Homework 7

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Contents

1 Intersections and Annihilators

Problem 1.1. Suppose V is finite-dimensional, and U, W are its subspaces. Prove that

$$(U \cap W)^0 = U^0 + W^0.$$

Solution. To begin with, we will show that

$$U^0 + W^0 \subseteq (U \cap W)^0.$$

First, we observe that, by definition, we have:

$$U^0 = \{\varphi \in V' : \varphi(u) = 0, \forall u \in U\}$$
$$W^0 = \{\psi \in V' : \psi(w) = 0, \forall w \in W\}$$

Meanwhile, we note that for any $\gamma \in (U \cap W)^0$, γ annihilate all of $u \in U \cap W$, but not necessarily all $u \in U$. Then, it follows that $U^0 \subseteq (U \cap W)^0$.

Similarly, we have that $W^0 \subseteq (U \cap W)^0$.

From here, suppose we have some $\varphi \in U^0$ and $\psi \in W^0$. Then, for some $a, b \in \mathbb{F}$, we observe the following:

$$\begin{aligned}(a\varphi + b\psi)(v) &= a\varphi(v) + b\psi(v) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

Then, we note that any $\gamma \in (U \cap W)^0$ can be expressed as a linear combination of $\varphi \in U^0$ and $\psi \in W^0$. In other words, we observe that for any $a\varphi + b\psi \in U^0 + W^0$, we also have that it's in $(U \cap W)^0$. Thus, $U^0 + W^0 \subseteq (U \cap W)^0$.

Now, to show that equality holds, we can simply show that $\dim(U \cap W)^0 = \dim U^0 + \dim W^0$. To do this, we introduce the following lemma:

Lemma 1.1.

$$U^0 \cap W^0 = (U + W)^0$$

Proof. We will first show that $U^0 \cap W^0 \subseteq (U + W)^0$.

To do this, suppose we have some $\gamma \in U^0 \cap W^0$. Then, by definition, we observe that $\gamma(u) = 0$ for all $u \in U$, and also that $\gamma(w) = 0$ for all $w \in W$.

Then, suppose we had some $v \in U + W$. By definition of $U + W$, we observe then that we can rewrite $v = u + w$, for some $u, w \in U, W$ respectively. From here, we observe the following:

$$\begin{aligned}\gamma(v) &= \gamma(u + w) \\ &= \gamma(u) + \gamma(w) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

So, we see that $\gamma \in (U + W)^0$ as well, and thus $U^0 \cap W^0 \subseteq (U + W)^0$.

Now, to show that $U^0 \cap W^0 \supseteq (U + W)^0$, we consider some $\gamma \in (U + W)^0$. We observe that, by definition, we have that for any $v \in U + W$,

$$\gamma(v) = 0$$

Now, as U, W are subspaces of V, they must contain the zero vector $\vec{0}$. Then, it follows that $u+\vec{0} = u \in U+W$, and $w + \vec{0} = w \in U + W$. Then, we observe that for any $u, w \in U, W$ we have:

$$\gamma(u) = 0$$
$$\gamma(w) = 0$$

In other words, we see that $\gamma \in U^0 \cap W^0$.

Therefore, we can conclude that we have equality as desired. □

Now, with this in mind, we note as well that:

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$
$$\dim(U \cap W) = \dim U + \dim W - \dim(U + W)$$

observe the following:

$$\begin{aligned}
\dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\
&= \dim U^0 + \dim W^0 - \dim(U + W)^0 \\
&= (\dim V - \dim U) + (\dim V - \dim W) - (\dim V - \dim(U + W)) \\
&= \dim V - \dim U - \dim W + \dim(U + W) \\
&= \dim V - (\dim U + \dim W - \dim(U + W)) \\
&= \dim V - \dim(U \cap W) \\
&= \dim(U \cap W)^0
\end{aligned}$$

Thus, since $\dim(U \cap W)^0 = \dim(U^0 + W^0)$, then, indeed, we see that equality holds as desired. ■

2 Null and Range

Problem 2.1. Suppose V, W are finite dimensional, and $T \in \mathcal{L}(V, W)$, and $\operatorname{null} T' = \operatorname{span}(\varphi)$ for some $\varphi \in W'$. Prove that $\operatorname{range} T = \operatorname{null} \varphi$.

Solution. To begin with, we will show that $\operatorname{range} T \subseteq \operatorname{null} \varphi$.

We first note that $\operatorname{null} T' = \operatorname{span}(\varphi)$. Now, we see then that this means that:

$$\begin{aligned} T'(\varphi) &= 0 \\ \varphi \circ T &= 0 \\ (\varphi \circ T)(v) &= 0 \\ \varphi(T(v)) &= 0 && \text{(for all } v \in V\text{)} \end{aligned}$$

Then, we see that, in fact, we have that for all $v \in V$, we have that $Tv \in \operatorname{null} \varphi$. Or, in other words, we have that $\operatorname{range} T \subseteq \operatorname{null} \varphi$.

