5.153   equilibrium

Origin   Inspired by the Irish Collegiate Programming Competition 2012 (equilibrium index)

Constraint

$$
\text{equilibrium} \left(
\begin{array}{l}
\text{VARIABLES,} \\
\text{INDEX1,} \\
\text{INDEX2,} \\
\text{EPSILON,} \\
\text{COEF1,} \\
\text{COEF2,} \\
\text{TOLERANCE,} \\
\text{CTR}
\end{array}
\right)
$$

Synonym   `balanced`.

Arguments

```
VARIABLES  :  collection(var−dvar)
INDEX1     :  dvar
INDEX2     :  dvar
EPSILON    :  int
COEF1      :  int
COEF2      :  int
TOLERANCE  :  int
CTR        :  atom
```

Restrictions

$|\text{VARIABLES}| \geq 1$

$\text{INDEX1} \geq 1$

$\text{INDEX1} \leq |\text{VARIABLES}|$

$\text{INDEX2} \geq 1$

$\text{INDEX2} \leq |\text{VARIABLES}|$

$\text{INDEX1} \leq \text{INDEX2}$

$\text{EPSILON} \geq 0$

$\text{EPSILON} \leq 2$

$\text{EPSILON} = \text{INDEX2} - \text{INDEX1}$

$\text{COEF1} \neq 0$

$\text{COEF2} \neq 0$

$\text{TOLERANCE} \geq 0$

$$\text{CTR} \in \begin{bmatrix} \text{among\_diff\_0,} \\ \text{and,} \\ \text{change,} \\ \text{deepest\_valley,} \\ \text{highest\_peak,} \\ \text{increasing\_nvalue,} \\ \text{inflexion,} \\ \text{longest\_change,} \\ \text{longest\_decreasing\_sequence,} \\ \text{longest\_increasing\_sequence,} \\ \text{max\_decreasing\_slope,} \\ \text{max\_increasing\_slope,} \\ \text{min\_decreasing\_slope,} \\ \text{min\_increasing\_slope,} \\ \text{min\_width\_peak,} \\ \text{min\_width\_valley,} \\ \text{peak,} \\ \text{sum\_ctr,} \\ \text{valley} \end{bmatrix}$$

Given $\texttt{VARIABLES} = \langle \texttt{VAR}_1, \texttt{VAR}_2, \ldots, \texttt{VAR}_{|\texttt{VARIABLES}|} \rangle$, enforce the following conditions:

- $\texttt{INDEX1} \geq 1$
- $\texttt{INDEX2} \geq 1$
- $\texttt{EPSILON} \geq 0$
- $\texttt{INDEX1} \leq \texttt{INDEX2}$
- $\texttt{COEF1} \neq 0$

- $\texttt{INDEX1} \leq |\texttt{VARIABLES}|$
- $\texttt{INDEX2} \leq |\texttt{VARIABLES}|$
- $\texttt{EPSILON} \leq 2$
- $\texttt{INDEX2} - \texttt{INDEX1} = \texttt{EPSILON}$
- $\texttt{TOLERANCE} \geq 0$
- $\texttt{COEF2} \neq 0$

Purpose

$$\begin{cases} \text{if } \texttt{CTR} = \texttt{change}: \\ \quad \texttt{change}(C_1, \langle \texttt{VAR}_1, \ldots, \texttt{VAR}_{\texttt{INDEX1}} \rangle, \neq) \\ \quad \texttt{change}(C_2, \langle \texttt{VAR}_{\texttt{INDEX2}}, \ldots, \texttt{VAR}_{|\texttt{VARIABLES}|} \rangle, \neq) \\ \text{if } \texttt{CTR} = \texttt{longest\_change}: \\ \quad \texttt{longest\_change}(C_1, \langle \texttt{VAR}_1, \ldots, \texttt{VAR}_{\texttt{INDEX1}} \rangle, \neq) \\ \quad \texttt{longest\_change}(C_2, \langle \texttt{VAR}_{\texttt{INDEX2}}, \ldots, \texttt{VAR}_{|\texttt{VARIABLES}|} \rangle, \neq) \\ \text{if } \texttt{CTR} = \texttt{sum\_ctr}: \\ \quad \texttt{sum\_ctr}(\langle \texttt{VAR}_1, \ldots, \texttt{VAR}_{\texttt{INDEX1}} \rangle, =, C_1) \\ \quad \texttt{sum\_ctr}(\langle \texttt{VAR}_{\texttt{INDEX2}}, \ldots, \texttt{VAR}_{|\texttt{VARIABLES}|} \rangle, =, C_2) \\ \text{otherwise}: \\ \quad \texttt{CTR}(C_1, \langle \texttt{VAR}_1, \ldots, \texttt{VAR}_{\texttt{INDEX1}} \rangle) \\ \quad \texttt{CTR}(C_2, \langle \texttt{VAR}_{\texttt{INDEX2}}, \ldots, \texttt{VAR}_{|\texttt{VARIABLES}|} \rangle) \end{cases}$$

$$|\texttt{COEF1} \cdot C_1 - \texttt{COEF2} \cdot C_2| \leq \texttt{TOLERANCE}$$

