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WYDAWNICTWA AGH

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Rozprawy
Monografie 40

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Systemy wspomagania decyzji oparte o zastosowanie zbiorów odniesienia

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DISSERTATIONS
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Decision Support Systems Based on Reference Sets

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Contents

Andrzej M.J. Skulimowski

DECISION SUPPORT SYSTEMS BASED ON REFERENCE SETS

Summary

In this monograph we present the theoretical foundations, interactive computer implementations, and some most characteristic applications of a new decision support methodology for the multicriteria decision problems, the multiple reference points and reference set method. We will pay a special attention to the solu tion of nonlinear and non-convex multicriteria decision problems which are motivated by the needs of numerous engineering problems, as well as are applicable to the portfolio optimization and other management problems.

The fundamental feature of the method here presented consists in the fact that the preference information is modelled by the different classes of reference points in the space of values of the vector objective function. The reference points are assumed to be defined independently from the preliminary problem formulation and must be of a special importance for the problem solution. Thus, it constitutes a generalization of the well-known single reference point method. Consequently, to select a compromise solution, the decision-maker should provide the preference information consisting of the following classes of reference points : the most desired optimization results (ideal points), those satisfactory, the values to be avoided (anti-ideal points), and the limits of optimality. The decision-maker is allowed to define further classes of reference points and consider them similarly within the same decision model. Moreover, one can si multaneously consider the constraints on the trade-offs between the criteria, and the criteria space constraints.

Chapters 1-3 have an introductory character. Chapter 2 contains most important definitions and theorems as an outline of the mathematical background of the theory presented in the subsequent Chapters. In the Chapter 3 we give some basic facts from the multicriteria optimization and multicriteria decision making. We discuss the methodology of multicriteria analysis, define the vector optimization and scalarization problems, present some basic scalarization techniques, and the foundations of utility

theory and outranking methods.

The main theoretical developments concerning the reference point method reside in the fourth and subsequent Chapters. The 4th Chapter is devoted to the classification of elements of the criteria space of a multicriteria optimization problem taking into account their situation with respect to the set of all available values of criteria. We pay a special attention to the class of here defined dominating points, since the well defined target reference points should be dominating for the multicriteria problem under consideration. In Chapter 5 we give a theoretical background of this approach, providing the optimality conditions for the scalarization methods applied further to solve static and dynamic multicriteria decision problems. In Chapter 6 we propose a method of aggregation of the above supplementary information which allows to reduce the compromise decision choice problem to the bicriteria trade-off between the distances to the sets of de sired and avoidable values of criteria, based on the idea of utility function estimation. We study the properties of the decision thus made, in particular it is nondominated and stable with respect to the change of reference points. The decision choice procedure may be generalized to make possible a selection of a subset of compromise alternatives. The above methodology may be applied to solving the decision problems formulated as discrete, linear, nonlinear, or nonconvex multicriteria optimization problems, as well as in multicriteria optimal control. The above decision model has been implemented as an interactive decision support system and applied to the dynamic portfolio optimization.

In Chapter 7 we describe a method to achieve a target set in multicriteria optimization problems in a systematic way by relaxing the soft constraints. We assume that the numerical bounds defining a part of constraints may be changed subject to some payments. The values of the cost function and a measure of proximity from the target reference set Q serve as auxiliary objectives which are taken into account by the decision-maker. The target set has the same meaning as in the preceding chapters and its construction based on reference points might be the same as in Chapter 6. We give a theoretical background for the analysis of this kind of problems in the case where the admissible decision subsets are convex sets defined by m inequality constraints, the criteria are monotone, and the target set is convex. An interactive procedure involving a visualization of the attainable and target sets is also outlined. In the following Chapter 8 we demonstrate that the proper description of conflicts and multiple functions of discrete production systems leads to the formulation of a multiple objective optimization problem. We present a multicriteria problem statement combining quantitative criteria with a qualitative preference structure, regarding the set of final states of the systems as a reference set. We propose a solution method combining a generalization of the multicriteria shortest-path method for variable-structure nets with the reference sets and reference trajectory approaches presented in the preceding chapters. Finally, we will analyze a real-life example of control of several interconnected industrial devices. The discussion of the results presented in the book is contained in Chapter 9, which concludes the monograph.

Andrzej M.J. Skulimowski

SYSTEMY WSPOMAGANIA DECYZJI OPARTE O ZASTOSOWANIE ZBIOROW ODNIESIENIA

Streszczenie

Metoda zbiorów odniesienia stanowi uogólnienie klasycznych metod punktu odniesienia stosowanych w wielokryterialnej analizie i wspomaganiu decyzji oraz wyników zawartych we wcześniejszych pracach autora zwiazanych z estymacją funkcji użyteczności w oparciu o analizę wielu punktów odniesienia. W metodzie tej proces wspomagania decyzji oparty jest o zdefiniowane przez (jednego lub więcej) decydentów lub dowolną liczbę niezależnych ekspertów takich wartości wskaźników jakości, które odpowiadają kolejno:

- pożądanym przez decydenta wartościom, które jednak niekoniecznie są dopuszczalne w danym problemie decyzyjnym;
- rozwiązaniom status-quo, utożsamianymi zazwyczaj z charakterystykami strategii dostępnych bez zastosowania jakiejkolwiek formalnej procedury optymalizacyjnej;
- strategiom oznaczającym *a priori* niepomyślny dla decydenta wynik procesu decyzyjnego; wskazywane są one w celu ich wstępnego wyeliminowania;
- granicom optymalności strategii, których przekroczenie może okazać się niekorzystne np. ze względów podatkowych, lub w związku z ograniczeniami prawnymi itp.

Zdefiniowane w ten sposób wartości wskaźników jakości noszą nazwę punktów odniesienia, a ich klasy - zbiorów odniesienia, które dały nazwę przedstawionej metodzie. Rozdziały 1-3 zawierają wprowadzenie do matematycznych metod optymalizacji wielokryterialnej oraz podstawy metodologii wielokryterialnej analizy decyzyjnej.
Własności zbiorów punktów dominujących, które są potencjalnymi punktami odniesienia, zaprezentowane zostały w rozdziale 4.

Podstawą stosowania powyższej metody wspomagania decyzji stanowi sformułowanie problemu optymalizacji wielokryterialnej, który w przypadku omawianych tu problemów jest z reguły nieliniowym problemem optymalizacji statycznej, problemem wielokryterialnego programowania dynamicznego lub problemem optymalizacji trajektorii. Uzupełniającą informacją o preferencjach są punkty odniesienia definiowane w procesie interakcyjnym i należące do wyszczególnionych wyżej klas. W trakcie ich definiowania każdy z punktów musi być zaszeregowany do jednej z klas, może się jednak okazać, że zbiory punktów odniesienia pochodzące z różnych źródeł informacji nie są ze sobą zgodne, co może być związane ze sprzecznymi zaleceniami różnych ekspertów. W celu usunięcia ewentualnych sprzeczności, w skład systemu wspomagania decyzji wchodzi procedura weryfikacyjna, która może dokonać przeklasyfikowania poszczególnych punktów, a także rozbić każdy z powyższych zbiorów na podklasy złożone z wzajemnie ze sobą nieporównywalnych punktów odniesienia. Następnie dokonywana jest agregacja całości informacji o preferencjach wprowadzonej do systemu, w wyniku czego powstają dwa zbiory odniesienia, z których jeden powinien być przybliżany, a drugi unikany w procesie poszukiwania strategii kompromisowej oraz formułowane są dodatkowe ograniczenia na wartości kryteriów. W metodzie przedstawionej w niniejszej monografii działania te polegają na stosowaniu metod skalaryzacji przez odległość.

Na tym etapie rozwiązania najważniejsze jest zapewnienie racjonalności procesu decyzyjnego, która sprowadza się do spełnienia warunków optymalności wektorowej rozwiazań generowanych metodami skalaryzacji przez odległość. Warunki optymalności takich metod, opracowane dla nowo wprowadzonych klas punktów odniesienia znaleźć można w rozdziale 5 niniejszej monografii. W procesie dialogowym decydentowi przedstawiane są kolejno proponowane rozwiązania kompromisowe, które w kolejnym kroku procedury mogą być modyfikowane w oparciu o subiektywne życzenia decydenta. Opis metody zbiorów odniesienia oraz jej implementacji komputerowej zawiera rozdział 6.

W rozdziale 7 zawarte jest uogólnienie metody zbiorów odniesienia na przypadek problemu ze zmiennymi ograniczeniami, które mogą być redefiniowane po poniesieniu pewnych dodatkowych kosztów. Uzyskane wyniki pozwalają na systematyczną analizę konfliktu pomiędzy wartościami optymalnymi możliwymi do uzyskania jako rozwiązania problemu, a ceną ewentualnego rozszerzenia ograniczeń. Innego rodzaju uogólnienie zawiera rozdział 8, podane jest w nim mianowicie sformułowanie i rozwiązanie problemu sterowania optymalnego systemów zdarzeń dyskretnych, gdzie oprócz ilościowej analizy kosztów sterowania i kosztów procesu możliwe jest uwzględnienie jakościowej informacji o celach procesu sterowania podanej w postaci zbiorów i trajektorii odniesienia.

Chapter 1

Preface

This monograph is devoted to one of the most rapidly developing fields of Operations Research, namely to the multicriteria decision support. A need for a more penetrative analysis of decision problems existed since the beginning of mankind; however, the theoretical and computational tools which made this possible appeared only in the last three decades, and became really accessible for everyone in the last several years.

There exist two basic types of decision problems. One is confronted with the solution of problems of the first type if the preferences of the decision-maker are clearly stated, but the information about the alternative characteristics is incomplete. The problems of the second type are characterized by the usually large amount of additional preference information, which should be acquired from experts and/or the decision-maker(s), evaluated, and used in an optimal way to select a compromise decision in a decision support procedure. The main difficulty in the first kind of problems consists in getting the appropriate information, which may often be a very hard task. As an example for this type of problems may serve the stock exchange investment decisions, where the criteria are well defined and the decision to be made is well defined provided that the future stock prices are known. Since the latter are *ex definitione* unknown, they are replaced by some stochastic estimation, and the decision problem becomes a typical decision-making problem with risk. In the second type of problems the difficulty consists in constructing an appropriate mathematical model of the decision situation using all relevant information available. This is the usual playground of multicriteria analysis, and the subject of the present monograph. Specifically, as a decision-making aid we use reference points, a preference modelling tool originating from the psychological decision theory, where it is called aspiration levels. One of the characteristic features of the approach here presented is the simultaneous use of multiple reference points, grouped in the reference sets, or classes of reference points. This allows an aggregation of information coming from different information sources, such as different experts, multiple decision-makers, or being put into the model as alternative wishes of a single decision-maker, in contradistinction to the classical reference point approach, where only a single reference point could be processed at one time.

The text of this monograph is organized as follows : Chapter 2 contains the most important definitions and theorems as an outline of the mathematical background of the theory presented in the subsequent Chapters. In the Chapter 3 we give some basic facts from the multicriteria optimization and multicriteria decision making. We discuss the methodology of multicriteria analysis, define the vector optimization and scalarization problems, present some basic scalarization techniques and the foundations of utility theory and outranking methods. The main theoretical developments concerning the reference point method reside in the fourth and subsequent Chapters. The Fourth Chapter is devoted to the classification of elements of the criteria space in a multicriteria optimization problem, taking into account their situation with respect to the set of all available values of criteria. We pay special attention to the class of here-defined dominating points, since for the multicriteria problem under consideration, the well defined target reference points should be dominating. We define some relevant subclasses of dominating points and derive some of their properties. In the Fifth Chapter this classification is used to formulate a series of theorems concerning the vector optimality of scalarization procedures used while applying reference points and reference sets in decision support systems. A complete scheme of such procedure is provided in Chapter 6. While using this method, the decision-maker should define at least one of four main classes of reference points: those available at the pre-decision stage, without any expected improvement; the target reference points representing the decision-maker's wishes, needs, and aspirations; anti-ideal points which should be avoided while surveying the admissible decisions, and the limits of optimality which define the criteria space constraints. Additionally, the decision-maker may define the constraints on the trade-off coefficients. Thus, all potentially relevant information to make a decision is represented in one model and may be used in the same decision support procedure. Chapter 7 contains an extension of the target set method to the case, where it is possible to relax the constraints in the original multicriteria problem, so that an attainment of some previously unattainable targets were possible. Constructive techniques of such attainment are provided for multicriteria problems with convex or monotone objectives, and convex or linear constraints. In the final Chapter 8 we describe an application of the reference set and reference trajectory approach to the optimal control of asynchronous discrete-event systems. The approach there presented constitutes the first attempt to an entire optimization of a discrete-event process from both points of view : optimality of the process itself, and the costs of control and supervision.

This book has been originated at the Institute of Automatic Control, the University of Mining and Metallurgy, Cracow, Poland, and written in part during the author's stay at the Institute of Automatic Control at the Swiss Federal Institute of Technology (ETH), Zurich, and at the Institute of Information Management, the University of St. Gallen, Switzerland. The author is grateful to Professor Henryk Górecki, the Director of the Institute of Automatic Control for his continuous encouragement during this work, Professor Mohammed M. Mansour, for the hospitality during the stay at the ETH Zurich, and to Professor Beat F. Schmid, for a fruitful collaboration on relevant applications of the decision-aid methods here presented. The support of the Swiss National Research Foundation, grants no. 12-30240.90 and 2100-037738.93/1 is also greatly appreciated.

Chapter 2

Mathematical Foundations of Multi-criteria Optimization

This introductory Chapter is devoted to the presentation of those mathematical topics which are especially relevant for the scope of this monograph. Except the Section 2.1, we omit the topics which either belong to the elementary mathematical course, or are very well explained in the professional literature, such as e.g. optimization techniques with scalar criteria, and optimal control.

2.1 Notation and Basic Definitions

Throughout this monograph we will use the standard mathematical notation. Let U be an arbitrary linear space. A real function d defined on U and fulfilling the conditions

(i) $\forall\, x \in U \qquad d(x,x) = 0,$

(ii) $\forall\, x, y \in U \qquad d(x,y) = d(y,x)$

(iii) $\forall x, y, z \in U \qquad d(x,y) + d(y,z) \leq d(x,z)$

will be called the distance in U. U together with d will be called a *metric space*. If, moreover, d fulfills the condition

(iv) $\forall\, x, y, u \in U \qquad d(x+u, y+u) = d(x,y),$

then U will be called the *linear metric space*, and the distance d *invariant*. If the function d fulfilling the above properties may be defined as

$$d(x,y) := \eta(x-y),$$

where η is a real function defined on U such that

$$\forall x \in U \ \forall t \in I\!R \ \ \eta(tx) = \mid t \mid \eta(x)$$

then $\eta : U \to I\!R_+$ will be called *norm* and denoted by $\| \, . \, \|$, while U will be called (real) *normed space* (complex normed spaces may be defined analogously). By an open ball with center x and radius r in a normed space U we will mean the set

$$k_r(x) := \{y \in U : \| x - y \| < r\}. \tag{2.1}$$

Closed balls will be defined analogously as

$$K_r(x) := \{y \in U : \| x - y \| \leq r\}. \tag{2.2}$$

By the *base spanning* U we will mean the subset E of U such that

(i) $\forall x \in U$ x can be represented as a linear combination of elements of E;
(ii) the property (i) is fulfilled by no essential subset of E.

The number of elements of the base will be called the *dimension* of U and denoted by $\dim(U)$. Now, let X be a subspace of U. By the *codimension* of X, denoted by $\mathrm{codim}(X)$, we will mean the dimension of the quotient space U/X, by definition U/X is the set of equivalence classes of the following equivalence relation R

$$xRy \Leftrightarrow x - y \in X,$$

with the linear operations defined by the operations on representants of equivalence classes of R. Any subspace H of U such that $\mathrm{codim}(H) = 1$ will be called a *hyperplane*.

The function $f : U \to V$, where both U and V are normed spaces is called *Lipschitz-continuous*, or *lipschitzian*, with the constant $\tau > 0$ iff

$$\forall x, y \in U \ \ \| f(x) - f(y) \| \leq \tau \| x - y \| .$$

By the *Cauchy sequence* in the normed space we will mean any sequence $\{x_n\} \subset U$ such that

$$\forall \varepsilon > 0 \ \exists M \in N \ \forall m, n > M \quad \| x_m - x_n \| < \varepsilon.$$

U will be called *complete*, iff every Cauchy sequence has a limit in U. A subset $V \subset U$ is called *dense* in U iff

$$\forall x \in U \ \ \forall \varepsilon > 0 \ \ \exists v \in V \quad \| x - v \| < \varepsilon.$$

U will be called *separable* iff there exists a countable and dense subset of U. Although many specific results in this monograph are proved for finite-dimensional Euclidean spaces $I\!R^N$, in the general setting the functions to be optimized are Banach space valued, by definition U is a *Banach space* iff U is normed, complete, and separable.

Let $Cl(U)$ denote the family of closed subsets of the Banach space U. For any $V, W \in Cl(U)$ one can define the arithmetic operations

$$\begin{aligned} V + W &:= \{v + w : v \in V, w \in W\} \\ V - W &:= \{v - w : v \in V, w \in W\} \\ tV &:= \{tv : v \in V\}, \text{ for any } t \in I\!R \text{ or } t \in \mathbb{C}. \end{aligned}$$

Remark that $V+W-W$ is rarely equal to V, unless $W=\{w\}$ for a $w \in U$, therefore $Cl(U)$ is not a linear space. However, $Cl(U)$ endowed with the following metric

$$d_H(V,W) := \inf\{r>0 : V \subset W + k_r(0) \text{ and } W \subset V + k_r(0)\} \tag{2.3}$$

is a complete metric space. The function d_H is called the *Hausdorff distance*.

Any subset V of U such that $tV \subset V$ for any $t \in \mathbb{R}$ such that $\mid t \mid \leq 1$ (or $t \in \mathbb{C}$ if U is a complex linear space) is called *balanced*.

Let X be an arbitrary topological space, and let G be a function defined on X with values in $Cl(U)$. Then G will be called *multifunction*. G will be called a *Hausdorff-continuous* multifunction iff G is continuous in the Hausdorff metric in $Cl(U)$.

2.2 Convexity and Generalized Convexity

2.2.1 Convex, pseudoconvex and quasiconvex functions

Let A be a nonempty subset of a linear space E, and Y a partially ordered linear space. A is *convex*, by definition, iff $\forall a,b \in A \;\; [a,b] \subset A$. Recall now that the *epigraph* of a function $f : A \to Y$ is the set

$$epi(f) := \{(x,y) \in A \times Y \;\; : f(x) \leq y\}.$$

Definition 2.1. *A function $f : E \to Y$ is called convex if its epigraph is convex.* ■

The convexity and generalized convexity play an important role while formulating optimality conditions. Unlike the convexity, the generalized convexity conditions are defined for functions rather than for sets.

Definition 2.2. *The function $f : E \to Y$ is called quasiconvex iff all level sets of f, $L_\alpha := \{x \in E \;\; : f(x) \leq \alpha\}$ are convex.* ■

Definition 2.3. *A (Fréchet) differentiable function $f : E \to Y$ is called pseudoconvex on E iff*

$$\forall x \in E \; \forall y \in E \quad [(\delta f(x), y-x) \geq 0 \;\Rightarrow\; f(y) \geq f(x)],$$

where $\delta f(x)$ denotes the Fréchet derivative of f at x. ■

From Definition 2.2 it follows the following relation between pseudo- and quasiconvexity:

Proposition 2.1. *If f is pseudoconvex then all level sets of f,*

$$L_\alpha := \{x \in E : f(x) \leq \alpha\}$$

are convex. What follows, every pseudoconvex function is also quasiconvex. ■

Another useful generalization of convexity was given by Ky Fan (1953):

Definition 2.4. *The function $f : E \to Y$ is called convex-like at $x_0 \in E$ iff*

$$\forall x \in E \ \exists \xi(x_0, x) \in E \ : \ f(\xi(x_0, x)) \leq \frac{1}{2}(f(x_0) + f(x)).$$

■

Classical convexity of subsets of U will be in the sequel often used jointly with the following important fact, known as the geometric version of the *Hahn-Banach Theorem.*

Theorem 2.1. *Let X and Y be two convex subsets of a topological linear space U, X-open, Y-closed, and such that their intersection is empty. Then there exists a hyperplane H which separates X and Y, i.e. $H \cap X = \emptyset$, and $H \cap Y \subset H$.* ■

2.3 Invexity

Further important generalization of the notion of convexity, called invexity, has been introduced by Hanson (1981), and investigated later by Craven (1981), Martin (1985), Mond and Weir (1986), Ben-Israel and Mond (1986), and many others. Introduced originally for differentiable real functions, it has been later generalized to vector functions (Craven and Glover, 1985), non-differentiable functions, and multifunctions (Sach and Craven, 1991). As the most important ideas of convexity appear already in the differentiable case, while the aforementioned generalizations have rather a technical character, here we will confine ourselves to the differentiable invexity, with one exception for regular functions in the sense of Clarke (1983).

We will start from the classical definition of invexity for differentiable scalar functions.

Definition 2.5. *A differentiable function $f : I\!R^n \supset A \to I\!R$ is called invex at $x_0 \in A$ iff*

$$\forall \ x \in A \ \exists \ \eta(x, x_0) \in I\!R^n \ : \ f(x) - f(x_0) - \eta(x, x_0)^T \delta f(x_0) \geq 0. \tag{2.4}$$

■

Observe that the above condition (2.4) is actually more general than convexity, since by setting $\eta(x, x_0) := x - x_0$ we obtain the class of convex differentiable functions.

An important characterization of differentiable invexity is given by the following

Theorem 2.2. *(Craven and Glover, 1985; Ben-Israel and Mond, 1986). The function $f \in D^1(I\!R^N, I\!R)$ is invex iff every stationary point off is a global minimum. In particular, if f has no stationary point then it is invex.* ■

The differentiability assumption in the definition of invexity can be weakened by assuming the property called *regularity in the sense of Clarke* on the set $A \subset I\!R^n$ (see Clarke, 1983) which consists in fulfilling the following conditions (i) and (ii) :

(i) $\forall\, x_0 \in A\ \forall v \in I\!R^n\ \ \exists f'(x_0; v) := \lim_{t\to 0^+}((f(x_0+tv) - f(x_0))/t,$

and

(ii) $f'(x_0; v) = f^0(x_0; v) := \lim_{t\to 0^+} \sup_{y\to x_0}((f(y+tv) - f(y))/t.$

Suppose now that f is lipschitzian with the constant τ on $A \subset I\!R^n$. One can easily observe that $\parallel f^0(x_0; v) \parallel \leq \tau \parallel v \parallel$, therefore $f^0(x_0; v)$ (but not always $f'(x_0; v)$) is well-defined on A for a lipschitzian f, as well as so is the generalized gradient $\partial f(x_0)$, by definition :

$$\partial f(x_0) := \{q \in I\!R^n\ :\ \forall\, v \in I\!R^n\ q^T v \leq f^0(x_0; v)\}.$$

Invexity can now be further generalized for non-differentiable but regular vector-valued functions by setting :

Definition 2.6. *Suppose that $f \in C(I\!R^n, I\!R^k)$ and the directional derivative of f at $x_0, f'(x_0; .)$, exists at x_0. Then we call f invex iff*

$$\forall x \in A\ \exists \eta(x, x_0) \in I\!R^n\ :\ f(x) - f(x_0) \leq f'(x_0; \eta(x, x_0)). \tag{2.5}$$

■

Analogously to the definitions of quasi-convexity and pseudo-convexity, one defines the quasi-, respectively, pseudo-invex functions :

Definition 2.7. *A differentiable function $f : I\!R^n \supset D \to I\!R^n$ satisfying*

$$\forall x \in D\ \exists \eta(x, x_0) \in I\!R^n\ :\ \eta(x, x_0)^T \delta f(x_0) \geq 0 \Rightarrow f(x) - f(x_0) \leq 0, \tag{2.6}$$

or

$$\forall x \in D\ \exists \eta(x, x_0) \in I\!R^n\ :\ f(x) - f(x_0) \leq 0 \Rightarrow \eta(x, x_0)^T \delta f(x_0) \leq 0, \tag{2.7}$$

will be called pseudo-invex, or quasi-invex, respectively. ■

Proposition 2.2. *(Martin, 1985) The following relations hold :*

(i) f *- pseudo-convex or* f *- pseudo-invex* $\Rightarrow$ f *- invex;*

(ii) f *- pseudo-convex* $\Rightarrow$ f *- quasi-convex* $\Rightarrow$ f *- quasi-invex;*

(iii) f *- convex* $\Rightarrow$ f *- pseudo-convex* $\Rightarrow$ f *- invex* $\Rightarrow$ f *- quasi-invex.* ■

Moreover, there are examples of invex functions which are not quasi-convex and quasi-invex functions which are not invex (cf. e.g. Ben-Israel and Mond, 1986).

2.4 Mathematical Formulation of the Multicriteria Optimization Problem

We will consider the general problem of the multicriteria optimization

$$(F : U_d \to E) \to \min(\theta) \tag{2.8}$$

where the set of the admissible controls U_d is a subset of a linear space U, the goal space E is a partially ordered Banach space with a closed convex cone θ. Moreover we assume that the admissible set $F(U_d)$ is nonempty and closed.

A control u_{opt} is said to be *nondominated* (other often used terms : Pareto-optimal, efficient, non-inferior, or θ-optimal) if and only if

$$(F(u_{opt}) - \theta) \cap F(U_d) = \{F(u_{opt})\} \tag{2.9}$$

The condition (2.6) means that no element of the admissible set is better than u_{opt} in the sense of the partial order relation induced further on (cf. e.g. Sawaragi, Nakayama, and Tanino, 1985).

The set of all nondominated alternatives for the problem (2.8) will be denoted by $P(U_d, \theta)$, and the set of all nondominated values of F by $FP(U_d, \theta)$. There holds the following evident relation :

$$FP(U_d, \theta) = F(P(U_d, \theta)).$$

If no misunderstanding arises, we shall write simply $P(U_d)$ and $FP(U_d)$, respectively.

Let us recall that a subset θ of a linear space E is called a *convex cone* if and only if

$$\forall t_1 \geq 0, \forall t_2 \geq 0, \forall x_1, x_2 \in \theta \quad (t_1x_1 + t_2x_2) \in \theta, \tag{2.10}$$

i.e. iff every nonnegative linear combination of elements of θ belongs to θ. The cone is called nontrivial if it is not the whole space E and contains at least two different points. If θ is nontrivial and

$$\theta \cap (-\theta) = \{0\} \tag{2.11}$$

then θ will be called a *sharp* cone. A sharp and convex cone will be called *pointed*.

We say that an ordering relation R in E is *consistent with the linear structure* if for every $x_1, x_2, y_1, y_2 \in E$ and every real numbers t_1, t_2 there holds

$$x_1Ex_2 \text{ and } y_1Ey_2 \Leftrightarrow (t_1x_1 + t_2x_1)R\ (t_1y_1 + t_2y_2) \tag{2.12}$$

To every convex cone θ there corresponds the relation in E defined by

$$x_1 \leq_\theta x_2 \Leftrightarrow x_2 - x_1 \in \theta \tag{2.13}$$

which induces a partial order in E satisfying the condition (2.12). Conversely, the set θ_R defined by the formula

$$\theta_R := \{x \in E : 0 \; R \; x\} \tag{2.14}$$

satisfies (2.10) and is called the positive cone associated to the relation R.

From (2.9) and (2.13) it follows that the set of all nondominated values of F is a minorant of the relation $\leq_\theta$ in the set $F(U_d)$.

Remark that the relations of partial order induced by convex cones are generalization of the natural order in $I\!R^n$ defined as follows

$$x \leq y \Leftrightarrow \forall i = 1, ...n \;\; x_i \leq y_i \tag{2.15}$$

where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. Observe that (2.15) is equivalent to the relation $y - x \in I\!R^n_+$. The positive orthant $I\!R^n_+$ satisfies evidently all properties of convex cones. The relation (2.15) plays the principal role in the classical problems of multicriteria optimization which can be reduced to the simultaneous minimization of scalar functions

$$(F_1, F_2, ..., F_n) \to \min . \tag{2.16}$$

One can consider also the relation of the strict order

$$x <_\theta y \Leftrightarrow y - x \in \theta \setminus \{0\} \tag{2.17}$$

and, assuming that $int(\theta) \neq \emptyset$, the relation of the strong partial order

$$x <<_\theta y \Leftrightarrow y - x \in int(\theta). \tag{2.18}$$

For the case of the natural cone $I\!R^n_+$, the relations equivalent to (2.17) and (2.18) may be formulated respectively as follows

$$x < y \Leftrightarrow \forall \; 1 \leq i \leq n \quad x_i \leq y_i \text{ and } \exists j, 1 \leq j \leq n, \; x_j < y_j$$

$$x << y \Leftrightarrow \forall \; 1 \leq i \leq n \quad x_i < y_i. \tag{2.19}$$

The elements of the minorant of the relation (2.17) are frequently called *weakly nondominated solutions*, *weakly θ-optimal points*, or weak Pareto solutions.

Another type of solutions related to the vector optimization problem (2.8), called *properly nondominated solutions*, or *proper efficient points* has been considered by Geoffrion (1968) for the case $\theta = I\!R^n_+$, and generalized further by Borwein (1977), Benson (1979), Henig (1982), and White (1983). A nondominated element $x \in F(U_d)$ is *properly nondominated* in problem (2.8), iff it exists an open convex cone θ_1 such that $\theta \subset \theta_1 \cup \{0\}$ and x is θ -optimal. Properly nondominated points play an important role as potential compromise solutions to vector optimization problems since the values of criteria characterizing them may not be improved without a substantial worsening of at least one other criterion (cf. Geoffrion, 1968).

Imposing additional constraints on prices of some criteria expressed by the units of other criteria for problems of type (2.16), allows us to consider more general partial order relations induced by convex cones different from $\mathbb{R}^n_+$. For example, to a bicriteria problem

$$(F = (F_1, F_2) : U_d \to \mathbb{R}^2) \to \min(\mathbb{R}^2_+) \tag{2.20}$$

there exists a constraint defined by the formula

$$F_1(u_{opt}) \leq aF_2(u), \quad a > 0. \tag{2.21}$$

All nondominated points of the problem (2.20) satisfying (2.21) are nondominated with respect to the partial order induced by the convex cone

$$\chi := \{(t_1, t_2) \in \mathbb{R}^2 : t_1 + at_2 \leq 0,\ t_1 \geq 0\} \tag{2.22}$$

(cf. Yu and Leitmann, 1974b).

Numerous multicriteria decision support procedures apply the preference information concerning the trade-off coefficients between the criteria or - in the above methodology - the constraints on the prices of one criterion expressed by the value of another criterion, for all, or only some of suitable pairs of criteria (cf. Nakayama and Furukawa, 1985; Cogger and Yu, 1985). One of the most important features of the decision support methodology presented in this monograph is the possibility of combining trade-off information coded equivalently in form of a convex cone with the sets of reference criteria values in one preference model.

The problems with the infinite-dimensional goal space constitute another domain of the applications of the theory of minimization of the functional F with respect to the cone θ. In this situation the definition of various partial orders in E using convex cones becomes especially relevant for the properties of the problem and for the subsequent decision-choice procedures.

2.4.1 Sufficient conditions of optimality for invex functions

Let $f \in D^1(\mathbb{R}^n, \mathbb{R}^k)$ and $g \in D^1(\mathbb{R}^n, \mathbb{R}^m)$, and consider the following class of vector optimization problems :

$$(f : U \subset \mathbb{R}^n \to \mathbb{R}^k) \to \min(\theta)$$

$$\text{s.t. } U := \{x \in \mathbb{R}^n \ : \ g(x) \leq 0\}, \tag{2.23}$$

where θ is a pointed convex cone with non-empty interior in $\mathbb{R}^k$.

Numerous optimality conditions obtained for scalar invex functions may be generalized for the optimization of vector invex functions. In particular, the Theorem 2.1 has been generalized by Tanaka, Fukushima and Ibaraki (1989) to yield the following:

Theorem 2.3. *The regular function $f \in C(I\!R^n, I\!R)$ is invex iff every point x_0 such that $0 \in \partial f(x_0)$, where $\partial f(x_0)$ is the generalized gradient of f at x_0, is a global Pareto minimum of f.* ■

Hanson (1981) proved that the convexity assumption for f and g in the Kuhn-Tucker conditions may be replaced by the invexity of f and g. Let us recall the following notation :

$$I(x) := \{i \in \{1, ..m\} \ : \ g(x) = 0\},$$

and the Slater's constraint qualification :

$$\exists x \in U : g(x) < 0. \tag{2.24}$$

Let us formulate now the well-known Kuhn-Tucker theorem for the scalar problem (2.23) (i.e. for $k = 1$) and invex f :

Theorem 2.4. *Suppose that the functions f and g in problem (2.23) are invex with respect to the same $\eta(x, y)$, and there exist $x_0 \in U$, and $\phi = (\phi_1, ...\phi_m) \in I\!R^m_+$, such that*

$$\delta f(x_0) + \sum_{i \in I(x_0)} \phi_i \delta g_i(x_0) = 0 \tag{2.25}$$

and

$$\phi^T g(x_0) = 0, \tag{2.26}$$

Then x_0 is the global minimum of f. ■

Recall that the fulfillment of the conditions (2.25)-(2.26) is always necessary for the optimality of x_0 .

Another field of vector optimization, where the invexity and other generalized convexity notions proved especially useful, is the multiobjective duality theory. Since this subject is rather loosely connected with the scope of the present monograph, the reader is referred e.g. to Sawaragi, Nakayama, Tanino (1985), Weir (1988b), Egudo, Weir, and Mond (1992), and others.

Chapter 3

A Brief Introduction to Multicriteria Optimization and Multicriteria Decision Theory

During the last two decades the development of the multicriteria optimization methods has been one of the most significant within the theory of Operations Research. The term "multicriteria", as well as its synonym terms, such as the "multiobjective" or "vector" optimization, indicate the fact that the basic problem consists in finding the set of optimal solutions with respect to the vector index of quality F, defined on a set of all admissible strategies U.

In the case of scalar criteria the notion, "optimal element" is unambiguous, and it is understood as an argument of optimum (minimum or maximum) of the criterion function on the set of admissible decisions. For the multicriteria optimization, however, the notion of optimality should be defined more precisely since the elements of the decision space which optimize one of the criteria need not optimize the others. In the sequel, we will consider only those approaches to the solution of multicriteria problems which lead to the choice of so-called nondominated solutions. The selection of problems presented in this introductory section does not pretend to give a complete presentation of the theory of multicriteria optimization; its aim is to provide only the basic information necessary to understand the further chapters. More detailed information can be found in papers mentioned in the References section. For more complete information, especially regarding purely theoretical questions, and the other approaches to multicriteria decision-making see e.g. the books and expository papers of Hwang and Yoon (1981), White (1982), Haimes and Changkong (1983), Sawaragi, Nakayama and Tanino (1985), Luc (1989), Stadler and Dauer and Stadler (1979,1986), Yu (1985), Steuer (1986), Liebermann (1988), Sakawa and Seo (1993), and others.

3.1 A General Characterization of the Scope and Methodology of Multicriteria Decision Analysis

Numerous practical questions concerning almost any kind of human activity can be mathematically formulated as multicriteria optimization problems. For instance, the problem of choosing one from many possible realizations of the technical projects can be reduced to the simultaneous minimization of cost and time of realization of the project. In general, these criteria may be conflicting, i.e. to reduce the time of realization we have to increase the costs. Thus we have to concentrate on the necessary compromise. From the alternatives taken into our consideration we should first eliminate those which are at the same time more expensive and longer lasting than an other alternative.

In nontrivial multicriteria problems (i.e. those which cannot be immediately reduced to the optimization of a scalar function) the criteria are non-comparable and conflicting in the sense that when considered as N scalar functions, their separate optima are achieved at different points of the set of all admissible decisions. Hence, it is usually impossible to choose a solution which is optimal with respect to all criteria at the same time. We can distinguish, however, all elements of the decision set for which it is impossible to improve the value of the fixed criterion without worsening some of the others. Such points are called nondominated (also: cone optimal, Pareto optimal, efficient). The set of all nondominated decisions is called the nondominated set (or efficient, or Pareto set). In practice, the set of all nondominated values of criteria, called also the set of compromises, is more frequently used while selecting a compromise decision than the set of nondominated decisions in the decision space because of its direct relation to the selection objectives.

The knowledge of the set of all nondominated solutions gives us the possibility of choosing the most satisfactory solutions based upon the supplementary information inaccessible at the stage of formulation of the problem. On the other hand, however, this set can rarely be defined analytically, and its approximation obtained numerically requires much computational effort, for a higher dimensional criteria space, and it is often hard to be presented to the decision-maker in a convenient way.

The second principal question of multicriteria decision analysis following the formulation of the multicriteria optimization problem is how to choose a nondominated solution based upon the additional information about the utility of particular solutions. One of the most intuitive approaches is the method which consists in defining in goal space a point, or a subset consisting of those points, whose coordinates are equal to the desired values of criteria. Then, one is looking for a solution which is closest to this point or subset, in the sense of certain proximity measure defined in the criteria space. This group of methods originated from the early multicriteria decision-making approaches using a single target point and a distance function as the proximity measure. The above-mentioned idea has been referred to with the common name of compromise programming (Zeleny, 1973; Yu, 1973; Yu and Leitmann 1974a), goal programming - for the special case where the l_1 norm has been used as the distance in the criteria space (Charnes and Cooper, 1977), or the method of "point de mire" (Roy, 1975). These approaches are closely connected to the scalarization technique called norm scalarization (Rolewicz, 1975), or distance scalarization, or to the reference point method (Wierzbicki, 1977, 1980). A generalization of this approach is the main subject of this monograph. Some other most common methods of the multicriteria choice will be discussed in the sequel.

