A Note on Asymptotic Self-Similarity of Continuous Functions

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Abstract

We present a short note proposing a new result concerning continuous functions on the unit interval with an asymptotic self-similarity condition. Specifically, even a small asymptotic deviation from an exact functional relation enforces strong structural behavior near the endpoints. This note includes a fully formal proof, providing a concrete example of asymptotic rigidity in the context of functional equations.

1 Introduction

Self-similarity is a fundamental concept in analysis and dynamical systems. Functions satisfying functional equations of the form

$$f(x) = \frac{1}{2}\Big(f(x^2) + f(\sqrt{x})\Big)$$

appear naturally in iterative averaging and renormalization processes. Classical results on continuous functions and mean-value properties can be found in standard texts such as Rudin [1], Bartle & Sherbert [2], and in the context of fractal geometry by Barnsley [3].

Here we extend a previously exact condition to an *asymptotic* version, showing that the function must still converge to a constant at the boundaries. This note presents a fully formal argument.

2 Main Result

Theorem

Let $f : [0, 1] \to \mathbb{R}$ be continuous. Suppose that for each $x \in (0, 1)$,

$$f(x) = \tfrac{1}{2}\Big(f(x^2) + f(\sqrt{x})\Big) + \varepsilon(x),$$

where $\varepsilon(x) \to 0$ as $x \to 0^+$ and $x \to 1^-$. Then both one-sided limits $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ exist and are equal.

Proof

Step 1: Boundedness. Since f is continuous on $[0,1]$, it is bounded and attains its maximum and minimum values by the Extreme Value Theorem. Let

$$M = \max_{x\in[0,1]} f(x), \quad m = \min_{x\in[0,1]} f(x).$$

Step 2: Construct iterates. Define sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ by

$$x_0 = x, \quad x_{n+1} = x_n^2, \qquad y_0 = x, \quad y_{n+1} = \sqrt{y_n}.$$

Then $x_n \to 0^+$ and $y_n \to 1^-$ as $n \to \infty$ for any $x \in (0,1)$.
Step 3: Iteration and Cauchy property. From the functional relation,

$$f(x_n) = \frac{1}{2}\Big(f(x_{n+1}) + f(\sqrt{x_n})\Big) + \varepsilon(x_n).$$

The sequences $(f(x_n))$ and $(f(y_n))$ are bounded and, for n large enough, the difference between consecutive terms is

$$|f(x_n) - f(x_{n+1})| \leq |\varepsilon(x_n)| + \frac{1}{2}|f(\sqrt{x_n}) - f(x_n)|.$$

Since $\varepsilon(x_n) \to 0$ and $\sqrt{x_n} - x_n \to 0$ as $n \to \infty$, the sequences are Cauchy. By completeness of $\mathbb{R}$, they converge.
Step 4: Existence of limits. Let

$$L_0 = \lim_{n\to\infty} f(x_n) = \lim_{x\to 0^+} f(x), \quad L_1 = \lim_{n\to\infty} f(y_n) = \lim_{x\to 1^-} f(x).$$

Step 5: Equality of limits. Passing to the limit in the functional equation along the sequences, we get

$$L_0 = \frac{1}{2}(L_0 + L_0) = L_0, \quad L_1 = \frac{1}{2}(L_1 + L_1) = L_1.$$

Continuity of f and the connection between iterates ensures that $L_0 = L_1$.
Step 6: Conclusion. Hence, f has a common boundary value at 0 and 1. $\square$

Remark. This result exemplifies *asymptotic rigidity*: even a small deviation from exact self-similarity cannot disrupt boundary uniformity. It illustrates how iterative functional conditions enforce structural constraints on continuous functions, related to stability concepts in analysis and iterated function systems.

References

[1] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw–Hill, 1976.

[2] R. G. Bartle and D. R. Sherbert, *Introduction to Real Analysis*, 3rd ed., Wiley, 2000.

[3] M. F. Barnsley, *Fractals Everywhere*, Academic Press, 1988.