# Neighbourhood and Number of States dependence of the Transient Period and Cluster Patterns in Cyclic Cellular Automata

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**ABSTRACT**

Cellular automata provide a valuable platform for exploring complex and emergent behaviour. We explore the sensitivity to different neighbourhoods and different number of cellular states of the Cyclic Cellular Automata. This model produces complex kaleidoscopic repeating patterns a short transient period after being initialised randomly. We use component labelling metrics and appropriately chosen ratios to characterise the different model regimes for nearest, next-nearest and Moore neighbourhoods and for a range of allowable cellular states ranging $Q = 2, 3, ..., 20$. We discuss the emergent macroscopic behaviours averaged over many independent trajectories through model space and the transition in behaviours observed around the critical $Q^* \approx 4 - 5$.

**KEY WORDS**
cyclic automata; cellular automata; complexity; emergence; spirals.

## 1 Introduction

The Cyclic Cellular Automaton model was first suggested by Griffeath [12] who investigated a two dimensional square grid with Moore neighbourhood and up to ten distinct cellular states. We extend this model to different neighbourhoods and up to twenty distinct allowable cellular states.

Cellular automata CA) models [28] have a long history for showing complex macroscopic phenomena and behaviours that are not trivially anticipated from their localised microscopic rules. Wolfram [35] and many other researchers have reported emergent complexity from such models in one dimension systems [33] or in two dimensional automata [25] including the well known Game of Life(GoL) system [10]. Conway's GoL has been extended to have a whole family of different microscopic rules [15] and also to have more than just two states [17].

Generally these models are investigated computationally and can be simulated with a range of synchronous and asynchronous update approaches [4, 23] and data structures [29].

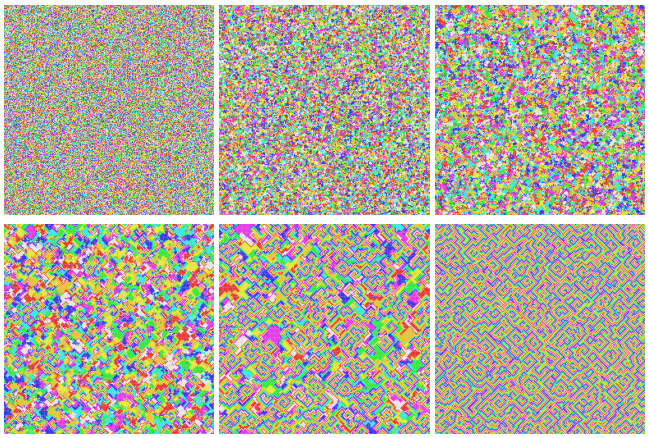

Figure 1: $Q = 8, N = 256^2$ Cyclic CA at times 0, 4, 8, 16, 24, 128

CA exhibit sophisticated growth behaviours [22, 32] and also even more complex physical phenomena [11] such as phase transitions [31], non-equilibrium thermodynamical properties and sometimes also universal properties [5, 34].

Automata that aim to mimic some physical properties or behaviours can be studies on a range of different lattice structures, beyond simple two dimensional square lattices [2, 24].

Certain forms of fluid dynamics [3, 36] and lattice gases [19, 21] can be modelled using CA systems. Other applications of CA systems include: artificial life models [1]; game theoretic models [9]; non linear ratchet systems [13]; predator-prey ecological models [16]; and other growth and decay systems [7]. These models often spontaneously exhibit spatially rich structures such as spirals [18].

The Cyclic Cellular Automata (CCA) model was first identified by Griffeath [12] as a self organising system in which long range structure emerges from a random initial state including spirals [26]. Various studies have been reported in the literature on the CCA [27, 30] with most work on the square two dimensional lattice , the Moore neighbourhood and a limited number of different allowable species states [6, 8]. Figure 1 shows the Cyclic Cellular Automaton model at various times with 8 allowable states.

