Algebraic Number Theory, a Computational Approach

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Preface

This book is based on notes the author created for a one-semester undergraduate course on Algebraic Number Theory, which the author taught at Harvard during Spring 2004 and Spring 2005. This book was mainly inspired by the [SD01, Ch. 1] and Cassels's article *Global Fields* in [Cas67]

Please send any typos or corrections to `wstein@gmail.com`.

Acknowledgement: This book closely builds on Swinnerton-Dyer's book [SD01] and Cassels's article [Cas67]. Many of the students of Math 129 at Harvard during Spring 2004 and 2005 made helpful comments: Jennifer Balakrishnan, Peter Behrooz, Jonathan Bloom, David Escott Jayce Getz, Michael Hamburg, Deniz Kural, Danielle li, Andrew Ostergaard, Gregory Price, Grant Schoenebeck, Jennifer Sinnott, Stephen Walker, Daniel Weissman, and Inna Zakharevich in 2004; Mauro Braunstein, Steven Byrnes, William Fithian, Frank Kelly, Alison Miller, Nizameddin Ordulu, Corina Patrascu, Anatoly Preygel, Emily Riehl, Gary Sivek, Steven Sivek, Kaloyan Slavov, Gregory Valiant, and Yan Zhang in 2005. Also the course assistants Matt Bainbridge and Andrei Jorza made many helpful comments. The mathemtical software [S$^+$11], [PAR], and [BCP97] were used in writing this book.

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Chapter 1

Introduction

1.1 Mathematical background

In addition to general mathematical maturity, this book assumes you have the following background:

- Basics of finite group theory
- Commutative rings, ideals, quotient rings
- Some elementary number theory
- Basic Galois theory of fields
- Point set topology
- Basic of topological rings, groups, and measure theory

For example, if you have never worked with finite groups before, you should read another book first. If you haven't seen much elementary ring theory, there is still hope, but you will have to do some additional reading and exercises. We will briefly review the basics of the Galois theory of number fields.

Some of the homework problems involve using a computer, but there are examples which you can build on. We will not assume that you have a programming background or know much about algorithms. Most of the book uses Sage `http://sagemath.org`, which is free open source mathematical software. The following is an example Sage session:

```
sage: 2 + 2
4
sage: k.<a> = NumberField(x^2 + 1); k
Number Field in a with defining polynomial x^2 + 1
```

1.2 What is algebraic number theory?

A number field K is a finite degree algebraic extension of the rational numbers $\mathbf{Q}$. The primitive element theorem from Galois theory asserts that every such extension can be represented as the set of all polynomials of degree at most $d = [K : \mathbf{Q}] = \dim_{\mathbf{Q}} K$ in a single algebraic number α:

$$K = \mathbf{Q}(\alpha) = \left\{ \sum_{n=0}^{m} a_n \alpha^n : a_n \in \mathbf{Q} \right\}.$$

Here α is a root of a polynomial with coefficients in $\mathbf{Q}$.

Algebraic number theory involves using techniques from (mostly commutative) algebra and finite group theory to gain a deeper understanding of the arithmetic of number fields and related objects (e.g., functions fields, elliptic curves, etc.). The main objects that we study in this book are number fields, rings of integers of number fields, unit groups, ideal class groups, norms, traces, discriminants, prime ideals, Hilbert and other class fields and associated reciprocity laws, zeta and L-functions, and algorithms for computing each of the above.

1.2.1 Topics in this book

These are some of the main topics that are discussed in this book:

- Rings of integers of number fields
- Unique factorization of ideals in Dedekind domains
- Structure of the group of units of the ring of integers
- Finiteness of the group of equivalence classes of ideals of the ring of integers (the "class group")
- Decomposition and inertia groups, Frobenius elements
- Ramification
- Discriminant and different
- Quadratic and biquadratic fields
- Cyclotomic fields (and applications)
- How to use a computer to compute with many of the above objects (both algorithms and actual use of software).
- Valuations on fields
- Completions (p-adic fields)
- Adeles and Ideles

Note that we will not do anything nontrivial with zeta functions or L-functions.

1.3 Some applications of algebraic number theory

The following examples illustrate that learning algebraic number theory as soon as possible is an excellent investment of your time.

1. **Integer factorization** using the number field sieve. The number field sieve is the asymptotically fastest known algorithm for factoring general large integers (that don't have too special of a form). Recently, in December 2003, the number field sieve was used to factor the RSA-576 $10000 challenge:

$$\begin{aligned}
&188198812920607963838697239461650439807163563379417382700 7\ldots\\
&\ldots 6335642298885971523466548531906060650474304531738801130339\ldots\\
&\ldots 671619969232120573403187955065699622130516875930765025705 9\\
&= 398075086424064937397125500550386491199064362342526708406\ldots\\
&\quad\ldots 385189575946388957261768583317\\
&\qquad\times 472772146107435302536223071973048224632914695302097116\ldots\\
&\qquad\quad\ldots 459852171130520711256363590397527
\end{aligned}$$

(The ... indicates that the newline should be removed, not that there are missing digits.)

2. **Primality test:** Agrawal and his students Saxena and Kayal from India found in 2002 the first ever deterministic polynomial-time (in the number of digits) primality test. There methods involve arithmetic in quotients of $(\mathbf{Z}/n\mathbf{Z})[x]$, which are best understood in the context of algebraic number theory. For example, Lenstra, Bernstein, and others have done that and improved the algorithm significantly.

3. **Deeper point of view** on questions in number theory:

   (a) Pell's Equation $(x^2 - dy^2 = 1) \implies$ Units in real quadratic fields $\implies$ Unit groups in number fields

   (b) Diophantine Equations $\implies$ For which n does $x^n + y^n = z^n$ have a nontrivial solution?

   (c) Integer Factorization $\implies$ Factorization of ideals

   (d) Riemann Hypothesis $\implies$ Generalized Riemann Hypothesis

   (e) Deeper proof of Gauss's quadratic reciprocity law in terms of arithmetic of cyclotomic fields $\mathbf{Q}(e^{2\pi i/n})$, which leads to class field theory.

4. Wiles's proof of **Fermat's Last Theorem**, i.e., that the equation $x^n + y^n = z^n$ has no solutions with x, y, z, n all positive integers and $n \geq 3$, uses methods from algebraic number theory extensively, in addition to many other deep techniques. Attempts to prove Fermat's Last Theorem long ago were hugely influential in the development of algebraic number theory by Dedekind, Hilbert, Kummer, Kronecker, and others.

5. **Arithmetic geometry:** This is a huge field that studies solutions to polynomial equations that lie in arithmetically interesting rings, such as the integers or number fields. A famous major triumph of arithmetic geometry is Faltings's proof of Mordell's Conjecture.

Theorem 1.3.1 (Faltings)**.** *Let X be a nonsingular plane algebraic curve over a number field K. Assume that the manifold $X(\mathbf{C})$ of complex solutions to X has genus at least* 2 *(i.e., $X(\mathbf{C})$ is topologically a donut with two holes). Then the set $X(K)$ of points on X with coordinates in K is finite.*

For example, Theorem 1.3.1 implies that for any $n \geq 4$ and any number field K, there are only finitely many solutions in K to $x^n + y^n = 1$.

A major open problem in arithmetic geometry is the *Birch and Swinnerton-Dyer conjecture*. An *elliptic curves E* is an algebraic curve with at least one point with coordinates in K such that the set of complex points $E(\mathbf{C})$ is a topological torus. The Birch and Swinnerton-Dyer conjecture gives a criterion for whether or not $E(K)$ is infinite in terms of analytic properties of the L-function $L(E, s)$.

Chapter 2

Basic Commutative Algebra

The commutative algebra in this chapter provides a foundation for understanding the more refined number-theoretic structures associated to number fields.

First we prove the structure theorem for finitely generated abelian groups. Then we establish the standard properties of Noetherian rings and modules, including a proof of the Hilbert basis theorem. We also observe that finitely generated abelian groups are Noetherian $\mathbf{Z}$-modules. After establishing properties of Noetherian rings, we consider rings of algebraic integers and discuss some of their properties.

2.1 Finitely Generated Abelian Groups

Finitely generated abelian groups arise all over algebraic number theory. For example, they will appear in this book as class groups, unit groups, and the underlying additive groups of rings of integers, and as Mordell-Weil groups of elliptic curves.

In this section, we prove the structure theorem for finitely generated abelian groups, since it will be crucial for much of what we will do later.

Let $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$ denote the ring of (rational) integers, and for each positive integer n let $\mathbf{Z}/n\mathbf{Z}$ denote the ring of integers modulo n, which is a cyclic abelian group of order n under addition.

Definition 2.1.1 (Finitely Generated)**.** A group G is *finitely generated* if there exists $g_1, \ldots, g_n \in G$ such that every element of G can be expressed as a finite product of positive or negative powers of the g_i.

For example, the group $\mathbf{Z}$ is finitely generated, since it is generated by 1.

Theorem 2.1.2 (Structure Theorem for Abelian Groups)**.** *Let G be a finitely generated abelian group. Then there is an isomorphism*

$$G \cong (\mathbf{Z}/n_1\mathbf{Z}) \oplus (\mathbf{Z}/n_2\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/n_s\mathbf{Z}) \oplus \mathbf{Z}^r,$$

where $n_1 > 1$ and $n_1 \mid n_2 \mid \cdots \mid n_s$. Furthermore, the n_i and r are uniquely determined by G.

We will prove the theorem as follows. We first remark that any subgroup of a finitely generated free abelian group is finitely generated. Then we see that finitely generated abelian groups can be presented as quotients of finite rank free abelian groups, and such a presentation can be reinterpreted in terms of matrices over the integers. Next we describe how to use row and column operations over the integers to show that every matrix over the integers is equivalent to one in a canonical diagonal form, called the Smith normal form. We obtain a proof of the theorem by reinterpreting Smith normal form in terms of groups. Finally, we observe by a simple argument that the representation in the theorem is necessarily unique.

Proposition 2.1.3. *If H is a subgroup of a finitely generated abelian group then H is finitely generated.*

The key reason that this is true is that G is a finitely generated module over the principal ideal domain $\mathbf{Z}$. We will give a complete proof of a beautiful generalization of Proposition 2.1.3 in the context of Noetherian rings in Section 2.2, but will not prove this proposition here.

Corollary 2.1.4. *Suppose G is a finitely generated abelian group. Then there are finitely generated free abelian groups F_1 and F_2 and a homomorphism $\psi : F_2 \to F_1$ such that $G \cong F_1/\psi(F_2)$.*

Proof. Let $x_1, \ldots, x_m$ be generators for G. Let $F_1 = \mathbf{Z}^m$ and let $\varphi : F_1 \to G$ be the map that sends the ith generator $(0, 0, \ldots, 1, \ldots, 0)$ of $\mathbf{Z}^m$ to x_i. Then φ is a surjective homomorphism, and by Proposition 2.1.3 the kernel $\ker(\varphi)$ of φ is a finitely generated abelian group. Let $F_2 = \mathbf{Z}^n$ and fix a surjective homomorphism $\psi : F_2 \to \ker(\varphi)$. Then $F_1/\psi(F_2)$ is isomorphic to G. □

Suppose G is a nonzero finitely generated abelian group. By the corollary, there are free abelian groups F_1 and F_2 and a homomorphism $\psi : F_2 \to F_1$ such that $G \approx F_1/\psi(F_2)$. Choosing a basis for F_1 and F_2, we obtain isomorphisms $F_1 \approx \mathbf{Z}^n$ and $F_2 \approx \mathbf{Z}^m$ for integers n and m. We can thus view $\psi : F_2 \to F_1$ as being given by left multiplication by the $n \times m$ matrix A whose columns are the images of the generators of F_2 in $\mathbf{Z}^n$. The *cokernel* of this homomorphism is the quotient of $\mathbf{Z}^n$ by the image of A (the $\mathbf{Z}$-span of the columns of A), and this cokernel is isomorphic to G.

By augmenting A with zero columns or adding (standard basis) rows to A, we may assume that $m = n$. For example, we would replace

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} \text{ by } \begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix}$$

and would replace

$$\begin{pmatrix} 4 & 2 \end{pmatrix} \text{ by } \begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix}.$$

The following proposition implies that we may choose a bases for F_1 and F_2 such that the matrix of A is diagonal, so that the structure of the cokernel of A will be easy to understand.

Proposition 2.1.5 (Smith normal form)**.** *Suppose A is an $n \times n$ integer matrix. Then there exist invertible integer matrices P and Q such that $A' = PAQ$ is a diagonal matrix with entries $n_1, n_2, \ldots, n_s, 0, \ldots, 0$, where $s \geq 0$, $n_1 > 1$ and $n_1 \mid n_2 \mid \ldots \mid n_s$. Here P and Q are invertible as integer matrices, so* $\det(P)$ *and* $\det(Q)$ *are ± 1. The matrix A' is called the* Smith normal form *of A.*

We will see in the proof of Theorem 2.1.2 that A' is uniquely determined by A. An example of a matrix in Smith normal form is

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The matrix P will be a product of matrices that define elementary row operations and Q will be a product corresponding to elementary column operations. The elementary row and column operations over $\mathbf{Z}$ are as follows:

1. [**Add multiple**] Add an integer multiple of one row to another (or a multiple of one column to another).

2. [**Swap**] Interchange two rows or two columns.

3. [**Rescale**] Multiply a row by -1.

Each of these operations is given by left or right multiplying by an invertible matrix E with integer entries, where E is the result of applying the given operation to the identity matrix, and E is invertible because each operation can be reversed using another row or column operation over the integers.

To see that the proposition must be true, assume $A \neq 0$ and perform the following steps (compare [Art91, pg. 459]):

1. By permuting rows and columns, move a nonzero entry of A with smallest absolute value to the upper left corner of A. Now attempt to make all other entries in the first row and column 0 by adding multiples of row or column 1 to other rows (see step 2 below). If an operation produces a nonzero entry in the matrix with absolute value smaller than $|a_{11}|$, start the process over by permuting rows and columns to move that entry to the upper left corner of A. Since the integers $|a_{11}|$ are a decreasing sequence of positive integers, we will not have to move an entry to the upper left corner infinitely often.

2. Suppose a_{i1} is a nonzero entry in the first column, with $i > 1$. Using the division algorithm, write $a_{i1} = a_{11}q + r$, with $0 \leq r < a_{11}$. Now add $-q$ times the first row to the ith row. If $r > 0$, then go to step 1 (so that an entry with absolute value at most r is the upper left corner). Since we will only perform step 1 finitely many times, we may assume $r = 0$. Repeating this procedure we set all entries in the first column (except a_{11}) to 0. A similar process using column operations sets each entry in the first row (except a_{11}) to 0.

3. We may now assume that a_{11} is the only nonzero entry in the first row and column. If some entry a_{ij} of A is not divisible by a_{11}, add the column of A containing a_{ij} to the first column, thus producing an entry in the first column that is nonzero. When we perform step 2, the remainder r will be greater than 0. Permuting rows and columns results in a smaller $|a_{11}|$. Since $|a_{11}|$ can only shrink finitely many times, eventually we will get to a point where every a_{ij} is divisible by a_{11}. If a_{11} is negative, multiple the first row by -1.

After performing the above operations, the first row and column of A are zero except for a_{11} which is positive and divides all other entries of A. We repeat the above steps for the matrix B obtained from A by deleting the first row and column. The upper left entry of the resulting matrix will be divisible by a_{11}, since every entry of B is. Repeating the argument inductively proves the proposition. □

Example 2.1.6. The matrix $\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$ has Smith normal form to $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, and the matrix $\begin{pmatrix} 1 & 4 & 9 \\ 16 & 25 & 36 \\ 49 & 64 & 81 \end{pmatrix}$ has Smith normal form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 72 \end{pmatrix}$. As a double check, note that the determinants of a matrix and its Smith normal form match, up to sign. This is because

$$\det(PAQ) = \det(P)\det(A)\det(Q) = \pm\det(A).$$

We compute each of the above Smith forms using SAGE, along with the corresponding transformation matrices. *Warning: Currently in Sage the entries down the diagonal are reversed from the discussion above.* First the 2×2 matrix.

```
sage: A = matrix(ZZ, 2, [-1,2, -3,4])
sage: S, U, V = A.smith_form(); S
[2 0]
[0 1]
sage: U*A*V
[2 0]
[0 1]
sage: U
[ 1 -1]
[ 0  1]
sage: V
[4 1]
[3 1]
```

The SAGE `matrix` command takes as input the base ring, the number of rows, and the entries. Next we compute with a 3×3 matrix.

```
sage: A = matrix(ZZ, 3, [1,4,9,  16,25,36, 49,64,81])
sage: S, U, V = A.smith_form(); S
[72  0  0]
[ 0  3  0]
[ 0  0  1]
sage: U*A*V
[72  0  0]
[ 0  3  0]
[ 0  0  1]
sage: U
[  1 -20 -17]
[  0   1  -1]
[  0   0   1]
sage: V
[  93   74   47]
[-156 -125  -79]
[  67   54   34]
```

Finally we compute the Smith form of a matrix of rank 2:

```
sage: m = matrix(ZZ, 3, [2..10]); m
[ 2  3  4]
[ 5  6  7]
[ 8  9 10]
sage: m.smith_form()[0]
[0 0 0]
[0 3 0]
[0 0 1]
```

Theorem 2.1.2. Suppose G is a finitely generated abelian group, which we may assume is nonzero. As in the paragraph before Proposition 2.1.5, we use Corollary 2.1.4 to write G as a the cokernel of an $n \times n$ integer matrix A. By Proposition 2.1.5 there are isomorphisms $Q : \mathbf{Z}^n \to \mathbf{Z}^n$ and $P : \mathbf{Z}^n \to \mathbf{Z}^n$ such that $A' = PAQ$ is a diagonal matrix with entries $n_1, n_2, \ldots, n_s, 0, \ldots, 0$, where $n_1 > 1$ and $n_1 \mid n_2 \mid \ldots \mid n_s$. Then G is isomorphic to the cokernel of the diagonal matrix A', so

$$G \cong (\mathbf{Z}/n_1\mathbf{Z}) \oplus (\mathbf{Z}/n_2\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/n_s\mathbf{Z}) \oplus \mathbf{Z}^r, \tag{2.1.1}$$

as claimed. The n_i are determined by G, because n_i is the smallest positive integer n such that nG requires at most $s + r - i$ generators. We see from the representation (2.1.1) of G as a product that n_i has this property and that no smaller positive integer does.

□

2.2 Noetherian Rings and Modules

Let R be a commutative ring with unit element. We will frequently work with R-modules, which are like vector spaces but over a ring.

More precisely, an *R-module* is an additive abelian group M equipped with a map $R \times M \to M$ such that for all $r, r' \in R$ and all $m, m' \in M$ we have $(rr')m = r(r'm)$, $(r + r')m = rm + r'm$, $r(m + m') = rm + rm'$, and $1m = m$. A *submodule* is a subgroup of M that is preserved by the action of R. An *ideal* in a ring R is an R-submodule $I \subset R$, where we view R as a module over itself.

Example 2.2.1. The set of abelian groups are in natural bijection with $\mathbf{Z}$-modules.

A *homomorphism* of R-modules $\varphi : M \to N$ is a abelian group homomorphism such that for any $r \in R$ and $m \in M$ we have $\varphi(rm) = r\varphi(m)$. A *short exact sequence* of R-modules

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is a specific choice of injective homomorphism $f : L \to M$ and a surjective homomorphism $g : M \to N$ such that $\mathrm{im}(f) = \ker(g)$.

Example 2.2.2. The sequence

$$0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

is an exact sequence, where the first map sends 1 to 2, and the second is the natural quotient map.

Definition 2.2.3 (Noetherian)**.** An R-module M is *noetherian* if every submodule of M is finitely generated. A ring R is *noetherian* if R is noetherian as a module over itself, i.e., if every ideal of R is finitely generated.

Notice that any submodule M' of a noetherian module M is also noetherian. Indeed, if every submodule of M is finitely generated then so is every submodule of M', since submodules of M' are also submodules of M.

Definition 2.2.4 (Ascending chain condition)**.** An R-module M satisfies the *ascending chain condition* if every sequences $M_1 \subset M_2 \subset M_3 \subset \cdots$ of submodules of M eventually stabilizes, i.e., there is some n such that $M_n = M_{n+1} = M_{n+2} = \cdots$.

We will use the notion of maximal element below. If $\mathcal{X}$ is a set of subsets of a set S, ordered by inclusion, then a *maximal element* $A \in \mathcal{X}$ is a set so that no superset of A is contained in $\mathcal{X}$. Note that it is *not* necessary that A contain every other element of $\mathcal{X}$, and that $\mathcal{X}$ could contain many maximal elements.

Proposition 2.2.5. *If M is an R-module, then the following are equivalent:*

1. *M is noetherian,*

2. *M satisfies the ascending chain condition, and*

3. *Every nonempty set of submodules of M contains at least one maximal element.*

Proof. $1 \implies 2$**:** Suppose $M_1 \subset M_2 \subset \cdots$ is a sequence of submodules of M. Then $M_\infty = \cup_{n=1}^{\infty} M_n$ is a submodule of M. Since M is noetherian and M_∞ is a submodule of M, there is a finite set $a_1, \ldots, a_m$ of generators for M_∞. Each a_i must be contained in some M_j, so there is an n such that $a_1, \ldots, a_m \in M_n$. But then $M_k = M_n$ for all $k \geq n$, which proves that the chain of M_i stabilizes, so the ascending chain condition holds for M.
$2 \implies 3$**:** Suppose 3 were false, so there exists a nonempty set S of submodules of M that does not contain a maximal element. We will use S to construct an infinite ascending chain of submodules of M that does not stabilize. Note that S is infinite, otherwise it would contain a maximal element. Let M_1 be any element of S. Then there is an M_2 in S that contains M_1, otherwise S would contain the maximal element M_1. Continuing inductively in this way we find an M_3 in S that properly contains M_2, etc., and we produce an infinite ascending chain of submodules of M, which contradicts the ascending chain condition.
$3 \implies 1$**:** Suppose 1 is false, so there is a submodule M' of M that is not finitely generated. We will show that the set S of all finitely generated submodules of M' does not have a maximal element, which will be a contradiction. Suppose S does have a maximal element L. Since L is finitely generated and $L \subset M'$, and M' is not finitely generated, there is an $a \in M'$ such that $a \notin L$. Then $L' = L + Ra$ is an element of S that strictly contains the presumed maximal element L, a contradiction. □

Lemma 2.2.6. *If*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is a short exact sequence of R-modules, then M is noetherian if and only if both L and N are noetherian.

Proof. First suppose that M is noetherian. Then L is a submodule of M, so L is noetherian. If N' is a submodule of N, then the inverse image of N' in M is a submodule of M, so it is finitely generated, hence its image N' is finitely generated. Thus N is noetherian as well.

Next assume nothing about M, but suppose that both L and N are noetherian. If M' is a submodule of M, then $M_0 = \varphi(L) \cap M'$ is isomorphic to a submodule of the noetherian module L, so M_0 is generated by finitely many elements $a_1, \ldots, a_n$. The quotient M'/M_0 is isomorphic (via g) to a submodule of the noetherian module N, so M'/M_0 is generated by finitely many elements $b_1, \ldots, b_m$. For each $i \leq m$, let c_i be a lift of b_i to M', modulo M_0. Then the elements $a_1, \ldots, a_n, c_1, \ldots, c_m$ generate M', for if $x \in M'$, then there is some element $y \in M_0$ such that $x - y$ is an R-linear combination of the c_i, and y is an R-linear combination of the a_i. □

Proposition 2.2.7. *Suppose R is a noetherian ring. Then an R-module M is noetherian if and only if it is finitely generated.*

Proof. If M is noetherian then every submodule of M is finitely generated so M is finitely generated. Conversely, suppose M is finitely generated, say by elements $a_1, \ldots, a_n$. Then there is a surjective homomorphism from $R^n = R \oplus \cdots \oplus R$ to M that sends $(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in ith factor) to a_i. Using Lemma 2.2.6 and exact sequences of R-modules such as $0 \to R \to R \oplus R \to R \to 0$, we see inductively that R^n is noetherian. Again by Lemma 2.2.6, homomorphic images of noetherian modules are noetherian, so M is noetherian. □

Lemma 2.2.8. *Suppose $\varphi : R \to S$ is a surjective homomorphism of rings and R is noetherian. Then S is noetherian.*

Proof. The kernel of φ is an ideal I in R, and we have an exact sequence

$$0 \to I \to R \to S \to 0$$

with R noetherian. This is an exact sequence of R-modules, where S has the R-module structure induced from φ (if $r \in R$ and $s \in S$, then $rs = \varphi(r)s$). By Lemma 2.2.6, it follows that S is a noetherian R-modules. Suppose J is an ideal of S. Since J is an R-submodule of S, if we view J as an R-module, then J is finitely generated. Since R acts on J through S, the R-generators of J are also S-generators of J, so J is finitely generated as an ideal. Thus S is noetherian. □

Theorem 2.2.9 (Hilbert Basis Theorem)**.** *If R is a noetherian ring and S is finitely generated as a ring over R, then S is noetherian. In particular, for any n the polynomial ring $R[x_1, \ldots, x_n]$ and any of its quotients are noetherian.*

Proof. Assume first that we have already shown that for any n the polynomial ring $R[x_1, \ldots, x_n]$ is noetherian. Suppose S is finitely generated as a ring over R, so there are generators $s_1, \ldots, s_n$ for S. Then the map $x_i \mapsto s_i$ extends uniquely to a surjective homomorphism $\pi : R[x_1, \ldots, x_n] \to S$, and Lemma 2.2.8 implies that S is noetherian.

The rings $R[x_1, \ldots, x_n]$ and $(R[x_1, \ldots, x_{n-1}])[x_n]$ are isomorphic, so it suffices to prove that if R is noetherian then $R[x]$ is also noetherian. (Our proof follows [Art91, §12.5].) Thus suppose I is an ideal of $R[x]$ and that R is noetherian. We will show that I is finitely generated.

Let A be the set of leading coefficients of polynomials in I. (The leading coefficient of a polynomial is the coefficient of highest degree, or 0 if the polynomial is 0; thus $3x^7 + 5x^2 - 4$ has leading coefficient 3.) We will first show that A is an ideal of R. Suppose $a, b \in A$ are nonzero with $a + b \neq 0$. Then there are polynomials f and g in I with leading coefficients a and b. If $\deg(f) \leq \deg(g)$, then $a + b$ is the leading coefficient of $x^{\deg(g)-\deg(f)} f + g$, so $a + b \in A$. Suppose $r \in R$ and $a \in A$ with $ra \neq 0$. Then ra is the leading coefficient of rf, so $ra \in A$. Thus A is an ideal in R.

Since R is noetherian and A is an ideal, there exist nonzero $a_1, \ldots, a_n$ that generate A as an ideal. Since A is the set of leading coefficients of elements of I, and the a_j are in A, we can choose for each $j \leq n$ an element $f_j \in I$ with leading coefficient a_j. By multipying the f_j by some power of x, we may assume that the f_j all have the same degree $d \geq 1$.

Let $S_{<d}$ be the set of elements of I that have degree strictly less than d. This set is closed under addition and under multiplication by elements of R, so $S_{<d}$ is a module over R. The module $S_{<d}$ is the submodule of the R-module of polynomials of degree less than n, which is noetherian because it is generated by $1, x, \ldots, x^{n-1}$. Thus $S_{<d}$ is finitely generated, and we may choose generators $h_1, \ldots, h_m$ for $S_{<d}$.

We finish by proving using induction on the degree that every $g \in I$ is an $R[x]$-linear combination of $f_1, \ldots, f_n, h_1, \ldots h_m$. If $g \in I$ has degree 0, then $g \in S_{<d}$, since $d \geq 1$, so g is a linear combination of $h_1, \ldots, h_m$. Next suppose $g \in I$ has degree e, and that we have proven the statement for all elements of I of degree $< e$. If $e \leq d$, then $g \in S_{<d}$, so g is in the $R[x]$-ideal generated by $h_1, \ldots, h_m$. Next suppose that $e \geq d$. Then the leading coefficient b of g lies in the ideal A of leading coefficients of elements of I, so there exist $r_i \in R$ such that $b = r_1 a_1 + \cdots + r_n a_n$. Since f_i has leading coefficient a_i, the difference $g - x^{e-d} r_i f_i$ has degree less than the degree e of g. By induction $g - x^{e-d} r_i f_i$ is an $R[x]$ linear combination of $f_1, \ldots, f_n, h_1, \ldots h_m$, so g is also an $R[x]$ linear combination of $f_1, \ldots, f_n, h_1, \ldots h_m$. Since each f_i and h_j lies in I, it follows that I is generated by $f_1, \ldots, f_n, h_1, \ldots h_m$, so I is finitely generated, as required. □

Properties of noetherian rings and modules will be crucial in the rest of this course. We have proved above that noetherian rings have many desirable properties.

2.2.1 The Ring Z is noetherian

The ring $\mathbf{Z}$ of integers is noetherian because every ideal of $\mathbf{Z}$ is generated by one element.

Proposition 2.2.10. *Every ideal of the ring* $\mathbf{Z}$ *of integers is principal.*

Proof. Suppose I is a nonzero ideal in $\mathbf{Z}$. Let d the least positive element of I. Suppose that $a \in I$ is any nonzero element of I. Using the division algorithm, write $a = dq + r$, where q is an integer and $0 \leq r < d$. We have $r = a - dq \in I$ and $r < d$, so our assumption that d is minimal implies that $r = 0$, so $a = dq$ is in the ideal generated by d. Thus I is the principal ideal generated by d. □

Example 2.2.11. Let $I = (12, 18)$ be the ideal of $\mathbf{Z}$ generated by 12 and 18. If $n = 12a + 18b \in I$, with $a, b \in \mathbf{Z}$, then $6 \mid n$, since $6 \mid 12$ and $6 \mid 18$. Also, $6 = 18 - 12 \in I$, so $I = (6)$.

The ring $\mathbf{Z}$ in SAGE is ZZ, which is Noetherian.

```
sage: ZZ.is_noetherian()
True
```

We create the ideal I in SAGE as follows, and note that it is principal:

```
sage: I = ideal(12,18); I
Principal ideal (6) of Integer Ring
sage: I.is_principal()
True
```

We could also create I as follows:

```
sage: ZZ.ideal(12,18)
Principal ideal (6) of Integer Ring
```

Proposition 2.2.7 and 2.2.10 together imply that any finitely generated abelian group is noetherian. This means that subgroups of finitely generated abelian groups are finitely generated, which provides the missing step in our proof of the structure theorem for finitely generated abelian groups.

2.3 Rings of Algebraic Integers

In this section we will learn about rings of algebraic integers and discuss some of their properties. We will prove that the ring of integers $\mathcal{O}_K$ of a number field is noetherian.

Fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$. Thus $\overline{\mathbf{Q}}$ is an infinite field extension of $\mathbf{Q}$ with the property that every polynomial $f \in \mathbf{Q}[x]$ splits as a product of linear factors in $\overline{\mathbf{Q}}[x]$. One choice of $\overline{\mathbf{Q}}$ is the subfield of the complex numbers $\mathbf{C}$ generated by all roots in $\mathbf{C}$ of all polynomials with coefficients in $\mathbf{Q}$. Note that any two choices of $\overline{\mathbf{Q}}$ are isomorphic, but there will be many isomorphisms between them.

An *algebraic integer* is an element of $\overline{\mathbf{Q}}$.

Definition 2.3.1 (Algebraic Integer)**.** An element $\alpha \in \overline{\mathbf{Q}}$ is an *algebraic integer* if it is a root of some monic polynomial with coefficients in $\mathbf{Z}$.

For example, $\sqrt{2}$ is an algebraic integer, since it is a root of $x^2 - 2$, but one can prove $1/2$ is not an algebraic integer, since one can show that it is not the root of any monic polynomial over $\mathbf{Z}$. Also π and e are *not* algebraic numbers (they are *transcendental*).

Example 2.3.2. We compute some minimal polynomials in SAGE. The minimal polynomial of $1/2$:

```
sage: (1/2).minpoly()
x - 1/2
```

We construct a root a of $x^2 - 2$ and compute its minimal polynomial:

Finally

```
sage: k.<a> = NumberField(x^2 - 2)
sage: a^2 - 2
0
sage: a.charpoly()
x^2 - 2
```

we compute the minimal polynomial of $\sqrt{2}/2+3$, which is not integral:

```
sage: (a/2 + 3).charpoly()
x^2 - 6*x + 17/2
```

The only elements of $\mathbf{Q}$ that are algebraic integers are the usual integers $\mathbf{Z}$. However, there are elements of $\overline{\mathbf{Q}}$ that have denominators when written down, but are still algebraic integers. For example,

$$\alpha = \frac{1+\sqrt{5}}{2}$$

is an algebraic integer, since it is a root of the monic polynomial $x^2 - x - 1$. We verify this using SAGE below, though of course this is easy to do by hand (you should try much more complicated examples in SAGE).

```
sage: k.<a> = QuadraticField(5)
sage: a^2
5
sage: alpha = (1 + a)/2
sage: alpha.charpoly()
x^2 - x - 1
sage: alpha.is_integral()
True
```

Definition 2.3.3 (Minimal Polynomial)**.** The *minimal polynomial* of $\alpha \in \overline{\mathbf{Q}}$ is the monic polynomial $f \in \mathbf{Q}[x]$ of least positive degree such that $f(\alpha) = 0$.

It is a consequence of Lemma 2.3.5 that the minimal polynomial α is unique. The minimal polynomial of $1/2$ is $x - 1/2$, and the minimal polynomial of $\sqrt[3]{2}$ is $x^3 - 2$.