Example

$(\langle 4, 4, 3, 6, 2 \rangle, 2, 4, 2, 1, 1, 0, \texttt{sum\_ctr})$
$(\langle -2, 5, -2, 6, -1, 0, -3, 5, -7, 6, -1, 7, 0 \rangle, 5, 5, 0, 1, 1, 0, \texttt{sum\_ctr})$
$(\langle -2, 5, -2, 6, -1, 0, -3, 5, -7, 6, -1, 7, 0 \rangle, 11, 11, 0, 1, 1, 0, \texttt{sum\_ctr})$
$(\langle 0, 3, 2, 6, 2, 2, 5, 8, 7, 6, 7, 3 \rangle, 5, 7, 2, 1, 1, 0, \texttt{peak})$
$(\langle 0, 5, 3, 8, 2, 2, 5, 5, 8, 7, 2, 7, 3 \rangle, 7, 7, 0, 1, 1, 0, \texttt{change})$

The first example, $\texttt{equilibrium}(\langle 4_1, 4_2, 3_3, 6_4, 2_5 \rangle, 2, 4, 2, 1, 1, 0, \texttt{sum\_ctr})$, holds since:

- $\texttt{INDEX1} = 2 \geq 1$,
- $\texttt{INDEX2} = 4 \geq 1$,
- $\texttt{EPSILON} = 2 \geq 0$,
- $\texttt{INDEX1} = 2 \leq \texttt{INDEX2} = 4$,
- $C_1 = 4_1 + 4_2 = 8$,
- $\texttt{INDEX2} - \texttt{INDEX1} = \texttt{EPSILON} = 2$,

- $\texttt{INDEX1} = 2 \leq |\texttt{VARIABLES}| = 5$,
- $\texttt{INDEX2} = 4 \leq |\texttt{VARIABLES}| = 5$,
- $\texttt{EPSILON} = 2 \leq 2$,
- $\texttt{TOLERANCE} = 0 \geq 0$,
- $C_2 = 6_4 + 2_5 = 8$,
- $|1 \cdot 8 - 1 \cdot 8| \leq \texttt{TOLERANCE} = 0$.

$$|1 \cdot 8 - 1 \cdot 8| \leq \texttt{TOLERANCE} = 0$$

	4_1	4_2	3_3	6_4	2_5	VARIABLES
	4	**8**	11	17	15	sum on prefixes
19	15	11	**8**	2		sum on suffixes
4_1	4_2	3_3	6_4	2_5		VARIABLES

$$\texttt{EPSILON} = 2$$

Figure 5.326: Illustration of the first example of the **Example** slot

The second example, $\texttt{equilibrium}(\langle -2_1, 5_2, -2_3, 6_4, -1_5, 0_6, -3_7, 5_8, -7_9, 6_{10}, -1_{11}, 7_{12}, 0_{13} \rangle,$ $5, 5, 0, 1, 1, 0, \texttt{sum\_ctr})$, holds since:

- $\texttt{INDEX1} = 5 \geq 1$,
- $\texttt{INDEX2} = 5 \geq 1$,
- $\texttt{EPSILON} = 0 \geq 0$,
- $\texttt{INDEX1} = 5 \leq \texttt{INDEX2} = 5$,
- $C_1 = -2_1 + 5_2 - 2_3 + 6_4 - 1_5 = 6$,
- $\texttt{INDEX2} - \texttt{INDEX1} = \texttt{EPSILON} = 0$,

- $\texttt{INDEX1} = 5 \leq |\texttt{VARIABLES}| = 13$,
- $\texttt{INDEX2} = 5 \leq |\texttt{VARIABLES}| = 13$,
- $\texttt{EPSILON} = 0 \leq 2$,
- $\texttt{TOLERANCE} = 0 \geq 0$,
- $C_2 = -1_5 + 0_6 - 3_7 + 5_8 - 7_9 + 6_{10} - 1_{11} + 7_{12} + 0_{13} = 6$,
- $|1 \cdot 6 - 1 \cdot 6| \leq \texttt{TOLERANCE} = 0$.