3.2 Solutions to the Multicriteria Problems

Some ideas on solving the multicriteria optimization had been presented for the first time in the late XIX century independently by Vilfredo Pareto (1896), an Italian mathematician and economist, and Edgeworth (1881). However, only the work of Pareto has been widely known until recently, hence the term "Pareto optimality" became the standard. Below we present remarks concerning methods of solution of the multicriteria problems. Those remarks are referred only to the decision situations in which the optimal solution in the sense of Pareto can be reached. We can distinguish the following stages of solving such problems:

a) choice of the mathematical formulation of the problem; it contains the definition of the decision variables and indices of quality, taking into account constraints in the control space;

b) checking the well-posedness of the problem : existence of the nondominated solutions and properties of the set of all compromise solutions;

c) approximation of the set of compromise solutions and choice of the solution from this set based on the information on the position of all nondominated points;

or alternatively:

c') applying the additional preference information of the decision-maker during an interactive process of choosing the solution from among the alternatives proposed by the experts (the experts may be replaced by the suitably programmed computer application),

c") choice of the method which replaces the original multicriteria problem by the scalar optimization problem which solution coincides with a nondominated solution to the original problem.

Now we will briefly discuss the above-mentioned questions providing the necessary references. Similarly as above, in the mathematical formulation of a multicriteria optimization problem, we will distinguish four main substages:

– definition of the decision space (continuous or discrete, inclusion of the system dynamics, etc.);

– choice of the optimization criteria;

– definition of constraints;

– checking the consistency of the optimization problem so arisen.

As for the choice of the mathematical description, we want to stress that the systematic study of the problem started relatively a short time ago. The methodology of construction of optimization criteria was presented in books written by Germeier (1976), Zeleny (1982), Roy (1985), and in the paper of Zionts (1987). The problems concerning the definition and structure of constraints for multicriteria problems has been considered by the author in an earlier paper (1987) and in Chapter 7 of this monograph. Namely, a suitable formulation of the problem plays a decisive role for the possibility of the subsequent numerical solution of practical multicriteria problems, in particular, in the optimization of large economical systems. Observe that the satisfaction of the set of constraints put on the values of the criteria is superior with respect to the optimality of the solution. Hence, after the formulation of the problem one should take into consideration the duality between criteria and constraints. In the case of constraints which cannot be satisfied, the distance between the admissible values of criteria and the set determined by the inaccessible constraints can serve as a measure of satisfaction of demands.

It may happen that a correctly formulated (in the formal sense) problem does not admit any solutions. This is an illustration of the necessity of checking the existence of nondominated points for the problem just formulated. The existence theory plays an important role in multicriteria optimization, especially for the dynamic problems of which criterion functions are defined in an implicit form (cf. Cesari and Suryanarayana, 1978; Olech, 1969). If the set of scalar criteria $\{F_1, ..., F_\omega\}$ is closed, it suffices to verify whether there exists a minimum of each one separately. It is worth to note that the set of all compromises can be nonempty even though there does not exist any minimum of each coordinate F of the goal function (cf. e.g. Wierzbicki, 1977). In such cases, however, the set of compromises is unbounded, which makes the use of the scalarization methods more difficult or even impossible.

Except for the discrete optimization problems (see Kung, Luccio, and Preparata, 1975), it is seldom possible to find the approximation of the set of all nondominated solutions or the set of compromises using only an analytic method, though this is often desired for further stages of the decision-making process. On the other hand, the use of numerical methods may be time-consuming, but if it is justified, we may admit one of the following possibilities:

a) approximation of the set of compromises by the sets of compromises found for a sequence of subsets of the set of admissible decisions,

b) making use of a suitable scalarization procedure, one can find a number of nondominated points such that the precision of the approximation of the whole set of nondominated points were sufficiently small.

The key problem in finding an approximation of the set of nondominated points is its numerical stability (Armann, 1988), implied by the fact that small perturbations of the attainable set may result in discontinuous jumps of the Pareto surface. This problem is addressed also in Skulimowski (1985c).

The visualization of attainable sets for linear problems may be based on the explicit knowledge of nondominated facets of the attainable polyhedron (see Winkels, 1983; Korhonen and Laakso, 1986; Korhonen, 1987, 1988). To obtain a convenient graphic visualization for nonlinear problems one can apply an approximation in the smooth form, which can be achieved by means of spline approximation for either one of the both above discrete approximation approaches (cf. Polak, 1976; Jahn and Merkel, 1992).

The problem of determining the set of all nondominated criteria values is relevant not only with regard to estimating the potential domain of a concrete decision to be made, but it is also relevant for studying the properties of solutions sets of vector optimization problems. Therefore the very question is not only of great practical importance but of theoretical importance as well.

3.3 Multicriteria Optimization, an Outlook

As it has already been mentioned earlier in this Chapter, the current state of research on the particular classes of multicriteria optimization problems is not uniformly advanced, imitating to some extent the hitherto development of scalar optimization. For instance, the problems with linear goal functions and linear constraints have been exhaustively studied (c.f. Yu and Zeleny, 1975; Słowiński, 1984ab; Dauer, 1987; Armand and Malivert, 1991; Armand, 1993, and others), compared to the theory of nonconvex problems which was initiated by the paper on local Pareto optima of Wan (1975), but still possesses many fundamental open problems. Another disproportion is connected with the number of criteria considered while the theory and algorithms of the bicriteria optimization are well elaborated (cf. Gearhart, 1979; Payne and others (1975); the optimization with the infinite-dimensional goal space has not been sufficiently worked out yet. Further, the study of stochastic multicriteria problems is still at its initial stage, concentrating on problems related with the stochastic or probability dominance (cf. Wrather and Yu, 1982), which may be especially useful in portfolio theory. The relations between stochastic and multicriteria optimization constitute still a challenging open problem.

It is convenient to distinguish the class of dynamic multicriteria problems with the discrete set of alternatives, so-called multistage decision processes, because of usually different methods of solving them. Namely, the methods used there are based, as a rule, on the generalization of the Bellman optimality principle to so-called multicriteria dynamic programming, employing the shortest-path techniques. Such problems were studied by Villareal and Karwan, 1982; Henig, 1983,1985; and many others. Another new class of multicriteria problems based on the generalized dynamic programming is the optimal control of discrete-event systems defined in the paper of Skulimowski (1991). These problems are studied in more detail in Chapter 8.

The methods of the game theory (cf. Haurie, 1973; Germeier, 1976; Tołwiński, 1986; Haurie and Tołwiński, 1990; Wierzbicki, 1988; and others), especially the methods of the cooperative many-person games are closely related to the multicriteria optimization. The idea of Pareto-equilibrium, which is crucial in the theory of cooperative games, is nothing else but the nondominated solution to the multicriteria problem which arises when considering the criteria of all players jointly.

The development of the multicriteria optimization is a part of the deeper process which consists of extending the domain of the mathematical programming applications to the optimization of economical, social and biological processes. These problems demand new methods of solving, which take into consideration such features as complexity, multipolarity, conflict of interest and autonomy of subsystems. This development is stimulated by the continuous increase of the degree of complication of the technical systems and production processes controlled with respect to many performance criteria and has been made possible by the progress in computer technology, which provides the tools for the effective solving the practical multicriteria decision problems.

3.3.1 Scalarization methods

The *scalarization* of multicriteria problems consists of replacing the vector goal function by the parameterized family of scalar functions of which optima are nondominated for the original multicriteria problem. By the scalarization problem we mean the solution of a scalar problem chosen from a given family of scalarizing functions, as well as the process of choice itself. The choice of the parameter is equivalent to the choice of the solution from the set of all nondominated solutions. The scalarizing function chosen may play a role of the value, or utility function for the original multicriteria problem. Sometimes, by a solution to the scalarization problem one means all solutions to the parameterized scalar problems and the arising approximation of the nondominated set.

In this subsection we will show how to use use a scalarization method to obtain an inner approximation of the set of all nondominated control. Specifically, we will use the arguments of minima in the optimization of the family of functionals to parameterize a subset of the set $P(U_d, \theta)$.

The most frequently used scalarization method for the problem

$$(F : U_d \to I\!R^n) \to \min(I\!R^n_+) \tag{3.1}$$

is the positive convex combination of the criteria

$$F_x(u) := \sum_{i=1}^{n} x_i F_i(u), \tag{3.2}$$

where $u \in U_d, x_i > 0$ for $1 \le i \le n$ and $\sum_{i=1}^{n} x_i = 1$. The coefficients $x_i, 1 \le i \le n$, are called weights or prices and refer to the particular coordinates of the criterion function. Every solution u_x of the problem $F_x \to \min$ is obviously nondominated for the problem $F \to \min(I\!R^n_+)$, since if there is a solution $w \in P(U_d, I\!R^n_+)$ dominating u_x , i.e. such that

$$F_i(w) \le F_i(u_x) \text{ for } 1 \le i \le n$$

and

$$F_j(w) < F_j(u_x) \text{ for some } j,$$

then

$$\sum_{i=1}^{n} x_i F_i(w) < \sum_{i=1}^{n} x_i F_i(u_x),$$

which contradicts the minimality of u_x.

Because of the form of the scalarizing function, which is constructed similarly to the Lagrange function, the above-described method is called sometimes *the method of the Lagrange multipliers.*

This is why for solving relevant classes of multicriteria optimization problems, it is more convenient to use other scalarization methods employing non-linear functions, which aggregate the coordinates of the goal function. Among them the most widespread are functions based upon the use of a distance or a norm in the goal space.

Methods of scalarization by distance are those for which the scalarizing functions have the form

$$F_d(u) := d(q, F(u)), \tag{3.3}$$

where d is a metric in the goal space. This requires that the goal space must be metric or metrizable, which is true for all problems considered in this monograph. If q is a fixed unattainable element of the goal space which dominates at least one point from $F(U_d)$, then (under some additional assumptions) the point closest to q in the above metric is nondominated for the former multicriteria optimization. In this case the scalarization consists of the minimization of the function F_d defined for the family of metrics. If the point q is attainable, however, then a nondominated solution can be obtained by the maximization of the function F_d in the set of all attainable points which dominate q. In the space of type L_p, both finite- and infinite-dimensional, the metric scalarizing function may be defined equivalently as the p-th power of the norm, i.e.

$$F_p(u) := \| q - F(u) \|_p^p.$$

We will call this method the *scalarization by norm.* For instance, as the scalarizing family for a finite-dimensional multicriteria optimization problem with respect to the natural partial order in $I\!R^n$ we may apply the family of functionals of the form

$$N_p(u, w) := \sum_{i=1}^{n} w_i(F_i(u)) - q_i)^p, \ w \in I\!R^n_+ \setminus \{0\}, \ 1 \le p < \infty, \tag{3.4}$$

and restrict the set of compromise solutions considered to those elements which minimize one of the functions $N_p(u, w)$.
The norm and distance scalarization techniques play an essential role while using the reference set approach presented in the further Chapters.

The third family of the most commonly used scalarization techniques constitutes the so-called scalarization by criterion function levels. The original problem (3.1) is there replaced by the parameterized problem

$$F_i(u) \to \min \ s.t. \ F_j(u) \le x_j, \ \text{for fixed } i \in \{1, ..n\}, \ j = 1, ..i-1, i+1, ..n, \tag{3.5}$$

where $x_j, j = 1, ..i-1, i+1, ..n$, are certain values selected from the range of the function F_j . The main problem arising while using this type of scalarizing functions consists in the fact that the minimizing solutions to (3.5) may be merely weakly nondominated. More information about this method may be found in White (1985) and in any of the monographs listed at the beginning of this Chapter.

3.3.2 Sufficient Conditions for the Pareto optimality

Now let us formulate the analogue of the Theorem 2.1 for the optimization of vector functions.

Theorem 3.1 *Assume that for certain $x_0 \in U$ f and g_i, $i \in I(x_0)$ are invex with respect to the same η, and the Slater constraints classification (2.21) for the problem (2.20) holds. Then x_0 is a global proper θ-minimal point of f iff*

$$\sum_{j=1}^{k} \lambda_j \nabla f_j(x_0) + \sum_{i \in I(x_0)} \phi_i \nabla g_i(x_0) = 0 \tag{3.6}$$

for certain $\lambda \in int(\theta^*), \phi \in I\!R^m_+$.

The Slater's condition in Theorem 3.1, as well as the differentiability requirements, may be omitted, assuming that

Theorem 3.2 *(Egudo, Weir, Mond; 1992). The point $x_0 \in U$ is nondominated for (2.20) iff there exist $\lambda \in I\!R^k$ and $\phi \ge 0$ such that*

$$\sum_{j=1}^{k} \lambda_j \partial f_j(x_0) \quad + \quad \sum_{i \in Q(x_0)} \phi_i \partial g_i(x_0) \subset [S(x_0)]^*, \tag{3.7}$$

where

$$Q(x_0) := \{i \in [1:m] : g_i(x_0) = 0 \text{ and } \exists u \in U \text{ such that } g_i(u) < 0\},$$

and

$$S(x_0) := \bigcap_{j \in J(x_0)} \{\rho \in I\!R^n : \exists T_j > 0 \text{ such that } \forall t \in (0, T_j]\ f_i(x_0 + t\rho) = f_i(x_0)\}$$

with

$$J(x_0) := \bigcup_{1 \le i \le k} J_i(x_0) \cap \{j \in [1:m] : \forall u \in U \ \ g_j(u) = 0\},$$

$$J_i(x_0) := \{q \in [1:k] \setminus \{i\} : \forall u \in U([\forall j, j \neq i,\ f_j(u) \le f_j(x_0)] \Rightarrow f_q(u) \le f_q(x_0))\}. \ \blacksquare$$

A rationale for the above construction of index sets, which replaces the usual constraint qualification conditions, can be found in the paper of Egudo, Weir, Mond (1992).

Another field of vector optimization where the invexity and other generalized convexity notions proved especially useful is the multiobjective duality theory. Since this subject is rather loosely connected with the scope of the present monograph, the reader is referred e.g. to Sawaragi, Nakayama, Tanino (1985), Weir (1988b), Egudo, Weir, and Mond (1992), and others.

3.4 The Additional Preference Information

While selecting a compromise solution to the multicriteria optimization problems, we have usually at our disposal some additional information about our preferences, i.e. we have to admit a priori certain preference model (cf. Vincke, 1982).

In the usual decision situations the choice procedure to be performed based on a preference model consists of the following steps:

– considering all possible constraints on the values of the criteria;
– finding the arguments of minimum of the scalar utility function defined in the goal space,

or

– establishing the importance hierarchy of all criteria and then iterative performing of some single-criteria optimization procedures in the set of decisions obtained as solutions of the previous optimization step and successively restricted.

The latter approach is called the lexicographical optimization method (see e.g. Waltz, 1967; Podinovskij and Gavrilov, 1975).

3.4.1 Utility and value functions

In this subsection we will discuss in more detail the questions connected with the analysis of *utility* and *value functions*.

The idea of the analysis of utility for solutions of the multicriteria optimization consists in linear ordering the nondominated solutions according to an estimation of an ordering function from an implicitly given or incomplete preference structure in the goal space (cf. Arrow, 1958; Debreu, 1959; Fishburn, 1970). In their famous monograph (1976), Keeney and Raiffa consider the situation where to every admissible value $x = (x_1, ..., x_n)$ of the vector criterion F there correspond some known prices $a_{ij}(x)$ of increase in the i-th criterion at the cost of the j-th one. This procedure serves as a base for the analysis of so-called multiattribute utility functions.

The multiattribute utility theory bases on the assumption that to any element y of the criteria space E there may be assigned a real number $v(y)$ denoting the utility of the criteria values $x = (F_1(u), ...F_N(u))$. To comply with the vector minimization problem (3.1), and to preserve the intuitive meaning of distance minimization, here we will assume that the smaller values of v correspond to the higher utility of its arguments. The function v is called the *utility* or *value function* (see e.g. Keeney and Raiffa, 1976; Fishburn, 1970; or Barron, von Winterfeldt and Fischer, 1984, for a detailed presentation of the theory and terminology).

By the well-known theorem due to Debreu (1959), for every partial order $\leq_E$ in the goal space E there exists a scalar function v defined on E such that the relation

$$x <_v y \Leftrightarrow v(x) \leq v(y) \tag{3.8}$$

is a linear order relation in E which is consistent with the partial order "$\leq_E$". This means, by definition, that the relation $<_v$ holds for all $x, y \in E$ such that $x \leq y$. Thus the choice of the solution of the multicriteria problem is reduced to the questions of finding an extremum of the function v in the set $F(U_d)$.

In the sequel a utility or value function v will be assumed order representing (cf. Wierzbicki, 1986), i.e.

$$\forall x, y \in E (x \leq_\theta y \Rightarrow v(x) \leq v(y)). \tag{3.9}$$

Consequently, the minimal value of v can be achieved only on the nondominated set $FP(U_d)$ and determines the best-compromise solution to multicriteria optimization problems. The above property assures the compliance with the original multicriteria problem, i.e. it is guaranteed that the extremum of the estimated function v is reached at a certain nondominated point of $F(U_d)$. In practice, this condition may sometimes be not satisfied; in such cases one has to restrict the optimization of v to the set $FP(U_d)$ or to admit an assumption of the priority of the optimization of the function v over the optimization of the vector criterion F.

The main difficulty in the multicriteria decision-making consists in the fact that v is unknown, or difficult to formalize. To estimate v one should use all available information concerning the preferences of the specific values in the criteria space, gained usually during a dialog with the decision-maker (see e.g. von Nitzsch and Weber, 1988, who propose an interactive utility assessment algorithm for a personal computer). Nevertheless, the information on hand is often insufficient to evaluate the global estimate of v, nor is such estimate necessary to select a single compromise alternative. One is usually satisfied with a local estimate of which global minimum on $FP(U_d)$ coincides with the minimum of the hypothetical global estimate.

Numerous methods exist to estimate, explicitly or implicitly, the utility function and select thus a compromise solution (see e.g. Bouyssou, 1984; Krzysztofowicz and Koch, 1989). As further examples of functions of utility may serve various functions aggregating the criteria $F_i, 1 \leq i \leq n$, and also the function of the distance from a fixed point $q \in E$ or a set Q contained in the space of values of the criteria (Krzysztofowicz, 1986). The positive linear combination of criteria (cf. the Lagrange function (3.2)),

$$G(u) := \sum_{i=1}^{n} w_i F_i(u), \text{ with } w_i > 0 \text{ for } i = 1, ...n,$$

is a particularly important type of the utility function. Let us remark that most interactive algorithms of multicriteria optimization are based on the consideration of local preferences of the decision-maker in the strictly defined way which corresponds to the rule of constructing the utility function (cf. Dyer, 1972; Geoffrion, Dyer and Feinberg, 1973). There exists a close relation between the construction of a class of value functions satisfying (3.7) and scalarization methods.

Summarizing, the multicriteria decision making problem for (3.1) consists in finding the value function v and solving the minimization problem

$$(v : F(U) \to I\!R) \to \min, \tag{3.10}$$

while from (3.7) it follows that arg $\min\{v(x) : x \in F(U)\} \subset FP(U)$.

The level sets of v in E are the equivalence classes of the *indifference relation* in E. By definition, we say that $x, y \in E$ are indifferent with respect to v, and denote it by xIy iff $v(x) = v(y)$. Hence, if the problem (3.9) admits non-unique solutions, then they all are mutually indifferent.

The function v discussed above is always a scalar value (or utility) function. The vector-valued utility and value functions have been considered by Skulimowski (1985c). It follows from the above paper that the analysis of vector utility (or value) functions is equivalent to multistage multicriteria decision processes.

The utility or value function analysis plays a crucial role for the principal subject of this monograph, i.e. for the design of the decision support systems using reference set and multiple reference points approaches, where to the reference points in the criteria space one associates the values of certain utility function. Besides of the approach presented by Górecki and Skulimowski (1986), this problem has been preliminarily considered by Troutt (1988), Ballestero and Romero (1991), and Paolucci and Pesenti (1992). It refers additionally to the notion of preference strength or intensity (Cook and Kress, 1986; Farquhar and Keller, 1989), allowing to avoid its conceptual difficulties.

3.5 Multicriteria Optimal Control and Trajectory Optimization

In this section we present the formulations of multicriteria optimal control and trajectory optimization problems for systems of which dynamics is described by the difference or differential equations, and some related problems relevant for the subject of this book.

3.5.1 Multicriteria optimal control

A general formulation of the multicriteria optimal control problem for dynamical system described by difference equations may be presented as follows:

$$x(t+1) = f(x(t), u(t)), \tag{3.11}$$

with

$$u(t) \in U(t) \text{ for } t_0 \leq t \leq T,\ \ x_0 \in X_0, \tag{3.12}$$

where $x(t) \in I\!R^n$,$u(t) \in I\!R^k$, and the constraint set $U(t)$ is a closed and convex subset of $I\!R^k$ for $t_0 \leq t \leq T$. The classical multicriteria optimization task for the above system consists in simultaneous minimization of the functions

$$J(u) := (J_1(u), ..., J_N(u)) : I\!R^{k(T+1)} \to I\!R^N \tag{3.13}$$

for the terminal time T. This formulation usually leads to finding a Pareto-optimal control u_{opt} - a sequence of decisions $(u_{t_0}, u_{t_0+1}, ..., u_T)$ for the corresponding moments of time, satisfying some additional conditions resulting from the consideration of the decision-maker's preferences which were not included in the preliminary problem formulation.

The solution of the multicriteria optimal control problems described by the ordinary differential equations of the form

$$\dot{x}= f(x, t, u),\ t_0 \leq t \leq T,\ x(t) \in I\!R^n, \tag{3.14}$$

s.t.

$$u(t) \in U(t) \subset I\!R^k$$

$$x(t_0) := x_0 \in X_0,$$

bases on solving the suitably chosen scalarization problems for the simultaneously minimized criteria

$$(J_1(x, t, u), ..., J_N(x, t, u)) \to \min . \tag{3.15}$$

Each scalar problem is an optimal control problem with the aggregated criterion function $h(J_1(x,T,u),...,J_N(x,T,u),\lambda)$, where $\lambda \in I\!R^{N-1}$ is the scalarization parameter. If the criteria (3.13) are assumed to be absolutely continuous functions of x, t, and u, it is possible to regard them as additional state variables in the scalar optimal control problem $J_1(x,t,u) \to \min$ by setting

$$\dot{x}_{n+1} = \frac{d}{dt}J_2(x,t,u), ..., \dot{x}_{n+N-1} = \frac{d}{dt}J_N(x,t,u). \tag{3.16}$$

This technique is closely related to the scalarization by objective function levels for the static problems. It is particularly useful when the criteria $J_1, ...J_N$ are integrals of certain known functions.

To select a compromise trajectory for both types of problems, continuous and discrete, one can apply the here-presented multiple reference point method and distance scalarization approach (cf. the following Chapters 4-5)). The choice of the multiple reference points method to solve the dynamical problems may be justified by its great flexibility and case of adaptability for the time-dependent objectives. However, several particular problems which do not occur in static vector optimization, of the theoretical as well as of computational nature, had to be additionally solved.

In general, it is possible to consider the values of criteria either as trajectory objectives on an interval $[t_1, T]$, with $t_0 \leq t_1 \leq T$, or as the vector of the final values of J at the terminal time T. The minimal-time problems (i.e. the time t is one of the objectives to be minimized) can be considered within the same methodological framework, as well as the free-time and infinite-time problems.

3.5.2 The aggregation of preference information for different moments of time

As we have already mentioned, the decision-maker may be interested in achieving certain intermediate levels of objectives before the termination of the optimized process. Such preference information may be expressed as the reference points defined for different moments of time from the interval $[t_1, T]$, or as the reference trajectories defined as the functions $q_i(t)$ for all $t \in [t_1, T]$. Roughly speaking, the difficulty arising while considering the reference points defined on a discrete subset of $[t_1, T]$ consists in finding the appropriate approximated trajectories, while the reference trajectories should be fragmented according to their situation with respect to the attainable criteria set at time t. Consequently, they may be interpreted as avoidable or approached values. Aspiration levels in a dynamic setting have been also considered by Östermark (1988). Another method for analyzing the time preferences applying a sequence of utility functions has been presented by Streufert (1993) for an economic choice problem.

Multicriteria decision support based on similar principles as the above-sketched reference trajectory approach has been also applied to the optimal control of asynchronous discrete-event systems with a finite number of states (cf. Skulimowski, 1991).

3.5.3 Approximation of the set of nondominated criteria values

To approximate the set of nondominated points in the continuous optimal control problems we derived a stochastic approximation technique (Skulimowski et al., 1988) consisting of random generation of sample values of criteria using the bang-bang extremals, finding their nondominated subset, and refining the approximation until the assumed density (max-min of distance between single points) is obtained. Finally, we use a spline approximation to obtain a continuous lower bound of the discrete Pareto set obtained at the previous stage. In the above-cited report we pointed out several properties of the above method; specifically, we proved that the sequence of discrete approximations is convergent to the set of weakly nondominated points while the spline surfaces actually approximate the Pareto set.

3.5.4 The output trajectory optimization problem

In contradistinction to the above classical formulation of the multicriteria optimal control problem (3.10)-(3.11) or (3.12)-(3.14), a trajectory optimization problem (see Wierzbicki, 1980) involves the consideration of the values of $F(x(t), u(t))$ for the intermediate values of t, in the sense that they should be optimal for each moment of time. Thus this problem is transformed, in fact, into the multicriteria problem with a number of objectives equal to the number of time moments considered. The additional preference information may also be given in form of conditions concerning a subinterval of $[t_0, T]$, as e.g.

$$F(u(t)) \in Q(t) \text{ for } t \in [t_1, T], \tag{3.17}$$

where Q is so-called reference multifunction for (3.11)-(3.12), and $t_1 \in [t_0, T]$. An approach to solving a trajectory optimization problem, which has been admitted for the management of an artificial atmosphere and for multi-stage portfolio optimization (Skulimowski, 1982; 1994), consists in finding a finite sequence of time instances $t \in [0, T], i = 1, ...k$, such that the original problem is equivalent to the multicriteria optimization problem with the objectives being the momentary values of $F(x(t), u(t))$ for $t := t_i, i = 1, ..k$. These real-life cases led to formulate the following general problem : given a trajectory optimization problem for an ordinary differential equation (in $I\!R^n$), find the sequence of time moments $t_i \in [0, T], i = 1, ..k$, such that the original problem can be replaced by a multicriteria optimization problem with the objectives being the momentary values of the output trajectory for $t_i, i = 1, ..k$. Simple evidence from the theory of differential operators points out that finite sets of "characteristic points" exist for linear stationary equations and a class of output trajectories (Skulimowski, 1995), but the general problem remains open.

3.6 Outranking Methods and Discrete Choice Models

For most practitioners, especially from the areas of management and economic theory, multicriteria decision analysis means a choice from a discrete set of alternatives characterized by multiple attributes, i.e. $U = \{u_1, ...u_k\}$, $F(u_i) := f_i$,$f_i = (f_{i1}, ..., f_{iN})$, for $i = 1, ...k$. A number of methods exists to solve such problems (cf. e.g. D'Avignon and Vincke, 1988; Oral and Kettani, 1990). The emphasis is put here not on the use of multicriteria optimization techniques, as the difficulty does not consist in a large number of alternatives. Instead, one is trying to apply the additional information about preferences gained usually during an interactive procedure to establish a ranking of alternatives (therefrom the origin of the term *outranking*). This means that although the choice problem may consist in finding a single best-compromise alternative, this group of methods usually allows also to find a second-best, and next-best compromise solution, as well as to select an optimal subset of the set $P(U)$.

Two main approaches found a widespread application to solve real-life problems. The first is based on the seminal work of Roy (1972), who proposed the representation of preference relations in a digraph and derived the method called ELECTRE to find the kernel of the preference relation, being at the same time the subset of best compromises. The subsequent versions of this method, up to ELECTRE VII, used different additional preference information, and allowed for a larger class of discrete choice problems to be analyzed. Many other outranking methods, although using different preference information and developed for other purposes, apply the general ideas proposed by Roy. The second approach, the so-called analytic hierarchy process is due to Saaty (cf. e.g. Saaty, 1987, 1990; or the collective work edited by Golden, Vasil and Harker, 1989). It is sometimes referred to (although imprecisely) as the pairwise comparison method since the ranking of alternatives is derived based on comparing pairwise the alternatives at the first step, and deriving the relative importance weights $w_1, ...w_N$ for all attributes as the eigenvalues of the comparison matrix at the second decision step. Then the final ranking is obtained by calculating the value of the Lagrange function

$$\phi(u_i) = \sum_{j=1}^{N} w_j f_{ij},$$

where $f_{ij} = f_j(u_i)$, for all alternatives u_i, and selecting that one which is characterized by the maximal value of ϕ.

Chapter 4

The Structure of Sets of Reference Points

Let $X \subset E$ be the set of values of the objective F in a vector optimization problem $(F : U \to E) \to min$, where the minimum of F is taken with respect to the partial order introduced by a convex cone θ such as formulated in Chapter 3. A dominating point with respect to X is an element y of $E \setminus (X + \theta)$ such that $(y+\theta) \cap X \neq \emptyset$. Our interest in dominating points is motivated by the role of reference points played by these points in multicriteria decision making. We will distinguish several classes of dominating points taking into account their relation with respect to $X = F(U)$. We will investigate the relations between the subsets of the class of dominating points, and their properties, paying a special attention to the case where $E = I\!R^k$ and to the classes of so–called ideal and strictly dominating points. An ideal point x^* for the vector optimization problem $(F : U \to E) \to min(\theta)$, where θ is a convex cone introducing the partial order in the criteria space E, is defined as a maximal element of

$$TD(F(U), \theta) := \{y \in E : \ F(U) \subset y + \theta\}.$$

If $E = I\!R^N$, and θ is the positive orthant then the coordinates of x^* can be calculated as the global minima of F_i, for $i = 1, ..N$. In this Chapter we will investigate the properties of ideal points from the point of view of their applicability in multicriteria decision aid. We will examine their existence and uniqueness, as well as the properties of totally dominating points and local ideal points defined by Skulimowski (1992). It turns out that the uniqueness of ideal points for all θ–bounded subsets can only be guaranteed in the criteria space ordered by a cone isomorphic to $I\!R^N_+$. Based on the properties of local ideal points, we propose a class of scalarization methods for non-convex vector optimization problems that use multiple reference points. An algorithm of verification to which subclass of dominating points an element of E belongs will also be given. Finally, we point out the consequences of the above results for the methodology of multicriteria decision analysis, in particular to the methods based on reference points and reference sets.

4.1 Preliminaries

Our considerations will refer to multicriteria decision – making ($MCDM$) for the vector optimization problems of the form

$$(F : U \to E) \to min(\theta), \tag{4.1}$$

where U is the set of admissible decisions, E is the space of criteria – a Banach space partially ordered by a closed, convex and pointed cone θ, and F is a vector objective to be minimized with respect to the partial order introduced by θ. In the sequel F and U will play no separate role as we will concentrate our attention on the set $X := F(U) \subset E$ and its relation to dominating reference points occurring in decision–making procedures.

To select a compromise solution for the problem of type (4.1) one may define reference points, i.e. the elements of the criteria space being of a special importance to the decision–maker. A subsequent consideration of information related to the reference points may consist in finding a solution closest or farthest to a reference point, depending on its interpretation as a vector of desirable, or non–desirable values of criteria. A more detailed study of the theory of reference points and their applications may be found in the following two Chapters, as well as in Górecki and Skulimowski (1986), and Skulimowski (1992).

In this Chapter we will study the structure of the sets of dominating points – a class of potential reference points which may then be used in distance minimization and other achievement procedures. Based on their relation with respect to X and on the scope of applications in decision–making procedures, we will distinguish partly, totally, and strictly dominating points. We will also discuss the properties of ideal points, local ideal points, and local partly dominating points which are subsets of totally and partly dominating points, respectively. We will prove several theorems concerning the properties of each class of points and their mutual relations. An algorithm of verification to which subclass of dominating points belongs an element of the criteria space will also be given. Finally, we will present the applications of the classification introduced to formulate a set of sufficient conditions of optimality in distance scalarization.

Before passing to the classification of dominating points we will collect some basic notions and definitions used in vector optimization. Let us recall that a subset θ of a linear space E is called a convex cone iff each positive linear combination of elements of θ belongs to θ, i.e. iff $\forall x, y \in \theta \;\; \forall s, t \in I\!R_+ \;\; sx + ty \in \theta$. Each convex cone θ defines the relation $x \leq_\theta y \Leftrightarrow y - x \in \theta$, the partial order $\leq_\theta$ in E.

If two points, x and y fulfill the above relation then we say that x *dominates* y, or that y is *dominated* by x. A cone $\theta \in E$ is called *non–degenerated* iff it contains a base of E. We say that θ is *pointed* iff θ is non–trivial and $\theta \cap (-\theta) = \{0\}$. An element y of X fulfilling the condition $(y - \theta) \cap X = \{y\}$ is called θ–minimal or nondominated in X. Thus, $(-\theta)$–minimal point is called θ–maximal. The set of all θ–minimal points in X is denoted by $P(X, \theta)$.

Now we will present several definitions related to dominating reference points.

Definition 4.1. *A dominating point with respect to X is an element y of $E \setminus (X+\theta)$ such that $(y+\theta) \cap \neq \emptyset$. The set of dominating points will be denoted by $D(X,\theta)$.* ■

Definition 4.2. *A point $x \in E$ such that*

$$X \subset x + \theta \tag{4.2}$$

is called a totally dominating point for X. The set of totally dominating points is denoted by $TD(X,\theta)$. ■

As we have already mentioned, the coordinates of ideal points in multicriteria optimization problems with $\theta = \mathbb{R}^N_+$, express the best values of criteria on the same set of admissible decisions calculated separately. Here we present a more abstract definition based on the notion of totally dominating points.

Definition 4.3. *Any $(-\theta)$–optimal element of $TD(X,\theta)$ is called an ideal point for X. The set of ideal points is denoted by $x^*(X,\theta)$.* ■

This notion is attractive because each solution of the distance scalarization problem

$$\inf\{\parallel x^* - F(q) \parallel : q \in U\},$$

is nondominated (even properly nondominated – see e.g. Jahn (1984)) for the vector optimization problem (4.1), provided that the norm $\parallel . \parallel$ in E is strongly monotonically increasing with respect to the partial order introduced by θ, i.e. if $0 \leq_\theta x \leq_\theta y$, $x \neq y$ implies $\parallel x \parallel < \parallel y \parallel$.

The notion of an ideal point, sometimes called also the utopia point, belongs to one of those most frequently used in multicriteria optimization, cf. e.g. the papers of Jahn (1984), Wierzbicki (1986), Yu (1973), Yu and Leitmann (1974b), or Zeleny (1973b). For multicriteria problems with the natural (coordinatewise) partial order in the criteria space $\mathbb{R}^N$ the coordinates of the ideal point x^* are defined as the global infima of the criterion functions F_i, for $i = 1, ..N$, $F = (F_1, \ldots F_N)$. Therefore they are uniquely determined if the set of values of each F_i is bounded from below. In a non–trivial multicriteria decision–making problem the ideal point does not belong to the set of admissible values of criteria, $F(U)$, since in such a case the set of nondominated points would be reduced to x^*. Instead, the elements of the criteria space with the coordinates equal or better than those of x^* – i. e. the above defined totally dominating points – serve often as reference points in various multicriteria decision–making algorithms. As a result of a numerical procedure calculating the minima of non–convex criterion functions one may also get a set of points of the criteria space associated to the local minima of F_i. These points, called here the local ideal points, may play an important role in improving the accuracy of the nondominated set approximation using distance scalarization techniques.

Besides of totally dominating and ideal points we will distinguish the class of partly dominating points.

Definition 4.4. *A point $y \in E$ such that $(y + \theta) \cap P(X, \theta) \neq \emptyset$ will be called a partly dominating point for X. The set of partly dominating points will be denoted by $PD(X, \theta)$.* ■

Remark that previous results regarding the use of reference points in MCDM methods based on distance–minimization touched upon the ideal, or totally dominating points for X, with only some corollaries regarding partly dominating points. Here, we will additionally study the properties of a new class of dominating points, called strictly dominating.

Definition 4.5. *An element x of E is called a strictly dominating point for X iff $x \in PD(X, \theta)$ and*

$$P((x + \theta) \cap X, \theta) = P(X, \theta) \cap (x + \theta). \tag{4.3}$$

The set of strictly dominating points for x will be denoted by $SD(X, \theta)$. ■

Observe that the inclusion "$\subset$" is always satisfied and the condition (4.3) means that no point of $P(X, \theta) \cap (x + \theta)$ is dominated by another attainable point. The condition $x \in PD(X, \theta)$ assures that none side of (4.3) may be empty.

A motivation for introducing this condition may be explained by a wish to ensure that the set $(x + \theta) \cap X$ does not contain new nondominated points created by the constraints $z \leq_\theta x,\ z \in X$. The notion of strictly dominating points, plays an important role in distance scalarization as well as in the theory of local ideal points presented in Sec. 4.4. The reader is referred to the next Chapter for more details concerning distance minimization with respect to reference points and reference sets.

Throughout this Chapter we will assume that X is *θ–closed* and *θ–complete*, i.e. the set $X + \theta$ is assumed closed and

$$\forall x \in X \ \ \exists y \in P(X, \theta) : \ y \leq_\theta x. \tag{4.4}$$

To guarantee the existence of totally dominating points we will make the assumption that the set X is *θ–bounded*, by definition it means that there exists a point $y \in E$ such that $X \subset y + \theta$. Directly from this definition it follows that X is θ–bounded iff $TD(X, \theta)$ is non–empty.

The properties of dominating points will be discussed in a more detailed way in Secs. 4.5 and 4.6 Now let us make the following simple observation.

Proposition 4.1. *If X is θ–bounded and $P(X, \theta) \neq \emptyset$ then*

$$TD(X, \theta) \subset SD(X, \theta) \subset PD(X, \theta) \subset D(X, \theta). \tag{4.5}$$

■

An example of sets of totally, strictly, and partly dominating points for a bicriteria optimization problem with the natural partial order is shown in Fig. 4.1.