Example 2.3.4. We compute the minimal polynomial of a number expressed in terms of $\sqrt[4]{2}$:

```
sage: k.<a> = NumberField(x^4 - 2)
sage: a^4
2
sage: (a^2 + 3).minpoly()
x^2 - 6*x + 7
```

Lemma 2.3.5. *Suppose $\alpha \in \overline{\mathbf{Q}}$. Then the minimal polynomial of α divides any polynomial h such that $h(\alpha) = 0$.*

Proof. Let f be a minimal polynomial of α. If $h(\alpha) = 0$, use the division algorithm to write $h = qf + r$, where $0 \leq \deg(r) < \deg(f)$. We have

$$r(\alpha) = h(\alpha) - q(\alpha)f(\alpha) = 0,$$

so α is a root of r. However, f is the monic polynomial of least positive degree with root α, so $r = 0$. □

Lemma 2.3.6. *If α is an algebraic integer, then the minimal polynomial of α has coefficients in $\mathbf{Z}$.*

Proof. Suppose $f \in \mathbf{Q}[x]$ is the minimal polynomial of α. Since α is an algebraic integer, there is a polynomial $g \in \mathbf{Z}[x]$ that is monic such that $g(\alpha) = 0$. By Lemma 2.3.5, we have $g = fh$, for some monic $h \in \mathbf{Q}[x]$. If $f \not\in \mathbf{Z}[x]$, then some prime p divides the denominator of some coefficient of f. Let p^i be the largest power of p that divides some denominator of some coefficient f, and likewise let p^j be the largest power of p that divides some denominator of a coefficient of h. Then $p^{i+j}g = (p^i f)(p^j h)$, and if we reduce both sides modulo p, then the left hand side is 0 but the right hand side is a product of two nonzero polynomials in $\mathbf{F}_p[x]$, hence nonzero, a contradiction. □

Proposition 2.3.7. *An element $\alpha \in \overline{\mathbf{Q}}$ is integral if and only if $\mathbf{Z}[\alpha]$ is finitely generated as a $\mathbf{Z}$-module.*

Proof. Suppose α is integral and let $f \in \mathbf{Z}[x]$ be the monic minimal polynomial of α (that $f \in \mathbf{Z}[x]$ is Lemma 2.3.6). Then $\mathbf{Z}[\alpha]$ is generated by $1, \alpha, \alpha^2, \ldots, \alpha^{d-1}$, where d is the degree of f. Conversely, suppose $\alpha \in \overline{\mathbf{Q}}$ is such that $\mathbf{Z}[\alpha]$ is finitely generated, say by elements $f_1(\alpha), \ldots, f_n(\alpha)$. Let d be any integer bigger than the degrees of all f_i. Then there exist integers a_i such that $\alpha^d = \sum_{i=1}^n a_i f_i(\alpha)$, hence α satisfies the monic polynomial $x^d - \sum_{i=1}^n a_i f_i(x) \in \mathbf{Z}[x]$, so α is integral. □

Example 2.3.8. The rational number $\alpha = 1/2$ is not integral. Note that $G = \mathbf{Z}[1/2]$ is not a finitely generated $\mathbf{Z}$-module, since G is infinite and $G/2G = 0$. (You can see that $G/2G = 0$ implies that G is not finitely generated, by assuming that G is finitely generated, using the structure theorem to write G as a product of cyclic groups, and noting that G has nontrivial 2-torsion.)

Proposition 2.3.9. *The set $\overline{\mathbf{Z}}$ of all algebraic integers is a ring, i.e., the sum and product of two algebraic integers is again an algebraic integer.*

Proof. Suppose $\alpha, \beta \in \overline{\mathbf{Z}}$, and let m, n be the degrees of the minimal polynomials of α, β, respectively. Then $1, \alpha, \ldots, \alpha^{m-1}$ span $\mathbf{Z}[\alpha]$ and $1, \beta, \ldots, \beta^{n-1}$ span $\mathbf{Z}[\beta]$ as $\mathbf{Z}$-module. Thus the elements $\alpha^i\beta^j$ for $i \leq m, j \leq n$ span $\mathbf{Z}[\alpha, \beta]$. Since $\mathbf{Z}[\alpha+\beta]$ is a submodule of the finitely-generated module $\mathbf{Z}[\alpha, \beta]$, it is finitely generated, so $\alpha + \beta$ is integral. Likewise, $\mathbf{Z}[\alpha\beta]$ is a submodule of $\mathbf{Z}[\alpha, \beta]$, so it is also finitely generated and $\alpha\beta$ is integral. □

Example 2.3.10. We illustrate an example of a sum and product of two algebraic integers being an algebraic integer. We first make the relative number field obtained by adjoining a root of $x^3 - 5$ to the field $\mathbf{Q}(\sqrt{2})$:

```
sage: k.<a, b> = NumberField([x^2 - 2, x^3 - 5])
sage: k
Number Field in a with defining polynomial x^2 + -2 over its base field
```

Here a and b are roots of $x^2 - 2$ and $x^3 - 5$, respectively.

```
sage: a^2
2
sage: b^3
5
```

We compute the minimal polynomial of the sum and product of $\sqrt[3]{5}$ and $\sqrt{2}$. The command `absolute_minpoly` gives the minimal polynomial of the element over the rational numbers.

```
sage: (a+b).absolute_minpoly()
x^6 - 6*x^4 - 10*x^3 + 12*x^2 - 60*x + 17
sage: (a*b).absolute_minpoly()
x^6 - 200
```

Of course the minimal polynomial of the product is $\sqrt[3]{5}\sqrt{2}$ is trivial to compute by hand. The minimal polynomial of $\alpha = \sqrt[3]{5} + \sqrt{2}$ could be computed by hand by computing the determinant of the matrix given by left multiplication of α on this basis:

$$1, \sqrt{2}, \sqrt[3]{5}, \sqrt[3]{5}\sqrt{2}, \sqrt[3]{5}^2, \sqrt[3]{5}^2\sqrt{2}.$$

The following is an alternative more symbolic way to compute the minimal polynomials above, though it is not provably correct. We compute α to 100 bits precision (via the `n` command), then use the LLL algorithm (via the `algdep` command) to heuristically find a linear relation between the first 6 powers of α.

```
sage: a = 5^(1/3); b = sqrt(2)
sage: c = a+b; c
5^(1/3) + sqrt(2)
sage: (a+b).n(100).algdep(6)
x^6 - 6*x^4 - 10*x^3 + 12*x^2 - 60*x + 17
sage: (a*b).n(100).algdep(6)
x^6 - 200
```

Definition 2.3.11 (Number field)**.** A *number field* is a field K that contains the rational numbers $\mathbf{Q}$ such that the degree $[K:\mathbf{Q}] = \dim_{\mathbf{Q}}(K)$ is finite.

If K is a number field, then by the primitive element theorem there is an $\alpha \in K$ so that $K = \mathbf{Q}(\alpha)$. Let $f(x) \in \mathbf{Q}[x]$ be the minimal polynomial of α. For any fixed choice of $\overline{\mathbf{Q}}$, there is some $\alpha' \in \overline{\mathbf{Q}}$ such that $f(\alpha') = 0$. The map $K \to \overline{\mathbf{Q}}$ that sends α to α' defines an embedding $K \hookrightarrow \overline{\mathbf{Q}}$. Thus any number field can be embedded (in $[K:\mathbf{Q}]$ possible ways) in any fixed choice $\overline{\mathbf{Q}}$ of an algebraic closure of $\mathbf{Q}$.

Definition 2.3.12 (Ring of Integers)**.** The *ring of integers* of a number field K is the ring

$$\mathcal{O}_K = \{x \in K : \ x \text{ satisfies a monic polynomial with integer coefficients }\}.$$

Note that $\mathcal{O}_K$ is a ring, because if we fix an embedding of K into $\overline{\mathbf{Q}}$, then

$$\mathcal{O}_K = K \cap \overline{\mathbf{Z}}.$$

The field $\mathbf{Q}$ of rational numbers is a number field of degree 1, and the ring of integers of $\mathbf{Q}$ is $\mathbf{Z}$. The field $K = \mathbf{Q}(i)$ of Gaussian integers has degree 2 and $\mathcal{O}_K = \mathbf{Z}[i]$. The field $K = \mathbf{Q}(\sqrt{5})$ has ring of integers $\mathcal{O}_K = \mathbf{Z}[(1+\sqrt{5})/2]$. Note that the Golden ratio $(1+\sqrt{5})/2$ satisfies $x^2 - x - 1$. The ring of integers of $K = \mathbf{Q}(\sqrt[3]{9})$ is $\mathbf{Z}[\sqrt[3]{3}]$, where $\sqrt[3]{3} = \frac{1}{3}(\sqrt[3]{9})^2$.

Definition 2.3.13 (Order)**.** An *order* in $\mathcal{O}_K$ is any subring R of $\mathcal{O}_K$ such that the quotient $\mathcal{O}_K/R$ of abelian groups is finite. (Note that R must contain 1 because it is a ring, and for us every ring has a 1.)

As noted above, $\mathbf{Z}[i]$ is the ring of integers of $\mathbf{Q}(i)$. For every nonzero integer n, the subring $\mathbf{Z} + ni\mathbf{Z}$ of $\mathbf{Z}[i]$ is an order. The subring $\mathbf{Z}$ of $\mathbf{Z}[i]$ is not an order, because $\mathbf{Z}$ does not have finite index in $\mathbf{Z}[i]$. Also the subgroup $2\mathbf{Z} + i\mathbf{Z}$ of $\mathbf{Z}[i]$ is not an order because it is not a ring.

We define the number field $\mathbf{Q}(i)$ and compute its ring of integers, which has discriminant -4.

```
sage: K.<i> = NumberField(x^2 + 1)
sage: OK = K.ring_of_integers(); OK
Order with module basis 1, i in Number Field in i with
defining polynomial x^2 + 1
sage: OK.discriminant()
-4
```

Next we compute the order $\mathbf{Z} + 3i\mathbf{Z}$.

```
sage: O3 = K.order(3*i); O3
Order with module basis 1, 3*i in Number Field in i with
defining polynomial x^2 + 1
sage: O3.gens()
[1, 3*i]
```

Notice that the distriminant is $-36 = -4 \cdot 3^2$:

```
sage: O3.discriminant()
-36
```

We test whether certain elements are in the order.

```
sage: 5 + 9*i in O3
True
sage: 1 + 2*i in O3
False
```

We will frequently consider orders because they are often much easier to write down explicitly than $\mathcal{O}_K$. For example, if $K = \mathbf{Q}(\alpha)$ and α is an algebraic integer, then $\mathbf{Z}[\alpha]$ is an order in $\mathcal{O}_K$, but frequently $\mathbf{Z}[\alpha] \neq \mathcal{O}_K$.

Example 2.3.14. In this example $[\mathcal{O}_K : \mathbf{Z}[a]] = 2197$. First we define the number field $K = \mathbf{Q}(a)$ where a is a root of $x^3 - 15x^2 - 94x - 3674$, then we compute the order $\mathbf{Z}[a]$ generated by a.

```
sage: K.<a> = NumberField(x^3 - 15*x^2 - 94*x - 3674)
sage: Oa = K.order(a); Oa
Order with module basis 1, a, a^2 in Number Field in a with defining
polynomial x^3 - 15*x^2 - 94*x - 3674
```

Next we compute the maximal order $\mathcal{O}_K$ of K with a basis, and compute that the index of $\mathbf{Z}[a]$ in $\mathcal{O}_K$ is $2197 = 13^3$.

```
sage: OK = K.maximal_order()
sage: OK.basis()
[25/169*a^2 + 10/169*a + 1/169, 5/13*a^2 + 1/13*a, a^2]
sage: Oa.index_in(OK)
2197
```

Lemma 2.3.15. *Let $\mathcal{O}_K$ be the ring of integers of a number field. Then $\mathcal{O}_K \cap \mathbf{Q} = \mathbf{Z}$ and $\mathbf{Q}\mathcal{O}_K = K$.*

Proof. Suppose $\alpha \in \mathcal{O}_K \cap \mathbf{Q}$ with $\alpha = a/b \in \mathbf{Q}$ in lowest terms and $b > 0$. Since α is integral, $\mathbf{Z}[a/b]$ is finitely generated as a module, so $b = 1$ (see Example 2.3.8).

To prove that $\mathbf{Q}\mathcal{O}_K = K$, suppose $\alpha \in K$, and let $f(x) \in \mathbf{Q}[x]$ be the minimal monic polynomial of α. For any positive integer d, the minimal monic polynomial of $d\alpha$ is $d^{\deg(f)} f(x/d)$, i.e., the polynomial obtained from $f(x)$ by multiplying the coefficient of $x^{\deg(f)}$ by 1, multiplying the coefficient of $x^{\deg(f)-1}$ by d, multiplying the coefficient of $x^{\deg(f)-2}$ by d^2, etc. If d is the least common multiple of the denominators of the coefficients of f, then the minimal monic polynomial of $d\alpha$ has integer coefficients, so $d\alpha$ is integral and $d\alpha \in \mathcal{O}_K$. This proves that $\mathbf{Q}\mathcal{O}_K = K$. □

2.4 Norms and Traces

In this section we develop some basic properties of norms, traces, and discriminants, and give more properties of rings of integers in the general context of Dedekind domains.

Before discussing norms and traces we introduce some notation for field extensions. If $K \subset L$ are number fields, we let $[L : K]$ denote the dimension of L viewed as a K-vector space. If K is a number field and $a \in \overline{\mathbf{Q}}$, let $K(a)$ be the extension of K generated by a, which is the smallest number field that contains both K and a. If $a \in \overline{\mathbf{Q}}$ then a has a minimal polynomial $f(x) \in \mathbf{Q}[x]$, and the *Galois conjugates* of a are the roots of f. The are called the Galois conjugates because the are the orbit of a under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

Example 2.4.1. The element $\sqrt{2}$ has minimal polynomial $x^2 - 2$ and the Galois conjugates are $\sqrt{2}$ and $-\sqrt{2}$. The cube root $\sqrt[3]{2}$ has minimial polynomial $x^3 - 2$ and three Galois conjugates $\sqrt[3]{2}, \zeta_3\sqrt[3]{2}, \zeta_3^2\sqrt[3]{2}$, where ζ_3 is a cube root of unity.

We create the extension $\mathbf{Q}(\zeta_3)(\sqrt[3]{2})$ in SAGE.

```
sage: L.<cuberoot2> = CyclotomicField(3).extension(x^3 - 2)
sage: cuberoot2^3
2
```

Then we list the Galois conjugates of $\sqrt[3]{2}$.

```
sage: cuberoot2.galois_conjugates()
[cuberoot2, (-zeta3 - 1)*cuberoot2, zeta3*cuberoot2]
```

Note that $\zeta_3^2 = -\zeta_3 - 1$:

```
sage: zeta3 = L.base_field().0
sage: zeta3^2
-zeta3 - 1
```

Suppose $K \subset L$ is an inclusion of number fields and let $a \in L$. Then left multiplication by a defines a K-linear transformation $\ell_a : L \to L$. (The transformation ℓ_a is K-linear because L is commutative.)

Definition 2.4.2 (Norm and Trace)**.** The *norm* and *trace* of a from L to K are

$$\mathrm{Norm}_{L/K}(a) = \det(\ell_a) \quad \text{ and } \quad \mathrm{tr}_{L/K}(a) = \mathrm{tr}(\ell_a).$$

We know from linear algebra that determinants are multiplicative and traces are additive, so for $a, b \in L$ we have

$$\mathrm{Norm}_{L/K}(ab) = \mathrm{Norm}_{L/K}(a) \cdot \mathrm{Norm}_{L/K}(b)$$

and

$$\mathrm{tr}_{L/K}(a + b) = \mathrm{tr}_{L/K}(a) + \mathrm{tr}_{L/K}(b).$$

Note that if $f \in \mathbf{Q}[x]$ is the characteristic polynomial of ℓ_a, then the constant term of f is $(-1)^{\deg(f)} \det(\ell_a)$, and the coefficient of $x^{\deg(f)-1}$ is $-\mathrm{tr}(\ell_a)$.

Proposition 2.4.3. *Let $a \in L$ and let $\sigma_1, \ldots, \sigma_d$, where $d = [L : K]$, be the distinct field embeddings $L \hookrightarrow \overline{\mathbf{Q}}$ that fix every element of K. Then*

$$\mathrm{Norm}_{L/K}(a) = \prod_{i=1}^{d} \sigma_i(a) \quad \textit{and} \quad \mathrm{tr}_{L/K}(a) = \sum_{i=1}^{d} \sigma_i(a).$$

Proof. We prove the proposition by computing the characteristic polynomial F of a. Let $f \in K[x]$ be the minimal polynomial of a over K, and note that f has distinct roots and is irreducible, since it is the polynomial in $K[x]$ of least degree that is satisfied by a and K has characteristic 0. Since f is irreducible, we have $K(a) = K[x]/(f)$, so $[K(a) : K] = \deg(f)$. Also a satisfies a polynomial if and only if ℓ_a does, so the characteristic polynomial of ℓ_a acting on $K(a)$ is f. Let $b_1, \ldots, b_n$ be a basis for L over $K(a)$ and note that $1, \ldots, a^m$ is a basis for $K(a)/K$, where $m = \deg(f) - 1$. Then $a^i b_j$ is a basis for L over K, and left multiplication by a acts the same way on the span of $b_j, ab_j, \ldots, a^m b_j$ as on the span of $b_k, ab_k, \ldots, a^m b_k$, for any pair $j, k \leq n$. Thus the matrix of ℓ_a on L is a block direct sum of copies of the matrix of ℓ_a acting on $K(a)$, so the characteristic polynomial of ℓ_a on L is $f^{[L:K(a)]}$. The proposition follows because the roots of $f^{[L:K(a)]}$ are exactly the images $\sigma_i(a)$, with multiplicity $[L : K(a)]$ (since each embedding of $K(a)$ into $\overline{\mathbf{Q}}$ extends in exactly $[L : K(a)]$ ways to L). □

It is important in Proposition 2.4.3 that the product and sum be over *all* the images $\sigma_i(a)$, not over just the distinct images. For example, if $a = 1 \in L$, then $\mathrm{Tr}_{L/K}(a) = [L : K]$, whereas the sum of the distinct conjugates of a is 1.

The following corollary asserts that the norm and trace behave well in towers.

Corollary 2.4.4. *Suppose $K \subset L \subset M$ is a tower of number fields, and let $a \in M$. Then*

$$\mathrm{Norm}_{M/K}(a) = \mathrm{Norm}_{L/K}(\mathrm{Norm}_{M/L}(a)) \quad \text{and} \quad \mathrm{tr}_{M/K}(a) = \mathrm{tr}_{L/K}(\mathrm{tr}_{M/L}(a)).$$

Proof. For the first equation, both sides are the product of $\sigma_i(a)$, where σ_i runs through the embeddings of M into $\overline{\mathbf{Q}}$ that fix K. To see this, suppose $\sigma : L \to \overline{\mathbf{Q}}$ fixes K. If σ' is an extension of σ to M, and $\tau_1, \ldots, \tau_d$ are the embeddings of M into $\overline{\mathbf{Q}}$ that fix L, then $\sigma'\tau_1, \ldots, \sigma'\tau_d$ are exactly the extensions of σ to M. For the second statement, both sides are the sum of the $\sigma_i(a)$. □

The norm and trace down to $\mathbf{Q}$ of an algebraic integer a is an element of $\mathbf{Z}$, because the minimal polynomial of a has integer coefficients, and the characteristic polynomial of a is a power of the minimal polynomial, as we saw in the proof of Proposition 2.4.3.

Proposition 2.4.5. *Let K be a number field. The ring of integers $\mathcal{O}_K$ is a lattice in K, i.e., $\mathbf{Q}\mathcal{O}_K = K$ and $\mathcal{O}_K$ is an abelian group of rank $[K : \mathbf{Q}]$.*

Proof. We saw in Lemma 2.3.15 that $\mathbf{Q}\mathcal{O}_K = K$. Thus there exists a basis $a_1, \ldots, a_n$ for K, where each a_i is in $\mathcal{O}_K$. Suppose that as $x = \sum_{i=1}^n c_i a_i \in \mathcal{O}_K$ varies over all elements of $\mathcal{O}_K$ the denominators of the coefficients c_i are arbitrarily large. Then subtracting off integer multiples of the a_i, we see that as $x = \sum_{i=1}^n c_i a_i \in \mathcal{O}_K$ varies over elements of $\mathcal{O}_K$ with c_i between 0 and 1, the denominators of the c_i are also arbitrarily large. This implies that there are infinitely many elements of $\mathcal{O}_K$ in the bounded subset

$$S = \{c_1 a_1 + \cdots + c_n a_n : c_i \in \mathbf{Q},\ 0 \le c_i \le 1\} \subset K.$$

Thus for any $\varepsilon > 0$, there are elements $a, b \in \mathcal{O}_K$ such that the coefficients of $a - b$ are all less than ε (otherwise the elements of $\mathcal{O}_K$ would all be a "distance" of least ε from each other, so only finitely many of them would fit in S).

As mentioned above, the norms of elements of $\mathcal{O}_K$ are integers. Since the norm of an element is the determinant of left multiplication by that element, the norm is a homogenous polynomial of degree n in the indeterminate coefficients c_i, which is 0 only on the element 0. If the c_i get arbitrarily small for elements of $\mathcal{O}_K$, then the values of the norm polynomial get arbitrarily small, which would imply that there are elements of $\mathcal{O}_K$ with positive norm too small to be in $\mathbf{Z}$, a contradiction. So the set S contains only finitely many elements of $\mathcal{O}_K$. Thus the denominators of the c_i are bounded, so for some d, we have that $\mathcal{O}_K$ has finite index in $A = \frac{1}{d}\mathbf{Z}a_1 + \cdots + \frac{1}{d}\mathbf{Z}a_n$. Since A is isomorphic to $\mathbf{Z}^n$, it follows from the structure theorem for finitely generated abelian groups that $\mathcal{O}_K$ is isomorphic as a $\mathbf{Z}$-module to $\mathbf{Z}^n$, as claimed. □

Corollary 2.4.6. *The ring of integers $\mathcal{O}_K$ of a number field is noetherian.*

Proof. By Proposition 2.4.5, the ring $\mathcal{O}_K$ is finitely generated as a module over $\mathbf{Z}$, so it is certainly finitely generated as a ring over $\mathbf{Z}$. By Theorem 2.2.9, $\mathcal{O}_K$ is noetherian. □

2.5 Recognizing Algebraic Numbers using Lattice Basis Reduction (LLL)

Suppose you somehow compute a decimal approximation α to some rational number $\beta \in \mathbf{Q}$ and from this wish to recover β. For concreteness, say

$$\beta = 22/389 = 0.05655526992287917737789203084832904884318766066838046\ldots$$

and you compute

$$\alpha = 0.056555.$$

Now suppose given only α that you would like to recover β. A standard technique is to use continued fractions, which yields a sequence of good rational approximations for α; by truncating right before a surprisingly big partial quotient, we obtain β:

```
sage: v = continued_fraction(0.056555)
sage: continued_fraction(0.056555)
[0, 17, 1, 2, 6, 1, 23, 1, 1, 1, 1, 1, 2]
sage: convergents([0, 17, 1, 2, 6, 1])
[0, 1/17, 1/18, 3/53, 19/336, 22/389]
```

Generalizing this, suppose next that somehow you numerically approximate an algebraic number, e.g., by evaluating a special function and get a decimal approximation $\alpha \in \mathbf{C}$ to an algebraic number $\beta \in \overline{\mathbf{Q}}$. For concreteness, suppose $\beta = \frac{1}{3} + \sqrt[4]{3}$:

```
sage: N(1/3 + 3^(1/4), digits=50)
1.64940734628582579415255223513033238849340192353916
```

Now suppose you very much want to find the (rescaled) minimal polynomial $f(x) \in \mathbf{Z}[x]$ of β just given this numerical approximation α. This is of great value even without proof, since often in practice once you know a potential minimal polynomial you can verify that it is in fact right. Exactly this situation arises in the explicit construction of class fields (a more advanced topic in number theory) and in the construction of Heegner points on elliptic curves. As we will see, the LLL algorithm provides a polynomial time way to solve this problem, assuming α has been computed to sufficient precision.

2.5.1 LLL Reduced Basis

Given a basis $b_1, \ldots, b_n$ for $\mathbf{R}^n$, the *Gramm-Schmidt orthogonalization* process produces an orthogonal basis $b_1^*, \ldots, b_n^*$ for $\mathbf{R}^n$ as follows. Define inductively

$$b_i^* = b_i - \sum_{j<i} \mu_{i,j} b_j^*$$

where

$$\mu_{i,j} = \frac{b_i \cdot b_j^*}{b_j^* \cdot b_j^*}.$$

Example 2.5.1. We compute the Gramm-Schmidt orthogonal basis of the rows of a matrix. Note that no square roots are introduced in the process; there would be square roots if we constructed an orthonormal basis.

```
sage: A = matrix(ZZ, 2, [1,2, 3,4]); A
[1 2]
[3 4]
sage: Bstar, mu = A.gramm_schmidt()
```

The rows of the matrix B^* are obtained from the rows of A by the Gramm-Schmidt procedure.

```
sage: Bstar
[   1    2]
[ 4/5 -2/5]
sage: mu
[   0    0]
[11/5    0]
```

A *lattice* $L \subset \mathbf{R}^n$ is a subgroup that is free of rank n such that $\mathbf{R}L = \mathbf{R}^n$.

Definition 2.5.2 (LLL-reduced basis)**.** The basis $b_1, \ldots, b_n$ for a lattice $L \subset \mathbf{R}^n$ is *LLL reduced* if for all i, j,

$$|\mu_{i,j}| \leq \frac{1}{2}$$

and for each $i \geq 2$,

$$|b_i^*|^2 \geq \left(\frac{3}{4} - \mu_{i,i-1}^2\right) |b_{i-1}^*|^2$$

For example, the basis $b_1 = (1,2)$, $b_2 = (3,4)$ for a lattice L is *not* LLL reduced because $b_1^* = b_1$ and

$$\mu_{2,1} = \frac{b_2 \cdot b_1^*}{b_1^* \cdot b_1^*} = \frac{11}{5} > \frac{1}{2}.$$

However, the basis $b_1 = (1,0)$, $b_2 = (0,2)$ for L is LLL reduced, since

$$\mu_{2,1} = \frac{b_2 \cdot b_1^*}{b_1^* \cdot b_1^*} = 0,$$

and

$$2^2 \geq (3/4) \cdot 1^2.$$

```
sage: A = matrix(ZZ, 2, [1,2, 3,4])
sage: A.LLL()
[1 0]
[0 2]
```

2.5.2 What LLL really means

The following theorem is not too difficult to prove.

Let $b_1, \ldots, b_n$ be an LLL reduced basis for a lattice $L \subset \mathbf{R}^n$. Let $d(L)$ denote the absolute value of the determinant of any matrix whose rows are basis for L. Then the vectors b_i are "nearly orthogonal" and "short" in the sense of the following theorem:

Theorem 2.5.3. *We have*

1. $d(L) \leq \prod_{i=1}^n |b_i| \leq 2^{n(n-1)/4} d(L)$,
2. *For* $1 \leq j \leq i \leq n$*, we have*
$$|b_j| \leq 2^{(i-1)/2} |b_i^*|.$$
3. *The vector* b_1 *is very short in the sense that*
$$|b_1| \leq 2^{(n-1)/4} d(L)^{1/n}$$
and for every nonzero $x \in L$ *we have*
$$|b_1| \leq 2^{(n-1)/2} |x|.$$
4. *More generally, for any linearly independent* $x_1, \ldots, x_t \in L$*, we have*
$$|b_j| \leq 2^{(n-1)/2} \max(|x_1|, \ldots, |x_t|)$$
for $1 \leq j \leq t$*.*

Perhaps the most amazing thing about the idea of an LLL reduced basis is that there is an algorithm (in fact many) that given a basis for a lattice L produce an LLL reduced basis for L, and do so *quickly*, i.e., in polynomial time in the number of digits of the input. The current optimal implementation (and practically optimal algorithms) for computing LLL reduced basis are due to Damien Stehle, and are included standard in Magma in Sage. Stehle's code is amazing – it can LLL reduce a random lattice in $\mathbf{R}^n$ for $n < 1000$ in a matter of minutes!!

```
sage: A = random_matrix(ZZ, 200)
sage: t = cputime()
sage: B = A.LLL()
sage: cputime(t)     # random output
3.0494159999999999
```

There is even a very fast variant of Stehle's implementation that computes a basis for L that is very likely LLL reduced but may in rare cases fail to be LLL reduced.

```
sage: t = cputime()
sage: B = A.LLL(algorithm="fpLLL:fast")   # not tested
sage: cputime(t)      # random output
0.96842699999999837
```

2.5.3 Applying LLL

The LLL definition and algorithm has many application in number theory, e.g., to cracking lattice-based cryptosystems, to enumerating all short vectors in a lattice, to finding relations between decimal approximations to complex numbers, to very fast univariate polynomial factorization in $\mathbf{Z}[x]$ and more generally in $K[x]$ where K is a number fields, and to computation of kernels and images of integer matrices. LLL can also be used to solve the problem of recognizing algebraic numbers mentioned at the beginning of Section 2.5.

Suppose as above that α is a decimal approximation to some algebraic number β, and to for simplicity assume that $\alpha \in \mathbf{R}$ (the general case of $\alpha \in \mathbf{C}$ is described in [**?**]). We finish by explaining how to use LLL to find a polynomial $f(x) \in \mathbf{Z}[x]$ such that $f(\alpha)$ is small, hence has a shot at being the minimal polynomial of β.

Given a real number decimal approximation α, an integer d (the degree), and an integer K (a function of the precision to which α is known), the following steps produce a polynomial $f(x) \in \mathbf{Z}[x]$ of degree at most d such that $f(\alpha)$ is small.

1. Form the lattice in $\mathbf{R}^{d+2}$ with basis the rows of the matrix A whose first $(d+1)\times(d+1)$ part is the identity matrix, and whose last column has entries

$$K, \lfloor K\alpha \rfloor, \lfloor K\alpha^2 \rfloor, \ldots \lfloor K\alpha^d \rfloor. \tag{2.5.1}$$

(Note this matrix is $(d+1) \times (d+2)$ so the lattice is not of full rank in $\mathbf{R}^{d+2}$, which isn't a problem, since the LLL definition also makes sense for less vectors.)

2. Compute an LLL reduced basis for the $\mathbf{Z}$-span of the rows of A, and let B be the corresponding matrix. Let $b_1 = (a_0, a_1, \ldots, a_{d+1})$ be the first row of B and notice that B is obtained from A by left multiplication by an invertible integer matrix. Thus $a_0, \ldots, a_d$ are the linear combination of the (2.5.1) that equals a_{d+1}. Moreover, since B is LLL reduced we expect that a_{d+1} is relatively small.

3. Output $f(x) = a_0 + a_1 x + \cdots a_d x^d$. We have that $f(\alpha) \sim a_{d+1}/K$, which is small. Thus $f(x)$ may be a very good candidate for the minimal polynomial of β (the algebraic number we are approximating), assuming d was chosen minimally and α was computed to sufficient precision.

The following is a complete implementation of the above algorithm in Sage:

```
def myalgdep(a, d, K=10^6):
    aa = [floor(K*a^i) for i in range(d+1)]
    A = identity_matrix(ZZ, d+1)
    B = matrix(ZZ, d+1, 1, aa)
    A = A.augment(B)
    L = A.LLL()
    v = L[0][:-1].list()
    return ZZ['x'](v)
```

Here is an example of using it:

```
sage: R.<x> = RDF[]
sage: f = 2*x^3 - 3*x^2 + 10*x - 4
sage: a = f.roots()[0][0]; a
sage: myalgdep(a, 3, 10^6)       # not tested
2*x^3 - 3*x^2 + 10*x - 4
```

Chapter 3

Dedekind Domains and Unique Factorization of Ideals

Unique factorization into irreducible elements frequently fails for rings of integers of number fields. In this chapter we will deduce a central property of the ring of integers $\mathcal{O}_K$ of an algebraic number field, namely that every nonzero *ideal* factors uniquely as a products of prime ideals. Along the way, we will introduce fractional ideals and prove that they form a free abelian group under multiplication. Factorization of *elements* of $\mathcal{O}_K$ (and much more!) is governed by the class group of $\mathcal{O}_K$, which is the quotient of the group of fractional ideals by the principal fractional ideals (see Chapter 7).

3.1 Dedekind Domains

Recall (Corollary 2.4.6) that we proved that the ring of integers $\mathcal{O}_K$ of a number field is noetherian, as follows. As we saw before using norms, the ring $\mathcal{O}_K$ is finitely generated as a module over $\mathbf{Z}$, so it is certainly finitely generated as a ring over $\mathbf{Z}$. By the Hilbert Basis Theorem, $\mathcal{O}_K$ is noetherian.

If R is an integral domain, the *field of fractions* $\mathrm{Frac}(R)$ of R is the field of all equivalence classes of formal quotients a/b, where $a, b \in R$ with $b \neq 0$, and $a/b \sim c/d$ if $ad = bc$. For example, the field of fractions of $\mathbf{Z}$ is (canonically isomorphic to) $\mathbf{Q}$ and the field of fractions of $\mathbf{Z}[(1+\sqrt{5})/2]$ is $\mathbf{Q}(\sqrt{5})$. The field of fractions of the ring $\mathcal{O}_K$ of integers of a number field K is just the number field K.

Example 3.1.1. We compute the fraction fields mentioned above.

```
sage: Frac(ZZ)
Rational Field
```

In Sage the `Frac` command usually returns a field canonically isomorphic to the fraction field (not a formal construction).

```
sage: K.<a> = QuadraticField(5)
sage: OK = K.ring_of_integers(); OK
Order with module basis 1/2*a + 1/2, a in Number Field
in a with defining polynomial x^2 - 5
sage: Frac(OK)
Number Field in a with defining polynomial x^2 - 5
```

The fraction field of an *order* – i.e., a subring of $\mathcal{O}_K$ of finite index – is also the number field again.

```
sage: O2 = K.order(2*a); O2
Order with module basis 1, 2*a in Number Field
in a with defining polynomial x^2 - 5
sage: Frac(O2)
Number Field in a with defining polynomial x^2 - 5
```

Definition 3.1.2 (Integrally Closed)**.** An integral domain R is *integrally closed in its field of fractions* if whenever α is in the field of fractions of R and α satisfies a monic polynomial $f \in R[x]$, then $\alpha \in R$.

