5.153   equilibrium

Origin   Inspired by the Irish Collegiate Programming Competition 2012 (equilibrium index)

Constraint

$$
\text{equilibrium} \left(
\begin{array}{l}
\text{VARIABLES,} \\
\text{INDEX1,} \\
\text{INDEX2,} \\
\text{EPSILON,} \\
\text{COEF1,} \\
\text{COEF2,} \\
\text{TOLERANCE,} \\
\text{CTR}
\end{array}
\right)
$$

Synonym   balanced.

Arguments
```
VARIABLES  :  collection(var−dvar)
INDEX1     :  dvar
INDEX2     :  dvar
EPSILON    :  int
COEF1      :  int
COEF2      :  int
TOLERANCE  :  int
CTR        :  atom
```

Restrictions

$|\text{VARIABLES}| \geq 1$

$\text{INDEX1} \geq 1$

$\text{INDEX1} \leq |\text{VARIABLES}|$

$\text{INDEX2} \geq 1$

$\text{INDEX2} \leq |\text{VARIABLES}|$

$\text{INDEX1} \leq \text{INDEX2}$

$\text{EPSILON} \geq 0$

$\text{EPSILON} \leq 2$

$\text{EPSILON} = \text{INDEX2} - \text{INDEX1}$

$\text{COEF1} \neq 0$

$\text{COEF2} \neq 0$

$\text{TOLERANCE} \geq 0$

$$
\text{CTR} \in \left[
\begin{array}{l}
\text{among\_diff\_0}, \\
\text{and}, \\
\text{change}, \\
\text{deepest\_valley}, \\
\text{highest\_peak}, \\
\text{increasing\_nvalue}, \\
\text{inflexion}, \\
\text{longest\_change}, \\
\text{longest\_decreasing\_sequence}, \\
\text{longest\_increasing\_sequence}, \\
\text{max\_decreasing\_slope}, \\
\text{max\_increasing\_slope}, \\
\text{min\_decreasing\_slope}, \\
\text{min\_increasing\_slope}, \\
\text{min\_width\_peak}, \\
\text{min\_width\_valley}, \\
\text{peak}, \\
\text{sum\_ctr}, \\
\text{valley}
\end{array}
\right]
$$

Purpose

Given $\texttt{VARIABLES} = \langle \texttt{VAR}_1, \texttt{VAR}_2, \ldots, \texttt{VAR}_{|\texttt{VARIABLES}|} \rangle$, enforce the following conditions:

- $\texttt{INDEX1} \geq 1$
- $\texttt{INDEX2} \geq 1$
- $\texttt{EPSILON} \geq 0$
- $\texttt{INDEX1} \leq \texttt{INDEX2}$
- $\texttt{COEF1} \neq 0$

- $\texttt{INDEX1} \leq |\texttt{VARIABLES}|$
- $\texttt{INDEX2} \leq |\texttt{VARIABLES}|$
- $\texttt{EPSILON} \leq 2$
- $\texttt{INDEX2} - \texttt{INDEX1} = \texttt{EPSILON}$
- $\texttt{TOLERANCE} \geq 0$
- $\texttt{COEF2} \neq 0$

$$
\begin{cases}
\text{if } \texttt{CTR} = \texttt{change} : \\
\quad \texttt{change}(C_1, \langle \texttt{VAR}_1, \ldots, \texttt{VAR}_{\texttt{INDEX1}} \rangle, \neq) \\
\quad \texttt{change}(C_2, \langle \texttt{VAR}_{\texttt{INDEX2}}, \ldots, \texttt{VAR}_{|\texttt{VARIABLES}|} \rangle, \neq) \\
\text{if } \texttt{CTR} = \texttt{longest\_change} : \\
\quad \texttt{longest\_change}(C_1, \langle \texttt{VAR}_1, \ldots, \texttt{VAR}_{\texttt{INDEX1}} \rangle, \neq) \\
\quad \texttt{longest\_change}(C_2, \langle \texttt{VAR}_{\texttt{INDEX2}}, \ldots, \texttt{VAR}_{|\texttt{VARIABLES}|} \rangle, \neq) \\
\text{if } \texttt{CTR} = \texttt{sum\_ctr} : \\
\quad \texttt{sum\_ctr}(\langle \texttt{VAR}_1, \ldots, \texttt{VAR}_{\texttt{INDEX1}} \rangle, =, C_1) \\
\quad \texttt{sum\_ctr}(\langle \texttt{VAR}_{\texttt{INDEX2}}, \ldots, \texttt{VAR}_{|\texttt{VARIABLES}|} \rangle, =, C_2) \\
\text{otherwise} : \\
\quad \texttt{CTR}(C_1, \langle \texttt{VAR}_1, \ldots, \texttt{VAR}_{\texttt{INDEX1}} \rangle) \\
\quad \texttt{CTR}(C_2, \langle \texttt{VAR}_{\texttt{INDEX2}}, \ldots, \texttt{VAR}_{|\texttt{VARIABLES}|} \rangle)
\end{cases}
$$

