Math 135: Homework 3

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Problems

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3 Relations and Functions

Problem 3.12. Assume that f and g are functions and show that

$$f \subseteq g \iff (\operatorname{dom} f \subseteq \operatorname{dom} g) \wedge (\forall x \in \operatorname{dom} f)\, f(x) = g(x).$$

Solution. Let us suppose that f and g are functions

First, we will prove the forward direction.

Proof. Suppose that $f \subseteq g$.

From here, let us take some $x \in \operatorname{dom} f$. Then, by definition, there exists some $f(x)$ such that $\langle x, f(x)\rangle \in f$. Then, since $f \subseteq g$, it follows that $\langle x, f(x)\rangle \in g$ as well. Thus, we see that $x \in \operatorname{dom} g$ by definition. Therefore, we have that $\operatorname{dom} f \subseteq \operatorname{dom} g$.

Next, we want to show that $\forall x \in \operatorname{dom} f$, we have $f(x) = g(x)$.

To do this, we observe from the last step that by definition of $x \in \operatorname{dom} g$, there exists some $g(x)$ such that $\langle x, g(x)\rangle \in g$. However, we note that $\langle x, f(x)\rangle \in g$ as well. Then, as g is a function and we have that $\langle x, f(x)\rangle$ and $\langle x, g(x)\rangle$ are both in g, it must be that $f(x) = g(x)$ for all $x \in \operatorname{dom} f$. □

Now, we will prove the backwards direction.

Proof. Let us suppose that $\operatorname{dom} f \subseteq \operatorname{dom} g$, and that $(\forall x \in \operatorname{dom} f) f(x) = g(x)$. We want to show that $f \subseteq g$.

To do this, let us take some $\langle x, f(x)\rangle \in f$. Then, $x \in \operatorname{dom} f$, and by our assumption, $x \in \operatorname{dom} g$ as well. Then, this means that there exists some $g(x)$ such that $\langle x, g(x)\rangle \in g$.

However, we note that by our assumption, $f(x) = g(x)$ for all $x \in \operatorname{dom} f$, so in fact we have $\langle x, g(x)\rangle = \langle x, f(x)\rangle \in g$.

Thus, $f \subseteq g$ as desired. □

■

Problem 3.14a. Assume that f and g are functions. Show that $f \cap g$ is a function.

Solution. Let us suppose that f and g are functions.

Now, by definition, we have that $f \cap g$ consists of all ordered pairs in both f and g.

So, let us take some $\langle a, b\rangle \in f \cap g$ and $\langle a, c\rangle \in f \cap g$. Note then that $\langle a, b\rangle$ and $\langle a, c\rangle$ must both be in f. And since f is a function, by definition, it must be that $b = c$, and thus $f \cap g$ is a function as well. ■

Problem 3.14b. Assume that f and g are functions. Show that $f \cup g$ is a function if and only if $f(x) = g(x)$ for every x in $(\operatorname{dom} f) \cap (\operatorname{dom} g)$.

Solution. Suppose that f and g are functions.

Now, we will prove the forward direction.

Proof. Suppose that $f \cup g$ is a function. Then, let us take some $x \in (\operatorname{dom} f) \cap (\operatorname{dom} g)$. Then, this means that there exists some $\langle x, f(x)\rangle \in f$ and $\langle x, g(x)\rangle \in g$.

Now, since $f \cup g$ consists of all ordered pairs in f or g, we note then that $\langle x, f(x)\rangle$ and $\langle x, g(x)\rangle$ are both in $f \cup g$. However. since $f \cup g$ is a function, then by definition, we have that $f(x) = g(x)$. □

Next, we'll show the backwards direction.

Proof. Suppose that $f(x) = g(x)$ for every $x \in (\operatorname{dom} f) \cap (\operatorname{dom} g)$.