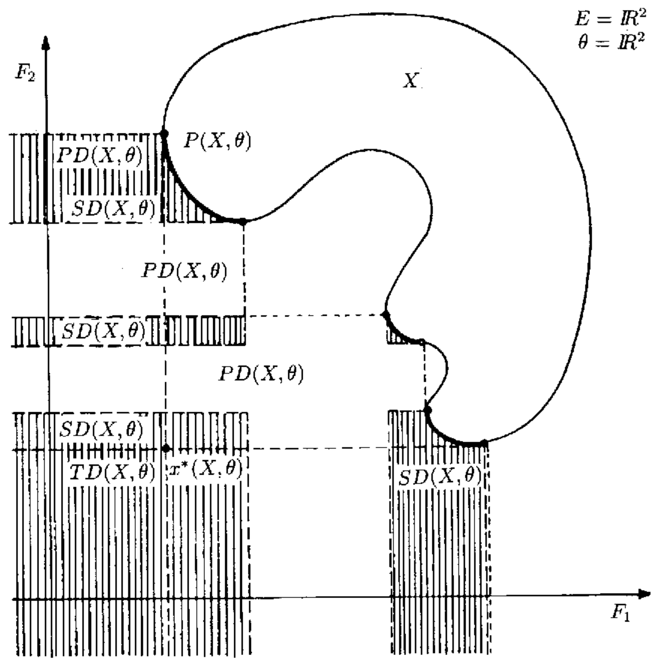

Fig. 4.1.
An example of the sets of totally, strictly, and partly dominating points in $\mathbb{R}^2$.

4.2 Properties of Totally Dominating Points

Before passing to the study of the properties of totally dominating and ideal points we will prove the following lemma.

Lemma 4.1. *The multifunction*

$$T : P(X,\theta) \ni a \to a - \theta \subset E$$

is continuous in the topology generated by the Hausdorff distance in the family of closed subsets of E. ∎

Proof. Let us take an arbitrary $\delta > 0$. If a and b are such that $\| a - b \| < \delta$ then the Hausdorff distance, d_H, of $T(a)$ and $T(b)$ can be estimated as follows:

$$d_H(T(a), T(b)) = d_H(a - \theta, b - \theta) = d_H(a - \theta, (b - a) + (a - \theta)) \leq \| a - b \| \leq \delta.$$

Hence T is Hausdorff – continuous. ■

Now we can show the following

Lemma 4.2. *If θ is closed then $TD(X, \theta)$ is closed.* ■

Proof. Let us take a sequence of totally dominating points $\{x_i\}_{i \in N}$ convergent to an $x \in E$. By Lemma 4.1 the multifunction

$$S(y) := (y + \theta) \cap (X + \theta)$$

is upper semicontinuous as an intersection of two continuous, closed–valued multifunctions and

$$S(x_i) = (x_i + \theta) \cap (X + \theta) = X + \theta$$

by the definition of $TD(X, \theta)$. Hence, it follows that

$$S(x) = (x + \theta) \cap (X + \theta)$$

equals to $X + \theta$ which is equivalent that $X \subset x + \theta$, or $x \in TD(X, \theta)$. ■

From Lemma 4.1 we derive a useful characterization of the set of totally dominating points.

Theorem 4.1. *If θ is closed and pointed, and $TD(X, \theta)$ is non–empty, then the latter set can be expressed in the form*

$$TD(X, \theta) = x^*(X, \theta) - \theta. \tag{4.6}$$

■

Proof. If $X \subset x + \theta$ then for each $z \in x - \theta$

$$x + \theta \subset z + \theta,$$

consequently, if $x \in TD(X, \theta)$ then $x - \theta \subset TD(X, \theta)$ which proves the inclusion "$\supset$". To prove that for each $z \in TD(X, \theta)$ there exists $v \in P(TD(X, \theta), -\theta)$ such that $z \in v - \theta$, which is equivalent to stating that $TD(X, \theta)$ is $(-\theta)$–complete, it is sufficient to observe that $TD(X, \theta)$ is closed and $(-\theta)$–bounded which – as justified by Sawaragi, Nakayama and Tanino (1985, Thm. 3.2.10)) – implies θ–completeness of $TD(X, \theta)$. The first property is a consequence of the fact that both sets θ and $X + \theta$ were assumed closed, while as any element of X can be taken a $(-\theta)$–totally dominating point for $TD(X, \theta)$. Therefore also the inclusion "$\subset$" holds which ends the proof. ■

The above theorem implies immediately the existence of ideal points for nonempty $TD(X, \theta)$.

Corollary 4.1. *Under the assumptions of Thm. 4.1 the set of ideal points is non–empty.* ■

4.3 Uniqueness of Ideal Points

The theorem given below relates the uniqueness of ideal points to the properties of θ.

Theorem 4.2. *Suppose that E is a linear space partially ordered by a closed, convex and pointed cone θ. Then the following conditions are equivalent:*

a) If the subset X of E is θ–bounded then the set of ideal points for X consists of a single point $x^*(X, \theta)$.

b) For every two points $x_1,\ x_2 \in E$ there exists the unique $z \in E$ such that

$$(x_1 + \theta) \cap (x_2 + \theta) = z + \theta \tag{4.7}$$

c) (E, θ) is a vector lattice, i.e. $\forall a, b \in E\ \ \exists a \wedge b := x^*(\{a, b\}, \theta)$. ■

Proof. Assume first that the condition (b) is satisfied and suppose that for a set $X \subset E$ there exists a pair of distinct totally dominating points z_1 and z_2 which both are ideal, i.e. they are θ–maximal in $TD(X, \theta)$. However, if $X \subset z_1 + \theta$ and $X \subset z_2 + \theta$ then, of course,

$$X \subset (z_1 + \theta) \cap (z_2 + \theta) = (z + \theta),$$

therefore $z \in TD(X, \theta)$. Thus we have found an element of $TD(X, \theta)$ which $(-\theta)$–dominates z_1 and z_2 – a contradiction with the assumption that both points were ideal.

To prove the remaining implications observe that (c) is immediately implied by (a), so it suffices to prove that (c) $\Rightarrow$ (b). Since for each pair of points $x_1,\ x_2 \in E$ there exists the unique ideal point x for the set $\{x_1, x_2\}$, then by Prop. 4.2

$$TD(\{x_1, x_2\}, \theta) = x - \theta. \tag{4.8}$$

On the other hand, $TD(\{x_i\}, \theta) = x_i - \theta$ for $i = 1, 2$, hence

$$TD(\{x_1, x_2\}, \theta) = (x_1 - \theta) \cap (x_2 - \theta). \tag{4.9}$$

Replacing θ by $(-\theta)$ we conclude from (4.8) and (4.9) that (4.7) is fulfilled for $(-\theta)$. Applying the already proved implication (b) $\Rightarrow$ (a) we conclude that there exists the unique ideal point y for $\{x_1, x_2\}$ with respect to $(-\theta)$. Similarly as in (4.8) and (4.9) we observe that

$$TD(\{x_1, x_2\}, (-\theta)) = x - (-\theta) = x + \theta$$

and

$$TD(\{x_1, x_2\}, (-\theta)) = (x_1 + \theta) \cap (x_2 + \theta)$$

Therefore $(x_1 + \theta) \cap (x_2 + \theta) = (x + \theta)$ and x is unique with this property. ■

From Theorem 4.2 it follows that to have the unique ideal point in a vector optimization problem, the intersection of any two translations of the ordering cone must exist and be congruent to this cone. It is easy to see that in $\mathbb{R}^N_+$ only the polyhedral cones generated by exactly N linearly independent vectors, so called *simplicial cones* (cf. e.g. Barker, 1981, or Fuchsteiner and Lusky, 1981), possess this property. This class of convex cones can be equivalently characterized as the isomorphic transformations of the natural positive cone $\mathbb{R}^N_+$.

Hence we conclude that in all situations where the ideal point is unique, its determination can always be reduced to finding N optimal (minimal or maximal) values of N scalar functions $G_i,\ i = 1,..N$, obtained as the transformation of the vector criterion F according to the formula

$$G = I^{-1} \circ F, \tag{4.10}$$

where $G = (G_1, \ldots G_N)$ and I is an automorphism of $\mathbb{R}^N$ such that $\theta = I(\mathbb{R}^N_+)$.

The above mentioned theorem implies that the class of convex cones which do not satisfy (4.7) is large in linear spaces of dimension greater than 2, e.g. (4.7) is not satisfied by the "ice–cream cone"

$$\theta := \{(x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 \leq z\}.$$

One can see that the set

$$Z := \theta \cap ((1, 0, 0) + \theta)$$

does not have a unique ideal point with respect to θ since the points $(0, 0, 0)$ and $(1, 0, 0)$ are evidently totally dominating, non–comparable, and the cone translated to a point belonging to $\theta \cup ((1, 0, 0) + \theta)$ and different from $(0, 0, 0)$ and $(1, 0, 0)$ may not contain the set Z.

4.4 Properties of Local Ideal Points

From Theorem 4.2 it follows that to find the unique ideal point for the problem (4.1) satisfying the assumptions of Theorem 4.2 it is necessary to calculate the global optimal values of the criterion functions (perhaps transformed according to (4.10)). However, while performing the numerical optimization procedures for non–convex functions we often deal with the local minima rather than with the global ones. It is natural to expect that the elements of the criteria space with all coordinates being local minima of the criterion functions will constitute local surrogates of ideal points to similar extent as are the local minima in the scalar optimization with respect to the global optimum. Unfortunately, it comes out that such points can even be attainable.

Consequently, some elements of the set

$$L(X) := \{y = (y_1, \ldots, y_N) \in I\!R^N : y_i \in V(F_i, U) \text{ for } i = 1, \ldots, N\} = \\ = \prod_{i=1}^{N} V(F_i, U), \tag{4.11}$$

where $X := F(U)$ and

$$V(F_i, U) := \{r \in I\!R : r \text{ is a local minimal value of } F_i \text{ on } U\},$$

cannot be admitted as reference points in distance minimization procedures without loosing the guarantee that the least–distance solutions are nondominated. Therefore we are interested in finding a subset $L^*(X)$ of $L(X)$ consisting exclusively of strictly dominating points, because the minimization of a strongly monotonically increasing norm with respect to a strictly dominating reference point will yield a nondominated solution (cf. Skulimowski, 1988). We call such points *local ideal points*.

From a decision–maker's point of view the most interesting local ideal points may be those located as close as possible to the set $F(U)$.

Definition 4.6. *The elements of the set*

$$PL(X) := P(L^*(X), (-\theta)) \tag{4.12}$$

are called the proper local ideal points. ■

Before giving a filtering criterion for local ideal points let us define the set of *local edge points* $R(X)$ in the criteria space for the vector optimization problem (4.1) with $E := I\!R^N$:

$$R(X) := \{x \in I\!R^N : x_j \in V(F_j, U) \text{ for exactly one } j \in \{1, ..N\} \text{ and} \\ x_i = \inf F_i(u) \;\text{ s. t. }\; u \in F_j^{-1}(x_j) \cap U \text{ for } i \neq j\}. \tag{4.13}$$

Let us observe that for each i to every element v of $V(F_i, U)$ there corresponds exactly one local edge point p and this correspondence is one–to–one.

The above considerations let us formulate the following :

Theorem 4.3. *Let $X \subset I\!R^N$ and let $S(X)$ be the nondominated subset of $R(X)$,*

$$S(X) := P(R(X), \theta),$$

and $T_i(X)$ – the set of those local optimal values of F_i which are represented in $S(X)$, i.e.

$$T_i(X) := \{v \in V(F_i, U) : \; \exists j \in \{1, ..N\} \; \exists x = (x_1, \ldots, x_N) \in S(X) \textit{ so that } \; x = v\}.$$

Then the local ideal points $L^*(X)$ can be obtained by the following construction:

$$L^*(X) = \{x \in I\!R^N : \forall i \in \{1,..N\}\ x_i \in T_i(X)\ and\ x \leq r_j(x_i)\}, \tag{4.14}$$

where $r(x_i)$ is the local edge point corresponding to the local optimal value x_i with the coordinates

$$\begin{aligned} r_i(x_i) &= x_i, \\ r_j(x_i) &= \inf\{F_j(u) : u \in F_i^{-1}(x_i) \cap U\}, \quad for\, i \neq j. \end{aligned}$$

■

From the above Theorem it immediately follows the algorithm for finding the local ideal points employing N independent global minimization procedures for each coordinate of F, and a selection of a nondominated subset of a discrete set.

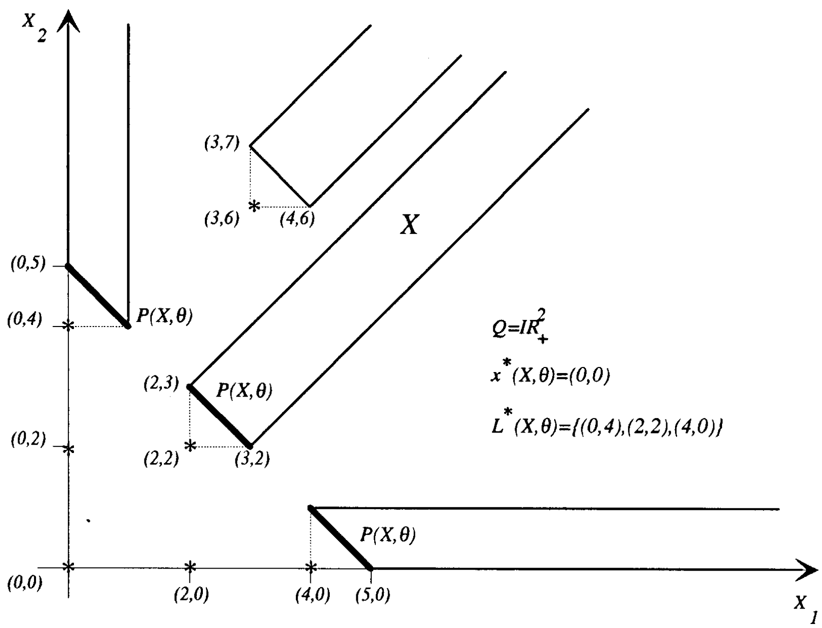

Fig. 4.2. *The construction of local ideal points in Example 4.1.*

Example 4.1. Let us consider the following bicriteria linear programming problem with disjunctive constraints (cf. Fig. 4.2):

$$((F_1, F_2) : I\!R^2 \supset X \rightarrow I\!R^2) \rightarrow min,$$

where $F_1(x_1, x_2) = x_1,\ F_2(x_1, x_2) = x_2$, and the decision set X is determined by the following inequalities

$$x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \geq 5$$

and

$$[x_1 - 1 \leq 0, \text{ or } x_2 - 1 \leq 0, \text{ or } (x_2 - x_1 - 1 \leq 0 \text{ and } x_1 - x_2 - 1 \leq 0),$$
$$\text{or } (-x_1 - x_2 + 10 \leq 0 \text{ and } x_2 - x_1 - 4 \leq 0 \text{ and } x_1 - x_2 + 2 \leq 0)].$$

It is easy to see that $X = F(X)$ and X consists of four disjoint regions X_1, X_2, X_3, X_4 bounded by the appropriate straight line intervals:

$$\begin{aligned}
X_1 &= \{x \in I\!R^2 : x_1 \geq 0,\ x_1 + x_2 \geq 5,\ x_1 - 1 \leq 0\}, \\
X_2 &= \{x \in I\!R^2 : x_1 + x_2 \geq 5,\ x_2 + x_1 - 1 \leq 0,\ x_1 - x_2 - 1 \leq 0\}, \\
X_3 &= \{x \in I\!R^2 : x_2 \geq 0,\ x_1 + x_2 \geq 5,\ x_2 - 1 \leq 0\}, \\
X_4 &= \{x \in I\!R^2 : -x_1 - x_2 + 10 \leq 0,\ x_2 - x_1 - 4 \leq 0,\ x_1 - x_2 + 2 \leq 0\}.
\end{aligned}$$

The procedure to find the sets of local ideal and proper ideal points can be presented as follows. We start from calculating all local minima of F_1, and F_2 ,

$$V(F_1, X) = \{4, 3, 2, 0\} \text{ and } V(F_2, X) = \{6, 4, 2, 0\}.$$

Thus we get the (global) ideal point $x^*(X, I\!R^N_+) = (0, 0)$ and the set

$$L(X) = V(F_1, X) \times V(F_2, X) = \{(4, 6), \ldots, (4, 0), (3, 6), \ldots, (2, 0), (0, 6), \ldots, (0, 0)\}$$

consisting of 16 points.

At each local minimum of F_i we calculate the minimal admissible values of remaining criteria $F_j, j \neq i$ (in our case $i, j = 1, 2$) in order to determine the set of local edge points

$$R(X) = \{(4, 1), (3, 7), (2, 3), (0, 5), (4, 6), (1, 4), (3, 2), (5, 0)\}.$$

The nondominated part of $R(X)$ consists of six points, namely

$$S(X) = \{(4, 1), (2, 3), (0, 5), (1, 4), (3, 2), (5, 0)\}.$$

Observe that the points $(3, 7)$ and $(4, 6)$ have thus been eliminated from $R(X)$. According to Theorem 4.1 now we have to find the sets $T_i(X)$ for $i = 1, 2$, containing those local minima of F_i which are represented in $S(X)$, i.e.

$$T_1(X) = \{4, 2, 0\} \text{ and } T_2(X) = \{4, 2, 0\}.$$

Following the formula (4.11) we find the set of local ideal points of F on X:

$$L^*(X) = \{(4, 0), (2, 0), (0, 0), (2, 2), (0, 2), (0, 4)\}.$$

Three elements of $L^*(X)$ are moreover proper local ideal points (cf. Definition 4.6), namely

$$PL(X) := \{(4, 0), (2, 2), (0, 4)\}. \qquad \blacksquare$$

A straightforward field of applications of local ideal points are distance scalarization methods. An approximation of the set of nondominated points by finding the least–distance solutions to a fixed totally dominating point with respect to a parameterized family of norms has some unpleasant features, namely :

- for a fixed discretization step of the norm parameter the accuracy of approximation is dependent on the distance between the reference point and the nondominated set : the more distant parts are surveyed, the lower is the accuracy;

- for non–convex problems the least–distance solutions for some values of norm parameters may not be unique, in particular they may belong to different connected components of the set $P(X, \theta)$;

- it is difficult, or even impossible to restrict a priori the family of norms so that the search for a compromise solution would be confined to the prescribed subset of $P(X, \theta)$ being of a special importance to the decision–maker.

The disadvantages of distance scalarization with respect to a single reference point may be eliminated by decomposing the set $F(U)$ into subregions associated to the proper local ideals and using a distance scalarization technique for the approximation of each subregion. Below we propose an algorithm of such scalarization procedure.

Algorithm 4.1.

Step 0. Transform, if necessary, the coordinate system in the criteria space by the transformation I^{-1} (cf. (4.10)) to obtain the natural $I\!R_+^N$–ordering in the transformed space.

Step 1. Calculate the local minimal values of the functions F_i, for $i = 1, \ldots, N$.

Step 2. Find those local minimal values which correspond to the nondominated edge points according to Theorem 4.1.

Step 3. Find the set $L^*(X)$ according to the construction presented in Theorem 4.1.

Step 4. Find $PL(X)$ – the $(-\theta)$–optimal part of $L^*(X)$.

Step 5. Perform the distance scalarization procedure for a parameterized family of norms with respect to each element of $PL(X)$. ∎

A variation of the Algorithm 4.1 can also be applied as a two–stage interactive multicriteria decision-making method: first the decision–maker selects interactively this proper local ideal point whose coordinates fits best his preferences, then the point thus selected is used as the reference point in an interactive distance minimization procedure.

4.5 Properties of Partly Dominating Points

As shown by the inclusion (4.5), partly dominating points constitute the largest subclass of dominating points which are considered in this paper. We will start from showing a dual nature of partly dominating points which is expressed by the following property.

Proposition 4.2. *The set of θ–maximal points of $PD(X,\theta)$ is the same as the set of θ–minimal points of X. i.e.*

$$P(PD(X,\theta),(-\theta)) = P(X,\theta). \tag{4.15}$$

Moreover,

$$PD(X,\theta) = \bigcup\{x-\theta : x \in P(X,\theta)\}. \tag{4.16}$$

■

Proof. By Def. 4.4 each θ–minimal point of X belongs to $PD(X,\theta)$, i.e.

$$P(X,\theta) \subset PD(X,\theta).$$

If y is dominated by an element of $P(X,\theta)$ then $(y+\theta) \cap P(X,\theta) = \emptyset$, consequently $y \notin PD(X,\theta)$ and

$$P(X,\theta) \subset P(PD(X,\theta),(-\theta)).$$

Conversely, if $z \in P(PD(X,\theta),(-\theta))$ then $z \leq_\theta x$ for certain $x \in P(X,\theta)$, hence and from (4.12) it follows that $z = x$.

To prove the relation (4.14) suppose that $y \in PD(X,\theta)$. By Def. 4.4. there exists $x \in P(X,\theta)$ such that $x \in y+\theta$, i.e. $y \in x-\theta$. If $x \in P(X,\theta)$ and $y \in x-\theta$ then $x-y \in \theta$, i.e. y dominates x, consequently, y is a partly dominating point for X. ■

Corollary 4.2. *If $P(X,\theta)$ and θ are closed then so is $PD(X,\theta)$.* ■

Proof. By (4.11) $PD(X,\theta)$ is expressed as the range of the closed–valued multifunction

$$T : P(X,\theta) \ni a \to a-\theta \subset E$$

defined on the closed set $P(X,\theta)$. For the closedness of $PD(X,\theta)$ it is sufficient to observe that by Lemma 4.1 T is Hausdorff–continuous so that the range of $T, PD(X,\theta)$, is closed. ■

Remark 4.1. *Let us note that, in general, the converse statement is not true, i.e. the closedness of $PD(X,\theta)$ does not imply that $P(X,\theta)$ is closed.* ■

We will also prove that for a large class of convex cones $PD(X,\theta)$ is connected.

Proposition 4.3. *Let X be an arbitrary subset of E. If θ is non–degenerated, (E,θ) is a Banach lattice (i. e. (E,θ) is a vector lattice and E is a Banach space), and $P(X,\theta)$ is non–empty then $PD(X,\theta)$ is connected.* ■

Proof. Let a and b be two arbitrary elements of $PD(X,\theta)$. Then by the Banach lattice assumption there exists $c \in E$ such that $c \leq_\theta a$ and $c \leq_\theta b$, moreover, for each $x \in [a,c]$ $x \leq_\theta \leq a$ while for each $y \in [b,c]$ $y \leq_\theta b$. If v is an element of $P(X,\theta)$ such that $a \leq_\theta v$ then for each $x \in [a,c]$ $x \leq_\theta v$, therefore

$$[a,c] \subset PD(X,\theta).$$

Similarly we conclude that

$$[b,c] \subset PD(X,\theta),$$

consequently, x and y are connected by the broken line $[a,c] \cup [c,b]$. ■

4.6 Relations between Strictly and Partly Dominating Points

A general relation between the subclasses of dominating points has been formulated as the inclusion (4.5) in Prop. 4.1. Now, we will study the properties of strictly dominating points in some special cases.

First, we will answer the question whether the sets $PD(X,\theta)$ and $SD(X,\theta)$ coincide for convex X. It comes out that it is not true in a general case, however such a property may be proved for bicriteria problems.

Theorem 4.4. *If $X \subset I\!R^2$ is θ–closed and θ–convex (i. e. $X+\theta$ is convex) then*

$$SD(X,\theta) = PD(X,\theta). \tag{4.17}$$

■

Proof. Let x be an arbitrary element of $PD(X,\theta)$. By Prop. 4.1. it suffices to prove that $x \in SD(X,\theta)$. Suppose that it exists a point $y \in P(X \cap (x+\theta),\theta)$ which is not θ–minimal in X. Then there exists $y_1 \in X$ such that $y \in y_1 + \theta$. Let us consider the quadrangle with the vertices x, y, y_1 and x_1, where x_1 is an arbitrary element of $P(X,\theta) \cap (x+\theta)$ which exists since we assumed that x is partly dominating. The points x_1 and y_1 are non–comparable because both are elements of $P(X \cap (x+\theta),\theta)$. Taking into account that $y_1 \leq_\theta y$, it follows that x_1 , y and y_1 are not collinear – otherwize x_1 would belong to $y+\theta$. Since X is θ–convex, the triangle $[x_1, y_1, y]$ is contained in $X+\theta$. On the other hand, x dominates x_1 and is non–comparable with y_1, hence we can similarly conclude that x_1, x, and y_1 form a non–degenerated triangle.

Observe that x may not be an element of $[x_1, y, y_1]$ which would imply that it belongs to $X + \theta$, therefore the quadrangle $[x, x_1, y, y_1]$ is convex and does not degenerate to a triangle. Consequently, the diagonal $[x_1, y_1]$ intersects the other one, $[x, y]$, at a point y_0 belonging to $X+\theta$. However, $[x, y]$ is contained in $(x+\theta)\cap(y-\theta)$, therefore $y_0 \in y - \theta$ and $y_0 \in X + \theta$ since

$$[x_1, y, y_1] \subset X + \theta,$$

which contradicts the assumption that y were θ–minimal in $X\cap(x+\theta)$ but dominated in X. Thus we conclude that each θ–minimal point in $X\cap(x+\theta)$ is θ–minimal in X, i.e.

$$P(X \cap (x + \theta), \theta) = P(X, \theta).$$

By definition, that means that $x \in SD(X, \theta)$. ■

Let us note that the above proof remains valid for such convex sets in E, $dimE >$ 2, that there exists a subspace $E_1 \subset E$, with $dimE_1 = 2$, containing simultaneously x, y, and the above defined points x_1 and y_1.

Theorem 4.4 may not be true when X is convex but the dimension of the criteria space E is greater than 2. An example of such situation for $E = I\!R^2$ and $\theta = I\!R^3_+$ is given below.

Example 4.2. Let us consider the attainable set $X = [a, b, c]$, $a = (1, 0, 0)$, $b = (1, -1, 0)$, $c = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and a dominating point $x = (0, 0, 0)$ (cf. Fig. 4.3). It is easy to see that

$$P(X, \theta) = [c, b]$$

and for $x = (0, 0, 0)$,

$$P(X, \theta) \cap (x + \theta) = [c, z],$$

where $z = (\frac{2}{3}, 0, \frac{1}{3})$. However,

$$R := P(X \cap (x + \theta), \theta) = [c, z] \cup [z, a],$$

where $a \notin P(X, \theta)$. Therefore $x \in SD(X, \theta)$, although it is an element of

$$PD(X, \theta) = [c, b] - I\!R^N_+.$$

One can show that

$$\begin{aligned} SD(X,\theta) =& PD(X,\theta) \cap \{(y, y_2, y_3) \in I\!R^3 : y_2 \leq x_2^*\} \cup \\ & \cup \{(z_1, z_2, z_3) \in I\!R^3 : z_1 \leq \ frac12, z_3 \leq \tfrac{1}{2}, z_3 = \tfrac{1}{2}\} = \qquad f \\ =& \{(y_1, y_2, y_3) \in I\!R_3 : y_1 \leq 1, y_2 \leq -1,\ y_3 \leq y_1 \ \text{ or } \ y_1 \leq \tfrac{1}{2},\ y_2 \leq \tfrac{1}{2},\ y_3 = \tfrac{1}{2}\}, \end{aligned} \tag{4.18}$$

where x_2^* is the second coordinate of the ideal point $x^*(X,\theta) = (-\frac{1}{2}, -1, 0)$. ■

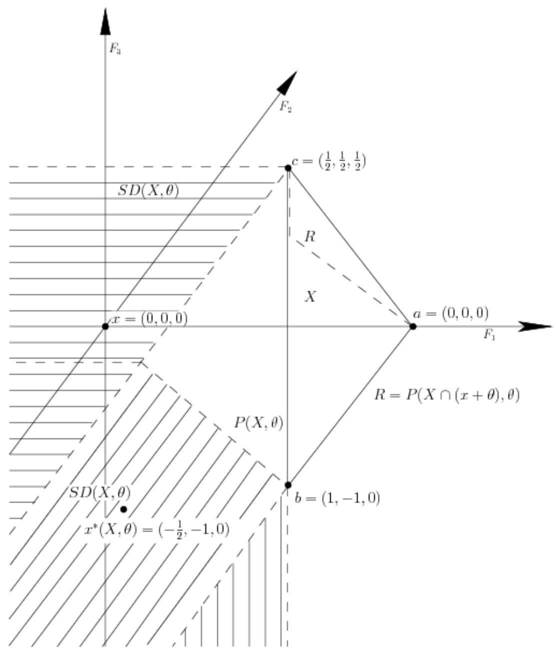

Fig 4.3. *An example of situation, where x is partly dominating, $X \subset I\!R^3$ is convex, but x is not strictly dominating.*

Let us note that, in general, topological properties of $SD(X,\theta)$ are determined by the properties of the whole set X, irrespectively of the properties of $P(X,\theta)$ itself. For instance, if in the above presented Example 4.2 we modify the set X by setting

$$X_1 := (X \setminus \{(z_1, z_2, z_3) \in I\!R^3 : r_1 < z_2 < r_2\}) \cup [q_1, q_2],$$

where $b_2 \le r_1 < r_2 \le c_2$ and $q_i,\ i = 1,2$, are the intersections of the planes $z_2 = r_i$ and the interval $[b,c]$, for $i = 1,2$, then the set $SD(X_1,\theta)$ fails to be closed, although so are X_1 and $P(X_1,\theta)$. However, one can show that in $I\!R^2$ $SD(X,\theta)$ is closed for closed $P(X,\theta)$, and connected, for connected $P(X,\theta)$.

Below we give a characterization of the set of strictly dominating points which will let us propose a constructive procedure of evaluating $SD(X,\theta)$ for $X \subset I\!R^2$.

4.7 Constructive Evaluation of $SD(X, \theta)$ for Bicriteria Problems

Although the set $SD(X, \theta)$ possesses interesting properties, its definition (cf. Def. 4.5.) does not suggest a constructive algorithm of finding $SD(X, \theta)$, or even of verifying whether a given point $x_0 \in E$ is strictly dominating. Here, we give a further characterization of $SD(X, \theta)$ which should be helpful in answering the above questions, especially when $E = I\!R^2$.

Lemma 4.3. *If X is a closed and connected subset of $I\!R^2$ then*

$$SD(X, \theta) = \{y \in PD(X, \theta) : P(\partial(y + \theta) \cap X, \theta) \subset P(X, \theta)\}. \tag{4.19}$$

■

Proof. Observe first that if θ degenerates to a half–line then for each $y \in I\!R^2$

$$(y + \theta) = \partial(y + \theta),$$

consequently

$$P(\partial(y + \theta) \cap X, \theta) = P((y + \theta) \cap X, \theta)$$

and $P((y + \theta) \cap X, \theta)$ contains at most one point, namely $P((y + \theta) \cap X, \theta) = \{x\}$, iff $y \in PD(X, \theta)$, and is empty elsewhere. Therefore in this case

$$SD(X, \theta) = PD(X, \theta)$$

and (4.19) is trivially satisfied. Thus, without a loss of generality we may suppose that θ contains a base of $I\!R^2$ and the coordinates of points in $I\!R^2$ are related to this base. Notice that from the connectedness of X it follows that if $\partial(y + \theta) \cap X = \emptyset$ for $y \in PD(X, \theta)$ then $X \not\subset y + \theta$ and $y \in TD(X, \theta)$. Thus we assume that $\partial(y + \theta) \cap (X + \theta) \neq \emptyset$. Each point $v \in \partial(y + \theta) \cap X$ maximizes one of the coordinates of points from $X \cap (y + \theta)$ therefore if

$$w \in P(\partial(y + \theta) \cap X, \theta)$$

then w belongs also to $P((y + \theta) \cap X, \theta)$.

If y is strictly dominating then $w \in P(X, \theta)$ which proves that $SD(X, \theta)$ is contained in the set defined as the right–hand side of (4.19).

Suppose now that y is such that the inclusion

$$P(\partial(y + \theta) \cap X, \theta) \subset P(X, \theta) \tag{4.20}$$

holds. Let us notice that $\partial(y + \theta)$ consists of two half–lines beginning at y which allows us to distinguish the following subcases:

a) $P(\partial(y+\theta) \cap X, \theta)$ consists of two points x_1 and x_2,

b) $P(\partial(y+\theta) \cap X, \theta)$ consists of one point.

Observe that in the case (a) it is sufficient to prove that if $x_1,\ x_2 \in P(X, \theta)$ then an element z of $(y+\theta) \cap X$ is θ–optimal in X iff it is so in $(y+\theta) \cap X$. Suppose that

$$z \in P((y+\theta) \cap X, \theta).$$

From basic geometric properties of convex, non–degenerated cones in $I\!R^2$ (cf. Fig. 4.4a) it follows that if x_1 and x_2 belong to $P(X, \theta)$ then any point z of $(y+\theta) \cap X$ is either dominated by x_1 or x_2, or the intersection of X and $z-\theta$ does not contain exterior points of $y+\theta$. The latter property is implied by the inclusion

$$z - \theta \subset (x_1 - \theta) \cup (x_2 - \theta) \cup (y+\theta) \cap (max(x_1, x_2) - \theta)$$

which holds for $z \in (y+\theta) \cap (max(x_1, x_2) - \theta)$, where

$$max(x_1, x_2) = (max(x_{11}, x_{21}), \max(x_{21}, x_{22})). \tag{4.21}$$

Consequently, if z is θ–optimal in $(y+\theta) \cap X$, it is also θ–minimal for X, which ends the proof of the case (a). In case (b) let us take an arbitrary $z \in P((y+\theta) \cap X, \theta)$, and let x_1' be a θ–minimal point of the intersection of $\partial(y+\theta)$ and X.

If we take as x_2' a point of $\partial(y+\theta)$ such that

(i) x_2' belongs to the half–line h_2 different from that containing x_1' (recall that we assumed that θ is non–degenerated and $\partial(y+\theta)$ consists of two half–lines, h_1 and h_2),

(ii) $z \leq_\theta \max(x_1', x_2')$, (cf. Fig. 4.4b),

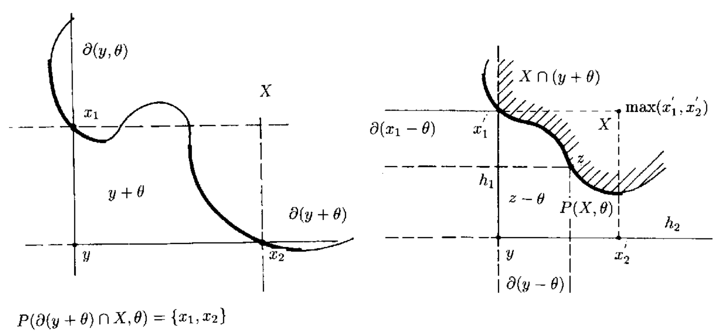

Fig. 4.4. *An illustration of the proof of Lemma 4.3: cases (a) (left), and (b) (right).*

then the same arguments as applied to x_1 and x_2 in case (a) imply that $z \in P(X,\theta)$. We only have to observe that if X is connected, $x_1^{'} \in P(X,\theta)$ and x_2' is defined as above, then the set

$$(\partial(x_1' - \theta) \setminus \partial(y-\theta)) \cup h_2$$

divides $I\!R^2$ into two disjoint parts in such a way that X is contained in this one which does not contain y.

Therefore x_2' may not be dominated by an element of X and $x_2' \in P(X \cup \{x_2'\}, \theta)$ which allows to consider x_2' in the same way as in the proof of the case (a), with $x_1 := x_1'$, $x_2 := x_2'$ and $X' := X \cup \{x_2'\}$. ■

From Lemma 4.3 and the preceding considerations it follows a characterization of strictly dominating points in $I\!R^2$:

Theorem 4.5. *If X is a closed and connected subset of $I\!R^2$ then $SD(X,\theta)$ can be represented as*

$$\bigcup \{pr_1(S_i) \times pr_2(S_j) : i,\ j \in I \cup \{0\}\} \ \cap\ PD(X,\theta), \tag{4.22}$$

where $(S_i)_{i\in I}$ is the set of connected components of $P(X,\theta)$, by definition $S_0 := TD(X,\theta)$, $(r_k)_{k=1,2}$ is a base of E spanning the cone θ, pr_k is the projection on the k–th axis in the system of coordinates (r_1,r_2). ■

The Theorem 4.5 implies the following procedure of verifying whether a reference point q in $I\!R^2$ is strictly dominating.

Algorithm 4.2.

Step 0. Convert the coordinates of X and q to those of a base (r_1,r_2) spanning θ.

Step 1. Calculate all local minima of F_1 and F_2.

Step 2. Find all local ideal points $x_1^* = (x_{1i}^*, x_{2i}^*)$ and the edge points

$$e_{1i} = (x_{1i}, e_{12i}) \text{ and } e_{2i} = (e_{21i}, x_{21})$$

associated to the connected components $\{S_i\}_{i\in I}$ of $P(X,\theta)$, where

$$e_{12i} := \inf\{x_2 : (x_{1i}, x_2) \in X\}$$

and

$$e_{21i} := \inf\{x_1 : (x_1, x_{2i}) \in X\}.$$

If the minimum of the j–th coordinate, $j \in \{1,2\}$, is not achieved then put $e_{kji} := \inf, k \in \{1,2\}$, so that for each machine representation of a real number $r \in I\!R,\ r \le \inf$.

Step 3. Find the global ideal point $x = (x_1^*, x_2^*)$ and the corresponding edge points

$$e_1 := (x_1, e_{12}) \text{ and } e_2 := (e_{21}, x_2).$$

If $q \leq x^*$ then $q \in TD(X, \theta)$, STOP.

If $q_1 \geq e_{21}$ or $q_2 \geq e_{12}$ then $q \notin PD(X, \theta)$ STOP.

If $q_1 < x_1^*$ then go to Step 5.

Step 4. Find i such that $x_{1i}^* \leq q_1$ and for each $j \neq i$: $x_{1j}^* < x_{1i}^*$ or $x_{1j}^* > q_1$.

If $e_{21i} < q_1$ or $q_2 \geq e_{12i}$ then $q \notin SD(X, \theta)$, STOP.

If ($e_{21i} > q_1$ or $e_{21i} = q_1$ and $e_{2i} \in S_i$) and $q_2 < x_2^*$ then $q \in SD(X, \theta)$, STOP.

Step 5. Find n such that $x_{2n}^* \leq q_2$ and for each $m \neq n$ $x_{2m}^* < x_{2n}^*$ or $x_{2n}^* > q_2$.

If $e_{12n} < q_2$ or $q_1 \geq e_{21n}$ then $q \notin SD(X, \theta)$, STOP.

If $i = n$ then verify whether $q \in X$, if it is so then $x \notin SD(X, \theta)$, STOP.

If $e_{12n} > q_2$ or $e_{12n} = q_2$ and $e_{1n} \in S_n$ then $q \in SD(X, \theta)$, STOP. ■

Remark that to find all local ideal points in Step 2 for a non–convex vector optimization problem, one has to determine all local minima of the objectives considered separately. Thus, in general, in this step one has to execute two global minimization procedures.

