Math 135: Homework 6

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Problems

4 Natural Numbers

Problem 4.19. Prove that if m is a natural number, and d is a nonzero number, then there exists numbers q and r such that $m = (dq) + r$, with $r < d$.

Solution. We will proceed by induction. First, let us construct a set A as follows:

$$A := \{n \in \omega : \exists q \exists r\, [(n = dq + r) \wedge (r < d)]\}.$$

Now, we first look at $n = 0$. We observe that if $n = 0$, then for any $d > 0$, we have $0 = 0d + 0$. And we note that $0 < d$ by definition of d. Thus, we see that, indeed, $n \in A$.

Next, suppose that $k \in A$. Then, we look at k^+. We note that by our induction hypothesis, we have $k = dq+r$. Then, this means that $k^+ = (dq + r)^+ = dq + r^+$.

Next, we note that $r < d$ by our induction hypothesis, and thus we have that $r^+ < d^+$. If $r^+ < d^+$, then by definition we have that $r <= d$.

Then there are two cases. In the first one, if $r < d$, then k^+ satisfy the condition as desired.

On the other hand, $r = d$, then we have the following:

$$k^+ = dq + r^+ = dq + d = d(q + 1) + 0.$$

And we see that since d is nonzero, it follows that $0 < d$. Thus, we see that indeed, $k^+ \in A$.

Therefore, we conclude that A is an inductive set as desired, and thus our claim holds. ■

Problem 4.20. Let A be a nonempty subset of ω such that $\bigcup A = A$. Show that $A = \omega$.

Solution. We want to show that A is an inductive subset of ω in order for them to be equal.

First, we observe that because $A \neq \emptyset$, then for any $a \in A$, we see that $0 \in a$. Then, $0 \in \bigcup A = A$. Thus, $0 \in A$ as desired.

Next, suppose that $k \in A$. Then, we look at $k^+ \in A$. We note that by our induction hypothesis, since $k \in A = \bigcup A$, then that means that there exists some $a \in A$ such that $k \in a$.

Then, that means that we have three options for k^+. First, if $a \in k^+$, we note that this can't happen because then this means that $a = k$ or $a \in k$ by definition of $k^+ = k \cup \{k\}$. If this happened, then we would violate trichotomy.

From here, if $k^+ = a$, then we see that $k^+ = a \in A$, so $k^+ \in A$ as desired. Finally, if $k^+ \in a$, then we observe that $k^+ \in \bigcup A = A$. Thus, we have $k^+ \in A$ too.

Therefore, we see that $k^+ \in A$, and thus A is indeed an inductive subset of ω. Therefore, we conclude that, in fact, $A = \omega$. ■

Problem 4.21. Show that no natural number is a subset of any of its elements.

Solution. First, we observe that if $n = 0$, then $n = \emptyset$. Then, since there are no elements in n, n can't be a subset of its elements.

Now, suppose that $n > 0$. Take $m \in n$. Now, suppose for the sake of contradiction that $n \subseteq m$. However, if this is the case, then it follows that $n \in m$. However, this would violate trichotomy then, and thus we conclude that $n \not\subseteq m$. ■

Problem 4.26. Assume that $n \in \omega$ and $f : n^+ \to \omega$. Show that $\operatorname{ran} f$ has a largest element.

Solution. We shall proceed by induction.

To begin with, let us define the set A to be as follows:

$$A \coloneqq \left\{n \in \omega : f : n^+ \to \omega,\ \ \operatorname{ran} f \text{ has a largest element}\right\}$$

First, we observe that for $n = 0$, then $n^+ = 1 = 0 \cup \{0\} = \emptyset \cup \{\emptyset\}$. Then, we note that $\operatorname{ran} f = \{f(0)\}$ will have only one element, and thus it follows that it has a largest element.

Now, suppose that $k \in A$. Then, we want to show that $k^+ \in A$ as well.

We observe that $(k^+)^+ = k^+ \cup \{k^+\}$. Now, we look at $f : k^+ \cup \{k^+\}$. By our induction hypothesis, $f\,[\![k^+]\!]$ contains a largest element K. So, we now have to consider $f(k^+)$.

For $f(k^+)$, we note that there are there cases for it:

1. $f(k^+) \in K$. In this case, we see that K is the largest element of $\operatorname{ran} f$.
2. $f(k^+) = K$. In this case as well, we see then that K is the largest element of $\operatorname{ran} f$.
3. $K \in f(k^+) = K'$. In this case then, we observe that the largest element of $\operatorname{ran} f$ will then be K'.

Thus, in all three cases, we see that, indeed, $\operatorname{ran} f$ has a maximum element. Thus, $k^+ \in A$. And therefore, by induction, we conclude that the claim is true. ■

Problem 4.27. Assume that A is a set, G is a function, and f_1 and f_2 map ω into A. Further assume that for each n in ω, both $f_1 \restriction n$ and $f_2 \restriction n$ belong to $\operatorname{dom} G$ and

$$f_1(n) = G(f_1 \restriction n) \wedge f_2(n) = G(f_2 \restriction n).$$

Show that $f_1 = f_2$.

