Math 110: Homework 12

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Contents

1 Injectivity and Surjectivity

Problem 1.1. Let $T \in \mathcal{L}(V, W)$. Prove that

1. T is injective if and only if T^* is surjective.
2. T^* is injective if and only if T is surjective.

Solution. To prove the first statement, we will work in reverse and suppose that T^* is surjective. From here, we recall that $\operatorname{range} T^* = (\operatorname{null} T)^\perp$. This means the following:

$$\begin{aligned} T^* \text{ is surjective} &\iff \operatorname{range} T^* = V \\ &\iff (\operatorname{null} T)^\perp = V \\ &\iff \operatorname{null} T = \{0\} \\ &\iff T \text{ is injective} \end{aligned}$$

For the second statement, we note that if we replaced T with T^*, and note that $(T^*)^* = T$, then we observe that first, $\operatorname{range} T = (\operatorname{null} T^*)^\perp$, and also that this means then that:

$$\begin{aligned} T \text{ is surjective} &\iff \operatorname{range} T = V \\ &\iff (\operatorname{null} T^*)^\perp = V \\ &\iff \operatorname{null} T^* = \{0\} \\ &\iff T^* \text{ is injective} \end{aligned}$$

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2 Self-Adjointness

Problem 2.1. Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

Solution. To begin with, we proceed with the forwards direction. Suppose that ST is self-adjoint. Then, by definition, we observe that $ST = (ST)^*$. Then, we observe the following:

$$\begin{aligned} ST &= (ST)^* \\ &= T^* S^* \end{aligned}$$

However, we note that T, S are also self-adjoint, meaning that $T^* = T$ and $S^* = S$. So, we have that $T^* S^* = TS$. Thus, we see that $ST = TS$ as desired.

For the backwards direction, suppose that $ST = TS$. Furthermore, we note that in fact, S, T are self-adjoint, so we have:

$$\begin{aligned} ST &= TS \\ &= T^* S^* \\ &= (ST)^* \end{aligned}$$

And since $ST = (ST)^*$, we see that indeed, ST is self-adjoint as desired. ■

3 Projector

Problem 3.1. Let $P \in \mathcal{L}(V)$ be such that $P^2 = P$. Prove that there is a subspace U of V such that $P_U = P$ if and only if P is self-adjoint.

Solution. We shall begin with the forward direction. Suppose that P is an orthogonal projection P_U onto a subspace U of V. From here, we observe the following:

$$\begin{aligned}
\langle P_U v, w\rangle &= \langle P_U v, P_U w + (I - P_U)w\rangle && \text{(Rewriting } Iw \text{ as } P_U + (I-P_U)w\text{)}\\
&= \langle P_U v, P_U w\rangle + \langle P_U v, (I - P_U)w\rangle && \text{(Properties of Inner Product)}\\
&= \langle P_U v, P_U w\rangle && (P_U v \in U\text{, but } w - P_U w \in U^\perp)\\
&= \langle P_U v, P_U w\rangle + \langle (I - P_U)v, P_U w\rangle\\
&= \langle P_U v + (I - P_U)v, P_U w\rangle\\
&= \langle v, P_U w\rangle
\end{aligned}$$

Thus, we see that, indeed, P is self-adjoint as desired.

For the backwards direction, let us suppose that P is self-adjoint. Then, by definition, we see that $P = P^* = P^2$. Furthermore, we have that $\langle Pv, w\rangle = \langle w, Pv\rangle$.

Now, with this in mind, let us denote $U = \operatorname{range} P$. From here, let us take $v, w \in V$ and we observe the following:

$$\begin{aligned}
\langle Pv, (I - P)w\rangle &= \langle (I - P)^* Pv, w\rangle\\
&= \langle (I - P)Pv, w\rangle\\
&= \left\langle Pv - P^2 v, w\right\rangle\\
&= \langle Pv - Pv, w\rangle\\
&= \langle 0, w\rangle\\
&= 0
\end{aligned}$$

Then, we note that since the inner product is equal to zero, this maens then that Pv and $(I-P)w$, for $v, w \in V$, are orthogonal to each other. Since v, w are arbitrary, we observe that this means that P is orthogonal to $I - P$.

From here, we note that for any $v \in V$, we have that $v = u + w = Pv + (I - P)v$. And we note that since we let $U = \operatorname{range} P$, then $Pv \in U$, and $(I - P)v \in U^\perp$. Thus, by definition, we observe that, indeed, P is an orthogonal projection P_U. ■

4 Anti-Hermitian

Problem 4.1. Let $n \in \mathbb{N}$ be fixed. Consider the real space $V := \operatorname{span}\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx\}$, equipped with the inner product space

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)\mathrm{d}x.$$

Show that the differentiation operator $D \in \mathcal{L}(V)$ satisfies $D^* = -D$.

Solution. To begin with, we note that all functions in V are periodic with a period of 2π. In other words, we have that $f(\pi) = f(-\pi)$ for $f \in V$.

