The Final Homework

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Contents

1 Self-Adjoint Operators and Minimal Polynomials

Problem 1.1. Let T be a self-adjoint operator on a finite-dimensional inner product space (real or complex) such that $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be the only eigenvalues of T.

Prove that $p(T) = 0$, where $p(\lambda) := (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$.

Solution. To begin with, we note that since T is a self-adjoint operator and its only eigenvalues are $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, then it has a minimal polynomial of the form:

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3).$$

From here, we recall that by definition of a minimal polynomial, we have that $p(T) = 0$. Then, since we define $p(\lambda) = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3)$, we see then that in fact, it is the minimal polynomial. Thus, we see that $p(T) = 0$ as desired. ■

Problem 1.2. Give a counterexample to this statement for an operator which is not self-adjoint.

Solution. Suppose that we have the following T with a matrix representation under some orthonormal basis as follows, with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$:

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

Now, we note that since the matrix representation of T^* is the conjugate transpose of T, we have then that

$$\mathcal{M}(T^*) = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

Now, since the two matrices aren't equal to each other, we see that, indeed, T is not self-adjoint. Now, we note that T is upper-triangular, so the eigenvalues are precisely $\lambda_1, \lambda_2, \lambda_3$.

Now, we note then that $p(T) = (T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I)$, we get:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 - \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_3 - \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 & 1 & 0 & 0 \\ 0 & \lambda_1 - \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 - \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_3 & 1 & 0 & 0 \\ 0 & \lambda_1 - \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_2 - \lambda_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And we note that this evaluates to the matrix

$$\begin{bmatrix} 0 & (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And we see that since it isn't equal to the zero matrix, we have then that $p(T) \neq 0$, and thus have found a counterexample as desired. We further note here that in order to annihilate T, we in fact need $p(z) = (z - \lambda_1)^2(z - \lambda_2)(z - \lambda_3)$.

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2 Positive Operators

Problem 2.1. Let $T \in \mathcal{L}(V)$. Show that the following is an inner product on V if and only if T is positive:

$$\langle v, u \rangle_T \coloneqq \langle Tv, u \rangle$$

Solution. Now, let us prove the backwards direction. Suppose that T is positive. This means then that, by definition, we have that $\langle Tv, v \rangle > 0$ for all $v \in V \setminus \{0\}$. Furthermore, we note that T must be self-adjoint.

Then, we will show that $\langle v, u \rangle_T \coloneqq \langle Tv, u \rangle$ is indeed an inner product:

1. **Positivity**: We first note that for all $v \in V \setminus \{0\}$, we have by definition that $\langle Tv, v \rangle > 0$. Then, we note that if $v = 0$, then $\langle Tv, v \rangle = \langle T(0), 0 \rangle = \langle 0, 0 \rangle = 0$. Thus, positivity holds.

2. **Definiteness**: We note that from the previous part, we see that if $v = 0$, then we have that $\langle v, v \rangle_T = \langle Tv, v \rangle = 0$. On the other hand, if $\langle v, v \rangle_T = \langle Tv, v \rangle = 0$, we recall then that since T is positive, this can only occur when $v = 0$ or else $\langle Tv, v \rangle > 0$. Thus, we have shown that $\langle Tv, v \rangle = 0$ if and only if $v = 0$.

3. **Additivity** We observe the following:

$$\begin{aligned} \langle u + v, w \rangle_T &= \langle T(u + v), w \rangle \\ &= \langle Tu + Tv, w \rangle \\ &= \langle Tu, w \rangle + \langle Tv, w \rangle \\ &= \langle u, w \rangle_T + \langle v, w \rangle_T \end{aligned}$$

   Thus, we have additivity (in the first slot).

4. **Homogeneity**: Consider some $\lambda \in \mathbb{F}$. Then, we see the following:

$$\begin{aligned} \langle \lambda v, w \rangle_T &= \langle T(\lambda v), w \rangle \\ &= \langle \lambda T(v), w \rangle \\ &= \lambda \langle Tv, w \rangle \\ &= \lambda \langle v, w \rangle_T \end{aligned}$$

   As such, we have homogeneity in the first slot as desired.

5. **Conjugate Symmetry**: For this, we see that

$$\begin{aligned} \langle v, w \rangle_T &= \langle Tv, w \rangle \\ &= \overline{\langle w, Tv \rangle} \\ &= \overline{\langle Tw, v \rangle} \\ &= \overline{\langle w, v \rangle}_T \end{aligned}$$

   Thus, we have conjugate symmetry as desired.

Therefore, we have shown that if T is positive, then indeed $\langle v, u \rangle_T = \langle Tv, u \rangle$ is an inner product.

Now we will prove the forward direction. Let us suppose that $\langle v, u \rangle_T = \langle Tv, u \rangle$ is an inner product on V.