Applying the above algorithm for disconnected X, we may get an erroneous result if y is situated as in Fig. 4.5. Namely, in this case the strict dominance of y will not be detected because Thm. 4.5 does not give a complete characterization of $SD(X, \theta)$ for disconnected X. However, as it has been discussed in Skulimowski (1988), only those elements of $SD(X, \theta)$ which are of the form given in Thm. 4.5 have the desired properties as reference points for distance minimization techniques.

Remark that from the situation shown in Fig. 4.5 it follows that if X is disconnected then the condition (4.19) may not be applied to the (connected) set $X + \theta$ instead of X.

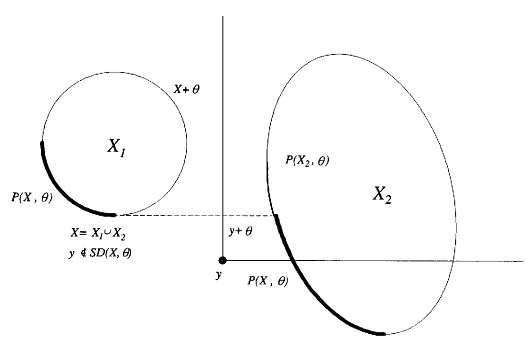

Fig. 4.5. *An example of situation, where Lemma 4.3 does not properly characterize $SD(X,\theta)$ ($y \notin SD(X,\theta)$ although 4.19 holds) for disconnected X.*

Let us note that an appropriate algorithm for verifying whether an element of $\mathbb{R}^3$, $n \geq 3$, is strictly dominating which could be based on examining the situation of a point in E with respect to the relative boundary of the efficient set would be essentially more difficult since the behavior of the efficient set is much more complicated in higher–dimensional spaces.

4.8 Conclusions – Impact on Multicriteria Decision Making

In Sec.4.3 we have shown that the existence of the unique ideal point may not be taken for granted for multicriteria decision–making problems with the preference structure other than congruent to a simplicial cone. In particular, applying a distance scalarization procedure with respect to different ideal points will usually result in different least–distance decisions. That means that the notion of compromise decision based on ideal points is in this case inconsistent. This problem, however, does not touch upon the bicriteria problems since all nontrivial convex cones in $\mathbb{R}^2$ which are not degenerated to a half–line are simplicial.

Furthermore, we concluded that in all situations where the ideal point is unique, its determination can always be reduced to finding global optima of N scalar functions according to (4.10). The notion of local ideal points introduced here can replace the ideal point in distance minimization procedures for multicriteria problems with non–convex criteria iff they are previously filtered as proposed in Theorem 4.3. Local ideal points may be used in the general framework of multiple reference points method as well as they may increase the accuracy of the approximation of the nondominated set which applies the distance scalarization techniques.

The relevance of the classification of dominating points here presented is exposed by the summary of the properties of distance–scalarization procedures presented in the concluding section of the next Chapter. One may there observe that the assumptions concerning the subclass of dominating points to which a reference point belongs play a fundamental role in the proofs of θ–optimality. Observe that the larger is the class of reference points, the stronger assumptions concerning the properties of the attainable set X and the norm in the criteria space E are required. For the largest class, $PD(X,\theta)$, one can give a transparent θ–optimality condition for the convex subsets of two–dimensional euclidean space only. As shown by the examples, where a least–distance solution fails to be θ–optimal, the higher–dimensional versions of theorems concerning the bicriteria case would require more sophisticated assumptions regarding X, θ, and the norm in E, so their potential field of applications in decision support systems may be more restricted.

Another difficulty which might be observed in conditions involving the use of strictly dominating points, consists in the fact that the boundaries of this set are influenced by the shape of X which may not be assumed a priori known. Hence arises a need for an efficient algorithm of determination to which class of dominating points given reference point belongs. A proposal for such an algorithm has been presented in Sec.4.6. It turns out that a classification of a single point may be executed in a short time, even if the criteria functions have a complicated form. However, since the global optimization procedures used to determine the local ideal points are usually based on randomization techniques, the evaluation of $SD(X, \theta)$ may have an approximate character. Moreover, let us notice that to find all local ideal points for a non–convex vector optimization problem one has to determine all local minima of the objectives considered separately. Thus to achieve this in a numerically efficient way, one can use a parallel machine and can assign the execution of each scalar minimization procedure to a different processor. Similarly, the decomposition of the nondominated set into subregions proposed in Algorithm 4.1 allows for a parallel approximation of each subregion, which may essentially increase the numerical efficiency of MCDM methods applying ideal points.

There remain still some open questions, such as a more constructive description of the set of strictly dominating points in $\mathbb{R}^k$, $k > 2$, or an investigation of properties of those reference points which might be used in distance–maximization techniques. We hope, however, that the theory presented in this Chapter may be helpful in understanding the nature of reference points which turn out to be one of the most relevant tools in decision support systems.

Chapter 5

Optimality Conditions for Decision Choice Methods Based on Reference Points and Proximity Measures

Usually, reference points are being defined independently from the original statement of multicriteria optimization problems, serving as an auxiliary source of preference information at the second stage of solution of multicriteria problems, namely at the stage of a compromise decision choice. However, if we cannot guarantee that posteriorly defined reference points dominate the ideal point, then the Pareto optimality of compromise solutions resulting from a distance minimization procedure is an open question. In this Chapter we investigate the conditions which should be satisfied by a reference point to ensure that the solution of the distance minimization problem is Pareto optimal. We apply the idea of strictly dominating points in E studied in the preceding Chapter, and prove that, under some additional conditions, if x is strictly dominating then distance scalarization with respect to x results in a Pareto optimal point. These results will be applied in design of decision support systems based on reference points.

5.1 Introduction

Distance minimization procedures are most common tools to generate compromise solutions to a vector optimization problem with the preference structure modelled by reference points. A variety of methods has been proposed by numerous authors making this approach one of most classical in vector optimization. The fundamental question which arises is as follows: under which assumptions the minimum of the distance scalarizing function

$$g(u) \; := \; d(x, F(u)) \; = \parallel x - F(u) \parallel, \tag{5.1}$$

where x is an element of the criteria space and the function d is a certain measure of proximity between u and x, exists and is nondominated in the set of decisions U.
A particular attention has been attracted by the case, where $d(x, y)$ is defined by certain norm in E, i.e. if $d(x, y) := \| x - y \|$. This problem has been studied by numerous authors at an early stage of development of multicriteria optimization - cf. e.g. Salukvadze (1971), Huang (1972), Dinkelbach and Duerr (1972), Yu (1973), Zeleny (1973), Wierzbicki (1975), Rolewicz (1975), Dinkelbach (1980) and others, who considered the case where x is an ideal point for $F(U)$, i.e. the vector in the criteria space with the coordinates equal to the optimal values of scalar criteria evaluated individually. So defined x dominates all attainable values of F. In this case, as well as in its simple generalization, where x dominates the ideal point, one can prove that under relatively weak assumptions concerning the set of attainable values of criteria, each point minimizing the function (5.1) is nondominated, or even properly nondominated (cf. Gearhart, 1979). Similar results can be obtained for abstract problems in Hilbert and Banach spaces ordered by a closed, convex, and pointed cone satisfying certain additional assumptions (cf. Rolewicz, 1975). Jahn (1984) presented a collection of general results on properties of scalarization methods including the norm scalarization as one of the subcases, while Zaharopol (1989) studied the monotonicity properties of norms. The above results have been applied in various, usually interactive, procedures supporting the decision-maker in the choice of a compromise decision (cf. Zeleny, 1974,1975, for the method of the displaced ideal; Kallio, Lewandowski and Orchard-Hays, 1980; Lotfi, 1989; Lotfi, Stewart and Zionts, 1992; Popchev, Metev and Yordanova, 1989, and Metev and Yordanova, 1993). Some of the reference point approaches are directly related to the goal programming and its generalizations (Ignizio, 1978, 1981; Romero, 1986; Martel, Aouni, 1990; Thore, Nagurney and Pau, 1992), using sometimes fuzzy reference points (cf. e.g. Rao, Tiwari, and Mohanty, 1988). Different distance functions have been used as the proximity measures, starting from the first approaches using the L_p distances (Yu and Leitmann, 1974a, cf. also Terlaky, 1989), through some attempts to use more complicated functions in so-called *composite programming* (Bardossy, Bogardi, Duckstein, 1985; Jeyakumar, Yano, 1993), until the advantages of the L_∞ distance (called also the Tchebyshev norm) and its modifications have been early recognized. This resulted in a variety of decision-choice methods applying this norm to calculate the least-distance solution, mostly for the linear multicriteria optimization problems (see Bowman, 1976; Sakawa in his SPOT method (1982); Wood, Greis and Steuer, 1982; Steuer and Choo, 1983; Kaliszewski, 1986; Yano, Sakawa, 1987). Later on, Wierzbicki (1986) proposed a similar approach based on a family of modified distance functions called by him *achievement functionals* which are defined so that the minimizing solution is always θ-optimal.

However, in distance scalarization there exist still open problems. In this Chapter we will pay our attention to the case where x is a point which dominates some but not all nondominated points. Elements of the criteria space of this property will be called partly dominating points. We will impose additional conditions on these points, which were defined in the preceding Chapter, to ensure that for a class of norms in the criteria space the minimum of the scalarizing function (5.1) will be admitted at a nondominated point. Namely, we will pay a special attention to the strictly dominating reference points. In the Subsection 5.3 we will apply the specific geometric properties of the set of strictly dominating points to derive a stronger sufficient condition for θ-optimality in bicriteria problems.

5.1.1 Basic definitions and properties

As in the previous Chapters, we will refer to the vector optimization problem

$$(F : U \to E) \to \min(\theta),$$

where U is an arbitrary set of admissible decisions, E is a linear space partially ordered by a closed, convex and pointed cone θ, and F is to be minimized with respect to the partial order introduced by θ. In further considerations F and U will play no separate role, hence we will concentrate our attention on the set $X := F(U) \subset E$ and its relation to the reference point x occurring in the function (5.1).

Recall that an element y of X fulfilling the condition $(y - \theta)) \cap X = \{y\}$ will be called θ-minimal or nondominated in X. A $(-\theta)$-minimal point will also be called θ-maximal. The set of all θ-minimal points in X will be denoted by $P(X, \theta)$. Similarly as in the preceding Chapter, throughout this Chapter we will assume that X is θ-closed and θ-complete, by definition this means that $X + \theta$ is closed and satisfies the property

$$\forall x \in X \;\; \exists v \in P(X, \theta) \quad : \quad v \leq_\theta x.$$

Recall that a subset X of E satisfying the above condition is sometimes called *externally stable* or *having the domination property*. This condition has been investigated by Benson (1983,1984) and Henig (1986), and it is a sufficient condition for the existence of solutions to the scalarization problem with the scalarizing function (5.1).

As previously, the set of totally dominating points will be denoted by $TD(X, \theta)$, the set of ideal points, by definition the set of $(-\theta)$-minimal elements of $TD(X, \theta)$, will be denoted by $x^*(X, \theta)$. Let us note that $TD(X, \theta)$ can be expressed in the form

$$TD(X, \theta) = x^*(X, \theta) - \theta.$$

A point $y \in E$ such that $(y + \theta) \cap P(X, \theta) \neq \emptyset$ will be called a partly dominating point for X. The set of partly dominating points will be denoted by $PD(X, \theta)$.

The necessary conditions for θ-minimality in distance-minimization derived in previous research touched upon the reference points being ideal, or totally dominating points for X, with some corollaries regarding partly dominating points with additional constraints in the criteria space. In this Chapter we will pay a special attention to the class of strictly dominating points, $SD(X, \theta)$, i.e. partly dominating elements of E fulfilling additionally the condition (4.5).

5.2 Distance Minimization with Respect to Dominating Reference Points

Now, we will present several new theorems on θ-optimality in scalarization via distance functions. Let us note that there exist scalarization methods based on transformed norms (cf. Wierzbicki, 1986) called there *achievement functions* which let avoid much of difficulties with classical distance functions. However, the evidence from the practice shows that in some cases the decision-maker's preferences can be modelled by the distance functions in a most adequate way. This is also the reason why we will not be concerned here on other features of scalarization methods such as completeness of characterization of properly efficient points, or computational issues - a distance function will be treated here as an estimation of a value function for vector optimization problem.

In our further considerations we will often refer to the following geometric optimality condition (5.2) introduced by Rolewicz.

Lemma 5.1. *(Rolewicz (1975). If the cone θ is closed, convex, and pointed, X is a θ-closed subset of E, and the norm in E is such that*

$$\forall x \in E \ \ \theta \cap (x - \theta) \subset k_{\|x\|}(0) \cup \{x\}, \tag{5.2}$$

where $k_{\|x\|}(0)$ denotes the open ball with center 0 and radius $\| x \|$, then for each $q \in TD(X, \theta)$ the scalarizing function $z \to \| q - z \|$ admits its infimum on X at a θ-minimal point.

The geometric interpretation of condition (5.2) for a translated cone $y + \theta$ is shown in Fig. 5.1.

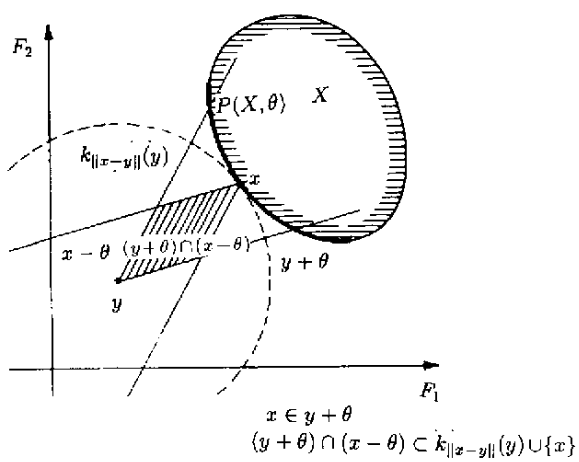

Fig. 5.1. *A geometric interpretation of the condition (5.2).*

Notice that if E is a Hilbert space, then Lemma 5.1. generalizes an earlier result of Wierzbicki (1975), who proved under same assumptions concerning θ, X, and x that the condition

$$\theta \subset \theta^*, \tag{5.3}$$

where

$$\theta^* := \{y \in E : \ \forall z \in \theta \ \ < y, z > \geq 0\}$$

is the dual cone, is sufficient for $\theta-$minimal- ity of least distance elements of X. Namely, it turns out that (5.2) and (5.3) are equivalent in the case of a Hilbert criteria space (cf. Rolewicz, 1975). Later on, Jahn (1984) proved a series of general theorems on scalarization methods, including distance scalarization, using the notion of strongly monotonically increasing functionals, by definition, $f : E \to I\!R$ is *strongly monotonically increasing* (s.m.i.) on E iff

$$x \leq_\theta y, \ x \neq y \Rightarrow f(x) < f(y). \tag{5.4}$$

Assuming that the norm in E is s.m.i. and $x_0 \in TD(X, \theta)$, one can easily prove that the least-distance solution to x_0 in E is θ-minimal, however, it is not hard to see that (5.2) holds iff the norm is s.m.i., therefore these theorems coincide.

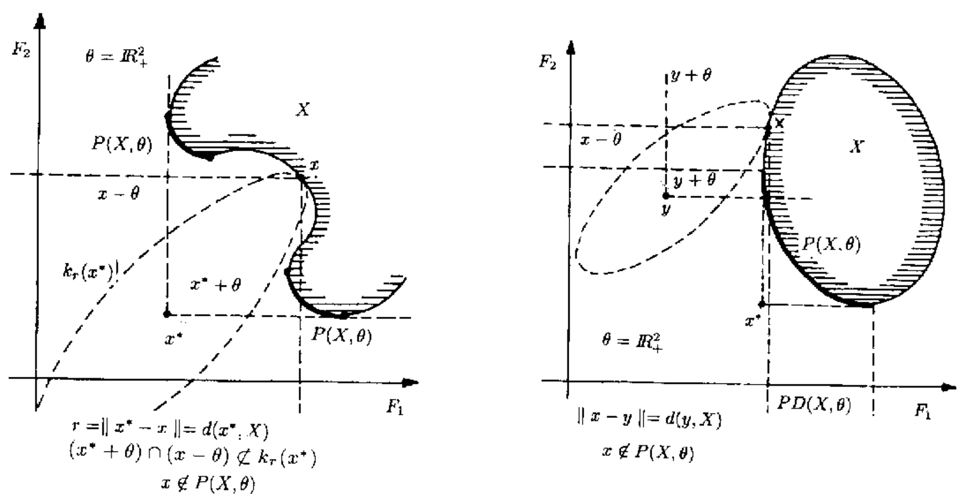

Fig. 5.2. *Examples of situations, where the least-distance element x in X to a dominating reference point x_0 need not be θ-minimal :*
a) as x_0 we took the ideal point $x^*(X, \theta)$ but (5.2) is not satisfied,
b) $x_0 := y \in PD(X, \theta)$, X is convex, but (5.2) does not hold.

Condition (5.2) is evidently not necessary, but when it is not satisfied, θ-minimality of points minimizing (5.1) depends on the shape of X, and on the situation of x with respect to X, which are usually not a priori known. Several examples of situations, where the least-distance element fails to be θ-minimal are presented in Figs. 5.2. and 5.3.

Remark 5.1. *Let us note that the statement "If $d(p, X) = \| p - x \|, x \in X, p \leq_\theta x$, and (5.2) holds then $x \in P(X, \theta)$" (cf. Rolewicz (1975), Thm. 1') may not be true when X is not convex, which is exemplified in Fig. 5.3.*

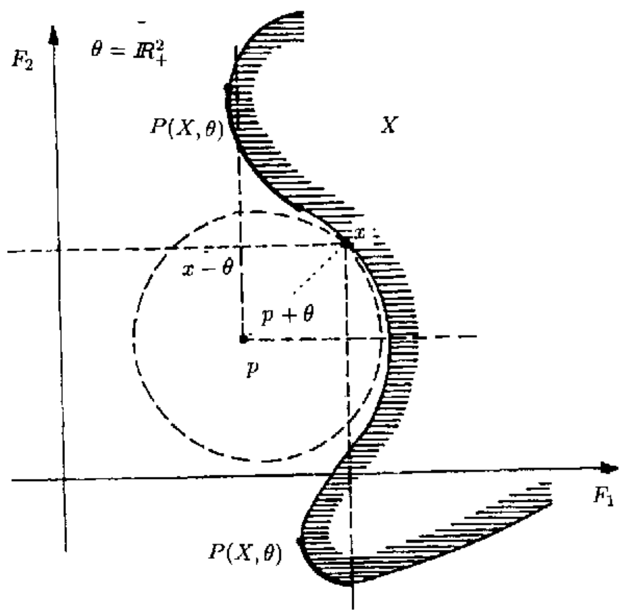

Fig. 5.3. *An example of situation, where (5.2) is fulfilled and a point x least-distant to p is situated within the set $p + \theta$, but x_0 is not θ-minimal.*

Now we will show that in the case $E := I\!R^2$ the condition (5.2) is too strong, namely one can prove the following theorem (cf. Skulimowski, 1988).

Lemma 5.2. *Suppose that $E = I\!R^2, X$ is θ-convex, and θ is an arbitrary closed and convex cone such that $x^*(X, \theta)$ and $P(X, \theta)$ are non-empty. If f is a non-constant convex function defined on E, having its global minimum attained at a point θ belonging to*

$$Z(X, \theta) := (x^*(X, \theta) + \theta) \cap PD(X, \theta) \tag{5.5}$$

then the minimum of f on X is attained at a θ-minimal point. ■

Proof. Let x be a point of X such that the minimum of f on X is attained at x. The function f is convex then for each point y from the interval $[q, x]$ $f(y) \leq f(x)$, and the function

$$f(t) := \bar{f}(tq + (1-t)x)$$

is convex and non-decreasing on [0,1].

Let $\bar{t}$ be the infimum of $t \in [0,1]$ such that $\bar{f}(t) = f(x)$ and let us denote $\bar{x} := \bar{f}(\bar{t})$. From the introductory assumptions it follows that the boundary of θ consists of two half-lines and $P(X,\theta)$ is a curve which separates dominated and dominating points in $x^*(X,\theta) + \theta$. If x is a dominated element of X then there exists

$$\bar{y} \in [q, \bar{x}) \cap P(X,\theta) \subset X,$$

consequently, $f(\bar{y}) \leq f(x)$ which leads to a contradiction with the minimality of $f(x)$. Therefore $x \in P(X,\theta)$. ■

Corollary 5.1. *Under the assumptions of Lemma 5.2 every solution to the scalarization problem (5.1) with the reference point q being an element of $Z(X,\theta)$, is θ-minimal.* ■

Corollary 5.2. *Suppose that $X = F(U)$ is a closed subset of $I\!R^2$. If $F = (F_1, F_2)$ and*

$$\inf\{F_1(u) : u \in U\} = \inf\{F_2(u) : u \in U\} = -\infty,$$

then for each convex function f having its global minimum attained at a partly dominating point of X (if $PD(X,\theta) \neq \emptyset$) the solution to the scalarization problem

$$(f : X \to I\!R) \to min$$

is θ-optimal. ■

Remark 5.2. *In the proof of Lemma 5.2 we used only the property that in convex bicriteria problems the set $P(X,\theta)$ is a curve dividing the set $x^*(X,\theta) + \theta$ in two disjoint subsets. Hence Lemma 5.2 and Corollary 5.1. remain true if $P(X,\theta)$ is an arbitrary surface dividing $x^*(X,\theta) + \theta$, irrespectively of the dimension of E and the convexity of $X + \theta$. Thus the above convexity assumption may be replaced by a quasiconvexity, or another suitable kind of generalized convexity.* ■

It turns out, however, that in $I\!R^2$ the condition (5.5) warrants that a solution to (5.1) is θ-optimal for all $y \in PD(X,\theta)$.

Theorem 5.1. *If $X \subset I\!R^2$ is θ-convex and θ-closed, θ is closed, convex, pointed, and satisfies condition (5.2), then any solution to the scalarization problem*

$$\| y - x \| \to \min, \ x \in X,$$

is θ-minimal for each partly dominating reference point y. ■

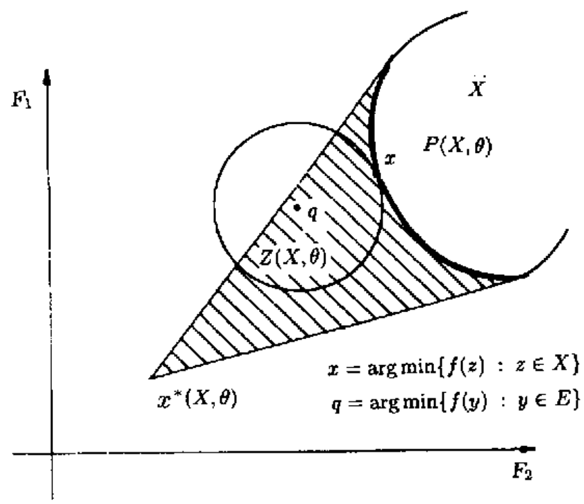

Fig. 5.4. *An illustration of the proof of Thm. 5.1.*

Proof. Let y be an arbitrary partly dominating point, and let x be an element of X such that $\| x - y \| = d(y, X)$. Suppose first that the cone θ does not degenerate to a half-line. The set $PD(X,\theta)$ can be decomposed into the disjoint union of sets

$$Z(X,\theta) \text{ (cf.5.5)}, \;\; A_0 := TD(X,\theta) \setminus x^*(X,\theta), \text{ and}$$

$$A_1 := \{(z_1, z_2) \in PD(X,\theta) \;:\; z_1 > x_1^*, \; z_2 < x_2^*\},$$

and

$$A_2 := \{(z_1, z_2) \in PD(X,\theta) \;:\; z_1 < x_1^*, \; z_2 > x_2^*\},$$

(some of them may be empty), where $x^*(X,\theta) = (x_1^*, x_2^*)$, and all coordinates are related to a base spanning θ.

Let us note that this theorem is already proved for a y belonging to $Z(X,\theta)$ or $TD(X,\theta)$ (cf. Lemmas 5.2 and 5.1, respectively). If $y \in A_1$ or $y \in A_2$ and $y \leq_\theta x$ then $x \in P(X,\theta)$ by Corollary 5.1. Suppose that $y \in A_1$, and x is non-comparable with y. The set of dominated points in X, non-comparable with any point of A_1 is separated from A_1 by sets $Z(X,\theta)$ and A_2, therefore the interval $[y, x]$ must have a common point v either with A_0 or A_2. Let w be a least-distance point to v in X. Of course,

$$\| w - v \| \leq \| x - v \|,$$

hence by the triangle inequality

$$\| y - w \| \leq \| y - x \|,$$

consequently w is also least-distant for y in X. If $v \in Z(X, \theta)$ then w is θ-minimal, if $v \in A_2$ then $v \leq_\theta w$ - otherwise w would be dominated by y which was excluded, hence $w \in P(X, \theta)$. Similarly we conclude that $w \in P(X, \theta)$ for $y \in A_2$, and $x \in P(X, \theta)$. If θ degenerates to a half-line then the above reasoning is true for any non-degenerated cone θ_1 containing θ and fulfilling the assumptions of this Theorem. Since $P(X, \theta_1)$ is contained in $P(X, \theta)$ then $w \in P(X, \theta)$ in this case as well. Therefore $x \in P(X, \theta)$. ∎

We will conclude this section with a general theorem giving a sufficient conditions for θ-optimality for distance minimizing points with respect to strictly dominating reference points and non-convex attainable sets X.

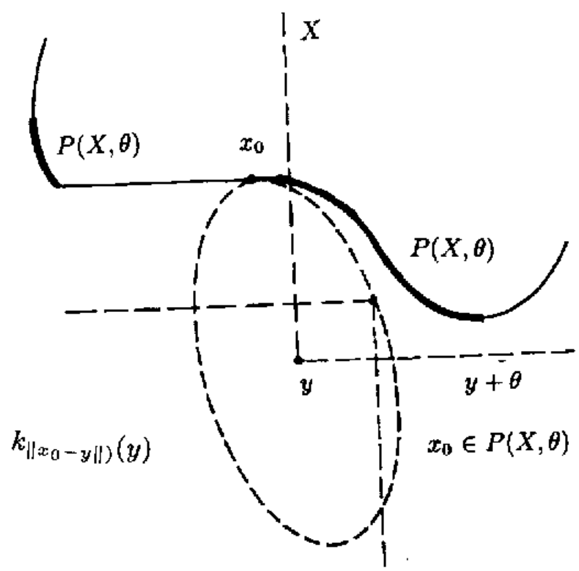

Fig. 5.5. *The condition (5.2) is not sufficient for θ-minimality of a least-distance solution x_0 to a reference point $y \in SD(X, \theta)$ without the additional constraints $x_0 \leq y$.*

Theorem 5.2. *Suppose that X is θ-closed, the cone θ is closed, convex, pointed, and satisfies condition (5.2). Then for each strictly dominating point y the solution to the scalarization problem*

$$\| y - x \| \to \min,\ x \in X,\ y \leq_\theta x, \tag{5.6}$$

is θ-optimal in X. ∎

Proof. Let us take an arbitrary $y \in SD(X, \theta)$. By the definition of strictly dominating points, all θ-minimal points in the set $X \cap (y + \theta)$ are elements of $P(X, \theta)$. Since y is the ideal point for $X \cap (y + \theta)$, then by Lemma 5.1 a least-distance element in this set with respect to y is θ-minimal in $X \cap (y + \theta)$, consequently it is θ-minimal in X. ∎

One can see that the above theorem may not be strengthened by removing the constraints $y \leq_\theta x$. An example when a least-distance point x to a strictly dominating reference point y fails to be θ-minimal can be easily found even in $I\!R^2$ as it is exemplified in Fig. 5.5.

The problem of removing the additional constraints in (5.6) will be dealt with in the next subsection.

5.3 A Sufficient Condition for θ-Optimality without Auxiliary Constraints

So far, we have shown that a distance scalarization procedure with respect to a strictly dominating reference point may lead to a dominated least-distance point even in $E = I\!R^2$. However, imposing an additional regularity condition on the distance in E we can prove the following Theorem 5.3. In its proof we apply information about the structure of the set of strictly dominating points which has been studied in a more detailed way in Chapter 4.

Theorem 5.3. *If $E = I\!R^2$, $X \subset E$ is connected, θ satisfies (5.2), for each $x \in E$, and $r > 0$, the balls $k_r(x)$ are balanced subsets of E, and for $i = 1, 2$ satisfy the condition*

$$arg\ max\{z_i \ : \ z \in k_r(x)\} \subset (x + \theta) \cap (x - \theta) \tag{5.7}$$

then a least-distance element x_0 in X to a strictly dominating reference point y is θ-optimal in X, irrespectively whether the additional constraints $y \leq_\theta x_0$ are fulfilled or not. The coordinates of $z = (z_1, z_2)$ must be related to a positively oriented basis (r_1, r_2) spanning a cone θ_1 such that $\theta_1 \supset \theta$. ∎

Proof. Let S be the set of connected components of $P(X, \theta)$, $S = \{S_i\}_{i \in I}$, where I is an ordered set, and let us associate to each S_i the local ideal point $x_i^* = (x_{i1}^*, x_{i2}^*)$ and the edge points $e_{1i} = (x_{1i}, e_{12i})$ and $e_{2i} = (e_{21i}, x_{i2})$, where

$$e_{12i} := \ \inf\{x_2 \in I\!R \ : \ (x_{i1}, x_2) \in X\}$$

and

$$e_{21i} := \inf\{x_1 \in I\!R \ : \ (x_1, x_{i2}) \in X\}.$$

We assume that the components S_i are ordered in such a way that if $i \prec j$ then $x_{i1} < x_{j1}$. We will admit the convention $\inf \emptyset := \infty$, but it may be shown that such situation may happen only if $i = \inf I$ for e_{12i} or $i = \sup I$ for e_{21i}. Without a loss of generality we can assume that θ is non-degenerated and let r_1 and r_2 be a base spanning θ.

In the preceding Chapter (cf. Thm. 4.5, formula (4.22)), we proved that $SD(X,\theta)$ can be represented as a union of sets

$$Z(S_i,\theta) \text{ (cf. (5.5))},$$

$$pr_1(TD(X,\theta)) \times pr_2(S_i),$$

$$pr_1(S_i) \times pr_2(TD(X,\theta)),$$

for $i \in I$, further,

$$pr_1(S_i) \times pr_2(S_j) \text{ for } i,j \in I,\ i \prec j,$$

and

$$TD(X,\theta),$$

where pr_k denotes the projection parallel to the k-th axis.

Therefore it is sufficient to prove this Theorem for each possible situation of a reference point $y = (y_1, y_2)$ within $SD(X,\theta)$. By Proposition 5.1 if $y \in Z(S_i,\theta)$ then a least-distance element x to y in X situated within the set $(x_1 + \theta)$ is θ-optimal. However, the curve consisting of S_i and the half-lines

$$h_1 := \{(t, e_{12i}) \ : \ t < x^*_{1i}\}$$

and

$$h_2 := \{(e_{21i}, t) \ : \ t < x^*_{2i}\}$$

separates the area of local partly dominating points of S, P_{i1}, and the remaining part of $I\!R^2, P_{i2}$. The interior of P_{i1} does not contain any other point of X, since e_{1i} and e_{2i} are either θ-optimal or are limits of θ-optimal sequences. The subset P_{i2} contains points of X and we will show that they are more distant from y than e_{1i} or e_{2i}. Suppose that $x = (x_1, x_2) \in S_i$ and consider the interval $[y, x]$ which in this case must intersect h_1 or h_2 in a point $z = (z_1, z_2)$. Without a loss of generality suppose that $z \in h_1$ and $z_2 = e_{12i}$, consequently $x_2 \geq e_{12i}$. According to (5.7), the maximum of the second coordinate over $k_{\|x-y\|}(y)$ must be achieved within $y + \theta$. At the same time, it follows from (5.7) and Proposition 5.1, that $k_{\|x-y\|}(y)$ has a common point with an element v of $P(X,\theta) \cap (y + \theta)$.

Observe now that for each θ-optimal point $v = (v_1, v_2)$ it holds

$$x^*_{i2} \leq v_2 \leq e_{12i},$$

therefore

$$\max\{z_2 \ : \ z \in k_{\|x-y\|}(y)\} = e_{12i},$$

while no element of $X \setminus ((y+\theta) \cap (x-\theta))$ may have its second coordinate greater or equal to this value. Hence we obtain a contradiction with the assumption that there exists a least distance element $x = (x_1, x_2) \in X \setminus (y+\theta)$ such that $x_2 \geq e_{12i}$.

Similarly we will prove this Theorem in the case where

$$y \in R_{ij} \ := pr_1(S_i) \ \times \ pr_2(S_j), \ i, j \in I, \ i \prec j,$$

where S_i and S_j are connected components of $P(X, \theta)$ and pr_k denotes the projection parallel to the k-th axis. Analogously as above we will define

$$h_1 \ := \ \{(t, e_{12i}) : t < x^*_{2i}\}$$

and

$$h_2 \ := \ \{(e_{21j}, t) \ : \ t < x^*_{2j}\},$$

and let us suppose that a least-distance element $x = (x_1, x_2)$ does not belong to $(y+\theta) \cap X$, so that the interval $[y, x]$ intersects h_1 at a point $z = (z_1, z_2)$. Now, it is sufficient to show that no element $v = (v_1, v_2)$ of $k_{\|x-y\|}(y) \cap (y+\theta)$ has its second coordinate greater than e_{12i} .

If $v \in P(X, \theta) \cap (y+\theta)$ then

$$x^*_{j2} < v_2 < e_{12i},$$

hence we need only to investigate the case then $v \notin P(X, \theta)$. If $v \in y+\theta$ then by the condition (5.2)

$$(v-\theta) \cap (y+\theta) \subset k_{\|y-v\|}(y) \subset k_{\|y-x\|}(y),$$

since $v \in k_{\|y-x\|}(y)$. By definition of R_{ij},

$$y_2 \in pr_1(S_i),$$

and

$$\forall w = (w_1, w_2) \in S_i \ : \ w_2 < e_{12i},$$

therefore if $v_2 > e_{12i}$ then $(v-\theta) \cap (y+\theta)$ would have a non–empty intersection with S_i, consequently an element of S_i would belong to the interior of $k_{\|y-x\|}(y)$ which is impossible, because we assumed that x is least–distant to y in X. Thus we get a contradiction with the assumption that $x \notin y + \theta$. Observing that in the situation where

$$[x, y] \cap h_2 \neq \emptyset$$

the proof is a repetition of that above let us end the proof in the case where $y \in R_{ij}$.

Same arguments (only one separating half–line is needed) prove θ–optimality of a least distance point x to a reference point y contained in the cartesian product of $TD(X, \theta)$ and a projection of a connected component of $P(X, \theta)$. Since θ–optimality of x in the case $y \in TD(X, \theta)$ we quoted as a classical result then all possible situation of y have been investigated and the proof of this Theorem is completed. ■

Based on the above Thm. 5.3. and Lemma 4.2 we can also provide a θ–optimality criterion for the disconnected case. Observing that:

(i) if θ is non–degenerated then $X + \theta$ is connected,

(ii) $A \subset B \Rightarrow SD(B, \theta) \subset SD(A, \theta)$,

(iii) $P(X, \theta) = P(X + \theta, \theta)$ i.e. X and $X + \theta$ have the same connected components of the set of θ–optimal points,

let us conclude that the set

$$\bigcup \{pr_1(S_i) \times pr_2(S_j) : \ i, \ j \in I \cup \{0\}\} \cap PD(X, \theta)$$

(cf. Theorem 4.5 in Chapter 4), where S_i, for $i \in I$, are connected components of $P(X, \theta)$, $S_0 := TD(X, \theta)$, and pr_i is the projection on the i–th axis in the base spanning θ, is equal to $SD(X + \theta, \theta)$ and, at the same time, it is contained in $SD(X, \theta)$. As a consequence, the following statement is true.

Corollary 5.3. *If X is a closed subset of $\mathbb{R}^2$, y is an element of $SD(X + \theta, \theta)$, and the distance and the ordering cone in $\mathbb{R}^2$ satisfy (5.11) and (5.13) then a least–distance solution to y in X is θ–optimal.* ■

It is easy to see that if X is not connected and

$$y \in SD(X, \theta) \setminus SD(X + \theta, \theta)$$

then a least–distance element to y in X may not be θ–optimal even if (5.2) and (5.7) are satisfied, as an example may serve the situation presented in Fig. 4.5.

It is interesting to know which norms in $\mathbb{R}^2$ satisfy the condition (5.7). It turns out that a simultaneous satisfaction of (5.2) and (5.7) is equivalent to another, more intuitive property, which may be called "symmetric monotonicity".

Lemma 5.3. *Suppose that $\| \, . \, \|$ is a norm in $\mathbb{R}^n$. Then the following conditions are equivalent:*

(i) $\| \, . \, \|$ *satisfies (5.2) and (5.7);*

(ii)
$$[\forall i \in \{1,\ldots,n\} \setminus \{j\} : \; y_i = z_i, \; | \, y_j \, | < | \, z_j \, |, \; \textit{and} \; \; y_j \; z_j \geq 0] \Rightarrow \| \, y \, \| < \| \, z \, \| \\ \textit{for all} \; \; y = (y_1,\ldots,y_n), z = (z_1,\ldots,z_n) \in I\!R^n, \; \textit{and} \; \; j \in \{1,\ldots,n\} \tag{5.8}$$

■

Proof. Assume first that $\| \, . \, \|$ satisfies both (5.2) and (5.7). If y_j and z_j are non–negative then the inequality $\| \, y \, \| < \| \, z \, \|$ is implied by the fact that $\| \, . \, \|$ is strictly monotonically increasing which is equivalent to the assumed condition (5.2). If y_j and z_j are both negative then (5.8) can be expressed as

$$z_j < y_j \Rightarrow \| \, y \, \| < \| \, z \, \| \; .$$

Suppose the contrary, i.e. let $x \; \in \; I\!R^n$ be such that

$$z_i \;\; = \;\; x_i) for (i = 1, \ldots, j-1, j+1, ..., n) and (x_j < y_j < 0,$$

but $\| \, z \, \| \leq \| \, x \, \|$, and let us consider the closed ball $K_r(0)$, where $r := \| \, z \, \|$. From the theory of normed spaces it follows that $K_r(0)$ intersected with any affine subspace H of $I\!R^n$ such that $H \cap \theta$ is a convex cone in H, is a ball in H. Moreover, if the maximum of the j–th coordinate over $K_r(0)$ has been achieved at a point $v \in \theta$, then the maximum of j–th coordinate over $K_r(0) \cap H$ is achieved at $w \in H \cap \theta$.