Proposition 3.1.3. *If K is any number field, then $\mathcal{O}_K$ is integrally closed. Also, the ring $\overline{\mathbf{Z}}$ of all algebraic integers (in a fixed choice of $\overline{\mathbf{Q}}$) is integrally closed.*

Proof. We first prove that $\overline{\mathbf{Z}}$ is integrally closed. Suppose $\alpha \in \overline{\mathbf{Q}}$ is integral over $\overline{\mathbf{Z}}$, so there is a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_i \in \overline{\mathbf{Z}}$ and $f(\alpha) = 0$. The a_i all lie in the ring of integers $\mathcal{O}_K$ of the number field $K = \mathbf{Q}(a_0, a_1, \ldots a_{n-1})$, and $\mathcal{O}_K$ is finitely generated as a $\mathbf{Z}$-module, so $\mathbf{Z}[a_0, \ldots, a_{n-1}]$ is finitely generated as a $\mathbf{Z}$-module. Since $f(\alpha) = 0$, we can write α^n as a $\mathbf{Z}[a_0, \ldots, a_{n-1}]$-linear combination of α^i for $i < n$, so the ring $\mathbf{Z}[a_0, \ldots, a_{n-1}, \alpha]$ is also finitely generated as a $\mathbf{Z}$-module. Thus $\mathbf{Z}[\alpha]$ is finitely generated as $\mathbf{Z}$-module because it is a submodule of a finitely generated $\mathbf{Z}$-module, which implies that α is integral over $\mathbf{Z}$.

Without loss we may assume that $K \subset \overline{\mathbf{Q}}$, so that $\mathcal{O}_K = \overline{\mathbf{Z}} \cap K$. Suppose $\alpha \in K$ is integral over $\mathcal{O}_K$. Then since $\overline{\mathbf{Z}}$ is integrally closed, α is an element of $\overline{\mathbf{Z}}$, so $\alpha \in K \cap \overline{\mathbf{Z}} = \mathcal{O}_K$, as required. □

Definition 3.1.4 (Dedekind Domain)**.** An integral domain R is a *Dedekind domain* if it is noetherian, integrally closed in its field of fractions, and every nonzero prime ideal of R is maximal.

The ring $\mathbf{Z} \oplus \mathbf{Z}$ is not a Dedekind domain because it is not an integral domain. The ring $\mathbf{Z}[\sqrt{5}]$ is not a Dedekind domain because it is not integrally closed in its field of fractions, as $(1 + \sqrt{5})/2$ is integrally over $\mathbf{Z}$ and lies in $\mathbf{Q}(\sqrt{5})$, but not in $\mathbf{Z}[\sqrt{5}]$. The ring $\mathbf{Z}$ is a Dedekind domain, as is any ring of integers $\mathcal{O}_K$ of a number field, as we will see below. Also, any field K is a Dedekind domain, since it is an integral domain, it is trivially integrally closed in itself, and there are no nonzero prime ideals so the condition that they be maximal is empty. The ring $\overline{\mathbf{Z}}$ is not noetherian, but it is integrally closed in its field of fraction, and every nonzero prime ideal is maximal.

Proposition 3.1.5. *The ring of integers $\mathcal{O}_K$ of a number field is a Dedekind domain.*

Proof. By Proposition 3.1.3, the ring $\mathcal{O}_K$ is integrally closed, and by Proposition 2.4.6 it is noetherian. Suppose that $\mathfrak{p}$ is a nonzero prime ideal of $\mathcal{O}_K$. Let $\alpha \in \mathfrak{p}$ be a nonzero element, and let $f(x) \in \mathbf{Z}[x]$ be the minimal polynomial of α. Then

$$f(\alpha) = \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0,$$

so $a_0 = -(\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha) \in \mathfrak{p}$. Since f is irreducible, a_0 is a nonzero element of $\mathbf{Z}$ that lies in $\mathfrak{p}$. Every element of the finitely generated abelian group $\mathcal{O}_K/\mathfrak{p}$ is killed by a_0, so $\mathcal{O}_K/\mathfrak{p}$ is a finite set. Since $\mathfrak{p}$ is prime, $\mathcal{O}_K/\mathfrak{p}$ is an integral domain. Every finite integral domain is a field (see Exercise 10), so $\mathfrak{p}$ is maximal, which completes the proof. □

If I and J are ideals in a ring R, the product IJ is the ideal generated by all products of elements in I with elements in J:

$$IJ = (ab : a \in I, b \in J) \subset R.$$

Note that the set of all products ab, with $a \in I$ and $b \in J$, need not be an ideal, so it is important to take the ideal generated by that set (see Exercise 11).

Definition 3.1.6 (Fractional Ideal)**.** A *fractional ideal* is a nonzero $\mathcal{O}_K$-submodule I of K that is finitely generated as an $\mathcal{O}_K$-module.

We will sometimes call a genuine ideal $I \subset \mathcal{O}_K$ an *integral ideal.* The notion of fractional ideal makes sense for an arbitrary Dedekind domain R – it is an R-module $I \subset K = \mathrm{Frac}(R)$ that is finitely generated as an R-module.

Example 3.1.7. We multiply two fractional ideals in SAGE:

```
sage: K.<a> = NumberField(x^2 + 23)
sage: I = K.fractional_ideal(2, 1/2*a - 1/2)
sage: J = I^2
sage: I
Fractional ideal (2, 1/2*a - 1/2) of Number Field ...
sage: J
Fractional ideal (4, 1/2*a + 3/2) of Number Field ...
sage: I*J
Fractional ideal (-1/2*a - 3/2) of Number Field ...
```

Since fractional ideals I are finitely generated, we can clear denominators of a generating set to see that there is some nonzero $\alpha \in K$ such that

$$\alpha I = J \subset \mathcal{O}_K$$

with J an integral ideal. Thus dividing by α, we see that every fractional ideal is of the form

$$aJ = \{ab : b \in J\}$$

for some $a \in K$ and integral ideal $J \subset \mathcal{O}_K$.

For example, the set $\frac{1}{2}\mathbf{Z}$ of rational numbers with denominator 1 or 2 is a fractional ideal of $\mathbf{Z}$.

Theorem 3.1.8. *The set of fractional ideals of a Dedekind domain R is an abelian group under ideal multiplication with identity element R.*

Note that fractional ideals are nonzero by definition, so it is not necessary to write "nonzero fractional ideals" in the statement of the theorem. We will *only* prove Theorem 3.1.8 in the case when $R = \mathcal{O}_K$ is the ring of integers of a number field K. Before proving Theorem 3.1.8 we prove a lemma. For the rest of this section $\mathcal{O}_K$ is the ring of integers of a number field K.

Definition 3.1.9 (Divides for Ideals)**.** Suppose that I, J are ideals of $\mathcal{O}_K$. Then we say that I *divides* J if $I \supset J$.

To see that this notion of divides is sensible, suppose $K = \mathbf{Q}$, so $\mathcal{O}_K = \mathbf{Z}$. Then $I = (n)$ and $J = (m)$ for some integer n and m, and I divides J means that $(n) \supset (m)$, i.e., that there exists an integer c such that $m = cn$, which exactly means that n divides m, as expected.

Lemma 3.1.10. *Suppose I is a nonzero ideal of $\mathcal{O}_K$. Then there exist prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdots \mathfrak{p}_n \subset I$, i.e., I divides a product of prime ideals.*

Proof. Let S be the set of nonzero ideals of $\mathcal{O}_K$ that do not satisfy the conclusion of the lemma. The key idea is to use that $\mathcal{O}_K$ is noetherian to show that S is the empty set. If S is nonempty, then since $\mathcal{O}_K$ is noetherian, there is an ideal $I \in S$ that is maximal as an element of S. If I were prime, then I would trivially contain a product of primes, so we may assume that I is not prime. Thus there exists $a, b \in \mathcal{O}_K$ such that $ab \in I$ but $a \notin I$ and $b \notin I$. Let $J_1 = I + (a)$ and $J_2 = I + (b)$. Then neither J_1 nor J_2 is in S, since I is maximal, so both J_1 and J_2 contain a product of prime ideals, say $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subset J_1$ and $\mathfrak{q}_1 \cdots \mathfrak{q}_s \subset J_2$. Then

$$\mathfrak{p}_1 \cdots \mathfrak{p}_r \cdot \mathfrak{q}_1 \cdots \mathfrak{q}_s \subset J_1 J_2 = I^2 + I(b) + (a)I + (ab) \subset I,$$

so I contains a product of primes. This is a contradiction, since we assumed $I \in S$. Thus S is empty, which completes the proof. $\square$

We are now ready to prove the theorem.

Proof of Theorem 3.1.8. Note that we will *only* prove Theorem 3.1.8 in the case when $R = \mathcal{O}_K$ is the ring of integers of a number field K.

The product of two fractional ideals is again finitely generated, so it is a fractional ideal, and $I\mathcal{O}_K = I$ for any ideal I, so to prove that the set of fractional ideals under multiplication is a group it suffices to show the existence of inverses. We will first prove that if $\mathfrak{p}$ is a prime ideal, then $\mathfrak{p}$ has an inverse, then we will prove that all nonzero integral ideals have inverses, and finally observe that every fractional ideal has an inverse. (Note: Once we know that the set of fractional ideals is a group, it will follows that inverses are unique; until then we will be careful to write "an" instead of "the".)

Suppose $\mathfrak{p}$ is a nonzero prime ideal of $\mathcal{O}_K$. We will show that the $\mathcal{O}_K$-module

$$I = \{a \in K : a\mathfrak{p} \subset \mathcal{O}_K\}$$

is a fractional ideal of $\mathcal{O}_K$ such that $I\mathfrak{p} = \mathcal{O}_K$, so that I is an inverse of $\mathfrak{p}$.

For the rest of the proof, fix a nonzero element $b \in \mathfrak{p}$. Since I is an $\mathcal{O}_K$-module, $bI \subset \mathcal{O}_K$ is an $\mathcal{O}_K$ ideal, hence I is a fractional ideal. Since $\mathcal{O}_K \subset I$ we have $\mathfrak{p} \subset I\mathfrak{p} \subset \mathcal{O}_K$, hence since $\mathfrak{p}$ is maximal, either $\mathfrak{p} = I\mathfrak{p}$ or $I\mathfrak{p} = \mathcal{O}_K$. If $I\mathfrak{p} = \mathcal{O}_K$, we are done since then I is an inverse of $\mathfrak{p}$. Thus suppose that $I\mathfrak{p} = \mathfrak{p}$. Our strategy is to show that there is some $d \in I$, with $d \notin \mathcal{O}_K$. Since $I\mathfrak{p} = \mathfrak{p}$, such a d would leave $\mathfrak{p}$ invariant, i.e., $d\mathfrak{p} \subset \mathfrak{p}$. Since $\mathfrak{p}$ is an $\mathcal{O}_K$-module we will see that it will follow that $d \in \mathcal{O}_K$, a contradiction.

By Lemma 3.1.10, we can choose a product $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$, with m minimal, with

$$\mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_m \subset (b) \subset \mathfrak{p}.$$

If no $\mathfrak{p}_i$ is contained in $\mathfrak{p}$, then we can choose for each i an $a_i \in \mathfrak{p}_i$ with $a_i \notin \mathfrak{p}$; but then $\prod a_i \in \mathfrak{p}$, which contradicts that $\mathfrak{p}$ is a prime ideal. Thus some $\mathfrak{p}_i$, say $\mathfrak{p}_1$, is contained in $\mathfrak{p}$, which implies that $\mathfrak{p}_1 = \mathfrak{p}$ since every nonzero prime ideal is maximal. Because m is minimal, $\mathfrak{p}_2 \cdots \mathfrak{p}_m$ is not a subset of (b), so there exists $c \in \mathfrak{p}_2 \cdots \mathfrak{p}_m$ that does not lie in (b). Then $\mathfrak{p}(c) \subset (b)$, so by definition of I we have $d = c/b \in I$. However, $d \notin \mathcal{O}_K$, since if it were then c would be in (b). We have thus found our element $d \in I$ that does not lie in $\mathcal{O}_K$.

To finish the proof that $\mathfrak{p}$ has an inverse, we observe that d preserves the $\mathcal{O}_K$-module $\mathfrak{p}$, and is hence in $\mathcal{O}_K$, a contradiction. More precisely, if $b_1, \ldots, b_n$ is a basis for $\mathfrak{p}$ as a $\mathbf{Z}$-module, then the action of d on $\mathfrak{p}$ is given by a matrix with entries in $\mathbf{Z}$, so the minimal polynomial of d has coefficients in $\mathbf{Z}$ (because d satisfies the minimal polynomial of ℓ_d, by the Cayley-Hamilton theorem – here we also use that $\mathbf{Q} \otimes \mathfrak{p} = K$, since $\mathcal{O}_K/\mathfrak{p}$ is a finite set). This implies that d is integral over $\mathbf{Z}$, so $d \in \mathcal{O}_K$, since $\mathcal{O}_K$ is integrally closed by Proposition 3.1.3. (Note how this argument depends strongly on the fact that $\mathcal{O}_K$ is integrally closed!)

So far we have proved that if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$, then

$$\mathfrak{p}^{-1} = \{a \in K : a\mathfrak{p} \subset \mathcal{O}_K\}$$

is the inverse of $\mathfrak{p}$ in the monoid of nonzero fractional ideals of $\mathcal{O}_K$. As mentioned after Definition 3.1.6, every nonzero fractional ideal is of the form aI for $a \in K$ and I an integral ideal, so since (a) has inverse $(1/a)$, it suffices to show that every integral ideal I has an inverse. If not, then there is a nonzero integral ideal I that is maximal among all nonzero integral ideals that do not have an inverse. Every ideal is contained in a maximal ideal, so there is a nonzero prime ideal $\mathfrak{p}$ such that $I \subset \mathfrak{p}$. Multiplying both sides of this inclusion by $\mathfrak{p}^{-1}$ and using that $\mathcal{O}_K \subset \mathfrak{p}^{-1}$, we see that

$$I \subset \mathfrak{p}^{-1}I \subset \mathfrak{p}^{-1}\mathfrak{p} = \mathcal{O}_K.$$

If $I = \mathfrak{p}^{-1}I$, then arguing as in the proof that $\mathfrak{p}^{-1}$ is an inverse of $\mathfrak{p}$, we see that each element of $\mathfrak{p}^{-1}$ preserves the finitely generated $\mathbf{Z}$-module I and is hence integral. But then $\mathfrak{p}^{-1} \subset \mathcal{O}_K$, which, upon multiplying both sides by $\mathfrak{p}$, implies that $\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^{-1} \subset \mathfrak{p}$, a contradiction. Thus $I \neq \mathfrak{p}^{-1}I$. Because I is maximal among ideals that do not have an inverse, the ideal $\mathfrak{p}^{-1}I$ does have an inverse J. Then $\mathfrak{p}^{-1}J$ is an inverse of I, since $(J\mathfrak{p}^{-1})I = J(\mathfrak{p}^{-1}I) = \mathcal{O}_K$. □

We can finally deduce the crucial Theorem 3.1.11, which will allow us to show that any nonzero ideal of a Dedekind domain can be expressed uniquely as a product of primes (up to order). Thus unique factorization holds for ideals in a Dedekind domain, and it is this unique factorization that initially motivated the introduction of ideals to mathematics over a century ago.

Theorem 3.1.11. *Suppose I is a nonzero integral ideal of $\mathcal{O}_K$. Then I can be written as a product*

$$I = \mathfrak{p}_1 \cdots \mathfrak{p}_n$$

of prime ideals of $\mathcal{O}_K$, and this representation is unique up to order.

Proof. Suppose I is an ideal that is maximal among the set of all ideals in $\mathcal{O}_K$ that can not be written as a product of primes. Every ideal is contained in a maximal ideal, so I is contained in a nonzero prime ideal $\mathfrak{p}$. If $I\mathfrak{p}^{-1} = I$, then by Theorem 3.1.8 we can cancel I from both sides of this equation to see that $\mathfrak{p}^{-1} = \mathcal{O}_K$, a contradiction. Since $\mathcal{O}_K \subset \mathfrak{p}^{-1}$, we have $I \subset I\mathfrak{p}^{-1}$, and by the above observation I is strictly contained in $I\mathfrak{p}^{-1}$. By our maximality assumption on I, there are maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that $I\mathfrak{p}^{-1} = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. Then $I = \mathfrak{p} \cdot \mathfrak{p}_1 \cdots \mathfrak{p}_n$, a contradiction. Thus every ideal can be written as a product of primes.

Suppose $\mathfrak{p}_1 \cdots \mathfrak{p}_n = \mathfrak{q}_1 \cdots \mathfrak{q}_m$. If no $\mathfrak{q}_i$ is contained in $\mathfrak{p}_1$, then for each i there is an $a_i \in \mathfrak{q}_i$ such that $a_i \not\in \mathfrak{p}_1$. But the product of the a_i is in $\mathfrak{p}_1 \cdots \mathfrak{p}_n$, which is a subset of $\mathfrak{p}_1$, which contradicts that $\mathfrak{p}_1$ is a prime ideal. Thus $\mathfrak{q}_i = \mathfrak{p}_1$ for some i. We can thus cancel $\mathfrak{q}_i$ and $\mathfrak{p}_1$ from both sides of the equation by multiplying both sides by the inverse. Repeating this argument finishes the proof of uniqueness. □

Theorem 3.1.12. *If I is a fractional ideal of $\mathcal{O}_K$ then there exists prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$, unique up to order, such that*

$$I = (\mathfrak{p}_1 \cdots \mathfrak{p}_n)(\mathfrak{q}_1 \cdots \mathfrak{q}_m)^{-1}.$$

Proof. We have $I = (a/b)J$ for some $a, b \in \mathcal{O}_K$ and integral ideal J. Applying Theorem 3.1.11 to (a), (b), and J gives an expression as claimed. For uniqueness, if one has two such product expressions, multiply through by the denominators and use the uniqueness part of Theorem 3.1.11 □

Example 3.1.13. The ring of integers of $K = \mathbf{Q}(\sqrt{-6})$ is $\mathcal{O}_K = \mathbf{Z}[\sqrt{-6}]$. We have

$$6 = -\sqrt{-6}\sqrt{-6} = 2 \cdot 3.$$

If $ab = \sqrt{-6}$, with $a, b \in \mathcal{O}_K$ and neither a unit, then $\mathrm{Norm}(a)\,\mathrm{Norm}(b) = 6$, so without loss $\mathrm{Norm}(a) = 2$ and $\mathrm{Norm}(b) = 3$. If $a = c + d\sqrt{-6}$, then $\mathrm{Norm}(a) = c^2 + 6d^2$; since the equation $c^2 + 6d^2 = 2$ has no solution with $c, d \in \mathbf{Z}$, there is no element in $\mathcal{O}_K$ with norm 2, so $\sqrt{-6}$ is irreducible. Also, $\sqrt{-6}$ is not a unit times 2 or times 3, since again the norms would not match up. Thus 6 can not be written uniquely as a product of irreducibles in $\mathcal{O}_K$. Theorem 3.1.12, however, implies that the principal ideal (6) can, however, be written uniquely as a product of prime ideals. An explicit decomposition is

$$(6) = (2, 2 + \sqrt{-6})^2 \cdot (3, 3 + \sqrt{-6})^2, \tag{3.1.1}$$

where each of the ideals $(2, 2 + \sqrt{-6})$ and $(3, 3 + \sqrt{-6})$ is prime. We will discuss algorithms for computing such a decomposition in detail in Chapter 4. The first idea is to write $(6) = (2)(3)$, and hence reduce to the case of writing the (p), for $p \in \mathbf{Z}$ prime, as a product of primes. Next one decomposes the finite (as a set) ring $\mathcal{O}_K/p\mathcal{O}_K$.

The factorization (3.1.1) can be compute using SAGE as follows:

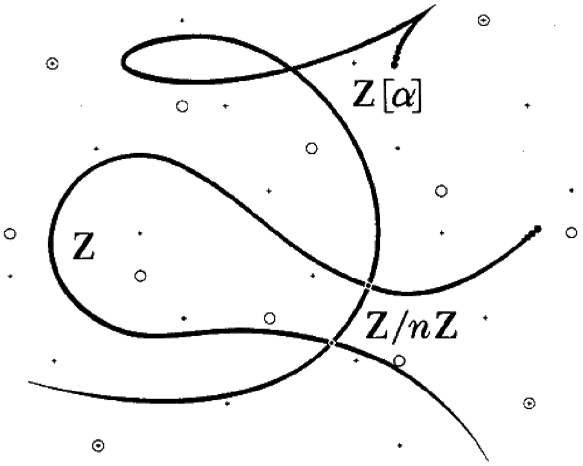
```
sage: K.<a> = NumberField(x^2 + 6); K
Number Field in a with defining polynomial x^2 + 6
sage: K.factor_integer(6)
(Fractional ideal (2, a) of Number Field ...)^2 *
(Fractional ideal (3, a) of Number Field ...)^2
```

Chapter 4

Factoring Primes

Let p be a prime and $\mathcal{O}_K$ the ring of integers of a number field. This chapter is about how to write $p\mathcal{O}_K$ as a product of prime ideals of $\mathcal{O}_K$. Paradoxically, computing the explicit prime ideal factorization of $p\mathcal{O}_K$ is easier than computing $\mathcal{O}_K$.

4.1 The Problem



A diagram from [LL93].

> "The obvious mathematical breakthrough would be development of an easy way to factor large prime numbers."
> – Bill Gates, *The Road Ahead*, 1st ed., pg 265

Bill Gates meant[1] factoring products of two primes, which would break the RSA cryptosystem (see e.g. [Ste09, §3.2]). However, perhaps Gates is an algebraic number theorist, and he really meant what he said: then we might imagine that he meant factorization of primes of $\mathbf{Z}$ in rings of integers of number fields. For example, $2^{16}+1 = 65537$ is a "large" prime, and in $\mathbf{Z}[i]$ we have

$$(65537) = (65537, 2^8 + i) \cdot (65537, 2^8 - i).$$

4.1.1 Geometric Intuition

Let $K = \mathbf{Q}(\alpha)$ be a number field, and let $\mathcal{O}_K$ be the ring of integers of K. To employ our geometric intuition, as the Lenstras did on the cover of [LL93], it is helpful to view $\mathcal{O}_K$ as a 1-dimensional scheme

$$X = \mathrm{Spec}(\mathcal{O}_K) = \{ \text{ all prime ideals of } \mathcal{O}_K \ \}$$

over

$$Y = \mathrm{Spec}(\mathbf{Z}) = \{(0)\} \cup \{p\mathbf{Z} : p \in \mathbf{Z}_{>0} \text{ is prime } \}.$$

There is a natural map $\pi : X \to Y$ that sends a prime ideal $\mathfrak{p} \in X$ to $\mathfrak{p} \cap \mathbf{Z} \in Y$. For example, if

$$\mathfrak{p} = (65537, 2^8 + i) \subset \mathbf{Z}[i],$$

then $\mathfrak{p} \cap \mathbf{Z} = (65537)$. For more on this viewpoint, see [Har77] and [EH00, Ch. 2].

If $p \in \mathbf{Z}$ is a prime number, then the ideal $p\mathcal{O}_K$ of $\mathcal{O}_K$ factors uniquely as a product $\prod \mathfrak{p}_i^{e_i}$, where the $\mathfrak{p}_i$ are maximal ideals of $\mathcal{O}_K$. We may imagine the decomposition of $p\mathcal{O}_K$ into prime ideals geometrically as the fiber $\pi^{-1}(p\mathbf{Z})$, where the exponents e_i are the multiplicities of the fibers. Notice that the elements of $\pi^{-1}(p\mathbf{Z})$ are the prime ideals of $\mathcal{O}_K$ that contain p, i.e., the primes that divide $p\mathcal{O}_K$. This chapter is about how to compute the $\mathfrak{p}_i$ and e_i.

Remark 4.1.1. More technically, in algebraic geometry one defines the inverse image of the point $p\mathbf{Z}$ to be the spectrum of the tensor product $\mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{Z}/\mathfrak{p}\mathbf{Z}$; by a generalization of the Chinese Remainder Theorem, we have

$$\mathcal{O}_K \otimes_{\mathbf{Z}} (\mathbf{Z}/\mathfrak{p}\mathbf{Z}) \cong \oplus \mathcal{O}_K/\mathfrak{p}_i^{e_i}.$$

[1] This quote is on page 265 of the first edition. In the second edition, on page 303, this sentence is changed to "The obvious mathematical breakthrough that would defeat our public key encryption would be the development of an easy way to factor large numbers." This is less nonsensical; however, fast factoring is *not* known to break all commonly used public-key cryptosystem. For example, there are cryptosystems based on the difficulty of computing discrete logarithms in $\mathbf{F}_p^*$ and on elliptic curves over $\mathbf{F}_p$, which (presumably) would not be broken even if one could factor large numbers quickly.

4.1.2 Examples

The following SAGE session shows the commands needed to compute the factorization of $p\mathcal{O}_K$ for K the number field defined by a root of $x^5 + 7x^4 + 3x^2 - x + 1$ and $p = 2$ and 5. We first create an element $f \in \mathbf{Q}[x]$ in SAGE:

```
sage: R.<x> = QQ[]
sage: f = x^5 + 7*x^4 + 3*x^2 - x + 1
```

Then we create the corresponding number field obtained by adjoining a root of f, and find its ring of integers.

```
sage: K.<a> = NumberField(f)
sage: OK = K.ring_of_integers()
sage: OK.basis()
[1, a, a^2, a^3, a^4]
```

We define the ideal $2\mathcal{O}_K$ and factor – it turns out to be prime.

```
sage: I = K.fractional_ideal(2); I
Fractional ideal (2)
sage: I.factor()
Fractional ideal (2)
sage: I.is_prime()
True
```

Finally we factor $5\mathcal{O}_K$, which factors as a product of three primes.

```
sage: I = K.fractional_ideal(5); I
Fractional ideal (5)
sage: I.factor()
(Fractional ideal (5, a^2 + 9*a + 2)) * (Fractional ideal (5, a + 2)) * (Fractional ideal (5, a
```

Notice that the polynomial f factors in a similar way:

```
sage: f.factor_mod(5)
(x + 2) * (x + 3)^2 * (x^2 + 4*x + 2)
```

Thus $2\mathcal{O}_K$ is already a prime ideal, and

$$5\mathcal{O}_K = (5, 2 + a) \cdot (5, 3 + a)^2 \cdot (5, 2 + 4a + a^2).$$

Notice that in this example $\mathcal{O}_K = \mathbf{Z}[a]$. (Warning: There are examples of $\mathcal{O}_K$ such that $\mathcal{O}_K \neq \mathbf{Z}[a]$ for any $a \in \mathcal{O}_K$, as Example 4.3.2 below illustrates.) When $\mathcal{O}_K = \mathbf{Z}[a]$ it is relatively easy to factor $p\mathcal{O}_K$, at least assuming one can factor polynomials in $\mathbf{F}_p[x]$. The following factorization gives a hint as to why:

$$x^5 + 7x^4 + 3x^2 - x + 1 \equiv (x+2)\cdot(x+3)^2\cdot(x^2+4x+2) \pmod{5}.$$

The exponent 2 of $(5, 3+a)^2$ in the factorization of $5\mathcal{O}_K$ above suggests "ramification", in the sense that the cover $X \to Y$ has less points (counting their "size", i.e., their residue class degree) in its fiber over 5 than it has generically:

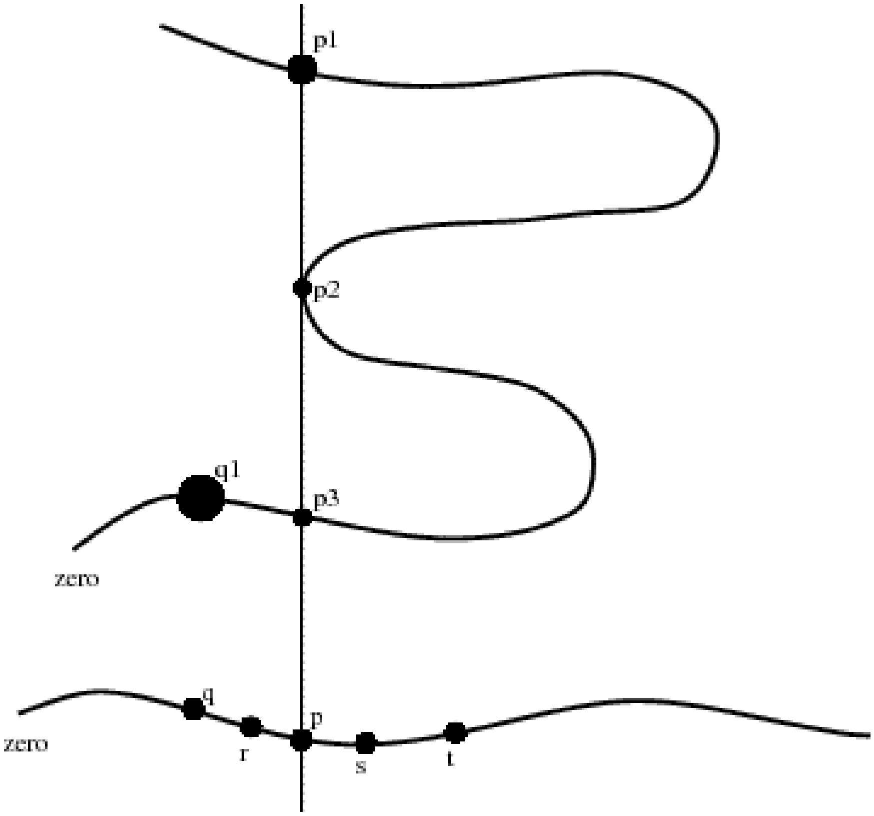


Diagram of $\mathrm{Spec}(\mathcal{O}_K) \to \mathrm{Spec}(\mathbf{Z})$

4.2 A Method for Factoring Primes that Often Works

Suppose $a \in \mathcal{O}_K$ is such that $K = \mathbf{Q}(a)$, and let $f(x) \in \mathbf{Z}[x]$ be the minimal polynomial of a. Then $\mathbf{Z}[a] \subset \mathcal{O}_K$, and we have a diagram of schemes

$$\begin{tikzcd}
\bigcup \mathrm{Spec}(\mathcal{O}_K/\mathfrak{p}_i^{e_i}) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(\mathcal{O}_K) \arrow[d] \\
\bigcup \mathrm{Spec}(\mathbf{F}_p[x]/(\overline{f}_i^{e_i})) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(\mathbf{Z}[a]) \arrow[d] \\
\mathrm{Spec}(\mathbf{F}_p) \arrow[r, hook] & \mathrm{Spec}(\mathbf{Z})
\end{tikzcd}$$

where $\overline{f} = \prod_i \overline{f}_i^{e_i}$ is the factorization of the image of f in $\mathbf{F}_p[x]$, and $p\mathcal{O}_K = \prod \mathfrak{p}_i^{e_i}$ is the factorization of $p\mathcal{O}_K$ in terms of prime ideals of $\mathcal{O}_K$. On the level of rings, the bottom horizontal map is the quotient map $\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \cong \mathbf{F}_p$. The middle horizontal map is induced by

$$\mathbf{Z}[x] \to \bigoplus_i \mathbf{F}_p[x]/(\overline{f}_i^{e_i}),$$

and the top horizontal map is induced by

$$\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K \cong \bigoplus \mathcal{O}_K/\mathfrak{p}_i^{e_i},$$

where the isomorphism is by the Chinese Remainder Theorem, which we will prove in Chapter 5. The left vertical maps come from the inclusions

$$\mathbf{F}_p \hookrightarrow \mathbf{F}_p[x]/(\overline{f}_i^{e_i}) \hookrightarrow \mathcal{O}_K/\mathfrak{p}_i^{e_i},$$

and the right from the inclusions $\mathbf{Z} \hookrightarrow \mathbf{Z}[a] \hookrightarrow \mathcal{O}_K$.

The cover $\pi : \mathrm{Spec}(\mathbf{Z}[a]) \to \mathrm{Spec}(\mathbf{Z})$ is easy to understand because it is defined by the single equation $f(x)$, in the sense that $\mathbf{Z}[a] \cong \mathbf{Z}[x]/(f(x))$. To give a maximal ideal $\mathfrak{p}$ of $\mathbf{Z}[a]$ such that $\pi(\mathfrak{p}) = p\mathbf{Z}$ is the same as giving a homomorphism $\varphi : \mathbf{Z}[x]/(f) \to \overline{\mathbf{F}}_p$ up to automorphisms of the image, which is in turn the same as giving a root of f in $\overline{\mathbf{F}}_p$ up to automorphism, which is the same as giving an irreducible factor of the reduction of f modulo p.

Lemma 4.2.1. *Suppose the index of $\mathbf{Z}[a]$ in $\mathcal{O}_K$ is coprime to p. Then the primes $\mathfrak{p}_i$ in the factorization of $p\mathbf{Z}[a]$ do not decompose further going from $\mathbf{Z}[a]$ to $\mathcal{O}_K$, so finding the prime ideals of $\mathbf{Z}[a]$ that contain p yields the primes that appear in the factorization of $p\mathcal{O}_K$.*

Proof. Fix a basis for $\mathcal{O}_K$ and for $\mathbf{Z}[a]$ as $\mathbf{Z}$-modules. Form the matrix A whose columns express each basis element of $\mathbf{Z}[a]$ as a $\mathbf{Z}$-linear combination of the basis for $\mathcal{O}_K$. Then

$$\det(A) = \pm[\mathcal{O}_K : \mathbf{Z}[a]]$$

is coprime to p, by hypothesis. Thus the reduction of A modulo p is invertible, so it defines an isomorphism $\mathbf{Z}[a]/p\mathbf{Z}[a] \cong \mathcal{O}_K/p\mathcal{O}_K$.

