Math 135: Homework 12

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Problems

7 Orderings and Ordinals

Problem 7.26. Show that every ordinal α is grounded, and that $\operatorname{rank}\alpha = \alpha$.

Solution. To begin with, we note by the Axiom of Regularity that every set is grounded. Then, it follows that since α is a set, it must thus be grounded as well.

Now, we know then that for every ordinal α, we have $\alpha \subseteq V_{\operatorname{rank}(\alpha)}$ and $\alpha \in V_{(\operatorname{rank}\alpha)^+}$. We now want to show that $\operatorname{rank}\alpha = \alpha$. To do this, we will proceed by transfinite induction.

Suppose that for ordinal α, we have that $\operatorname{rank}\beta = \beta$ for all $\beta < \alpha$.

First, we observe that for $\alpha = 0$, we have that $\alpha = 0 = \emptyset \subseteq \emptyset = V_0$. Also, we note that $V_{0^+} = \mathscr{P} V_0 = \{\emptyset\}$. So, we have $\alpha = 0 \in \{\emptyset\} = \mathscr{P} V_0 = V_{0^+}$. And since there exists no β such that $\beta \in 0 = \emptyset$, we see that, indeed, $\alpha = 0$ is the least ordinal such that $\alpha \subseteq V_\alpha$. In other words, $\alpha = \operatorname{rank}\alpha$ for $\alpha = 0$.

Now, we look at the case of α is the successor ordinal. That is, $\alpha = \beta^+$ for some $\beta < \alpha$. From here, we see that $\alpha = \beta^+ = \beta \cup \{\beta\}$. We note that if $\gamma \in \alpha$, then either $\gamma \in \beta$ or $\gamma = \beta$.

Now, we observe that

$$\begin{aligned}\operatorname{rank}\alpha &= \bigcup \{(\operatorname{rank}\gamma)^+ : \gamma \in \alpha\} \\ &= \bigcup \{\gamma^+ : \gamma \in \alpha\}\end{aligned}$$

But we see then that $\operatorname{rank}\alpha$ is simply $\beta^+ = \alpha$. In other words, $\operatorname{rank}\alpha = \alpha$.

Now, if α is a limit ordinal, we note that $\alpha = \bigcup_{\beta<\alpha} \beta$. Then, from here, we have that

$$\begin{aligned}\operatorname{rank}\alpha &= \operatorname{rank} \bigcup_{\beta<\alpha} \beta \\ &= \bigcup \left\{(\operatorname{rank}\beta)^+ : \beta < \alpha\right\} \\ &= \bigcup \{\beta^+ : \beta < \alpha\} \\ &= \alpha\end{aligned}$$

Thus, we see that $\operatorname{rank}\alpha = \alpha$. So, by transfinite induction, we have proven our claim as desired. ■

Problem 7.33. Assume that D is a transitive set. Let B be a set with the property that for any a in D,

$$a \subseteq B \implies a \in B.$$

Show that $D \subseteq B$.

Solution. Suppose that $D \subseteq B$ is false. Then, that means that there exists some element $a \in D$ such that $a \notin B$.

Now, because $a \in D$, then by transitivity, we note that for all $a' \in a$, we have that $a' \in D$. We note here that there exists some $a' \in a \in D \implies a' \in D$ such that $a' \notin B$, or else we'd have that $a \subseteq B \implies a \in B$.

So, let us define D' to be the set of $a \in D$ such that $a \notin B$. By definition, we observe then that $D' \setminus B \neq \emptyset$.

So, we note that there exists some $a \in D' \setminus B$ such that $a \cap D' \setminus B = \emptyset$. Then, we have the following:

$$a \cap (D' \setminus B) = (a \cap D') \setminus (a \cap B).$$

From here, we note that for $a \in D'$, there exists some $a' \in a$ such that $a' \in D'$ as well by our observation from earlier. So, we note that $a \cap D' \neq \emptyset$, and contains all $a' \in a$ such that $a' \notin B$.