Consider now the two dimensional affine subspace H_{ij} of $I\!R^n$ spanned by the i–th element of the base of $I\!R^n$ and passing through x and z. By (5.7) the maximum of i–th coordinate over the ball

$$K_r(0) \cap H_{ij}, \quad i \in \{1,\ldots,n\}, \; i \neq j,$$

is achieved at a point v such that $v_j > 0$. Of course, $v_i \geq z_i$, since $z \notin \theta$, hence also $v_i \geq x_i$. Therefore x is contained in the triangle $T = [z, v, c_{ij}]$, where

$$c_{ij} := (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n),$$

which by the convexity of the norm is contained in $K_r(0) \cap H_{ij}$.

Consequently, x would belong to $K_r(0) \cap H_{ij}$, but we assumed that $\| \, x \, \| \geq r$, i.e. x must belong to the boundary of $K_r(0)$. However, in this case x must also belong to the boundary of T, namely to the interval $[z, v]$, which is impossible, since we assumed that $z_i = x_i < v_i$. This contradiction implies that the assumptions made were sufficient for the inequality $\| \, x \, \| < \| \, z \, \|$, which ends the first part of the proof.

To prove that (5.8) implies (5.2) and (5.7), it is sufficient to show that the maxima of coordinates over $K_r(0)$ are not achieved on $I\!R^n \setminus \theta$. The equivalence of (5.2) with a subcase of (5.8) has already been noted in the first part of this proof. Let us take $k \in \{1,\ldots,n\}$, two elements of $I\!R^n$, x and y such that $x_i = y_i$ for $i \in \{1,\ldots,j-1,j+1,\ldots,n\}$, $j \neq k$, and $x_j < y_j \leq 0$. By (5.8) $\| \, x \, \| > \| \, y \, \|$, consequently $y \in k_{\|x\|}(0)$, and in certain neighborhood of y there are points of $k_{\|x\|}(0)$ such that their k–th coordinate is greater than $y_i = x_i$. Since this schema is true for all $x \in \mathbb{R}^n \setminus \theta$ then it follows that the maximum of k–th coordinate cannot be achieved at such point. ■

Let us remark that neither (5.2) implies (5.7) nor vice versa, an example of the distance satisfying (5.2) but not (5.7) is shown in Fig. 5.5, while the balls in a norm which satisfies (5.7) but fails to fulfill (5.2) are shown in Fig. 5.2.a and 5.2.b. Such norm can be defined e.g. by the formula:

$$n(x) := (0,5(x_1 + x_2)^2 + 2(x_1 - x_2)^2)^{1/2}.$$

Now it is easy to see that (5.8) is satisfied by distances generated by L_p norms, $1 \leq p < \infty$, i.e. functions of the form

$$L(x, p, w) := (\sum_{i=1}^{n} w_i \mid x_i \mid^p)^{1/p}, \tag{5.9}$$

where $w_i > 0$ for $1 \leq i \leq n$, x_i are the coordinates in $\mathbb{R}^n$ related to an arbitrary base spanning θ, and p is defined as above.

As the conclusion, we may formulate the following.

Corollary 5.4. *If the distance in $\mathbb{R}^2$ is generated by the norm which can be expressed by the formula (5.9) in the coordinates related to the cone θ, then a least–distance element in X to a reference point $y \in SD(X + \theta, \theta)$ is θ–optimal.* ■

5.4 Distance Minimization with Respect to Convex Reference Sets

Instead of a single point we may also consider arbitrary closed subsets of an ordered Banach space as sets of the desired values of the vector criterion. We will address the problem of reducing the distance–minimization procedure to take into account the minimal subset of the reference set. Next, we list some conditions which warrant that the solution of a generalized distance–minimization problem is nondominated. Some particular cases of reference sets are also studied.

The main problems which arise while introducing this extension of distance scalarization require the following:

- determination of cases where the reference set can be replaced by a reference point, and
- identification of assumptions that guarantee the solutions chosen is nondominated.

The above problems are studied in the Subsections. 5.4.1 and 5.4.2. To make possible the solution of trajectory optimization problems we will assume that the goal space is a partially ordered Banach space. We will also use the notation

$$d(p, Q) := \inf\{\| p - q \| : \ q \in Q\},$$
$$d(P, Q) := \inf\{\| p - q \| : \ p \in P, \ q \in Q\},$$

where $p \in E$, $Q \subset E$, $P \subset E$.

5.4.1 Reference sets determined by reference points

The distance scalarization method sketched above makes use of additional information concerning preferences which is represented by the distance in the space of goals and the reference point q, both of them influence the choice of a nondominated strategy in a fundamental way. Reference point q may usually be treated as a minimal aspiration level which is to be approximated by nondominated solutions of vector optimzation problem, i.e. any point q' less than q would be at least equally desired as a potential solution to vector optimization problem. Therefore when taking into account an aspiration level q as a source of additional preference information, we implicitly deal with the reference set

$$Q := q - \theta. \tag{5.10}$$

The main problem that now arises may be formulated as follows:
Is the distance minimization procedure with respect to Q always equivalent to minimizing the distance from q?

An answer is given by the following

Theorem 5.4. *If the closed and convex cone θ in the Banach space E fulfils the condition (5.2) then for an arbitrary totally dominating point q and θ–closed set $F(U)$*

$$d(F(U), q - \theta) = d(F(U), q). \tag{5.11}$$

If, moreover, the set $F(U)$ is θ–convex then the relation (5.11) holds for every q belonging to $PD(F, U, \theta)$.

Proof. First we assume that $q \in TD(F(U), \theta)$ and that $F(U)$ is an arbitrary θ–closed subset of E. Let y be an element of $F(U)$ such that the distance $r := \| y - q \|$ is minimal in $F(U)$. The first part of the theorem will be proved if we show that y is the unique element of intersection of the sets $F(U)$ and $K_r(q) - \theta$. Furthermore, by the inclusion

$$F(U) \subset q + \theta$$

it is sufficient to prove that no attainable element different from y belongs to the set

$$W := (K_r(q) - \theta) \cap (q + \theta).$$

Condition (5.2) of Lemma (5.1) implies that

$$W \subset K_r(q) \cap (q + \theta). \tag{5.12}$$

However, we assumed that the only attainable element of the ball $K_r(q)$ is the point y. Therefore (5.12) completes the proof in the case where $q \in TD(F(U), \theta)$.

To prove the theorem in the case where q is a partly dominating point for the θ–convex set $F(U)$ we will use the geometrical version of the Hahn–Banach theorem. Let q, r, and y be such as in the first part of the proof. The sets $K_r(q)$ and $F(U) + \theta$ are convex and y is the only element of their intersection so there exists a hyperplane H dividing the space E into two halfspaces E_1 and E_2 such that $F(U) \subset E_1$ and $K_r(q) \subset E_2$. By the condition (5.2) H separates also $F(U)$ and $y - \theta$. Similarly as previously we conclude that $y + \theta \subset E_1$. Hence

$$(y + \theta) \cap H = \{y\}. \tag{5.13}$$

By definition of H, the hyperplane

$$H_1 := H + (q - y)$$

fulfils the condition

$$H_1 \cap (q - \theta) = \{q\}.$$

Moreover, each ball of radius r and center $z \in H_1$ is separated from E_1 by the hyperplane H, so that

$$d(H, H_1) = \| y - q \| = r. \tag{5.14}$$

The formulas (5.2) and (5.14) let us conclude that each ball of radius r and center $x \in (q - \theta) \setminus \{q\}$ is contained in the interior of E_2. Hence

$$F(U) \cap (q - \theta + K_r(0)) = F(U) \cap K_r(q), \tag{5.15}$$

which is equivalent to the second part of the thesis. ■

The theorem proved above constitutes a link between the classical distance scalarization and applying reference sets for selecting a solution to multicriteria optimization problems. The consideration of the reference set $Q = q - \theta$ was motivated by an implicit assumption that if q is a reference point for the multicriteria optimization problems then so is each point p better than q.

Another example of a reference set was proposed by Górecki (1982) who considered a cuboid in the space $\mathbb{R}^N$ as the set of desirable values of N simultaneously minimized scalar criteria. This concept can be generalized for an arbitrary linear goal space, namely, we will now consider reference subsets of the form

$$Q := (q_1 - \theta) \cap (q_2 + \theta), \quad q_2 \leq_\theta q_1. \tag{5.16}$$

Points q_1 and q_2 will be called the upper, and the lower reference level, respectively. Taking into account only the lower reference level q_2 may not be consistent with the θ–optimality required for the solutions of multicriteria optimzation problems. However, such "limits of optimality" often occur in hierarchical multicriteria decision making, where the set Q is defined by a superior decision center or independent experts, which is explained in the following Chapter. Examples of decision situations of the above type which have arisen in control of technological processes and in dynamical resource allocation have been presented by Górecki and Skulimowski (1986, 1988).

Considering (5.16) as the reference set raises the following problem, similar as for the reference set defined by the formula (5.10):
Under which assumptions the procedure of minimizing the distance from a non–attainable set Q is equivalent to finding this θ–optimal solution, which realizes the minimal distance from the upper reference level q_1?

An answer to this question turns out to be an easy corollary from Theorem 5.4.

Theorem 5.5. *Under the assumptions of Theorem 5.4 with respect to the upper reference level q_1 and the attainable set $F(U)$, the reference set Q defined by the formula (5.16) satisfies the following relation:*

$$d(F(U), Q) = d(F(U), q_1). \tag{5.17}$$

Proof. Let $r := d(q_1, F(U))$ and let Q_r be the closed r–neighborhood of Q, by definition

$$Q_r := Q + K_r(0) = (q_1 - \theta) \cap (q_2 + \theta) + K_r(0).$$

It is obvious that Q_r is contained in the set

$$V_r := (q_1 - \theta) + K_r(0).$$

According to Theorem 5.4 V_r has no common points with the reachable set $F(U)$, except for, possibly, the point $y \in F(U)$ which realizes the distance between q_1 and $F(U)$. ∎

5.4.2 Conditions of θ–optimality

A basic acceptability criteria for a multicriteria optimization method is θ–optimality of the obtained solutions. Now we will give some conditions assuring the consistency of generalized distance scalarization with the underlying vector optimization problem.

We will start this section with the following statement easily implied by Theorem 5.2:

Theorem 5.6. *Suppose that E is a Banach space with the partial order defined by a closed convex cone θ satisfying (5.2). Then the following alternative conditions (i)* *and (ii) imply that the solution to the multicriteria optimization problem obtained via minimizing the distance from a closed set $Q \subset E$ is nondominated:*

(i) $Q \subset TD(F(U), \theta)$, $F(U)$ *is* θ*–closed,*

(ii) $Q \subset PD(F(U), \theta)$, $F(U) \subset I\!R^2$, $F(U)$ *is* θ*–closed and* θ*–convex.*

Proof. Let us assume that the condition (i) is fulfilled. Since the set Q is closed, there exists a point $q_0 \in Q$ such that

$$d(F(U), Q) = d(F(U), q_0).$$

By assumption (i) q_0 is an element of $TD(F(U), \theta)$ and it follows from Lemma 5.1 that if $y_0 \in F(U)$ minimizes the distance from q_0 then y_0 is nondominated. Condition (ii) can be dealt with in the same manner. ■

Less evident to prove is the θ–optimality of attainable points realizing the distance from a reference set which has nonempty intersection with the set $PD(F(U), \theta)$, but is not contained in it.

The following theorem concerning reference sets of type (5.16) has been proved by Skulimowski (1985a):

Theorem 5.7. *Let us assume that the attainable set X is θ–closed and θ–convex, the cone $\theta \subset E$ satisfies the condition (5.2), and the reference set Q defined by the formula (5.1b) contains at least one partly dominating point for X. Then there exists a nondominated point $y \in X$ realizing the distance between Q and X.*

Before formulating an analogous theorem for arbitrary reference sets we will define the conical hull of a set $Q \subset E$.

Definition 5.1 *Let Q be a bounded subset of E. The smallest set of form $(q_1 + \theta) \cap (q_2 - \theta)$ containing Q will be called the conical hull of Q and denoted by $\theta(Q)$.*

The existence of conical hulls, which is implied by the assumed properties of θ is discussed in the above mentioned paper (Skulimowski, 1985a).

We will close this section by the formerly announced

Theorem 5.8. *Let us assume that the attainable set $F(U)$ is θ–closed and θ–convex, θ is a closed convex cone satisfying (5.2)), and a reference set Q possesses the following properties (j) and (jj):*

(i) $\theta(Q) \cap PD(F(U), \theta) \neq \emptyset$,

(ii) $Q \cap H_1 \subset PD(F(U), \theta)$,

where H_1 is a hyperplane supporting $\theta(Q)$ at a point q closest to the set $F(U)$. Under the above assumptions each element of the attainable set $F(U)$ realizing the distance from Q is nondominated.

Proof of Theorem 4.3 is contained in the paper by Skulimowski (1985a).

5.5 Discussion

In this Chapter we concentrated our attention on the conditions for θ–optimality which were sufficient but not necessary. However, as it might be observed in numerous examples, the set of elements of the criteria space which may be used as the reference points in distance scalarization is influenced by the shape of X which may not be assumed a priori known. Therefore the results here presented may be classified as an improvement of the estimation from below of the set of potential reference points, for a fixed norm and a partial order in E.

Another question which appears in relation to the distance scalarization is the design of constructive algorithms to calculate the solutions of scalar optimization problems so arisen. Starting from the fundamental paper of Wolfe (1976), these questions, which are closely related to the projection–on–the–set problems, have been extensively studied by numerous researchers (cf. Stadje, 1982; Ariola, Laratta, Menchi, 1984, who considered the projection onto a polyhedron; Komiya, 1984, for the general problem in vector lattices; Phelps, 1985; Sehgal, Singh, Smithson, 1987; Fitzpatrick, 1989; Owens, Sreedharan, 1989; White, 1990; Dax, 1990,1993; Al-Sultan, Murty, 1992, Yang, 1993; finally, Li, Pardalos and Han, 1992, who apply the Gauss-Seidel method to calculate the minimum norm element, and others). Some specialized problems, such as an efficient algorithm to find the nearest points in two polytopes (Górecki, Madeja, Skulimowski, 1988; Sekitani, Yamamoto, 1993), finding the nearest neighbors in a discrete set (Guibas, Stolfi, 1983), or the computational complexity of norm maximization (Bodlaender et al., 1990) are also well–documented.

Remark that the relevance of distance minimization and θ–optimality problems here considered does not restrict to solving the multicriteria optimization problems. The least–distance regularizing solutions are also extensively used for the solution of ill–posed linear programming problems (cf. Betke, Henk, 1991), and other ill–posed problems in mathematical programming, optimal control, and data analysis. The problems considered in Section 5.4 are motivated by the fact that in numerous real-life multiple criteria optimization problems it is not sufficient to consider a single reference point. The desired values of criteria form rather a subset of the space of goals. In the above presented approach to the reference set method some basic properties characterizing the scope of possible application of the method have been given. The results there presented are supplemented by the methods of choice of an solution to the multicriteria optimization problems in the case where the reference set has a nonempty intersection with the attainable set which have been given in Górecki and Skulimowski (1986). Another perspective for future research lies in considering fuzzy or random reference sets. The construction of reference sets based on the elementary preference information gained from the decision–maker is presented in the following Chapter 6.

Computer implementations for linear reference sets in a finite dimensional goal–space can be obtained as an extension of existing reference point algorithms. New advantages, such as parallel modification of vertices of the reference polyhedron suggest their special usefulness in complicated interactive multicriteria decision making (cf. Górecki and Skulimowski, 1983). A procedure package for the numerical analysis of trajectory optimization problems has been evaluated recently and applied to optimization of a water reservoir system, multi–stage portfolio management, and other problems.

There are still open questions, such as the problem of removing the additional constraints occurring in Thm. 5.2 in the case where the dimension of the criteria space is greater than 2. Some questions such as proper or weak efficiency of the least–distance solutions or the completeness of characterization of $P(X, \theta)$ by distance functions associated to different reference points were not studied for the brevity's sake. A special class of scalarizing functions based on reference points which have been considered by Wierzbicki (1986) requires a separate treatment since they constitute an entirely different approach to scalarization problems related to the reference point method. Those functions are defined in such a way that θ–optimality of their minima is implied by the form of the functions. In our approach, however, a distance function or a family of them, as well as reference points, are imposed as a value function by the decision–maker. Then we have to verify whether the scalar minimization problem arisen has a θ–optimal solution. If the answer is positive we execute a distance–minimization procedure. Since the values of the distance functions are usually easily interpretable by the decision–makers, such situations occur frequently while solving real–life problems. Therefore, it is hoped that the results here obtained may be helpful to design decision support systems based on multiple reference points and reference sets described in the following Chapter.

Chapter 6

Application of Multiple Reference Points and Reference Sets to Multicriteria Decision Support

In this Chapter we present the foundations of the multiple reference points and reference set method of solving multicriteria decision problems and discuss its properties. To select a compromise solution to the multicriteria optimization problem, the decision–maker is requested to provide the additional information consisting of several families of reference points representing the values of criteria being of a special importance for the problem solution. The constraints on the trade–offs between the criteria can be considered simultaneously. The method applies the results of Chapters 4 and 5 concerning the optimality properties of distance scalarization procedures for different classes of reference sets to verify the well–posedness of the additional preference information. The supplementary information concerning the preferences in the criteria space is aggregated in such a manner, so that the problem is reduced to the bicriteria trade–off between the proximity measures to the sets of desired and avoidable values of criteria. The solution process is interactive, the decision-maker may define at each step new reference points, trade–offs, and search directions. Finally, we discuss an application of the above decision models to the dynamic portfolio optimization based on filtering the information available at each decision step and using the criteria of maximal revenue, minimal risk, and maximal investment flexibility to evaluate the optimal compromise decision. The method may also be applied to the trajectory optimization problems, discrete–event systems, and multicriteria optimal control problems.

6.1 Introduction

Similarly as in the previous Chapters, we will study the solutions to the vector optimization problems

$$(F : U \to E) \to \min(\theta). \tag{6.1}$$

U and E are called the decision space and the criteria space, respectively, $F = (F_1, F_2, \ldots, F_N)$ is the vector objective function, and θ is a closed and convex cone which introduces the partial order in E. In the most common case where $\theta := \mathbb{R}^N_+$, the corresponding partial order is the natural (coordinatewise) partial order in $\mathbb{R}^N$. The decision space U may be a list of discrete alternatives, a subset of a Euclidean space $\mathbb{R}^k$, or a subset of a function space. The criteria space E is assumed to be a partially ordered linear metric space, and the minimum in (6.1) relates to this partial order. The sets of nondominated decisions and nondominated values of criteria will be denoted by $P(U, \theta)$, and $FP(U, \theta)$, respectively.

Multicriteria decision problems here considered are decomposed into the following subproblems:

– finding a collection of Pareto–optimal solutions,

and

– selecting a compromise decision therefrom.

Solution choice methods applied to the Pareto–optimal subset of U use additional knowledge about the decision–maker's preferences not included in the formulation of the problem (6.1).

This Chapter presents a method of deriving preference structures from a set of points in the criteria space, called *reference points*, and their application to select a compromise solution to (6.1). A reference point is an element of the criteria space representing the values of criteria being of a special importance to the decision–maker. When regarded as a potential solution to the original decision problem, it may be evaluated as a "desired", "acceptable", or "wrong" optimization result. Using suitable mathematical methods one can find the element $u \in U$ so that its values of criteria, $F(u)$, most closely approximate the desired results, or are most distant to a "wrong" one.

The methods based on reference points constitute one of the most important classes of multicriteria decision-making procedures. The idea of reference points representing the desired (or ideal) values of criteria has been extensively studied in the past, cf. e.g. Wierzbicki (1986), Yu (1973), Yu and Leitmann (1974b), Zeleny (1973, 1974), and many others. The drawback of the classical approach based on a single reference point consists mainly in the fact that the decision–maker is allowed to define one such point at a time only. This leads to an inconsistency if two different criteria values are equally desired, but a separate approximation of each one of them results in different admissible solutions. The decision maker's judgment about the compromise solutions so generated touches upon, in fact, the choice of a scalarizing function.

However, to pass to the next interactive step, the decision maker has to change the reference point (cf. e.g. the approaches presented by Olbrisch, 1986; Bogetoft, Hallefjord, Kok, 1988). Moreover, the classical single reference point methods neglect usually the existence of the criteria space–constraints.

The above remarks can be regarded as a motivation to introduce the reference set method as a generalization of the reference point approach based on the distance scalarization with respect to a target subset of the criteria space (Skulimowski, 1985a). Following an earlier paper of Górecki and Skulimowski (1986), in the present Chapter we extend the reference point and target set methods to the situation where the preference information may be expressed as several classes of reference points which are to be taken into account simultaneously. Thus, besides of the target reference points we consider the anti–ideal reference points (or failure levels), achievement of which may be regarded as the failure, the solutions available at the pre–decision stage (or status–quo points), and the lower bounds of optimality. Additionally, each of the above classes of reference points may be split into subclasses.

We assume that reference points can be determined by the decision–maker, or are results of experts' judgments. The evaluations of experts are assumed independent from each other and from the constraints occurring in the multicriteria problem formulation, and have the same credibility to the decision–maker. Separate consideration of each point with any solution choice method would usually lead to specifying a set of non–comparable solutions, each of them being chosen based on only a part of all the information available. In order to integrate all preference information in one decision model, we propose a method of taking into account all reference points simultaneously. The method bases on aggregation of reference points into sets of desired and avoidable values, and on considering some of them as the criteria–space constraints. Thus we obtain a model of preferences consisting of a family of reference sets in the criteria space, and distance functions modelling the decision–maker's wish to reach one of the sets and to avoid another. To select a compromise solution, we find the set D containing the elements of E nondominated with respect to both distance functions. Then we confine the search for a compromise solution to the intersection of D and of the set of nondominated points to the problem (6.1).

6.2 The Decision–Making Problem Statement

In the present Chapter we will study the problem of selecting a single solution to (6.1) based on the additional information provided by experts or the decision–maker. Namely, the reference points will serve to define the function $v(x)$ called the *utility* or *value function* (cf. Section 3.4.1). We will show that v thus obtained is *strongly monotonically increasing*, i.e.

$$\forall x, y \in E \ (x \leq_\theta y,\ x \neq y \Rightarrow v(x) < v(y)),$$

consequently, the minimal value of v can be achieved only on the nondominated set $FP(U, \theta)$ and determines the best-compromise solution to (6.1). Taking into account that the information on hand is often insufficient to evaluate the global estimate of v, nor is such estimate necessary to select a single compromise alternative, one is usually satisfied with a local estimate which minimum on $FP(U, \theta)$ coincides with the minimum of the hypothetical global estimate. Thus, the multicriteria decision making problem for (6.1) consists in finding or estimating v, and solving the minimization problem

$$(v : F(U) \to I\!R) \to \min, \tag{6.2}$$

while from the strong monotonicity property it follows that

$$\arg\min\{v(x) : x \in F(U)\} \subset FP(U).$$

The level sets of v in E are the equivalence classes of the indifference relation in E (cf. Section 3.4.1).

The main features of the here presented multiple reference point approach to solve (6.1)-(6.2) can be summarized as follows:

- After formulating the problem (6.1), the decision–maker makes statements concerning the selected elements of the criteria space (reference points). Thus, to select a compromise solution one uses exclusively the information actually available, in contradistinction to a large class of MCDM methods which require an evaluation of subsequently generated nondominated solutions which need not be of special interest to the decision–maker. However, if available, such evaluations can be additionally considered during the decision–making process.

- We assume that to each reference point one can associate a utility value which, however, need not be given explicitly. There is no need for a linearity or additivity assumptions which are replaced by the distance utility principle (cf. Sec. 6.4).

- The original vector optimization problem (6.1) will be replaced by the *limited optimization problem* with the criteria space constraints Q,

  $$(F : U \cap F^{-1}(Q) \to E) \to \min, \; F_{opt} \in Q, \tag{6.3}$$

  where $Q \subset E$ is the set of values of F, where the minimization of F makes sense, and F_{opt} is the compromise value of F. The lower frontier of the region Q is determined by a class of reference points called the *bounds of optimality.* This assumption expresses the limited ability of criteria functions to model the real–life human preference structures, i.e. the same objective may be wished to optimize within certain limits; as soon as these bounds are achieved, their further improvement is not desired any more.

- Each iteration of the decision–making process consists of two stages: at the first stage all additional information available is used to confine the set of nondominated points satisfying additional requirements. At the second one a compromise solution is generated and presented to the decision–maker who can accept it or propose an improvement. If such improvement is possible the procedure calculates it, otherwise the decision–maker is given a choice either to accept the present solution, or to modify the reference points, constraints, or criteria. This is illustrated in Fig. 6.1.

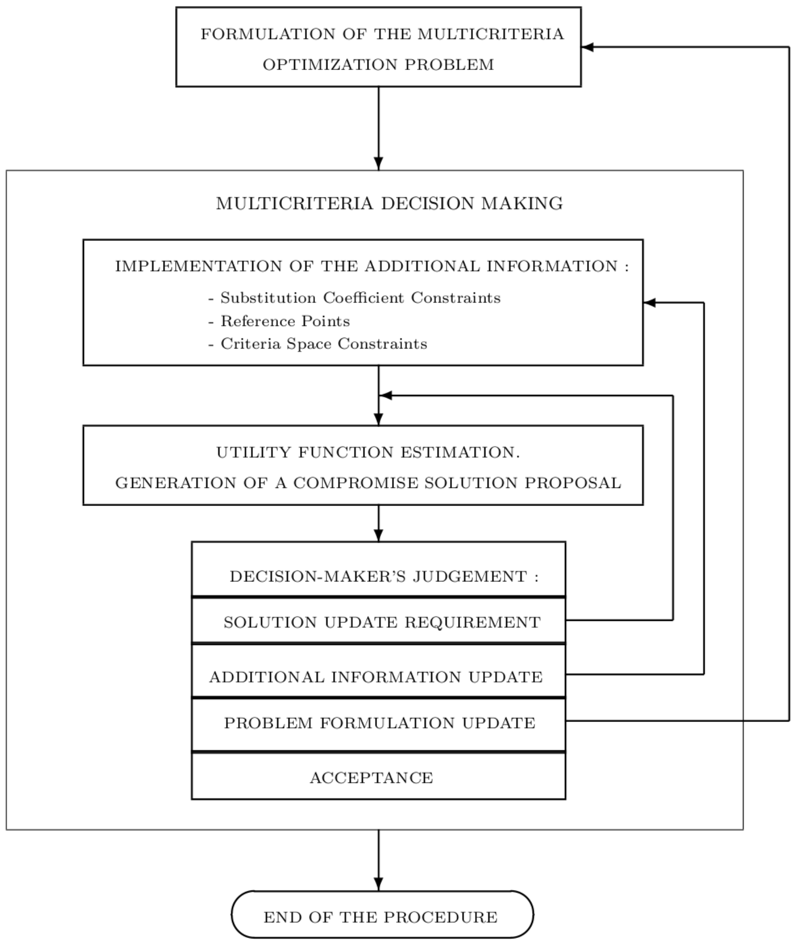

Fig. 6.1. *A scheme of multicriteria decision support based on reference sets.*

– Observing that the constraints on the substitution coefficients between the criteria $F_i,\ i = 1, ...N$, can be expressed equivalently as the partial order introduced by a convex cone which may be different from $\mathbb{R}^N_+$, we conclude that the above formulation of the problem (6.1) lets us consider the trade–off constraints and the reference points in one preference model jointly.

6.3 Utility Interpretation of Reference Points

According to the previous remarks, distinguished points or sets of points in the criteria space will serve as a source of additional information while modelling the decision maker's preferences. All these points are called *reference points*. Their nature, however, is usually heterogeneous, even if occurring in the same optimization problem. This results in their different treatment in the decision making procedures.

Each reference point is characterized by two different types of information:

– its meaning to the decision–maker, which is set up a priori by experts involved in the decision–aid, usually without taking into account the constraints in the underlying vector optimization problem;

and

– the relation to the set of attainable criteria values in the vector optimization problem (6.1).

The latter relation may be taken into account after gaining some information about the location of the attainable set $F(U)$ in the criteria space and may result in re–classifying the first class of relations.

Both kinds of information generate different classifications of reference points. First, we will consider this one which is based on external information conveyed to the decision maker by experts. This classification is based on the experience with the real–life problem solving which has been reported in an earlier publication (Górecki and Skulimowski, 1986). Motivated by the notation there proposed, we denote the classes of reference points consecutively by $A_0,\ A_1,\ A_2, \ldots,$ although the present notation differs from that contained in the above quoted paper.

A_0 – **Bounds of optimality** – the reference points which determine the lower bound of the region Q where the optimization of criteria makes sense (cf. the optimization problem with the criteria space constraints (6.3)).

The existence of these reference points illustrate a limited scope of assumption that every point dominating a target point is still desired by the decision–maker.
Estimated utility: To the elements of A_0 we will assign the same deterministic utility value as to the target points (see below), i.e. $v(A_0) = a_1 > 0$.

A_1 – **Target points** – the elements of E which model the ideal solutions desired by the decision–maker.

The coordinates of target points express the desired values of criteria without paying attention to constraints and properties of the objective functions. Whenever possible,

the target points should be achieved or surpassed by the solutions to (6.3). If it is impossible, a compromise solution should be as close as possible to this set.
Other notation: aspiration levels, ideal points.
Estimated utility: To each element of A we assign the same utility a_1.

A_2 – **Status–Quo Solutions** – attainable values of criteria which must be surpassed during the decision–making process.

If multiple status–quo solutions are defined, it is sufficient to find a nondominated solution dominating at least one of them. Alternatively, the status–quo solutions may be satisfying solutions corresponding to the lower aspiration levels of the decision-maker. If both interpretations occur simultaneously then it is reasonable to split the class A_2 into two subclasses according to the scheme presented later. The interpretation of status–quo solutions can be (at least) twofold; either as

- the values corresponding to the alternatives chosen in one of previous decision problems which are still available without performing any improvement within an optimization process,

or as

- the experience-based judgments concerning the minimal satisfying levels of criteria ("bounds of satisfaction").

Other notation: reservation levels (Wierzbicki, 1986), required values (Weistroffer; 1983,1984), solutions available at the pre-decision stage (Górecki and Skulimowski, 1986).
Estimated utility: To the status–quo solutions we will associate a utility value a_2, with $a_1 < a_2$.

A_3 – **Anti–ideal reference points** – elements of the criteria space which express the wrong choice of solution.

These points should be avoided by the decision–maker, by choosing a solution most distant to A_3.
Other notation: failure levels, avoidance points.
Estimated utility: To each element of A_3 we will assign a utility value a_3, with $a_1 < a_2 < a_3$.

In some situations, e.g. when the information is provided by experts, a finer classification may be available. For instance, the required and satisfying values may be distinguished, the discrimination of more and less desired target points may be possible. If the reference points are defined as the interval subsets of E, one may similarly consider the upper and lower edges of the estimation intervals. In general, one can split a class A_i into the subclasses $A_{i,1}, \ldots, A_{i,k(i)}$ assigning to each subclass a utility value $a_{i,j}$, for $0 \leq i \leq 3$, $1 \leq j \leq k(i)$, with

$$a_{i,j-1} \leq a_{i,j} \leq a_{i,j+1}$$

if $a_{i,j-1}$ or $a_{i,j+1}$ exist. To simplify the consistency conditions (cf. Sec. 6.4), one should order the classes $A_{i,j}$ so that $a_{i,j} \leq a_{i,j+1}$ for $1 \leq j \leq k(i) - 1$.

Now, let us present an example of a real–life portfolio management problem illustrating the application of the above ideas.

Example 6.1. Regarding the portfolio management as a dynamical process, for the sake of simplicity we will consider only a single decision step at the time t and three most fundamental performance criteria:

– the total expected revenue on the interval $[t, t+1]$, $(F_1(t, u))$,

– the agregated measure of risk $F_2(t, u)$,

and

– the amount of liquidity at $t + 1$, immediatly available to the decision–maker, $F_3(t, u)$.

The reference points for the above problem may be defined according to the following rules:

– The ideal reference points correspond to the fully successful termination of the investment process. They are characterized by the high values of F_1 and F_3 and the low rate of risk (F_2).

– The status quo solutions may usually be defined as the utility–equivalent values of F_1 for different levels of risk corresponding to the initial state of the portfolio.

– The financial evaluation of the failure values may correspond to the loss of the bank credibility, bankruptcy, or a mandatory sale of shares. It should be derived by financial experts basing on the appropriate economic analysis.

– The bounds of optimality are in form of the obvious criteria space constraints, namely $F_2 \geq 0$, $F_1 \geq F_3$. Other bounds of optimality may result from different legislative regulations concerning the economic activity. ■

Another example refers to an industrial design problem.

Example 6.2. An industrial company is planning to construct a new model of an elevator. Each considerable alternative is described by the set of noncompensatory criteria, including the reliability coefficient, the amount of capital necessary to start the production, the minimal time of evaluation of the project, the lifting capacity, the maximal range of the arm, the cost of production per unit, and others.

To make a compromise decision the following information is available:

- the parameters of the elevators already produced by this company (the set A_2),
- the parameters of the products which are expected to sell best at the market, the parameters of best elevators produced by other companies, or both (A_1),
- the bounds for the optimal values of parameters which are justified by the expected scope of the use of elevators (A_0),
- the parameters of elevators which turned out a market failure for the companies producing them (A_3).

To consider all that information jointly, the choice of a decision should be made according to the rules described in the next section. ■

After defining the reference points as the utility indicators, without paying attention to the actually attainable values of F in the problem (6.1), it is necessary to investigate the mutual situation of the points specified, and of the attainable set $F(U)$. At this stage of the decision process the decision–maker may redefine the utility of certain points, define new points, or new subclasses of reference points. The consistency of the information structure thus arisen will be checked automatically by the decision support system according to the procedure presented in the next subsection. The updates may also be undertaken automatically, or they may be performed by the decision–maker within an interactive process. The relation of reference points to the attainable set is discussed in detail in Skulimowski (1989).

6.4 The Consistency of Reference Points

In this section we provide the consistency definitions regarding the fundamental properties of multicriteria decision–making algorithms.

Definition 6.1. *A decision–making process will be called rational iff it results in selecting a nondominated solution to the multicriteria optimization problem (6.1).* ■

Only a rational solution selecting procedure ensures the accordance of the additional information structure used for choosing a compromise solution with the original multicriteria optimization problem. Thus, with a few exceptions (cf. e.g. the decision–making method STEM of Benayoun et al., 1971, or the so–called entropy target point approach of Hallefjord and Jörnsten, 1986), only rational procedures are subject of a series consideration as a tool for decision support.

Definition 6.2. *A decision–making process based on utility estimation is consistent, iff*

$$\forall x, y \in E \;\; [\tilde{v}(x) \leq \tilde{v}(y) \Rightarrow x \leq_\theta y \;\;\; or \;\;\; x \sim y], \tag{6.4}$$

where $\tilde{v}(x)$ is an estimate of the utility v at x, and $x \sim y$ denote the noncomparability relation of x and y. ■

Additionally, while performing in the utility estimation based on reference points one has to assure that the values estimated comply with the metric utility principle, and the common–sense meaning of the reference points. This additional property will be called here the consistency with the problem (6.1) and will be investigated in in the following subsections.

6.4.1 The internal and mutual consistency of reference sets

Recall that the main idea of the multiple reference points approach here presented consists in the fact all the elements of the class of reference points A_i correspond to the same value of the estimated utility function v. It is easy to see that to comply with the strong monotonicity assumption, it is necessary that no different elements of A_i are mutually comparable. We will call this property the internal consistency of the class A_i. By definition, the set of reference points is *internally consistent* iff

$$\forall q_1, q_2 \in A_i \;\;\; q_1 \;\; and \;\; q_2 \;\; are\ non\text{–}comparable. \tag{6.5}$$

If the reference set A_i is a result of aggregation of different experts judgments then it may happen that it is not internally consistent. However, applying any nondominated subset selection algorithm, it is easy to find a subset of A_i consisting of non–comparable points, either Pareto–minimal, $P(A_i, \theta)$, or Pareto–maximal, $P(A_i, (-\theta))$, and use one of them instead of A_i. Moreover, each one of the sets A_i, $0 \leq i \leq K$, where $K + 1$ is the number of classes of reference points, should be well-defined with respect to all other reference points. This requirement can be formulated as an assumption that each element of A_j should be dominated by an element of A_{j+1}, for $0 \leq j \leq K - 1$, i.e.

$$\forall x \in A_j \; \exists y \in A_{j+1} : \; x \leq_\theta y. \tag{6.6}$$

In order to obtain the desired properties of the level sets of $\tilde{v}$, we have to impose an additional condition (6.7), symmetric to (6.6):

$$\forall x \in A_{j+1} \; \exists y \in A_j : x \leq_\theta y, \tag{6.7}$$

which allows us to formulate the following definition:

Definition 6.3. *The reference classes A_j, and A_{j+1} satisfying the above conditions (6.6)-(6.7) will be called mutually consistent.* ■

Checking and correcting the mutual consistency is an essential part of the multicriteria decision–making algorithm based on the reference set method which is justified by the following theorem.

Theorem 6.1. *If all classes of reference points A_i for the problem (6.1)–(6.2) are both internally and mutually consistent, then the solution process described in the following Sec.6.5 is consistent.* ■

More details on this subject can be found in Sec. 6.5.

6.4.2 Rationality and the relation to the vector optimization problem

As noticed in the introductory remarks of this section, rationality is a fundamental property of the multicriteria decision–making methods which should be verified in the first order of importance.

Applying reference points and measures of proximity of feasible solutions from them as a model of decision–maker's preferences is a rational procedure, provided that a set of assumptions concerning the properties of distance functions and location of points in the criteria space is fulfilled. The sufficient and necessary conditions for the Pareto–optimality in distance scalarization have been studied earlier in more detail, see e.g. Wierzbicki (1986), Skulimowski (1988).