Let $\overline{\mathbf{F}}_p$ denote a fixed algebraic closure of $\mathbf{F}_p$; thus $\overline{\mathbf{F}}_p$ is an algebraically closed field of characteristic p, over which all polynomials in $\mathbf{F}_p[x]$ factor into linear factors. Any homomorphism $\mathcal{O}_K \to \overline{\mathbf{F}}_p$ sends p to 0, so is the composition of a homomorphism $\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K$ with a homomorphism $\mathcal{O}_K/p\mathcal{O}_K \to \overline{\mathbf{F}}_p$. Since $\mathcal{O}_K/p\mathcal{O}_K \cong \mathbf{Z}[a]/p\mathbf{Z}[a]$, the homomorphisms $\mathcal{O}_K \to \overline{\mathbf{F}}_p$ are in bijection with the homomorphisms $\mathbf{Z}[a] \to \overline{\mathbf{F}}_p$. The homomorphisms $\mathbf{Z}[a] \to \overline{\mathbf{F}}_p$ are in bijection with the roots of the reduction modulo p of the minimal polynomial of a in $\overline{\mathbf{F}}_p$. □

Remark 4.2.2. Here is a "high-brow" proof of Lemma 4.2.1. By hypothesis we have an exact sequence of abelian groups

$$0 \to \mathbf{Z}[a] \to \mathcal{O}_K \to H \to 0,$$

where H is a finite abelian group of order coprime to p. Tensor product is right exact, and there is an exact sequence

$$\mathrm{Tor}_1(H, \mathbf{F}_p) \to \mathbf{Z}[a] \otimes \mathbf{F}_p \to \mathcal{O}_K \otimes \mathbf{F}_p \to H \otimes \mathbf{F}_p \to 0,$$

and $\mathrm{Tor}_1(H, \mathbf{F}_p) = 0$ (since H has no p-torsion), so $\mathbf{Z}[a] \otimes \mathbf{F}_p \cong \mathcal{O}_K \otimes \mathbf{F}_p$.

As suggested in the proof of the lemma, we find all homomorphisms $\mathcal{O}_K \to \overline{\mathbf{F}}_p$ by finding all homomorphism $\mathbf{Z}[a] \to \overline{\mathbf{F}}_p$. In terms of ideals, if $\mathfrak{p} = (f(a), p)\mathbf{Z}[a]$ is a maximal ideal of $\mathbf{Z}[a]$, then the ideal $\mathfrak{p}' = (f(a), p)\mathcal{O}_K$ of $\mathcal{O}_K$ is also maximal, since

$$\mathcal{O}_K/\mathfrak{p}' \cong (\mathcal{O}_K/p\mathcal{O}_K)/(f(\tilde{a})) \cong (\mathbf{Z}[a]/p\mathbf{Z}[a])/(f(\tilde{a})) \subset \overline{\mathbf{F}}_p,$$

where $\tilde{a}$ denotes the image of a in $\mathcal{O}_K/p\mathcal{O}_K$.

We formalize the above discussion in the following theorem (note: we will not prove that the powers are e_i here):

Theorem 4.2.3. *Let $f \in \mathbf{Z}[x]$ be the minimal polynomial of a over $\mathbf{Z}$. Suppose that $p \nmid [\mathcal{O}_K : \mathbf{Z}[a]]$ is a prime. Let*

$$\overline{f} = \prod_{i=1}^{t} \overline{f}_i^{e_i} \in \mathbf{F}_p[x]$$

where the $\overline{f}_i$ are distinct monic irreducible polynomials. Let $\mathfrak{p}_i = (p, f_i(a))$ where $f_i \in \mathbf{Z}[x]$ is a lift of $\overline{f}_i$ in $\mathbf{F}_p[x]$. Then

$$p\mathcal{O}_K = \prod_{i=1}^{t} \mathfrak{p}_i^{e_i}.$$

We return to the example from above, in which $K = \mathbf{Q}(a)$, where a is a root of $f = x^5 + 7x^4 + 3x^2 - x + 1$. According to SAGE, the ring of integers $\mathcal{O}_K$ has discriminant $2945785 = 5 \cdot 353 \cdot 1669$:

```
sage: K.<a> = NumberField(x^5 + 7*x^4 + 3*x^2 - x + 1)
sage: D = K.discriminant(); D
2945785
sage: factor(D)
5 * 353 * 1669
```

The order $\mathbf{Z}[a]$ has the same discriminant as $f(x)$, which is the same as the discriminant of $\mathcal{O}_K$, so $\mathbf{Z}[a] = \mathcal{O}_K$ and we can apply the above theorem. (Here we use that the index of $\mathbf{Z}[a]$ in $\mathcal{O}_K$ is the square of the quotient of their discriminants, a fact we will prove later in Section 6.2.)

```
sage: R.<x> = QQ[]
sage: discriminant(x^5 + 7*x^4 + 3*x^2 - x + 1)
2945785
```

We have

$$x^5 + 7x^4 + 3x^2 - x + 1 \equiv (x + 2) \cdot (x + 3)^2 \cdot (x^2 + 4x + 2) \pmod 5,$$

which yields the factorization of $5\mathcal{O}_K$ given before the theorem.

If we replace a by $b = 7a$, then the index of $\mathbf{Z}[b]$ in $\mathcal{O}_K$ will be a power of 7, which is coprime to 5, so the above method will still work.

```
sage: K.<a> = NumberField(x^5 + 7*x^4 + 3*x^2 - x + 1)
sage: f = (7*a).minpoly(’x’)
sage: f
x^5 + 49*x^4 + 1029*x^2 - 2401*x + 16807
sage: f.disc()
235050861175510968365785
sage: factor(f.disc() / K.disc())
7^20
sage: f.factor_mod(5)
(x + 4) * (x + 1)^2 * (x^2 + 3*x + 3)
```

Thus 5 factors in $\mathcal{O}_K$ as

$$5\mathcal{O}_K = (5, 7a + 1)^2 \cdot (5, 7a + 4) \cdot (5, (7a)^2 + 3(7a) + 3).$$

If we replace a by $b = 5a$ and try the above algorithm with $\mathbf{Z}[b]$, then the method fails because the index of $\mathbf{Z}[b]$ in $\mathcal{O}_K$ is divisible by 5.

```
sage: K.<a> = NumberField(x^5 + 7*x^4 + 3*x^2 - x + 1)
sage: f = (5*a).minpoly('x')
sage: f
x^5 + 35*x^4 + 375*x^2 - 625*x + 3125
sage: f.factor_mod(5)
x^5
```

4.3 A General Method

There are numbers fields K such that $\mathcal{O}_K$ is not of the form $\mathbf{Z}[a]$ for any $a \in K$. Even worse, Dedekind found a field K such that $2 \mid [\mathcal{O}_K : \mathbf{Z}[a]]$ for *all* $a \in \mathcal{O}_K$, so there is no choice of a such that Theorem 4.2.3 can be used to factor 2 for K (see Example 4.3.2 below).

4.3.1 Inessential Discriminant Divisors

Definition 4.3.1. A prime p is an *inessential discriminant divisor* if $p \mid [\mathcal{O}_K : \mathbf{Z}[a]]$ for *every* $a \in \mathcal{O}_K$.

See Example 6.2.6 below for why it is called an inessential "discriminant divisor" instead of an inessential "index divisor".

Since $[\mathcal{O}_K : \mathbf{Z}[a]]^2$ is the absolute value of $\mathrm{Disc}(f(x))/\mathrm{Disc}(\mathcal{O}_K)$, where $f(x)$ is the characteristic polynomial of $f(x)$, an inessential discriminant divisor divides the discriminant of the characteristic polynomial of any element of $\mathcal{O}_K$.

Example 4.3.2 (Dedekind). Let $K = \mathbf{Q}(a)$ be the cubic field defined by a root a of the polynomial $f = x^3 + x^2 - 2x + 8$. We will use SAGE to show that 2 is an inessential discriminant divisor for K.

```
sage: K.<a> = NumberField(x^3 + x^2 - 2*x + 8); K
Number Field in a with defining polynomial x^3 + x^2 - 2*x + 8
sage: K.factor_integer(2)
(Fractional ideal (1/2*a^2 - 1/2*a + 1)) *
(Fractional ideal (a^2 - 2*a + 3)) *
(Fractional ideal (3/2*a^2 - 5/2*a + 4))
```

Thus $2\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$, with the $\mathfrak{p}_i$ distinct, and one sees directly from the above expressions that $\mathcal{O}_K/\mathfrak{p}_i \cong \mathbf{F}_2$ for each i. If $\mathcal{O}_K = \mathbf{Z}[a]$ for some $a \in \mathcal{O}_K$ with minimal polynomial f, then $\overline{f}(x) \in \mathbf{F}_2[x]$ must be a product of three *distinct* linear factors, which is impossible, since the only linear polynomials in $\mathbf{F}_2[x]$ are x and $x + 1$.

4.3.2 Remarks on Ideal Factorization in General

Recall (from Definition 2.3.13) that an *order* in $\mathcal{O}_K$ is a subring $\mathcal{O}$ of $\mathcal{O}_K$ that has finite index in $\mathcal{O}_K$. For example, if $\mathcal{O}_K = \mathbf{Z}[i]$, then $\mathcal{O} = \mathbf{Z} + 5\mathbf{Z}[i]$ is an order in $\mathcal{O}_K$, and as an abelian group $\mathcal{O}_K/\mathcal{O}$ is cyclic of order 5.

Most algebraic number theory books do not describe an algorithm for decomposing primes in the general case. Fortunately, Cohen's book [Coh93, Ch. 6] does describe how to solve the general problem, in more than one way. The algorithms are nontrivial, and occupy a substantial part of Chapter 6 of Cohen's book. Our goal for the rest of this section is to give a hint as to what goes into them.

The general solutions to prime ideal factorization are somewhat surprising, since the algorithms are much more sophisticated than the one suggested by Theorem 4.2.3. However, these complicated algorithms all run very quickly in practice, even without assuming the maximal order is already known. In fact, they avoid computing $\mathcal{O}_K$ altogether, and instead compute only an order $\mathcal{O}$ that is *p-maximal*, i.e., is such that $p \nmid [\mathcal{O}_K : \mathcal{O}]$.

For simplicity we consider the following slightly easier problem whose solution illustrates the key ideas needed in the general case.

Problem 4.3.3. Let $\mathcal{O}$ be any order in $\mathcal{O}_K$ and let p be a prime of $\mathbf{Z}$. Find the prime ideals of $\mathcal{O}$ that contain p.

Given a prime p that we wish to factor in $\mathcal{O}_K$, we first find a p-maximal order $\mathcal{O}$. We then use a solution to Problem 4.3.3 to find the prime ideals $\mathfrak{p}$ of $\mathcal{O}$ that contain p. Second, we find the exponents e such that $\mathfrak{p}^e$ exactly divides $p\mathcal{O}$. The resulting factorization in $\mathcal{O}$ completely determines the factorization of $p\mathcal{O}_K$.

A p-maximal order can be found reasonably quickly in practice using algorithms called "round 2" and "round 4". To compute $\mathcal{O}_K$, given an order $\mathbf{Z}[\alpha] \subset \mathcal{O}_K$, one takes a sum of p-maximal orders, one for every p such that p^2 divides $\mathrm{Disc}(\mathbf{Z}[\alpha])$. The time-consuming part of this computation is finding the primes p such that $p^2 \mid \mathrm{Disc}(\mathbf{Z}[\alpha])$, not finding the p-maximal orders. This example illustrates that a fast algorithm for factoring integers would not only break the RSA cryptosystems, but would massively speed up computation of the ring of integers of a number field.

Remark 4.3.4. The MathSciNet review of [BL94] by J. Buhler contains the following:

> A result of Chistov says that finding the ring of integers $\mathcal{O}_K$ in an algebraic number field K is equivalent, under certain polynomial time reductions, to the problem of finding the largest squarefree divisor of a positive integer. No feasible (i.e., polynomial time) algorithm is known for the latter problem, and it is possible that it is no easier than the more general problem of factoring integers.

Thus it appears that computing the ring $\mathcal{O}_K$ is quite hard.

4.3.3 Finding a p-Maximal Order

Before describing the general factorization algorithm, we sketch some of the theory behind the general algorithms for computing a p-maximal order $\mathcal{O}$ in $\mathcal{O}_K$. The main input is the following theorem:

Theorem 4.3.5 (Pohst-Zassenhaus). *Let $\mathcal{O}$ be an order in the the ring of integers $\mathcal{O}_K$ of a number field, let $p \in \mathbf{Z}$ be a prime, and let*

$$I_p = \{x \in \mathcal{O} : x^m \in p\mathcal{O} \text{ for some } m \geq 1\ \} \subset \mathcal{O}$$

be the radical of $p\mathcal{O}$, which is an ideal of $\mathcal{O}$. Let

$$\mathcal{O}' = \{x \in K : xI_p \subset I_p\}.$$

Then $\mathcal{O}'$ is an order and either $\mathcal{O}' = \mathcal{O}$, in which case $\mathcal{O}$ is p-maximal, or $\mathcal{O} \subset \mathcal{O}'$ and p divides $[\mathcal{O}' : \mathcal{O}]$.

Proof. We prove here only that $[\mathcal{O}' : \mathcal{O}] \mid p^n$, where n is the degree of K. We have $p \in I_p$, so if $x \in \mathcal{O}'$, then $xp \in I_p \subset \mathcal{O}$, which implies that $x \in \frac{1}{p}\mathcal{O}$. Since $(\frac{1}{p}\mathcal{O})/\mathcal{O}$ is of order p^n, the claim follows.

To complete the proof, we would show that if $\mathcal{O}' = \mathcal{O}$, then $\mathcal{O}$ is already p-maximal. See [Coh93, §6.1.1] for the rest if this proof. □

After deciding on how to represent elements of K and orders and ideals in K, one can give an efficient algorithm to compute the $\mathcal{O}'$ of the theorem. The algorithm mainly involves linear algebra over finite fields. It is complicated to describe, but efficient in practice, and is conceptually simple—just compute $\mathcal{O}'$. The trick for reducing the computation of $\mathcal{O}'$ to linear algebra is the following lemma:

Lemma 4.3.6. *Define a homomorphism $\psi : \mathcal{O} \hookrightarrow \mathrm{End}(I_p/pI_p)$ given by sending $\alpha \in \mathcal{O}$ to left multiplication by the reduction of α modulo p. Then*

$$\mathcal{O}' = \frac{1}{p}\,\mathrm{Ker}(\psi).$$

Proof. If $x \in \mathcal{O}'$, then $xI_p \subset I_P$, so $\psi(x)$ is the 0 endomorphism. Conversely, if $\psi(x)$ acts as 0 on I_p/pI_p, then clearly $xI_p \subset I_p$. □

Note that to give an algorithm one must also figure out how to explicitly compute I_p/pI_p and the kernel of this map (see the next section for more details).

4.3.4 General Factorization Algorithm of Buchman-Lenstra

We finally give an algorithm to factor $p\mathcal{O}_K$ in general. This is a summary of the algorithm described in more detail in [Coh93, §6.2].

Algorithm 4.3.7 (Factoring a Finite Separable Algebra). Let A be a finite separable algebra over $\mathbf{F}_p$. This algorithm either shows that A is a field or finds a nontrivial idempotent in A, i.e., an $\varepsilon \in A$ such that $\varepsilon^2 = \varepsilon$ with $\varepsilon \neq 0$ and $\varepsilon \neq 1$.

1. The dimension of the kernel V of the map $x \mapsto x^p - x$ is equal to k. This is because abstractly we have that $A \approx A_1 \times \cdots \times A_k$, with each A_i a finite field extension of $\mathbf{F}_p$.

2. If $k = 1$ we are done. Terminate.

3. Otherwise, choose $\alpha \in V$ with $\alpha \not\in \mathbf{F}_p$. (Think of $\mathbf{F}_p$ as the diagonal embedding of $\mathbf{F}_p$ in $A_1 \times \cdots \times A_k$). Compute powers of α and find the minimal polynomial $m(X)$ of α.

4. Since $V \approx \mathbf{F}_p \times \cdots \times F_p$ (k factors), the polynomial $m(X)$ is a square-free product of linear factors, that has degree > 1 since $\alpha \not\in \mathbf{F}_p$. Thus we can compute a splitting $m(X) = m_1(X) \cdot m_2(X)$, where both $m_i(X)$ have positive degree.

5. Use the Euclidean algorithm in $\mathbf{F}_p[X]$ to find $U_1(X)$ and $U_2(X)$ such that
$$U_1 m_1 + U_2 m_2 = 1.$$

6. Let $\varepsilon = (U_1 m_1)(\alpha)$. Then we have
$$U_1 m_1 U_1 m_1 + U_2 m_2 U_1 m_1 = U_1 m_1,$$
so since $(m_1 m_2)(\alpha) = m(\alpha) = 01$, we have $\varepsilon^2 = \varepsilon$. Also, since $\gcd(U_1, m_2) = \gcd(U_2, m_1) = 1$, we have $\varepsilon \neq 0$ and $\varepsilon \neq 1$.

Given Algorithm 4.3.7, we compute an idempotent $\varepsilon \in A$, and observe that
$$A \cong \mathrm{Ker}(1 - \varepsilon) \oplus \mathrm{Ker}(\varepsilon).$$
Since $(1 - \varepsilon) + \varepsilon = 1$, we see that $(1 - \varepsilon)v + \varepsilon v = v$, so that the sume of the two kernels equals A. Also, if v is in the intersection of the two kernels, then $\varepsilon(v) = 0$ and $(1 - \varepsilon)(v) = 0$, so $0 = (1 - \varepsilon)(v) = v - \varepsilon(v) = v$, so the sum is direct.

Remark 4.3.8. The beginning of [Coh93, §6.2.4] suggests that one can just randomly find an $\alpha \in A$ such that $A \cong \mathbf{F}_p[x]/(m(x))$ where m is the minimal polynomial of α. This is usually the case, but is *wrong in general*, since there need *not* be an $\alpha \in A$ such that $A \cong \mathbf{F}_p[\alpha]$. For example, let $p = 2$ and K be as in Example 4.3.2. Then $A \cong \mathbf{F}_2 \times \mathbf{F}_2 \times \mathbf{F}_2$, which as a ring is not generated by a single element, since there are only 2 distinct linear polynomials over $\mathbf{F}_2[x]$.

Algorithm 4.3.9 (Factoring a General Prime Ideal)**.** Let $K = \mathbf{Q}(a)$ be a number field given by an algebraic integer a as a root of its minimal monic polynomial f of degree n. We assume that an order $\mathcal{O}$ has been given by a basis $w_1, \ldots, w_n$ and that $\mathcal{O}$ that contains $\mathbf{Z}[a]$. For any prime $p \in \mathbf{Z}$, the following algorithm computes the set of maximal ideals of $\mathcal{O}$ that contain p.

1. [Check if easy] If $p \nmid \mathrm{disc}(\mathbf{Z}[a])/\mathrm{disc}(\mathcal{O})$ (so $p \nmid [\mathcal{O} : \mathbf{Z}[a]]$), then using Theorem 4.2.3 we factor $p\mathcal{O}$.

2. [Compute radical] Let I be the *radical* of $p\mathcal{O}$, which is the ideal of elements $x \in \mathcal{O}$ such that $x^m \in p\mathcal{O}$ for some positive integer m. Note that $p\mathcal{O} \subset I$, i.e., $I \mid p\mathcal{O}$; also I is the product of the primes that divide p, without multiplicity. Using linear algebra over the finite field $\mathbf{F}_p$, we compute a basis for $I/p\mathcal{O}$ by computing the abelian subgroup of $\mathcal{O}/p\mathcal{O}$ of all nilpotent elements. This computes I, since $p\mathcal{O} \subset I$.

3. [Compute quotient by radical] Compute an $\mathbf{F}_p$ basis for

$$A = \mathcal{O}/I = (\mathcal{O}/p\mathcal{O})/(I/p\mathcal{O}).$$

The second equality comes from the fact that $p\mathcal{O} \subset I$. Note that $\mathcal{O}/p\mathcal{O}$ is obtained by simply reducing the basis $w_1, \ldots, w_n$ modulo p. Thus this step entirely involves linear algebra modulo p.

4. [Decompose quotient] The ring A is isomorphic to the quotient of $\mathcal{O}$ by a radical ideal, so it decomposes as a product $A \cong A_1 \times \cdots \times A_k$ of finite fields. We find such a decomposition explicitly using Algorithm 4.3.7.

5. [Compute the maximal ideals over p] Each maximal ideal $\mathfrak{p}_i$ lying over p is the kernel of one of the compositions

$$\mathcal{O} \to A \approx A_1 \times \cdots \times A_k \to A_i.$$

Algorithm 4.3.9 finds all primes of $\mathcal{O}$ that contain the radical I of $p\mathcal{O}$. Every such prime clearly contains p, so to see that the algorithm is correct, we prove that the primes $\mathfrak{p}$ of $\mathcal{O}$ that contain p also contain I. If $\mathfrak{p}$ is a prime of $\mathcal{O}$ that contains p, then $p\mathcal{O} \subset \mathfrak{p}$. If $x \in I$ then $x^m \in p\mathcal{O}$ for some m, so $x^m \in \mathfrak{p}$ which implies that $x \in \mathfrak{p}$ by primality of $\mathfrak{p}$. Thus $\mathfrak{p}$ contains I, as required. Note that we do not find the powers of primes that divide p in Algorithm 4.3.9; that's left to another algorithm that we will not discuss in this book.

Algorithm 4.3.9 was invented by J. Buchmann and H. W. Lenstra, though their paper seems to have never been published; however, the algorithm is described in detail in [Coh93, §6.2.5]. Incidentally, this chapter is based on Chapters 4 and 6 of [Coh93], which is highly recommended, and goes into much more detail about these algorithms.

Chapter 5

The Chinese Remainder Theorem

We prove the Chinese Remainder Theorem (CRT) for commutative rings and discuss how to compute with it. We also apply the Chinese Remainder Theorem to prove that every ideal in $\mathcal{O}_K$ is generated by two elements and determine the structure of $\mathfrak{p}^n/\mathfrak{p}^{n+1}$, where $\mathfrak{p}$ is a nonzero prime ideal of $\mathcal{O}_K$.

5.1 The Chinese Remainder Theorem

5.1.1 CRT in the Integers

The Chinese Remainder Theorem from elementary number theory asserts that if $n_1, \ldots, n_r$ are integers that are coprime in pairs, and $a_1, \ldots, a_r$ are integers, then there exists an integer a such that $a \equiv a_i \pmod{n_i}$ for each $i = 1, \ldots, r$. Here "coprime in pairs" means that $\gcd(n_i, n_j) = 1$ whenever $i \neq j$; it does *not* mean that $\gcd(n_1, \ldots, n_r) = 1$, though it implies this. In terms of rings, the Chinese Remainder Theorem (CRT) asserts that the natural map

$$\mathbf{Z}/(n_1 \cdots n_r)\mathbf{Z} \to (\mathbf{Z}/n_1\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/n_r\mathbf{Z}) \tag{5.1.1}$$

that sends $a \in \mathbf{Z}$ to its reduction modulo each n_i, is an isomorphism.

This map is *not* an isomorphism if the n_i are not coprime. Indeed, the cardinality of the image of the left hand side of (5.1.1) is $\mathrm{lcm}(n_1, \ldots, n_r)$, since it is the image of a cyclic group and $\mathrm{lcm}(n_1, \ldots, n_r)$ is the largest order of an element of the right hand side, whereas the cardinality of the right hand side is $n_1 \cdots n_r$.

The isomorphism (5.1.1) can alternatively be viewed as asserting that any system of linear congruences

$$x \equiv a_1 \pmod{n_1}, \quad x \equiv a_2 \pmod{n_2}, \quad \cdots, x \equiv a_r \pmod{n_r}$$

with pairwise coprime moduli has a unique solution modulo $n_1 \ldots n_r$.

Before proving the CRT in more generalize, we prove (5.1.1). There is a natural map

$$\phi : \mathbf{Z} \to (\mathbf{Z}/n_1\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/n_r\mathbf{Z})$$

given by projection onto each factor. It's kernel is

$$n_1\mathbf{Z} \cap \cdots \cap n_r\mathbf{Z}.$$

If n and m are integers, then $n\mathbf{Z} \cap m\mathbf{Z}$ is the set of multiples of both n and m, so $n\mathbf{Z} \cap m\mathbf{Z} = \mathrm{lcm}(n, m)\mathbf{Z}$. Since the n_i are coprime,

$$n_1\mathbf{Z} \cap \cdots \cap n_r\mathbf{Z} = n_1 \ldots n_r\mathbf{Z}.$$

Thus we have proved there is an inclusion

$$i : \mathbf{Z}/(n_1 \cdots n_r)\mathbf{Z} \hookrightarrow (\mathbf{Z}/n_1\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/n_r\mathbf{Z}). \tag{5.1.2}$$

This is half of the CRT; the other half is to prove that this map is surjective. In this case, it is clear that i is also surjective, because i is an injective map between sets of the same cardinality. We will, however, give a proof of surjectivity that doesn't use finiteness of the above two sets.

To prove surjectivity of i, note that since the n_i are coprime in pairs,

$$\gcd(n_1, n_2 \ldots n_r) = 1,$$

so there exists integers x, y such that

$$xn_1 + yn_2 \cdots n_r = 1.$$

To complete the proof, observe that $yn_2 \cdots n_r = 1 - xn_1$ is congruent to 1 modulo n_1 and 0 modulo $n_2 \cdots n_r$. Thus $(1, 0, \ldots, 0) = i(yn_2 \cdots n_r)$ is in the image of i. By a similar argument, we see that $(0, 1, \ldots, 0)$ and the other similar elements are all in the image of i, so i is surjective, which proves CRT.

5.1.2 CRT in General

Recall that *all rings in this book are commutative with unity.*

Definition 5.1.1 (Coprime)**.** Ideals I and J are *coprime* if $I + J = (1)$.

If I and J are nonzero ideals in the ring of integers of a number field, then they are coprime precisely when the prime ideals that appear in their two (unique) factorizations are disjoint.

Lemma 5.1.2. *If I and J are coprime ideals in a ring R, then $I \cap J = IJ$.*

Proof. Choose $x \in I$ and $y \in J$ such that $x + y = 1$. If $c \in I \cap J$ then

$$c = c \cdot 1 = c \cdot (x + y) = cx + cy \in IJ + IJ = IJ,$$

so $I \cap J \subset IJ$. The other inclusion is obvious by definition of ideal. □

Lemma 5.1.3. *Suppose $I_1, \ldots, I_s$ are pairwise coprime ideals. Then I_1 is coprime to the product $I_2 \cdots I_s$.*

Proof. It suffices to prove the lemma in the case $s = 3$, since the general case then follows from induction. By assumption, there are $x_1 \in I_1, y_2 \in I_2$ and $a_1 \in I_1, b_3 \in I_3$ such

$$x_1 + y_2 = 1 \qquad \text{and} \qquad a_1 + b_3 = 1.$$

Multiplying these two relations yields

$$x_1 a_1 + x_1 b_3 + y_2 a_1 + y_2 b_3 = 1 \cdot 1 = 1.$$

The first three terms are in I_1 and the last term is in $I_2 I_3 = I_2 \cap I_3$ (by Lemma 5.1.2), so I_1 is coprime to $I_2 I_3$. □

Next we prove the general Chinese Remainder Theorem. We will apply this result with $R = \mathcal{O}_K$ in the rest of this chapter.

Theorem 5.1.4 (Chinese Remainder Theorem)**.** *Suppose $I_1, \ldots, I_r$ are nonzero ideals of a ring R such I_m and I_n are coprime for any $m \neq n$. Then the natural homomorphism $R \to \bigoplus_{n=1}^{r} R/I_n$ induces an isomorphism*

$$\psi : R/\prod_{n=1}^{r} I_n \to \bigoplus_{n=1}^{r} R/I_n.$$

Thus given any $a_n \in R$, for $n = 1, \ldots, r$, there exists some $a \in R$ such that $a \equiv a_n \pmod{I_n}$ for $n = 1, \ldots, r$; moreover, a is unique modulo $\prod_{n=1}^{r} I_n$.

Proof. Let $\varphi : R \to \bigoplus_{n=1}^{r} R/I_n$ be the natural map induced by reduction modulo the I_n. An inductive application of Lemma 5.1.2 implies that the kernel $\cap_{n=1}^{r} I_n$ of φ is equal to $\prod_{n=1}^{r} I_n$, so the map ψ of the theorem is injective.

Each projection $R \to R/I_n$ is surjective, so to prove that ψ is surjective, it suffices to show that $(1, 0, \ldots, 0)$ is in the image of φ, and similarly for the other factors. By Lemma 5.1.3, $J = \prod_{n=2}^{r} I_n$ is coprime to I_1, so there exists $x \in I_1$ and $y \in J$ such that $x + y = 1$. Then $y = 1 - x$ maps to 1 in R/I_1 and to 0 in R/J, hence to 0 in R/I_n for each $n \geq 2$, since $J \subset I_n$. □

5.2 Structural Applications of the CRT

The next lemma is an application of the Chinese Remainder Theorem. We will use it to prove that every ideal of $\mathcal{O}_K$ can be generated by two elements. Suppose that I is a nonzero integral ideals of $\mathcal{O}_K$. If $a \in I$, then $(a) \subset I$, so I divides (a) and the quotient $(a)I^{-1}$ is an integral ideal. The following lemma asserts that (a) can be chosen so the quotient $(a)I^{-1}$ is coprime to any given ideal.

Lemma 5.2.1. *If I and J are nonzero integral ideals in $\mathcal{O}_K$, then there exists an $a \in I$ such that the integral ideal $(a)I^{-1}$ is coprime to J.*

Before we give the proof in general, note that the lemma is trivial when I is principal, since if $I=(b)$, just take $a=b$, and then $(a)I^{-1}=(a)(a^{-1})=(1)$ is coprime to every ideal.

Proof. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ be the prime divisors of J. For each n, let v_n be the largest power of $\mathfrak{p}_n$ that divides I. Since $\mathfrak{p}_n^{v_n}\neq\mathfrak{p}_n^{v_n+1}$, we can choose an element $a_n\in\mathfrak{p}_n^{v_n}$ that is not in $\mathfrak{p}_n^{v_n+1}$. By Theorem 5.1.4 applied to the $r+1$ coprime integral ideals

$$\mathfrak{p}_1^{v_1+1},\ldots,\mathfrak{p}_r^{v_r+1},\, I\cdot\left(\prod\mathfrak{p}_n^{v_n}\right)^{-1},$$

there exists $a\in\mathcal{O}_K$ such that

$$a\equiv a_n\pmod{\mathfrak{p}_n^{v_n+1}}$$

for all $n=1,\ldots,r$ and also

$$a\equiv 0\quad\left(\text{mod }I\cdot\left(\prod\mathfrak{p}_n^{v_n}\right)^{-1}\right).$$

To complete the proof we show that $(a)I^{-1}$ is not divisible by any $\mathfrak{p}_n$, or equivalently, that each $\mathfrak{p}_n^{v_n}$ exactly divides (a). First we show that $\mathfrak{p}_n^{v_n}$ divides (a). Because $a\equiv a_n\pmod{\mathfrak{p}_n^{v_n+1}}$, there exists $b\in\mathfrak{p}_n^{v_n+1}$ such that $a=a_n+b$. Since $a_n\in\mathfrak{p}_n^{v_n}$ and $b\in\mathfrak{p}_n^{v_n+1}\subset\mathfrak{p}_n^{v_n}$, it follows that $a\in\mathfrak{p}_n^{v_n}$, so $\mathfrak{p}_n^{v_n}$ divides (a). Now assume for the sake of contradiction that $\mathfrak{p}_n^{v_n+1}$ divides (a); then $a_n=a-b\in\mathfrak{p}_n^{v_n+1}$, which contradicts that we chose $a_n\notin\mathfrak{p}_n^{v_n+1}$. Thus $\mathfrak{p}_n^{v_n+1}$ does not divide (a), as claimed. □

Suppose I is a nonzero ideal of $\mathcal{O}_K$. As an abelian group $\mathcal{O}_K$ is free of rank equal to the degree $[K:\mathbf{Q}]$ of K, and I is of finite index in $\mathcal{O}_K$, so I can be generated as an abelian group, hence as an ideal, by $[K:\mathbf{Q}]$ generators. The following proposition asserts something much better, namely that I can be generated *as an ideal* in $\mathcal{O}_K$ by at most two elements.

Proposition 5.2.2. *Suppose I is a fractional ideal in the ring $\mathcal{O}_K$ of integers of a number field. Then there exist $a,b\in K$ such that $I=(a,b)=\{\alpha a+\beta b:\alpha,\beta\in\mathcal{O}_K\}$.*

Proof. If $I=(0)$, then I is generated by 1 element and we are done. If I is not an integral ideal, then there is $x\in K$ such that xI is an integral ideal, and the number of generators of xI is the same as the number of generators of I, so we may assume that I is an integral ideal.

Let a be *any* nonzero element of the integral ideal I. We will show that there is some $b\in I$ such that $I=(a,b)$. Let $J=(a)$. By Lemma 5.2.1, there exists $b\in I$ such that $(b)I^{-1}$ is coprime to (a). Since $a,b\in I$, we have $I\mid(a)$ and $I\mid(b)$, so $I\mid(a,b)$. Suppose $\mathfrak{p}^n\mid(a,b)$ with $\mathfrak{p}$ prime and $n\geq 1$. Then $\mathfrak{p}^n\mid(a)$ and $\mathfrak{p}^n\mid(b)$, so $\mathfrak{p}\nmid(b)I^{-1}$, since $(b)I^{-1}$ is coprime to (a). We have $\mathfrak{p}^n\mid(b)=I\cdot(b)I^{-1}$ and $\mathfrak{p}\nmid(b)I^{-1}$, so $\mathfrak{p}^n\mid I$. Thus by unique factorization of ideals in $\mathcal{O}_K$ we have that $(a,b)\mid I$. Sine $I\mid(a,b)$ we conclude that $I=(a,b)$, as claimed. □

We can also use Theorem 5.1.4 to determine the $\mathcal{O}_K$-module structure of $\mathfrak{p}^n/\mathfrak{p}^{n+1}$.

Proposition 5.2.3. *Let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}_K$, and let $n \geq 0$ be an integer. Then $\mathfrak{p}^n/\mathfrak{p}^{n+1} \cong \mathcal{O}_K/\mathfrak{p}$ as $\mathcal{O}_K$-modules.*

Proof. [1] Since $\mathfrak{p}^n \neq \mathfrak{p}^{n+1}$, by unique factorization, there is an element $b \in \mathfrak{p}^n$ such that $b \notin \mathfrak{p}^{n+1}$. Let $\varphi : \mathcal{O}_K \to \mathfrak{p}^n/\mathfrak{p}^{n+1}$ be the $\mathcal{O}_K$-module morphism defined by $\varphi(a) = ab$. The kernel of φ is $\mathfrak{p}$ since clearly $\varphi(\mathfrak{p}) = 0$ and if $\varphi(a) = 0$ then $ab \in \mathfrak{p}^{n+1}$, so $\mathfrak{p}^{n+1} \mid (a)(b)$, so $\mathfrak{p} \mid (a)$, since $\mathfrak{p}^{n+1}$ does not divide (b). Thus φ induces an injective $\mathcal{O}_K$-module homomorphism $\mathcal{O}_K/\mathfrak{p} \hookrightarrow \mathfrak{p}^n/\mathfrak{p}^{n+1}$.

