## 5.98 cumulative_product

DESCRIPTION LINKS GRAPH

**Origin**

Derived from cumulative.

**Constraint**

cumulative_product(TASKS, LIMIT)

**Arguments**

$$\begin{array}{lll} \texttt{TASKS} & : & \texttt{collection}\begin{pmatrix} \texttt{origin}-\texttt{dvar}, \\ \texttt{duration}-\texttt{dvar}, \\ \texttt{end}-\texttt{dvar}, \\ \texttt{height}-\texttt{dvar} \end{pmatrix} \\ \texttt{LIMIT} & : & \texttt{int} \end{array}$$

**Restrictions**

require_at_least(2, TASKS, [origin, duration, end])
required(TASKS, height)
TASKS.duration $\geq 0$
TASKS.origin $\leq$ TASKS.end
TASKS.height $\geq 1$
LIMIT $\geq 0$

**Purpose**

Consider a set $\mathcal{T}$ of tasks described by the TASKS collection. The cumulative_product constraint forces that at each point in time, the product of the heights of the set of tasks that overlap that point, does not exceed a given limit. A task overlaps a point $i$ if and only if (1) its origin is less than or equal to $i$, and (2) its end is strictly greater than $i$. It also imposes for each task of $\mathcal{T}$ the constraint origin + duration = end.

**Example**

$$\left(\left\langle\begin{array}{llll} \texttt{origin}-1 & \texttt{duration}-3 & \texttt{end}-4 & \texttt{height}-1, \\ \texttt{origin}-2 & \texttt{duration}-9 & \texttt{end}-11 & \texttt{height}-2, \\ \texttt{origin}-3 & \texttt{duration}-10 & \texttt{end}-13 & \texttt{height}-1, \\ \texttt{origin}-6 & \texttt{duration}-6 & \texttt{end}-12 & \texttt{height}-1, \\ \texttt{origin}-7 & \texttt{duration}-2 & \texttt{end}-9 & \texttt{height}-3 \end{array}\right\rangle, 6\right)$$

Figure 5.237 shows the solution associated with the example. To each task of the cumulative_product constraint corresponds a set of rectangles coloured with the same colour: the sum of the lengths of the rectangles corresponds to the duration of the task, while the height of the rectangles (i.e., all the rectangles associated with a task have the same height) corresponds to the height of the task. The profile corresponding to the product of the heights of the tasks that overlap a given point is depicted by a thick red line. The cumulative_product constraint holds since at each point in time the product of the heights of the tasks that overlap that point is not strictly greater than the upper limit 6 enforced by the last argument of the cumulative_product constraint.

TASKS

| | | | | |
|---|---|---|---|---|
| ① | o − 1 | d − 3 | e − 4 | h − 1 |
| ② | o − 2 | d − 9 | e − 11 | h − 2 |
| ③ | o − 3 | d − 10 | e − 13 | h − 1 |
| ④ | o − 6 | d − 6 | e − 12 | h − 1 |
| ⑤ | o − 7 | d − 2 | e − 9 | h − 3 |

( o for origin, d for duration, e for end, h for height )