One of the most important features of the solution methodology described in this Chapter consists in the fact that the utility estimation at the reference points is (or should be) independent from the constraints in the vector optimization problem (6.1). Thus, the same preference model may be used for different decision problems, or for problems with a variable set of controls (cf. Chapter 7). However, the reference points from the classes $A_i,\ i = 0, \ldots 3$, do have a predetermined interpretation referring to the attainable set, which is contained already in their verbal description. A characterization of an ideal situation, where the initial decision–maker's judgments comply with the actual situation of attainable values is given below as the Conditions $1-4$. We assume that the mutual consistency conditions from the previous subsection are already fulfilled.

Condition 1. The target reference points should have a non–empty intersection with the set of non-attainable strictly dominating points.

Condition 2. The status quo solutions should be attainable.

Condition 3. The anti-ideal reference points should be dominated by at least one attainable point, or they should be non-comparable with $FP(U)$.

Condition 4. The lower limits of optimality should be partly dominating (cf. Chapter 5) or non–comparable.

An example of reference points fulfilling the Conditions 1 – 4 for certain multicriteria optimization problem, and satisfying the internal and mutual consistency assumptions is shown in Fig. 6.2.

After estimating the shape of the attainable set $F(U)$ in the problem (6.1) it may happen that the actual situation of reference points differs from that presented above. Then, it is necessary to reformulate the experts' or decision–maker's judgments according to the general rule that the rationality of the compromise solution is superior to the intuitive interpretation of the reference points defined. Thus, we get the *a posteriori* defined set of reference points. The redefinition process may be performed automatically, or the decision–maker may manually manipulate the reference values.

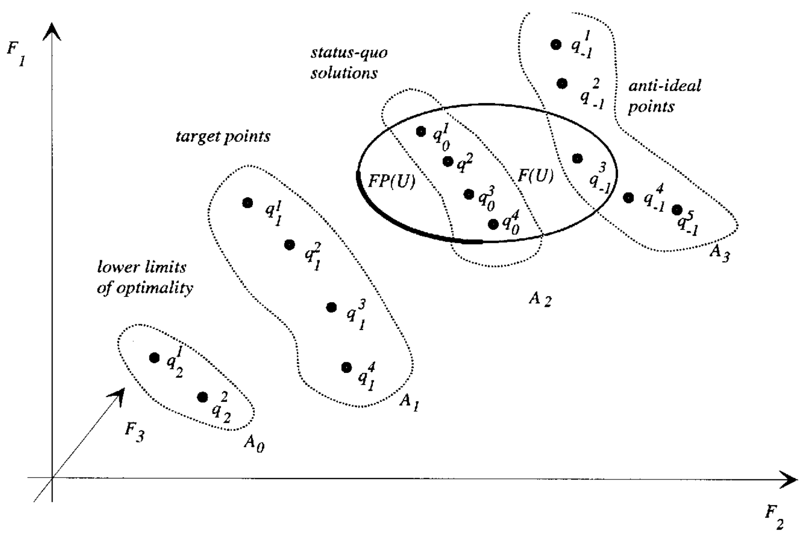

Fig. 6.2. *An example of well–defined reference points for a multicriteria optimization problem.*

6.5 The Metric Utility Principle

Having defined the goals, and/or avoidance region in the criteria space we have to formalize the meaning of "approaching the target" or "avoiding the dangerous decisions".

From the definition of the sets $A_0, \ldots, A_3$ it may be derived the following verbal description of the decision–maker's interpretation of the classes of reference points :

– the set of target points A_1 should be approached,

– the set of anti–ideal points A_3 should be avoided,

– the set of status–quo solutions A_2 should be exceeded,

– the solutions better than the bounds of optimality A_0 should be avoided.

The utility estimation process will consist of three steps:

– the approximation of the level sets of v,

– the determination of the estimation domain E where the utility function estimate $\tilde{v}$ is defined,

– the interpolation of $\tilde{v}$ in the regions bounded by the level sets.

As the first step to estimate the utility function one should approximate those level sets of $\tilde{v}$ which correspond to the values associated to the sets A_i. The level sets so found divide the criteria space E into the regions

$$R_i := \{x \in E : a_i \leq \tilde{v}(x) < a_{i-1}\}, \text{ for } i = 1, \ldots, K \tag{6.8}$$

with $a_0 := -\infty$, $a_K := +\infty$. From the assumed continuity of v it follows that the regions R_j are disjoint, i. e. $R_i \cap R_j = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, k$, therefore it exists the unique

$$j := \min\{i : R_i \cap FP(U, \theta) \neq \emptyset\}. \tag{6.9}$$

The utility values in R_j are highest possible for the elements of $F(U)$, so it is reasonable to restrict the further search for a compromise solution to the set $R_j \cap FP(U, \theta)$.

Hence one can conclude that to a complete model of the decision–maker's preferences one has to find two distance functions, the first one modelling the degree of avoidance of A_j, the second one modelling the achievement of A_{j+1} within the region R_j. The measures of proximity from the sets A_j take a form of seminorm–like functions g_{-j} (for distance maximization), and g_{+j} (for distance minimization), defined as follows:

$$g_{-j}(x) := d(x, Q_-) := \inf\{\| x - r \| : r \in Q_-\}, \tag{6.10}$$

$$g_{+j}(x) := d(x, Q_+) := \inf\{\| x - s \| : r \in Q_+\}, \tag{6.11}$$

where $Q_- := A_j - \theta$, and $Q_+ := A_j + \theta$ for j determined by (6.9), and the norm in E is strongly monotonically increasing on θ. Thus, the compromise solution should belong to the intersection of two sets of nondominated points: first one to the initial problem (6.1), the second one to the bicriteria problem

$$[(-g_{-j}, g_{+j}) : R_j \to I\!R^2] \to \min(I\!R^2_+). \tag{6.12}$$

To evaluate such intersection it is necessary to find the attainable elements which minimize g_{+j} or maximize g_{-j}. This is equivalent to applying the distance scalarization with respect to the reference set which has been defined in Skulimowski (1985a) and investigated further in the preceding Chapters.

To choose the compromise solution from the set $FP(U, \theta)$ we will use the bicriteria trade–off method based on the following theorem:

Theorem 6.2. *For each $\lambda > 0$ the utility estimation function in the region R_j,*

$$G_j(x) := g_{+j}(x) - \lambda g_{-j}(x)$$

is strongly monotonically increasing. Consequently, the set $P(G_j, F(U,\theta), I\!R^2_+)$ is contained in $FP(U,\theta)$ and

$$\arg\min\{G_j(F(u) : \ u \in U\} \subset P(F,U,\theta),$$

i.e. $G_j \circ F$ can be used as a scalarizing function for the problem (6.1).

Proof. Based on the optimality conditions given in Skulimowski (1987), it is sufficient to observe that the functions $g_{+j}(x)$ and $-g_{-j}(x)$ are both monotonically increasing in R_j. Then so is any their positive linear combination. ■

For the basic decision–making problem with four classes of reference points $A_0, \ldots, A_3$, the solution thus obtained fulfills the following compromise rule: it is as distant as possible from the set A_3, exceeds one of elements of A_2, it is as close as possible to the convex hull of the set $A_1 + \theta$, and does not exceed any element of A_0 provided that the assumptions of Sec.6.4 are satisfied. Varying the positive trade–off parameter λ and the scaling coefficients $w \in I\!R^N$ contained in the definition of norms serving to define g_{+j} and g_{-j} (cf. (6.10)-(6.11)), lets us interactively modify the resulting compromise solution $x_c \in U$ and $F_{opt} = F(x_c)$ within a consistent decision process.

6.6 An Interactive Decision–Making Algorithm

To present an application of the above described issues in a decision–choice procedure, let us zoom the central box "Utility function estimation. Generation of a compromise solution proposal" from the Fig. 6.1. The results are shown in Fig. 6.3.

The numerical methods applied to generate the nondominated solutions within the decision–making procedure presented in Secs. 6.4–6.5 use Pareto–optimality test and an a posteriori correction of results, so that the solution proposed to the decision–maker is actually nondominated.

As the input information for the procedure, the user in requested to define the constraints, criteria and the additional preference information:

- for the continuous optimal control problems with the terminal–time values of objectives:
  the reference values of criteria representing the most desired optimization results (ideal points), those satisfactory, the limits of optimality, and the values identified with a failure which should therefore be avoided;

- for the optimal control problems with trajectory objectives:
  the reference trajectories in the criteria space corresponding to the above listed classes of reference points, or the reference multifunction $Q(t) \subset I\!R^N$.

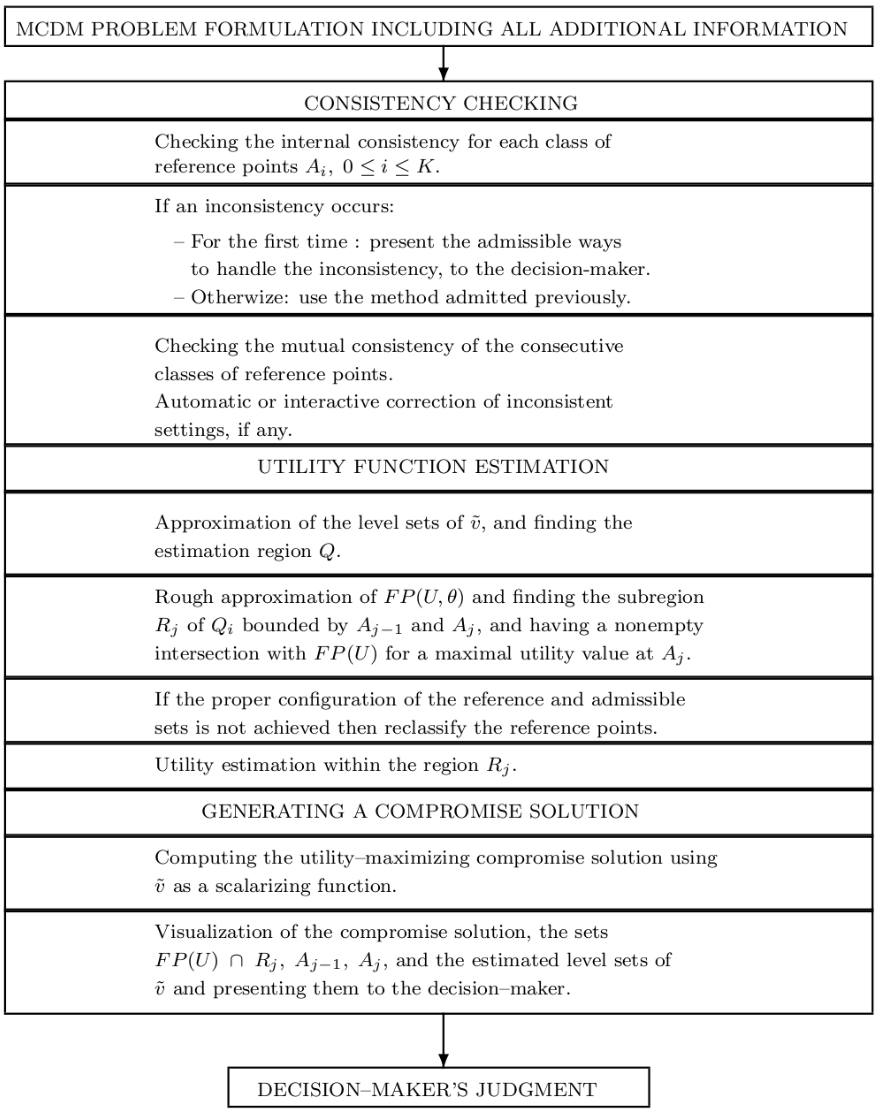

Fig. 6.3. *A scheme of establishing the preference model and generating the compromise solutions in decision–support algorithm based on reference sets.*

Moreover, in all above cases the user can define separately a priori constraints on the trade–offs between the criteria which may be equivalently considered as the preference structure defined by the convex cone θ constructed as in Yu and Leitmann

(1974a). The supplementary information concerning the decision–maker's preferences is aggregated in such a way, so that the time dependent reference subsets of the criteria space could be derived such as in Sec. 6.5. Thus we get two convex and compact subsets of the criteria space (multifunctions for the optimal control problems) representing the desired and avoidable values of criteria for each moment of time.

6.6.1 The logical scheme of decision support procedure for nonlinear static problems

After starting the program the user is prompted to define the problem to be solved. The description of the problem may be created at this moment, or it may be chosen from the list of problem files created previously, since each problem definition may be saved as a *.prm file. In both cases the problem design window will be shown on the screen, making possible entering new data or editing the information already available therein.

Before starting the problem definition, the problem type must be chosen from the following pick–up list:

"DISCRETE",
"LINEAR" (problem 6.1 with linear objectives),
"NONLINEAR" (problem 6.1 with arbitrary continuous objectives),
"CONTINUOUS-TIME CONTROL" (problem 3.15–3.17),
"DISCRETE-TIME CONTROL" (problem 3.13-3.14),
"ASYNCHRONOUS CONTROL" (discrete-event systems - see Chapter 8),
"DYNAMIC PROGRAMMING" (multicriteria shortest path).

The selection of one of the above problem types influences the (automatic) choice of the consistency checking procedure, and the layout of the problem design window. Below we will describe in more detail the structure of the model file for linear and nonlinear optimization problems. The menu of the model design window is shown in Fig 6.4.

FILE	TYPE	VARIABLES	CONSTRAINTS	OBJECTIVES
SAVE	TYPES	LIST OF	LINEAR	LINEAR
SAVE AS	LIST	VARIABLES	POLYNOMIAL	CONVEX
PRINT		AND THEIR	NONLINEAR	POLYNOMIAL
DELETE		DESCRIPTIONS	DISCRETE	NONLINEAR
COMPUTE			CONTROL	INTEGRAL
QUIT			DISJUNCTIVE	TRAJECTORY

Fig. 6.4. *A scheme of the model editing menu.*

The problem definition for continuous (linear and nonlinear) static optimization problems contains the following information:

A. Definition of variables.

Any definition of functions (constraints or objectives) must be preceded by the definition of variables. After choosing the "VARIABLES" option from the menu a database-like edit screen with the fields

– "variable symbol",
– "variable description"

and

– "interval"

will be shown in a window. By default, the variables will be consecutively named $X_1, X_2, \ldots, X_N$, although any names consisting of maximal six alphanumeric characters may be chosen. The "description" field may contain up to one line verbal interpretation of the variable. This interpretation appears in the info box at the bottom of the design screen any time when the appropriate variable is used. In the "interval" field one can enter the simple constraints of the form

$$X_i \leq a_i, \;\; X_i \geq b_i \quad (a_i \geq b_i),$$

or both. No input in this field means that there are no constraints of this type. The maximal number of variables is 1000.

B. The list of constraints.

   Any function of the previously defined variables may be created using the standard C/C++ –language syntax and the following operators:

   ABS, LOG(a,.), EXP, SQRT, SIN, COS, TG, CTG, ASIN, ACOS, ATG, ACTG, ENT(x) (largest integer smaller than x), INTG(a,b,<expression>,variable) – integral of a function from a to b with respect to 'variable' and others. Having defined a function, there appear the following operators '$\leq$ 0', '$\geq$ 0', and others '= 0'. The appropriate one can be chosen using the space bar and confirming the choice by pressing ENTER. There is no upper bound for the number of constraints.

C. Definition of objectives.

   While editing the functions one can mark, copy, and paste at a new place any of the expressions previously defined.

D. The state–space constraints.

E. The constraints on the trade–off coefficients.

F. The initial reference points.

   Here, the user may define the initial reference values of criteria representing the most desired optimization results (ideal points), those satisfactory, the limits of optimality, and the values identified with a failure which should therefore be avoided. The reference points can be redefined during the interaction with, or without affecting their initial values stored in the model file.

6.6.2 Solution of multicriteria optimal control problems

The criteria $J_1(x,t,u),\ldots,J_N(x,t,u)$ for the continuous–time optimal control systems are assumed to be absolute continuous functions of x, t, and u, so that it is possible to regard them as additional state variables. It is possible to consider the values of criteria either as trajectory objectives on an interval $[t_1, T]$, with $t_0 \leq t_1 < T$, or as the vector of the final values of J at the terminal time T. To approximate the set of nondominated points in the continuous optimal control problems we derive a stochastic approximation technique as outlined in Sec. 3.5.3. The role of a reference set is played by a reference multifunction (see Skulimowski, 1983b), an example of which is shown in Fig. 6.5.

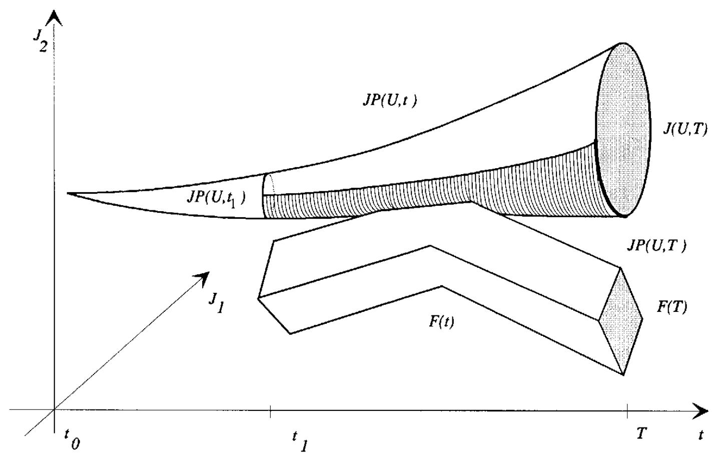

Fig. 6.5. *An example of the reference multifunction application in a multicriteria optimal control problem.*

The solution process is interactive, the results of each step are illustrated graphically, the decision–maker may use a pointing device to pick up the relevant values, or define numerically new reference points, trade–offs, and search directions. Additionally, for multi-step solutions with the time horizon not strictly defined, we propose a learning scheme which allows for examining the currently defined reference points versus the a posteriori judgments related to the solutions obtained in the past with similar reference objectives. The method is now being implemented in the MS WINDOWS^{TM} environment.

A discrete–time version of the above procedure is analyzed in Chapter 8.

6.7 Discussion

A reference point q does not itself contain sufficient preference information to make possible a selection of a compromise solution. Therefore the preference structure in multicriteria problems with a reference point has to be completed by a definition of a distance between q and the set of attainable criteria values. However, it often happens that no explicit information about the proximity or achievement measure is available, so that the decision support procedures applying single reference points, even changed interactively, did either contain an ad hoc component (cf. e.g. a magic role assigned sometimes to the set of minimal points in $F(U)$ with respect to the L_p norms, $1 \leq p \leq \infty$, so–called *compromise set*), or extensively use other preference information. On the other side, the approach presented by Jacquet–Lagrèze and Siskos (1982), called by the authors the *additive utility* approach (UTA), consisted in assessing an additive utility function using stochastic estimation of different samples of utility throughout the criteria space. It has been developed further by Jacquet–Lagrèze, Mèziani, and Słowiński (1987), Słowiński (1988), and Despotis, Yannacopoulos and Zopounidis (1990), who allowed the assessment of piecewise linear utilities. The approach here presented may be regarded as a compromise between the classical reference point and utility assessment models: the preference information is available for some distinguished points in the criteria space E (as for reference points, but not for arbitrary samples in E), but it is sufficient to estimate a family of value functions for the decision–making problem (no *ad hoc* or *a priori* known proximity measure is necessary). It generalizes previous decision models for status–quo solutions and target points introduced by Górecki (1982) and investigated by Górecki and Skulimowski (1988, 1989), the over-and under- achievement approach of Weistroffer (1984), the aspiration–reservation levels model of Ogryczak and Lahoda (1992), the bi–reference procedure of Michalowski and Szapiro (1989,1992, cf. also Michalowski, 1988)), as well as other extensions of the reference point method (Skulimowski, 1985a; Ogryczak, Studziński and Zorychta, 1989,1992; Inuiguchi and Kume, 1991). Similarly as the dynamical versions of reference point methods, in particular the extensions of the DIDASS method (Grauer, 1983; Lewandowski, Rogowski, Kręglewski, 1985; Rogowski, 1989; Lewandowski and Wierzbicki, 1989; Wierzbicki, 1988; and Rogowski, Sobczyk, and Wierzbicki, 1989), the present method can be applied for multicriteria optimal control problems and output trajectory optimization, while the reference sets are replaced by time varying reference multifunctions. The reference set approach can be easily embedded as intelligent user interface in the general Decision Support Systems framework (for the latter topics see e.g. Buchanan and Daellenbach (1987), Finlay (1990), Buchanan and McKinnon (1991).

Chapter 7

An Interactive Modification of Constraints To Attain a Target Reference Set

In this chapter we describe a method to achieve the target set in multicriteria optimization problems in a systematic way by relaxing the soft constraints in the primary problem formulation. We assume that the numerical bounds defining a part of constraints may be changed subject to some payments, i.e. a change of constraints from a preliminarily disposed control set U to a new one V bears the cost $v(U, V)$. The values of the cost function v and of a measure of proximity from the target reference set Q serve as auxiliary objectives which are taken into account by the decision-maker during the decision making procedure. The target set has the same meaning as in preceding chapters and its construction based on reference points might be the same as of the reference set in Chapter 6. We give a theoretical background for the analysis of this kind of problems in the case where the admissible decision subsets are convex sets defined by m inequality constraints, the criteria are monotonic functions and the target set is convex. An interactive procedure involving a visualization of the attainable and target sets is outlined in the final part of this chapter. An analogous procedure may also be applied to relax the constraints so that the distance to the set of avoidable strategies were maximal.

7.1 Introduction

We assumed that for the multiple criteria optimization problems considered in the previous chapters one could specify the reference points as the values of criteria being of a special importance to the decision-maker, interpreted later (see Chapter 6) in terms of a decision-maker's utility function. As previously, we assume that the reference points are defined independently from the multicriteria problem formulation. Therefore it may happen that the unattainability of a desired reference point is equivalent to the decision-maker's failure. Once a reference point, or - more generally - a reference target set is defined, reaching it or coming closer than it is possible with the initial constraints, may be more important to the decision-maker than preserving the optimality of solutions thus found with respect to the original multicriteria optimization problem. In such cases he/she may want to apply a method allowing to relax the constraints in order to reach some previously unattainable points.

The idea of improving the solution to a constrained optimization problem is probably so old as the optimization theory itself. Especially attractive may seem the attainment of the previously unattainable points in the criteria space of a multicriteria optimization problem, where the separate optima of each criterion function are usually not achieved at the same point, so the unattainable ideal point or other reference points attract the attention of the decision-maker. Thus, speaking about reference points, and taking into account the fact that some constraints are often formulated ad hoc or are interchangeable with the criterion functions, it is desirable to propose the methods to influence the shape of the set of attainable criteria values by controlling the constraints in the decision space.

In earlier approaches, the researchers suggested sometimes that after a more penetrative analysis of the problem some constraints in the original problem formulation may be removed or relaxed without any loss, making thus possible an improvement of the solution (cf. Dubov and Shmulyan, 1973). Based on the observation that the previously uncertain or non-rigid constraints may be precised while or after a problem solution, Seiford and Yu (1979) introduced the notion of a potential solution to a linear multicriteria problem with parameterized constraints, while Werners (1987) assumed that the constraints may be fuzzy. A method for solving scalar linear programming problems with flexible constraints related to the penalty function technique has been proposed by Grossman (1983). The desired properties of the solution of an optimization problem may also result from a thorough design of the optimal system, meant usually as a selection of constraints during an optimization process (cf. Tokarev 1971ab, and Ovsiannikov and Tokarev, 1971). This observation plays also a fundamental role for the theory of multi-level optimization problems (see e.g. Candler and Townsley (1982), Bard and Falk (1982), Bard (1983,1988), Benson (1989)), multitest multicriteria linear programming (Glushkov et al., 1982), the multi-stage consequence analysis (Skulimowski, 1985b), the input optimization (Zlobec, 1986,1987), de Novo programming (Hessel and Zeleny, 1987), to name only some main approaches. An important issue in multi-level optimization, where the constraints are selected at a higher level, while the final solution is to be found in so designed decision set at a lower level, is the efficiency of solutions with respect to the objectives employed at all optimization levels (cf. Marcotte and Savard, 1991; Wen and Hsu, 1992) and consequent trade-off between the results of all optimization levels. This question is also relevant for the design of the interactive procedure outlined in Sec.7.5, where the minimal proximity to the target set should be obtained at a minimal cost of changing the constraints.

None of the approaches above allows for a sequential analysis of a simultaneous acquisition of new feasible decisions and selling some of those already at hand, starting from a fixed decision set, and assuming that the change of constraints may bear certain cost or profit in the case of sale. Here, following an earlier paper (Skulimowski, 1987), we will present an interactive approach to solve the multicriteria decision-making problem with soft constraints by attaining or approaching the target reference set and a flexible management of available decisions. Namely, we assume that in the decision space U there is distinguished a family of subsets S which is interpreted as a family of all admissible domains of optimization and a real function $v(A, B)$, where $A, B \in S$, is the cost of changing the decision set A being just at the decision-maker's disposal to a desired one B. The statement of the problem is given in Sec.7.2. In the next section we will discuss the properties of the structure (S, v) occurring in some practical cases. We will study in detail the case where S is the family of subsets of finite dimensional real vector space defined by convex constraints. In Sec.7.4 the theoretical background for finding the admissible subsets attainable from a given set U_0 and intersecting a convex set Q is described for a special case of S and v .

The additional information represented by the structure (S, v) and the set of target points Q has been applied in Sec.7.5 to develop an interactive procedure for solving multicriteria problems with convex constraints and linear criteria. An analogous procedure may also be applied to relax the constraints so that the distance to the set of avoidable strategies were maximal.

The interactive procedure may be outlined as follows. At the preliminary stage of the procedure, the minimal value $v_{\min}$ of the constraint change cost function v ensuring the achievement of at least one element of the reference set is computed. If the value of $v_{\min}$ is admissible for the decision-maker then one of the achieved points is chosen and the procedure terminates. Otherwise, payments v_i less than v_{min} are interactively proposed to the decision-maker together with the points u_i closest to the target set Q in one of new decision sets obtained after paying the amount of v_i for changing the constraints. The minimal value of the distance (or another proximity measure) from $Q, d_i(Q)$, is the auxiliary objective for the decision-maker who can accept it or not. Thus the decision-making problem can be regarded as a trade-off between v and $d_i(Q)$. While performing this procedure the decision-maker is allowed to modify (to some extent) the target reference set Q. We will provide conditions which will ensure convergence of the solution method proposed. To facilitate the decision-maker-computer interaction we propose a package of graphical subroutines including the visualization of the attainable set for the preliminary and new constraints in two and three criteria case, the visualization of the target set, and the indication of the least-distance solution. An outline of the algorithm, which can also be applied to solve multicriteria design problems, is presented in the final part of this chapter.

7.1.1 Basic notations and assumptions

Throughout this chapter we will use the same standard mathematical notation as in the whole book. In particular, recall that $K_r(x)$ denotes the closed ball with radius

r and center x, and

$$d(x, A) := \inf\{\| x - a \| : a \in A\}$$

is the distance between a subset $A \subset I\!R^N$ and a point x.

To concentrate on the main ideas of the method presented, we assume that the partial order in the criteria space is introduced by the natural cone $I\!R^N_+ := (I\!R_+)^N$, where $I\!R_+$ is the set of non-negative real numbers, although the method may be directly applied to multicriteria problems with any linear partial order in the criteria space. To denote the order relation we will use the common notation "$\leq$", i.e.

$$x := (x_1, ..., x_N) \leq y := (y_1, ..., y_N) \text{ iff } x_i \leq y_i \text{ for } i = 1, ..N.$$

By the attainable set in the criteria space for the multicriteria optimization problem $(F : U \to I\!R^N) \to \min$ we will mean the set $F(U)$. Its nondominated part will be denoted by $FP(U)$, while the set of nondominated decisions will be denoted by $P(U, F)$, both with respect to the partial order relation "$\leq$". Obviously, $FP(U) = F(P(U, F))$. All other terms and symbols are explained in the text where they are first used.

7.2 Statement of The Target Attainment Problem

The decision situation concerned may be presented as follows. The multicriteria optimization problem is initially formulated as

$$(F = (F_1, ..., F_N) : U_0 \to I\!R^N \to) \min . \tag{7.1}$$

The additional preference structure consists of desired reference points or the limits of optimality which, in turn, are represented as the reference target set Q in the criteria space $I\!R^N$. During the decision-making process the decision-maker endeavors to reach an element of Q or come to it as close as possible. To achieve this goal, an initial feasible decision set U_0 can be modified to another set U_1 belonging to the family of admissible domains of optimization S, provided that a specified amount $v(U_0, U_1)$ is paid for this modification.

The problem of reaching Q at minimal cost by changing the constraints can be stated as

$$\begin{aligned} &(S \ni V \to v(U_0, V) \in I\!R) \to \min \qquad \text{s.t.} \\ &V \in S_0(Q) := \{U \in S : \ \ F(U) \cap Q \neq \emptyset\}. \end{aligned} \tag{7.2}$$

Let us denote the minimal value of v in the problem (7.2) by $v_{\min}$. If the payment of $v_{\min}$ for changing the constraints is not admissible for the decision-maker, or if $v_{\min} = \infty$, then the following formulation makes sense

$$(S \ni V \to d(F(V), Q)) \to \min$$

s.t.

$$V \in \{W \in S : \ v(U_0, W) \leq v_0\}, \ 0 \leq v_0 \leq v_{\min}, \tag{7.3}$$

where

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$$

is the interstice between sets A and B, and v_0 is an admissible value of v.

Following Skulimowski (1985a), let us assume that V is a solution to the problem (7.2) also in the case if instead of reaching a target point $q \in Q$ on V, one can reach q' such that $q' \leq q$. Assuming moreover that the sets Q and $F(V)$, for all $V \in S$, are closed and Q is $I\!R^N_+$ -complete (cf.5.1.1), we may represent Q in the form

$$Q = Q_0 - I\!R^N_+,$$

where Q_0 is the set of maximal points of Q, i.e.

$$Q_0 := P(Q, -I\!R^N_+).$$

From the above assumptions it follows that if $F(V) \cap Q \neq \emptyset$ then also $FP(V) \cap Q \neq \theta$.

In a more general setting, assuming that the decision-maker is satisfied with approaching the target set to the distance d_0, (7.2) may be formulated as

$$(S \ni V \to v(U_0, V)) \to \min$$

s.t.

$$V \in \{W \in S : d(F(W), Q) \leq d_0\} \ := S(Q, d_0) \tag{7.4}$$

Let us observe that the formulation of the problems (7.3) and (7.4) is dual in the sense that each optimal solution to one of them is weakly nondominated in the bicriteria problem

$$(S \ni V \to (v(U_0, V), d(F(V), Q))) \to \min. \tag{7.5}$$

Let us note that the solution to the initial multicriteria problem (7.1) is implicitly contained in the set of solutions to the problems (7.3) - (7.5). Namely, according to the decision-maker's requirements concerning the Pareto-optimality and a least-distance location to Q, a compromise solution to (7.5) should be an element $u_{opt} \in V_{opt}$ such that

$$d(F(u_{opt}), Q) = d(F(V_{opt}), Q),$$

where V_{opt} is an optimal subset in problems (7.3) or (7.4).

If Q has a non-empty intersection with $F(V_{opt})$, then from an initial remark concerning the form of Q it follows that one can always choose a nondominated decision u_{opt}. To ensure that u_{opt} is nondominated when $d(FP(V_{opt}), Q) > 0$, some further assumptions concerning the situation of $F(V_{opt})$ must be made. Following the conditions listed in Chapter 5, it suffices that $F(V_{opt})$ is contained in $q^* + I\!R_+^N$, where $q^* := (q_1^*, ..., q_N^*)$ is the anti-ideal point of Q , i.e.

$$q_i^* := \max\{q_i \ : \ q = (q_1, ..., q_i, ...q_N) \in Q\}, \ 1 \le i \le N.$$

If $F(V_{opt}) + I\!R_+^N$ is convex then it also suffices that

$$F(V_{opt}) \subset Q_0 + I\!R_+^N. \tag{7.6}$$

Some other conditions ensuring the efficiency of the solution admitted have been given in Chapter 5.

Hence it follows that the final goal of the decision-maker is to find a pair (u_{opt}, V_{opt}) such that

$$u_{opt} \in P(V_{opt}, F),$$

$$v(U_0, V_{opt}) \le v_{ad},$$

$$d(F(V_{opt}), Q) \le d_{ad},$$

and

$$d(F(u_{opt}), Q) \text{ is minimal for all } u \in V_{opt}, \tag{7.7}$$

where v_{ad} and d_{ad} are the maximal admissible values of v and d respectively. The choice of the set V_{opt} will be carried over as a trade-off problem between the values of v and d during an interactive procedure, while the proposed compromise strategy u_{opt} fulfills (7.7) unless a separate procedure must be used to obtain a nondominated decision.

7.3 Properties of The Variable Constraint Family

The assumed structure (S, v) in the family of subsets of the decision space must be an adequate representation of real-life decision problems with non-rigidly defined constraint levels.

7.3.1 Properties of the admissible subset family S

Let $Cl(U)$ denote the family of closed subsets of the decision space U. The Hausdorff distance

$$d_H(A, B) := \inf\{r : A \subset K_r(B) \text{ and } B \subset K_r(A)\}, \tag{7.8}$$

where $A, B \in Cl(U), K_r(X) := \{x \in U : d(x, X) \leq r\}$, is a metric in $Cl(U)$ which generates the topology in $Cl(U)$ called the Hausdorff topology. If U is complete then $Cl(U)$ with the Hausdorff metric is also complete. A similar implication holds for the compactness and separability properties, as well (cf. e.g. Castaign and Valadier, 1977).

To ensure the existence of the above formulated optimization problems in the family of admissible subsets, we will make the following assumptions regarding the subset family S :

SF 1. S is a closed subfamily of $Cl(U)$, the topology in S is the Hausdorff topology induced from $Cl(U)$;

SF 2. (connectivity). If A and B belong to S then there exists a continuous multifunction $G : [0, 1] \to Cl(U)$ such that $G(0) = A$ and $G(1) = B$.

For the formal purposes we will also assume that $\emptyset \in S$.

Example 7.1. Let $M = 1$ and W be the family of affine half-spaces in $\mathbb{R}^n$. Then S is a family of subsets of $\mathbb{R}^n$ defined by at most N linear inequality constraints, or k equality, and $N - 2k, k \leq N/2$, linear inequality constraint. Of course, this subfamily is closed and each two half-spaces are continuously deformable on each other in the sense of SF 2.

In the sequel the subfamily W will always refer to the class of constraints admissible in the optimization problem just considered.

The specificity of the optimization problems may require making additional assumptions concerning the family of admissible decision subsets. All those assumptions or only some of them may appear jointly:

AA1. $\forall \ V \in S \ : \ V$ is bounded ;

AA2. $\exists \, U_0 \in S \ \ \forall V \in S : U_0 \subset V$.

If in the space of decisions U there exists a measure μ, one may also require that

AA3. $\exists \, 0 \leq c_1 \leq c_2 \leq \infty \ \ \forall \, V \in S, c_1 \leq \mu(V) \leq c_2$.

The admission of the additional assumption AA2 or the existence of positive reals c_1 or c_2 in the assumption AA3 is often necessary to avoid the situation, where single points can be admitted as decision sets, which usually is impossible, or not reasonable in real-life problems. On the other hand, the upper limitation c_2 in AA3 together with the assumptions concerning the cost measure, ensures the existence of solutions to the arising optimization problems (cf. Morris, 1979).

7.3.2 Properties of the cost function v

According to the previous remarks, the cost function v is defined on S with values in $\mathbb{R}_+ \cup \{\infty\}$. The value of $v(A, B)$ may be interpreted as the cost of making available the decisions from a set B, provided that the decisions from the set A had been already at hand. Referring to the real-life situations again, we will impose the following assumptions concerning v :

CF 1. v is continuous with respect to the Hausdorff topology in S^2 ;

CF 2. $X \subset Y, X \in S \Rightarrow [v(X,Y) \leq 0$ and $-v(X,Y) \leq v(Y,X) \leq 0]$;

CF 3. $\forall\, V,X,Y \in S\ \ [V \subset X \subset Y \Rightarrow v(V,Y) \leq v(V,X) + v(X,Y)]$;

CF 4. $\forall A,V,X \in S : [A \cap V = \emptyset$ and $A \cap X = \emptyset \Rightarrow v(V,X) = v(V \cup A, X \cup A)]$.

The continuity of v is assumed mainly to assure the regularity of the arising optimization problems, but it seems also to be justified by the structure of v in real-life problems. The assumption CF 2 excludes the situation, where cyclic oscillations between sets X and Y may be profitable to the decision-maker. It means also that if the decision-maker may already choose a decision from the set V, he/she is not obliged to choose the solution which is optimal in V, but is free to restrict his/her range of decision to a subset X of V, $X \in S$, which may bring certain profit in terms of v and losses in terms of F. Thus we allow to consider the creation of additional constraints which might be useful in modelling some conflicting situations, as well as it is possible to "sell" a part of the decision set without exchanging it immediately to previously non-admissible decisions. The triangle inequality for v assumed in CF 3 means that to be able to select a decision from the set Y starting from V by reaching first the set X may give worse results than a direct attainment of Y. On the other hand, to reach a Y from V it might be necessary to pass through one or more intermediate optimization domains $X_1, X_2, ..$, i.e. the time factor may be involved. CF 4 is, roughly speaking, an invariance property concerning the sum of disjoint sets.

Observe that assumptions CF 1 - CF 4 are independent from each other and can therefore be regarded as axioms characterizing v. For $V \in S, A \notin S$, we may define an extension ν of v by setting

$$\nu(V,A) := \inf\{v(V,X) : X \in S,\ A \subset X\}.$$

Analogously, if $B \notin S, V \in S$, we may define

$$\nu(B,V) := \inf\{v(Y,V) : Y \in S, Y \subset B\}.$$

Obviously, ν is well-defined, $\nu_{|S} = v$, and if v possesses the properties CF 1 - CF 4, so does ν, so in the sequel, whenever necessary, we may regard v as already extended.