Next, we will show that they are in fact equal.

To do this, we will show that their dimensions are equal.

We have two cases to consider for this. First, consider $\varphi \neq 0$. Then, we note that since $\varphi \in W'$, then it means that $\varphi \in \mathcal{L}(W, \mathbb{F})$. From here, since $\varphi \neq 0$, it follows that $\dim \operatorname{range}(\varphi) = \dim \mathbb{F} = 1$.

Now, we observe:

$$\begin{aligned} \dim W &= \dim \operatorname{range} \varphi + \dim \operatorname{null} \varphi \\ \dim \operatorname{null} \varphi &= \dim W - \dim \operatorname{range} \varphi \\ &= \dim W - 1 \end{aligned}$$

Next, we look at $\dim \operatorname{range} T$. To do this, we recall that since V, W are finite dimensional, it follows then that $\dim \operatorname{range} T = \dim \operatorname{range} T'$. Furthermore, we observe that since $\operatorname{null} T' = \operatorname{span}(\varphi)$, and $\varphi \neq 0$, then $\dim \operatorname{null} T' = 1$, as it's the span of a single vector. Furthermore, note that $T' \in \mathcal{L}(W', V')$, and that since W is finite-dimensional, we have $\dim W' = \dim W$.

Then, with this in mind, we observe the following:

$$\begin{aligned} \dim W' &= \dim \operatorname{range} T' + \dim \operatorname{null} T' \\ \dim \operatorname{range} T' &= \dim W' - \dim \operatorname{null} T' \\ \dim \operatorname{range} T &= \dim W - \dim \operatorname{null} T' \\ &= \dim W - 1 \end{aligned}$$

So, we see that, in fact, we have $\dim \varphi = \dim \operatorname{range} T$ for $\varphi \neq 0$.

Now, in the case where $\varphi = 0$, we note that $\dim \operatorname{range} \varphi = 0$, so we have that $\dim \operatorname{null} \varphi = \dim W$. Meanwhile, since $\varphi = 0$, then $\dim \operatorname{span}(\varphi) = 0$, meaning that $\dim \operatorname{null} T' = 0$, so $\dim \operatorname{range} T' = \dim \operatorname{range} T = \dim W$.

Thus, their dimensions are equal.

Then, since we see that $\dim \operatorname{range} T = \dim \operatorname{null} \varphi$, and also that $\operatorname{range} T \subseteq \operatorname{null} \varphi$, we can conclude that equality holds. ■

Problem 2.2. Give an example of such a pair $T, \varphi \neq 0$ for $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$.

Solution. Suppose we have $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$.

Since T is a linear map from V to W, we know that it has some matrix representation $\mathcal{M}(T)$. Now, we define T to have the matrix representation as follows:

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Meanwhile, we let φ to be as follow:

$$\mathcal{M}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Then, we note that $\operatorname{null} \varphi = \operatorname{span}\{(0,1,0),(0,0,1)\}$. Meanwhile, $\operatorname{range} T = \operatorname{span}\{(0,1,0),(0,0,1)\}$. Thus, we see that, in fact, $\operatorname{null} \varphi = \operatorname{range} T$ as desired. ■

3 Primer on Lagrange Interpolation

Problem 3.1. Let $p \in \mathscr{P}_n(\mathbb{C})$ for some n and suppose there exists distinct real numbers $x_0, \ldots, x_n$ such that $p(x_j) \in \mathbb{R}$ for all $j = 0, \ldots, n$. Prove that all the coefficients of p are real.

Solution. From Question 4, we know that we can construct a unique Langrage Interpolating Polynomial p such that $p(x_j) \in \mathbb{R}$ for each $x_0, \ldots, x_n$.

Let us denote each $p(x_j) = y_j$, and we note that $y_j \in \mathbb{R}$.

Now, we can in fact construct a polynomial $p \in \mathscr{P}_n(\mathbb{C})$ as follows:

$$p(x) \coloneqq \sum_{j=0}^{n} y_j p_j(x),$$

where we define p_j as:

$$p_j(x) \coloneqq \prod_{k=0, k\neq j}^{n} \frac{(x - x_k)}{(x_j - x_k)}$$

Notice here that at x_j, $p_j(x_j) = 1$ and $p_k(x_j) = 0$ for all $k \neq j$. Then, $p(x_j) = y_j$ as desired.