$$|\texttt{COEF1} \cdot C_1 - \texttt{COEF2} \cdot C_2| \leq \texttt{TOLERANCE}$$

Example

$(\langle 4, 4, 3, 6, 2 \rangle, 2, 4, 2, 1, 1, 0, \texttt{sum\_ctr})$
$(\langle -2, 5, -2, 6, -1, 0, -3, 5, -7, 6, -1, 7, 0 \rangle, 5, 5, 0, 1, 1, 0, \texttt{sum\_ctr})$
$(\langle -2, 5, -2, 6, -1, 0, -3, 5, -7, 6, -1, 7, 0 \rangle, 11, 11, 0, 1, 1, 0, \texttt{sum\_ctr})$
$(\langle 0, 3, 2, 6, 2, 2, 5, 8, 7, 6, 7, 3 \rangle, 5, 7, 2, 1, 1, 0, \texttt{peak})$
$(\langle 0, 5, 3, 8, 2, 2, 5, 5, 8, 7, 2, 7, 3 \rangle, 7, 7, 0, 1, 1, 0, \texttt{change})$

The first example, $\texttt{equilibrium}(\langle 4_1, 4_2, 3_3, 6_4, 2_5\rangle, \mathbf{2}, \mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \texttt{sum\_ctr})$, holds since:

- $\texttt{INDEX1} = \mathbf{2} \geq 1$,
- $\texttt{INDEX2} = \mathbf{4} \geq 1$,
- $\texttt{EPSILON} = \mathbf{2} \geq 0$,
- $\texttt{INDEX1} = \mathbf{2} \leq \texttt{INDEX2} = \mathbf{4}$,
- $C_1 = 4_1 + 4_2 = \mathbf{8}$,
- $\texttt{INDEX2} - \texttt{INDEX1} = \texttt{EPSILON} = \mathbf{2}$,

- $\texttt{INDEX1} = \mathbf{2} \leq |\texttt{VARIABLES}| = 5$,
- $\texttt{INDEX2} = \mathbf{4} \leq |\texttt{VARIABLES}| = 5$,
- $\texttt{EPSILON} = \mathbf{2} \leq 2$,
- $\texttt{TOLERANCE} = \mathbf{0} \geq 0$,
- $C_2 = 6_4 + 2_5 = \mathbf{8}$,
- $|\mathbf{1} \cdot \mathbf{8} - \mathbf{1} \cdot \mathbf{8}| \leq \texttt{TOLERANCE} = \mathbf{0}$.

Figure 5.343: Illustration of the first example of the **Example** slot

The second example, $\texttt{equilibrium}(\langle -2_1, 5_2, -2_3, 6_4, -1_5, 0_6, -3_7, 5_8, -7_9, 6_{10}, -1_{11}, 7_{12}, 0_{13}\rangle,$ $\mathbf{5}, \mathbf{5}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \texttt{sum\_ctr})$, holds since:

- $\texttt{INDEX1} = \mathbf{5} \geq 1$,
- $\texttt{INDEX2} = \mathbf{5} \geq 1$,
- $\texttt{EPSILON} = \mathbf{0} \geq 0$,
- $\texttt{INDEX1} = \mathbf{5} \leq \texttt{INDEX2} = \mathbf{5}$,
- $C_1 = -2_1 + 5_2 - 2_3 + 6_4 - 1_5 = \mathbf{6}$,
- $\texttt{INDEX2} - \texttt{INDEX1} = \texttt{EPSILON} = \mathbf{0}$,

- $\texttt{INDEX1} = \mathbf{5} \leq |\texttt{VARIABLES}| = 13$,
- $\texttt{INDEX2} = \mathbf{5} \leq |\texttt{VARIABLES}| = 13$,
- $\texttt{EPSILON} = \mathbf{0} \leq 2$,
- $\texttt{TOLERANCE} = \mathbf{0} \geq 0$,
- $C_2 = -1_5 + 0_6 - 3_7 + 5_8 - 7_9 + 6_{10} - 1_{11} + 7_{12} + 0_{13} = \mathbf{6}$,
- $|\mathbf{1} \cdot \mathbf{6} - \mathbf{1} \cdot \mathbf{6}| \leq \texttt{TOLERANCE} = \mathbf{0}$.

