Math 110: Homework 10

Michael Pham

Fall 2023

Contents

1 Inner Product Validity 3

2 Orthogonal Subspace and Orthonormal Basis 4

3 Gram-Schmidt Pain 5

4 An Orthonormal List of Vectors 6

5 Deja Vu? Almost. 8

1 Inner Product Validity

Problem 1.1. Determine whether or not the function taking the pair $((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbb{R}^3 \times \mathbb{R}^3$ to $x_1y_1 + x_2y_2 - 3x_2y_3 + 3x_3y_2 + x_3y_3$ is an inner product.

Solution. We will provide a counterexample to show that such a function isn't an inner product.

To do this, consider the pair $((0, 1, 0), (0, 0, 1)) \in \mathbb{R}^3 \times \mathbb{R}^3$. Then, we observe that the inner product will be:

$$\begin{aligned}\langle (0,1,0),(0,0,1)\rangle &= (0)(0) + (1)(0) - 3(1)(1) + 3(0)(0) + (0)(1) \\ &= -3\end{aligned}$$

On the other hand, observe that the pair $((0, 0, 1), (0, 1, 0))$ will get us:

$$\begin{aligned}\langle (0,0,1),(0,1,0)\rangle &= (0)(0) + 0(1) - 3(1)(0) + 3(1)(1) + (0)(0) \\ &= 3\end{aligned}$$

But, note that $\overline{\langle (0,0,1),(0,1,0)\rangle} = \overline{3} = 3$. So, we have that $\langle (0,1,0),(0,0,1)\rangle \neq \overline{\langle (0,0,1),(0,1,0)\rangle}$. Thus, this function fails to satisfy the conjugate symmetry requirement of an inner product. ■

2 Orthogonal Subspace and Orthonormal Basis

Problem 2.1. Consider a complex vector space $V = \operatorname{span}(1, \cos x, \sin x, \cos 2x, \sin 2x)$ with an inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(t)\overline{g(t)}\mathrm{d}t.$$

Let U be the subspace of odd functions in V. What is $U^{\perp}$? Find an orthonormal basis for U and $U^{\perp}$.

Solution. To begin with, we note that each $1, \cos x, \sin x, \cos 2x, \sin 2x$ are linearly independent. To do this, we note that $\cos 2x = \cos^2 x - \sin^2 x$, and $\sin 2x = 2\sin x\cos x$ by the addition formula for $\cos$ and $\sin$. From here, we note the following:

$$\begin{aligned} a_0(1) + a_1\cos x + a_2\sin x + a_3\cos 2x + a_4\sin 2x &= 0 \\ a_0 + a_1\cos x + a_2\sin x + a_3(\cos^2 x - \sin^2 x) + a_4(2\sin x\cos x) &= 0 \end{aligned}$$

and we see that this only holds when $a_0 = \cdots = a_4 = 0$. In other words, they are linearly independent.

Then, from here, we note that since U consists of the odd functions in V, then a basis for it must be $\sin x, \sin 2x$. From here, we can orthonormalise it using the Gram-Schmidt process we get the following vectors:

$$e_1 = \frac{\sin x}{\|\sin x\|} = \frac{\sin x}{\sqrt{\langle \sin x, \sin x\rangle}} = \frac{\sin x}{\sqrt{\int_{-\pi}^{\pi}\sin x\overline{\sin x}\mathrm{d}x}} = \frac{\sin x}{\sqrt{\pi}}$$

$$e_2 = \frac{\sin 2x - \left\langle \sin 2x, \frac{\sin x}{\sqrt{\pi}}\right\rangle\frac{\sin x}{\sqrt{\pi}}}{\left\|\sin 2x - \left\langle \sin 2x, \frac{\sin x}{\pi}\right\rangle\frac{\sin x}{\sqrt{\pi}}\right\|} = \frac{\sin 2x - \int_{-\pi}^{\pi}\frac{2\sin^2 x\cos x}{\sqrt{\pi}}\mathrm{d}x\frac{\sin x}{\sqrt{\pi}}}{\left\|\sin 2x - \int_{-\pi}^{\pi}\frac{2\sin^2 x\cos x}{\sqrt{\pi}}\mathrm{d}x\frac{\sin x}{\sqrt{\pi}}\right\|} = \frac{\sin 2x}{\|\sin 2x\|} = \frac{\sin 2x}{\sqrt{\int_{-\pi}^{\pi}\sin 2x\overline{\sin 2x}\mathrm{d}t}} = \frac{\sin 2x}{\sqrt{\pi}}$$

From here, we note that $V = U \oplus U^{\perp}$. Furthermore, since $\dim V = \dim U \oplus \dim U^{\perp}$, we see then that since $\dim U = 2$, and $1, \cos x, \cos 2x \notin U$ and also that these three vectors are also linearly independent, it follows then that a basis for $U^{\perp}$ would be $1, \cos x, \cos 2x$.