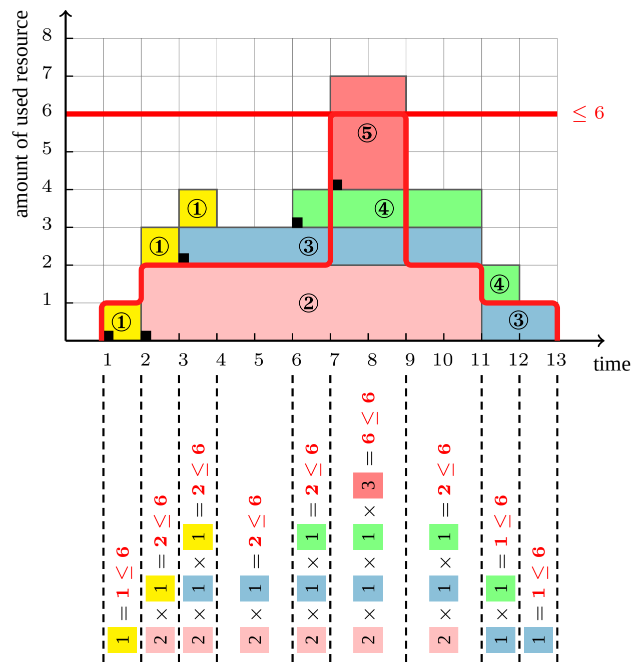



Figure 5.237: Resource consumption profile in red corresponding to the product of the heights of the five tasks of the **Example** slot

**Typical**

$|\texttt{TASKS}| > 1$
$\textsf{range}(\texttt{TASKS.origin}) > 1$
$\textsf{range}(\texttt{TASKS.duration}) > 1$
$\textsf{range}(\texttt{TASKS.end}) > 1$
$\textsf{range}(\texttt{TASKS.height}) > 1$
$\texttt{TASKS.duration} > 0$
$\texttt{LIMIT} < \textsf{prod}(\texttt{TASKS.height})$

**Symmetries**

- Items of TASKS are permutable.
- TASKS.height can be decreased to any value $\geq 0$.
- One and the same constant can be added to the origin and end attributes of all items of TASKS.
- LIMIT can be increased.

**Arg. properties**

Contractible wrt. TASKS.

**Reformulation**

The cumulative_product constraint can be expressed in term of a set of reified constraints and of |TASKS| constraints of the form $h_1 \cdot h_2 \cdot \cdots \cdot h_{|\mathtt{TASKS}|} \leq l$:

1. For each pair of tasks $\mathtt{TASKS}[i], \mathtt{TASKS}[j]$ $(i, j \in [1, |\mathtt{TASKS}|])$ of the TASKS collection we create a variable $H_{ij}$ which is set to the height of task $\mathtt{TASKS}[j]$ if task $\mathtt{TASKS}[j]$ overlaps the origin attribute of task $\mathtt{TASKS}[i]$, and to 1 otherwise:
   - If $i = j$:
     - $H_{ij} = \mathtt{TASKS}[i].\mathtt{height}$.
   - If $i \neq j$:
     - $H_{ij} = \mathtt{TASKS}[j].\mathtt{height} \vee H_{ij} = 1$.
     - $((\mathtt{TASKS}[j].\mathtt{origin} \leq \mathtt{TASKS}[i].\mathtt{origin} \wedge$
       $\mathtt{TASKS}[j].\mathtt{end} > \mathtt{TASKS}[i].\mathtt{origin}) \wedge (H_{ij} = \mathtt{TASKS}[j].\mathtt{height})) \vee$
       $((\mathtt{TASKS}[j].\mathtt{origin} > \mathtt{TASKS}[i].\mathtt{origin} \vee$
       $\mathtt{TASKS}[j].\mathtt{end} \leq \mathtt{TASKS}[i].\mathtt{origin}) \wedge (H_{ij} = 1))$
2. For each task $\mathtt{TASKS}[i]$ $(i \in [1, |\mathtt{TASKS}|])$ we impose a constraint of the form $H_{i1} \cdot H_{i2} \cdot \cdots \cdot H_{i|\mathtt{TASKS}|} \leq \mathtt{LIMIT}$.

**See also**

**common keyword:** cumulative *(resource constraint)*.

**used in graph description:** product_ctr.

**Keywords**

**characteristic of a constraint:** product.

**constraint type:** scheduling constraint, resource constraint, temporal constraint.

**filtering:** compulsory part.

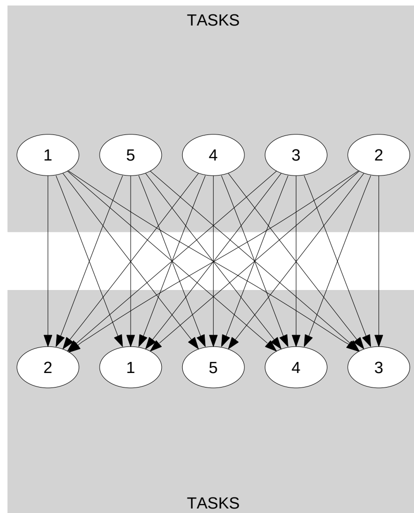
**modelling:** zero-duration task.