In this present work we use extensive computer simulations to explore how the CCA model behaves when the lattice geometry and neighbourhood size are varied. We speculate that the kaleidoscopic patterns observed and reported for the model are due to cyclic repetitions of the Q-long cycle of Q different species present as they adapt and adjust to fit long range structures into the available lattice space and geometry. We experiment with nearest, next-nearest and Moore neighbourhoods on the square lattice as well as also hexagonal and triangular lattice geometries. We vary Q over a range of $Q = 2, 3, ..., 20$ and demonstrate a transition in behaviour at around $Q^* \approx 4 - 5$ depending upon the particular lattice.

Our article is structured as follows: In Section 2 we summarise the Cyclic Cellular Automaton model and the procedure we used for the numerical experiments – the results of which are presented in Section 3. We discuss some of the metrics and macroscopic patterns we observed in Section 4 and offer some conclusions and areas for further work in Section 5.

## 2 Method

The cyclic cellular automaton (CCA) model is initialised with an equal proportion of the Q different species being investigated. An update algorithm is applied to all the cells simultaneously based upon their current value and that of their neighbours. The size and shape of the neighbourhood can be varied and also depends upon the lattice being studied. We investigated square lattices with nearest(N1), next-nearest(N2) and Moore (both N1 and N2) neighbourhoods and we also investigated hexagonal and triangular systems. We adopt the common convention and refer to a hexagonal system with 6 nearest neighbours even although it consists of triangle) and a triangular lattice with 3 nearest neighbour which when drawn looks as though it is made of hexagons. The hexagonal and triangular lattice are mutual adjoint structures and so crystallographers and physicists often refer to these the other way around in different literature communities. The square lattice is its own adjoint - when bonds and sites are swapped.

Algorithm 1 shows the algorithmic process for the numerical experiments reported.

The model is implemented using a simple array structure to hold the individual cells and a small integer value suffices to model the range of species studied.

Since we needed to average over multiple independent configurations (least ten) we implemented a custom simulation program rather than using a commodity simulator. The model code was generated using a fluent simulation interface as described in [14] and generated Java code which ran adequately on a multi-cored desktop computer. There is scope for improving the simulation performance using data parallel computing techniques as described in [20] however.

**Algorithm 1** Q-State Cyclic Cellular Automaton Model.

```
choose lattice size, shape, eg square 256^2
choose neighbourhood N eg Nearest
choose number of allowed states Q
for all sample runs eg 10 or 100 do
  initialise N sites randomly
  for all steps, eg 200 do
    for all cells i = 1..N do
      newvValue = (oldValue + 1) modulo Q
      for all neighbouring site j ∈ N do
        if sites i and j are same species then
          record cell i to take newValue
        end if
      end for
    end for
    update all cells i
    record bond populations and cluster sizes
  end for
end for
normalise averaged measurements
```

## 3 Selected Results

It is useful to study snapshots of the model configuration to make sense of the measurements. Figure 1 shows how the model typically evolves in time when it is first initialised randomly with each of the Q species having an equal population. After initialisation small clumps of species form and how large these become, appears to depend upon Q. Unlike other models, the CCA model quite rapidly forms these initial clusters which them give rise to long range kaleidoscopic patterns of interleaving and periodically repeating structures. We have artificially coloured the states and have tried to choose colours and shades that are visually distinct.

Most of the work reported in this present paper made use of a simulated system size of $N = 128^2$ and we found that the transient early phase usually lasted less than 50 time steps.

The configurations shown in Figure 2 were all run for 200 time steps and show how the behaviour changes with varying the number of states Q. These snapshots are all for a Moore neighbourhood and a square lattice. The range of Q we investigated was from 2 up to 20. At low $Q < 5$ only relatively short length scales appear and although clumps and cluster arrange them selves with repeating patterns – which oscillate through the characteristic Q cycles of the CCA model - they remain relatively small with respect to the lattice length. At higher Q values we see the repetition structure forming over longer length scales and it appears to grow as we run the systems for more time steps. As Q increases we obtain very large scale repetitive structures that will typically have all Q states present in spiral patterns.