From here, once more, by Gram-Schmidt, we observe that an orthonormal basis for $U^{\perp}$ would be:

$$\begin{aligned} e_3 &= \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_{-\pi}^{\pi}1\mathrm{d}x}} = \frac{1}{\sqrt{2\pi}} \\ e_2 &= \frac{\cos x - \langle \cos x, e_3\rangle e_3}{\|\cos x - \langle \cos x, e_3\rangle e_3\|} = \frac{\cos x}{\sqrt{\pi}} \\ e_3 &= \frac{\cos 2x - \langle \cos 2x, e_3\rangle e_3 - \langle \cos 2x, e_4\rangle e_4}{\|\cos 2x - \langle \cos 2x, e_3\rangle e_3 - \langle \cos 2x, e_4\rangle e_4\|} = \frac{\cos 2x}{\sqrt{\pi}}. \end{aligned}$$

So, an orthonormal basis for U would be $\frac{\sin x}{\sqrt{\pi}}$ and $\frac{\sin 2x}{\sqrt{\pi}}$.

Meanwhile, an orthonormal basis for $U^{\perp}$ would be $\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}$. ■

3 Gram-Schmidt Pain

Problem 3.1. Consider the space $\mathscr{P}_3(\mathbb{R})$ with an inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x.$$

Use the Gram-Schmidt algorithm to orthonormalize the basis $1, x, x^2, x^3$.

Solution. Using Gram-Schmidt, we get the following:

$$
\begin{aligned}
e_1 &= \frac{1}{\|1\|} = \frac{1}{\sqrt{\langle 1, 1\rangle}} = \frac{1}{\sqrt{\int_{-1}^{1} 1(1)\mathrm{d}x}} = \frac{1}{\sqrt{2}} \\
e_2 &= \frac{x - \left\langle x, \frac{1}{\sqrt{2}}\right\rangle \frac{1}{\sqrt{2}}}{\left\|x - \left\langle x, \frac{1}{\sqrt{2}}\right\rangle \frac{1}{\sqrt{2}}\right\|} = \frac{x}{\|x\|} = \frac{x}{\sqrt{\langle x, x\rangle}} = \frac{x}{\sqrt{\int_{-1}^{1} x^2\mathrm{d}x}} = \sqrt{\frac{3}{2}}x \\
e_3 &= \frac{x^2 - \left\langle x^2, \frac{1}{\sqrt{2}}\right\rangle \frac{1}{\sqrt{2}} - \left\langle x^2, \sqrt{\frac{3}{2}}x\right\rangle \sqrt{\frac{3}{2}}x}{\left\|x^2 - \left\langle x^2, \frac{1}{\sqrt{2}}\right\rangle \frac{1}{\sqrt{2}} - \left\langle x^2, \sqrt{\frac{3}{2}}x\right\rangle \sqrt{\frac{3}{2}}x\right\|} = \frac{x^2 - \frac{1}{3}}{\left\|x^2 - \frac{1}{3}\right\|} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \\
e_4 &= \frac{x^3 - \langle x^3, e_1\rangle e_1 - \langle x^3, e_2\rangle e_2 - \langle x^3, e_3\rangle e_3}{\|x^3 - \langle x^3, e_1\rangle e_1 - \langle x^3, e_2\rangle e_2 - \langle x^3, e_3\rangle e_3\|} = \frac{x^3 - \frac{3x}{5}}{\left\|x^3 - \frac{3x}{5}\right\|} = \sqrt{\frac{175}{8}}\left(x^3 - \frac{3x}{5}\right)
\end{aligned}
$$

Thus, we have e_1, e_2, e_3, e_4 defined as above as the orthonormalised basis for $\mathscr{P}_3(\mathbb{R})$ under the standard basis. ∎

4 An Orthonormal List of Vectors

Problem 4.1. Let $e_1, \dots, e_m$ be an orthonormal list of vectors. Prove that $v \in \operatorname{span}(e_1, \dots, e_m)$ if and only if

$$\|v\|^2 = \sum_{j=1}^{m} |\langle v, e_j \rangle|^2.$$

Solution. To begin with, we will proceed with the forward direction. Let us suppose that $v \in \operatorname{span}(e_1, \dots, e_m)$. We note then that since $e_1, \dots, e_m$ are orthonormal, then they are also linearly independent.

Then, let us define the subspace $U \coloneqq \operatorname{span}(e_1, \dots, e_m)$. And since $e_1, \dots, e_m$ are linearly independent, they in fact form a basis for U.