Then, let us take some $\langle x, y\rangle \in f \cup g$ and $\langle x, z\rangle \in f \cup g$. We have four cases to consider:

- First, we note that if both $\langle x, y\rangle$ and $\langle x, z\rangle$ are in f, then, since f is a function, it follows that $y = z$.
- Similarly, if $\langle x, y\rangle$ and $\langle x, z\rangle$ are both in g, it follows that because g is a function, $y = z$.
- Next, if $\langle x, y\rangle \in f$ and $\langle x, z\rangle \in g$, we first note then that $x \in (\operatorname{dom} f) \cap (\operatorname{dom} g)$. From here, by our assumption, we have that $f(x) = g(x)$ for all $x \in (\operatorname{dom} f) \cap (\operatorname{dom} g)$. Thus, we have that $\langle x, y\rangle = \langle x, f(x)\rangle = \langle x, g(x)\rangle = \{x, z\}$. Thus, we have that $y = z$ as desired.
- Finally, if $\langle x, y\rangle \in g$ and $\langle x, z\rangle \in f$, we can follow a similar argument as above to conclude that $y = z$ as well.

Therefore, we conclude that $\forall x \forall y \forall z$, we have that if $\langle x, y\rangle \in f \cup g$ and $\langle x, z\rangle \in f \cup g$, then $y = z$, which is precisely what it means for $f \cup g$ to be a function. □

■

Problem 3.18. Let R be the set

$$\{\langle 0, 1\rangle, \langle 0, 2\rangle, \langle 0, 3\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}.$$

Evaluate the following: $R \circ R$, $R \restriction \{1\}$, $R^{-1} \restriction \{1\}$, $R [\![\{1\}]\!]$, and $R^{-1} [\![\{1\}]\!]$.

Solution. First, we note that we have

$$R^{-1} = \{\langle 1, 0\rangle, \langle 2, 0\rangle, \langle 3, 0\rangle, \langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle\}.$$

We have the following:

- $R \circ R = \{\langle 0, 2\rangle, \langle 0, 3\rangle, \langle 1, 3\rangle\}$.
- $R \restriction \{1\} = \{\langle 1, 2\rangle, \langle 1, 3\rangle\}$.
- $R^{-1} \restriction \{1\} = \{\langle 1, 0\rangle\}$.
- $R [\![\{1\}]\!] = \{2, 3\}$.
- $R^{-1} [\![\{1\}]\!] = \{0\}$.

■

Problem 3.21. Show that $(R \circ S) \circ T = R \circ (S \circ T)$ for any set R, S, and T.

Solution. We shall proceed as follows:

$$\begin{aligned}
\langle w, z\rangle \in (R \circ S) \circ T &\iff \exists x(\langle w, x\rangle \in T \wedge \langle x, z\rangle \in R \circ S) \\
&\iff \exists x(\langle w, x\rangle \in T \wedge \exists y(\langle x, y\rangle \in S \wedge \langle y, z\rangle \in R)) \\
&\iff \exists x \exists y(\langle w, x\rangle \in T \wedge (\langle x, y\rangle \in S \wedge \langle y, z\rangle \in R)) \\
&\iff \exists y \exists x((\langle w, x\rangle \in T \wedge \langle x, y\rangle \in S) \wedge \langle y, z\rangle \in R) \\
&\iff \exists y(\exists x(\langle w, x\rangle \in T \wedge \langle x, y\rangle \in S) \wedge \langle y, z\rangle \in R) \\
&\iff \exists y(\langle w, y\rangle \in (S \circ T) \wedge \langle y, z\rangle \in R) \\
&\iff \langle w, z\rangle \in R \circ (S \circ T)
\end{aligned}$$

Thus, we see that, indeed, $(R \circ S) \circ T = R \circ (S \circ T)$ as desired. ■

Problem 3.22a. For any set, show that $A \subseteq B \implies F[\![A]\!] \subseteq F[\![B]\!]$.