Solution. In order for $f_1 = f_2$, we require that for all $n \in \omega$, we have that $f_1(n) = f_2(n)$. So, we will proceed by induction.

Let us define a set A to be as follows:

$$A \coloneqq \{n \in \omega : f_1(n) = f_2(n)\}\,.$$

Now, we first show that $0 \in A$. To do this, we observe that $f_1(0) = G(f_1 \restriction 0)$. However, we note that $0 = \emptyset$. Then, by definition of $\restriction$, we have that $f_1 \restriction 0 = \{\langle u, v\rangle : \langle u, v\rangle \in f_1 \wedge u \in \emptyset\}$. Then, we see that in fact, $f_1 \restriction 0 = \emptyset$. The same applies to $f_2(0)$.

Then, with this in mind, we see that $f_1(0) = G(f_1 \restriction 0) = G(\emptyset) = G(f_2 \restriction 0) = f_2(0)$. Thus, $0 \in A$.

Next, suppose $k \in A$. Then, we look at k^+.

We note that $f_1(k^+) = f_1(k \cup \{k\}) = G(f_1 \restriction (k \cup \{k\})) = G(f_1 \restriction k) \cup G(f_1 \restriction \{k\})$.

Similarly, $f_2(k^+) = f_2(k \cup \{k\}) = G(f_2 \restriction (k \cup \{k\})) = G(f_2 \restriction k) \cup G(f_2 \restriction \{k\})$.

By our induction hypothesis, we observe that $f_1(k) = G(f_1 \restriction k) = G(f_2 \restriction k) = f_2(k)$.

Furthermore, we note that $G(f_1 \restriction \{k\}) = f_1(\{k\}) = \langle k, f_1(k)\rangle$, and $G(f_2 \restriction \{k\}) = f_2(\{k\}) = \langle k, f_2(k)\rangle$. And by our induction hypothesis, we have that $\langle k, f_1(k)\rangle = \langle k, f_2(k)\rangle$. So, $G(f_1 \restriction \{k\}) = G(f_2 \restriction \{k\})$.

Putting this all together then, we see that $f_1(k^+) = G(f_1 \restriction k) \cup G(f_1 \restriction \{k\}) = G(f_2 \restriction k) \cup G(f_2 \restriction \{k\}) = f_2(k^+)$.

Thus, we conclude that $k^+ \in A$, and thus A is inductive. Therefore, we see that, indeed, $f_1 = f_2$ as desired. ■

5 Construction of the Real Numbers

Problem 5.1. Is there a function $F : \mathbb{Z} \to \mathbb{Z}$ satisfying the equation

$$F\left([\langle m, n\rangle]\right) = [\langle m + n, n\rangle]?$$

Solution. No.

For our counterexample, let us first define the following function:

$$G : (\omega \times \omega) \to (\omega \times \omega) : \langle m, n\rangle \mapsto \langle m + n, n\rangle$$

Then, if we show that G isn't compatible with $\sim$, then such a function F as described above won't exist.

So, let us look at $\langle 1, 0\rangle$ and $\langle 3, 2\rangle$. First, we see that $\langle 1, 0\rangle \sim \langle 3, 2\rangle$. However, $G\left(\langle 1, 0\rangle\right) = \langle 1 + 0, 0\rangle = \langle 1, 0\rangle \not\sim \langle 5, 2\rangle = \langle 3 + 2, 2\rangle = G\left(\langle 3, 2\rangle\right)$.

Thus, we see that G is not compatible with $\sim$, and thus F does not exist. ■

Problem 5.7. Show that $a \cdot_Z (-b) = (-a) \cdot_Z b = -(a \cdot_Z b)$.

Solution. First, let $a = [\langle m, n\rangle]$ and $b = [\langle p, q\rangle]$.

Then, we observe the following:

$$\begin{aligned}
a \cdot_Z (-b) &= [\langle m, n\rangle] \cdot_Z [\langle q, p\rangle] \\
&= [\langle mq + np, mp + nq\rangle] \\
(-a) \cdot_Z b &= [\langle n, m\rangle] \cdot_Z [\langle p, q\rangle] \\
&= [\langle np + mq, nq + mp\rangle] \\
&= [\langle mq + np, mp + nq\rangle]
\end{aligned}$$

So, we see that $a \cdot_Z (-b) = (-a) \cdot_Z b$. Next, we observe the following:

$$\begin{aligned}
a \cdot_Z b + a \cdot_Z (-b) &= [\langle m, n\rangle] \cdot_Z [\langle p, q\rangle] + [\langle mq + np, mp + nq\rangle] \\
&= [\langle mp + nq, mq + np\rangle] + [\langle mq + np, mp + nq\rangle] \\
&= [\langle mp + nq + mq + np, mq + np + mp + nq\rangle] \\
&= 0_Z
\end{aligned}$$

Thus, we see that $a \cdot_Z (-b)$ is the unique additive inverse of $a \cdot_Z b$. So, $a \cdot_Z (-b) = -(a \cdot_Z b)$ as desired.