Now, with this in mind, we observe the following:

$$\begin{aligned}
\langle Df, g \rangle &= \int_{-\pi}^{\pi} f'(x)g(x)\mathrm{d}x \\
&= f(x)g(x)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)g'(x)\mathrm{d}x \\
&= f(\pi)g(\pi) - f(-\pi)g(-\pi) - \int_{-\pi}^{\pi} f(x)g'(x)\mathrm{d}x \\
&= f(\pi)g(\pi) - f(\pi)g(\pi) - \int_{-\pi}^{\pi} f(x)g'(x)\mathrm{d}x \\
&= -\int_{-\pi}^{\pi} f(x)g'(x)\mathrm{d}x \\
&= \langle f, (-D)g \rangle
\end{aligned}$$

Thus, we observe that, indeed, we have that $D^* = -D$ as desired. ■

5 A Normal Problem

Problem 5.1. Suppose that T is normal. Prove that, for any $\lambda \in \mathbb{F}$ and any $k \in \mathbb{N}$, we have:

$$\operatorname{null}(T - \lambda I)^k = \operatorname{null}(T - \lambda I).$$

Solution. We note that since T is normal, then we have that $S = (T - \lambda I)$ is normal as well. From here, we will first introduce the following lemma, then instead shall prove a more general result.

To begin with, we introduce the following lemma:

Lemma 5.1. Given that T is a normal operator, and for any $k \in \mathbb{N}$ then we have:

$$(TT^*)^k = T^k(T^*)^k$$

Proof. We shall proceed with induction.

To begin with, we observe that for the base case of $k = 1$, we have:

$$(TT^*)^1 = TT^* = T^1(T^*)^1.$$

Thus, our claim holds for $k = 1$. For $k = 2$, we see that

$$\begin{aligned}(TT^*)^2 &= TT^*TT^* \\ &= TTT^*T^* \\ &= T^2(T^*)^2\end{aligned}$$

Now, suppose that for $n = k > 2$, our claim holds. We now will show that it holds for $n = k + 1$ as well. To do this, we first observe the following:

$$\begin{aligned}(TT^*)^{k+1} &= (TT^*)^k(TT^*) \\ &= T^k(T^*)^k(T^*T) \\ &= T^k(T^*)^{k+1}T\end{aligned}$$

Then from here, we can simply switch the T and T^* with each other repeatedly, moving the T inside until we eventually get $T^{k+1}(T^*)^{k+1}$. This is possible due to T being normal.

Thus, we have proven our claim. □

We note here that $(T^*T)^k = (T^*)^kT^k$; we simply replace T with T^* in the lemma above (and use the fact that $(T^*)^* = T$) to see that this is the case.

We will also introduce the following observation:

Remark 5.2. If we have some T which is self-adjoint, and we consider some v such that $T^kv = 0$, then we see the following:

$$\begin{aligned}\langle T^kv, T^{k-2}v\rangle &= \langle T^{k-1}v, T^{k-1}v\rangle \\ &= 0\end{aligned}$$

Then, we observe that $T^{k-1}v$ must be equal to 0, since $\langle v, v\rangle = 0$ if and only if $v = 0$.

Then, from here, we can simply keep on repeating this process recursively until we hit $Tv = 0$.

Now, we will prove a more generalised lemma:

Lemma 5.3. Given that T is a normal operator, then $\operatorname{null} T^k = \operatorname{null} T$.

Proof. To do this, we first show that $\operatorname{null} T \subseteq \operatorname{null} T^k$.

For this inclusion, we note that for any $v \in \operatorname{null} T$, we have that $Tv = 0$. Then, it follows that $T^k = T^{k-1}(Tv) = T^{k-1}(0) = 0$. Thus, we see that the inclusion holds.

Next, for the other inclusion, we want to show that $\operatorname{null} T^k \subseteq \operatorname{null} T$. To do this, let us consider some $v \in \operatorname{null} T^k$.

Then, from here, we have that $T^k v = 0$. Then, this means that $(T^*T)^k v = (T^*)^k T^k v = (T^*)^k 0 = 0$.

Thus, we have that $(T^*T)^k v = 0$.

Now, we note here that since T is normal, we have that $(T^*T)^* = T^*(T^*)^* = T^*T$. In other words, T^*T is self-adjoint.

From our observation in Remark 5.2, we observe then that, in fact, we also have that $T^*Tv = 0$. Then, with this in mind, we do the following:

$$\begin{aligned} \langle T^*Tv, v\rangle &= \langle Tv, Tv\rangle \\ &= 0 \end{aligned}$$

And since $\langle v, v\rangle = 0$ if and only if $v = 0$, we see that, in fact, we have that $Tv = 0$. In other words, given that $v \in \operatorname{null} T^k$, we have that $v \in \operatorname{null} T$ as well.

Thus, we have that $\operatorname{null} T^k \subseteq \operatorname{null} T$.

Then, since we have these two inclusions, we can in fact conclude that they are equal. □

Now, with this lemma proven, we note that since S is normal, then we can apply the lemma above to S and thus we have proven our claim as desired. ■