First, we will show that T is self-adjoint. That is, $T = T^*$. To do this, we observe the following:

$$\begin{aligned} \langle v, u \rangle_T &= \overline{\langle u, v \rangle_T} \\ \langle Tv, u \rangle &= \overline{\langle Tu, v \rangle} \\ &= \langle v, Tu \rangle \\ &= \langle T^* v, u \rangle . \end{aligned}$$

So, we have that $\langle Tv, u\rangle = \langle T^*v, u\rangle$. To show that this is true for all $u, v \in V$ and verify that $T = T^*$, we proceed as follows:

$$\begin{aligned}\langle Tv, u\rangle - \langle T^*v, u\rangle &= 0\\ \langle Tv - T^*v, u\rangle &= 0\\ \langle (T - T^*)v, u\rangle &= 0\end{aligned}$$

Then, let us set $u = (T - T^*)v$. Then, we get

$$\langle (T - T^*)v, (T - T^*)v\rangle = 0$$

Then, we note here that since $\langle x, x\rangle = 0$ if and only if $x = 0$, we have then that $(T - T^*)v = 0$. However, this holds for all $v \in V$, it follows then that we must have that $T - T^* = 0 \implies T = T^*$. Thus, we see that T is self-adjoint as desired.

From here, we have that for $\langle v, v\rangle_T = \langle Tv, v\rangle \geq 0$, for all $v \in V$. Thus, we see that T is a non-negative operator.

Furthermore, we note that $\langle v, v\rangle_T = \langle Tv, v\rangle = 0$ if and only if $v = 0$. Then, it follows that $\langle v, v\rangle_T = \langle Tv, v\rangle > 0$ for all $v \in V \setminus \{0\}$. Thus, we observe that, indeed, T is positive as desired. ■

3 Non-negative Operators

Problem 3.1. Show that the operator $T = -D^2$ is non-negative on the space $V := \text{span}(1, \cos x, \sin x)$ over $\mathbb{R}$, with the inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)\mathrm{d}x.$$

Solution. To begin with, we will show that T is self-adjoint. To do this, we can orthonormalise our basis to get the following orthonormal basis: $\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}$. So, we get then that the matrix representation for T must be:

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, we note that since T is in fact a diagonal matrix, its conjugate transpose (and thus its adjoint) T^* is equal to T. Thus, since $T = T^*$, we have that it is self-adjoint.

Next, we want to show that $\langle Tf, g\rangle \geq 0$ for all $f, g \in V$. First, we make the following observation:

$$\begin{aligned} -(\sin x)'' &= \sin x \\ -(\cos x)'' &= \cos x \\ -(1)'' &= 0 \end{aligned}$$

In other words, for $f = \cos x$ or $\sin x$, we see that $Tf = f$. Next, we recall that since $1, \cos x, \sin x$ are all orthogonal to each other with respect to the given inner product above, it follows then that the their inner products must be equal to zero. Thus, we have to consider the following three cases:

1. $\langle T(1), 1\rangle$,
2. $\langle T(\sin x), \sin x\rangle$,
3. $\langle T(\cos x), \cos x\rangle$.

In the first case, we note that since $T(1) = 0$, it follows then that $\langle T(1), 1\rangle = 0$. For the other two cases, we observe that:

$$\begin{aligned} \langle T(\sin x), \sin x\rangle &= \int_{-\pi}^{\pi} \sin^2(x)\mathrm{d}x \\ &= \pi \\ \langle T(\cos x), \cos x\rangle &= \int_{-\pi}^{\pi} \cos^2(x)\mathrm{d}x \\ &= \pi \end{aligned}$$

Thus, we see that, indeed, since T is both self-adjoint and $\langle Tf, g\rangle \geq 0$ for all $f \in V$, it follows then that it is in fact a non-negative operator as desired. ∎

Problem 3.2. Find its square root operator, $\sqrt{T}$.

Solution. Recall that the matrix representation for T under our orthonormal basis is

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, we let $R = \sqrt{T}$ to have the following matrix representation:

$$\mathcal{M}(R) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And we note that, indeed, R is non-negative as desired, and $R^2 = T$. ■

Problem 3.3. Find a self-adjoint operator $R \neq \sqrt{T}$ such that $R^2 = T$.

Solution. We recall that $T(1) = 0$, and $Tf = f$ for $f = \sin x$ or $\cos x$.

Then, from here, we observe that we can define $R = D^2$. We observe then that $R(1) = 0 = T(1)$. So, we have then that $R^2(1) = R(R(1)) = R(0) = 0 = T(1)$.

Meanwhile, for $f = \sin x$ or $\cos x$, we have that $Rf = -f$. Then, $R^2 f = R(Rf) = R(-f) = -Rf = f = Tf$, as desired.

To check for self-adjointness, we observe the following:

$$\begin{aligned} \langle Rf, g \rangle &= \int_{-\pi}^{\pi} f''(x)g(x)\mathrm{d}x \\ &= f'(x)g(x)\big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)g'(x)\mathrm{d}x \\ &= -\int_{-\pi}^{\pi} f'(x)g'(x)\mathrm{d}x \\ &= -\left(f(x)g'(x)\big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)g''(x)\mathrm{d}x \right) \\ &= \int_{-\pi}^{\pi} f(x)g''(x)\mathrm{d}x \\ &= \langle f, Rg \rangle \end{aligned}$$

However, we note that this isn't $\sqrt{T}$ since it fails the non-negativity check. Specifically, we note that if we had:

$$\begin{aligned} \langle R(\sin x), \sin x \rangle &= \int_{-\pi}^{\pi} -\sin^2(x)\mathrm{d}x \\ &= -\int_{-\pi}^{\pi} \sin^2(x)\mathrm{d}x \\ &= -\pi \ngeq 0. \end{aligned}$$

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Problem 3.4. Find a non-self-adjoint operator S such that $S^*S = T$.

