Math 135: Homework 7

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Problems

6 Cardinal Numbers and the Axiom of Choice

Problem 6.1. Show that the equation

$$f(m,n) = 2^m(2n+1) - 1$$

defines a one-to-one correspondence between $\omega \times \omega$ and ω.

Solution. We will want to show that $f(m,n)$ is both one-to-one and onto.

First, we will show that it is one-to-one. Suppose that we have $f(m,n) = f(m',n')$. We want to show then that $(m,n) = (m',n')$.

To do this, we observe the following:

$$\begin{aligned} f(m,n) &= 2^m(2n+1) - 1 \\ f(m',n') &= 2^{m'}(2n'+1) - 1 \end{aligned}$$

Then, we have:

$$\begin{aligned} f(m,n) &= f(m',n') \\ 2^m(2n+1) - 1 &= 2^{m'}(2n'+1) - 1 \\ 2^{m-m'}(2n+1) &= 2n'+1 \end{aligned}$$

From here, we note that for any $n \in \omega$, $2n+1$ and $2n'+1$ are both odd numbers; they can't have 2 as one of their factors. In other words, we require for $2^{m-m'} = 2^0 = 1$ for the equality above to be true.

Then, this yields us:

$$\begin{aligned} m - m' &= 0 \\ m &= m' \end{aligned}$$

Furthermore, since this is the case, we have:

$$\begin{aligned} 2n+1 &= 2n'+1 \\ 2n - 2n' &= 0 \\ 2(n-n') &= 0 \\ n - n' &= 0 \\ n &= n' \end{aligned}$$

Thus, we have $m = m'$ and $n = n'$; in other words, we have that if $f(m,n) = f(m',n')$, then $(m,n) = (m',n')$ as desired. Thus, f is indeed one-to-one.

Next, to show onto, we want to show that for any $k \in \omega$, there exists $(m,n) \in \omega \times \omega$ such that $f(m,n) = k$.

For the case of $k = 0$, we observe that if we let $m = n = 0$, then we have:

$$\begin{aligned} 2^0(2(0)+1) - 1 &= 1(0+1) - 1 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

So, there exists m, n such that $f(m,n) = k = 0$.

And for $k = 1$, we observe that if we let $m = 1$ and $n = 0$, then we have:

$$\begin{aligned} 2^1(2(0)+1) - 1 &= 2(0+1) - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

So, there exists m, n such that $f(m, n) = k = 1$.

Now, for $k > 1$, we note that by the Fundamental Theorem of Arithmetic, k has some unique prime factorisation:

$$k = \prod_{i=1}^{j} p_i^{n_i},$$

where $p_1 < p_2 < \cdots < p_n$, and the n_i are positive integers.

Note that this prime factorisation will contain a 2^m term, where m is non-negative (with $m = 0$ if k is odd). Then, the product of the remaining primes in the unique factorisation of k will be an odd number; i.e. there exists some $n \in \omega$ such that $2n + 1 = \prod_{i=2}^{j} p_i^{n_i}$.

Now with this in mind, we first note that all $k' \in \omega$ must have some unique prime factorisation which we can rewrite as $k' = 2^m(2n+1)$, for some $m, n \in \omega$.

And if this is the case, then we have that for all $k \in \omega$, we have $k = k' - 1 = 2^m(2n+1) - 1$. Thus, we have shown that for all $k \in \omega$, there exists $(m, n) \in \omega \times \omega$ such that $f(m, n) = k$. Thus, f is indeed onto.

Therefore, we can conclude that f is a one-to-one correspondence between $\omega \times \omega$ and ω. ■

Problem 6.2. Show that in Fig. 32 we have:

$$\begin{aligned} J(m,n) &= [1 + 2 + \cdots + (m+n)] + m \\ &= \frac{1}{2}\left[(m+n)^2 + 3m + n\right] \end{aligned}$$

Solution. We want to show that J is a one-to-one correspondence between $\omega \times \omega$ and ω.

First, we show that J is one-to-one. To do this, let us suppose that $J(m, n) = J(m', n')$. Now, suppose for the sake of contradiction that $(m, n) \neq (m', n')$.

Without loss of generality, we suppose that $m + n < m' + n'$.