It remains to show that φ is surjective, and this is where we will use Theorem 5.1.4. Suppose $c \in \mathfrak{p}^n$. By Theorem 5.1.4 there exists $d \in \mathcal{O}_K$ such that

$$d \equiv c \pmod{\mathfrak{p}^{n+1}} \qquad \text{and} \qquad d \equiv 0 \pmod{(b)/\mathfrak{p}^n}.$$

We have $\mathfrak{p}^n \mid (d)$ since $d \in \mathfrak{p}^n$ and $(b)/\mathfrak{p}^n \mid (d)$ by the second displayed condition, so since $\mathfrak{p} \nmid (b)/\mathfrak{p}^n$, we have $(b) = \mathfrak{p}^n \cdot (b)/\mathfrak{p}^n \mid (d)$, hence $d/b \in \mathcal{O}_K$. Finally

$$\varphi\left(\frac{d}{b}\right) \quad = \quad \frac{d}{b} \cdot d \pmod{\mathfrak{p}^{n+1}} \quad = \quad b \pmod{p^{n+1}} \quad = \quad c \pmod{p^{n+1}},$$

so φ is surjective. □

5.3 Computing Using the CRT

In order to explicitly compute an a as given by the Theorem 5.1.4, usually one first precomputes elements $v_1, \ldots, v_r \in R$ such that $v_1 \mapsto (1, 0, \ldots, 0)$, $v_2 \mapsto (0, 1, \ldots, 0)$, etc. Then given any $a_n \in R$, for $n = 1, \ldots, r$, we obtain an $a \in R$ with $a_n \equiv a \pmod{I_n}$ by taking

$$a = a_1 v_1 + \cdots + a_r v_r.$$

How to compute the v_i depends on the ring R. It reduces to the following problem: Given coprimes ideals $I, J \subset R$, find $x \in I$ and $y \in J$ such that $x + y = 1$. If R is torsion free and of finite rank as a $\mathbf{Z}$-module, so $R \approx \mathbf{Z}^n$, then I, J can be represented by giving a basis in terms of a basis for R, and finding x, y such that $x + y = 1$ can then be reduced to a problem in linear algebra over $\mathbf{Z}$. More precisely, let A be the matrix whose columns are the concatenation of a basis for I with a basis for J. Suppose $v \in \mathbf{Z}^n$ corresponds to $1 \in \mathbf{Z}^n$. Then finding x, y such that $x + y = 1$ is equivalent to finding a solution $z \in \mathbf{Z}^n$ to the matrix equation $Az = v$. This latter linear algebra problem can be solved using Hermite normal form (see [Coh93, §4.7.1]), which is a generalization over $\mathbf{Z}$ of reduced row echelon form.

[[rewrite this to use Sage.]]

[1] Proof from [SD01, pg. 13].

5.3.1 Magma

The Magma command ChineseRemainderTheorem implements the algorithm suggested by Theorem 5.1.4. In the following example, we compute a prime over (3) and a prime over (5) of the ring of integers of $\mathbf{Q}(\sqrt[3]{2})$, and find an element of $\mathcal{O}_K$ that is congruent to $\sqrt[3]{2}$ modulo one prime and 1 modulo the other.

```
> R<x> := PolynomialRing(RationalField());
> K<a> := NumberField(x^3-2);
> OK := RingOfIntegers(K);
> I := Factorization(3*OK)[1][1];
> J := Factorization(5*OK)[1][1];
> I;
Prime Ideal of OK
Two element generators:
    [3, 0, 0]
    [4, 1, 0]
> J;
Prime Ideal of OK
Two element generators:
    [5, 0, 0]
    [7, 1, 0]
> b := ChineseRemainderTheorem(I, J, OK!a, OK!1);
> K!b;
-4
> b - a in I;
true
> b - 1 in J;
true
```

5.3.2 PARI

There is also a CRT algorithm for number fields in PARI, but it is more cumbersome to use. First we defined $\mathbf{Q}(\sqrt[3]{2})$ and factor the ideals (3) and (5).

```
? f = x^3 - 2;
? k = nfinit(f);
? i = idealfactor(k,3);
? j = idealfactor(k,5);
```

Next we form matrix whose rows correspond to a product of two primes, one dividing 3 and one dividing 5:

```
? m = matrix(2,2);
? m[1,] = i[1,];
? m[1,2] = 1;
? m[2,] = j[1,];
```

Note that we set `m[1,2] = 1`, so the exponent is 1 instead of 3. We apply the CRT to obtain a lift in terms of the basis for $\mathcal{O}_K$.

```
? ?idealchinese
idealchinese(nf,x,y): x being a prime ideal factorization and y
a vector of elements, gives an element b such that
v_p(b-y_p)>=v_p(x) for all prime ideals p dividing x,
and v_p(b)>=0 for all other p.
? idealchinese(k, m, [x,1])
[0, 0, -1]~
? nfbasis(f)
[1, x, x^2]
```

Thus PARI finds the lift $-(\sqrt[3]{2})^2$, and we finish by verifying that this lift is correct. I couldn't figure out how to test for ideal membership in PARI, so here we just check that the prime ideal plus the element is not the unit ideal, which since the ideal is prime, implies membership.

```
? idealadd(k, i[1,1], -x^2 - x)
[3 1 2]
[0 1 0]
[0 0 1]
? idealadd(k, j[1,1], -x^2-1)
[5 2 1]
[0 1 0]
[0 0 1]
```

Chapter 6

Discrimants and Norms

In this chapter we give a geometric interpretation of the discriminant of an order in a number field. We also define norms of ideals and prove that the norm function is multiplicative. Discriminants of orders and norms of ideals will play a crucial role in our proof of finiteness of the class group in the next chapter.

6.1 Viewing $\mathcal{O}_K$ as a Lattice in a Real Vector Space

Let K be a number field of degree n. By the primitive element theorem, $K = \mathbf{Q}(\alpha)$ for some α, so we can write $K \cong \mathbf{Q}[x]/(f)$, where $f \in \mathbf{Q}[x]$ is the minimal polynomial of α. Because $\mathbf{C}$ is algebraically closed and f is irreducible, it has exactly $n = [K : \mathbf{Q}]$ complex roots. Each of these roots $z \in \mathbf{C}$ induces a homomorphism $\mathbf{Q}[x] \to \mathbf{C}$ given by $x \mapsto z$, whose kernel is the ideal (f). Thus we obtain n embeddings of $K \cong \mathbf{Q}[x]/(f)$ into $\mathbf{C}$:

$$\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbf{C}.$$

Example 6.1.1. We compute the embeddings listed above for $K = \mathbf{Q}(\sqrt[3]{2})$.

```
sage: K = QQ[2^(1/3)]; K
Number Field in a with defining polynomial x^3 - 2
sage: K.complex_embeddings()
[Ring morphism: ...
  Defn: a |--> -0.629960524947 - 1.09112363597*I,
 Ring morphism: ...
  Defn: a |--> -0.629960524947 + 1.09112363597*I,
 Ring morphism: ...
  Defn: a |--> 1.25992104989]
```

Let $\sigma : K \hookrightarrow \mathbf{C}^n$ be the map $a \mapsto (\sigma_1(a), \ldots, \sigma_n(a))$, and let $V = \mathbf{R}\sigma(K)$ be the $\mathbf{R}$-span of the image $\sigma(K)$ of K inside $\mathbf{C}^n$.

Lemma 6.1.2. *Suppose $L \subset \mathbf{R}^n$ is a subgroup of the vector space $\mathbf{R}^n$. Then the induced topology on L is discrete if and only if for every $H > 0$ the set*

$$X_H = \{v \in L : \max\{|v_1|, \dots, |v_n|\} \le H\}$$

is finite.

Proof. If L is not discrete, then there is a point $x \in L$ such that for every $\varepsilon > 0$ there is $y \in L$ such that $0 < |x - y| < \varepsilon$. By choosing smaller and smaller ε, we find infinitely many elements $x - y \in L$ all of whose coordinates are smaller than 1. The set X_1 is thus not finite. Thus if the sets X_H are all finite, L must be discrete.

Next assume that L is discrete and let $H > 0$ be any positive number. Then for every $x \in X_H$ there is an open ball B_x that contains x but no other element of L. Since X_H is closed and bounded, it is compact, so the open covering $\cup B_x$ of X_H has a finite subcover, which implies that X_H is finite, as claimed. □

Lemma 6.1.3. *If L if a free abelian group that is discrete in a finite-dimensional real vector space V and $\mathbf{R}L = V$, then the rank of L equals the dimension of V.*

Proof. Let $x_1, \dots, x_m \in L$ be an $\mathbf{R}$-vector space basis for $\mathbf{R}L$, and consider the $\mathbf{Z}$-submodule $M = \mathbf{Z}x_1 + \cdots + \mathbf{Z}x_m$ of L. If the quotient L/M is infinite, then there are infinitely many distinct elements of L that all lie in a fundamental domain for M, so Lemma 6.1.2 implies that L is not discrete. This is a contradiction, so L/M is finite, and the rank of L is $m = \dim(\mathbf{R}L)$, as claimed. □

Proposition 6.1.4. *The $\mathbf{R}$-vector space $V = \mathbf{R}\sigma(K)$ spanned by the image $\sigma(K)$ of K has dimension n.*

Proof. We prove this by showing that the image $\sigma(\mathcal{O}_K)$ is discrete. If $\sigma(\mathcal{O}_K)$ were not discrete it would contain elements all of whose coordinates are simultaneously arbitrarily small. The norm of an element $a \in \mathcal{O}_K$ is the product of the entries of $\sigma(a)$, so the norms of nonzero elements of $\mathcal{O}_K$ would go to 0. This is a contradiction, since the norms of nonzero elements of $\mathcal{O}_K$ are nonzero integers.

Since $\sigma(\mathcal{O}_K)$ is discrete in $\mathbf{C}^n$, Lemma 6.1.3 implies that $\dim(V)$ equals the rank of $\sigma(\mathcal{O}_K)$. Since σ is injective, $\dim(V)$ is the rank of $\mathcal{O}_K$, which equals n by Proposition 2.4.5. □

6.1.1 The Volume of $\mathcal{O}_K$

Since $\sigma(\mathcal{O}_K)$ is a lattice in V, the volume of $V/\sigma(\mathcal{O}_K)$ is finite. Suppose $w_1, \dots, w_n$ is a basis for $\mathcal{O}_K$. Then if A is the matrix whose ith row is $\sigma(w_i)$, then we define the *volume* of $V/\sigma(\mathcal{O}_K)$ to be $|\det(A)|$.

Example 6.1.5. The ring $\mathcal{O}_K = \mathbf{Z}[i]$ of integers of $K = \mathbf{Q}(i)$ has $\mathbf{Z}$-basis $w_1 = 1$, $w_2 = i$. The map $\sigma : K \to \mathbf{C}^2$ is given by

$$\sigma(a + bi) = (a + bi, a - bi) \in \mathbf{C}^2.$$

The image $\sigma(\mathcal{O}_K)$ is spanned by $(1,1)$ and $(i,-i)$. The volume determinant is

$$\left|\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\right| = |-2i| = 2.$$

Let $\mathcal{O}_K = \mathbf{Z}[\sqrt{2}]$ be the ring of integers of $K = \mathbf{Q}(\sqrt{2})$. The map σ is

$$\sigma(a + b\sqrt{2}) = (a + b\sqrt{2}, a - b\sqrt{2}) \in \mathbf{R}^2,$$

and

$$A = \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix},$$

which has determinant $-2\sqrt{2}$, so the volume of $V/\sigma(\mathcal{O}_K)$ is $2\sqrt{2}$.

As the above example illustrates, the volume $V/\sigma(\mathcal{O}_K)$ need not be an integer.

6.2 Discriminants

Suppose $w_1, \ldots, w_n$ are a basis for $\mathcal{O}_K$ as a $\mathbf{Z}$-module, which we view as a $\mathbf{Q}$-vector space. Let $\sigma : K \hookrightarrow \mathbf{C}^n$ be the embedding $\sigma(a) = (\sigma_1(a), \ldots, \sigma_n(a))$, where $\sigma_1, \ldots, \sigma_n$ are the distinct embeddings of K into $\mathbf{C}$. Let A be the matrix whose rows are $\sigma(w_1), \ldots, \sigma(w_n)$. The quantity $\det(A)$ depends on the ordering of the w_i, and need not be an integer.

If we consider $\det(A)^2$ instead, we obtain a number that does not depend on ordering; moreover, as we will see, it is an integer. Note that

$$\begin{aligned} \det(A)^2 &= \det(AA) = \det(A)\det(A) = \det(A)\det(A^t) = \det(AA^t) \\ &= \det\left(\sum_{k=1,\ldots,n} \sigma_k(w_i)\sigma_k(w_j)\right) = \det\left(\sum_{k=1,\ldots,n} \sigma_k(w_i w_j)\right) \\ &= \det(\mathrm{Tr}(w_i w_j)_{1 \le i,j \le n}), \end{aligned}$$

so $\det(A)^2$ can be defined purely in terms of the trace without mentioning the embeddings σ_i. Also, changing our choice of basis for $\mathcal{O}_K$ is the same as left multiplying A by an integer matrix U of determinant ± 1; this does not change the squared determinant, since $\det(UA)^2 = \det(U)^2 \det(A)^2 = \det(A)^2$. Thus $\det(A)^2 \in \mathbf{Z}$ is well defined as a quantity associated to $\mathcal{O}_K$.

If we view K as a $\mathbf{Q}$-vector space, then $(x, y) \mapsto \mathrm{Tr}(xy)$ defines a bilinear pairing $K \times K \to \mathbf{Q}$ on K, which we call the *trace pairing*. The following lemma asserts that this pairing is nondegenerate, so $\det(\mathrm{Tr}(w_i w_j)) \neq 0$ hence $\det(A) \neq 0$.

Lemma 6.2.1. *The trace pairing is nondegenerate.*

Proof. If the trace pairing is degenerate, then there exists $a \in K$ such that for every $b \in K$ we have $\mathrm{Tr}(ab) = 0$. In particularly, taking $b = a^{-1}$ we see that $0 = \mathrm{Tr}(aa^{-1}) = \mathrm{Tr}(1) = [K : \mathbf{Q}] > 0$, which is absurd. □

Definition 6.2.2 (Discriminant)**.** Suppose $a_1, \ldots, a_n$ is any $\mathbf{Q}$-basis of K. The *discriminant* of $a_1, \ldots, a_n$ is

$$\mathrm{Disc}(a_1, \ldots, a_n) = \det(\mathrm{Tr}(a_i a_j)_{1 \leq i,j \leq n}) \in \mathbf{Q}.$$

The *discriminant* $\mathrm{Disc}(\mathcal{O})$ of an order $\mathcal{O}$ in $\mathcal{O}_K$ is the discriminant of any basis for $\mathcal{O}$. The *discriminant* $d_K = \mathrm{Disc}(K)$ of the number field K is the discriminant of $\mathcal{O}_K$. Note that these discriminants are all nonzero by Lemma 6.2.1.

Remark 6.2.3. It is also standard to define the discriminant of a monic polynomial to be the product of the differences of the roots. If $\alpha \in \mathcal{O}_K$ with $\mathbf{Z}[\alpha]$ of finite index in $\mathcal{O}_K$, and f is the minimal polynomial of α, then $\mathrm{Disc}(f) = \mathrm{Disc}(\mathbf{Z}[\alpha])$. To see this, note that if we choose the basis $1, \alpha, \ldots, \alpha^{n-1}$ for $\mathbf{Z}[\alpha]$, then both discriminants are the square of the same Vandermonde determinant.

Example 6.2.4. In SAGE, we compute the discriminant of a number field or order using the discriminant command:

```
sage: K.<a> = NumberField(x^2 - 5)
sage: K.discriminant()
5
```

This also works for orders (notice the square factor below, which will be explained by Proposition 6.2.5):

```
sage: R = K.order([7*a]); R
Order in Number Field in a with defining polynomial x^2 - 5
sage: factor(R.discriminant())
2^2 * 5 * 7^2
```

Warning: In MAGMA $\mathrm{Disc}(K)$ is defined to be the discriminant of the polynomial you happened to use to define K.

```
> K := NumberField(x^2-5);
> Discriminant(K);
20
```

This is an intentional choice done for efficiency reasons, since computing the maximal order can take a long time. Nonetheless, it conflicts with standard mathematical usage, so beware.

The following proposition asserts that the discriminant of an order $\mathcal{O}$ in $\mathcal{O}_K$ is bigger than $\mathrm{disc}(\mathcal{O}_K)$ by a factor of the square of the index.

Proposition 6.2.5. *Suppose $\mathcal{O}$ is an order in $\mathcal{O}_K$. Then*

$$\mathrm{Disc}(\mathcal{O}) = \mathrm{Disc}(\mathcal{O}_K) \cdot [\mathcal{O}_K : \mathcal{O}]^2.$$

Proof. Let A be a matrix whose rows are the images via σ of a basis for $\mathcal{O}_K$, and let B be a matrix whose rows are the images via σ of a basis for $\mathcal{O}$. Since $\mathcal{O} \subset \mathcal{O}_K$ has finite index, there is an integer matrix C such that $CA = B$, and $|\det(C)| = [\mathcal{O}_K : \mathcal{O}]$. Then

$$\mathrm{Disc}(\mathcal{O}) = \det(B)^2 = \det(CA)^2 = \det(C)^2 \det(A)^2 = [\mathcal{O}_K : \mathcal{O}]^2 \cdot \mathrm{Disc}(\mathcal{O}_K).$$

□

Example 6.2.6. Let K be a number field and consider the quantity

$$D(K) = \gcd\{\mathrm{Disc}(\alpha) : \alpha \in \mathcal{O}_K \text{ and } [\mathcal{O}_K : \mathbf{Z}[\alpha]] < \infty\}.$$

One might hope that $D(K)$ is equal to the discriminant $\mathrm{Disc}(\mathcal{O}_K)$ of K, but this is not the case in general. Recall Example 4.3.2, in which we considered the field K generated by a root of $f = x^3 + x^2 - 2x + 8$. In that example, the discriminant of $\mathcal{O}_K$ is -503 with 503 prime:

```
sage: K.<a> = NumberField(x^3 + x^2 - 2*x + 8)
sage: factor(K.discriminant())
-1 * 503
```

For every $\alpha \in \mathcal{O}_K$, we have $2 \mid [\mathcal{O}_K : \mathbf{Z}[\alpha]]$, since $\mathcal{O}_K$ fails to be monogenic at 2. By Proposition 6.2.5, the discriminant of $\mathbf{Z}[\alpha]$ is divisible by 4 for all α, so $\mathrm{Disc}(\alpha)$ is also divisible by 4. This is why 2 is called an "inessential *discriminant* divisor".

Proposition 6.2.5 gives an algorithm for computing $\mathcal{O}_K$, albeit a slow one. Given K, find some order $\mathcal{O} \subset K$, and compute $d = \mathrm{Disc}(\mathcal{O})$. Factor d, and use the factorization to write $d = s \cdot f^2$, where f^2 is the largest square that divides d. Then the index of $\mathcal{O}$ in $\mathcal{O}_K$ is a divisor of f, and we (tediously) can enumerate all rings R with $\mathcal{O} \subset R \subset K$ and $[R : \mathcal{O}] \mid f$, until we find the largest one all of whose elements are integral. A much better algorithm is to proceed exactly as just described, except use the ideas of Section 4.3.3 to find a p-maximal order for each prime divisor of f, then add these p-maximal orders together.

Example 6.2.7. Consider the ring $\mathcal{O}_K = \mathbf{Z}[(1+\sqrt{5})/2]$ of integers of $K = \mathbf{Q}(\sqrt{5})$. The discriminant of the basis $1, a = (1+\sqrt{5})/2$ is

$$\mathrm{Disc}(\mathcal{O}_K) = \left| \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \right| = 5.$$

Let $\mathcal{O} = \mathbf{Z}[\sqrt{5}]$ be the order generated by $\sqrt{5}$. Then $\mathcal{O}$ has basis $1, \sqrt{5}$, so

$$\mathrm{Disc}(\mathcal{O}) = \left| \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix} \right| = 20 = [\mathcal{O}_K : \mathcal{O}]^2 \cdot 5,$$

hence $[\mathcal{O}_K : \mathcal{O}] = 2$.

Example 6.2.8. Consider the cubic field $K = \mathbf{Q}(\sqrt[3]{2})$, and let $\mathcal{O}$ be the order $\mathbf{Z}[\sqrt[3]{2}]$. Relative to the base $1, \sqrt[3]{2}, (\sqrt[3]{2})^2$ for $\mathcal{O}$, the matrix of the trace pairing is

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{pmatrix}.$$

Thus

$$\mathrm{disc}(\mathcal{O}) = \det(A) = 108 = 2^2 \cdot 3^3.$$

Suppose we do not know that the ring of integers $\mathcal{O}_K$ is equal to $\mathcal{O}$. By Proposition 6.2.5, we have

$$\mathrm{Disc}(\mathcal{O}_K) \cdot [\mathcal{O}_K : \mathcal{O}]^2 = 2^2 \cdot 3^3,$$

so $3 \mid \mathrm{disc}(\mathcal{O}_K)$, and $[\mathcal{O}_K : \mathcal{O}] \mid 6$. Thus to prove $\mathcal{O} = \mathcal{O}_K$ it suffices to prove that $\mathcal{O}$ is 2-maximal and 3-maximal, which could be accomplished as described in Section 4.3.3.

6.3 Norms of Ideals

In this section we extend the notion of norm to ideals. This will be helpful in the next chapter, where we will prove that the group of fractional ideals modulo principal fractional ideals of a number field is finite by showing that every ideal is equivalent to an ideal with norm at most some bound. This is enough, because as we will see below there are only finitely many ideals of bounded norm.

Definition 6.3.1 (Lattice Index)**.** If L and M are two lattices in a vector space V, then the *lattice index* $[L : M]$ is by definition the absolute value of the determinant of any linear automorphism A of V such that $A(L) = M$.

For example, if $L = 2\mathbf{Z}$ and $M = 10\mathbf{Z}$, then

$$[L : M] = [2\mathbf{Z} : 10\mathbf{Z}] = \det([5]) = 5,$$

since 5 multiplies $2\mathbf{Z}$ onto $10\mathbf{Z}$.

The lattice index has the following properties:

- If $M \subset L$, then $[L : M] = \#(L/M)$.
- If M, L, N are any lattices in V, then

$$[L : N] = [L : M] \cdot [M : N].$$

Definition 6.3.2 (Norm of Fractional Ideal)**.** Suppose I is a fractional ideal of $\mathcal{O}_K$. The *norm* of I is the lattice index

$$\mathrm{Norm}(I) = [\mathcal{O}_K : I] \in \mathbf{Q}_{\geq 0},$$

or 0 if $I = 0$.

Note that if I is an integral ideal, then $\mathrm{Norm}(I) = \#(\mathcal{O}_K/I)$.

Lemma 6.3.3. *Suppose $a \in K$ and I is an integral ideal. Then*

$$\mathrm{Norm}(aI) = |\,\mathrm{Norm}_{K/\mathbf{Q}}(a)|\,\mathrm{Norm}(I).$$

Proof. By properties of the lattice index mentioned above we have

$$[\mathcal{O}_K : aI] = [\mathcal{O}_K : I] \cdot [I : aI] = \mathrm{Norm}(I) \cdot |\,\mathrm{Norm}_{K/\mathbf{Q}}(a)|.$$

Here we have used that $[I : aI] = |\,\mathrm{Norm}_{K/\mathbf{Q}}(a)|$, which is because left multiplication ℓ_a by a is an automorphism of K that sends I onto aI, so

$$[I : aI] = |\det(\ell_a)| = |\,\mathrm{Norm}_{K/\mathbf{Q}}(a)|.$$

□

Proposition 6.3.4. *If I and J are fractional ideals, then*

$$\mathrm{Norm}(IJ) = \mathrm{Norm}(I) \cdot \mathrm{Norm}(J).$$

Proof. By Lemma 6.3.3, it suffices to prove this when I and J are integral ideals. If I and J are coprime, then Theorem 5.1.4 (the Chinese Remainder Theorem) implies that $\mathrm{Norm}(IJ) = \mathrm{Norm}(I) \cdot \mathrm{Norm}(J)$. Thus we reduce to the case when $I = \mathfrak{p}^m$ and $J = \mathfrak{p}^k$ for some prime ideal $\mathfrak{p}$ and integers m, k. By Proposition 5.2.3, which is a consequence of CRT, the filtration of $\mathcal{O}_K/\mathfrak{p}^n$ given by powers of $\mathfrak{p}$ has successive quotients isomorphic to $\mathcal{O}_K/\mathfrak{p}$. Thus we see that $\#(\mathcal{O}_K/\mathfrak{p}^n) = \#(\mathcal{O}_K/\mathfrak{p})^n$, which proves that $\mathrm{Norm}(\mathfrak{p}^n) = \mathrm{Norm}(\mathfrak{p})^n$. □

Example 6.3.5. We compute some ideal norms using SAGE.

```
sage: K.<a> = NumberField(x^2 - 5)
sage: I = K.fractional_ideal(a)
sage: I.norm()
5
sage: J = K.fractional_ideal(17)
sage: J.norm()
289
```

We can also use functional notation:

```
sage: norm(I*J)
1445
```

We will use the following proposition in the next chapter when we prove finiteness of class groups.

Proposition 6.3.6. *Fix a number field K. Let B be a positive integer. There are only finitely many integral ideals I of $\mathcal{O}_K$ with norm at most B.*

Proof. An integral ideal I is a subgroup of $\mathcal{O}_K$ of index equal to the norm of I. If G is any finitely generated abelian group, then there are only finitely many subgroups of G of index at most B, since the subgroups of index dividing an integer n are all subgroups of G that contain nG, and the group G/nG is finite. □

Chapter 7

Finiteness of the Class Group

Frequently $\mathcal{O}_K$ is not a principal ideal domain. This chapter is about a way to understand how badly $\mathcal{O}_K$ fails to be a principal ideal domain. The class group of $\mathcal{O}_K$ measures this failure. As one sees in a course on Class Field Theory, the class group and its generalizations also yield deep insight into the extensions of K that are Galois with abelian Galois group.

7.1 The Class Group

Definition 7.1.1 (Class Group)**.** Let $\mathcal{O}_K$ be the ring of integers of a number field K. The *class group* C_K of K is the group of fractional ideals modulo the sugroup of principal fractional ideals (a), for $a \in K$.

Note that if we let $\mathrm{Div}(\mathcal{O}_K)$ denote the group of fractional ideals, then we have an exact sequence

$$0 \to \mathcal{O}_K^* \to K^* \to \mathrm{Div}(\mathcal{O}_K) \to C_K \to 0.$$

That the class group C_K is finite follows from the first part of the following theorem and the fact that there are only finitely many ideals of norm less than a given integer (Proposition 6.3.6).

Theorem 7.1.2 (Finiteness of the Class Group)**.** *Let K be a number field. There is a constant $C_{r,s}$ that depends only on the number r, s of real and pairs of complex conjugate embeddings of K such that every ideal class of $\mathcal{O}_K$ contains an integral ideal of norm at most $C_{r,s}\sqrt{|d_K|}$, where $d_K = \mathrm{Disc}(\mathcal{O}_K)$. Thus by Proposition 6.3.6 the class group C_K of K is finite. One can choose $C_{r,s}$ such that every ideal class in C_K contains an integral ideal of norm at most*

$$\sqrt{|d_K|} \cdot \left(\frac{4}{\pi}\right)^s \frac{n!}{n^n}.$$

The explicit bound in the theorem is called the *Minkowski bound.* There are other better bounds, but they depend on unproven conjectures.

The following two examples illustrate how to apply Theorem 7.1.2 to compute C_K in simple cases.

Example 7.1.3. Let $K = \mathbf{Q}[i]$. Then $n = 2$, $s = 1$, and $|d_K| = 4$, so the Minkowski bound is

$$\sqrt{4} \cdot \left(\frac{4}{\pi}\right)^1 \frac{2!}{2^2} = \frac{4}{\pi} < 2.$$

Thus every fractional ideal is equivalent to an ideal of norm 1. Since (1) is the only ideal of norm 1, every ideal is principal, so C_K is trivial.

Example 7.1.4. Let $K = \mathbf{Q}(\sqrt{10})$. We have $\mathcal{O}_K = \mathbf{Z}[\sqrt{10}]$, so $n = 2$, $s = 0$, $|d_K| = 40$, and the Minkowski bound is

$$\sqrt{40} \cdot \left(\frac{4}{\pi}\right)^0 \cdot \frac{2!}{2^2} = 2 \cdot \sqrt{10} \cdot \frac{1}{2} = \sqrt{10} = 3.162277\ldots.$$

We compute the Minkowski bound in SAGE as follows:

```
sage: K = QQ[sqrt(10)]; K
Number Field in sqrt10 with defining polynomial x^2 - 10
sage: B = K.minkowski_bound(); B
sqrt(10)
sage: B.n()
3.16227766016838
```

Theorem 7.1.2 implies that every ideal class has a representative that is an integral ideal of norm 1, 2, or 3. The ideal $2\mathcal{O}_K$ is ramified in $\mathcal{O}_K$, so

$$2\mathcal{O}_K = (2, \sqrt{10}).$$

If $(2, \sqrt{10})$ were principal, say (α), then $\alpha = a + b\sqrt{10}$ would have norm ± 2. Then the equation

$$x^2 - 10y^2 = \pm 2, \tag{7.1.1}$$

would have an integer solution. But the squares mod 5 are $0, \pm 1$, so (7.1.1) has no solutions. Thus $(2, \sqrt{10})$ defines a nontrivial element of the class group, and it has order 2 since its square is the principal ideal $2\mathcal{O}_K$. Thus $2 \mid \#C_K$.

To find the integral ideals of norm 3, we factor $x^2 - 10$ modulo 3, and see that

$$3\mathcal{O}_K = (3, 2 + \sqrt{10}) \cdot (3, 4 + \sqrt{10}).$$

If either of the prime divisors of $3\mathcal{O}_K$ were principal, then the equation $x^2 - 10y^2 = \pm 3$ would have an integer solution. Since it does not have one mod 5, the prime divisors of $3\mathcal{O}_K$ are both nontrivial elements of the class group. Let

$$\alpha = \frac{4 + \sqrt{10}}{2 + \sqrt{10}} = \frac{1}{3} \cdot (1 + \sqrt{10}).$$

Then

$$(3, 2+\sqrt{10}) \cdot (\alpha) = (3\alpha, 4+\sqrt{10}) = (1+\sqrt{10}, 4+\sqrt{10}) = (3, 4+\sqrt{10}),$$

so the classes over 3 are equal.

In summary, we now know that every element of C_K is equivalent to one of

$$(1), \quad (2, \sqrt{10}), \quad \text{or} \quad (3, 2+\sqrt{10}).$$

Thus the class group is a group of order at most 3 that contains an element of order 2. Thus it must have order 2. We verify this in SAGE below, where we also check that $(3, 2+\sqrt{10})$ generates the class group.

```
sage: K.<sqrt10> = QQ[sqrt(10)]; K
Number Field in sqrt10 with defining polynomial x^2 - 10
sage: G = K.class_group(); G
Class group of order 2 with structure C2 of Number Field ...
sage: G.0
Fractional ideal class (3, sqrt10 + 1)
sage: G.0^2
Trivial principal fractional ideal class
sage: G.0 == G( (3, 2 + sqrt10) )
True
```

Before proving Theorem 7.1.2, we prove a few lemmas. The strategy of the proof is to start with any nonzero ideal I, and prove that there is some nonzero $a \in K$, with very small norm, such that aI is an integral ideal. Then $\mathrm{Norm}(aI) = \mathrm{Norm}_{K/\mathbf{Q}}(a)\,\mathrm{Norm}(I)$ will be small, since $\mathrm{Norm}_{K/\mathbf{Q}}(a)$ is small. The trick is to determine precisely how small an a we can choose subject to the condition that aI is an integral ideal, i.e., that $a \in I^{-1}$.

Let S be a subset of $V = \mathbf{R}^n$. Then S is *convex* if whenever $x, y \in S$ then the line connecting x and y lies entirely in S. We say that S is *symmetric about the origin* if whenever $x \in S$ then $-x \in S$ also. If L is a lattice in the real vector space $V = \mathbf{R}^n$, then the *volume* of V/L is the volume of the compact real manifold V/L, which is the same thing as the absolute value of the determinant of any matrix whose rows form a basis for L.

Lemma 7.1.5 (Blichfeld)**.** *Let L be a lattice in $V = \mathbf{R}^n$, and let S be a bounded closed convex subset of V that is symmetric about the origin. If $\mathrm{Vol}(S) \geq 2^n\,\mathrm{Vol}(V/L)$, then S contains a nonzero element of L.*

Proof. First assume that $\mathrm{Vol}(S) > 2^n \cdot \mathrm{Vol}(V/L)$. If the map $\pi : \frac{1}{2}S \to V/L$ is injective, then

$$\frac{1}{2^n}\,\mathrm{Vol}(S) = \mathrm{Vol}\left(\frac{1}{2}S\right) \leq \mathrm{Vol}(V/L),$$

a contradiction. Thus π is not injective, so there exist $P_1 \neq P_2 \in \frac{1}{2}S$ such that $P_1 - P_2 \in L$. Because S is symmetric about the origin, $-P_2 \in \frac{1}{2}S$. By convexity, the average $\frac{1}{2}(P_1 - P_2)$ of P_1 and $-P_2$ is also in $\frac{1}{2}S$. Thus $0 \neq P_1 - P_2 \in S \cap L$, as claimed.