Figure 5.344: Illustration of the second and third examples of the **Example** slot

The third example, $\texttt{equilibrium}(\langle -2_1, 5_2, -2_3, 6_4, -1_5, 0_6, -3_7, 5_8, -7_9, 6_{10}, -1_{11}, 7_{12}, 0_{13}\rangle,$ $\mathbf{11}, \mathbf{11}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \texttt{sum\_ctr})$, holds since:

- INDEX1 = $\mathbf{11} \geq 1$,
- INDEX2 = $\mathbf{11} \geq 1$,
- EPSILON = $\mathbf{0} \geq 0$,
- INDEX1 = $\mathbf{11} \leq$ INDEX2 = $\mathbf{11}$,
- $C_1 = -2_1 + 5_2 - 2_3 + 6_4 - 1_5 + 0_6 - 3_7 + 5_8 - 7_9 + 6_{10} - 1_{\mathbf{11}} = \mathbf{6}$,
- INDEX2 − INDEX1 = EPSILON = $\mathbf{0}$,

- INDEX1 = $\mathbf{11} \leq$ |VARIABLES| = 13,
- INDEX2 = $\mathbf{11} \leq$ |VARIABLES| = 13,
- EPSILON = $\mathbf{0} \leq 2$,
- TOLERANCE = $0 \geq 0$,
- $C_2 = -1_{\mathbf{11}} + 7_{12} + 0_{13} = \mathbf{6}$,
- $|\mathbf{1} \cdot \mathbf{6} - \mathbf{1} \cdot \mathbf{6}| \leq$ TOLERANCE = $\mathbf{0}$.

The fourth example, equilibrium($\langle 0_1, 3_2, 2_3, 6_4, 2_{\mathbf{5}}, 2_6, 5_{\mathbf{7}}, 8_8, 7_9, 6_{10}, 7_{11}, 3_{12} \rangle$, $\mathbf{5}, \mathbf{7}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{0}$, peak), holds since:

- INDEX1 = $\mathbf{5} \geq 1$,
- INDEX2 = $\mathbf{7} \geq 1$,
- EPSILON = $\mathbf{2} \geq 0$,
- INDEX1 = $\mathbf{5} \leq$ INDEX2 = $\mathbf{7}$,
- the sequence 0_1 3_2 2_3 6_4 $2_{\mathbf{5}}$ contains $\mathbf{2}$ peaks,
- INDEX2 − INDEX1 = EPSILON = $\mathbf{2}$,

- INDEX1 = $5 \leq$ |VARIABLES| = 12,
- INDEX2 = $7 \leq$ |VARIABLES| = 12,
- EPSILON = $\mathbf{2} \leq 2$,
- TOLERANCE = $0 \geq 0$,
- The sequence $5_{\mathbf{7}}$ 8_8 7_9 6_{10} 7_{11} 3_{12} contains $\mathbf{2}$ peaks,
- $|\mathbf{1} \cdot \mathbf{2} - \mathbf{1} \cdot \mathbf{2}| \leq$ TOLERANCE = $\mathbf{0}$.

$$|\mathbf{1} \cdot \mathbf{2} - \mathbf{1} \cdot \mathbf{2}| \leq \text{TOLERANCE} = \mathbf{0}$$

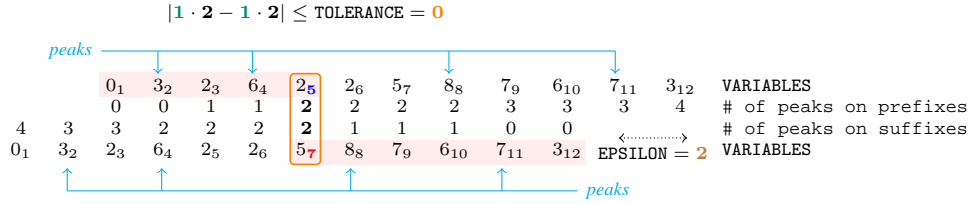|													
peaks													
		0_1	3_2	2_3	6_4	$2_{\mathbf{5}}$	2_6	5_7	8_8	7_9	6_{10}	7_{11}	3_{12}	VARIABLES
		0	0	1	1	**2**	2	2	2	3	3	3	4	# of peaks on prefixes
4	3	3	2	2	2	**2**	1	1	1	0	0		# of peaks on suffixes
0_1	3_2	2_3	6_4	2_5	2_6	$5_{\mathbf{7}}$	8_8	7_9	6_{10}	7_{11}	3_{12}	EPSILON = 2	VARIABLES
peaks													

