \mathscr{P}_3(\mathbb{R}) \to \mathscr{P}_2(\mathbb{R})$ is defined by the formula

$$(Tf)(x) = 4xf''(x) - f'(x).$$

Check that $T \in \mathcal{L}(\mathscr{P}_3(\mathbb{R}), \mathscr{P}_2(\mathbb{R}))$, and find a basis for the null space and range of T.

Solution. Let us define $V := \mathscr{P}_3(\mathbb{R})$, and $W := \mathscr{P}_2(\mathbb{R})$.

First, we check whether T is indeed linear. That is, $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$:

$$\begin{aligned} T(\alpha f + \beta g) &= 4x(\alpha f + \beta g)'' - (\alpha f + \beta g)' \\ &= 4x(\alpha f'' + \beta g'') - (\alpha f' + \beta g') \\ &= 4x\alpha f'' + 4x\beta g'' - \alpha f' - \beta g' \\ &= \alpha 4xf'' - \alpha f' + \beta 4xg'' - \beta g' \\ &= \alpha(4xf'' - f') + \beta(4xg'' - g') \\ &= \alpha T(f) + \beta T(g). \end{aligned}$$

Thus, we see that T is indeed linear. Next, we check to see whether it truly sends a polynomial $p \in \mathscr{P}_3(\mathbb{R})$ to $\mathscr{P}_2(\mathbb{R})$. To do this, we observe that for some $p = ax^3 + bx^2 + cx + d$, we have:

$$\begin{aligned} (Tp)(x) &= (4xp'' - p')(x) \\ &= 4x(6ax + 2b) - (3ax^2 + 2bx + c) \\ &= 24ax^2 + 8bx - 3ax^2 - 2bx - c \\ &= 21ax^2 + 6bx - c \end{aligned}$$

We see that T does send any $p \in \mathscr{P}_3(\mathbb{R}) \to \mathscr{P}_2(\mathbb{R})$. Thus, indeed, we have that $T \in \mathcal{L}(V, W)$.

From here, let us consider the following basis vectors of $\mathscr{P}_3(\mathbb{R})$:

1. $f_1 = 1$
2. $f_2 = x$
3. $f_3 = x^2$
4. $f_4 = x^3$.

We will now look at how T transform each of these vectors:

$$\begin{aligned} (Tf_1)(x) &= 4x(0) - 0 \\ &= 0 \\ (Tf_2)(x) &= 4x(0) - 1 \\ &= -1 \\ (Tf_3)(x) &= 4x(2) - 2x \\ &= 8x - 2x \\ &= 6x \\ (Tf_4)(x) &= 4x(6x) - 3x^2 \\ &= 24x^2 - 3x^2 \\ &= 21x^2 \end{aligned}$$

We observe that $\operatorname{null} T$ consists of vectors $v \in V$ such that $Tv = 0$. In this case, we observe that any $v \in \operatorname{span}(1)$ satisfies this condition. Thus, we have a basis for the null space of T as $\{1\}$.

For the basis for the range of T, we observe that any polynomial $p \in \operatorname{range} T$ will be of the form $21ax^2+6bx-c$. We note that this means that any polynomial in $\operatorname{range} T$ can be written as a linear combination of $21x^2, 6x, -1$. Furthermore, we note that they are linearly independent. Thus, we have that $\left\{-1, 6x, 21x^2\right\}$ forms a basis for $\operatorname{range} T$.

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4 Matrix Representation

Let $T : f(x) \mapsto (x-1)^2 f'''(x) - 3(x-1)f''(x) + f'(x)$. Write down the following matrix representations:

Problem 4.1. T as a map in $\mathcal{L}(\mathscr{P}_3, \mathscr{P}_2)$ using the standard monomial bases for both the domain and codomain.

Solution. We observe that T maps the basis vectors of $\mathscr{P}_3$ as follows:

$$\begin{aligned}
1 &\mapsto (x-1)^2(0) - 3(x-1)(0) + 0 \\
&= 0 \\
x &\mapsto (x-1)^2(0) - 3(x-1)(0) + 1 \\
&= 1 \\
x^2 &\mapsto (x-1)^2(0) - 3(x-1)(2) + 2x \\
&= -6x + 6 + 2x \\
&= -4x + 6 \\
x^3 &\mapsto (x-1)^2(6) - 3(x-1)(6x) + 3x^2 \\
&= 6(x^2 - 2x + 1) - 18(x^2 - x) + 3x^2 \\
&= -9x^2 + 6x + 6
\end{aligned}$$

So, we know that the matrix representation for T is as follows:

$$\begin{bmatrix} 0 & 1 & 6 & 6 \\ 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -9 \end{bmatrix}$$

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Problem 4.2. T as a map in $\mathcal{L}(\mathscr{P}_3, \mathscr{P}_3)$ using the standard monomial bases for both the domain and codomain.