$$|1 \cdot 6 - 1 \cdot 6| \leq \texttt{TOLERANCE} = 0 \qquad |1 \cdot 6 - 1 \cdot 6| \leq \texttt{TOLERANCE} = 0$$

-2_1	5_2	-2_3	6_4	-1_5	0_6	-3_7	5_8	-7_9	6_{10}	-1_{11}	7_{12}	0_{13}	VARIABLES
-2	3	1	7	**6**	6	3	8	1	7	**6**	13	13	sum on prefixes
13	15	10	12	**6**	7	7	10	5	12	**6**	7	0	sum on suffixes
-2_1	5_2	-2_3	6_4	-1_5	0_6	-3_7	5_8	-7_9	6_{10}	-1_{11}	7_{12}	0_{13}	VARIABLES

$$\texttt{EPSILON} = 0 \qquad\qquad\qquad \texttt{EPSILON} = 0$$

Figure 5.327: Illustration of the second and third examples of the **Example** slot

The third example, $\texttt{equilibrium}(\langle -2_1, 5_2, -2_3, 6_4, -1_5, 0_6, -3_7, 5_8, -7_9, 6_{10}, -1_{11}, 7_{12}, 0_{13} \rangle,$ $11, 11, 0, 1, 1, 0, \texttt{sum\_ctr})$, holds since:

- INDEX1 $= \mathbf{11} \geq 1$,
- INDEX2 $= \mathbf{11} \geq 1$,
- EPSILON $= \mathbf{0} \geq 0$,
- INDEX1 $= \mathbf{11} \leq$ INDEX2 $= \mathbf{11}$,
- $C_1 = -2_1 + 5_2 - 2_3 + 6_4 - 1_5 + 0_6 - 3_7 + 5_8 - 7_9 + 6_{10} - 1_{\mathbf{11}} = \mathbf{6}$,
- INDEX2 $-$ INDEX1 $=$ EPSILON $= \mathbf{0}$,

- INDEX1 $= \mathbf{11} \leq |\text{VARIABLES}| = 13$,
- INDEX2 $= \mathbf{11} \leq |\text{VARIABLES}| = 13$,
- EPSILON $= \mathbf{0} \leq 2$,
- TOLERANCE $= 0 \geq 0$,
- $C_2 = -1_{\mathbf{11}} + 7_{12} + 0_{13} = \mathbf{6}$,
- $|\mathbf{1} \cdot \mathbf{6} - \mathbf{1} \cdot \mathbf{6}| \leq$ TOLERANCE $= \mathbf{0}$.

The fourth example, $\texttt{equilibrium}(\langle 0_1, 3_2, 2_3, 6_4, 2_{\mathbf{5}}, 2_6, 5_{\mathbf{7}}, 8_8, 7_9, 6_{10}, 7_{11}, 3_{12} \rangle,$ $\mathbf{5}, \mathbf{7}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \texttt{peak})$, holds since:

- INDEX1 $= \mathbf{5} \geq 1$,
- INDEX2 $= \mathbf{7} \geq 1$,
- EPSILON $= \mathbf{2} \geq 0$,
- INDEX1 $= \mathbf{5} \leq$ INDEX2 $= \mathbf{7}$,
- the sequence 0_1 3_2 2_3 6_4 $2_{\mathbf{5}}$ contains $\mathbf{2}$ peaks,
- INDEX2 $-$ INDEX1 $=$ EPSILON $= \mathbf{2}$,

- INDEX1 $= 5 \leq |\text{VARIABLES}| = 12$,
- INDEX2 $= 7 \leq |\text{VARIABLES}| = 12$,
- EPSILON $= \mathbf{2} \leq 2$,
- TOLERANCE $= 0 \geq 0$,
- The sequence $5_{\mathbf{7}}$ 8_8 7_9 6_{10} 7_{11} 3_{12} contains $\mathbf{2}$ peaks,
- $|\mathbf{1} \cdot \mathbf{2} - \mathbf{1} \cdot \mathbf{2}| \leq$ TOLERANCE $= \mathbf{0}$.
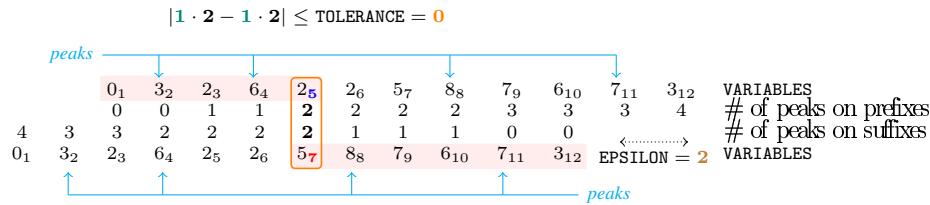
Figure 5.328: Illustration of the fourth example of the **Example** slot