The following basic facts are implied by CF 2 - CF 4 :

Proposition 7.1. *Suppose that $A,V,X \in S$ and $A \subset V \subset X$. Then*

$$v(A,V) \leq v(A,X) \tag{7.9}$$

and

$$v(X,A) \leq v(V,A). \tag{7.10}$$

Proof. The inequalities (7.9) and (7.10) are immediate consequences of the postulates CF 2 and CF 3 :

$$v(A,V) \leq v(A,X) + v(X,V) \leq v(A,X),$$

$$v(X,A) \leq v(X,V) + v(V,A) \leq v(V,A). \qquad \blacksquare$$

Proposition 7.2. *For arbitrary $A, B, C \in S$ such that $A \subset B$ and $B \cup C \in S$ it holds*

$$v(A, B \cup C) \leq v(A,B) + v(\emptyset, C).$$

Proof. From CF 3 and CF 4 it follows that

$$v(A, B\cup C) < v(A,B) + v(B, B\cup C) = v(A,B) + v(\emptyset, C\setminus B) \leq v(A,B) + v(\emptyset, C). \quad \blacksquare$$

7.4 Optimization in The Family of Admissible Decision Sets

In this Section our considerations will be restricted to the admissible sets defined by m inequality constraints of the form $g(u) \leq 0$, where $g = (g_1, ...g_m)$ is a differentiable convex function and $u \in I\!R^k$.

As the family S we will consider an m-parameter family of subsets

$$V_c := \{u \in I\!R^k : g(u) \leq c\}, \tag{7.11}$$

where $c = (c_1, ..., c_m) \in I\!R^m$.

Instead of the cost function $\bar{v}$ defined on S^2, it would be more convenient to consider the function $\bar{v}$ defined on the parameter space $I\!R^m \times I\!R^m$ and such that

$$v := \bar{v} \circ j,$$

where

$$j : S^2 \ni (V,U) \to (\inf\{r : V \subset V_r\},\ \inf\{s : U \subset V_s,\}) \in I\!R^m \times I\!R^m. \tag{7.12}$$

The infimum operator in (7.11), for $x, y \in I\!R^m, x = (x_1, ...x_m), y = (y_1, ...y_m)$, is defined as :

$$\inf\{x,y\} := (\min\{x_1,y_1\}, ..., \min\{x_m,y_m\}),$$

Analogously, for $X \subset I\!R^m, x = (x_1, ...x_m) \in X$,

$$\inf X := (\inf\{x_1 : x \in X\}, ..., \inf\{x_m : x \in X\}) = x^*(X, I\!R^m_+).$$

According to the following diagram, to prove that the optimization of v can be equivalently substituted by the optimization of $\bar{v}$ defined on $I\!R^{2m}$ it is necessary to show that j is continuous and well-defined.

$$\begin{array}{ccc} S^2 & \xrightarrow{v} & I\!R \\ j\downarrow & \nearrow \bar{v} & \\ & I\!R^{2m} & \end{array}$$

Diagram 7.1.

Lemma 7.1. *Suppose that $V \subset I\!R^n$ is defined by (7.11) for certain $s_0 \in I\!R^m$, and $V \neq \emptyset$. Then the mapping*

$$i : S \ni V \to \inf\{r : V \subset V_r\} \tag{7.13}$$

is well-defined. ■

Proof. To prove that for given nonempty $V \in S$ the index

$$r_0 := \inf\{r \in I\!R^m : V \subset V_r\}$$

exists and is unique, let us observe that for each $V_r \in S$

$$V_r := \bigcap\{V_{r(i)} \quad : 1 \leq i \leq m\},$$

where

$$r = (r(1), ..., r(m)) \text{ and } V_{r(i)} := \{x : g_i(x) \leq r(i)\}.$$

Suppose that $V \subset V_r$ and $V \subset V_s$ for arbitrary non-comparable $s, r \in I\!R^m$. Then

$$\begin{aligned} V \subset V_r \cap V_s \quad &= \quad \bigcap\{V_{r(i)} : 1 \leq i \leq m\} \ \cap \ \bigcap\{V_{s(i)} \ : 1 \leq i \leq m\} = \\ &= \quad \bigcap\{V_{r(i)} \cap V_{s(i)} \ : 1 \leq i \leq m\}. \end{aligned}$$

By definition,

$$V_{r(i)} \cap V_{s(i)} = V_{\min\{r(i), s(i)\}},$$

hence $V \subset V_{\inf\{r,s\}}$. Furthermore, if $V \neq \emptyset$ then under assumptions of this lemma $i(V)$ is bounded below by $s_0 := (s_{01}, ...s_{0n})$, where $s_{0i} := \sup\{g_i(x) : x \in V\}$. Consequently, for a $V \neq \emptyset$ there always exists the unique $r_0 = \inf\{r : V \subset V_r\}$, so that $r_0 = s_0$ and the mapping i and the expression (7.12) are well-defined. ■

To ensure the satisfaction of the properties CF 1 - CF 4 by the cost function v generated by $\bar{v}$, we have to impose several assumptions on $\bar{v}$.

Lemma 7.2. *Let the function $\bar{v} : I\!R^m \times I\!R^m \to I\!R$ possesses the following properties:*

(i) *$\bar{v}$ is continuous,*

(ii) *for $c_1 \leq c_2 \leq c_3$ $\bar{v}(c_1, c_2) \leq \bar{v}(c_1, c_2) + \bar{v}(c_2, c_3)$,*

(iii) $c_2 \leq c_1$ *implies* - $\bar{v}(c_2, c_1) \leq \bar{v}(c_1, c_2) \leq 0$,

and

(iv) *for each* $x \in I\!R^m$ *the set* V_x *defined by (7.11) is compact.*

Then $v : S^2 \to I\!R^m$ *defined as the superposition*

$$v := \bar{v} \circ j,$$

where j *is defined by (7.13), fulfills the conditions CF 1 - CF 4.* ■

Proof. The properties CF 2 and CF 3 are implied immediately by the assumptions of the lemma. CF 4 plays no role since no subset Y of the form

$$Y = X \cup A; \quad A \cap X = \emptyset; \quad A, X \in S$$

may be defined by the constraints (7.11). We only need to prove the continuity of v. Since $v = \bar{v} \circ j$, it is sufficient to prove that the mapping i defined by (7.12) is continuous.

Let us fix a $V = V_s \in S$ for an $s \in I\!R^m$, and let us take an arbitrary $\varepsilon > 0$. We have to prove that there exists $\mu > 0$ such that for any $V_r \in S$, $d_H(V_r, V_s) < \mu$ implies $\| r - s \| \leq \varepsilon$.

We have assumed that the function g is continuous at every point x of V_s , hence

$$\forall x \in V_s \;\; \exists \mu(x) > 0 : \;\; [y \in K_{\mu(x)}(x) \Rightarrow \| g(x) - g(y) \| \leq \varepsilon/m].$$

Since V_s has been assumed compact, then there exist $p \in N$ and $x_1, ...x_p \in V_s$ such that

$$V_s \subset K_{\mu(x_1)}(x_1) \cup ... \cup K_{\mu(x_p)}(x_p).$$

Taking $\mu_1 := \min\{\mu(x_1), ..., \mu(x_p)\}$, we conclude that

$$\forall x \in V_s \quad [z \in K_{\mu_1}(x) \Rightarrow \| g(z) - g(x) \| < \varepsilon/m]. \tag{7.14}$$

From (7.11) and (7.14) it follows that for every $z \in V_s + K_{\mu_1}(0)$,

$$g(z) \leq s + e\varepsilon/m, \tag{7.15}$$

where $e = (1, ..., 1)^T \in I\!R^m$. If for certain $r \geq s$

$$V_r \subset V_s + K_{\mu(x_1)}(0)$$

then from (7.15) it follows that

$$g(V_r) \leq s + \varepsilon e/m,$$

i.e. $r \leq s + \varepsilon e/m$. Consequently, $\| r - s \| < \varepsilon$. If $r \leq s, V_r \neq \emptyset$, and

$$V_s \subset V_r + K_{\mu_2}(0) \text{ for a } \mu_2 > 0,$$

then replacing s by r and V_s by V_r in (7.15) one can prove similarly that

$$g(V_s) \leq r + \varepsilon e/m,$$

i.e. in this case it holds again $\| r - s \| < \varepsilon$. If neither $r \leq s$, nor $s \leq r$, and, by assumption, $d_H(V_r, V_s) < \mu$, then from the definition of the Hausdorff distance there exist $\mu_1, \mu_2 > 0$ such that

$$V_r \subset V_s + K_{\mu_1}(0) \text{ and } V_s \subset V_r + K_{\mu_2}(0). \tag{7.16}$$

Using the same construction as for the cases $s \leq r$ and $r \leq s$ we can select μ_1 and μ_2 such that (7.16) implies

$$r \leq s + e\varepsilon/m \quad \text{and} \quad s \leq r + e\varepsilon/m,$$

which is equivalent to

$$s_i - \varepsilon/m \leq r_i \leq s_i + \varepsilon/m, \text{ for } i = 1, ...m.$$

What follows, is the inequality $\| r - s \| \leq \varepsilon$ and the proof is complete. ∎

From the above lemma we conclude that having defined on $I\!R^m \times I\!R^m$ a function $\bar{v}$ fulfilling the assumptions of Lemma 7.2., the cost measure v fulfills all assumed properties and the optimization of $\bar{v}$ on the family of subsets S can be equivalently replaced by the optimization of $\bar{v}$ on the set $\{c \in I\!R^m : \ V_c \neq \emptyset\}$. All subproblems of (7.2) - (7.5) involving the use of v as an objective or a constraint can now be reformulated in terms of $\bar{v}$.

In particular, let $C_0(Q)$ be the set of parameters such that $c \in C_0(Q)$ implies $F(V_c) \in S_0(Q)$ (cf. (7.2)), i.e.

$$C_0(Q) := \{c \in I\!R^m \ : F(V_c) \cap Q \neq \emptyset\},$$

and let

$$\begin{aligned} C(Q, d_0) \quad &:= \quad \{c \in I\!R^m \ : F(V_c) \in S(Q, d_0)\} = \\ &= \quad \{c \in I\!R^m \ : d(F(V_c), Q) \leq d_0\}. \end{aligned}$$

Then the following lemma is true:

Lemma 7.3. *Assume that in the problem (7.1) the functions $g_i, i = 1, ..m$, are convex, the criterion function F fulfills the condition*

$$\forall w, z \in U \ : F([w, z]) = [F(w), F(z)], \tag{7.17}$$

and the reference set Q is convex. Then the set $C_0(Q)$ is convex or empty. If, moreover, the distance d in the criteria space is convex, so are the sets $C(Q, d_0)$ for any $d_0 \leq 0$.

Proof. Denote $U_c := F(V_c)$ for a $c \in R^m$, and suppose that

$$U_a \cap Q \neq \emptyset \quad \text{and} \quad U_b \cap Q \neq \emptyset \quad \text{for} \quad a \neq b,$$

i.e. a and b are elements of $C_0(Q)$. Then there exist $x \in F(V_a) \cap Q$ and $y \in F(V_b) \cap Q$. Now, let u_1 and u_2 be arbitrary elements of sets $F^{-1}(x) \cap V_a$ and $F^{-1}(y) \cap V_b$, respectively. The function g has been assumed convex, hence

$$\begin{aligned} g(tu_1 + (1-t)u_2) &\leq tg(u_1) + (1-t)g(u_2) \leq \\ &\leq ta + (1-t)b := c(t), \text{ for } 0 \leq t \leq 1. \end{aligned}$$

By definition, it means that

$$u(t) := u_1 t + (1-t)u_2$$

is an element of $V_{c(t)}$, for $0 \leq t \leq 1$. Moreover, since (7.16) holds and Q is assumed convex then

$$F(u(t)) = F(u_1 t + (1-t)u_2) \in [F(u_1), F(u_2)] = [x, y] \subset Q, \text{ for } 0 \leq t \leq 1.$$

From the above inclusion it follows that for each $u(t) \in V_{c(t)}, 0 \leq t \leq 1$,

$$F(u(t)) \in F((V_{c(t)}) \cap Q$$

which ends the first part of the proof.

To prove that for each $d_0 > 0$ $C(Q, d_0)$ is convex, it is sufficient to observe that under the above assumptions the set

$$Q(d_0) := \{x \in I\!R^N \ : d(x, Q) \leq d_0\} \tag{7.18}$$

is convex. Substituting $Q(d_0)$ instead of Q in the above reasoning we get the thesis of this lemma. ■

Corollary 7.1. *If F is linear, and g and Q are convex then the optimization problems (7.2) and (7.4) restricted to the family S have a convex optimization domain.*

Remark 7.1. *The condition (7.17) cannot be replaced by the weaker assumption, namely that F is convex. This is implied by the fact that the inverse image of an interval by a convex function need not be convex.*

To complete the characterization of the optimization problems (7.2) - (7.5) we shall investigate the properties of the cost measure $\bar{v}$. However, motivated by the real-life applications, fewer observations concerning the properties of $\bar{v}$ can be made in the general case, e.g. there is no relation between the convexity of $\bar{v}$ and of the constraint function g. Therefore we will concentrate our attention on the cost functions which can be represented in the form

$$v(V_a, V_b) = \bar{v}(a, b) = h(b - a). \tag{7.19}$$

From the assumptions CF 1 - CF 4 and the Lemmas 7.1.-7.2 it follows directly that to define v properly, h should be continuous, increasing (i.e. $x \leq y$ implies $h(x) \leq h(y)$), $h(x) \leq 0$ for x ≤ 0, $-h(-x) \leq h(x) \leq 0$ for $x \leq 0$, and for each $x, y \in I\!R^m$, $x \leq y$, the inequality $h(x+y) \leq h(x) + h(y)$ should hold. Such properties possesses e.g. the following function h:

Example 7.2. Let $\| \, . \, \|$ be a l_p norm on R^m. Then it is easy to see that the function h defined as follows

$$h(x) = \begin{cases} \| x \| & \text{iff } x \geq 0 \\ - \| x \| & \text{iff } x \leq 0 \\ \sum_{1 \leq i \leq m} x_i & \text{elsewhere} \end{cases}$$

fulfills all above properties. ■

As a corollary from the above considerations we may derive the following

Theorem 7.1. *Suppose that g is a continuous function from $I\!R^k$ to $I\!R^m$, Q is convex, the cost function v can be represented by the formula (7.19), and the sets V are compact. Then the solution to the minimization problems (7.3), (7.4) and (7.5) exists. The problem (7.4) may be regarded as a minimization of the convex function on the subset of $I\!R^m_+$ -maximal points of the set*

$$X_0 := \{x \in I\!R^m : h(x) \leq v_0\}.$$

The optimization problem (7.4) is equivalent to the minimization of a monotone function over a closed and convex set $C(Q, d_0)$, consequently, its minimum is achieved on the subset of Pareto-minimal points of $C(Q, d_0)$. If, moreover, the objective F satisfies (7.17) and h is convex, then both (7.3) and (7.4) become convex programming problems. If either Q or h are strictly convex then any solution of (7.3) or (7.4) is unique and nondominated in the bicriteria problem (7.5). ■

Proof. The thesis of the theorem is implied directly by the Lemmas 7.1 - 7.3. ■

Other approaches to the solution of problems (7.4)-(7.5) may be used in case where the family of admissible decision subsets is defined by the constraints imposed on measure, or if one considers a family of discrete sets only. In the first case the theory of optimization in the space of subsets of the measure space (cf. e.g. Morris, 1979; Garkavi and Kaminski, 1980; Tanaka and Maruyama, 1984; Zalmai, 1989) can be applied. In the second case, the problems (7.3) - (7.4) are similar to the problem of finding an optimal coalition of players in a cooperative game.

7.5 An Outline of the Interactive Procedure

So far, we have assumed that while selecting a compromise solution to the problem (7.1), the decision-maker takes into account the criteria $F_1, ...F_N$ in the initial multicriteria problem formulation, and the preference structure consisting of a reference target set Q and two second-level objectives : the cost v being paid for changing the set of decisions, and the value of distance from the target set Q.

A balance between the above criteria can be found during an interactive procedure. Its first step consists in finding the value of $d_1 := d(F(U_0), Q)$, and the solution to the problem (7.4), $v_{\min}$, for given U_0, v, Q, and the measure of proximity d in the criteria space. Then a least-distance point $F(u_0)$ and a least-distance solution $u_0 \in U_0$ are presented to the decision-maker. Depending on the specificity of the problem, at each step of the procedure a set of least-distance decisions corresponding to various weights used in the distance function can be generated and presented to the decision-maker.

The procedure terminates if $F(u_0)$ and d_1 are satisfactory. Otherwise, if the payment for changing constraints to V_{opt}, $v_{\min}$, is admissible then the decision set U_1 corresponding to the value of $v_{\min}$ is found and a solution u_1 belonging to the set $FP(U_1) \cap Q$ is to be chosen by the decision-maker. If neither u_0 and d_1 nor u_1 and $v_{\min}$ are satisfactory then the decision-maker is inquired whether the target set Q should be modified and what is the new admissible value of the distance from Q. We assume that the modification may concern only the Pareto-maximal part Q_0 of Q in such a way that for each point q of the modified set Q_1 there exists an element q' of Q_0 such that $q' \leq q$. According to the preliminary assumptions, a new target set may not be more distant to the set $F(U_0\ )$. Assuming a new admissible value for the distance from Q and modifying Q_0 may give methodologically the same result : in the next step the new target set

$$Q_1 := \{x \in I\!R^N : d(x, Q_{01}) \leq d_1 = d_1 - d_{01}\}$$

fulfills all the necessary properties assumed for Q and it is closer to $F(U_0)$.

After setting a new set of target points we repeat the above procedure with Q_1 instead of Q. In the i-th step, the set of target points is extended to

$$Q_i := \{x \in I\!R^N \ : d(x, Q_{0i}) \leq d_i\}, \tag{7.20}$$

unless a satisfying decision has been admitted in this step.

The procedure is going on until an admissible level of v and the corresponding compromise decision u_c are found by stepwise expanding the set of target points.

To facilitate the decision-maker's task during the interaction one may apply the analogous visualization procedure as proposed in Chapter 6 to visualize the attainable and target sets and indicate graphically the least-distance solution at each step of the procedure. A spline approximation approach of Polak (1976; see also Jahn and Merkel, 1992) to plot the sets $F(U_i)$ and $FP(U_0)$ has been extended to three-dimensional criteria space. A four dimensional extension, based on the investigation of three dimensional sections of $F(U)$ and Q may also be included in the system. Some limitations which concern the algorithm are presented in the next subsection.
An interactive algorithm may now be summarized as follow:

Algorithm 7.1.

Step 0. Set $V := U_0$. Set the target set Q by defining k_0 reference points, $Q_0 := \{q_1, q_2, ...q_{k0}\}$, as in Chapter 6. Verify the correct location of reference points; with respect to each other and to the attainable set $P(U)$ so that the conditions warranting the Pareto-optimality of solutions were satisfied. If the definition of Q_0 is not correct then the diagnosis of the problem is generated and communicated to the decision-maker, together with the appropriate hints concerning the redefinition of Q_0. The procedure passes to the next step after the problem has been formulated properly (The detailed algorithm for this step is described in Chapter 6).

Step 1. Visualize the sets $F(V), FP(V)$, and Q using the spline approximation algorithm for the subsets of $I\!R^p,\ p = 2, 3$.

Step 2. Minimize the proximity measure d between Q and $F(V)$. Terminate the procedure if the obtained value of $d(Q, F(V))$ and the corresponding least-distance solution are admissible for the decision-maker.
Otherwise **go to** Step 3.

Step 3. Find the value of $v_{\min}$ and the corresponding set W by solving the problem (7.4). If the value of $v_{\min}$ is admissible then select a nondominated point belonging to $F(W) \cap Q$ and terminate the procedure.

Step 4. Set the new admissible value d_i of the distance from the target set and modify the set Q_{0i} according to (7.20).
Set $Q := Q_{0i}, V := W$.
Go to Step 1. ■

The following subroutines described in the preceding Chapters of this monograph have been applied in the above algorithm :

(Step 0) - interactive procedure for the correct setting of a target set Q;
(Step 1) - a graphical procedure for the visual representation of the sets $F(U), FP(U)$, and Q;
(Step 2) - a distance minimization procedure finding a least-distance point to a polyhedral target set;
(Step 3) - a convex programming procedure based on Thm. 7.4.1. It includes a verification of constraints occurring in problem (7.4)-(7.5) by checking the existence of solutions to the system of inequalities $g(u) \leq c$, for $F(u) \in Q$.

7.5.1 The scope of application of the decision support algorithm

Compared to the general problem statement (7.1) - (7.5), the class of problems which can be solved making use of the above procedure has been restricted by several simplifying assumptions motivated by the availability of numerical methods:

1. The constraint functions $g_i, i \in [1:m]$, are pseudoconvex, the criteria F_j, $j \in [1:n]$ are piecewise linear and monotone.

2. The target set Q is defined by a finite number of reference points $q_1, ...q_p$ in the following way

$$Q := env[\bigcup_{i=1}^{p}(q_i - I\!R_+^N)],$$

   where $env(A)$ denotes the convex hull of A (cf. Chapter 6).

3. The family of admissible decision subsets is defined by the constraints (7.11) and parameterized by the right-side constraint levels; all admissible subsets are bounded.

4. The cost measure v is defined by the function h (cf. (7.18));

5. The visualization procedure may be applied for multicriteria problems with the number of the criteria less than four. The proximity measure d in the criteria space is convex (e.g. the Euclidean norm).

The proposed procedure may serve as a universal decision support tool in various real-life multicriteria decision problems. Its earlier implementations proved already useful in the design of a technical system, multi-stage portfolio optimization, and personnel selection.

Chapter 8

Optimal Control of Discrete – Event Systems Based on Reference Sets and Reference Trajectories

Few attempts (cf. Passino and Antsaklis, 1989; Skulimowski, 1991; Sengupta and Lafortune, 1992) exist to optimize the overall activity of a discrete event system. Although a necessity of applying optimal control strategies is evident in numerous real–life applications, the formulation of an optimal control problem in the classical sense for such systems was difficult due to the differentiated character of factors which are to be taken into account. Following the approach presented in the above mentioned paper (Skulimowski, 1991), in this chapter we demonstrate that the proper description of conflicts and multiple functions of discrete production systems leads to the formulation of a multiple objective optimization problem. Then we will present a multicriteria problem statement combining quantitative criteria with a qualitative preference structure defined on the set of final states considered as a reference set. We will propose solution method for feedback systems which will be based on a generalization of the multicriteria shortest–path method for variable structure nets as well as on the reference points and reference trajectory approaches. Finally, we will analyze a real–life example of control of several interconnected industrial devices.

8.1 Introduction

Proper mathematical description of a discrete–event process is the base for an apt formulation of optimal control problems for such processes. The modelling approaches using formalisms closely related to those of continuous dynamical systems would therefore be best suited for an adaptation of existing control methods for continuous and discrete–time control systems. For those reasons, we will apply here the approach of Ramadge and Wonham (1987), where a discrete–event system is described as an automaton

$$P = (Q, S, \delta, Q_0, Q_m) \tag{8.1}$$

with:

Q – the set of states,
S – the finite set of operations,
$\delta : S \times Q \to Q$ – the state transition function,
Q_0 – the set of potential initial states,

and

Q_m – the set of marked (final) states.

A pair of states (q_1, q_2) such that $q_2 = \delta(s, q_1)$ will be called an event. The above system is often interpreted as e.g. a discrete production process, or a transportation system. Observe that, except the discrete character of the sets occurring in the definition of P, this description is identical to that of a general semidynamical system.

A discrete event system P (DES) can be controlled by choosing an appropriate operation s from the set of admissible operations at a given state q, $S(q)$. The choice of control may be restricted by hard constraints related to the structure of the system as well as by specifying temporary constraints, so called control patterns, in order to achieve a prescribed system's behavior. As an example may serve a urban traffic system, where the hard constraints are related to the road system, while the control patterns may be interpreted as the system of red and green lights on the crossroads. More precisely, the control patterns are functions $\gamma :\to \{0, 1\}$ such that for an arbitrary $x \in Q$ the transition $x \to \delta(x, s)$ is allowed if and only if $s \in S(q)$ and $\gamma(s)$ equals 1. To obtain a well–posed control problem it is necessary to impose a regularity conditions on control patterns, assuming that for each $q \in Q$ there exists at least one $s \in S(q)$ such that $\gamma(s) = 1$. Thus the control is performed at two control levels: the constraints represented by γ–as are introduced by a superordinated level, while for the final choice of a concrete realization of the process a decision–maker at the lower control level may utilize its own control variable u. The actions of the superordinated level, which in the sequel will be called *supervisor*, may depend on the occurrence of events in the system according to a feedback law.

The idea of supervision via control patterns is based on two assumptions: first, in real–life systems it is not possible to eliminate the freedom of subordinated decision–makers in full, second, in case of complicated systems the acquisition and processing of all the data necessary to determine a sequence of actions for each simultaneous realization of the process would exceed the supervisor capability, or it would imply a very high cost of control. Hence a sequence of control patterns $\gamma_1, \ldots, \gamma_n$ and lower–level controls $u_1, \ldots, u_n$ jointly determine a realization of the process passing through a sequence of states $q_1, \ldots, q_n$. Since the only possible result of applying a control u

is the selection of an appropriate operation s, we will identify the controls $u_1, \ldots, u_n$ with the corresponding operations $s_1, \ldots, s_n$.

The actions of both control levels may be oriented towards optimizing certain performance functionals given explicitly or, more often, implicitly. Their goals are usually conflicting, since the better is the satisfaction of the supervisor goals, the less permissive are control patterns and less freedom is left to subordinated decision–makers. Consequently, the lower–level objectives are optimized on a smaller set and their values worsen. For instance, as indicated by practical experience, if in the above mentioned traffic problem only the forward and right directions are allowed then the safety or fluency of traffic as a whole may increase, although these regulations may be inconvenient for each single traffic participant. Besides of the objectives connected with the step–by–step realization of the process, one can also consider a preference structure defined on the set of final states of the realizations which may express e.g. the quality of the manufacturing, or mandatory intermediate stages of production processes.

As we already mentioned, few methodological attempts exist to optimize the overall activity of a discrete event system. Optimal control strategies, as applied in numerous real–life applications, have been usually derived using heuristics, or a non–discrete–event method which could be transplanted directly for a specific case. A general formulation of the optimal control problem in the classical sense has appeared rather difficult due to the differentiated character of factors which are to be taken into account. The goal of this chapter is to demonstrate that the proper description of conflicts and multiple functions of a DES leads to a formulation of a multiple objective optimization problem which can be solved efficiently using the reference sets approach. Namely, we will present a multicriteria optimal control problem statement combining a quantitative criteria related to the events' and states' labels with qualitative preference structure defined on the set of final states Q_m. We will propose a solution method for open–loop and feedback system based on a generalization of the multicriteria shortest–path method as well as on the reference points and reference trajectory approaches. Optimization problems related to the design of DES and discrete–event controllers will also be mentioned. Finally, we will analyze a real–life example of control of several industrial devices connected through buffers and we will discuss the computational properties of the method proposed.

8.2 Statement of the Multicriteria Optimization Problem

Before we pass to the statement of the multicriteria optimization problem, we will describe some quantitative and qualitative preference structures which may occur in real–life applications of the theory of controlled DES.

8.2.1 Quantitative description of a discrete event system

First of all, occurrence of every event (q_1, q_2) may bear certain cost, say $\psi(q_1, q_2)$. In the light of the above remarks concerning the system P, it would be justified to assume that $\psi(q_1, q_2)$ depends on the initial state of the event, q_1, and on this operation s which brings the state q_1 to the other one, q_2, i.e. we assume that

$$\psi(q_1, q_2) = \psi(q_1, \delta(s, q_1)) := f(s, q_1). \tag{8.2}$$

On the other hand, imposing any control pattern γ may also be connected with a cost $g(\gamma)$, representing the amount necessary to use the controller and to implement the control in P. In general, both values, f and g may be dependent on a sequence of previous states, operations, and control patterns. The total cost of implementing the control patterns $\gamma_1, \ldots, \gamma_n$ may be expressed as a function $G(\gamma_1, \ldots, \gamma_n)$, while the occurrence of a sequence of events is related to the amount of $F(q_0, s_1, \ldots, s_n)$, where $q_0 \in Q$ is an initial state of the process.

Motivated by practical applications, we will admit the following simplifying assumptions:

(i) The total cost of a sequence of operations $s_1, \ldots, s_n$ starting from an initial state q_0 can be expressed as a linear combination of costs of single operations with the coefficients depending on the starting state of each operation, i.e,

$$\begin{aligned} F(q_0, s_1, \ldots, s_n) =& \sum_{i=1}^{n} f(s_i, q_{i-1}) = \sum_{i=1}^{n} f_1(s_i)\ a_i(q_{i-1}) = \\ =& \sum_{i=1}^{n} f_1(s_i) f_{2i}(s_{i-1}, s_{i-2}, \ldots, s_1, q_0). \end{aligned} \tag{8.3}$$

In addition, for the analysis of some industrial applications it may be assumed that f_{2i} depends only on the initial state q_0, and not on order in which the subsequent operations occur; in this case we get

$$f_{2i}(s_{i_1}, s_{i_1}, \ldots, q_0) = f_{2i}(q_0) = a_i(q_0),$$

and

$$F(q_0, s_1, \ldots, s_n) = \sum_{i=1}^{n} f_1(s_i) a_i(q_0). \tag{8.4}$$

(ii) The cost of applying a control γ_i depends only on the previous control, γ_{i-1}, i.e. $g(\gamma_i) = g_1(\gamma_i, \gamma_{i-1})$, $i = 1, \ldots, n$, $\gamma_0 := (1, ..., 1)$. The cost of a sequence of control patterns, $G(\gamma_1, \ldots, \gamma_n)$ may be expressed as

$$G(\gamma_1, \ldots, \gamma_n) = \sum_{i=1}^{n} g_1(\gamma_1, \gamma_{i-1}). \tag{8.5}$$

Moreover, it is justified to assume that to the value of $g_1(\gamma, \gamma')$ contribute the costs of maintaining the control of each operation and of switching them on or off. Hencefrom g_1 may be subdivided into two terms, g_2 and g_3, in the following way

$$g_1(\gamma_i, \gamma_{i-1}) = g_2(\gamma_i) + g_3(\gamma_i, \gamma_{i-1}),$$

where

$$g_2(\gamma_i) = \sum_{j=1}^{k} [m_e(s_j)\ \gamma_i(s_j) + m_d(s_j)\ (1 - \gamma_i(s_j))],$$

$$g_3(\gamma_i, \gamma_{i-1}) = \sum_{j=1}^{k} [\max(\gamma_i(s_j) - \gamma_{i-1}(s_j), 0)\ e(s_j) + \max(\gamma_{i-1}(s_j) - \gamma_i(s_j), 0)\ d(s_j)],$$

and $m_e(s_j),\ m_d(s_j),\ e(s_j)$, and $d(s_j)$ are the costs of maintaining active, maintaining forbidden, enabling, and disabling a single operation s_j, respectively. If, in particular, only disabling an operation requires an active action of the supervisor and bear certain control cost, i.e. if

$$m_e(s_j) = e(s_j) = 0 \quad \text{for all} \quad j = 1, ...k,$$

then $g_1(\gamma_i, \gamma_{i-1})$ may be rewritten in a simplified form

$$g_1(\gamma_i, \gamma_{i-1}) = \sum_{j=1}^{k} [m_d(s_j)\ (1 - \gamma(s_j)) + \max(\gamma_{i-1}(s_j) - \gamma_i(s_j), 0)\ d(s_j)]. \tag{8.6}$$

Notice that the index j in the above formulas refers to the fixed order the elements of S are listed, and it is in no way related to the current realization of the process.

In the general case, a transition from γ to γ' may be considered as a change of constraints in a control problem, so it could be treated with a similar method as proposed in the preceding Chapter of this monograph.

From the above considerations it follows that under the simplifying assumptions made, the total cost of performing a string of operations $s_1, \ldots, s_n$ and applying the associated control patterns supervisor's $\gamma_1, \ldots, \gamma_n$ may be expressed as

$$\begin{aligned} \tilde{H}(s_1, \ldots, s_n, \gamma_1, \ldots, \gamma_n, q_0) =& \alpha\ F(q_0, s_1, \ldots, s_n) + \beta\ G(\gamma_1, \ldots, \gamma_n) = \\ =& \alpha \sum_{i=1}^{n} f_1(s_i)\ a_i(q_0) + \beta \sum_{i=1}^{n} g_1(\gamma_i, \gamma_{i-1}), \end{aligned} \tag{8.7}$$

where α and β are positive trade–off coefficients between the functions F and G. Since for most open–loop DES control problems, α and β should be assumed a priori unknown, at this stage we get a bicriteria optimization problem $(F, G) \to \min$.

8.2.2 Optimal control of closed–loop discrete event systems

Assume now that the supervisor for the above defined DES can also be described as an automaton (X, S, ξ, x_0), where X is the set of supervisor's states, the set of operations S is the same as for the process (8.1), ξ is the transfer function, $\xi : S \times X \to X$, and x_0 is an initial state from where the state transitions start. In the above quoted paper of Ramadge and Wonham (1987), the authors pointed out that for a large class of DES there exists a feedback control consisting of a function ϕ mapping the supervisor states into the control patterns, while the transitions between the supervisor states are imposed by the same operations as in the process controlled. In this setting, the supervisor is nothing else as a mechanical device automatically reflecting the actions of the process participants which have no influence on its behavior, so the whole activity of the system can be optimized exclusively from the subordinated system's point of view.

Observing that from an introductory assumption it follows that an operation s determines the control pattern γ at any state of the system uniquely, that is the function

$$\pi : S \ni s \to \phi(\xi(x, s)) = \gamma \in \{0, 1\}^S$$

is well–defined for any fixed supervisor's state x, we may identify the control patterns $\gamma_1, \ldots, \gamma_n$ in the formula (8.7) with the substrings of operations $s_1, \ldots, s_n$. Consequently, (8.7) may be rewritten as

$$\begin{aligned} H(s_1, \ldots, s_n, q_0) :=& \tilde{H}(s_1, \ldots, s_n, \pi(s_1, x_1), \ldots, \pi(s_n, x_n), q_0) = \\ =& \alpha \sum_{i=1}^{n} f_1(\pi(s_i, x_i))\, a_i(q_0) + \beta \sum_{i=1}^{n} g_1(\gamma_i, \gamma_{i-1}). \end{aligned} \tag{8.8}$$

The above defined function H, along with F and G will serve as a general form of objectives in closed–loop DES optimal control problems investigated in this Chapter. The other functions of a similar form as (8.8) may represent the values of time necessary to execute each operation, or to implement a control and other important factors. We assume that if the functions F_i and G_i have pairwise the same meaning then they may be aggregated to the function H. Finally, we will get the following vector optimization problem:

$$\begin{gathered} (F_1(s_1, \ldots, s_n, q_0), \ldots, F_M(s_1, \ldots, s_n, q_0), \\ (G_1(s_1, \ldots, s_n, q_0), \ldots, G_M(s_1, \ldots, s_n, q_0)) \to \min \end{gathered}$$

or

$$(H_1(s_1, \ldots, s_n, q_0), \ldots, H_M(s_1, \ldots, s_n, q_0)) \to \min \tag{8.9}$$

if the functions F_i and G_i may be aggregated to a function of type (8.8).

A dual problem to (8.9) may be considered, for the case where the supervisor transfer function ξ is to be defined to meet specific requirements. In this setting, the sequence of events occurring in the system may not be assumed a priori known. As a consequence, we get the worst case analysis problem to design ξ. If the parameters of the supervisor's structure thus constructed are not acceptable, one may allow for only a partial satisfaction of the supervisor's goals, provided that the actions of the subordinated level are optimal from its own point of view.

8.3 Qualitative Preference Structures in the Sets of Final States and Trajectories of the System

Besides of the above presented quantitative characteristics of states, events, and control patterns, the choice of a sequence of operations may be influenced by qualitative factors related to the lower–level decision–maker's wish to attain certain final state $q \in Q$, or to execute the set of operations along a desired trajectory $q_1, \ldots, q_n$. Some of this additional information which should be implemented in the final problem formulation may appear as a preference structure in the set of states or trajectories, in the form of constraints, or as a specification of the terminal state. In the further part of this section we shall discuss these problems.

8.3.1 Preference structures in the set of marked states

In the theory of DES, the marked states may not necessarily be the terminal states for the realizations of the process. However, the attainment of an element of Q_m is equivalent to achieving certain goal, irrespective from the fact whether the process is going on or not. Marked states may represent the quality of realization of the process, e.g. the quality of a product, or various, more or less desired destinations in traffic control problems.

This interpretation of marked states motivated us to consider a general preference relation on the set Q_m, i.e. Q_m is assumed to be a partially ordered set with the partial order relation " $\prec$ ". The relation " $\prec$ " may be defined explicitly, by assigning certain vector attributes v to each marked state q, or it may be derived during an interactive procedure from the qualitative judgments of persons involved in the decision–making process. In the latter case, one can posteriorly find the minimal k, and for each $q \in Q$ the appropriate vector $V(q) \in \mathbb{R}^k$, so that " $\prec$ " could be identified with the natural partial order in $\mathbb{R}^k$ restricted to the set $\{V(q_1), \ldots, V(q_m)\}$.

Such description of the set of marked states does not contradict the qualitative nature of the preference structure on Q_m – no explicit trade–offs between the coordinates of $v(q)$ will be considered. Instead, we will analyze the interference of " $\prec$ " and the partial order R_1 in the set of process realizations arising from the preferences introduced by objectives (8.8) or (8.9). To make it possible, " $\prec$ " will be regarded in a natural way as a partial order on the set of trajectories terminating at the elements of Q_m. More precisely, if p_n and r_n are two elements of Q_m such that $p_n \prec r_n$ then we will define the relation R_2 in the set of trajectories in the following way

$$\{p_1, \ldots, p_n\} \;\; R_2 \;\; \{r_1, \ldots, r_n\} \Leftrightarrow p_n \prec r_n. \tag{8.10}$$

Consequently, any compromise solution $\{p_1, \ldots, p_n\}$ to the optimal control problem should belong to the set of nondominated elements with respect to the relation $R := R_1 \cap R_2$ defined as follows:

$$\begin{aligned} \{p_1, \ldots, p_n\} \; R \; \{r_1, \ldots, r_n\} \Leftrightarrow (F_1, \ldots, F_k, G_1, \ldots, G_k) \; (\{p_1, \ldots, p_n\}) \leq \\ \leq (F_1, \ldots, F_k, G_1, \ldots, G_k)(\{r_1, \ldots, r_n\}) \;\text{ and }\; p_n \prec r_n. \end{aligned} \tag{8.11}$$

8.3.2 Reference trajectories and preferred system's realizations

Another preference structure may be associated to *reference trajectories.* Specifically, the lower–level decision-maker may wish to arrive at a given marked state by the specified route called the reference trajectory. In general, to each possible final state p may lead a privileged path $p = [p_{1k}, \ldots, p_{n(k)k}]$, with $p_{n(k)k} = p_k$, or a set P of such paths. Three basic situations are now possible:

(i) this path is reachable and nondominated with respect to the criteria (8.9);

(ii) p is reachable but there exists another path which ensures the better values of the optimization criteria (8.9);

(iii) p is non–attainable due to the hard constraints and supervisor's actions.