The figures show that the spirals interfere with one another and overlap and with time larger ones dominate smaller ones and the "kinks" are ironed out of these long range spatial

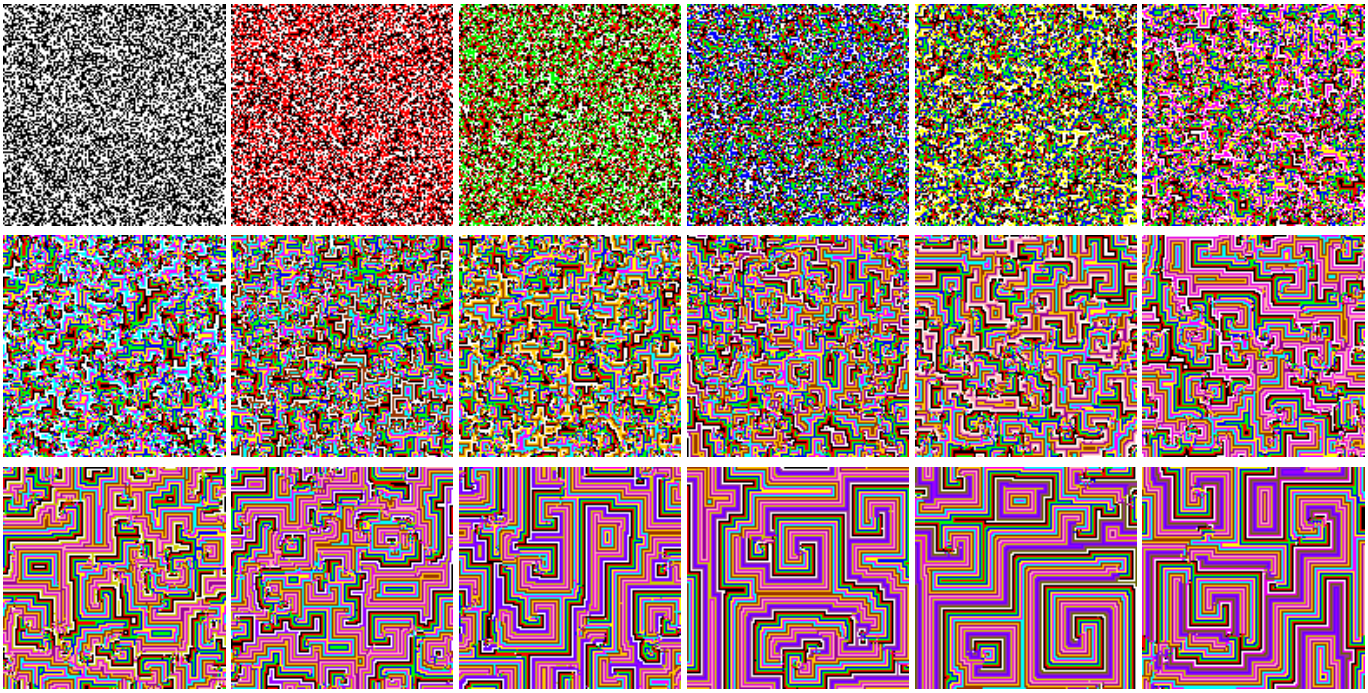

Figure 2: $N = 128^2, t = 200$ Cyclic CA with $Q = 2, 3, ..., 19$

structures.

The prevailing neighbourhood conditions affects the patterns. Moore neighbourhood with 8 neighbours gives rise to square shaped spiral patterns. If with use Nearest neighbour (N1) only we obtain diamond shaped spirals reflecting the symmetry of the four nearest neighbours of the square lattice. Choosing only next-nearest neighbours (N2) gives rise to checkerboard repetitions. Similarly in hexagonal and triangular lattice we observe spiral patterns that reflect the underpinning symmetry of the lattice.

There are various properties we can measure. One useful metric is the fraction of the bonds that have the same species at both ends. For a lattice of N sites there are 2N such bonds for N1 and N2 square lattices and 4N such bonds using a Moore neighbourhood and 3 or 1.5 for hexagonal and triangular with nearest neighbours. We normalise accordingly and obtain the plots shown in Figure 3. We have shown both periodic and fixed boundary conditions. There is a subtle difference between these with some shifting of the crossover points for the five neighbourhood and lattice shown.