Next assume that $\mathrm{Vol}(S) = 2^n \cdot \mathrm{Vol}(V/L)$. Then for all $\varepsilon > 0$ there is $0 \neq Q_\varepsilon \in L \cap (1+\varepsilon)S$, since $\mathrm{Vol}((1+\varepsilon)S) > \mathrm{Vol}(S) = 2^n \cdot \mathrm{Vol}(V/L)$. If $\varepsilon < 1$ then the Q_ε are all in $L \cap 2S$, which is finite since $2S$ is bounded and L is discrete. Hence there exists nonzero $Q = Q_\varepsilon \in L \cap (1+\varepsilon)S$ for arbitrarily small ε. Since S is closed, $Q \in L \cap S$. □

Lemma 7.1.6. *If L_1 and L_2 are lattices in V, then*

$$\mathrm{Vol}(V/L_2) = \mathrm{Vol}(V/L_1) \cdot [L_1 : L_2].$$

Proof. Let A be an automorphism of V such that $A(L_1) = L_2$. Then A defines an isomorphism of real manifolds $V/L_1 \to V/L_2$ that changes volume by a factor of $|\det(A)| = [L_1 : L_2]$. The claimed formula then follows, since $[L_1 : L_2] = |\det(A)|$, by definition. □

Fix a number field K with ring of integers $\mathcal{O}_K$.

Let $\sigma_1, \ldots, \sigma_r$ be the real embeddings of K and $\sigma_{r+1}, \ldots, \sigma_{r+s}$ be half the complex embeddings of K, with one representative of each pair of complex conjugate embeddings. Let $\sigma : K \to V = \mathbf{R}^n$ be the embedding

$$\begin{aligned}\sigma(x) = \big(&\sigma_1(x), \sigma_2(x), \ldots, \sigma_r(x),\\ &\mathrm{Re}(\sigma_{r+1}(x)), \ldots, \mathrm{Re}(\sigma_{r+s}(x)), \mathrm{Im}(\sigma_{r+1}(x)), \ldots, \mathrm{Im}(\sigma_{r+s}(x))\big),\end{aligned}$$

Note that this σ is *not* exactly the same as the one at the beginning of Section 6.2 if $s > 0$.

Lemma 7.1.7.

$$\mathrm{Vol}(V/\sigma(\mathcal{O}_K)) = 2^{-s}\sqrt{|d_K|}.$$

Proof. Let $L = \sigma(\mathcal{O}_K)$. From a basis $w_1, \ldots, w_n$ for $\mathcal{O}_K$ we obtain a matrix A whose ith row is

$$(\sigma_1(w_i), \cdots, \sigma_r(w_i), \mathrm{Re}(\sigma_{r+1}(w_i)), \ldots, \mathrm{Re}(\sigma_{r+s}(w_i)), \mathrm{Im}(\sigma_{r+1}(w_i)), \ldots, \mathrm{Im}(\sigma_{r+s}(w_i)))$$

and whose determinant has absolute value equal to the volume of V/L. By doing the following three column operations, we obtain a matrix whose rows are exactly the images of the w_i under *all* embeddings of K into $\mathbf{C}$, which is the matrix that came up when we defined $d_K = \mathrm{Disc}(\mathcal{O}_K)$ in Section 6.2.

1. Add $i = \sqrt{-1}$ times each column with entries $\mathrm{Im}(\sigma_{r+j}(w_i))$ to the column with entries $\mathrm{Re}(\sigma_{r+j}(w_i))$.
2. Multiply all columns with entries $\mathrm{Im}(\sigma_{r+j}(w_i))$ by $-2i$, thus changing the determinant by $(-2i)^s$.

3. Add each columns that now has entries $\mathrm{Re}(\sigma_{r+j}(w_i)) + i\mathrm{Im}(\sigma_{r+j}(w_i))$ to the the column with entries $-2i\mathrm{Im}(\sigma_{r+j}(w_i))$ to obtain columns $\mathrm{Re}(\sigma_{r+j}(w_i)) - i\mathrm{Im}(\sigma_{r+j}(w_i))$.

Recalling the definition of discriminant, we see that if B is the matrix constructed by doing the above three operations to A, then $|\det(B)^2| = |d_K|$. Thus

$$\mathrm{Vol}(V/L) = |\det(A)| = |(-2i)^{-s} \cdot \det(B)| = 2^{-s}\sqrt{|d_K|}.$$

□

Lemma 7.1.8. *If I is a fractional $\mathcal{O}_K$-ideal, then $\sigma(I)$ is a lattice in V and*

$$\mathrm{Vol}(V/\sigma(I)) = 2^{-s}\sqrt{|d_K|} \cdot \mathrm{Norm}(I).$$

Proof. Since $\sigma(\mathcal{O}_K)$ has rank n as an abelian group, and Lemma 7.1.7 implies that $\sigma(\mathcal{O}_K)$ also spans V, it follows that $\sigma(\mathcal{O}_K)$ is a lattice in V. For some nonzero integer m we have $m\mathcal{O}_K \subset I \subset \frac{1}{m}\mathcal{O}_K$, so $\sigma(I)$ is also a lattice in V. To prove the displayed volume formula, combine Lemmas 7.1.6–7.1.7 to get

$$\mathrm{Vol}(V/\sigma(I)) = \mathrm{Vol}(V/\sigma(\mathcal{O}_K)) \cdot [\mathcal{O}_K : I] = 2^{-s}\sqrt{|d_K|}\,\mathrm{Norm}(I).$$

□

Proof of Theorem 7.1.2. Let K be a number field with ring of integers $\mathcal{O}_K$, let $\sigma : K \hookrightarrow V \cong \mathbf{R}^n$ be as above, and let $f : V \to \mathbf{R}$ be the function defined by

$$f(x_1, \ldots, x_n) = |x_1 \cdots x_r \cdot (x_{r+1}^2 + x_{(r+1)+s}^2) \cdots (x_{r+s}^2 + x_n^2)|.$$

Notice that if $x \in K$ then $f(\sigma(x)) = |\mathrm{Norm}_{K/\mathbf{Q}}(x)|$, and for any $a \in \mathbf{R}$,

$$f(ax_1, \ldots, ax_n) = |a|^n f(x_1, \ldots, x_n).$$

Let $S \subset V$ be a fixed choice of closed, bounded, convex, subset with positive volume that is symmetric with respect to the origin and has positive volume. Since S is closed and bounded,

$$M = \max\{f(x) : x \in S\}$$

exists.

Suppose I is any fractional ideal of $\mathcal{O}_K$. Our goal is to prove that there is an integral ideal aI with small norm. We will do this by finding an appropriate $a \in I^{-1}$. By Lemma 7.1.8,

$$c = \mathrm{Vol}(V/\sigma(I^{-1})) = 2^{-s}\sqrt{|d_K|} \cdot \mathrm{Norm}(I)^{-1} = \frac{2^{-s}\sqrt{|d_K|}}{\mathrm{Norm}(I)}.$$

Let $\lambda = 2 \cdot \left(\frac{c}{v}\right)^{1/n}$, where $v = \mathrm{Vol}(S)$. Then

$$\mathrm{Vol}(\lambda S) = \lambda^n \,\mathrm{Vol}(S) = 2^n \frac{c}{v} \cdot v = 2^n \cdot c = 2^n \,\mathrm{Vol}(V/\sigma(I^{-1})),$$

so by Lemma 7.1.5 there exists $0 \neq b \in \sigma(I^{-1}) \cap \lambda S$. Let $a \in I^{-1}$ be such that $\sigma(a) = b$. Since M is the largest norm of an element of S, the largest norm of an element of $\sigma(I^{-1}) \cap \lambda S$ is at most $\lambda^n M$, so

$$|\operatorname{Norm}_{K/\mathbf{Q}}(a)| \leq \lambda^n M.$$

Since $a \in I^{-1}$, we have $aI \subset \mathcal{O}_K$, so aI is an integral ideal of $\mathcal{O}_K$ that is equivalent to I, and

$$\begin{aligned}
\operatorname{Norm}(aI) &= |\operatorname{Norm}_{K/\mathbf{Q}}(a)| \cdot \operatorname{Norm}(I) \\
&\leq \lambda^n M \cdot \operatorname{Norm}(I) \\
&\leq 2^n \frac{c}{v} M \cdot \operatorname{Norm}(I) \\
&= 2^n \cdot 2^{-s} \sqrt{|d_K|} \cdot M \cdot v^{-1} \\
&= 2^{r+s} \sqrt{|d_K|} \cdot M \cdot v^{-1}.
\end{aligned}$$

Notice that the right hand side is independent of I. It depends only on r, s, $|d_K|$, and our choice of S. This completes the proof of the theorem, except for the assertion that S can be chosen to give the claim at the end of the theorem, which we leave as an exercise. □

Corollary 7.1.9. *Suppose that $K \neq \mathbf{Q}$ is a number field. Then $|d_K| > 1$.*

Proof. Applying Theorem 7.1.2 to the unit ideal, we get the bound

$$1 \leq \sqrt{|d_K|} \cdot \left(\frac{4}{\pi}\right)^s \frac{n!}{n^n}.$$

Thus

$$\sqrt{|d_K|} \geq \left(\frac{\pi}{4}\right)^s \frac{n^n}{n!},$$

and the right hand quantity is strictly bigger than 1 for any $s \leq n/2$ and any $n > 1$ (exercise). □

A prime p ramifies in $\mathcal{O}_K$ if and only if $d \mid d_K$, so the corollary implies that every nontrivial extension of $\mathbf{Q}$ is ramified at some prime.

7.2 Class Number 1

The fields of class number 1 are exactly the fields for which $\mathcal{O}_K$ is a principal ideal domain. How many such number fields are there? We still don't know.

Conjecture 7.2.1. *There are infinitely many number fields K such that the class group of K has order 1.*

For example, if we consider real quadratic fields $K = \mathbf{Q}(\sqrt{d})$, with d positive and square free, many class numbers are probably 1, as suggested by the MAGMA output below. It looks like 1's will keep appearing infinitely often, and indeed Cohen and Lenstra conjecture that they do ([CL84]).

```
sage: for d in [2..1000]:
...       if is_fundamental_discriminant(d):
...           h = QuadraticField(d, 'a').class_number()
...           if h == 1:
...               print d,
5 8 12 13 17 21 24 28 29 33 37 41 44 53 56 57 61 69
73 76 77 88 89 92 93 97 101 109 113 124 129 133 137
141 149 152 157 161 172 173 177 181 184 188 193 197
201 209 213 217 233 236 237 241 248 249 253 268 269
277 281 284 293 301 309 313 317 329 332 337 341 344
349 353 373 376 381 389 393 397 409 412 413 417 421
428 433 437 449 453 457 461 472 489 497 501 508 509
517 521 524 536 537 541 553 556 557 569 573 581 589
593 597 601 604 613 617 632 633 641 649 652 653 661
664 668 669 673 677 681 701 709 713 716 717 721 737
749 753 757 764 769 773 781 789 796 797 809 813 821
824 829 844 849 853 856 857 869 877 881 889 893 908
913 917 921 929 933 937 941 953 956 973 977 989 997
```

In contrast, if we look at class numbers of quadratic imaginary fields, only a few at the beginning have class number 1.

```
sage: for d in [-1,-2..-1000]:
...       if is_fundamental_discriminant(d):
...           h = QuadraticField(d, 'a').class_number()
...           if h == 1:
...               print d,
-3 -4 -7 -8 -11 -19 -43 -67 -163
```

It is a theorem that was proved independently and in different ways by Heegner, Stark, and Baker that the above list of 9 fields is the complete list with class number 1. More generally, it is possible, using deep work of Gross, Zagier, and Goldfeld involving zeta functions and elliptic curves, to enumerate all quadratic number fields with a given class number (this may however not be practical – see Mark Watkins Ph.D. thesis).

7.3 More About Computing Class Groups

If $\mathfrak{p}$ is a prime of $\mathcal{O}_K$, then the intersection $\mathfrak{p} \cap \mathbf{Z} = p\mathbf{Z}$ is a prime ideal of $\mathbf{Z}$. We say that $\mathfrak{p}$ *lies over* $p \in \mathbf{Z}$. Note $\mathfrak{p}$ lies over $p \in \mathbf{Z}$ if and only if $\mathfrak{p}$ is one of the prime factors in the factorization of the ideal $p\mathcal{O}_K$. Geometrically, $\mathfrak{p}$ is a point of

$\mathrm{Spec}(\mathcal{O}_K)$ that lies over the point $p\mathbf{Z}$ of $\mathrm{Spec}(\mathbf{Z})$ under the map induced by the inclusion $\mathbf{Z} \hookrightarrow \mathcal{O}_K$.

Lemma 7.3.1. *Let K be a number field with ring of integers $\mathcal{O}_K$. Then the class group $\mathrm{Cl}(K)$ is generated by the prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ lying over primes $p \in \mathbf{Z}$ with $p \leq B_K = \sqrt{|d_K|} \cdot \left(\frac{4}{\pi}\right)^s \cdot \frac{n!}{n^n}$, where s is the number of complex conjugate pairs of embeddings $K \hookrightarrow \mathbf{C}$.*

Proof. Theorem 7.1.2 asserts that every ideal class in $\mathrm{Cl}(K)$ is represented by an ideal I with $\mathrm{Norm}(I) \leq B_K$. Write $I = \prod_{i=1}^{m} \mathfrak{p}_i^{e_i}$, with each $e_i \geq 1$. Then by multiplicativity of the norm, each $\mathfrak{p}_i$ also satisfies $\mathrm{Norm}(\mathfrak{p}_i) \leq B_K$. If $\mathfrak{p}_i \cap \mathbf{Z} = p\mathbf{Z}$, then $p \mid \mathrm{Norm}(\mathfrak{p}_i)$, since p is the residue characteristic of $\mathcal{O}_K/\mathfrak{p}$, so $p \leq B_K$. Thus I is a product of primes $\mathfrak{p}$ that satisfies the norm bound of the lemma. □

This is a sketch of how to compute $\mathrm{Cl}(K)$:

1. Use the algorithms of Chapter 4 to list all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ that appear in the factorization of a prime $p \in \mathbf{Z}$ with $p \leq B_K$.

2. Find the group generated by the ideal classes $[\mathfrak{p}]$, where the $\mathfrak{p}$ are the prime ideals found in step 1. (In general, this step can become fairly complicated.)

The following three examples illustrate computation of $\mathrm{Cl}(K)$ for $K = \mathbf{Q}(i), \mathbf{Q}(\sqrt{5})$ and $\mathbf{Q}(\sqrt{-6})$.

Example 7.3.2. We compute the class group of $K = \mathbf{Q}(i)$. We have

$$n = 2, \quad r = 0, \quad s = 1, \quad d_K = -4,$$

so

$$B_K = \sqrt{4} \cdot \left(\frac{4}{\pi}\right)^1 \cdot \left(\frac{2!}{2^2}\right) = \frac{8}{\pi} < 3.$$

Thus $\mathrm{Cl}(K)$ is generated by the prime divisors of 2. We have

$$2\mathcal{O}_K = (1+i)^2,$$

so $\mathrm{Cl}(K)$ is generated by the principal prime ideal $\mathfrak{p} = (1+i)$. Thus $\mathrm{Cl}(K) = 0$ is trivial.

Example 7.3.3. We compute the class group of $K = \mathbf{Q}(\sqrt{5})$. We have

$$n = 2, \quad r = 2, \quad s = 0, \quad d_K = 5,$$

so

$$B = \sqrt{5} \cdot \left(\frac{4}{\pi}\right)^0 \cdot \left(\frac{2!}{2^2}\right) < 3.$$

Thus $\mathrm{Cl}(K)$ is generated by the primes that divide 2. We have $\mathcal{O}_K = \mathbf{Z}[\gamma]$, where $\gamma = \frac{1+\sqrt{5}}{2}$ satisfies $x^2 - x - 1$. The polynomial $x^2 - x - 1$ is irreducible mod 2, so $2\mathcal{O}_K$ is prime. Since it is principal, we see that $\mathrm{Cl}(K) = 1$ is trivial.

Example 7.3.4. In this example, we compute the class group of $K = \mathbf{Q}(\sqrt{-6})$. We have

$$n = 2, \quad r = 0, \quad s = 1, \quad d_K = -24,$$

so

$$B = \sqrt{24} \cdot \frac{4}{\pi} \cdot \left(\frac{2!}{2^2}\right) \sim 3.1.$$

Thus $\mathrm{Cl}(K)$ is generated by the prime ideals lying over 2 and 3. We have $\mathcal{O}_K = \mathbf{Z}[\sqrt{-6}]$, and $\sqrt{-6}$ satisfies $x^2+6 = 0$. Factoring x^2+6 modulo 2 and 3 we see that the class group is generated by the prime ideals

$$\mathfrak{p}_2 = (2, \sqrt{-6}) \qquad \text{and} \qquad \mathfrak{p}_3 = (3, \sqrt{-6}).$$

Also, $\mathfrak{p}_2^2 = 2\mathcal{O}_K$ and $\mathfrak{p}_3^2 = 3\mathcal{O}_K$, so $\mathfrak{p}_2$ and $\mathfrak{p}_3$ define elements of order dividing 2 in $\mathrm{Cl}(K)$.

Is either $\mathfrak{p}_2$ or $\mathfrak{p}_3$ principal? Fortunately, there is an easier norm trick that allows us to decide. Suppose $\mathfrak{p}_2 = (\alpha)$, where $\alpha = a + b\sqrt{-6}$. Then

$$2 = \mathrm{Norm}(\mathfrak{p}_2) = |\,\mathrm{Norm}(\alpha)| = (a + b\sqrt{-6})(a - b\sqrt{-6}) = a^2 + 6b^2.$$

Trying the first few values of $a, b \in \mathbf{Z}$, we see that this equation has no solutions, so $\mathfrak{p}_2$ can not be principal. By a similar argument, we see that $\mathfrak{p}_3$ is not principal either. Thus $\mathfrak{p}_2$ and $\mathfrak{p}_3$ define elements of order 2 in $\mathrm{Cl}(K)$.

Does the class of $\mathfrak{p}_2$ equal the class of $\mathfrak{p}_3$? Since $\mathfrak{p}_2$ and $\mathfrak{p}_3$ define classes of order 2, we can decide this by finding the class of $\mathfrak{p}_2 \cdot \mathfrak{p}_3$. We have

$$\mathfrak{p}_2 \cdot \mathfrak{p}_3 = (2, \sqrt{-6}) \cdot (3, \sqrt{-6}) = (6, 2\sqrt{-6}, 3\sqrt{-6}) \subset (\sqrt{-6}).$$

The ideals on both sides of the inclusion have norm 6, so by multiplicativity of the norm, they must be the same ideal. Thus $\mathfrak{p}_2 \cdot \mathfrak{p}_3 = (\sqrt{-6})$ is principal, so $\mathfrak{p}_2$ and $\mathfrak{p}_3$ represent the same element of $\mathrm{Cl}(K)$. We conclude that

$$\mathrm{Cl}(K) = \langle \mathfrak{p}_2 \rangle = \mathbf{Z}/2\mathbf{Z}.$$

Chapter 8

Dirichlet's Unit Theorem

In this chapter we will prove Dirichlet's unit theorem, which is a structure theorem for the group of units of the ring of integers of a number field. The answer is remarkably simple: if K has r real and s pairs of complex conjugate embeddings, then

$$\mathcal{O}_K^* \approx \mathbf{Z}^{r+s-1} \times T,$$

where T is a finite cyclic group.

Many questions can be encoded as questions about the structure of the group of units. For example, Dirichlet's unit theorem explains the structure the integer solutions (x, y) to Pell's equation $x^2 - dy^2 = 1$ (see Section 8.2.1).

8.1 The Group of Units

Definition 8.1.1 (Unit Group)**.** The *group of units* U_K associated to a number field K is the group of elements of $\mathcal{O}_K$ that have an inverse in $\mathcal{O}_K$.

Theorem 8.1.2 (Dirichlet)**.** *The group U_K is the product of a finite cyclic group of roots of unity with a free abelian group of rank $r+s-1$, where r is the number of real embeddings of K and s is the number of complex conjugate pairs of embeddings.*

(Note that we will prove a generalization of Theorem 8.1.2 in Section 12.1 below.)

We prove the theorem by defining a map $\varphi : U_K \to \mathbf{R}^{r+s}$, and showing that the kernel of φ is finite and the image of φ is a lattice in a hyperplane in $\mathbf{R}^{r+s}$. The trickiest part of the proof is showing that the image of φ spans a hyperplane, and we do this by a clever application of Blichfeld's Lemma 7.1.5.

Remark 8.1.3. Theorem 8.1.2 is due to Dirichlet who lived 1805–1859. Thomas Hirst described Dirichlet thus:

> He is a rather tall, lanky-looking man, with moustache and beard about to turn grey with a somewhat harsh voice and rather deaf. He was unwashed, with his cup of coffee and cigar. One of his failings is forgetting time, he pulls his watch out, finds it past three, and runs out without even finishing the sentence.

Koch wrote that:

> ... important parts of mathematics were influenced by Dirichlet. His proofs characteristically started with surprisingly simple observations, followed by extremely sharp analysis of the remaining problem.

I think Koch's observation nicely describes the proof we will give of Theorem 8.1.2.

Units have a simple characterization in terms of their norm.

Proposition 8.1.4. *An element $a \in \mathcal{O}_K$ is a unit if and only if* $\mathrm{Norm}_{K/\mathbf{Q}}(a) = \pm 1$.

Proof. Write $\mathrm{Norm} = \mathrm{Norm}_{K/\mathbf{Q}}$. If a is a unit, then a^{-1} is also a unit, and $1 = \mathrm{Norm}(a)\,\mathrm{Norm}(a^{-1})$. Since both $\mathrm{Norm}(a)$ and $\mathrm{Norm}(a^{-1})$ are integers, it follows that $\mathrm{Norm}(a) = \pm 1$. Conversely, if $a \in \mathcal{O}_K$ and $\mathrm{Norm}(a) = \pm 1$, then the equation $aa^{-1} = 1 = \pm\mathrm{Norm}(a)$ implies that $a^{-1} = \pm\mathrm{Norm}(a)/a$. But $\mathrm{Norm}(a)$ is the product of the images of a in $\mathbf{C}$ by all embeddings of K into $\mathbf{C}$, so $\mathrm{Norm}(a)/a$ is also a product of images of a in $\mathbf{C}$, hence a product of algebraic integers, hence an algebraic integer. Thus $a^{-1} \in K \cap \overline{\mathbf{Z}} = \mathcal{O}_K$, which proves that a is a unit. □

Let r be the number of real and s the number of complex conjugate embeddings of K into $\mathbf{C}$, so $n = [K : \mathbf{Q}] = r + 2s$. Define the *log embedding*

$$\varphi : U_K \to \mathbf{R}^{r+s}$$

by

$$\varphi(a) = (\log|\sigma_1(a)|, \ldots, \log|\sigma_{r+s}(a)|).$$

(Here $|z|$ is the usual absolute value of $z = x + iy \in \mathbf{C}$, so $|z| = \sqrt{x^2 + y^2}$.)

Lemma 8.1.5. *The image of φ lies in the hyperplane*

$$H = \{(x_1, \ldots, x_{r+s}) \in \mathbf{R}^{r+s} : x_1 + \cdots + x_r + 2x_{r+1} + \cdots + 2x_{r+s} = 0\}. \quad (8.1.1)$$

Proof. If $a \in U_K$, then by Proposition 8.1.4,

$$\left(\prod_{i=1}^{r} |\sigma_i(a)|\right) \cdot \left(\prod_{i=r+1}^{r+s} |\sigma_i(a)|^2\right) = |\operatorname{Norm}_{K/\mathbf{Q}}(a)| = 1.$$

Taking logs of both sides proves the lemma. □

Lemma 8.1.6. *The kernel of φ is finite.*

Proof. We have

$$\begin{aligned}\operatorname{Ker}(\varphi) &\subset \{a \in \mathcal{O}_K : |\sigma_i(a)| = 1 \text{ for } i = 1, \ldots, r+s\}\\ &\subset \sigma(\mathcal{O}_K) \cap X,\end{aligned}$$

where X is the bounded subset of $\mathbf{R}^{r+s}$ of elements all of whose coordinates have absolute value at most 1. Since $\sigma(\mathcal{O}_K)$ is a lattice (see Proposition 2.4.5), the intersection $\sigma(\mathcal{O}_K) \cap X$ is finite, so $\operatorname{Ker}(\varphi)$ is finite. □

Lemma 8.1.7. *The kernel of φ is a finite cyclic group.*

Proof. Lemma 8.1.6 implies that $\ker(\varphi)$ is a finite group. It is a general fact that any finite subgroup G of the multiplicative group K^* of a field is cyclic. (Proof: If n is the exponent of G, then every element of G is a root of the polynomial $x^n - 1$. A polynomial of degree n over a field has at most n roots, so G has order at most n, hence G is cyclic of order n.) □

To prove Theorem 8.1.2, it suffices to prove that $\operatorname{Im}(\varphi)$ is a lattice in the hyperplane H of (8.1.1), which we view as a vector space of dimension $r + s - 1$.

Define an embedding

$$\sigma : K \hookrightarrow \mathbf{R}^n \quad (8.1.2)$$

given by $\sigma(x) = (\sigma_1(x), \ldots, \sigma_{r+s}(x))$, where we view $\mathbf{C} \cong \mathbf{R} \times \mathbf{R}$ via $a + bi \mapsto (a, b)$. Thus this is the embedding

$$\begin{aligned}x \mapsto \bigl(&\sigma_1(x), \sigma_2(x), \ldots, \sigma_r(x),\\ &\operatorname{Re}(\sigma_{r+1}(x)), \operatorname{Im}(\sigma_{r+1}(x)), \ldots, \operatorname{Re}(\sigma_{r+s}(x)), \operatorname{Im}(\sigma_{r+s}(x))\bigr).\end{aligned}$$

Lemma 8.1.8. *The image $\varphi : U_K \to \mathbf{R}^{r+s}$ is discrete.*

Proof. We will show that for any bounded subset X of $\mathbf{R}^{r+s}$, the intersection $\varphi(U_K) \cap X$ is finite. If X is bounded, then for any $u \in Y = \varphi^{-1}(X) \subset U_K$ the coordinates of $\sigma(u)$ are bounded, since $|\log(x)|$ is bounded on bounded subsets of $[1, \infty)$. Thus $\sigma(Y)$ is a bounded subset of $\mathbf{R}^n$. Since $\sigma(Y) \subset \sigma(\mathcal{O}_K)$, and $\sigma(\mathcal{O}_K)$ is a lattice in $\mathbf{R}^n$, it follows that $\sigma(Y)$ is finite; moreover, σ is injective, so Y is finite. Thus $\varphi(U_K) \cap X \subset \varphi(Y) \cap X$ is finite. □

We will use the following lemma in our proof of Theorem 8.1.2.

Lemma 8.1.9. *Let $n \geq 2$ be an integer, suppose $w_1, \ldots, w_n \in \mathbf{R}$ are not all equal, and suppose $A, B \in \mathbf{R}$ are positive. Then there exist $d_1, \ldots, d_n \in \mathbf{R}_{>0}$ such that*

$$|w_1 \log(d_1) + \cdots + w_n \log(d_n)| > B$$

and $d_1 \cdots d_n = A$.

Proof. Order the w_i so that $w_1 \neq 0$. By hypothesis there exists a w_j such that $w_j \neq w_1$, and again re-ordering we may assume that $j = 2$. Set $d_3 = \cdots = d_{r+s} = 1$. Suppose d_1, d_2 are any positive real numbers with $d_1 d_2 = A$. Since $\log(1) = 0$,

$$\begin{aligned}\left|\sum_{i=1}^{n} w_i \log(d_i)\right| &= |w_1 \log(d_1) + w_2 \log(d_2)| \\ &= |w_1 \log(d_1) + w_2 \log(A/d_1)| \\ &= |(w_1 - w_2)\log(d_1) + w_2 \log(A)|\end{aligned}$$

Since $w_1 \neq w_2$, we have $|(w_1 - w_2)\log(d_1) + w_2 \log(A)| \to \infty$ as $d_1 \to \infty$. It is thus possible to choose the d_i as in the lemma. □

Proof of Theorem 8.1.2. By Lemma 8.1.8, the image $\varphi(U_K)$ is discrete, so it remains to show that $\varphi(U_K)$ spans H. Let W be the $\mathbf{R}$-span of the image $\varphi(U_K)$, and note that W is a subspace of H, by Lemma 8.1.5. We will show that $W = H$ indirectly by showing that if $v \notin H^{\perp}$, where $\perp$ is the orthogonal complement with respect to the dot product on $\mathbf{R}^{r+s}$, then $v \notin W^{\perp}$. This will show that $W^{\perp} \subset H^{\perp}$, hence that $H \subset W$, as required.

Thus suppose $z = (z_1, \ldots, z_{r+s}) \notin H^{\perp}$. Define a function $f : K^* \to \mathbf{R}$ by

$$f(x) = z_1 \log|\sigma_1(x)| + \cdots + z_{r+s} \log|\sigma_{r+s}(x)|. \tag{8.1.3}$$

Note that $f(U_K) = \{0\}$ if and only if $z \in W^{\perp}$, so to show that $z \notin W^{\perp}$ we show that there exists some $u \in U_K$ with $f(u) \neq 0$.

Let

$$A = \sqrt{|d_K|} \cdot \left(\frac{2}{\pi}\right)^s \in \mathbf{R}_{>0}.$$

Choose any positive real numbers $c_1, \ldots, c_{r+s} \in \mathbf{R}_{>0}$ such that

$$c_1 \cdots c_r \cdot (c_{r+1} \cdots c_{r+s})^2 = A.$$

Let

$$\begin{aligned} S = \{(x_1, \ldots, x_n) \in \mathbf{R}^n :& \\ &|x_i| \leq c_i \text{ for } 1 \leq i \leq r, \\ &|x_i^2 + x_{i+s}^2| \leq c_i^2 \text{ for } r < i \leq r+s\} \subset \mathbf{R}^n. \end{aligned}$$

Then S is closed, bounded, convex, symmetric with respect to the origin, and of dimension $r+2s$, since S is a product of r intervals and s discs, each of which has these properties. Viewing S as a product of intervals and discs, we see that the volume of S is

$$\mathrm{Vol}(S) = \prod_{i=1}^{r}(2c_i)\cdot\prod_{i=1}^{s}(\pi c_i^2) = 2^r\cdot\pi^s\cdot A.$$

Recall Blichfeldt's Lemma 7.1.5, which asserts that if L is a lattice and S is closed, bounded, etc., and has volume at least $2^n\cdot\mathrm{Vol}(V/L)$, then $S\cap L$ contains a nonzero element. To apply this lemma, we take $L=\sigma(\mathcal{O}_K)\subset\mathbf{R}^n$, where σ is as in (8.1.2). By Lemma 7.1.7, we have $\mathrm{Vol}(\mathbf{R}^n/L) = 2^{-s}\sqrt{|d_K|}$. To check the hypothesis of Blichfeld's lemma, note that

$$\mathrm{Vol}(S) = 2^{r+s}\sqrt{|d_K|} = 2^n 2^{-s}\sqrt{|d_K|} = 2^n\,\mathrm{Vol}(\mathbf{R}^n/L).$$

Thus there exists a nonzero element x in $S\cap\sigma(\mathcal{O}_K)$. Let $a\in\mathcal{O}_K$ with $\sigma(a)=x$, then $\sigma(a)\in S$, so $|\sigma_i(a)|\leq c_i$ for $1\leq i\leq r+s$. We then have

$$\begin{aligned}|\operatorname{Norm}_{K/\mathbf{Q}}(a)| &= \left|\prod_{i=1}^{r+2s}\sigma_i(a)\right|\\ &= \prod_{i=1}^{r}|\sigma_i(a)|\cdot\prod_{i=r+1}^{s}|\sigma_i(a)|^2\\ &\leq c_1\cdots c_r\cdot(c_{r+1}\cdots c_{r+s})^2 = A.\end{aligned}$$

Since $a\in\mathcal{O}_K$ is nonzero, we also have

$$|\operatorname{Norm}_{K/\mathbf{Q}}(a)|\geq 1.$$

Moreover, if for any $i\leq r$, we have $|\sigma_i(a)|<\frac{c_i}{A}$, then

$$1\leq|\operatorname{Norm}_{K/\mathbf{Q}}(a)| < c_1\cdots\frac{c_i}{A}\cdots c_r\cdot(c_{r+1}\cdots c_{r+s})^2 = \frac{A}{A}=1,$$

a contradiction, so $|\sigma_i(a)|\geq\frac{c_i}{A}$ for $i=1,\ldots,r$. Likewise, $|\sigma_i(a)|^2\geq\frac{c_i^2}{A}$, for $i=r+1,\ldots,r+s$. Rewriting this we have

$$\frac{c_i}{|\sigma_i(a)|}\leq A \quad\text{for } i\leq r \quad\text{and}\quad \left(\frac{c_i}{|\sigma_i(a)|}\right)^2\geq A \quad\text{for } i=r+1,\ldots,r+s. \qquad (8.1.4)$$

Recall that our overall strategy is to use an appropriately chosen a to construct a unit $u\in U_K$ such $f(u)\neq 0$. First, let $b_1,\ldots,b_m$ be representative generators for the finitely many nonzero principal ideals of $\mathcal{O}_K$ of norm at most A. Since $|\operatorname{Norm}_{K/\mathbf{Q}}(a)|\leq A$, we have $(a)=(b_j)$, for some j, so there is a unit $u\in\mathcal{O}_K$ such that $a=ub_j$.

Let

$$t = t_{c_1,\ldots,c_{r+s}} = z_1\log(c_1)+\cdots+z_{r+s}\log(c_{r+s}),$$

and recall $f : K^* \to \mathbf{R}$ defined in (8.1.3) above. We first show that

$$|f(u) - t| \leq B_j = |f(b_j)| + \log(A) \cdot \left(\sum_{i=1}^{r} |z_i| + \frac{1}{2} \cdot \sum_{i=r+1}^{s} |z_i| \right).$$

We have

$$\begin{aligned} |f(u) - t| &= |f(a) - f(b_j) - t| \\ &\leq |f(b_j)| + |t - f(a)| \\ &= |f(b_j)| + |z_1(\log(c_1) - \log(|\sigma_1(a)|)) + \cdots + z_{r+s}(\log(c_{r+s}) - \log(|\sigma_{r+s}(a)|))| \\ &= |f(b_j)| + |z_1 \cdot \log(c_1/|\sigma_1(a)|) + \cdots + \frac{z_{r+s}}{2} \cdot \log((c_{r+s}/|\sigma_{r+s}(a)|)^2)| \\ &\leq |f(b_j)| + \log(A) \cdot \left(\sum_{i=1}^{r} |z_i| + \frac{1}{2} \cdot \sum_{i=r+1}^{s} |z_i| \right). \end{aligned}$$

In the last step we use (8.1.4).