Figure 5.345: Illustration of the fourth example of the **Example** slot

The fifth example, equilibrium($\langle 0_1, 5_2, 3_3, 8_4, 2_5, 2_6, 5_{\mathbf{7}}, 5_8, 8_9, 7_{10}, 2_{11}, 7_{12}, 3_{13} \rangle$, $\mathbf{7}, \mathbf{7}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}$, change), holds since:

- INDEX1 = $\mathbf{7} \geq 1$,
- INDEX2 = $\mathbf{7} \geq 1$,
- EPSILON = $\mathbf{0} \geq 0$,
- INDEX1 = $\mathbf{7} \leq$ INDEX2 = $\mathbf{7}$,
- the sequence $0_1, 5_2, 3_3, 8_4, 2_5, 2_6, 5_{\mathbf{7}}$ contains $\mathbf{5}$ changes,
- INDEX2 − INDEX1 = EPSILON = $\mathbf{0}$,

- INDEX1 = $\mathbf{7} \leq$ |VARIABLES| = 12,
- INDEX2 = $\mathbf{7} \leq$ |VARIABLES| = 12,
- EPSILON = $\mathbf{0} \leq 2$,
- TOLERANCE = $0 \geq 0$,
- The sequence $5_{\mathbf{7}}, 5_8, 8_9, 7_{10}, 2_{11}, 7_{12}, 3_{13}$ contains $\mathbf{5}$ changes,
- $|\mathbf{1} \cdot \mathbf{5} - \mathbf{1} \cdot \mathbf{5}| \leq$ TOLERANCE = $\mathbf{0}$.

$$|1 \cdot 5 - 1 \cdot 5| \leq \texttt{TOLERANCE} = 0$$

$0_1 \neq$	$5_2 \neq$	$3_3 \neq$	$8_4 \neq$	2_5	$2_6 \neq$	5_7	$5_8 \neq$	$8_9 \neq$	$7_{10} \neq$	$2_{11} \neq$	$7_{12} \neq$	3_{13}	VARIABLES
0	1	2	3	4	4	**5**	5	6	7	8	9	10	# of changes on prefixes
10	9	8	7	6	6	**5**	5	4	3	2	1	0	# of changes on suffixes
$0_1 \neq$	$5_2 \neq$	$3_3 \neq$	$8_4 \neq$	2_5	$2_6 \neq$	5_7	$5_8 \neq$	$8_9 \neq$	$7_{10} \neq$	$2_{11} \neq$	$7_{12} \neq$	3_{13}	VARIABLES
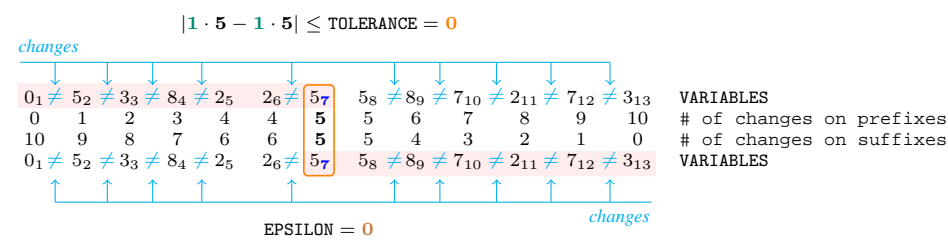
$$\texttt{EPSILON} = 0$$

Figure 5.346: Illustration of the fifth example of the **Example** slot

Typical	
	$|\texttt{VARIABLES}| > 2$
	$\texttt{INDEX1} > 1$
	$\texttt{INDEX1} < |\texttt{VARIABLES}|$
	$\texttt{INDEX2} > 1$
	$\texttt{INDEX2} < |\texttt{VARIABLES}|$
	$\texttt{COEF1} = 1$
	$\texttt{COEF2} = 1$
	$\texttt{EPSILON} = 1$
	$\texttt{TOLERANCE} = 0$

See also   **root concept:** balance.

Keywords   **characteristic of a constraint:** automaton with counters.