Now, since $v \in U$, then it means that v can be expressed as a unique linear combination of the vectors e_i, for $i = 1, \dots, m$. In other words, we have:

$$v = a_1 e_1 + \dots + a_m e_m.$$

From here, we observe the following:

$$\begin{aligned}
\sum_{j=1}^{m} |\langle v, e_j \rangle|^2 &= \sum_{j=1}^{m} |\langle a_1 e_1 + \dots + a_m e_m, e_j \rangle|^2 \\
&= \sum_{j=1}^{m} \left| \sum_{k=1}^{m} \langle a_k e_k, e_j \rangle \right|^2 \\
&= \sum_{j=1}^{m} \left| \sum_{k=1}^{m} a_k \langle e_k, e_j \rangle \right|^2 \\
&= \sum_{j=1}^{m} |a_j|^2 \\
&= |a_1|^2 + \dots + |a_m|^2 \\
&= \|a_1 e_1 + \dots + a_m e_m\|^2 \\
&= \|v\|^2
\end{aligned}$$

For the backward direction, let us suppose that

$$\|v\|^2 = \sum_{j=1}^{m} |\langle v, e_j \rangle|^2.$$

From here, let us define $w = v - \sum_{j=1}^{m} \langle v, e_j \rangle e_j$. Then, we observe that this vector is in fact orthogonal to each $e_1, \dots, e_m$. In other words, $w \perp \operatorname{span}(e_1, \dots, e_m)$. Then, from here, we see that, in fact, we have:

$$\begin{aligned}
v &= \sum_{j=1}^{m} \langle v, e_j \rangle e_j + w \\
\|v\|^2 &= \left\| \sum_{j=1}^{m} \langle v, e_j \rangle e_j \right\|^2 + \|w\|^2 && \text{(Pythagorean Theorem)} \\
&= \sum_{j=1}^{m} |\langle v, e_j \rangle|^2 + \|w\|^2 .
\end{aligned}$$

However, we note here that we assumed that

$$\|v\|^2 = \sum_{j=1}^{m} |\langle v, e_j \rangle|^2.$$

So, we now have

$$\begin{aligned}
\sum_{j=1}^{m} |\langle v, e_j \rangle|^2 &= \sum_{j=1}^{m} |\langle v, e_j \rangle|^2 + \|w\|^2 \\
\|w\|^2 &= 0 \\
\|w\| &= 0 \\
w &= 0
\end{aligned}$$

Thus, we have that

$$v = \sum_{j=1}^{m} \langle v, e_j \rangle \, e_j.$$

In other words, we have that $v \in \operatorname{span}(e_1, \ldots, e_m)$ as desired. ■

5 Deja Vu? Almost.

Problem 5.1. Suppose that $e_1, \ldots, e_n$ is a list of vectors in V of length 1 such that, for all $v \in V$, we have:

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \ldots + |\langle v, e_n\rangle|^2.$$

Prove that $e_1, \ldots, e_n$ is an orthonormal basis of V.

Solution. To begin with, we will show that $e_1, \ldots, e_n$ is in fact an orthonormal list of vectors in V. To do this, we first note that we have:

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \ldots + |\langle v, e_n\rangle|^2.$$

Now, without loss of generality, let $v = \alpha e_1$, where $\alpha \in \mathbb{F}$. Then, we observe the following:

$$\begin{aligned}
\|v\|^2 &= |\langle v, e_1\rangle|^2 + \ldots + |\langle v, e_n\rangle|^2 \\
\|\alpha e_1\|^2 &= |\langle \alpha e_1, e_1\rangle|^2 + \ldots + |\langle \alpha e_1, e_n\rangle|^2 \\
\|\alpha e_1\|^2 &= (\|\alpha e_1\| \, \|e_1\|)^2 + \ldots + |\langle \alpha e_1, e_n\rangle|^2 && \text{(Cauchy-Schwarz Inequality)} \\
\|\alpha e_1\|^2 &= \|\alpha e_1\|^2 + |\langle \alpha e_1, e_2\rangle|^2 + \ldots + |\langle \alpha e_1, e_n\rangle|^2 \\
0 &= |\langle \alpha e_1, e_2\rangle|^2 + \ldots + |\langle \alpha e_1, e_n\rangle|^2
\end{aligned}$$

From here, we note that since $\langle \cdot, \cdot\rangle$ satisfies non-negativity, this implies then that $|\langle \cdot, \cdot\rangle|^2 \geq 0$ as well. Then, it follows that in order for $\sum_{i=2}^{n} |\langle \alpha e_1, e_i\rangle|^2 = 0$, we must have that $\langle \alpha e_1, e_i\rangle = 0$ as well.

Thus, we have that each vectors e_1 is orthogonal to the other vectors in $\{e_2, \ldots, e_n\}$. We note that we can let $v = \alpha e_i$, for $i = 1, \ldots, n$ and we will get that e_i is orthogonal to the other vectors.

Then, we have established that $e_1, \ldots, e_n$ is a list of orthogonal vectors in V. Furthermore, since each of their length is 1, it follows then that they are in fact orthonormal.

From here, using our result from Question 4, we see that in fact we have that any $v \in V$ must be in $\operatorname{span}(e_1, \ldots, e_n)$. Then, we see that $e_1, \ldots, e_m$ spans V. Furthermore, since it is an orthonormal list of vectors, they must be linearly independent as well.

Therefore, we can conclude that, indeed, $e_1, \ldots, e_n$ is an orthonormal basis of V as desired. ∎