Let $B = \max_j B_j$, and note that B does not depend on the choice of the c_i; in fact, it only depends on the field K. Moreover, for any choice of the c_i as above, we have

$$|f(u) - t| \leq B.$$

If we can choose positive real numbers c_i such that

$$\begin{aligned} c_1 \cdots c_r \cdot (c_{r+1} \cdots c_{r+s})^2 &= A \\ |t_{c_1,\ldots,c_{r+s}}| &> B, \end{aligned}$$

then the fact that $|f(u) - t| \leq B$ would then imply that $|f(u)| > 0$, which is exactly what we aimed to prove.

If $r + s = 1$, then we are trying to prove that $\varphi(U_K)$ is a lattice in $\mathbf{R}^0 = \mathbf{R}^{r+s-1}$, which is automatically true, so assume $r + s > 1$. To finish the proof, we explain how to use Lemma 8.1.9 to choose c_i such that $|t| > B$. We have

$$\begin{aligned} &z_1 \log(c_1) + \cdots + z_{r+s} \log(c_{r+s}) = \\ &\qquad z_1 \log(c_1) + \cdots + z_r \log(c_r) + \frac{1}{2} \cdot z_{r+1} \log(c_{r+1}^2) + \cdots + \frac{1}{2} \cdot z_{r+s} \log(c_{r+s}^2) \\ &\qquad = w_1 \log(d_1) + \cdots + w_r \log(d_r) + w_{r+1} \log(d_{r+1}) + \cdots + \cdot w_{r+s} \log(d_{r+s}), \end{aligned}$$

where $w_i = z_i$ and $d_i = c_i$ for $i \leq r$, and $w_i = \frac{1}{2} z_i$ and $d_i = c_i^2$ for $r < i \leq r + s$, The condition that $z \notin H^\perp$ is that the w_i are not all the same, and in our new coordinates the lemma is equivalent to showing that $|\sum_{i=1}^{r+s} w_i \log(d_i)| > B$, subject to the condition that $\prod_{i=1}^{r+s} d_i = A$. But this is exactly what Lemma 8.1.9 shows. It is thus possible to find a unit u such that $|f(u)| > 0$. Thus $z \notin W^\perp$, so $W^\perp \subset Z^\perp$, whence $Z \subset W$, which finishes the proof Theorem 8.1.2. □

8.2 Examples with Sage

8.2.1 Pell's Equation

The so-called "Pell's equation" is $x^2 - dy^2 = 1$ with $d > 0$ square free, and we seek integer solutions x, y to this equation. If $x + y\sqrt{d} \in K = \mathbf{Q}(\sqrt{d})$, then

$$\mathrm{Norm}(x + y\sqrt{d}) = (x + y\sqrt{d})(x - y\sqrt{d}) = x^2 - dy^2.$$

Thus if (x, y) are integers such that $x^2 - dy^2 = 1$, then $\alpha = x + \sqrt{d}y \in \mathcal{O}_K$ has norm 1, so by Proposition 8.1.4 we have $\alpha \in U_K$. The integer solutions to Pell's equation thus form a finite-index subgroup of the group of units in the ring of integers of $\mathbf{Q}(\sqrt{d})$. Dirichlet's unit theorem implies that for any d the solutions to Pell's equation with x, y not both negative forms an infinite cyclic group, which is a fact that takes substantial work to prove using only elementary number theory (for example, using continued fractions).

We first solve Pell's equation $x^2 - 5y^2 = 1$ with $d = 5$ by finding the units of the ring of integers of $\mathbf{Q}(\sqrt{5})$ using Sage.

```
sage: K.<sqrt5> = QuadraticField(5)
sage: G = K.unit_group(); G
Unit group with structure C2 x Z of Number Field in sqrt5 with
defining polynomial x^2 - 5
sage: G.0
-1
sage: u = G.1; u
1/2*sqrt5 - 1/2
```

The subgroup of cubes gives us the units with integer x, y (not both negative).

```
sage: u, u^2, u^3, u^4, u^5, u^6
(1/2*sqrt5 - 1/2, -1/2*sqrt5 + 3/2, sqrt5 - 2, -3/2*sqrt5 + 7/2,
 5/2*sqrt5 - 11/2, -4*sqrt5 + 9)
sage: [list(v^i) for i in [0..9]]
[[1, 0], [-2, 1], [9, -4], [-38, 17], [161, -72], [-682, 305], [2889,
 -1292], [-12238, 5473], [51841, -23184], [-219602, 98209]]
```

A great article about Pell's equation is [Len02]. The MathSciNet review begins: "This wonderful article begins with history and some elementary facts and proceeds to greater and greater depth about the existence of solutions to Pell equations and then later the algorithmic issues of finding those solutions. The cattle problem is discussed, as are modern smooth number methods for solving Pell equations and the algorithmic issues of representing very large solutions in a reasonable way."

The simplest solutions to Pell's equation can be huge, even when d is quite small. Read Lenstra's paper for some examples from over two thousand years ago. Here is one example for $d = 10000019$.

```
sage: K.<a> = QuadraticField(next_prime(10^7))
sage: G = K.unit_group(); G.1
163580259880346328225592238121094625499142677693142915506747253000
340064100365767872890438816249271266423998175030309436575610631639
272377601680603795883791477817611974184075445702823789975945910042
889569323816504809803 9*a -
```

```
517286692885814967470170672368346798303629034373575202975075605058
714958080893991274427903448098643836512878351227856269086856679078
304979321047765031073345259902622712059164969008633603603640331175
6634562204182936222240930
```

Exercise 8.2.1. Let U be the group of units $x + y\sqrt{5}$ of the ring of integers of $K = \mathbf{Q}(\sqrt{5})$.

1. Prove that the set S of units $x + y\sqrt{5} \in U$ with $x, y \in \mathbf{Z}$ is a subgroup of U. (The main point is to show that the inverse of a unit with $x, y \in \mathbf{Z}$ again has coefficients in $\mathbf{Z}$.)

2. Let U^3 denote the subgroup of cubes of elements of U. Prove that $S = U^3$ by showing that $U^3 \subset S \subsetneq U$ and that there are no groups H with $U^3 \subsetneq H \subsetneq U$.

8.2.2 Examples with Various Signatures

In this section we give examples for various (r, s) pairs. First we consider $K = \mathbf{Q}(i)$.

```
sage: K.<a> = QuadraticField(-1)
sage: K.signature()
(0, 1)
sage: U = K.unit_group(); U
Unit group with structure C4 of Number Field in a with
defining polynomial x^2 + 1
sage: U.0
-a
```

The `signature` method returns the number of real and complex conjugate embeddings of K into $\mathbf{C}$. The `unit_group` method, which we used above, returns the unit group U_K as an abstract abelian group and a homomorphism $U_K \to \mathcal{O}_K$.

Next we consider $K = \mathbf{Q}(\sqrt[3]{2})$.

```
sage: R.<x> = QQ[]
sage: K.<a> = NumberField(x^3 - 2)
sage: K.signature()
(1, 1)
sage: U = K.unit_group(); U
Unit group with structure C2 x Z of Number Field in a with
defining polynomial x^3 - 2
sage: U.gens()
[-1, a - 1]
sage: u = U.1; u
a - 1
```

Below we use the `places` command, which returns the real embeddings and representatives for the complex conjugate embeddings. We use the places to define the log map φ, which plays such a big role in this chapter.

```
sage: S = K.places(prec=53); S
[Ring morphism:
  From: Number Field in a with defining polynomial x^3 - 2
  To:   Real Double Field
  Defn: a |--> 1.25992104989, Ring morphism:
  From: Number Field in a with defining polynomial x^3 - 2
  To:   Complex Double Field
  Defn: a |--> -0.629960524947 + 1.09112363597*I]
sage: phi = lambda z : [log(abs(sigma(z))) for sigma in S]
sage: phi(u)
[-1.34737734833, 0.673688674165]
sage: phi(K(-1))
[0.0, 0.0]
```

Note that $\varphi : U_K \to \mathbf{R}^2$, and the image lands in the 1-dimensional subspace of (x_1, x_2) such that $x_1 + 2x_2 = 0$. Also, note that $\varphi(-1) = 0$.

Let's try a field such that $r + s - 1 = 2$. First, one with $r = 0$ and $s = 3$:

```
sage: K.<a> = NumberField(x^6 + x + 1)
sage: K.signature()
(0, 3)
sage: U = K.unit_group(); U
Unit group with structure C2 x Z x Z of Number Field in a with
defining polynomial x^6 + x + 1
sage: u1 = U.1; u1
a
sage: u2 = U.2; u2
a^3 + a
sage: S = K.places(prec=53)
sage: phi = lambda z : [log(abs(sigma(z))) for sigma in S]
sage: phi(u1)
[-0.167415483286, 0.0486439097527, 0.118771573533]
sage: phi(u2)
[0.306785708923, -1.07251465055, 0.765728941626]
sage: phi(K(-1))
[0.0, 0.0, 0.0]
sage: sum(phi(u1))
-2.63677968348e-15
sage: sum(phi(u2))
-5.10702591328e-15
```

Notice that the log image of u_1 is clearly not a real multiple of the log image of u_2 (e.g., the scalar would have to be positive because of the first coefficient, but negative because of the second). This illustrates the fact that the log images of u_1 and u_2 span a two-dimensional space.

Next we compute a field with $r = 3$ and $s = 0$. (A field with $s = 0$ is called totally real.)

```
sage: K.<a> = NumberField(x^3 + x^2 - 5*x - 1)
sage: K.signature()
(3, 0)
sage: U = K.unit_group(); U
Unit group with structure C2 x Z x Z of Number Field in a with
defining polynomial x^3 + x^2 - 5*x - 1
sage: u1 = U.1; u
a - 1
sage: u2 = U.2; u2
a
sage: S = K.places(prec=53)
sage: phi = lambda z : [log(abs(sigma(z))) for sigma in S]
sage: phi(u1)
[-0.774767022346, -0.392848724581, 1.16761574693]
sage: phi(u2)
[0.996681204093, -1.64022415032, 0.643542946229]
```

A field with $r = 0$ is called totally complex. For example, the *cyclotomic fields* $\mathbf{Q}(\zeta_n)$ are totally complex, where ζ_n is a primitive nth root of unity. The degree of $\mathbf{Q}(\zeta_n)$ over $\mathbf{Q}$ is $\varphi(n)$ and $r = 0$, so $s = \varphi(n)/2$ (assuming $n > 2$).

```
sage: K.<a> = CyclotomicField(11); K
Cyclotomic Field of order 11 and degree 10
sage: K.signature()
(0, 5)
sage: U = K.unit_group(); U
Unit group with structure C22 x Z x Z x Z x Z of Cyclotomic Field
of order 11 and degree 10
sage: u = U.1; u
a^9 + a^7 + a^5 + a^3 + a + 1
sage: S = K.places(prec=20)
sage: phi = lambda z : [log(abs(sigma(z))) for sigma in S]
sage: phi(u)
[1.2566, 0.18533, -0.26981, -0.52028, -0.65179]
sage: for u in U.gens():
...       print phi(u)
[0.00000, 0.00000, 0.00000, -9.5367e-7, -9.5367e-7]
[1.2566, 0.18533, -0.26981, -0.52028, -0.65179]
[0.26981, 0.52029, -0.18533, 0.65180, -1.2566]
[0.65180, 0.26981, -1.2566, -0.18533, 0.52028]
[-0.084484, -1.1721, -0.33496, 0.60477, 0.98675]
```

How far can we go computing unit groups of cyclotomic fields directly with Sage?

```
sage: time U = CyclotomicField(11).unit_group()
Time: CPU 0.13 s, Wall: 0.13 s
sage: time U = CyclotomicField(13).unit_group()
Time: CPU 0.24 s, Wall: 0.24 s
sage: time U = CyclotomicField(17).unit_group()
Time: CPU 0.98 s, Wall: 0.98 s
sage: time U = CyclotomicField(23).unit_group()
.... I waited a few minutes and gave up....
```

However, if you are willing to assume some conjectures (something related to the Generalized Riemann Hypothesis), you can go further:

```
sage: proof.number_field(False)
sage: time U = CyclotomicField(11).unit_group()
CPU times: user 0.08 s, sys: 0.00 s, total: 0.09 s
Wall time: 0.09 s
sage: time U = CyclotomicField(13).unit_group()
CPU times: user 0.11 s, sys: 0.00 s, total: 0.12 s
Wall time: 0.12 s
sage: time U = CyclotomicField(17).unit_group()
CPU times: user 0.52 s, sys: 0.00 s, total: 0.53 s
Wall time: 0.53 s
sage: time U = CyclotomicField(23).unit_group()
CPU times: user 2.42 s, sys: 0.02 s, total: 2.44 s
Wall time: 2.44 s
sage: time U = CyclotomicField(29).unit_group()
CPU times: user 21.07 s, sys: 1.06 s, total: 22.13 s
Wall time: 22.14 s
```

The generators of the units for $\mathbf{Q}(\zeta_{29})$ are

$$\begin{aligned}
u_0 &= -\zeta_{29}^3\\
u_1 &= \zeta_{29}^{26} + \zeta_{29}^{25} + \zeta_{29}^{22} + \zeta_{29}^{21} + \zeta_{29}^{19} + \zeta_{29}^{18} + \zeta_{29}^{15} + \zeta_{29}^{14} + \zeta_{29}^{11} + \zeta_{29}^{8} + \zeta_{29}^{7} + \zeta_{29}^{4} + \zeta_{29}^{3} + \zeta_{29} + 1\\
u_2 &= \zeta_{29}^{14} + \zeta_{29}^{3}\\
u_3 &= \zeta_{29}^{3} + 1\\
u_4 &= \zeta_{29}^{26} + \zeta_{29}^{20} + \zeta_{29}^{3}\\
u_5 &= \zeta_{29}^{22} + \zeta_{29}^{11} + \zeta_{29}^{2}\\
u_6 &= \zeta_{29}^{10} + \zeta_{29}^{9} + \zeta_{29}^{8}\\
u_7 &= \zeta_{29}^{23} + \zeta_{29}\\
u_8 &= \zeta_{29}^{17} + \zeta_{29}^{11}\\
u_9 &= \zeta_{29}^{22} + \zeta_{29}^{3}\\
u_{10} &= \zeta_{29}^{24} + \zeta_{29}^{19} + \zeta_{29}^{5} + 1\\
u_{11} &= \zeta_{29}^{19} + \zeta_{29}^{6}\\
u_{12} &= \zeta_{29}^{27} + \zeta_{29}^{19} + \zeta_{29}^{11} + \zeta_{29}^{6} + \zeta_{29}^{3}\\
u_{13} &= \zeta_{29}^{26} + \zeta_{29}^{15} + \zeta_{29}^{4}
\end{aligned}$$

There are better ways to compute units in cyclotomic fields than to just use general purpose software. For example, there are explicit *cyclotomic units* that can be written down and generate a finite subgroup of U_K. See [Was97, Ch. 8], which would be a great book to read now that you've got this far in the present book. Also, using the theorem explained in that book, it is probably possible to make the `unit_group` command in Sage for cyclotomic fields extremely fast, which would be an interesting project for a reader who also likes to code.

Chapter 9

Decomposition and Inertia Groups

In this chapter we will study extra structure in the case when K is Galois over $\mathbf{Q}$. We will learn about Frobenius elements, the Artin symbol, decomposition groups, and how the Galois group of K is related to Galois groups of residue class fields. These are the basic structures needed to attach L-function to representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, which will play a central role in the next few chapters.

9.1 Galois Extensions

In this section we give a survey (no proofs) of the basic facts about Galois extensions of $\mathbf{Q}$ that will be needed in the rest of this chapter.

Definition 9.1.1 (Galois)**.** An extension K/L of number fields is *Galois* if

$$\#\,\mathrm{Aut}(K/L) = [K : L],$$

where $\mathrm{Aut}(K/L)$ is the group of automorphisms of K that fix L. We write

$$\mathrm{Gal}(K/L) = \mathrm{Aut}(K/L).$$

For example, if $K \subset \mathbf{C}$ is a number field embedded in the complex numbers, then K is *Galois* over $\mathbf{Q}$ if every field homomorphism $K \to \mathbf{C}$ has image K. As another example, any quadratic extension K/L is Galois over L, since it is of the form $L(\sqrt{a})$, for some $a \in L$, and the nontrivial automorphism is induced by $\sqrt{a} \mapsto -\sqrt{a}$, so there is always one nontrivial automorphism. If $f \in L[x]$ is an irreducible cubic polynomial, and a is a root of f, then one proves in a course on Galois theory that $L(a)$ is Galois over L if and only if the discriminant of f is a perfect square in L. "Random" number fields of degree bigger than 2 are rarely Galois.

If $K \subset \mathbf{C}$ is a number field, then the *Galois closure* K^{gc} of K in $\mathbf{C}$ is the field generated by all images of K under all embeddings in $\mathbf{C}$ (more generally, if K/L

is an extension, the Galois closure of K over L is the field generated by images of embeddings $K \to \mathbf{C}$ that are the identity map on L). If $K = \mathbf{Q}(a)$, then K^{gc} is the field generated by all of the conjugates of a, and is hence Galois over $\mathbf{Q}$, since the image under an embedding of any polynomial in the conjugates of a is again a polynomial in conjugates of a.

How much bigger can the degree of K^{gc} be as compared to the degree of $K = \mathbf{Q}(a)$? There is an embedding of $\mathrm{Gal}(K^{\mathrm{gc}}/\mathbf{Q})$ into the group of permutations of the conjugates of a. If a has n conjugates, then this is an embedding $\mathrm{Gal}(K^{\mathrm{gc}}/\mathbf{Q}) \hookrightarrow S_n$, where S_n is the symmetric group on n symbols, which has order $n!$. Thus the degree of the K^{gc} over $\mathbf{Q}$ is a divisor of $n!$. Also $\mathrm{Gal}(K^{\mathrm{gc}}/\mathbf{Q})$ is a transitive subgroup of S_n, which constrains the possibilities further. When $n = 2$, we recover the fact that quadratic extensions are Galois. When $n = 3$, we see that the Galois closure of a cubic extension is either the cubic extension or a quadratic extension of the cubic extension. One can show that the Galois closure of a cubic extension is obtained by adjoining the square root of the discriminant, which is why an irreducible cubic defines a Galois extension if and only if the discriminant is a perfect square.

For an extension K of $\mathbf{Q}$ of degree 5, it is "frequently" the case that the Galois closure has degree 120, and in fact it is an interesting problem to enumerate examples of degree 5 extension in which the Galois closure has degree smaller than 120. For example, the only possibilities for the order of a transitive proper subgroup of S_5 are 5, 10, 20, and 60; there are also proper subgroups of S_5 order $2, 3, 4, 6, 8, 12$, and 24, but none are transitive.

Let n be a positive integer. Consider the field $K = \mathbf{Q}(\zeta_n)$, where $\zeta_n = e^{2\pi i/n}$ is a primitive nth root of unity. If $\sigma : K \to \mathbf{C}$ is an embedding, then $\sigma(\zeta_n)$ is also an nth root of unity, and the group of nth roots of unity is cyclic, so $\sigma(\zeta_n) = \zeta_n^m$ for some m which is invertible modulo n. Thus K is Galois and $\mathrm{Gal}(K/\mathbf{Q}) \hookrightarrow (\mathbf{Z}/n\mathbf{Z})^*$. However, $[K : \mathbf{Q}] = \varphi(n)$, so this map is an isomorphism. (Remark: Taking a limit using the maps $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Gal}(\mathbf{Q}(\zeta_{p^r})/\mathbf{Q})$, we obtain a homomorphism $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{Z}_p^*$, which is called the *p-adic cyclotomic character*.)

Compositums of Galois extensions are Galois. For example, the biquadratic field $K = \mathbf{Q}(\sqrt{5}, \sqrt{-1})$ is a Galois extension of $\mathbf{Q}$ of degree 4, which is the compositum of the Galois extensions $\mathbf{Q}(\sqrt{5})$ and $\mathbf{Q}(\sqrt{-1})$ of $\mathbf{Q}$.

Fix a number field K that is Galois over a subfield L. Then the Galois group $G = \mathrm{Gal}(K/L)$ acts on many of the object that we have associated to K, including:

- the integers $\mathcal{O}_K$,
- the units U_K,
- the group of fractional ideals of $\mathcal{O}_K$,
- the class group $\mathrm{Cl}(K)$, and
- the set $S_{\mathfrak{p}}$ of prime ideals lying over a given nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_L$, i.e., the prime divisors of $\mathfrak{p}\mathcal{O}_K$.

In the next section we will be concerned with the action of $\mathrm{Gal}(K/L)$ on $S_{\mathfrak{p}}$, though actions on each of the other objects, especially $\mathrm{Cl}(K)$, are also of great interest. Understanding the action of $\mathrm{Gal}(K/L)$ on $S_{\mathfrak{p}}$ will enable us to associate, in a natural way, a holomorphic L-function to any complex representation $\mathrm{Gal}(K/L) \to \mathrm{GL}_n(\mathbf{C})$.

9.2 Decomposition of Primes: $efg = n$

If $I \subset \mathcal{O}_K$ is any ideal in the ring of integers of a Galois extension K of $\mathbf{Q}$ and $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$, then

$$\sigma(I) = \{\sigma(x) : x \in I\}$$

is also an ideal of $\mathcal{O}_K$.

Fix a prime $\mathfrak{p} \subset \mathcal{O}_K$ and write $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g}$, so $S_{\mathfrak{p}} = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_g\}$.

Definition 9.2.1 (Residue class degree)**.** Suppose $\mathfrak{P}$ is a prime of $\mathcal{O}_K$ lying over $\mathfrak{p}$. Then the *residue class degree* of $\mathfrak{P}$ is

$$f_{\mathfrak{P}/\mathfrak{p}} = [\mathcal{O}_K/\mathfrak{P} : \mathcal{O}_L/\mathfrak{p}],$$

i.e., the degree of the extension of residue class fields.

If $M/K/L$ is a tower of field extensions and $\mathfrak{q}$ is a prime of M over $\mathfrak{P}$, then

$$f_{\mathfrak{q}/\mathfrak{p}} = [\mathcal{O}_M/\mathfrak{q} : \mathcal{O}_L/\mathfrak{p}] = [\mathcal{O}_M/\mathfrak{q} : \mathcal{O}_K/\mathfrak{P}] \cdot [\mathcal{O}_K/\mathfrak{P} : \mathcal{O}_L/\mathfrak{p}] = f_{\mathfrak{q}/\mathfrak{P}} \cdot f_{\mathfrak{P}/\mathfrak{p}},$$

so the residue class degree is multiplicative in towers.

Note that if $\sigma \in \mathrm{Gal}(K/L)$ and $\mathfrak{P} \in S_p$, then σ induces an isomorphism of finite fields $\mathcal{O}_K/\mathfrak{P} \to \mathcal{O}_K/\sigma(\mathfrak{P})$ that fixes the common subfield $\mathcal{O}_L/\mathfrak{p}$. Thus the residue class degrees of $\mathfrak{P}$ and $\sigma(\mathfrak{P})$ are the same. In fact, much more is true.

Theorem 9.2.2. *Suppose K/L is a Galois extension of number fields, and let $\mathfrak{p}$ be a prime of $\mathcal{O}_L$. Write $\mathfrak{p}\mathcal{O}_K = \prod_{i=1}^{g} \mathfrak{P}_i^{e_i}$, and let $f_i = f_{\mathfrak{P}_i/\mathfrak{p}}$. Then $G = \mathrm{Gal}(K/L)$ acts transitively on the set $S_{\mathfrak{p}}$ of primes $\mathfrak{P}_i$, and*

$$e_1 = \cdots = e_g, \qquad f_1 = \cdots = f_g.$$

Morever, if we let e be the common value of the e_i, f the common value of the f_i, and $n = [K : L]$, then

$$efg = n.$$

Proof. For simplicity, we will give the proof only in the case $L = \mathbf{Q}$, but the proof works in general. Suppose $p \in \mathbf{Z}$ and $p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$, and $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_g\}$. We will first prove that G acts transitively on S. Let $\mathfrak{p} = \mathfrak{p}_i$ for some i. Recall that we proved long ago, using the Chinese Remainder Theorem (Theorem 5.1.4) that there exists $a \in \mathfrak{p}$ such that $(a)/\mathfrak{p}$ is an integral ideal that is coprime to $p\mathcal{O}_K$. The product

$$I = \prod_{\sigma \in G} \sigma((a)/\mathfrak{p}) = \prod_{\sigma \in G} \frac{(\sigma(a))\mathcal{O}_K}{\sigma(\mathfrak{p})} = \frac{(\mathrm{Norm}_{K/\mathbf{Q}}(a))\mathcal{O}_K}{\displaystyle\prod_{\sigma \in G} \sigma(\mathfrak{p})} \tag{9.2.1}$$

is a nonzero integral $\mathcal{O}_K$ ideal since it is a product of nonzero integral $\mathcal{O}_K$ ideals. Since $a \in \mathfrak{p}$ we have that $\mathrm{Norm}_{K/\mathbf{Q}}(a) \in \mathfrak{p} \cap \mathbf{Z} = p\mathbf{Z}$. Thus the numerator of the rightmost expression in (9.2.1) is divisible by $p\mathcal{O}_K$. Also, because $(a)/\mathfrak{p}$ is coprime to $p\mathcal{O}_K$, each $\sigma((a)/\mathfrak{p})$ is coprime to $p\mathcal{O}_K$ as well. Thus I is coprime to $p\mathcal{O}_K$. Thus the denominator of the rightmost expression in (9.2.1) must also be divisibly by $p\mathcal{O}_K$ in order to cancel the $p\mathcal{O}_K$ in the numerator. Thus we have shown that for any i,

$$\prod_{j=1}^{g} \mathfrak{p}_j^{e_j} = p\mathcal{O}_K \;\Big|\; \prod_{\sigma \in G} \sigma(\mathfrak{p}_i).$$

By unique factorization, since every $\mathfrak{p}_j$ appears in the left hand side, we must have that for each j there is a σ with $\sigma(\mathfrak{p}_i) = \mathfrak{p}_j$.

Choose some j and suppose that $k \neq j$ is another index. Because G acts transitively, there exists $\sigma \in G$ such that $\sigma(\mathfrak{p}_k) = \mathfrak{p}_j$. Applying σ to the factorization $p\mathcal{O}_K = \prod_{i=1}^{g} \mathfrak{p}_i^{e_i}$, we see that

$$\prod_{i=1}^{g} \mathfrak{p}_i^{e_i} = \prod_{i=1}^{g} \sigma(\mathfrak{p}_i)^{e_i}.$$

Taking $\mathrm{ord}_{\mathfrak{p}_j}$ on both sides and using unique factorization, we get $e_j = e_k$. Thus $e_1 = e_2 = \cdots = e_g$.

As was mentioned right before the statement of the theorem, for any $\sigma \in G$ we have $\mathcal{O}_K/\mathfrak{p}_i \cong \mathcal{O}_K/\sigma(\mathfrak{p}_i)$, so by transitivity $f_1 = f_2 = \cdots = f_g$. We have, upon apply CRT and that $\#(\mathcal{O}_K/(\mathfrak{p}^m)) = \#(\mathcal{O}_K/\mathfrak{p})^m$, that

$$\begin{aligned}[K : \mathbf{Q}] &= \dim_{\mathbf{Z}} \mathcal{O}_K = \dim_{\mathbf{F}_p} \mathcal{O}_K/p\mathcal{O}_K \\ &= \dim_{\mathbf{F}_p} \left(\bigoplus_{i=1}^{g} \mathcal{O}_K/\mathfrak{p}_i^{e_i} \right) = \sum_{i=1}^{g} e_i f_i = efg,\end{aligned}$$

which completes the proof. □

The rest of this section illustrates the theorem for quadratic fields and a cubic field and its Galois closure.

9.2.1 Quadratic Extensions

Suppose $K/\mathbf{Q}$ is a quadratic field. Then K is Galois, so for each prime $p \in \mathbf{Z}$ we have $2 = efg$. There are exactly three possibilities:

- **Ramified:** $e = 2$, $f = g = 1$: The prime p ramifies in $\mathcal{O}_K$, so $p\mathcal{O}_K = \mathfrak{p}^2$. There are only finitely many such primes, since if $f(x)$ is the minimal polynomial of a generator for $\mathcal{O}_K$, then p ramifies if and only if $f(x)$ has a multiple root modulo p. However, $f(x)$ has a multiple root modulo p if and only if p divides the discriminant of $f(x)$, which is nonzero because $f(x)$ is irreducible over $\mathbf{Z}$. (This argument shows there are only finitely many ramified primes in any number field. In fact, the ramified primes are exactly the ones that divide the discriminant.)

- **Inert:** $e = 1$, $f = 2$, $g = 1$: The prime p is inert in $\mathcal{O}_K$, so $p\mathcal{O}_K = \mathfrak{p}$ is prime. It is a nontrivial theorem that this happens half of the time, as we will see illustrated below for a particular example.

- **Split:** $e = f = 1$, $g = 2$: The prime p splits in $\mathcal{O}_K$, in the sense that $p\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2$ with $\mathfrak{p}_1 \neq \mathfrak{p}_2$. This happens the other half of the time.

For example, let $K = \mathbf{Q}(\sqrt{5})$, so $\mathcal{O}_K = \mathbf{Z}[\gamma]$, where $\gamma = (1+\sqrt{5})/2$. Then $p = 5$ is ramified, since $5\mathcal{O}_K = (\sqrt{5})^2$. More generally, the order $\mathbf{Z}[\sqrt{5}]$ has index 2 in $\mathcal{O}_K$, so for any prime $p \neq 2$ we can determine the factorization of p in $\mathcal{O}_K$ by finding the factorization of the polynomial $x^2 - 5 \in \mathbf{F}_p[x]$. The polynomial $x^2 - 5$ splits as a product of two distinct factors in $\mathbf{F}_p[x]$ if and only if $e = f = 1$ and $g = 2$. For $p \neq 2, 5$ this is the case if and only if 5 is a square in $\mathbf{F}_p$, i.e., if $\left(\frac{5}{p}\right) = 1$, where $\left(\frac{5}{p}\right)$ is $+1$ if 5 is a square mod p and -1 if 5 is not. By quadratic reciprocity,

$$\left(\frac{5}{p}\right) = (-1)^{\frac{5-1}{2}\cdot\frac{p-1}{2}} \cdot \left(\frac{p}{5}\right) = \left(\frac{p}{5}\right) = \begin{cases} +1 & \text{if } p \equiv \pm 1 \pmod 5 \\ -1 & \text{if } p \equiv \pm 2 \pmod 5. \end{cases}$$

Thus whether p splits or is inert in $\mathcal{O}_K$ is determined by the residue class of p modulo 5. It is a theorem of Dirichlet, which was massively generalized by Chebotarev, that $p \equiv \pm 1$ half the time and $p \equiv \pm 2$ the other half the time.

9.2.2 The Cube Root of Two

Suppose $K/\mathbf{Q}$ is not Galois. Then e_i, f_i, and g are defined for each prime $p \in \mathbf{Z}$, but we need not have $e_1 = \cdots = e_g$ or $f_1 = \cdots = f_g$. We do still have that $\sum_{i=1}^{g} e_i f_i = n$, by the Chinese Remainder Theorem.

For example, let $K = \mathbf{Q}(\sqrt[3]{2})$. We know that $\mathcal{O}_K = \mathbf{Z}[\sqrt[3]{2}]$. Thus $2\mathcal{O}_K = (\sqrt[3]{2})^3$, so for 2 we have $e = 3$ and $f = g = 1$.

Working modulo 5 we have

$$x^3 - 2 = (x+2)(x^2 + 3x + 4) \in \mathbf{F}_5[x],$$

and the quadratic factor is irreducible. Thus

$$5\mathcal{O}_K = (5, \sqrt[3]{2} + 2) \cdot (5, \sqrt[3]{2}^2 + 3\sqrt[3]{2} + 4).$$

Thus here $e_1 = e_2 = 1$, $f_1 = 1$, $f_2 = 2$, and $g = 2$. Thus when K is not Galois we need not have that the f_i are all equal.

9.3 The Decomposition Group

Suppose K is a number field that is Galois over $\mathbf{Q}$ with group $G = \mathrm{Gal}(K/\mathbf{Q})$. Fix a prime $\mathfrak{p} \subset \mathcal{O}_K$ lying over $p \in \mathbf{Z}$.

Definition 9.3.1 (Decomposition group)**.** The *decomposition group* of $\mathfrak{p}$ is the subgroup

$$D_\mathfrak{p} = \{\sigma \in G : \sigma(\mathfrak{p}) = \mathfrak{p}\} \subset G.$$

Note that $D_\mathfrak{p}$ is the stabilizer of $\mathfrak{p}$ for the action of G on the set of primes lying over p.

It also makes sense to define decomposition groups for relative extensions K/L, but for simplicity and to fix ideas in this section we only define decomposition groups for a Galois extension $K/\mathbf{Q}$.

Let $k_\mathfrak{p} = \mathcal{O}_K/\mathfrak{p}$ denote the residue class field of $\mathfrak{p}$. In this section we will prove that there is an exact sequence

$$1 \to I_\mathfrak{p} \to D_\mathfrak{p} \to \mathrm{Gal}(k_\mathfrak{p}/\mathbf{F}_p) \to 1,$$

where $I_\mathfrak{p}$ is the *inertia subgroup* of $D_\mathfrak{p}$, and $\#I_\mathfrak{p} = e$, where e is the exponent of $\mathfrak{p}$ in the factorization of $p\mathcal{O}_K$. The most interesting part of the proof is showing that the natural map $D_\mathfrak{p} \to \mathrm{Gal}(k_\mathfrak{p}/\mathbf{F}_p)$ is surjective.

We will also discuss the structure of $D_\mathfrak{p}$ and introduce Frobenius elements, which play a crucial role in understanding Galois representations.

Recall from Theorem 9.2.2 that G acts transitively on the set of primes $\mathfrak{p}$ lying over p. The orbit-stabilizer theorem implies that $[G : D_\mathfrak{p}]$ equals the cardinality of the orbit of $\mathfrak{p}$, which by Theorem 9.2.2 equals the number g of primes lying over p, so $[G : D_\mathfrak{p}] = g$.

