Math 135: Homework 4

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Problems

3 Relations and Functions

Problem 3.32a. Show that R is symmetric if and only if $R^{-1} \subseteq R$.

Solution. First, we begin with the forward direction.

Proof. To begin with, suppose that R is symmetric.

Then, for some element $\langle y, x\rangle \in R^{-1}$, we note then that $\langle x, y\rangle \in R$. However, because R is symmetric, it follows then that $\langle y, x\rangle \in R$ as well, thus showing that $R^{-1} \subseteq R$. □

For the other direction, we proceed as follows:

Proof. Suppose that $R^{-1} \subseteq R$. Then, suppose that $\langle x, y\rangle \in R$. Then, this means that $\langle y, x\rangle \in R^{-1}$. But because $R^{-1} \subseteq R$, it follows then that $\langle y, x\rangle \in R$ as well. Thus, R is symmetric. □

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Problem 3.32b. Show that R is transitive if and only if $R \circ R \subseteq R$.

Solution. First, we prove the forward direction.

Proof. Suppose that R is transitive. Then, for some $\langle x, z\rangle \in R \circ R$, we see that there exists some y such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$. However, since $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then by transitivity of R, $\langle x, z\rangle \in R$; therefore, $R \circ R \subseteq R$. □

For the other direction, we observe the following:

Proof. Suppose that $R \circ R \subseteq R$. Then, let us suppose that there exists some $\langle x, y\rangle$ and $\langle y, z\rangle$ in R. Then, we see that $\langle x, z\rangle \in R \circ R$. And since $R \circ R \subseteq R$, it follows then that $\langle x, z\rangle \in R$, thus showing that R is transitive as desired. □

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Problem 3.33. Show that R is a symmetric and transitive relation if and only if $R = R^{-1} \circ R$.

Solution. To begin with, we proceed with the forward direction.

Proof. Suppose that R is a symmetric and transitive relation.

Then, let us take some element $\langle x, y\rangle \in R$. Then, since R is symmetric, we have that $\langle y, x\rangle \in R$. And by transitivity, we see that $\langle x, x\rangle \in R$ as well.

Then, since $\langle y, x\rangle \in R$, then this means that $\langle x, y\rangle \in R^{-1}$. From here, note that $\langle x, x\rangle \in R$ and $\langle x, y\rangle \in R^{-1}$, so we have that $\langle x, y\rangle \in R^{-1} \circ R$; $R \subseteq R^{-1} \circ R$.

Then, for some $\langle x, z\rangle \in R^{-1} \circ R$, we note that there exists some y such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R^{-1}$. Since $\langle y, z\rangle \in R^{-1}$, it follows that $\langle z, y\rangle \in R$, and thus by symmetry, $\langle y, z\rangle \in R$ as well. Then, we have $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$; therefore, by transitivity, we have that $\langle x, y\rangle \in R$. Therefore, $R^{-1} \circ R \subseteq R$.

Thus, we conclude $R = R^{-1} \circ R$. □

Now, we prove the backwards direction.

Proof. Suppose that $R = R^{-1} \circ R$.

Then, suppose we had some $\langle x, z\rangle \in R$. Then, by our assumption, $\langle x, z\rangle \in R^{-1} \circ R$ too. This means that there exists a y such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R^{-1}$. Then, $\langle z, y\rangle \in R$ and $\langle y, x\rangle \in R^{-1}$. So, by definition, we have that $\langle z, x\rangle \in R^{-1} \circ R = R$. Thus, we see that R is indeed symmetric.

For transitivity, suppose that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$. Then, we see that $\langle z, y\rangle \in R$ by symmetry, meaning that $\langle y, z\rangle \in R^{-1}$.

Then, since $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R^{-1}$, then by definition, $\langle x, z\rangle \in R^{-1} \circ R = R$. Thus, $\langle x, z\rangle \in R$ as well, meaning that R is transitive as desired. □

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Problem 3.36. Assume that $f : A \to B$ and that R is an equivalence relation on B. Define Q to be

$$\{\langle x, y\rangle \in A \times A : \langle f(x), f(y)\rangle \in R\}.$$

Show that Q is an equivalence relation on A.

Solution. First, reflexivity: suppose that $x \in A$ and $f(x) \in B$. Then, we see that $\langle f(x), f(x)\rangle \in R$, and thus $\langle x, x\rangle \in Q$ as well. Therefore, we see that Q is reflexive as desired.

Next, we will show that Q is symmetric. Suppose that $\langle x, y\rangle \in Q$. Then, we observe that $\langle f(x), f(y)\rangle \in R$, so we know that $\langle f(y), f(x)\rangle \in R$ as well, and thus $\langle y, x\rangle \in Q$. Therefore, Q is symmetric.