Solution. Suppose that we have some $y \in F[\![A]\!]$. Then, by definition, this means that there exists some $x \in A$ such that $\langle x, y\rangle \in F$.

However, we note that since $A \subseteq B$, $x \in B$ as well. Thus, we see that $y \in F[\![B]\!]$ by definition, and thus we have that $F[\![A]\!] \subseteq F[\![B]\!]$ as desired. ■

Problem 3.22b. For any set, show that $(F \circ G)[\![A]\!] = F[\![G[\![A]\!]]\!]$.

Solution. We will first show that $(F \circ G)[\![A]\!] \subseteq F[\![G[\![A]\!]]\!]$.

Let us take some $z \in (F \circ G)[\![A]\!]$. Then, by definition, we observe that there exists some $x \in A$ such that $\langle x, z\rangle \in F \circ G$.

Then, by definition of composition, there exists some y such that $\langle x, y\rangle \in G$ and $\langle y, z\rangle \in F$. So, we observe that $y \in G[\![A]\!]$, and thus $z \in F[\![G[\![A]\!]]\!]$.

Now, we will show the other inclusion.

Suppose that $z \in F[\![G[\![A]\!]]\!]$. Then, by definition, we see that there exists some $y \in G[\![A]\!]$ such that $\langle y, z\rangle \in F$. But then, this means that there exists some $x \in A$ such that $\langle x, y\rangle \in G$.

Putting this all together, we have then that there exists some $x \in A$ and there exists some y such that $\langle x, y\rangle \in G$ and $\langle y, z\rangle \in F$. In other words, we have $z \in (F \circ G)[\![A]\!]$ as desired. ■

Problem 3.22c. For any set, show that $Q \upharpoonright (A \cup B) = (Q \upharpoonright A) \cup (Q \upharpoonright B)$.

Solution. We proceed as follows:

$$\begin{aligned}
\langle x, y\rangle \in Q \upharpoonright (A \cup B) &\iff \langle x, y\rangle \in \{\langle u, v\rangle : (\langle u, v\rangle \in Q) \wedge (u \in A \cup B)\} \\
&\iff \langle x, y\rangle \in \{\langle u, v\rangle : (\langle u, v\rangle \in Q) \wedge (u \in A \vee u \in B)\} \\
&\iff \langle x, y\rangle \in \{\langle u, v\rangle : (\langle u, v\rangle \in Q \wedge u \in A) \vee (\langle u, v\rangle \in Q \wedge u \in B)\} \\
&\iff \langle x, y\rangle \in \{\langle u, v\rangle : \langle u, v\rangle \in Q \wedge u \in A\} \vee \langle x, y\rangle \in \{\langle u, v\rangle : \langle u, v\rangle \in Q \wedge u \in B\} \\
&\iff \langle x, y\rangle \in (Q \upharpoonright A) \cup (Q \upharpoonright B)
\end{aligned}$$

And thus, we see that $Q \upharpoonright (A \cup B) = (Q \upharpoonright A) \cup (Q \upharpoonright B)$ as desired. ■

Problem 3.29. Assume that $f : A \to B$ and define a function $G : B \to \mathscr{P} A$ by

$$G(b) := \{x \in A : f(x) = b\} .$$

Show that if f maps A onto B, then G is one-to-one. Does the converse hold?

Solution. First, we will show that if f maps A onto B, then G is one-to-one.

Proof. Suppose that f maps A onto B. Then, by definition, for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Then, we want to show that G is one-to-one. Suppose that we had $G(b) = G(b')$ for $b, b' \in B$. Because f is surjective, we note then that $G(b)$ and $G(b')$ are both non-empty. Now, we have the following:

$$G(b) = \{x \in A : f(x) = b\} = \{x \in A : f(x) = b'\} = G(b')$$

However, we note here that for these two sets to be equal, their elements must be the same. In other words, $\forall x(x \in G(b) \iff x \in G(b'))$.