| | |
|---|---|
| **Arc input(s)** | TASKS |
| **Arc generator** | *SELF* $\mapsto$ collection(tasks) |
| **Arc arity** | 1 |
| **Arc constraint(s)** | tasks.origin + tasks.duration = tasks.end |
| **Graph property(ies)** | **NARC** = \|TASKS\| |

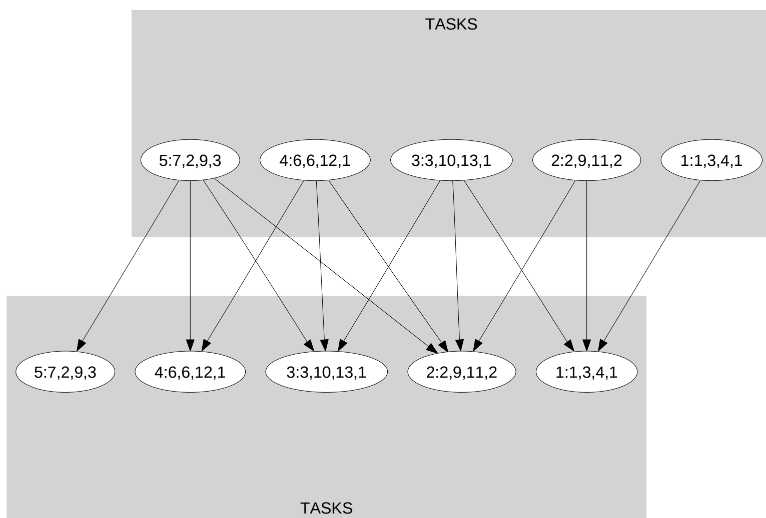
| | |
|---|---|
| **Arc input(s)** | TASKS TASKS |
| **Arc generator** | *PRODUCT* $\mapsto$ collection(tasks1, tasks2) |
| **Arc arity** | 2 |
| **Arc constraint(s)** | • tasks1.duration > 0<br>• tasks2.origin ≤ tasks1.origin<br>• tasks1.origin < tasks2.end |
| **Graph class** | • ACYCLIC<br>• BIPARTITE<br>• NO_LOOP |
| **Sets** | $\text{SUCC} \mapsto \left[\begin{array}{l}\texttt{source},\\ \texttt{variables} - \texttt{col}\left(\begin{array}{l}\texttt{VARIABLES}-\texttt{collection}(\texttt{var}-\texttt{dvar}),\\ [\texttt{item}(\texttt{var}-\texttt{ITEMS.height})]\end{array}\right)\end{array}\right]$ |
| **Constraint(s) on sets** | product_ctr(variables, ≤, LIMIT) |

**Graph model** Parts (A) and (B) of Figure 5.238 respectively show the initial and final graph associated with the second graph constraint of the **Example** slot. On the one hand, each source vertex of the final graph can be interpreted as a time point. On the other hand the successors of a source vertex correspond to those tasks that overlap that time point. The cumulative_product constraint holds since for each successor set $\mathcal{S}$ of the final graph the product of the heights of the tasks in $\mathcal{S}$ does not exceed the limit LIMIT $= 6$.

**Signature** Since TASKS is the maximum number of vertices of the final graph of the first graph constraint we can rewrite **NARC** = |TASKS| to **NARC** ≥ |TASKS|. This leads to simplify $\underline{\overline{\mathbf{NARC}}}$ to $\overline{\mathbf{NARC}}$.




**(A)**




**(B)**

Figure 5.238: Initial and final graph of the `cumulative_product` constraint