For transitivity, suppose that $\langle x, y\rangle \in Q$ and $\langle y, z\rangle \in Q$. Then, we have that $\langle f(x), f(y)\rangle$ and $\langle f(y), f(z)\rangle$ are both in R. But it follows then that $\langle f(x), f(z)\rangle \in R$. Then, we have $\langle x, z\rangle \in Q$ as well, and thus Q is transitive.

Thus, Q meets all requirements needed to be an equivalence relation on A. ■

Problem 3.37. Assume that Π is a partition of a set A. Define the relation R_Π to be as follows:

$$xR_\Pi y \iff (\exists B \in \Pi)(x \in B \wedge y \in B).$$

Show that R_Π is an equivalence relation on A.

Solution. First, we check reflexivity. For some $x \in A$, we observe that there exists some $B \in \Pi$ such that $x \in B \in \Pi$. Then, by definition, this means that there exists a $B \in \Pi$ such that $x \in B$ and $x \in B$; thus, $\langle x, x\rangle \in R_\Pi$, meaning it's reflexive.

For symmetry, we observe that for some $\langle x, y\rangle \in R_\Pi$, it means that there exists some $B \in \Pi$ such that $x \in B$ and $y \in B$. But this is the same as saying $(\exists B \in \Pi)\,(y \in B \wedge x \in B)$ by commutativity; thus, $\langle y, z\rangle \in R_\Pi$ as well, making it symmetric.

Finally, if $\langle x, y\rangle \in R_\Pi$ and $\langle y, z\rangle \in R_\Pi$, then $(\exists B \in \Pi)(x \in B \wedge y \in B) \wedge (\exists C \in \Pi)(y \in C \wedge z \in C)$. But then, this means that we have $y \in B \cap C$. Since each element in Π are disjoint by definition, it must be then that $B = C$. Thus, we have that $x \in B$ and $z \in B$, meaning that $\langle x, z\rangle \in R_\Pi$, making it transitive.

With all conditions met, we conclude that R_Π is an equivalence relation as claimed. ■

Problem 3.41a. Let $\mathbb{R}$ be the set of real numbers, and Q to be a relation on $\mathbb{R} \times \mathbb{R}$ by $\langle u, v\rangle Q\langle x, y\rangle \iff u + y = x + v$.

Show that Q is an equivalence relation on $\mathbb{R} \times \mathbb{R}$.

Solution. First, we show that Q is reflexive. Suppose that we have some $u \in Q$. Then, we see that $u + u = 2u = u + u$. Thus, $\langle u, u\rangle Q\langle u, u\rangle$, making it reflexive as desired.

Next, for symmetry, we observe that for $\langle u, v\rangle Q\langle x, y\rangle$, it follows that $u + y = x + v$. Then, this means that $x + v = u + y$ as well, and thus $\langle x, y\rangle Q\langle u, v\rangle$.

For transitivity, suppose $\langle u, v\rangle Q\langle w, x\rangle$ and $\langle w, x\rangle Q\langle y, z\rangle$. Then, we have the following:

$$\begin{aligned} u + x &= v + w \\ w + z &= x + y \\ u - y &= v - z \\ u + z &= y + v \end{aligned}$$

So by definition, we have that $\langle u, v\rangle Q\langle z, y\rangle$, meaning that Q is transitive.

Thus, we conclude that Q is indeed an equivalence relation on $\mathbb{R} \times \mathbb{R}$. ■

Problem 3.41b. Is there a function $G : (\mathbb{R} \times \mathbb{R})/Q \to (\mathbb{R} \times \mathbb{R})/Q$ satisfying the equation:

$$G\left([\langle x, y\rangle]_Q\right) = [\langle x + 2y, y + 2x\rangle]_Q?$$

Solution. First, let us define a function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that $\langle x, y\rangle \mapsto \langle x + 2y, y + 2x\rangle$.

Now, we observe that for any $\langle x, y\rangle \in Q$, we have $\langle a, b\rangle Q\langle c, d\rangle$ such that $a + d = b + c$. Note then that $2a + 2d = 2b + 2c$.

Then, we observe that $x+2y = \langle a, b\rangle+2\langle c, d\rangle = \langle a + 2c, b + 2d\rangle$ and $y+2x = \langle c, d\rangle+2\langle a, b\rangle = \langle c + 2a, d + 2b\rangle$.

With this in mind, we see that $a + 2c + d + 2b = c + 2a + b + 2d$; $\langle x + 2y, y + 2x\rangle \in Q$. In other words, $F(\langle a, b\rangle) QF(\langle c, d\rangle)$ as well, meaning that F is compatible with Q. And because F is compatible with Q, it follows that there exists a G that satisfies the equation as desired. ■

Problem 3.43. Assume that R is a linear ordering on a set A. Show that R^{-1} is also a linear ordering on A.

Solution. Suppose that R is a linear ordering on a set A.