Therefore, we have the following:

$$\begin{aligned}
a \cdot_Z (-b) &= (-a) \cdot_Z b \\
(-a) \cdot_Z b &= -(a \cdot_Z b)
\end{aligned}$$

Putting it together, we have $a \cdot_Z (-b) = (-a) \cdot_Z b = -(a \cdot_Z b)$. ■

Problem 5.9. Show that for all natural numbers m, n, we have:

$$[\langle m, n\rangle] = E(m) - E(n)$$

Solution. We proceed as follows:

$$\begin{aligned}[\langle m, n\rangle] &= [\langle m, 0\rangle] +_Z [\langle 0, n\rangle] \\ &= [\langle m, 0\rangle] - [\langle n, 0\rangle] \\ &= E(m) - E(n)\end{aligned}$$

■

Problem 5.14. Show that the ordering of rationals is dense. In other words, between any two rationals, there exists a third one:

$$p <_Q s \implies (\exists r)(p <_Q r <_Q s).$$

Solution. We proceed as follows: first, let $p = [\langle a, b\rangle]$ and $s = [\langle c, d\rangle]$ (with $b, d > 0_Z$).

Now, because $p <_Q s$, we note then that we have $[\langle a, b\rangle] <_Q [\langle c, d\rangle] \iff ad < cb$.

Now, we define r to be as follows:

$$\begin{aligned}r &\coloneqq (p +_Q s) \div [\langle 2, 1\rangle] \\ &= [\langle ad + cb, bd\rangle] \div \{\langle 2, 1\rangle\} \\ &= [\langle ad + cb, bd\rangle] \cdot_Q [\langle 1, 2\rangle] \\ &= [\langle ad + cb, 2bd\rangle]\end{aligned}$$

Then, we note that since $b, d > 0_Z$, we have the following:

1. $adb < cbb$, and
2. $add < cbd$.

With these inequalities in mind, we observe the following:

$$\begin{aligned}2abd = adb + adb < adb + cbb = b(ad + cb) \\ \iff [\langle a, b\rangle] <_Q [\langle ad + cb, 2bd\rangle]\end{aligned}$$

So, we have that $p <_Q r$ as desired.

We also observe the following:

$$\begin{aligned}d(ad + cb) = add + cbd < cbd + cbd = 2cbd \\ \iff [\langle ad + cb, 2bd\rangle] <_Q [\langle c, d\rangle]\end{aligned}$$

So, we have $r <_Q s$ as desired.

Thus, we have shown that indeed, there exists an r that satisfies our claim. ■

Problem 5.15. Show that $\bigcup A$ is closed downward and has no largest element.

Solution. First, we note that A is the set of real numbers.

Now, by definition, A is the set of all Dedekind cuts x.

So, we first will show that $\bigcup A$ is closed downwards.

To do this, let us look at some $q \in \bigcup A$. We note that for $q \in \bigcup A$, it means that there exists some $x \in A$ such that $q \in x \in A$.

Next, take some $r < q$. Then, by definition, because x is a Dedekind cut, it is closed downwards, and thus we have that $r \in x$ as well. But if this is the case, it follows then that $r \in \bigcup A$, and so we see that $\bigcup A$ is also closed downwards.

Now, we will show that it has no largest elements.

Let us take some $p \in \bigcup A$. Then, we see that there exists an x such that $p \in x \in A$. But, we note that since x is a Dedekind cut, then by definition, it has no largest member. So, there exists some $q \in x$ such that $p < q$.

However, we observe that if $q \in x$, then it follows that $q \in \bigcup A$ as well.

This means then that for any $p \in \bigcup A$, we can find some $q \in \bigcup A$ such that $p < q$; in other words, $\bigcup A$ does not have a largest element. ■

Problem 5.19. Assume that p is a positive rational number. Show that for any real number x there is a rational number q in x such that

$$p + q \notin x.$$

Solution. To prove this, we will introduce the following lemma:

Lemma (Archimedean Property with Rationals)**.** For any positive rational p and any rational q, there exists a natural n such that $np > q$ (this is the Archimedean Property)

Proof. We note that the proof for this lemma was completed in class previously. □

Now, we first note that for any real number x, there exists a rational r such that $r \notin x$.

Then, by our lemma, we observe that there exists some integer n such that $np > r$. We also note then that $np \notin x$.

Then, by the well-ordering of the naturals, there exists a least n such that $np \notin x$. But this means then that $(n-1)p \in x$.

So, let $q = (n-1)p$ and observe that:

$$\begin{aligned} p + q &= p + (n-1)p \\ &= p + np - p \\ &= np \notin x \end{aligned}$$

Thus, we have shown that there indeed exists such a q as desired. ■