Solution. We first recall that S^* is the conjugate transpose of S. Then, with this in mind, we find that the matrix representation of T with respect to the orthonormal basis from the previous parts is:

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, we want to find matrices S, S^* (who are not equal to each other, else S would be self-adjoint) such that their product is equal to T. In this case, we define S, S^* to be the following:

$$\mathcal{M}(S) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \mathcal{M}(S^*) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then, from here, we note that, indeed, we have

$$S^*S = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T.$$

■

4 Isometries

Problem 4.1. Let T_1 and T_2 be normal operators on an n-dimensional inner product space V. Suppose both have n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that there is an isometry $S \in \mathcal{L}(V)$ such that $T_1 = S^* T_2 S$.

Solution. To begin with, we note that since T_1, T_2 are normal operators on an n-dimensional inner product space V, and they both have n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, we observe then that the eigenvectors of T_1 and T_2 are orthogonal and thus form a basis for V.

Now, with this in mind, we note then that there exists an orthonormal basis $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ such that $T_1 e_i = \lambda_i e_i$ and $T_2 f_i = \lambda_i f_i$, for $i = 1, \ldots, n$.

From here, by the linear map lemma, we note then that we can define a unitary operator S such that $Se_i = f_i$. Note here that since S is a unitary operator, we know then that it's invertible, and that $S^{-1} = S^*$ and that $S^* f_i = e_i$.

Also, note that a unitary operator is simply an invertible isometry.

Then, with this in mind, we note that we get:

$$\begin{aligned} T_1 e_i &= S^* T_2 S e_i \\ &= S^* T_2 f_i \\ &= S^* \lambda_i f_i \\ &= \lambda_i S^* f_i \\ &= \lambda_i e_i \end{aligned}$$

And we note that since a linear map is defined by its action on a basis, since the above holds for $i = 1, \ldots, n$, we observe then that we in fact have $T_1 = S^* T_2 S$, where S is an isometry as desired. ■

5 Singular Values

Problem 5.1. Find the singular values of the operator $T \in \mathscr{P}_3(\mathbb{C}) : p(x) \mapsto 2xp'(x) - x^2p''(x)$ if the inner product on $\mathscr{P}_3(\mathbb{C})$ is defined as

$$\langle p, q \rangle := \int_{-1}^{1} p(x)\overline{q(x)}\mathrm{d}x.$$

Solution. To begin with, from previous homeworks, we can see that the orthonormal basis with respect to this inner product for the given space is: $\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right), \sqrt{\frac{175}{8}}\left(x^3 - \frac{3x}{5}\right)$. Let us denote this as e_1, e_2, e_3, e_4 respectively.

Then, from here, we observe the following:

1. $Te_1 = 0$.
2. $Te_2 = 2\sqrt{\frac{3}{2}}x = 2e_2$.
3. $Te_3 = 4\sqrt{\frac{45}{8}}x^2 - 2\sqrt{\frac{45}{8}}x^2 = 2\sqrt{\frac{45}{8}}x^2 = 2e_3 + \sqrt{5}e_1$.
4. $Te_4 = 2\sqrt{\frac{175}{8}}x(3x^2 - \frac{3}{5}) - \sqrt{\frac{175}{8}}x^2(6x) = -\frac{6}{5}\sqrt{\frac{175}{8}}x = -\sqrt{21}e_2$.

So, from here, we have then that

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 0 & \sqrt{5} & 0 \\ 0 & 2 & 0 & -\sqrt{21} \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we note that since $e_1, \ldots, e_4$ is an orthonormal basis, then the matrix representation of T^* is simply the conjugate transpose of T. Thus, we have:

$$\mathcal{M}(T^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \sqrt{5} & 0 & 2 & 0 \\ 0 & -\sqrt{21} & 0 & 0 \end{bmatrix}$$

From here, we observe then that we have:

$$\mathcal{M}(T^*T) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & -2\sqrt{21} \\ 0 & 0 & 9 & 0 \\ 0 & -2\sqrt{21} & 0 & 21 \end{bmatrix}$$

Next, we want to find the eigenvalues of T^*T. To do this, we will compute the determinant of $T^*T - \lambda I$. That is, we find the determinant of the following matrix:

$$\begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & 4-\lambda & 0 & -2\sqrt{21} \\ 0 & 0 & 9-\lambda & 0 \\ 0 & -2\sqrt{21} & 0 & 21-\lambda \end{bmatrix}$$

Through computations, we see that we get $-\lambda^2(9-\lambda)(25-\lambda)$. Then, we see that the eigenvalues of T^*T are $25, 9, 0, 0$.

Then, we note that the singular values are the square roots of the eigenvalues of T^*T. Thus, we have the singular values as $5, 3, 0, 0$. ■