In the case (i) the choice of p may be immediate provided that the final state $p_{n(k)k}$ is fixed. However, such situations happen rather rarely. Usually, we will assume that the reference trajectories impose another preference relation R_3 which will let us consider the cases (ii) and (iii) in a similar way. Such preferences may be connected with a measure of proximity μ to a given trajectory in the same sense as considered in Skulimowski (1985b). As an example of the function $\mu(p, r)$ may serve the relative number of common states of the trajectories p and r,

$$\mu(p, r) := L(p \cap r)/L(r), \tag{8.12}$$

where $L(q)$ is the number of (different) states the trajectory q comes through. The optimization task would then consist in finding a nondominated sequence of operations $s_1, \ldots, s_n$, so that the corresponding trajectory of the system were closest to p in the sense of μ. The appropriate preference relation R may be defined as

$$q \; R_3 \; r \Leftrightarrow \min\{\mu(p, q) : p \in P\} \leq \min\{\mu(p, r) : p \in P\}. \tag{8.13}$$

Observing that the set of nondominated trajectories X may be decomposed into a disjoint union of sets $X_1, \ldots, X_k$, so that each $p \in X_j$ has a common terminal state

$q_j \in Q_m$, the relation R_3 may be represented as the union of relations R_{3j} defined on the sets X_j, $j = 1, ..k$, i.e.

$$R_3 = R_{31} \cup \ldots \cup R_{3k}. \tag{8.14}$$

The value of m obtained on each subset may then be confronted with the hierarchy of q in Q_m defined by the relation "$\prec$". Following the above observation and the earlier remarks concerning the methods of consideration of the preference relation in Q_m, the choice of a compromise sequence of operations may be based on a bicriteria trade–off method between μ and an artificial quantitative criterion v associated to "$\prec$".

It is important to point out that the reference trajectories may be defined without any relation to the values of criteria (8.9) which their realization would bring up. This fact makes the occurrence of situations (ii) and (iii) realistic. The reference trajectories and the preference relation on the set of marked states constitute a separate preference structure which should be taken into account while solving a DES optimal control problem. However, it is justified to assume that both above qualitative structures are not conflicting if they occur in the same control problem. Namely, to define a reference trajectory one needs to specify its final state which may not be "bad" with respect to the relation "$\prec$", as it is a part of a "good" trajectory. According to (8.10), "$\prec$" orders the trajectories, but this relation is usually incomplete since many trajectories may end at one marked state. Therefore the additional preference structure in cases (ii) and (iii) may supplement "$\prec$", which results in a lexicographic partial order R_4:

$$\{p_1, \ldots, p_n\}\ R_4\ \{r_1, \ldots, r_n\} \Leftrightarrow p_n \prec r_n \quad \text{and} \quad \{p_1, \ldots, p_n\}\ R_{3n}\ \{r_1, \ldots, r_n\}$$

(cf. also (8.13)).

Let us note that reference trajectories make possible a uniform consideration of constraints imposed on the states of the process. In particular, if a trajectory has to pass through a given set of states in an arbitrary order, or preserving some logical conditions, this is equivalent to define an appropriate set of reference trajectories. At last, both reference points and reference trajectories can be defined at different levels of control. In this case we may have to consider conflicting preference structures on Q_m, and conflicting objectives μ and μ' related to subordinated system's reference trajectories and supervisor's preferred realization of the process.

A number of possible goals of a DES has not yet been completed, however we deem that the situation, where all of them occur in the same control problem jointly are exceptional, so the need for a detailed theory for the general case is rather irrelevant. Instead, each special case requires a penetrative analysis with a reference to a real–life application. In this chapter, we concentrate our attention on a class of simplified optimal control problems which may be solved via multicriteria shortest path techniques.

8.4 Aggregated Optimal Control Problem Formulation and Solution

Although the above general description of preference structures occurring in a controlled DES makes possible a complex study of a large class of discrete event processes, not all factors occurring in formulas (8.8) – (8.11) need to have a practical interpretation in a specific discrete control system.

Below we present a simplified model of a controlled closed–loop DES, oriented towards control of industrial and traffic processes. To generate all Pareto–optimal sequences of operations we propose a modified multicriteria shortest–path algorithm. A subsequent multicriteria decision–making problem consisting in finding the best–compromise strategy will be solved using the quasidistance to the reference trajectory μ, as an auxiliary criterion of choice.

This approach makes possible an application of the general reference point methodology described earlier in this monograph. Moreover, it lets us avoid a substantial part of the computational difficulties resulting from a great deal of nondominated discrete trajectories which are usually generated during a dynamic programming procedure.

8.4.1 Simplified problem formulation

To make possible an efficient application of multicriteria shortest–path methods, we will admit several simplifying assumptions. The list of these assumptions looks like as follows:

(a) The initial state q_0, the feedback function π, and the internal supervisor structure are fixed and known to the decision–maker.

(b) The optimal actions can be planned before the process starts (i.e. we do not optimize on–line control processes).

(c) If a lower–level criterion F_i can be matched with certain supervisor's criterion $G_{j(i)}$ so that the functions H_i can be formed then the trade–off coefficients between F_i and $G_{j(i)}$, α_i and β_i (cf. (8.8)), are known to the decision–maker, or they can be derived from the other trade–offs, and without any relation to the posterior realizations of the process.

(d) At least one reference trajectory $\tilde{q}$ is defined by the decision–maker, and the choice of a sequence of events constituting a best approximation of $\tilde{q}$ has a priority over the optimization of the functions $H_1, \ldots, H_M$. As a consequence, the minimization of μ, or another measure of proximity to $\tilde{q}$ will be executed on the set of nondominated sequences of operations before the minimization of H on the set of least–distance trajectories.

(e) The main goal of the system consists in achieving a specified marked state q_m. Consequently, to find a compromise trajectory, the relation " $\prec$ " on the set of marked states Q_m should be taken into account first, a provisional terminal state q_{m0} of the process is selected, then it follows the computation of the optimal sequence of operations $s_{10}, \ldots, s_{n0}$ with respect to the criteria $H_1, \ldots, H_M$, and μ. The values of these criteria for $s_{10}, \ldots, s_{n0}$ are then compared with the values of the function v ranking the elements of Q_m. Thus, the final choice of the best–compromise $s_{1c}, \ldots, s_{nc}$ and q_{mc} is the result of the trade–off between v and the values of $H_1, \ldots, H_M$.

The task which remained now is to define the scope of the optimization process.

As we have mentioned at the beginning of the previous section, the qualitative information used to formulate a DES control problem may have a form of constraints on the decision–maker's actions. Motivated by the real–life applications, one can distinguish the following classes of optimization problems:

(i) the number of events n is constrained;

(ii) the optimization of the criteria $H_2, \ldots, H_k$ is executed until the value of the distinguished criterion H_1 (e.g. time or money) exceeds a fixed value f_n;

(iii) the terminal state q_n is fixed or it belongs to a subset Q_n of Q_m.

Finally, one has to precise what we would like to optimize: the parameters of a single process realization, a set of subsequent realizations, or statistical characteristics of a great deal of processes. The assumption (e) answers partially the question concerning the termination criteria for a single step of the process.

Here, without a loss of generality, we will additionally admit the following assumption:

(f) The controlled process under consideration consists of mutually independent steps, each of them ends at certain marked state selected by the lower–level decision center. Reaching a chosen marked state q_{mc} terminates the optimization procedure, although the real–life process may go on.

The above assumptions require some further comments. Namely, from (a) it follows that design problems will not be considered in this chapter. Moreover, the coefficients $a_i(q_0)$ (cf. 8.7) can be treated as constant numbers. The required knowledge of the system's parameters restricts the scope of applicability of the presented algorithms to a class of deterministic production, or traffic control processes. The problems of finding an optimal structure of supervisor and the feedback mapping π have been considered by Ramadge and Wonham (1987), and Vaz and Wonham (1986).

The assumption (c) is equivalent to the observation that the subordinated and supervising decision centers have the objectives of the pairwise same nature. They may be exemplified as the durations of subsequent operations, energy consumption, or a risk measure. The privileged role of a reference trajectory in selecting a compromise sequence of operations may indicate that this trajectory is imposed by a superior decision center which otherwise would have no influence on the subordinated system's preferences. The general form of the objectives (8.9) can be preserved even if some of above matchings cannot be found; in this case it suffices to set $\alpha_i = 0$ or $\beta_i = 0$.

Overlooking the assumption (f), one can consider the learning phenomena, where the prior judgments concerning the choice of a compromise solution can be examined vs. the actual utility values characterizing this solution. In case of a remarkable difference, trade–off coefficients and parameters of the bargaining procedure may be updated and used to select the compromise trajectory in the next step. A study of such problems constitutes a challenge for future research.

Although the above assumptions may seem to have some restrictive character, for a sufficiently large class of real–life DES the quality of control can be adequately modeled within the framework here presented.

8.4.2 Applying multicriteria shortest–path methods to generate nondominated trajectories

Let P be a controlled DES described by (8.1) with the supervisor Y acting as a closed–loop controller and suppose that the assumptions (a) – (f) of the previous subsection are satisfied. The performance of the system is described by the criteria (8.4), (8.6), and (8.9), while the preferences concerning the choice of a nondominated trajectory are expressed by the relation " $\prec$ " generated by a vector function v and a reference trajectory $\tilde{q}$. Assume, moreover, that the set of operations S is ordered in a fixed way and let the terminal state q_{m0} be constant during each step of the algorithm.

The following numerical structures constitute an input to the multicriteria shortest–path method:

1. A computer representation of the system's network $G = (Q, E)$, where $e = (q_1, q_2) \in E$ iff there exists $s \in S$ such that $q_2 = \delta(s, q)$. Moreover, while listing the edges of G, the operation s causing the transfer from q_1 to q_2 must be explicitly indicated.

Remark 8.1. *The representation of G in the form of a list of successors and predecessors turned out especially useful for improving the computational efficiency of our Algorithm 8.1.*

2. The control patterns γ_i, $1 \le i \le k$, and the feedback map π are coded jointly as the list of pointers $p_1, \dots, p_{n(i)}$ to the forbidden successors of the i–th operation and the subsequent $\pi(i)$–th control pattern.
3. The numerical coefficients occurring in the simplified definition of the criteria $F_1, \dots, F_M$ and $G_1, \dots, G_N$ (cf. (8.4) and (8.6)), are provided in form of the matrices

$$A_1 := [f_{ij}]_{1 \le i \le k,\ 1 \le j \le M} \quad \text{with} \quad f_{ij} = f_{1j}(s_i)\ a_i(q_0),$$
$$A_2 := [m_{dij}]_{1 \le i \le k,\ 1 \le j \le M} \quad \text{with} \quad m_{dij} = m_{dij}\ (s_i),$$

and

$$A_3 := [d_{ij}]_{1 \leq i \leq k,\ 1 \leq j \leq M} \quad \text{with} \quad d_{ij} = d(s_i).$$

Without any loss of generality we will assume that all coefficients of $A_1,\ A_2$, and A_3 are non–negative, which will let us avoid the detection of a negative cycle in G.

4. The marked states Q_m are defined by listing their positions on the list of all states Q. The preference relation " $\prec$ " on Q_m is represented by the values of a vector function $v = (v_1, \ldots, v_1)$ defined on Q_m, and stored in the array V. At last, reference trajectories are specified as lists of elements of Q starting from q_0 and with the last element belonging to Q_m.

The proposed algorithm will be based on the well–known Dijkstra method allowing the simultaneous evaluation of all optimal paths starting at q_0 and terminating at the elements of Q_m. Some of the difficulties which had to be solved were implied by the varying structure of G and variable coefficients of A_2 and A_3, all depending on the control pattern previously implemented. The other problems were connected with a preliminary reduction of the set of nondominated trajectories in the case where this set grew exponentially. A draft of this algorithm is presented below.

Algorithm 8.1.

Step 1. Decompose the set Q into level sets $Q_1, \ldots, Q_p$ so that q_0 cannot be linked with an element of $Q_j,\ 1 \leq j \leq p$, by a path consisting of less than j edges. Set $Q_0 := \{q_0\},\ i := 2$.

Step 2. Compute the values of criteria for the shortest paths beginning at q_0 and terminating at the elements of Q_1.

Step 3*i*. For each $q \in Q_i$ find the set of transitions from q allowed by each of the control patterns $\gamma_i = \pi(s_i, x_i)$, where s_i is an operation governing the transition from an element of Q_{i-1} to q and x_i is the i–th supervisor's state.

Step 4*i*. For each $r \in Q_{i+1}$ find the characteristics of the shortest paths terminating at r in the reduced network found in the previous step. To update the values computed for all the predecessors q of r from Q_i find the appropriate values of $f_j(s_i, q)$ in the array A_1 and calculate

$$g_j(\gamma_i, \gamma_{i-1}) = g_j(\pi(s_i), \pi(s_{i-1}))$$

according to (8.6). Eliminate the dominated values of criteria and delete the related paths from the memory.

Step 5*i*. If a marked state q_{mp} occurs at the i–th level then compute the values of the function μ (cf. (8.12)) for all non–dominated paths terminating at

q_{mp} and choose the least–distance sequence of operations. The resulting criteria values (including μ) and least–distance trajectory are then stored as the p–th columns in the arrays B and C, respectively.

Step 6. If the shortest–path characteristics for all elements of Q_m are stored in B **Stop**, otherwise repeat the steps *3i*, *4i* and *5i* for the incremented value of i. ■

As a result of execution of the above algorithm one gets the arrays B and C containing the nondominated criteria values and the corresponding nondominated paths in G, closest to the reference trajectory $\tilde{q}$, respectively.

Although the procedure calculating the least–distance trajectory to $\tilde{q}$ for each marked state belongs to the decision–support part of the package, we included it in the preliminary part of the algorithm to reduce the number of nondominated paths which could be very large even for relatively small G. Thus we can store all nondominated least–distance paths in a rectangular array C with the dimensions equal to the number of marked states and the maximal length (provided that all edge labels are set to 1) of a path starting from q_0. An experimental implementation of this algorithm on a personal computer with a 32bit architecture and 90 Mhz clock processor proved efficient for relatively large problems.

8.4.3 Selecting a compromise trajectory

A preliminary selection of the compromise trajectory has been already made during the procedure generating nondominated paths. As an output of this procedure we obtained the set of nondominated trajectories, each of them terminating at a different marked state, and being closest to $\tilde{q}$ in the sense of μ from among all such trajectories.

All that remains is to choose the best compromise sequence of operations from the array C, based on the numerical characteristics contained in the matrices V and B. To achieve this goal, we propose the following interactive multicriteria decision–making procedure:

Algorithm 8.2.

Step 1. Decompose the relation " $\prec$ " into maximal well–ordered subsets (chains). Find the maximal and minimal elements of each chain and the number of elements dominated by each maximal element (rank).

Step 2. Find the global minimal value of μ over the set of all minimal values stored in B, μ_p, and the corresponding marked state q_{mp}.

Step 3. Present the vector of objective values (b_p, μ_p) to the decision–maker. If satisfied with this solution, find the appropriate sequence of operations; **Stop**, otherwise ask the decision–maker which of the criteria $F_1, \ldots, F_M$, $G_1, \ldots, G_N, v$ must be improved, compared to the values characterizing q_{mp}.

Step 4. Find the set of all marked states Q_r such that those characteristics of the corresponding shortest paths which have been specified in Step 3 are better than the appropriate values characterizing q_{mp}. If the values of v should also be improved, i.e. if the position of q_{mp} in the hierarchy imposed by " $\prec$ " is insufficient, then set

$$Q_r := Q_{r'} \cap Q_{r^*},$$

where the set $Q_{r'}$ contains the marked states dominating q_{mp} with respect to " $\prec$ ". To find Q_{r^*}, use the decomposition of " $\prec$ " into chains established in Step 1.

Step 5. If Q_r is empty then request the decision maker either to redefine the set of criteria which must be improved, avoiding those indicated as nonimprovable, or accept the characteristics of the current final state and optimal path. In the first case go to Step 4, in the latter one terminate the algorithm.

Otherwise set

$$\mu_p := \min\{\mu(\tilde{q}, \rho(q)) : \ q \in Q\},$$
$$q_{min} := \arg\min\{\mu(\tilde{q}, \rho(q)) : \ q \in Q\},$$

and

$$b_p := (F_1, \ldots, F_M, G_1, \ldots, G_N)\ (\rho(q_{min})),$$

where $\rho(q)$ is the least–distance path terminating at q.

Proceed to Step 3. ■

Remark that the Algorithm 8.2 is always convergent and that the interaction with the decision–maker can be easily replaced by an automatic choice of the compromise solution using a dialogue with an expert system.

A privileged role of the function μ may be justified by the special significance of reference trajectories which usually express the decision–maker's wishes in a most explicit way. Although any values of μ can be finally obtained, by generating first the solutions characterized by the smallest values of μ, the decision maker is encouraged to choose the trajectory closest to $\tilde{q}$. In real–life situations, the qualitative preference structures " $\prec$ " and $(\tilde{q}, \mu)$ may be defined by a superordinated decision center, so that the compromise solution chosen as a result of this algorithm would correspond to the interests of such higher–level decision center.

Below we present a simple example of the control of an industrial installations involving the use of the above ideas.

8.5 An Example of a Controlled Discrete Production Process

Let us consider a simple industrial system shown in Fig. 8.1:

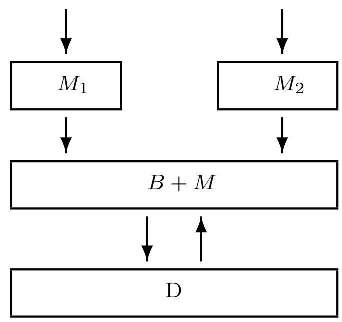

Fig. 8.1. *An example of a controlled DES.*

Two machines installed in parallel, M_1 and M_2, are producing two different products, P_1 and P_2, respectively. P_1 and P_2 are then stored in the buffer B which serves directly the montage line M. M assembles the final products T_1, or T_2 according to the external demands D. The products T_1 and T_2 can be assembled if P_1 or, respectively, both P_1 and P_2 are present in B. We assume that the choice of the final product is independent from the other events occurring in the system, while the goal of control consists in ensuring possibly fluent production, and optimization of certain criteria. The first goal is achieved by supervising the system by a discrete controller X which takes care of preserving the following qualitative control laws:

C1. M_1 and M_2 may not simultaneously load the buffer B.

C2. If B is empty then M_1 starts first.

C3. M_i must be in the passive state if P_i is already in B, $\;i = 1, 2$.

C4. There is no production if there is no external demand.

C5. The demands may not be cancelled before the buffer is empty.

M_1, M_2, B, and D constitute the only subsystems of the whole above system which are considered here. The sets of their states $Q(.)$ and admissible operations $S(.)$ are presented below:

$Q(M_i) = \{\text{passive, active}\} := \{0, 1\}$,
$S(M_i) = \{\text{off, on}\} := \{s_{i0}, s_{i1}\}, \;\; i = 1, 2;$
$Q(B) = \{\text{empty, product 1, products 1 and 2}\} := \{E, 1, 2\}$,

$S(B) = \{\text{load } P_1, \text{ load } P_2, \text{ use } P_1, \text{ use } P_1 \text{ and } P_2\} := \{b1, b2, b3, b4\};$

$Q(D) = \{\text{satisfied}, \text{ produce } T_1, \text{produce } T_2\} := \{0, 1, 2\}.$

$S(D) = \{\text{demand } T_1, \text{demand } T_2, \text{cancel demand } 1, \text{cancel demand } 2\} := \{d1, d2, d01, d02\}.$

The transition diagrams of each subsystem may be represented as in the Fig. 8.2.

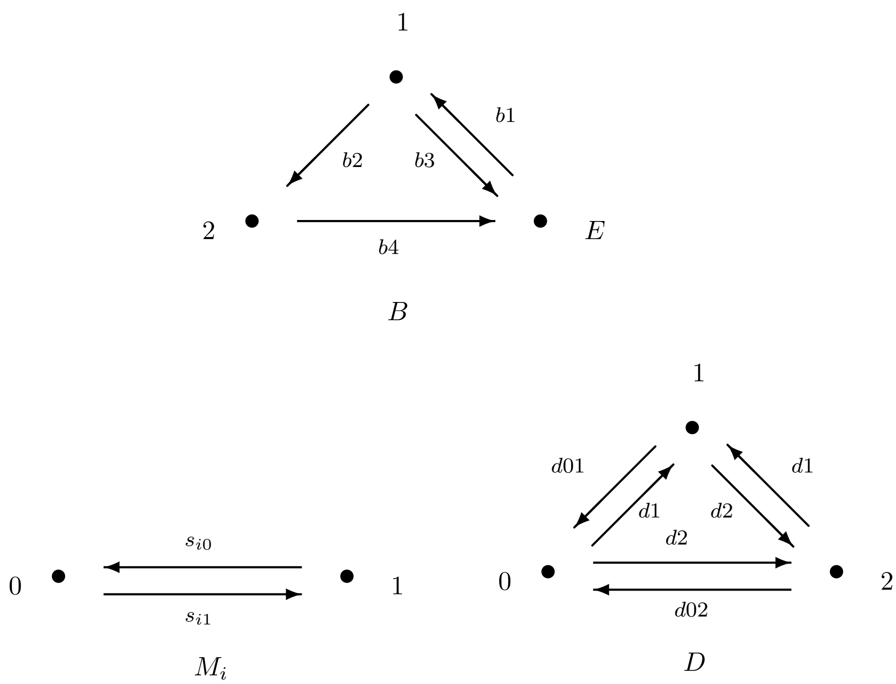

Fig. 8.2. *The state transitions in the subsystems M_i, $i = 1, 2$, B, and D.*

The entire production system can now be considered as the shuffle product (cf. Ramadge, 1989) of M_1, M_2, B, and D, with

$$Q = Q(M_1) \times Q(M_2) \times Q(B) \times Q(D)$$

and

$$S = S(Mi_1) \cup S(M_2) \cup S(B) \cup S(D).$$

The states of Q have the form $q = (M_1 M_2 B D)$, where the components' symbols should be regarded as a general form of their above listed states. Each operation $s \in S$ transfers the appropriate subsystem to a state specified in Fig.8.2, while all remaining components of the starting state remain unchanged. Therefore the investigation of the system's network G can be decomposed into a separate investigation of components and the relations between them.

The choice of an operation in G consists in selecting consecutively a system component, and an operation from its diagram. The choice of an operation is restricted by the control patterns imposed by the supervisor X. According to the preliminary remarks we will not be concerned here with the supervisor's design, assuming that it has been done in such a manner as presented by Ramadge and Wonham (1987). Consequently, all control patterns at each state q of our system are assumed known.

The next goal of the system consists in minimizing the criteria F_1, F_2, and G which can be interpreted as the costs of executing an operation, the necessary time, and the cost of control, respectively. The values of the functions F_1 and F_2 on the set S are presented in the table below.

Table 8.1.

S	s_{10}	s_{11}	s_{20}	s_{21}	b_1	b_2	b_3	b_4	d_1	d_3	d_{01}	d_{02}
F_1	0.1	1.0	0.1	1.0	0.5	0.5	1.0	1.5	0.1	0.1	0.0	0.0
F_2	0.0	1.0	0.0	0.5	0.1	0.1	0.5	1.0	0.2	0.3	0.0	0.0

To the cost of control G contribute the values of $m_d(s) = 0.1$ and $d(s) = 0.05$ for each $s \in S$ (cf.(8.6)). Based on the knowledge of the supervisor's states and the feedback function π, we will determine the values of G during the procedure described as Algorithm 8.1.

As the starting state we will admit $q_0 := (00E0)$. The marked states are those having as the last coordinate 0, i.e. we will optimize the process until the external demands of D are satisfied. In particular, $q_0 \in Q_m$, which indicates a periodic nature of the process. The qualitative preference information is given as the reference trajectory

$$\tilde{q} := ((01E1), (01E2), (0022), (00E0)),$$

and the relation " $\prec$ " on Q_m which can be defined as follows:

$$(00r0) \prec (01r0) \prec (11r0), \quad \text{and } (00r0) \prec (10r0) \prec (11r0),$$

where r is an arbitrary state of B. However, observing that the control laws $1-5$ do not allow that the buffer remains full after satisfying the demands of D, we find out that $r = E$ and $\sharp(Q_m) = 4$. Finally, the proximity measure μ from a given trajectory s to $\tilde{q}$ is defined according to (8.12) as

$$\mu(s, \tilde{q}) := L(s, \tilde{q})/L(s).$$

After performing the Algorithm 8.1 described in Sec. 8.4.2, we will find two shortest paths starting at q_0 and terminating at different elements of Q_m:

$$\rho_1 := \{(00E0, d_1), (00E1, s_{11}), (10E1, b_1), (1011, b_3), (10E1, d_{01}), (10E0)\},$$

and

$$\rho_2 := \{(00E0, d_1), (00E1, s_{11}), (10E1, b_1), (1011, s_{10}), (0011, b_3), (00E1, d_{01}), (00E0)\}.$$

It is easy to see that $\mu(\rho_1) = 0.33$, and $\mu(\rho_2) = 0.28$. The values of the criteria F_1 and F_2 (cf. Tab. 8.1) amount to:

$$(F_1, F_2)(\rho_1) = (2.5, 1.8),$$
$$(F_1, F_2)(\rho_2) = (2.6, 1.8).$$

To compute the values of G observe that the sets of admissible operations for the above trajectories have been defined by the following control patterns:

$$\begin{array}{llrl}
\gamma_0(s) = 0 & \text{iff} & s \in \{s_{11}, s_{21}, b_1, b_2\}, & (\gamma_i(s) \equiv 1 \quad \text{elsewhere}, \quad i = 0, 1, ..), \\
\gamma_1(s) = 0 & \text{iff} & s = s_{21}, & \gamma_1 := \pi(d_1), \\
\gamma_2(s) = 0 & \text{iff} & s = d_{01}, & \gamma_2 := \pi(d_1, s_{11}), \\
\gamma_3(s) = 0 & \text{iff} & s \in \{b_2, d_{01}\}, & \gamma_3 := \pi(d_1, s_{11}, b_1), \\
\gamma_4(s) = 0 & \text{iff} & s \in \{s_{11}, b_2, d_{01}\}, & \gamma_4 := \pi(d_1, s_{11}, b_1, s_{10}), \\
\gamma_5(s) = 0 & \text{iff} & s \in \{s_{10}, s_{21}, b_1\}, & \gamma_5 := \pi(d_1, s_{11}, b_1, s_{10}, b_3), \\
\gamma_4'(s) = 0 & \text{iff} & s \in \{s_{21}, b_1\}, & \gamma_4' := \pi(d_1, s_{11}, b_1, b_3).
\end{array}$$

Substituting the values of $m_d(s)$ and $d(s)$ in (8.6) for the above control patterns, one gets the values

$$G(\rho_1) = 1.2 \quad \text{and} \quad G(\rho_2) = 1.7.$$

Further, we will assume that F_2 and G have the same significance to the decision–maker (both contribute to the production costs) and can be aggregated to the function H (cf. (8.9)) with identical trade–off coefficients $\alpha = \beta = 1$. Therefore the quality of production will now be described by the factors

$$(F_1, H)(\rho_1) = (2.5, 3.0),$$

and

$$(F_1, H)(\rho_2) = (2.6, 3.5).$$

So far ρ_1 dominates ρ_2 but we still have to use the qualitative information contained in the relation " $\prec$ ". To consider " $\prec$ " during the final choice of the compromise alternative, we will introduce the *grading function* v which values can be interpreted as the penalties for an active machine after satisfying the production demands. Namely, we will set

$$v(00E0) := 0,$$
$$v(01E0) = v(10E0) := 1,$$

and

$$v(11E0) := 2.$$

This part of the decision–making procedure will usually be executed as an interactive algorithm described in Sec.8.4.3. The final choice will thus be based on the factors which were not previously contained in the problem formulation, so here we can arbitrarily assume the existence of the trade–off coefficients between v and F_1, and v and H equal to 0.2 and 1, respectively. As a result of this operation, the values of the criteria for the sequence ρ_1 increase to (2.7, 4.0), which leads to the choice of the sequence of operations ρ_2.

The above choice may be explained as an illustration of the role played by the security and energy–saving criteria, which impose, in turn, switching off all machines after termination of an external order despite of the fact that keeping the installation active might speed up the overall production process.

8.6 Conclusions

In this final Chapter we have outlined applications of the reference sets methodology to solve the variety of optimization problems occurring in controlled discrete–event systems. In Secs. 8.4 and 8.5 we proposed a solution method based on the multicriteria shortest path algorithm which has then been applied to find an optimal sequence of operations for control of an industrial installation. We have emphasized the role of conflicts, multiple goals, and uncertainties which led to formulation of the multicriteria discrete optimization problem.

The algorithms here presented do not pretend to be a general tool for evaluating the optimal trajectory in controlled DES – to an analysis of a general DES one would require rather a decision support system with a complicated internal logical structure making possible the choice of an appropriate optimization model for a given system. However, the procedure described in Sec. 8.4 may constitute a part of such decision support system. Moreover, the framework of automata theory the present approach is based on may not be regarded a general one. Interesting results for DES have also been obtained using the Petri net, or the max–algebra approaches. Nevertheless, the similarities between the description of an automaton and a controlled semidynamical system make it particularly attractive from the point of view of applying the classical notions of control theory to DES. The computational difficulties related to the exponential growth of the number of states for the multi–device interconnected systems or systems of queues may be removed after a more penetrative analysis of the model such as it has already be done in the papers of Ramadge (1989) and Vaz and Wonham (1986).

Chapter 9

Final Remarks on "How to Apply the Decision Support Tools ?"

The appearance of the computer-aided decision support contributed to a remarkable change in the practice as well as in the philosophy of decision-making. The traditional worth and the role played by the intuition and experience of the decision-maker remained relevant, but they are newly accompanied by the systematic analysis of the problem. This, in turn, implies the need for a proper mathematical model of the decision situation concerned and the representation of knowledge about the preference structure. Further, the notion of optimality has passed an instructive evolution : from a categorical minimum of a scalar objective function, through the notion of vector minimum in partially ordered criteria space, to the concept of the extended optimum, as considered in the subset selection or dynamical extension problems (Skulimowski, 1992).

The approach proposed in this monograph has been based on the idea of excluding any loss of information about the decision to be made. It follows from three underlying assumptions originated from the real-life experience:

- first, we assume that all information available should be applied jointly and as soon as possible during the decision-making process,
- second, we claim that mutually contradictory information should be aggregated, but not eliminated during the decision-making process,
- third, we assume that reference sets and trade-offs are sufficient and most suitable tools to describe most real-life preference structures.

The first assumption contradicts the usual approach used in the interactive decision support : partial information is used there to generate a decision, further information is processed only if it is not satisfactory to the decision-maker. The previous implementations of the reference point method use namely a sequential processing of reference points and concentrate rather on varying the coefficients of the proximity measure than on exploring the information associated with the reference points. In this setting, the decision finally made depends on the subjective sequence of processing the additional information. Moreover, such procedures prolong unnecessarily the decision-making process, and may lead to a loopy inconsistent process, since the convergence conditions for interactive procedures are usually based on too optimistic assumptions concerning the human rationality. The secret of success of this philosophy, despite of the above disadvantages may be explained by the well-known paradigm consisting in the fact that even a bad decision is often better than no decision. In addition, numerous decision-makers, especially acting as managers or politicians need an alibi - or any kind of decision support - for justifying voluntarily or randomly chosen decisions. This might explain the unusual popularity of some heuristic methods to select from among a discrete set of alternatives (cf. Sec. 3.6) used frequently without any consideration concerning their scope of applicability.

Decision support science is a part of mathematical modelling, specifically, of modelling the real-life decision processes. Therefore the best arguments to support our approach come from the practice, where it has proved useful e.g. to provide successful solutions to complicated portfolio optimization problems (Skulimowski, 1993), a field where one has to confront the competition of virtually all decision support techniques extensively tested in profit generation. Another important field of applications of the reference sets approach is design of technical systems, where reference points have a straightforward and intuitive interpretation (cf. Example 6.2). The extension of the reference set method proposed in Chapter 7 allows, in fact, to extend the interactive process to the problem formulation stage, and combines the problem statement and solution process in one interactive procedure. We expect that this novelty in multicriteria decision support will become a standard in the future advanced procedures. In Chapter 8 we have also shown an enormous family of application to optimal control of discrete-event systems.

More generally, we argue that the development trends in design of computer-aided decision support should lead to the intelligent and possibly universal systems characterized by the following properties:

- extensive use of knowledge bases storing the information about available decisions and various types of additional preference information : rules, reference characteristics of different kinds, criteria-space-constraints, trade-offs, information about the decisions previously made and their posterior evaluation, and other;

- information aggregation mechanisms, including the consistency checking and automatic inconsistency corrections, allowing to apply all preference information available simultaneously,

- learning schemes allowing to apply the past decision record to propose the compromise alternatives within interactive procedures;

- interactive support at the stage of the problem formulation;

- tools allowing the model extensions : from the single alternative to a subset selection problem; from the static problem to a dynamical model including the future consequences of the decision to be made (Skulimowski, 1985b), and their combinations;
- reductions of multicriteria decision problems to bi- or tricriteria trade-offs, their visualisation and graphical analysis.

Such systems may be referred to with the common name *flexible decision support systems* which can be understood both as the systems providing assistance in making a flexible decision, and as flexible systems to support decisions. Most of the above listed features are already fulfilled by the decision support systems proposed here, or in other publications of the author. Specifically, an implementation of the reference set method outlined in the Section 6.6 has been designed as a user-friendly decision support system allowing to consider different classes of reference sets, criteria-space constraints, and trade-offs in one model. The scope of potential applications of the methodology here presented has not been a priori restricted, and the decision-making procedure should provide a well-suited decision support for a large class of underlying multicriteria optimization problems. However, based on this methodology, one can design specialized decision support for solving e.g. industrial design, financial planning, or group decision and negotiation problems, in a similar way as the decision support for discrete-event processes presented in Chapter 8.

Having once formulated a multicriteria optimization problem and having selected the computer-aided decision support based on reference sets to choose a compromise decision, no specialized knowledge will be necessary to use the system. Graphical procedures will allow to define or modify the reference sets just by pointing them on the screen with a pointing device, and the resulting compromise decision will also be provided in the graphical form. An external expert may only be necessary to apply more penetrative consequence analysis techniques, or to provide the missing data.

Besides of the potential impact on real-life multicriteria decision methodology we contributed to the solution of some unsolved theoretical problems: in Chapter 6 we formulated a simple condition for the uniqueness of ideal points - a problem whose unverified improper solution has been sometimes mentioned in the literature (Tanino, 1988). In the same Chapter we introduced the notion of local ideal points, pointing out its relevance for decision support procedures in non-convex multicriteria problems. The notion of strictly dominating points introduced in Skulimowski (1988, 1989), and investigated further in Chapter 5 contributed to provide the solution to another classical problem in vector optimization, namely to the Pareto optimality of distance scalarizing procedures. The results there obtained can be easily applied to an arbitrary convex scalarizing function. Further, in Chapter 6 we gave a systematic approach to estimate the value function in the decision-making problem with the additional preference information given in form of multiple classes of reference points. This approach clarifies the questions arising while using the preference model consisting of well-defined reference points and a less precisely determined proximity measure.

Finally, let us mention that advanced mathematical results constituting a base for sophisticated decision support procedures need not be visible to the user of the system. In particular, the results on properties of dominating points and optimality conditions derived in Chapters 4 and 5, respectively, are merely used in the background to check the consistency of the reference points defined by the decision-maker, and - if necessary - to redefine them so that the consistency conditions were fulfilled. Eliminating these steps would, however, lead to a chaotic generation of dominated alternatives, and to the situation where the conditions for terminating the decision-making process were never fulfilled.

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Index

This index contains the list of terms and symbols followed by a reference to that page, where the appropriate definition appears. The reference to the formula or definition (if any) is given in parantheses.

List of Symbols

$[a, b]$ - interval with edge points a and b

$Cl(U)$ - family of all closed subsets of the topological space U

$d_H(A, B)$ - the Hausdorff distance of the sets A and B

$D^{(n)}(E, F)$ - n-differentiable functions from E to F

$env(A)$ - the convex hull of A

$[i : j]$ - integer interval between i and j

$k_r(x)$ - the open ball with center x and radius r

$K_r(x)$ - the closed ball with center x and radius r

$L^*(X)$ - local ideal points of X

$\mathbb{N}$ - the set of natural numbers (including zero)

pr_i - projection paralell to the i-th axis in a coordinate system in $\mathbb{R}^N$

$P(X, \theta)$ - the set of nondominated points in X with respect to the cone θ

$PD(X, \theta)$ - the set of partly dominating points to X with respect to θ

$\mid r \mid$ - the absolute value of r

$\mathbb{R}$ - the field of real numbers

$\mathbb{R}_+$ - the set of non-negative real numbers

$\mathbb{R}^N$ - N-dimensional Euclidean space

$R(X)$ - local edge points of X

$SD(X, \theta)$ - the set of strictly dominating points

$TD(X, \theta)$ - the set of totally dominating points

$\| x \|$ - the norm of x

$x^*(X, \theta)$ - the set of ideal points

$\delta f(x_0)$ - Fréchet derivative of f at x_0

$\partial f(x_0)$ - generalized gradient of f at x_0

$\nabla f(x_0)$ - the gradient of f at x_0