We observe that at low Q the hexagonal and Moore cases - having the highest numbers of neighbours show a flat and unchanging number of like-like bonds with time. The nearest and next nearest neighbour cases, with the same coordination number come next and initially drop to a then steady state value within around 40-50 time steps. The triangular lattice, with the smallest number of neighbours takes longer to reach (a smaller) steady like-like bond fraction.

There is a significant change as we increase Q. The middle two plots shows periodic and fixed boundaries when $Q = 10$. The system is able to establish better fitting repetitive structures with the triangular lattice and it rises, then falls before heading to a steady state. Similarly nearest and next-nearest both rise then fall. We show the fraction of like-like bonds on a logarithmic scale to emphasise the changes at different time scales. Straight line regions therefore correspond to exponentially falling trends. The Moore and hexagonal cases are the only two that show long term very steady state values.

Finally at higher Q the Q-repetitive structures show a different behaviour again. We plot the curves of the fraction of like-like bonds for $Q = 20$. In this case it is the Moore and hexagonal curves that show aberrant behaviours and the other curves appear to reach roughly steady states - although note the increased variances shown by the growth in plotted error bars.

There is clearly a significant change when we vary the spatial repeat length parameterised by Q. It is useful therefore to plot the fraction of like-like bonds as a function of Q.

Figure 4 shows the fraction of like-like bonds - this time on a linear scale - for the various values of Q and for a fixed Moore neighbourhood square lattice with periodic boundary conditions. The very low Q systems remain steady state but a peak appears at higher Q and occurs at later and later times and with varying sizes as Q is increased. We speculate that this behaviour is in fact periodic and is determined by the ratio of Q to the lattice length of the system.