Then, let us take $x \in G(b)$ such that $f(x) = b$. Since $G(b) = G(b')$, $x \in G(b')$ as well, meaning that $f(x) = b'$. Since f is a function, it follows then that since we have $f(x) = b$ and $f(x) = b'$, it must be that $b = b'$. Thus, we see that if $G(b) = G(b')$, then $b = b'$. Thus, G is one-to-one as desired. □

Now, we will examine whether the converse holds or not.

Proof. We claim that it doesn't.

Let us consider the set $A = \{x\}$, and $B = \{x, y\}$. Then, we define $G(x) = \{x\}$ and $G(y) = \emptyset$. By inspection, we see that G is indeed injective.

However, note here that while $f(x) = x$, there exists no $a \in A$ such that $f(a) = 2$; f does not map A onto B. □

■

Problem 3.30a. Assume that $F : \mathscr{P} A \to \mathscr{P} A$, and that F has the monotonicity property:

$$X \subseteq Y \subseteq A \implies F(X) \subseteq F(Y).$$

Define

$$B = \bigcap \{X \subseteq A : F(X) \subseteq X\} \qquad \text{and} \qquad C = \bigcup \{X \subseteq A : X \subseteq F(X)\} .$$

Show that $F(B) = B$ and $F(C) = C$.

Solution. We will first show that $F(B) = B$.

Proof. First, we will show that $F(B) \subseteq B$. To do this, let us define $\mathcal{B}$ to be the following:

$$\mathcal{B} = \{X \subseteq A : F(X) \subseteq X\} .$$

Then, we see that $B = \bigcap \mathcal{B}$.

Now, for all $X \in \mathcal{B}$, we observe that $B \subseteq X$ by definition of the intersection. Then, it follows that $F(B) \subseteq F(X)$ by monotonicity of F.

From here, we note that for every $X \in \mathcal{B}$, we have that $F(B) \subseteq F(X) \subseteq X$. Then, we have that for all $X \in \mathcal{B}$, $F(B) \subseteq \bigcap \{X : X \in \mathcal{B}\} = B$. Therefore, $F(B) \subseteq B$ as desired.

For this other inclusion, observe that since $F(B) \subseteq B \subseteq X \subseteq A$, then $F(F(B)) \subseteq F(B)$ by monotonicity. However, this means then that $F(B) \in \mathcal{B}$ by definition. Then, by definition of the intersection, any $x \in \bigcap \mathcal{B}$ is also in $F(B)$. Thus, we have that $\bigcap \mathcal{B} = B \subseteq F(B)$ as desired.

Therefore, we see that equality holds. □

Now we will prove that $F(C) = C$. Note that the proof is similar to the previous claim.

Proof. First, we will show $C \subseteq F(C)$. Let us define $\mathcal{C}$ to be

$$\mathcal{C} = \{X \subseteq A : X \subseteq F(X)\}.$$

Then, we have that $C = \bigcup \mathcal{C}$.

Now, for all $X \in \mathcal{C}$, we observe that $X \subseteq C$, and by monotonicity of F, $F(X) \subseteq F(C)$. And since $\mathcal{C}$ contains all X such that $X \subseteq F(X)$, we have that $X \subseteq F(X) \subseteq F(C)$.

So, for any $X \in \mathcal{C}$, we have that for any $x \in X$, $x \in F(C)$ as well. From here, by definition of $\bigcup \mathcal{C}$, this means that any $x \in \bigcup \mathcal{C}$ is in some $X \in \mathcal{C}$, and thus in $F(C)$ as desired. So, we have that $\bigcup \mathcal{C} = C \subseteq F(C)$.

For the other inclusion, we observe that since $C \subseteq F(C)$, then we have that $F(C) \subseteq F(F(C))$ by monotonicity of F. However, this means then that $F(C) \in \mathcal{C}$ by definition of $\mathcal{C}$.