Now, we note here that the denominator in p_j consists of real numerical values. Furthermore, the numerator consists of $n+1$ distinct linear terms which, which, when expanded, will result in a degree n polynomial. We note that this polynomial will also only have real coefficients by virtue of x_k being a real number.

Then, we observe that p_j will be a polynomial with only real coefficients. And thus $p(x)$ must contain only real coefficients as well since $y_j \in \mathbb{R}$, so $y_j p_j$ will have real coefficients only, and thus so will the sum of all $y_j p_j$.

Then, we observe that p has coefficients all real. Furthermore, by Question 4, we know that this p is in fact unique. ∎

4 Lagrange Interpolation

Problem 4.1 (Lagrange Interpolation). Prove *using linear algebra* that, given distinct data sites x_j and arbitrary data y_j, for $j = 0, \dots, n$, there exists a unique polynomial $p \in \mathscr{P}_n(\mathbb{R})$ such that $p(x_j) = y_j$.

Solution. First, we will begin by proving such a polynomial actually exists.

To do this, we can explicitly construct such a polynomial $p(x)$ as follows:

$$p(x) := \sum_{j=0}^{n} y_j p_j(x),$$

where we define p_j as:

$$p_j(x) := \prod_{k=0, k\neq j}^{n} \frac{(x - x_k)}{(x_j - x_k)}$$

Then, with this construction, we observe that at each x_j, we have that p_k, where $k \neq j$, will be equal to zero since the numerator will contain a $(x - x_j)$ term, so $p_k(x_j)$ will evaluate to zero. Meanwhile, we note that $p_j(x_j) = y_j$, as desired.

We note as well that for each of our $p_j(x)$, the denominator of our fraction contains numerical values as well. On the other hand, we note that the numerator of our fraction contains $n+1$ distinct linear terms, and thus they form a degree n polynomial. Furthermore, y_j is some constant. Then, $p(x)$, the sum of each of our $y_j p_j(x)$, must also be a degree n polynomial.

Thus, we have shown that there indeed exists $p \in \mathscr{P}_n(\mathbb{R})$ that satisfies our conditions.

Now, in order to show uniqueness, we will first show that our p_j's are linearly independent, and thus form a basis for $\mathscr{P}_n(\mathbb{R})$.

By definition, we observe that each $p_0, \dots, p_n$ is linearly independent if $a_0 p_0 + \dots + a_n p_n = 0$ only when $a_0 = \cdots = a_n = 0$ for all $x \in \mathbb{R}$.

With this in mind, we can construct the following system of equations:

$$\begin{aligned}
(a_0 p_0 + \dots + a_n p_n)(x_0) &= 0 \\
&\vdots \\
(a_0 p_0 + \dots + a_n p_n)(x_n) &= 0 \\
\\
a_0 p_0(x_0) + \dots + a_n p_n(x_0) &= 0 \\
&\vdots \\
a_0 p_0(x_n) + \dots + a_n p_n(x_n) &= 0 \\
\\
a_0 &= 0 \\
&\vdots \\
a_n &= 0
\end{aligned}$$

Thus, we see that $p_0, \dots, p_n$ are linearly independent. Furthermore, since there are $n+1$ polynomials, we see that they in fact form a basis for $\mathscr{P}_n(\mathbb{R})$.

This means then that any $p \in \mathscr{P}_n(\mathbb{R})$ can be written as a unique linear combination of our p_j's. We note now that as $y_0, \dots, y_j$ are all scalars, then $\sum_{j=0}^{n} y_j p_j$ must thus be a unique representation of p as desired. ■

5 Roots

Problem 5.1. Prove that every polynomial of odd degree with real coefficients has a real zero.

Solution. Let us suppose for the sake of contradiction that our polynomial has no real zeros.

Now, suppose that our polynomial has degree of $2n + 1$. Since our polynomial p must be of odd degree, $\deg p \geq 1$ (i.e. it can't be a constant polynomial). Then, we note that, by the Fundamental Theorem of Algebra, we know that our polynomial must have $2n + 1$ complex roots z (whose multiplicity can be greater than zero).

Then, with this in mind, we note that as p has real coefficients, then it follows that for each z_i that is a root of p, its conjugate $\overline{z_i}$ must be as well. Since p can't have a real zero, it must follow that $z_i \neq \overline{z_i}$.

From here, p can be iteratively divided by the real polynomial $(x - z)(x - \overline{z})$. Doing this process will then leave us with a single term $(x - z)$ remaining. However, we note that since $(x - z)$ is a root of p, then its complex conjugate $(x - \overline{z})$ must as well. But for this to be the case, we have that $z = \overline{z}$; p has a real zero. Thus, we have a contradiction.

Therefore, we can conclude that every polynomial of odd degree with real coefficients must have at least one real zero. ■