Homework 9

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Contents

1 Eigenvalues of Dual Maps

Problem 1.1. Suppose V is a complex finite-dimensional vector space, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{C}$. Prove or disprove: if λ is an eigenvalue of T, then it is also an eigenvalue of T'.

Solution. To begin with, let us denote $A = \mathcal{M}(T)$. Then, since V is a finite-dimensional vector space, with $T \in \mathcal{L}(V)$, then we note that $\mathcal{M}(T') = A^\intercal$.

From here, we note that since V is a complex, finite-dimensional vector space, then we know that T is triangularisable under some basis. In other words, for some basis of V, we have

$$\mathcal{M}(T) = A = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

We note here that since $\mathcal{M}(T)$ is an upper-triangular matrix, the eigenvalues of T are precisely the diagonal entries of the matrix $\mathrm{M}(T)$.

Then, with this in mind, we observe the following:

$$\mathcal{M}(T') = A^\intercal = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ * & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \lambda_n \end{bmatrix}$$

Then, since $\mathrm{M}(T')$ is a lower-triangular matrix, we note that its eigenvalues are precisely the diagonal entries. Thus, we note that if λ is an eigenvalue of T, then it must be an eigenvalue of T' as well. ■

Problem 1.2. Prove or disprove the converse.

Solution. We proceed similarly: let us denote $A = \mathcal{M}(T')$. Then, we note that $\mathcal{M}(T) = A^\intercal$.

Then, as before, we note that T' can be triangularised under some basis of V, yielding us the following matrix representation:

$$\mathcal{M}(T') = A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ * & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \lambda_n \end{bmatrix}$$

And since $\mathcal{M}(T')$ is a lower-triangular matrix, we note that the diagonal entries $\lambda_1, \ldots, \lambda_n$ are its eigenvalues.

Then, as before, we note that

$$\mathcal{M}(T) = A^\intercal = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

And since $\mathcal{M}(T)$ is an upper-triangular matrix, we see that its diagonal entries $\lambda_1, \ldots, \lambda_n$ are precisely its eigenvalues as well. Thus, if λ is an eigenvalue of T', it must also be an eigenvalue of T. ■

2 Partial Derivatives?!

Problem 2.1. Let V be the complex vector space of bivariate polynomials of total degree at most 2, and let T be a linear operator defined as such:

$$T : p \mapsto \frac{\partial p}{\partial x} - \frac{\partial p}{\partial y}.$$

Determine the minimal polynomial.

Solution. First, we note that a basis for V is: $1, x, y, x^2, y^2, xy$.

Then, we apply T on each of the basis vectors to see the following:

1. $T(1) = 0$,
2. $T(x) = 1$,
3. $T(y) = -1$,
4. $T(x^2) = 2x$,
5. $T(y^2) = -2y$,
6. $T(xy) = y - x$.

Then, we note that the matrix representation for T is as such:

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we note that $T^2(x^2) = 2$, which is non-zero. So the smallest power of T which annihilates everything is T^3. This is because every time we apply T onto some vector, it reduces the degree by one. As such, x^2, y^2, xy all have degree 2, so we need to apply T there times.

Thus, we see that the minimal polynomial must be $p(z) = z^3$. ■

Problem 2.2. Determine all eigenvalues.

Solution. We observe that since $\mathcal{T}$ is an upper-triangular matrix under our chosen basis, the eigenvalues are in fact the diagonal entries. In this case, all eigenvalues λ are 0. ■

Problem 2.3. Determine the corresponding eigenvectors.

Solution. First, we note that $\dim \operatorname{range} T = 3$. So, by the Rank-Nullity Theorem, we know that $\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T = 6 - 3 = 3$. Then, we must have three eigenvectors. The eigenvectors are some $p \in V$ such that $Tp = 0v = 0$. So, by inspection, we have the following:

1. 1,

2. $x + y$,
3. $2xy + x^2 + y^2$.

To confirm that these are our eigenvectors, we proceed as follows:

$$T(1) = 0$$

$$\begin{aligned} T(x+y) &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} T(2xy + x^2 + y^2) &= (2y + 2x) - (2x + 2y) \\ &= 2y + 2x - 2x - 2y \\ &= 0 \end{aligned}$$

■

3 Diagonalisability

Problem 3.1. Suppose that V is a finite-dimensional vector space. Prove or disprove: if two operators T, S from $\mathrm{L}(V)$ commute, then T is diagonalisable if and only if S is.

Solution. We provide the following counterexample: let us define $V = \mathbb{F}^3$. Then, let us define $T, S \in \mathcal{L}(\mathbb{F}^3)$ to have the following matrix representations:

$$\mathcal{M}(T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\mathcal{M}(S) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we note that $T = I$, and the identity matrix always commute. Then, we have that $TS = ST$ as desired. Furthermore, note that the identity matrix is diagonalisable.

However, we observe that S is not diagonalisable: we note that since $\mathcal{M}(S)$ is an upper-triangular matrix, its eigenvalues are the diagonal entries. In this case, $\lambda = 0$ is the only eigenvalue of S. Furthermore, we note that:

$$E(0, S) = \left\{(a, 0, 0) \in \mathbb{F}^3 : a \in \mathbb{F}\right\}.$$

Thus, we see that V does not have a basis consist of the eigenvectors of S, and thus S is not diagonalisable. ■

4 Checking a Bunch of Stuff

Problem 4.1. Let $V := \mathscr{P}_3(\mathbb{R})$ and let $T \in \mathcal{L}(V)$ be the operator $f(x) \mapsto f(x-1) + f(x+1)$. Is T triangularisable?

Solution. Let us consider a basis for V: $1, x, x^2, x^3$. Then, we note that the matrix representation of T under this basis is:

$$\mathrm{M}(T) = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Then, we see that $\mathrm{M}(T)$ is already upper-triangular, so T is indeed triangularisable. ∎

Problem 4.2. Is T diagonalisable?

Solution. To begin with, we can see that since $\mathcal{M}(T)$ is upper-triangular, its eigenvalues are the diagonal entries of the matrix. More specifically, we know that $\lambda = 2$ is the only eigenvalue of T.

From here, we note that the eigenvectors are non-zero vectors such that $(T - 2I)v = 0$. In other words, $v \in \operatorname{null}(T - 2I)$. We note as well that:

$$T - 2I = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, we see that $\dim \operatorname{range}(T-2I) = 2$. Now, by the Rank-Nullity theorem, we know that $\dim \operatorname{null}(T-2I) = \dim V - \dim \operatorname{range}(T-2I) = 4-2 = 2$. So, we know that we have two eigenvectors. From here, by inspection, we can also determine the eigenvectors to be:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

However, we note that V does not have a basis consisting of the eigenvectors of T, and thus T is not diagonalisable. ∎