Solution. For this, the matrix representation will be almost identical to the previous part. However, because we are mapping to $\mathscr{P}_3$, we need an extra row of 0's to indicate that the x^3 term always has a coefficient of 0.

Thus, we can represent T with the following matrix:

$$\begin{bmatrix} 0 & 1 & 6 & 6 \\ 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Problem 4.3. T as a map in $\mathcal{L}(\mathscr{P}_3, \mathscr{P}_3)$ using the shifted monomial bases $1, x-1, (x-1)^2, (x-1)^3$ for both the domain and codomain.

Solution. We first observe how T transform each of the basis vectors as follows:

$$\begin{aligned}
1 &\mapsto (x-1)^2(0) - 3(x-1)(0) + 0 \\
&= 0 \\
x-1 &\mapsto (x-1)^2(0) - 3(x-1)(0) + 1 \\
&= 1 \\
(x-1)^2 &\mapsto (x-1)^2(0) - 3(x-1)(2) + 2(x-1) \\
&= -6(x-1) + 2(x-1) \\
&= -4(x-1) \\
(x-1)^3 &\mapsto (x-1)^2(6) - 3(x-1)(6(x-1)) + 3(x-1)^2 \\
&= 6(x-1)^2 - 18(x-1)^2 + 3(x-1)^2 \\
&= -9(x-1)^2
\end{aligned}$$

Thus, the following is a matrix representation for T:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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5 Subspaces

Problem 5.1. Suppose V and W are finite-dimensional. Let v be a fixed vector in V, and consider

$$E_v \coloneqq \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

Show that E_v is a subspace of $\mathcal{L}(V, W)$.

Solution. To begin with, we check whether E_v contains the zero vector $0_{\mathcal{L}(V,W)}$. We note that this zero vector is simply the zero map; that is, the linear map T which maps all vectors $v \in V$ to $0 \in W$.

We observe that because $0_{\mathcal{L}(V,W)}$ sends all vectors $v \in V$ to $0 \in W$, then by definition of E_v, we know that it is contained in E_v.

Next, we want to check for closure under addition and scalar multiplication.

First, suppose we let v be some fixed vector within V. Next, we consider some $T, S \in E_v$. We now want to show that the map $S + T \in E_v$ as well for it to be closed under vector addition. Now, we observe the following:

$$\begin{aligned}(S+T)(v) &= S(v) + T(v)\\ &= 0 + 0\\ &= 0\end{aligned}$$

Thus, we see that $S + T$ is also in E_v. Now, we consider some $\lambda \in \mathbb{F}$, and see the following:

$$\begin{aligned}(\lambda T)(v) &= \lambda T(v)\\ &= \lambda(0)\\ &= 0\end{aligned}$$

Thus, we see that λT is in E_v too; it is closed under scalar multiplication as well.

Thus, we can conclude that E_v is indeed a subspace of $\mathcal{L}(V, W)$. ■

Problem 5.2. Suppose that $v \neq 0$. What is $\dim E_v$?

Solution. Let us consider the vector spaces V and W, with dimensions n and m respectively.

First, we note that since $v \neq 0$, then that means that we can extend it to be a basis of V, with the basis being $v, v_2, \ldots, v_n$. Furthermore, since W is finite-dimensional, then we know that it has a basis $w_1, \ldots, w_m$.

Now, from here, we note then that $\mathcal{L}(V, W)$ is thus isomorphic to $\mathbb{F}^{m,n}$ under these bases. So, every map $T \in \mathcal{L}(V, W)$ can be represented with a $m \times n$ matrix $\mathcal{M}(T)$ which has m rows and n columns. In other words, we have

$$\mathcal{M}(T) = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n}\\ \vdots & a_{2,2} & \cdots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ a_{m,1} & a_{m,2} & \cdots & a_{m,n}\end{bmatrix}$$

We note here that the i^{th} column of $\mathcal{M}(T)$ consists of the scalars required to write Tv_i as a linear combination of $w_1, \ldots, w_m$. However, since we have $Tv = 0$, this means then that v gets mapped to the zero vector 0_W. Furthermore, since we know that $w_1, \ldots, w_m$ are a basis for W, then they're linearly independent; thus, the only way to express 0_W as a linear combination of $w_1, \ldots, w_m$ is if all the scalars are zero.

In other words, we have that the first column of $\mathcal{M}(T)$ must be equal to zero. So, we know that $\mathcal{M}(T)$ must have the form

$$\mathcal{M}(T) = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,n} \\ \vdots & a_{2,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

From here, we see then that we have $m(n-1)$ entries that can take on any values. Thus, we have then that $\dim E_v = m(n-1) = \dim W(\dim V - 1)$. ■