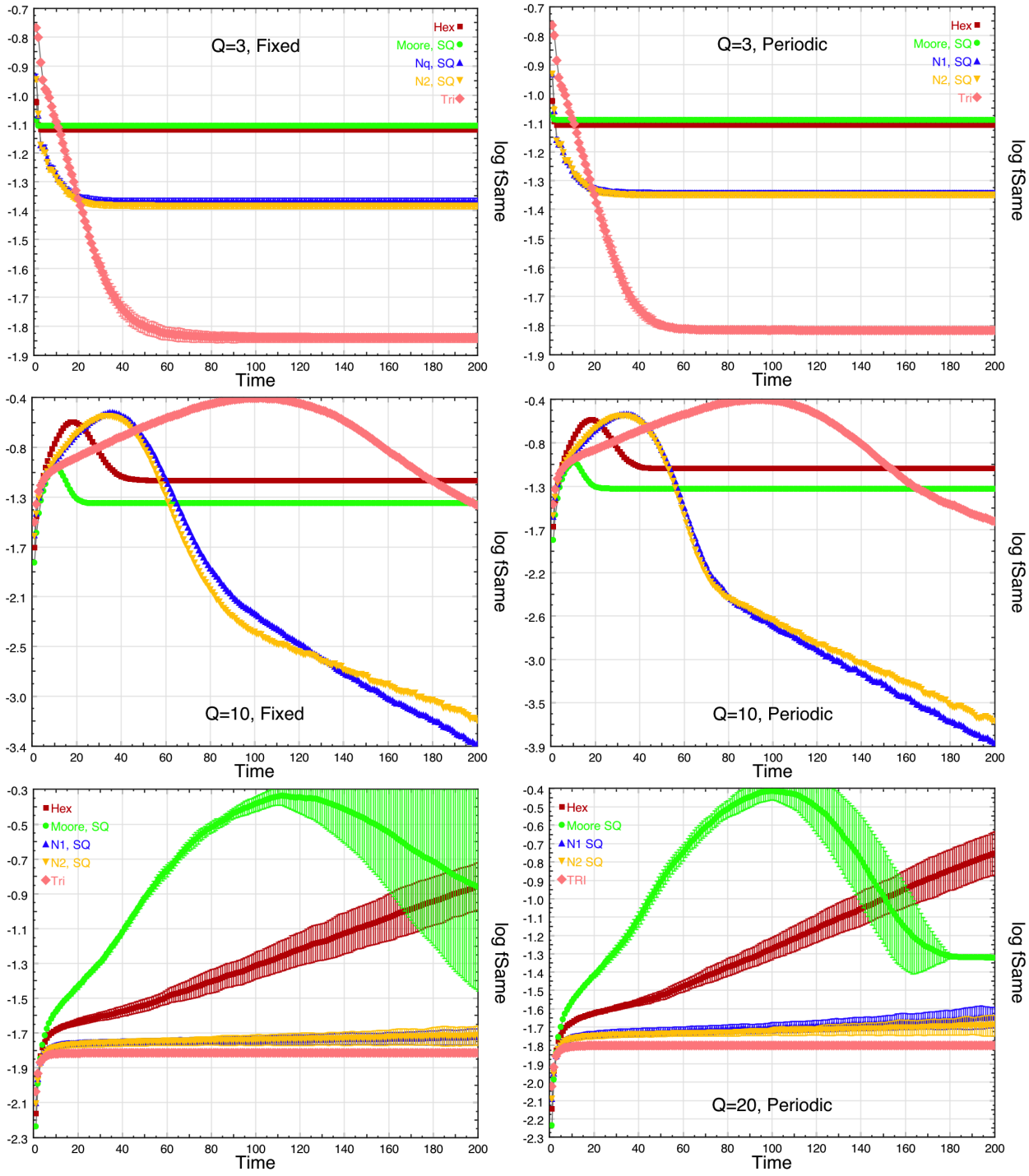


Figure 3: Log of the fraction of bonds that are the same plotted against time for fixed (left) and periodic (right) boundary conditions for the square $128^2$ system, averages over ten independent configurations, at Q=3,10,20 (top to bottom).

Other numerical metrics show similar transitional behaviour as we vary Q.

Figure 5 shows... shows the ratio between the smallest state population present and the highest. This is expected again to reflect the changing spatial structure as Q is varies. The plot shows how at low Q this curve remains constant but exhibits a structural transition with increasing Q as different repetitive structure adjust to fill the lattice. The position of the minima is quite noisy even when averaged over ten separate run configurations, but does exhibit a systemic variation with Q.

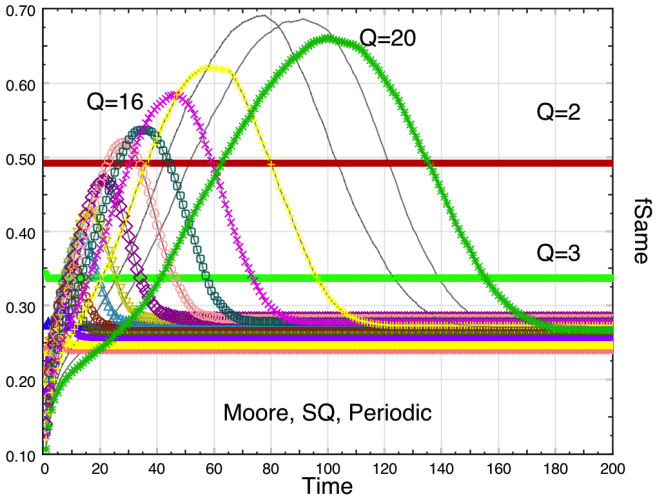


Figure 4: Fraction of bonds that are the same, plotted against $Q = 2, 3, ...20$ for Moore neighbourhood on square, periodic system.

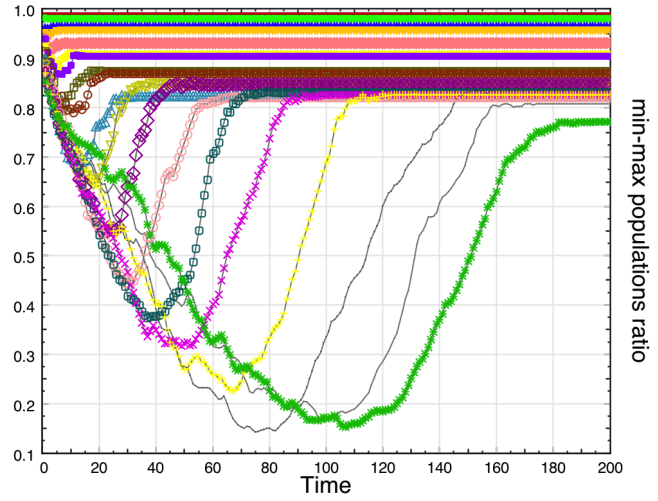


Figure 5: The ratio between minimum and maximum state populations plotted against time for the Moore neighbourhood with square lattice and periodic boundary conditions.