Let $\langle x, y\rangle \in R^{-1}$ and $\langle y, z\rangle \in R^{-1}$. Then, we have that $\langle y, x\rangle \in R$ and $\langle z, y\rangle \in R$. Then, we have that $\langle z, x\rangle \in R$ by transitivity, and thus $\langle x, z\rangle \in R^{-1}$. Thus, we see that R^{-1} is transitive.

For trichotomy, we observe that for any $x, y \in A$, we have exactly one of $\langle x, y\rangle \in R$, $x = y$, or $\langle y, x\rangle \in R$ occurring. But then this means that we have only one of $\langle y, x\rangle \in R^{-1}$, $y = x$, or $\langle x, y\rangle \in R^{-1}$ happening. Thus, R^{-1} satisfies trichotomy as well.

Therefore, with all condition satisfied, we see that R^{-1} is a linear ordering. ■

Problem 3.44. Assume that $<$ is a linear ordering on a set A. Assume that $f : A \to A$ and that f has the property that whenever $x < y$ then $f(x) < f(y)$.

Show that f is one-to-one and that whenever $f(x) < f(y)$, $x < y$.

Solution. To begin with, let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$. Then, we have three options for x_1 and x_2:

1. $x_1 < x_2$; in this case, we see that $f(x_1) < f(x_2)$, which would contradict the fact that $f(x_1) = f(x_2)$.
2. $x_2 < x_1$. But if this was the case, then we'd have $f(x_2) < f(x_1)$, again leading to contradiction.
3. $x_1 = x_2$. This must be true by trichotomy of $<$.

Thus, we see that if $f(x_1) = f(x_2)$, then $x_1 = x_2$; i.e. f is one-to-one.

Next, suppose that $f(x) < f(y)$. However, for the sake of contradiction, let us say that $x \not< y$. Then, we have two options:

1. $y < x$. But then this would mean that $f(y) < f(x)$, resulting in a contradiction.
2. $y = x$. But in this case, we have that $f(x) = f(y)$; again, this is a contradiction.

Therefore, by trichotomy, we conclude that $x < y$. ■

Problem 3.45. Assume that $<_A$ and $<_B$ are linear orderings on A and B respectively. Define the binary relation $<_L$ on $A \times B$ by:

$$\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle \iff a_1 <_A a_2 \vee (a_1 = a_2 \wedge b_1 <_B b_2)\,.$$

Show that $<_L$ is a linear ordering on $A \times B$.

Solution. First, we show that this is indeed transitive.

Suppose we have $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$ and $\langle a_2, b_2 \rangle <_L \langle a_3, b_3 \rangle$. Then, we have:

$$(a_1 <_A a_2 \vee (a_1 = a_2 \wedge b_1 <_B b_2)) \wedge (a_2 <_A a_3 \vee (a_2 = a_3 \wedge b_2 <_B b_3))$$

This can be rewritten as such:

$$\begin{aligned}&(a_1 <_A a_2 \wedge a_2 <_A a_3) \vee (a_1 <_A a_2 \wedge (a_2 = a_3 \wedge b_2 <_B b_3))\\ &\vee ((a_1 = a_2 \wedge b_1 <_B b_2) \wedge a_2 <_A a_3) \vee ((a_1 = a_2 \wedge b_1 <_B b_2) \wedge (a_2 = a_3 \wedge b_2 <_B b_3))\end{aligned}$$

But we note then that this is equivalent to:

$$(a_1 <_A a_3) \vee (a_1 <_A a_3) \vee (a_1 <_A a_3) \vee (a_1 = a_3 \wedge b_1 <_B b_3)$$

Or, in other words,

$$a_1 <_A a_3 \vee (a_1 = a_3 \wedge b_1 <_B b_3).$$

Then by definition, we see that indeed, $\langle a_1, b_1 \rangle <_L \langle a_3, b_3 \rangle$, thus making it transitive as desired.

To show trichotomy, we observe that for any ordered pairs $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle$, since A is a linear ordering, we must have one of the following:

1. $a_1 <_A a_2$; in this case, we see that $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$.
2. $a_2 <_A a_1$; then, we have $\langle a_2, b_2 \rangle <_L \langle a_1, b_1 \rangle$.
3. $a_1 = a_2$. If so, then we have three more cases to consider.

In the case that $a_1 = a_2$, then since B is a linear ordering, we have:

1. $b_1 <_B b_2$; in this case, we see that $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$.
2. $b_2 <_B b_1$; then, we have $\langle a_2, b_2 \rangle <_L \langle a_1, b_1 \rangle$.
3. $b_1 = b_2$. Since this is the case, we have that $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$, and we see that $\langle a_1, b_1 \rangle \not<_L \langle a_2, b_2 \rangle$.

Thus, we see that $<_L$ satisfies trichotomy, and therefore is a linear ordering. ■

Problem 3.30a. This is a homework question.

Solution. This is a solution.

Proof. This is some proof of another statement used in the overall solution. □

And this is the end of the solution. ■