However, we note that for $(a \cap D') \setminus (a \cap B) = \emptyset$, we require for $(a \cap D') \subseteq (a \cap B)$.

But this means then that for all $a' \in (a \cap D')$, we have that $a' \in (a \cap B) \implies (a' \in a \wedge a' \in B)$. But this is a contradiction.

Therefore, we conclude that $D \subseteq B$ as desired. ■

Problem 7.34. Assume that
$$\{x, \{x, y\}\} = \{u, \{u, v\}\}.$$
Show that $x = u$ and $y = v$.

Solution. We note that in order for $\{x, \{x, y\}\} = \{u, \{u, v\}\}$, we observe that for all $z \in \{x, \{x, y\}\}$, we must have $z \in \{u, \{u, v\}\}$ as well.

Now, we note that since $x \in \{x, \{x, y\}\}$, we must have that $x \in \{u, \{u, v\}\}$ too.

Then, either $x = u$ or $x = \{u, v\}$. Suppose first that $x = \{u, v\}$. Now, we note that since $\{x, y\} \in \{x, \{x, y\}\}$, then we have that $\{x, y\} \in \{u, \{u, v\}\}$.

Again, we have two cases:

1. $\{x, y\} = u$, or
2. $\{x, y\} = \{u, v\}$.

In the first case, if $\{x, y\} = u$, we note then that $x \in \{x, y\} = u \in \{u, v\} = x$. However, this contradicts Regularity, and thus we conclude that we can't have $\{x, y\} = u$.

Now, if $\{x, y\} = \{u, v\}$, we observe then that $x = \{u, v\} \in \{x, y\} = \{u, v\}$. And since we have $\{u, v\} \in \{u, v\}$, we see that this again contradicts Regularity.

So, we conclude that we can't have $x = \{u, v\}$.

Then, we have $x = u$. Again, in this case, we have two possibilities for $\{x, y\}$:

1. $\{x, y\} = u$, or
2. $\{x, y\} = \{u, v\}$.

In the case where $\{x, y\} = u$, we have then that $x \in \{x, y\} = u = x \implies x \in x$. This again contradicts Regularity, and thus we must have $\{x, y\} = \{u, v\}$.

Now, we see then that $x = u$ and $\{x, y\} = \{u, v\}$.

Then, we note that $\{y\} = \{x, y\} \setminus \{x\} = \{u, v\} \setminus \{u\} = \{v\}$. So, we have $\{y\} = \{v\} \implies y = v$. ■

Problem 7.39. Prove that a set is an ordinal number iff it is a transitive set of transitive sets.

Solution. First, we will prove the forward direction.

Proof. Let us suppose that α is an ordinal number. Then, we note that for all $\beta \in \alpha$, β is an ordinal.

From here, we note that since ordinals are transitive, it follows then that α is a transitive set of transitive sets. □

For the other direction, we proceed as follows:

Proof. Denote x to be a transitive set of transitive sets.

Now, we let α be the least ordinal such that $\alpha \notin x$; we know this exists due to the fact that there exists no set of all ordinals.

Now, we observe that in the case where $x \subseteq \alpha$, then x is a transitive set of ordinals; in other words, x is itself an ordinal.

If $x \not\subseteq \alpha$, we then note that $x \backslash \alpha \neq \emptyset$. Then, by Regularity, there exists some $y \in (x \backslash \alpha)$ such that $y \cap (x \backslash \alpha) = \emptyset$.

Now, we note that $y \in x$, and so $y \subseteq x$ by transitivity of x. Furthermore, note that y is transitive as well by our assumption.

We also note then that $y \subseteq \alpha$. However, since y is a transitive set of ordinals, we note that y is thus an ordinal.

From here, we see that $y \in x \setminus \alpha$, so it follows that $y \notin \alpha$. But this means then that either $\alpha = y$ or $\alpha \in y$.

In either cases, we note then that we get $\alpha \in x$, and thus a contradiction that $\alpha \notin x$.

Therefore, we conclude that, indeed, x is an ordinal. □

Thus, we have proven both directions as desired. ■