We can also study the component clusters of like-like species that form in the model.

Figure 6 shows the number of connected component clusters that are found in the system for the same series of systems as shown for Figure 4. Again at low $Q = 2, 3, 4$ (red, green and blue curves) the number of small clusters rapidly reaches a stable equilibrium value. For high Q we obtain a series of curves that start as nearly straight lines - indicating a power law on the log-log plot - but which each has its own cutoff point beyond which it remains steady.

The cutoff points could be used to determine a systematic variation law with Q but identifying cutoffs to good precision is

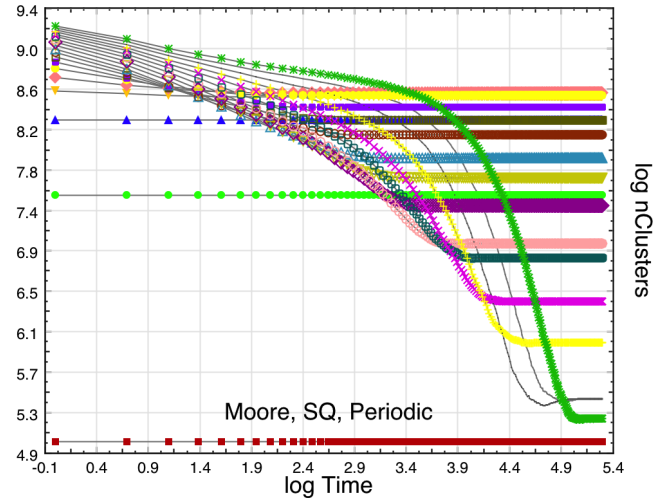


Figure 6: Log-log plot of the number of connected clusters plotted against time for the Moore neighbourhood with square lattice and periodic boundary conditions.

difficult numerically and instead we use the changing maxima in the fraction of like-like bonds to study this.

Returning to the fraction of like-like bonds counted in the system, we can identify the position and value of the peaks as seen in the data in Figure 4.

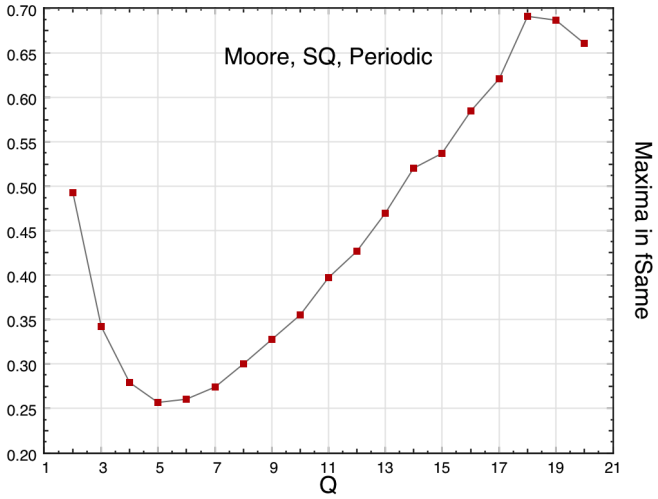


Figure 7: The maximum value of the fraction of the bonds that are the same plotted against Q for the Moore neighbourhood with square lattice and periodic boundary conditions.

Figure 7 shows a plot of the peak values plotted against Q. This gives us a means of identifying $Q^*$ - the critical Q value where the behaviour changes.

We can perform this analysis for all the neighbourhoods and lattice under consideration.