From here, by definition of $\bigcup \mathcal{C}$, we see then that $F(C) \subseteq \bigcup \mathcal{C} = C$.

Therefore, we can conclude that equality holds. □

■

Problem 3.30b. Show that if $F(X) = X$, then $B \subseteq X \subseteq C$.

Solution. We observe that if $F(X) = X$, then it follows that $F(X) \subseteq X$ and $X \subseteq F(X)$.

With this in mind, we observe that as $F(X) \subseteq X$, by definition of B, we have that $B \subseteq X$.

Similarly, since $X \subseteq F(X)$, then by definition of C, we have that $X \subseteq C$.

Putting it all together, we have that $B \subseteq X \subseteq C$ as desired. ■

Problem 3.31. Show that the first form of the Axiom of Choice is equivalent to the second form.

Solution. Before we begin, we will state the two forms of the Axiom of Choice:

1. For any relation R, there exists a function $H \subseteq R$ with $\operatorname{dom} H = \operatorname{dom} R$.
2. For any set I, and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\bigtimes_{i \in I} H(i) \neq \emptyset$.

Furthermore, we note that the definition of $\bigtimes_{i \in I} H(i)$ is:

$$\bigtimes_{i \in I} H(i) = \{f : f \text{ is a function with domain } I \text{ and } (\forall i \in I) f(i) \in H(i)\}.$$

First, we will prove the forward direction.

Proof. Let us assume the first form of the Axiom of Choice.

Now, to begin with, for any set I and any function H whose domain is I and $H(i) \neq \emptyset$ for all $i \in I$, suppose we had some relation $R \subseteq I \times \bigcup_{i\in I} H(i)$ with the following property:

$$\langle i, r\rangle \in R \iff r \in H(i).$$

Then from here, we see that by the first form of the Axiom of Choice, there exists a function $G \subseteq R$ with $\operatorname{dom} G = \operatorname{dom} R = I$. Then, we observe that for all $i \in I$, we have that $\langle i, G(i)\rangle \in G$, and thus $\langle i, G(i)\rangle \in R$.

By definition of R then, for all $i \in I$ we have $G(i) \in H(i)$, meaning that $G \in \bigtimes_{i\in I} H(i)$ by definition of $\bigtimes_{i\in I} H(i)$.

And since this is the case, we see then that $\bigtimes_{i\in I} H(i)$ can't be empty. Thus, we have proven that the first version of the Axiom of Choice implies the second. $\square$

Now, we will show the other direction.

Proof. Assume the second form of the Axiom of Choice. Then, let us define a relation R such that its domain is I. Next, we define a function H to be the following:

$$H : I \to \mathscr{P}(\operatorname{ran} R) \text{ where } i \mapsto H(i),$$

where $H(i)$ is defined to be the following:

$$H(i) = \{r \in \operatorname{ran} R : \langle i, r\rangle \in R\}.$$

From here, we see that the domain of H is I. Furthermore, $H(i) \neq \emptyset$ for all $i \in I$. Then, by the second form of the Axiom of Choice, we see that $\bigtimes_{i\in I} H(i) \neq \emptyset$.

With this in mind, we can choose some function $G \in \bigtimes_{i\in I} H(i)$. Note then that $\operatorname{dom} G = I = \operatorname{dom} R$.

Then, we want to show that $G \subseteq R$. To do this, let us take some $\langle i, G(i)\rangle \in G$. By definition, we note that $G(i) \in H(i)$ for all $i \in I$, meaning that $G(i) \in \operatorname{ran} R$. And since we have that $i \in \operatorname{dom} R$, we see that $\langle i, G(i)\rangle \in R$ for all $i \in I$. Thus, we can conclude then that $G \subseteq R$ with $\operatorname{dom} G = \operatorname{dom} R$ as desired. $\square$

Therefore, we can conclude that these two forms of the Axiom of Choice are indeed equivalent. $\blacksquare$