Then, we note that:

$$\begin{aligned} J(m,n) &= J(m',n') \\ \frac{1}{2}\left[(m+n)^2 + (m+n)\right] + m &= \frac{1}{2}\left[(m'+n')^2 + (m'+n')\right] + m' \\ m - m' &= \frac{1}{2}\left[(m'+n')^2 + (m'+n')\right] - \frac{1}{2}\left[(m+n)^2 + (m+n)\right] \\ &= \sum_{k=m+n+1}^{m'+n'} k \\ &< n' - n \end{aligned}$$

Then, with this in mind, we have:

$$\frac{1}{2}\left[(m+n)^2 + (m+n)\right] < \frac{1}{2}\left[(m'+n')^2 + (m'+n')\right]$$

But this contradicts with our assumption that $J(m,n) = J(m',n')$. Thus, we conclude that we must have $(m,n) = (m',n')$. Therefore, J is indeed one-to-one.

To show that it is onto, we note that ■

Problem 6.3. Find a one-to-one correspondence between the open unit interval $(0,1)$ and $\mathbb{R}$ that takes rationals to rationals and irrationals to irrationals.

Solution. We can construct a function as follows:

$$f(x) = \begin{cases} \dfrac{1}{x} - 2 & 0 < x \leq \dfrac{1}{2} \\ \dfrac{1}{x-1} + 2 & \dfrac{1}{2} < x < 1 \end{cases}$$

■

Problem 6.4. Construct a one-to-one correspondence between the closed unit interval

$$[0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\},$$

and the open unit interval $(0,1)$.

Solution. We can construct the following function, where $n \in \omega$:

$$f(x) = \begin{cases} \dfrac{1}{2} & x = 0 \\ \dfrac{1}{2^{n+2}} & x = \dfrac{1}{2^n} \\ x & \text{otherwise} \end{cases}$$

■

Problem 6.6. Let κ be a nonzero cardinal number. Show that there does not exist a set to which every set of cardinality κ belongs.

Solution. We can let $\kappa = 1$. Then, suppose for the sake of contradiction that there exists a set to which every set of cardinality $\kappa = 1$ belongs to.

We note that a set with cardinality $\kappa = 1$ is a singleton. And we have proven in a previous homework that the set of all singletons cannot exist. ■

Problem 6.7. Assume that A is finite and $f : A \to A$. Show that f is one-to-one iff $\operatorname{ran} f = A$.

Solution. First, suppose that f is one-to-one. Then, if $x \neq x'$, then $f(x) \neq f(x')$. This means then that every element in A must be mapped by f to some other element in A. Then, because A is finite, this means then that $\operatorname{ran} f = A$.

On the other hand, suppose that $\operatorname{ran} f = A$. Now, since A is finite, it has a cardinality of n, for some $n \in \omega$.

From here, let's suppose for the sake of contradiction that there exists some $y \in A$ such that for $x \neq x'$, we have $f(x) = f(x') = y$.

Then, there's $n-2$ elements left in the domain and $n-1$ in the range that need to be paired with each other. However, since f is a function, an element in $\operatorname{dom} f$ can't be mapped to two elements in $\operatorname{ran} f$. By the Pigeonhole Principle then, there's at least one element in A which doesn't have a pre-image.

Thus, we have a contradiction. So, we conclude that f is indeed one-to-one. ■

Problem 6.13. Show that a finite union of finite sets is finite.

Solution. We can proceed by induction.

First, we observe that for a set A with cardinality 0, we have then that $A = \emptyset$. Then, $\bigcup A = \emptyset$, so, indeed, we have that $\bigcup A$ is finite as well.

Next, suppose that our claim holds for set A with cardinality n.

Now, we look at A whose cardinality is n^+. Observe then that because A is finite, it follows that there exists a bijection between A and n^+, and thus some bijective function $f : n^+ \to A$.

Then, we have:

$$\begin{aligned}\bigcup A &= \bigcup_{k \in n^+} f(k) \\ &= \bigcup_{k \in n} f(k) \cup f(n)\end{aligned}$$

By our induction hypothesis, we have that $\bigcup_{k \in n} f(k)$ is finite. And since $f(n)$ is also finite, we thus have that $\bigcup_{k \in n} f(k) \cup f(n)$ is finite too.

Therefore, by induction, we conclude that the claim holds as desired. ■

Problem 6.14. Define a permutation of K to be any one-to-one function from K onto K. We can then define the factorial operation on cardinal numbers by the equation

$$\kappa! = \operatorname{card}\{f : f \text{ is a permutation of } K\}$$

where K is any set of cardinality κ. Show that $\kappa!$ is well defined.

Solution. content... ■