Figure 8 shows that the three square lattice structures with

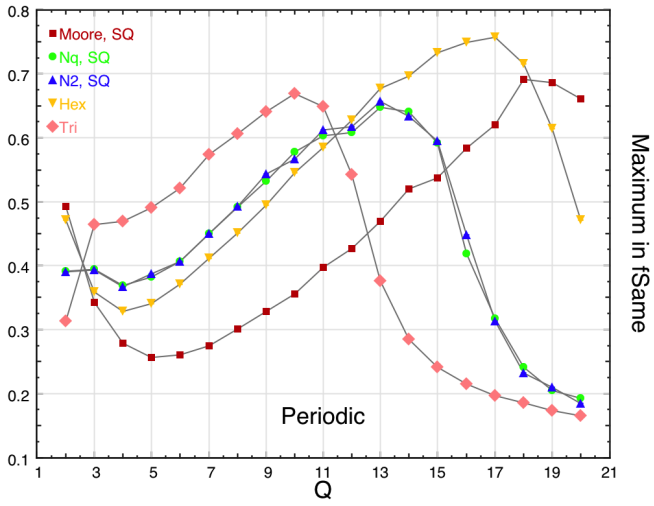


Figure 8: The maximum value of the fraction of the bonds that are the same plotted against Q for the various neighbour hoods tested.

Moore (red), nearest (green) and next-nearest (blue) all have similar shapes although the positions of the minima and maxima are different. The triangular is different in that it does not have an identifiable minimum trough but does still have an identifiable and significantly differently located maximum peak. The hexagonal system is closest to the Moore square system in shape but has a trough minimum nearer to that of the N1 and N2 square lattice cases.

# 4 Discussion

We have seen that the CCA model is dominated by the growth of Q-species repeat length structures. These can dominate if Q is high enough - compared to the neighbourhood distance that is being used. It appears that the triangular system is incapable of sustaining structure at low Q due to its lowest neighbourhood size. The remaining structures have broadly similar behaviours and have definite and finite critical Q values which appear to be $Q^* = 4$ for N1 and N2 on square lattices and also on hexagonal lattices, but $Q^* = 5$ for Moore neighbourhoods on square lattices.

We have been limited to lattices of length $L = 128$ sites in length due to the need to average over multiple independent configurations. It would be worthwhile to investigate larger systems to verify that what we are observing are structural transitions that are related to the ratio of Q and L.

It is not clear if running the systems for much longer will lead to further changes. We found that on observation no visual changes appeared top occur beyond 200 time steps. This may just be coincidental however and if data parallel processing technologies such as graphical processing units (GPUs) [20] were used, it should be possible to investigate larger simulated systems and for much longer numbers of time steps.

We have restricted our work to two dimensions. In principle the CCA model could be investigated in three dimensions or even on hyper-cubic lattice at higher dimensions. We speculate that the structural transitions we have observed my manifest themselves differently in higher dimensional systems.

The CCA appears to exhibit a very short transient period compared to other models that can be quenched from a “hot” random initial state. This would imply that some sort of nucleation process could be used to model the rapid onset growth of spatial structures. Theoretical apparatus such as fluctuation dissipation theory or droplet growth models could also be applied usefully to this model and compared with the computational experimental results presented here.

# 5 Conclusion

We have implemented an extension of Griffeath’s Cyclic Cellular Automaton model in three different lattice in two dimensions and have also studied nearest and next nearest neighbourhoods as well as the conventional Moore neighbourhood on the square lattice. We have found the system is somewhat insensitive to the boundary conditions but is very sensitive to the neighbourhood and lattice geometry applied.

We observed that there is a transition point in the number of species present at $Q^* = 5$ for the Moore neighbour hood on a square lattice and $Q^* = 4$ for the nearest and nest-nearest neighbourhoods on a square lattice as well as for nearest on a hexagonal lattice. It is arguable that the triangular system does not have a well defined transition above 2. In summary...

We believe these transitions are all related to the structural fitting in of kaleidoscopic patterns of layers of different cyclic species with a thickness determined by Q. Different lattice allow different length more or less easily dependent upon their neighbour hood sizes.

There is scope to study larger neighbourhood sizes with respect to Q and also to investigate whether similar phenomena occur in three or higher dimensional cyclic cellular automaton models.

This system has shown a surprisingly rich and complex behaviour on spatial and time scales that are readily simulated. The cyclic cellular automaton model is likely to prove a useful platform for investigating further spatial and structural transitions.

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