The fifth example, $\texttt{equilibrium}(\langle 0_1, 5_2, 3_3, 8_4, 2_5, 2_6, 5_{\mathbf{7}}, 5_8, 8_9, 7_{10}, 2_{11}, 7_{12}, 3_{13} \rangle,$ $\mathbf{7}, \mathbf{7}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \texttt{change})$, holds since:

- INDEX1 $= \mathbf{7} \geq 1$,
- INDEX2 $= \mathbf{7} \geq 1$,
- EPSILON $= \mathbf{0} \geq 0$,
- INDEX1 $= \mathbf{7} \leq$ INDEX2 $= \mathbf{7}$,
- the sequence $0_1, 5_2, 3_3, 8_4, 2_5, 2_6, 5_{\mathbf{7}}$ contains $\mathbf{5}$ changes,
- INDEX2 $-$ INDEX1 $=$ EPSILON $= \mathbf{0}$,

- INDEX1 $= \mathbf{7} \leq |\text{VARIABLES}| = 12$,
- INDEX2 $= \mathbf{7} \leq |\text{VARIABLES}| = 12$,
- EPSILON $= \mathbf{0} \leq 2$,
- TOLERANCE $= 0 \geq 0$,
- The sequence $5_{\mathbf{7}}, 5_8, 8_9, 7_{10}, 2_{11}, 7_{12}, 3_{13}$ contains $\mathbf{5}$ changes,
- $|\mathbf{1} \cdot \mathbf{5} - \mathbf{1} \cdot \mathbf{5}| \leq$ TOLERANCE $= \mathbf{0}$.

$$|\mathbf{1} \cdot \mathbf{5} - \mathbf{1} \cdot \mathbf{5}| \leq \texttt{TOLERANCE} = \mathbf{0}$$

changes													
$0_1 \neq$	$5_2 \neq$	$3_3 \neq$	$8_4 \neq$	2_5	$2_6 \neq$	5_7	$5_8 \neq$	$8_9 \neq$	$7_{10} \neq$	$2_{11} \neq$	$7_{12} \neq$	3_{13}	VARIABLES
0	1	2	3	4	4	**5**	5	6	7	8	9	10	# of changes on prefixes
10	9	8	7	6	6	**5**	5	4	3	2	1	0	# of changes on suffixes
$0_1 \neq$	$5_2 \neq$	$3_3 \neq$	$8_4 \neq$	2_5	$2_6 \neq$	5_7	$5_8 \neq$	$8_9 \neq$	$7_{10} \neq$	$2_{11} \neq$	$7_{12} \neq$	3_{13}	VARIABLES

changes
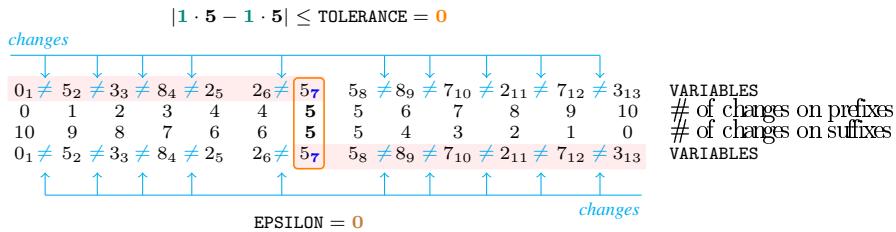
$$\texttt{EPSILON} = \mathbf{0}$$

Figure 5.329: Illustration of the fifth example of the **Example** slot

Typical

$|\texttt{VARIABLES}| > 2$
$\texttt{INDEX1} > 1$
$\texttt{INDEX1} < |\texttt{VARIABLES}|$
$\texttt{INDEX2} > 1$
$\texttt{INDEX2} < |\texttt{VARIABLES}|$
$\texttt{COEF1} = 1$
$\texttt{COEF2} = 1$
$\texttt{EPSILON} = 1$
$\texttt{TOLERANCE} = 0$

See also **root concept:** balance.

Keywords **characteristic of a constraint:** automaton with counters.