Math 135: Homework 8

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Problems

6 Cardinal Numbers and the Axiom of Choice

Problem 6.13. Show that a finite union of finite sets is finite.

Solution. We can proceed by induction.

First, we observe that for a set A with cardinality 0, we have then that $A = \emptyset$. Then, $\bigcup A = \emptyset$, so, indeed, we have that $\bigcup A$ is finite as well.

Next, suppose that our claim holds for set A with cardinality n.

Now, we look at A whose cardinality is n^+. Observe then that because A is finite, it follows that there exists a bijection between A and n^+, and thus some bijective function $f : n^+ \to A$.

Then, we have:

$$\begin{aligned} \bigcup A &= \bigcup_{k \in n^+} f(k) \\ &= \bigcup_{k \in n} f(k) \cup f(n) \end{aligned}$$

By our induction hypothesis, we have that $\bigcup_{k \in n} f(k)$ is finite. And since $f(n)$ is also finite, we thus have that $\bigcup_{k \in n} f(k) \cup f(n)$ is finite too.

Therefore, by induction, we conclude that the claim holds as desired. ■

Problem 6.14. Define a permutation of K to be any one-to-one function from K onto K. We can then define the factorial operation on cardinal numbers by the equation

$$\kappa! = \operatorname{card}\{f : f \text{ is a permutation of } K\}$$

where K is any set of cardinality κ. Show that $\kappa!$ is well defined.

Solution. Suppose we have sets K_0 and K_1. Let $\operatorname{card} K_0 = \kappa = \operatorname{card} K_1$.

Then, because the cardinalities of K_0 and K_1 are the same, we can thus construct a bijection between them.

To show that $\kappa!$ is well-defined, we will have to show then that there exists a bijection between the set of permutations of K_0 (which we will denote at K_0') and permutations of K_1 (which we denote as $K_1{}'$).

Then to do this, we recall that there exists a bijection g between K_0 and K_1. So, for each permutation f of K_1, we can first send this permutation to K_0 with our bijection g, which we then permute. After, we can send the permutation of K_0 back to K_1 using g^{-1}.

Thus, we observe then that there exists a bijection between the set of permutations f of K_0 and of K_1; i.e., we have shown that $\kappa!$ is well-defined. ■

Problem 6.15. Show that there is no set A with the property that for every set there is some member of A that dominates it.

Solution. Suppose for the sake of contradiction that such a set A did exist.

Then, let us consider the power set of the union of A. We denote this by S.

Then, we observe that $A \subset \mathscr{P} \bigcup A = S$. However, this means then that $A \prec S$ which is a contradiction, as then no element of A dominates S. ■

Problem 6.16. Show that for any set S we have $S \preccurlyeq {}^S 2$, but $S \not\approx {}^S 2$.

Solution. To begin with, denote $\operatorname{card} S = \lambda$. We note that $\operatorname{card} {}^S 2 = 2^\lambda$. Meanwhile, $\operatorname{card} S = \lambda$. So, we have that $\operatorname{card} S \leq \operatorname{card} {}^S 2$; in other words, $S \preccurlyeq {}^S 2$.

Now, we will show that $S \not\approx {}^S 2$. To do this, let us consider the following functions $F : S \to {}^S 2$ and $g(x) = 1 - F(x)(x)$. We note here that $g : S \to 2$.

Now, we observe that for some function $F(x)$ such that $F(x) = g$ for all $x \in S$, then we have the following:

$$\begin{aligned} F(x) = g &\implies F(x)(x) = 1 - F(x)(x) \\ &\implies 2F(x)(x) = 1 \\ &\implies F(x)(x) = \frac{1}{2} \end{aligned}$$

However, we note that since $F(x)$ should be a function from S to 2, it must be then that $F(x)$ has output of either 0 and 1; therefore, because g isn't in the image of F, $S \not\approx {}^S 2$. ■