Lemma 9.3.2. *The decomposition subgroups $D_\mathfrak{p}$ corresponding to primes $\mathfrak{p}$ lying over a given p are all conjugate as subgroups of G.*

Proof. We have for each $\sigma, \tau \in G$, that

$$\tau^{-1}\sigma\tau\mathfrak{p} = \mathfrak{p} \iff \sigma\tau\mathfrak{p} = \tau\mathfrak{p},$$

so

$$\sigma \in D_{\tau\mathfrak{p}} \iff \tau^{-1}\sigma\tau \in D_\mathfrak{p}.$$

Thus

$$\sigma \in D_\mathfrak{p} \iff \tau\sigma\tau^{-1} \in D_{\tau\mathfrak{p}}.$$

Thus $\tau D_\mathfrak{p} \tau^{-1} = D_{\tau\mathfrak{p}}$. □

The decomposition group is useful because it allows us to refine the extension $K/\mathbf{Q}$ into a tower of extensions, such that at each step in the tower we understand well the splitting behavior of the primes lying over p.

We characterize the fixed field of $D = D_\mathfrak{p}$ as follows.

Proposition 9.3.3. *The fixed field*

$$K^D = \{a \in K : \sigma(a) = a \text{ for all } \sigma \in D\}$$

of D is the smallest subfield $L \subset K$ such that the prime ideal $\mathfrak{p} \cap \mathcal{O}_L$ has $g(K/L) = 1$, i.e., there is a unique prime of $\mathcal{O}_K$ over $\mathfrak{p} \cap \mathcal{O}_L$.

Proof. First suppose $L = K^D$, and note that by Galois theory $\mathrm{Gal}(K/L) \cong D$, and by Theorem 9.2.2, the group D acts transitively on the primes of K lying over $\mathfrak{p} \cap \mathcal{O}_L$. One of these primes is $\mathfrak{p}$, and D fixes $\mathfrak{p}$ by definition, so there is only one prime of K lying over $\mathfrak{p} \cap \mathcal{O}_L$, i.e., $g = 1$. Conversely, if $L \subset K$ is such that $\mathfrak{p} \cap \mathcal{O}_L$ has $g = 1$, then $\mathrm{Gal}(K/L)$ fixes $\mathfrak{p}$ (since it is the only prime over $\mathfrak{p} \cap \mathcal{O}_L$), so $\mathrm{Gal}(K/L) \subset D$, hence $K^D \subset L$. □

Thus p does not split in going from K^D to K—it does some combination of ramifying and staying inert. To fill in more of the picture, the following proposition asserts that p splits completely and does not ramify in $K^D/\mathbf{Q}$.

Proposition 9.3.4. *Fix a finite Galois extension K of $\mathbf{Q}$, let $\mathfrak{p}$ be a prime lying over p with decomposition group D, and set $L = K^D$. Let $e = e(L/\mathbf{Q}), f = f(L/\mathbf{Q}), g = g(L/\mathbf{Q})$ be for $L/\mathbf{Q}$ and p. Then $e = f = 1$, $g = [L : \mathbf{Q}]$, $e(K/\mathbf{Q}) = e(K/L)$ and $f(K/\mathbf{Q}) = f(K/L)$.*

Proof. As mentioned right after Definition 9.3.1, the orbit-stabilizer theorem implies that $g(K/\mathbf{Q}) = [G : D]$, and by Galois theory $[G : D] = [L : \mathbf{Q}]$, so $g(K/\mathbf{Q}) = [L : \mathbf{Q}]$. Proposition 9.3.3,, $g(K/L) = 1$ so by Theorem 9.2.2,

$$\begin{aligned} e(K/L) \cdot f(K/L) &= [K : L] = [K : \mathbf{Q}]/[L : \mathbf{Q}] \\ &= \frac{e(K/\mathbf{Q}) \cdot f(K/\mathbf{Q}) \cdot g(K/\mathbf{Q})}{[L : \mathbf{Q}]} = e(K/\mathbf{Q}) \cdot f(K/\mathbf{Q}). \end{aligned}$$

Now $e(K/L) \leq e(K/\mathbf{Q})$ and $f(K/L) \leq f(K/\mathbf{Q})$, so we must have $e(K/L) = e(K/\mathbf{Q})$ and $f(K/L) = f(K/\mathbf{Q})$. Since $e(K/\mathbf{Q}) = e(K/L) \cdot e(L/\mathbf{Q})$ and $f(K/\mathbf{Q}) = f(K/L) \cdot f(L/\mathbf{Q})$, it follows that $e(L/\mathbf{Q}) = f(L/\mathbf{Q}) = 1$. □

9.3.1 Galois groups of finite fields

Each $\sigma \in D = D_\mathfrak{p}$ acts in a well-defined way on the finite field $k_\mathfrak{p} = \mathcal{O}_K/\mathfrak{p}$, so we obtain a homomorphism

$$\varphi : D_\mathfrak{p} \to \mathrm{Gal}(k_\mathfrak{p}/\mathbf{F}_p).$$

We pause for a moment and derive a few basic properties of $\mathrm{Gal}(k_\mathfrak{p}/\mathbf{F}_p)$, which are general properties of Galois groups for finite fields. Let $f = [k_\mathfrak{p} : \mathbf{F}_p]$.

The group $\mathrm{Gal}(k_\mathfrak{p}/\mathbf{F}_p)$ contains the element Frob_p defined by

$$\mathrm{Frob}_p(x) = x^p,$$

because $(xy)^p = x^p y^p$ and

$$(x+y)^p = x^p + px^{p-1}y + \cdots + y^p \equiv x^p + y^p \pmod{p}.$$

The group $k_{\mathfrak{p}}^*$ is cyclic (see proof of Lemma 8.1.7), so there is an element $a \in k_{\mathfrak{p}}^*$ of order $p^f - 1$, and $k_{\mathfrak{p}} = \mathbf{F}_p(a)$. Then $\mathrm{Frob}_p^n(a) = a^{p^n} = a$ if and only if $(p^f - 1) \mid p^n - 1$ which is the case precisely when $f \mid n$, so the order of Frob_p is f. Since the order of the automorphism group of a field extension is at most the degree of the extension, we conclude that $\mathrm{Aut}(k_{\mathfrak{p}}/\mathbf{F}_p)$ is generated by Frob_p. Also, since $\mathrm{Aut}(k_{\mathfrak{p}}/\mathbf{F}_p)$ has order equal to the degree, we conclude that $k_{\mathfrak{p}}/\mathbf{F}_p$ is Galois, with group $\mathrm{Gal}(k_{\mathfrak{p}}/\mathbf{F}_p)$ cyclic of order f generated by Frob_p. (Another general fact: Up to isomorphism there is exactly one finite field of each degree. Indeed, if there were two of degree f, then both could be characterized as the set of roots in the compositum of $x^{p^f} - 1$, hence they would be equal.)

9.3.2 The Exact Sequence

Because $D_{\mathfrak{p}}$ preserves $\mathfrak{p}$, there is a natural reduction homomorphism

$$\varphi : D_{\mathfrak{p}} \to \mathrm{Gal}(k_{\mathfrak{p}}/\mathbf{F}_p).$$

Theorem 9.3.5. *The homomorphism φ is surjective.*

Proof. Let $\tilde{a} \in k_{\mathfrak{p}}$ be an element such that $k_{\mathfrak{p}} = \mathbf{F}_p(\tilde{a})$. Lift $\tilde{a}$ to an algebraic integer $a \in \mathcal{O}_K$, and let $f = \prod_{\sigma \in D_p}(x - \sigma(a)) \in K^D[x]$ be the characteristic polynomial of a over K^D. Using Proposition 9.3.4 we see that f reduces to a multiple of the minimal polynomial $\tilde{f} = \prod(x - \widetilde{\sigma(a)}) \in \mathbf{F}_p[x]$ of $\tilde{a}$ (by the Proposition the coefficients of $\tilde{f}$ are in $\mathbf{F}_p$, and $\tilde{a}$ satisfies $\tilde{f}$). The roots of $\tilde{f}$ are of the form $\widetilde{\sigma(a)}$, and the element $\mathrm{Frob}_p(a)$ is also a root of $\tilde{f}$, so it is of the form $\widetilde{\sigma(a)}$. We conclude that the generator Frob_p of $\mathrm{Gal}(k_{\mathfrak{p}}/\mathbf{F}_p)$ is in the image of φ, which proves the theorem. □

Definition 9.3.6 (Inertia Group)**.** The *inertia group associated to* $\mathfrak{p}$ is the kernel $I_{\mathfrak{p}}$ of $D_{\mathfrak{p}} \to \mathrm{Gal}(k_{\mathfrak{p}}/\mathbf{F}_p)$.

We have an exact sequence of groups

$$1 \to I_{\mathfrak{p}} \to D_{\mathfrak{p}} \to \mathrm{Gal}(k_{\mathfrak{p}}/\mathbf{F}_p) \to 1. \tag{9.3.1}$$

The inertia group is a measure of how p ramifies in K.

Corollary 9.3.7. *We have $\#I_{\mathfrak{p}} = e(\mathfrak{p}/p)$, where $\mathfrak{p}$ is a prime of K over p.*

Proof. The sequence (9.3.1) implies that $\#I_{\mathfrak{p}} = (\#D_{\mathfrak{p}})/f(K/\mathbf{Q})$. Applying Propositions 9.3.3–9.3.4, we have

$$\#D_{\mathfrak{p}} = [K : L] = \frac{[K:\mathbf{Q}]}{g} = \frac{efg}{g} = ef.$$

Dividing both sides by $f = f(K/\mathbf{Q})$ proves the corollary. □

We have the following characterization of $I_{\mathfrak{p}}$.

Proposition 9.3.8. *Let $K/\mathbf{Q}$ be a Galois extension with group G, and let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ lying over a prime p. Then*

$$I_{\mathfrak{p}} = \{\sigma \in G : \sigma(a) \equiv a \pmod{\mathfrak{p}} \text{ for all } a \in \mathcal{O}_K\}.$$

Proof. By definition $I_{\mathfrak{p}} = \{\sigma \in D_{\mathfrak{p}} : \sigma(a) \equiv a \pmod{\mathfrak{p}}$ for all $a \in \mathcal{O}_K\}$, so it suffices to show that if $\sigma \notin D_{\mathfrak{p}}$, then there exists $a \in \mathcal{O}_K$ such that $\sigma(a) \not\equiv a \pmod{\mathfrak{p}}$. If $\sigma \notin D_{\mathfrak{p}}$, then $\sigma^{-1} \notin D_{\mathfrak{p}}$, so $\sigma^{-1}(\mathfrak{p}) \neq \mathfrak{p}$. Since both are maximal ideals, there exists $a \in \mathfrak{p}$ with $a \notin \sigma^{-1}(\mathfrak{p})$, i.e., $\sigma(a) \notin \mathfrak{p}$. Thus $\sigma(a) \not\equiv a \pmod{\mathfrak{p}}$. □

9.4 Frobenius Elements

Suppose that $K/\mathbf{Q}$ is a finite Galois extension with group G and p is a prime such that $e = 1$ (i.e., an unramified prime). Then $I = I_{\mathfrak{p}} = 1$ for any $\mathfrak{p} \mid p$, so the map φ of Theorem 9.3.5 is a canonical isomorphism $D_{\mathfrak{p}} \cong \mathrm{Gal}(k_{\mathfrak{p}}/\mathbf{F}_p)$. By Section 9.3.1, the group $\mathrm{Gal}(k_{\mathfrak{p}}/\mathbf{F}_p)$ is cyclic with canonical generator Frob_p. The *Frobenius element* corresponding to $\mathfrak{p}$ is $\mathrm{Frob}_{\mathfrak{p}} \in D_{\mathfrak{p}}$. It is the unique element of G such that for all $a \in \mathcal{O}_K$ we have

$$\mathrm{Frob}_{\mathfrak{p}}(a) \equiv a^p \pmod{\mathfrak{p}}.$$

(To see this argue as in the proof of Proposition 9.3.8.) Just as the primes $\mathfrak{p}$ and decomposition groups $D_{\mathfrak{p}}$ are all conjugate, the Frobenius elements corresponding to primes $\mathfrak{p} \mid p$ are all conjugate as elements of G.

Proposition 9.4.1. *For each $\sigma \in G$, we have*

$$\mathrm{Frob}_{\sigma\mathfrak{p}} = \sigma \,\mathrm{Frob}_{\mathfrak{p}}\, \sigma^{-1}.$$

In particular, the Frobenius elements lying over a given prime are all conjugate.

Proof. Fix $\sigma \in G$. For any $a \in \mathcal{O}_K$ we have $\mathrm{Frob}_{\mathfrak{p}}(\sigma^{-1}(a)) - \sigma^{-1}(a)^p \in \mathfrak{p}$. Applying σ to both sides, we see that $\sigma\,\mathrm{Frob}_{\mathfrak{p}}(\sigma^{-1}(a)) - a^p \in \sigma\mathfrak{p}$, so $\sigma\,\mathrm{Frob}_{\mathfrak{p}}\,\sigma^{-1} = \mathrm{Frob}_{\sigma\mathfrak{p}}$. □

Thus the conjugacy class of $\mathrm{Frob}_{\mathfrak{p}}$ in G is a well-defined function of p. For example, if G is abelian, then $\mathrm{Frob}_{\mathfrak{p}}$ does not depend on the choice of $\mathfrak{p}$ lying over p and we obtain a well defined symbol $\left(\frac{K/\mathbf{Q}}{p}\right) = \mathrm{Frob}_{\mathfrak{p}} \in G$ called the *Artin symbol*. It extends to a homomorphism from the free abelian group on unramified primes p to G. Class field theory (for $\mathbf{Q}$) sets up a natural bijection between abelian Galois extensions of $\mathbf{Q}$ and certain maps from certain subgroups of the group of fractional ideals for $\mathbf{Z}$. We have just described one direction of this bijection, which associates to an abelian extension the Artin symbol (which is a homomorphism). The Kronecker-Weber theorem asserts that the abelian extensions of $\mathbf{Q}$ are exactly the subfields of the fields $\mathbf{Q}(\zeta_n)$, as n varies over all positive integers. By Galois theory there is a correspondence between the subfields of the field $\mathbf{Q}(\zeta_n)$, which has Galois group $(\mathbf{Z}/n\mathbf{Z})^*$, and the subgroups of $(\mathbf{Z}/n\mathbf{Z})^*$, so giving an abelian extension K of $\mathbf{Q}$ is *exactly the same* as giving an integer n and a subgroup of $H \subset (\mathbf{Z}/n\mathbf{Z})^*$. The Artin reciprocity map $p \mapsto \left(\frac{K/\mathbf{Q}}{p}\right)$ is then $p \mapsto [p] \in (\mathbf{Z}/n\mathbf{Z})^*/H$.

9.5 Galois Representations, L-series and a Conjecture of Artin

The Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is an object of central importance in number theory, and we can interpreted much of number theory as the study of this group. A good way to study a group is to study how it acts on various objects, that is, to study its representations.

Endow $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with the topology which has as a basis of open neighborhoods of the origin the subgroups $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$, where K varies over finite Galois extensions of $\mathbf{Q}$. (Note: This is **not** the topology got by taking as a basis of open neighborhoods the collection of finite-index normal subgroups of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.) Fix a positive integer n and let $\mathrm{GL}_n(\mathbf{C})$ be the group of $n \times n$ invertible matrices over $\mathbf{C}$ with the discrete topology.

Definition 9.5.1. A *complex n-dimensional representation* of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is a continuous homomorphism

$$\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_n(\mathbf{C}).$$

For ρ to be continuous means that if K is the fixed field of $\mathrm{Ker}(\rho)$, then $K/\mathbf{Q}$ is a finite Galois extension. We have a diagram

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \arrow[rr, "\rho"] \arrow[dr] & & \mathrm{GL}_n(\mathbf{C}) \\
& \mathrm{Gal}(K/\mathbf{Q}) \arrow[ur, hook, "\rho'"'] &
\end{tikzcd}$$

Remark 9.5.2. That ρ is continuous implies that the image of ρ is finite, but the converse is not true. Using Zorn's lemma, one can show that there are homomorphisms $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \{\pm 1\}$ with image of order 2 that are not continuous, since they do not factor through the Galois group of any finite Galois extension.

Fix a Galois representation ρ and let K be the fixed field of $\ker(\rho)$, so ρ factors through $\mathrm{Gal}(K/\mathbf{Q})$. For each prime $p \in \mathbf{Z}$ that is not ramified in K, there is an element $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(K/\mathbf{Q})$ that is well-defined up to conjugation by elements of $\mathrm{Gal}(K/\mathbf{Q})$. This means that $\rho'(\mathrm{Frob}_p) \in \mathrm{GL}_n(\mathbf{C})$ is well-defined up to conjugation. Thus the characteristic polynomial $F_p(x) \in \mathbf{C}[x]$ of $\rho'(\mathrm{Frob}_p)$ is a well-defined invariant of p and ρ. Let

$$R_p(x) = x^{\deg(F_p)} \cdot F_p(1/x) = 1 + \cdots + \det(\mathrm{Frob}_p) \cdot x^{\deg(F_p)}$$

be the polynomial obtain by reversing the order of the coefficients of F_p. Following E. Artin [Art23, Art30], set

$$L(\rho, s) = \prod_{p \text{ unramified}} \frac{1}{R_p(p^{-s})}. \tag{9.5.1}$$

We view $L(\rho, s)$ as a function of a single complex variable s. One can prove that $L(\rho, s)$ is holomorphic on some right half plane, and extends to a meromorphic function on all $\mathbf{C}$.

Conjecture 9.5.3 (Artin)**.** *The L-function of any continuous representation*

$$\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_n(\mathbf{C})$$

is an entire function on all $\mathbf{C}$*, except possibly at* 1*.*

This conjecture asserts that there is some way to analytically continue $L(\rho, s)$ to the whole complex plane, except possibly at 1. (A standard fact from complex analysis is that this analytic continuation must be unique.) The simple pole at $s = 1$ corresponds to the trivial representation (the Riemann zeta function), and if $n \geq 2$ and ρ is irreducible, then the conjecture is that ρ extends to a holomorphic function on all $\mathbf{C}$.

The conjecture is known when $n = 1$. Assume for the rest of this paragraph that ρ is odd, i.e., if $c \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is complex conjugation, then $\det(\rho(c)) = -1$. When $n = 2$ and the image of ρ in $\mathrm{PGL}_2(\mathbf{C})$ is a solvable group, the conjecture is known, and is a deep theorem of Langlands and others (see [Lan80]), which played a crucial roll in Wiles's proof of Fermat's Last Theorem. When $n = 2$ and the image of ρ in $\mathrm{PGL}_2(\mathbf{C})$ is not solvable, the only possibility is that the projective image is isomorphic to the alternating group A_5. Because A_5 is the symmetry group of the icosahedron, these representations are called *icosahedral*. In this case, Joe Buhler's Harvard Ph.D. thesis [Buh78] gave the first example in which ρ was shown to satisfy Conjecture 9.5.3. There is a book [Fre94], which proves Artin's conjecture for 7 icosahedral representation (none of which are twists of each other). Kevin Buzzard and the author proved the conjecture for 8 more examples [BS02]. Subsequently, Richard Taylor, Kevin Buzzard, Nick Shepherd-Barron, and Mark Dickinson proved the conjecture for an infinite class of icosahedral Galois representations (disjoint from the examples) [BDSBT01]. The general problem for $n = 2$ is in fact now completely solved, due to recent work of Khare and Wintenberger [KW08] that proves Serre's conjecture.

Chapter 10

Elliptic Curves, Galois Representations, and L-functions

This chapter is about elliptic curves and the central role they play in algebraic number theory. Our approach will be less systematic and more a survey than the most of the rest of this book. The goal is to give you a glimpse of the forefront of research by assuming many basic facts that can be found in other books (see, e.g., [Sil92]).

10.1 Groups Attached to Elliptic Curves

Definition 10.1.1 (Elliptic Curve)**.** An *elliptic curve* over a field K is a genus one curve E defined over K equipped with a distinguished point $\mathcal{O} \in E(K)$.

We will not define genus in this book, except to note that a nonsingular curve over K has genus one if and only if over $\overline{K}$ it can be realized as a nonsingular plane cubic curve. Moreover, one can show (using the Riemann-Roch formula) that over any field a genus one curve with a rational point can always be defined by a projective cubic equation of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

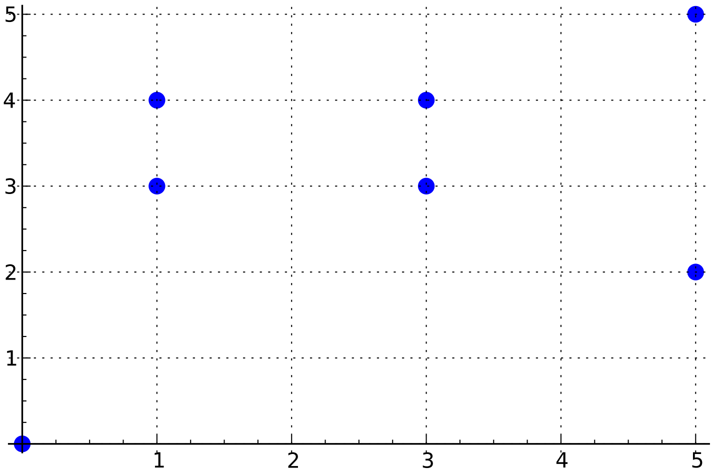
In affine coordinates this becomes

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{10.1.1}$$

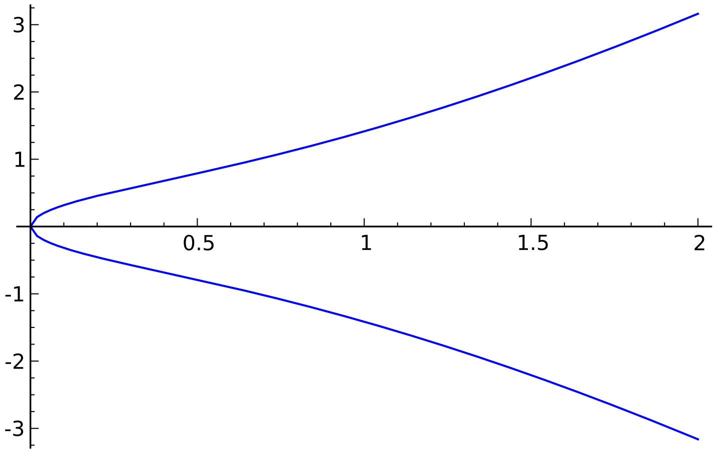
Thus one often presents an elliptic curve by giving a *Weierstrass equation* (10.1.1), though there are significant computational advantages to other equations for curves (e.g., Edwards coordinates – see work of Bernstein and Lange).

Using Sage we plot an elliptic curve over the finite field $\mathbf{F}_7$ and an elliptic curve curve defined over $\mathbf{Q}$.

```
sage: E = EllipticCurve(GF(7), [1,0])
sage: E.plot(pointsize=50, gridlines=True)
```



```
sage: E = EllipticCurve([1,0])
sage: E.plot()
```



Note that both plots above are of the affine equation $y^2 = x^3 + x$, and do not include the distinguished point $\mathcal{O}$, which lies at infinity.

10.1.1 Abelian Groups Attached to Elliptic Curves

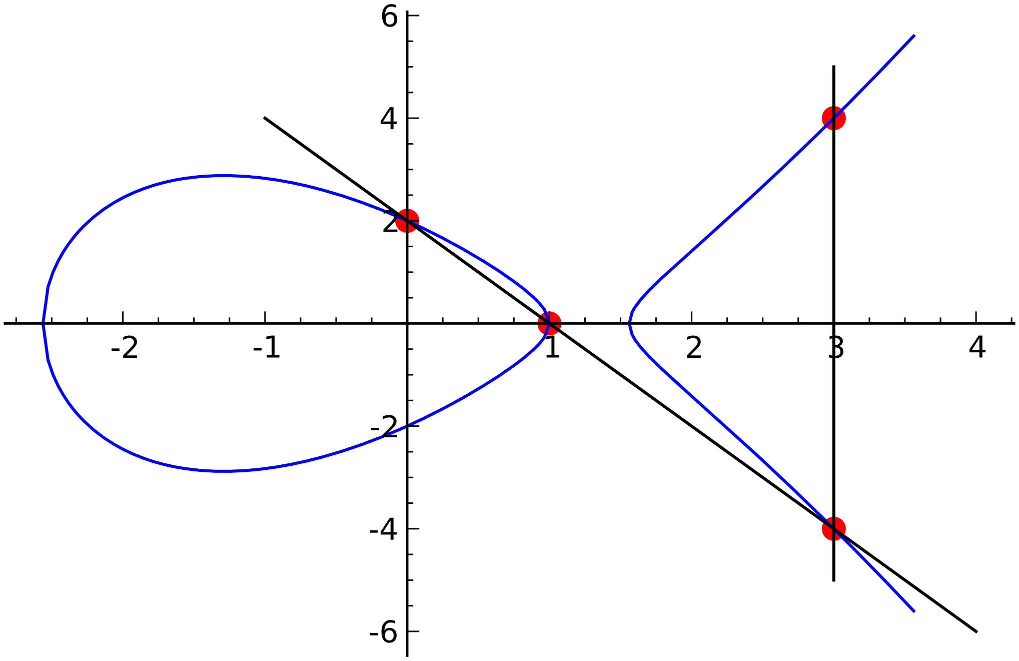
If E is an elliptic curve over K, then we give the set $E(K)$ of all K-rational points on E the structure of abelian group with identity element $\mathcal{O}$. If we embed E in the projective plane, then this group is determined by the condition that three points sum to the zero element $\mathcal{O}$ if and only if they lie on a common line.

For example on the curve $y^2 = x^3 - 5x + 4$, we have $(0, 2) + (1, 0) = (3, 4)$. This is because $(0,2)$, $(1,0)$, and $(3,-4)$ are on a common line (so sum to zero):

$$(0,2)+(1,0)+(3,-4)=\mathcal{O}$$

and $(3,4)$, $(3,-4)$, and $\mathcal{O}$ (the point at infinity on the curve) are also on a common line, so $(3,4)=-(3,-4)$. See the illustration below:

```
sage: E = EllipticCurve([-5,4])
sage: E(0,2) + E(1,0)
(3 : 4 : 1)
sage: G = E.plot()
sage: G += points([(0,2), (1,0), (3,4), (3,-4)], pointsize=50, color='red')
sage: G += line([(-1,4), (4,-6)], color='black')
sage: G += line([(3,-5),(3,5)], color='black')
sage: G
```



Iterating the group operation often leads quickly to very complicated points:

```
sage: 7*E(0,2)
(14100601873051200/48437552041038241 :
-1708700441870667845235922/10660394576906522772066289 : 1)
```

That the above condition—three points on a line sum to zero—defines an abelian group structure on $E(K)$ is not obvious. Depending on your perspective, the trickiest part is seeing that the operation satisfies the associative axiom. The best way to understand the group operation on $E(K)$ is to view $E(K)$ as being related to a class group. As a first observation, note that the ring

$$R=K[x,y]/(y^2+a_1xy+a_3y-(x^3+a_2x^2+a_4x+a_6))$$

is a Dedekind domain, so $\mathrm{Cl}(R)$ is defined, and every nonzero fractional ideal can be written uniquely in terms of prime ideals. When K is a perfect field, the prime ideals correspond to the Galois orbits of affine points of $E(\overline{K})$.

Let $\mathrm{Div}(E/K)$ be the free abelian group on the Galois orbits of points of $E(\overline{K})$, which as explained above is analogous to the group of fractional ideals of a number field (here we *do* include the point at infinity). We call the elements of $\mathrm{Div}(E/K)$ *divisors*. Let $\mathrm{Pic}(E/K)$ be the quotient of $\mathrm{Div}(E/K)$ by the principal divisors, i.e., the divisors associated to rational functions $f \in K(E)^*$ via

$$f \mapsto (f) = \sum_P \mathrm{ord}_P(f)[P].$$

Note that the principal divisor associated to f is analogous to the principal fractional ideal associated to a nonzero element of a number field. The definition of $\mathrm{ord}_P(f)$ is analogous to the "power of P that divides the principal ideal generated by f". Define the class group $\mathrm{Pic}(E/K)$ to be the quotient of the divisors by the principal divisors, so we have an exact sequence:

$$1 \to K(E)^*/K^* \to \mathrm{Div}(E/K) \to \mathrm{Pic}(E/K) \to 0.$$

A key difference between elliptic curves and algebraic number fields is that the principal divisors in the context of elliptic curves all have degree 0, i.e., the sum of the coefficients of the divisor (f) is always 0. This might be a familiar fact to you: the number of zeros of a nonzero rational function on a projective curve equals the number of poles, counted with multiplicity. If we let $\mathrm{Div}^0(E/K)$ denote the subgroup of divisors of degree 0, then we have an exact sequence

$$0 \to K(E)^*/K^* \to \mathrm{Div}^0(E/K) \to \mathrm{Pic}^0(E/K) \to 0.$$

To connect this with the group law on $E(K)$, note that there is a natural map

$$E(K) \to \mathrm{Pic}^0(E/K), \qquad P \mapsto [P - \mathcal{O}].$$

Using the Riemann-Roch theorem, one can prove that this map is a bijection, which is moreover an isomorphism of abelian groups. Thus really when we discuss the group of K-rational points on an E, we are talking about the class group $\mathrm{Pic}^0(E/K)$.

Recall that we proved (Theorem 7.1.2) that the class group $\mathrm{Cl}(\mathcal{O}_K)$ of a number field is finite. The group $\mathrm{Pic}^0(E/K) = E(K)$ of an elliptic curve can be either finite (e.g., for $y^2 + y = x^3 - x + 1$) or infinite (e.g., for $y^2 + y = x^3 - x$), and determining which is the case for any particular curve is one of the central unsolved problems in number theory.

The Mordell-Weil theorem (see Chapter 12) asserts that if E is an elliptic curve over a number field K, then there is a nonnegative integer r such that

$$E(\mathbf{Q}) \approx \mathbf{Z}^r \oplus T, \tag{10.1.2}$$

where T is a finite group. This is similar to Dirichlet's unit theorem, which gives the structure of the unit group of the ring of integers of a number field. The main difference is that T need not be cyclic, and computing r appears to be much more difficult than just finding the number of real and complex roots of a polynomial!

```
sage: EllipticCurve([0,0,1,-1,1]).rank()
0
sage: EllipticCurve([0,0,1,-1,0]).rank()
1
```

Also, if L/K is an arbitrary extension of fields, and E is an elliptic curve over K, then there is a natural inclusion homomorphism $E(K) \hookrightarrow E(L)$. Thus instead of just obtaining one group attached to an elliptic curve, we obtain a whole collection, one for each extension of L. Even more generally, if S/K is an arbitrary scheme, then $E(S)$ is a group, and the association $S \mapsto E(S)$ defines a functor from the category of schemes to the category of groups. Thus each elliptic curve gives rise to map:

$$\{\text{Schemes over } K\} \longrightarrow \{\text{Abelian Groups}\}$$

10.1.2 A Formula for Adding Points

We close this section with an explicit formula for adding two points in $E(K)$. If E is an elliptic curve over a field K, given by an equation $y^2 = x^3 + ax + b$, then we can compute the group addition using the following algorithm.

Algorithm 10.1.2 (Elliptic Curve Group Law)**.** Given $P_1, P_2 \in E(K)$, this algorithm computes the sum $R = P_1 + P_2 \in E(K)$.

1. [One Point $\mathcal{O}$] If $P_1 = \mathcal{O}$ set $R = P_2$ or if $P_2 = \mathcal{O}$ set $R = P_1$ and terminate. Otherwise write $P_i = (x_i, y_i)$.

2. [Negatives] If $x_1 = x_2$ and $y_1 = -y_2$, set $R = \mathcal{O}$ and terminate.

3. [Compute λ] Set $\lambda = \begin{cases} (3x_1^2 + a)/(2y_1) & \text{if } P_1 = P_2, \\ (y_1 - y_2)/(x_1 - x_2) & \text{otherwise.} \end{cases}$
   Note: If $y_1 = 0$ and $P_1 = P_2$, output $\mathcal{O}$ and terminate.

4. [Compute Sum] Then $R = \left(\lambda^2 - x_1 - x_2, -\lambda x_3 - \nu\right)$, where $\nu = y_1 - \lambda x_1$ and x_3 is the x coordinate of R.

10.1.3 Other Groups

There are other abelian groups attached to elliptic curves, such as the torsion subgroup $E(K)_{\text{tor}}$ of elements of $E(K)$ of finite order. The torsion subgroup is (isomorphic to) the group T that appeared in Equation (10.1.2) above). When K is a number field, there is a group called the Shafarevich-Tate group $\text{Ш}(E/K)$ attached to E, which plays a role similar to that of the class group of a number field (though it is an open problem to prove that $\text{Ш}(E/K)$ is finite in general). The definition of $\text{Ш}(E/K)$ involves Galois cohomology, so we wait until Chapter 11 to define it. There are also component groups attached to E, one for each prime of $\mathcal{O}_K$. These groups all come together in the Birch and Swinnerton-Dyer conjecture (see `http://wstein.org/books/bsd/`).

10.2 Galois Representations Attached to Elliptic Curves

Let E be an elliptic curve over a number field K. In this section we attach representations of $G_K = \mathrm{Gal}(\overline{K}/K)$ to E, and use them to define an L-function $L(E, s)$. This L-function is yet another generalization of the Riemann Zeta function, that is different from the L-functions attached to complex representations $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_n(\mathbf{C})$, which we encountered before in Section 9.5.

Fix an integer n. The group structure on E is defined by algebraic formulas with coefficients that are elements of K, so the subgroup

$$E[n] = \{R \in E(\overline{K}) : nR = \mathcal{O}\}$$

is invariant under the action of G_K. We thus obtain a homomorphism

$$\overline{\rho}_{E,n} : G_K \to \mathrm{Aut}(E[n]).$$

```
sage: E = EllipticCurve([1,1]); E
Elliptic Curve defined by y^2 = x^3 + x + 1 over Rational Field
sage: R.<x> = QQ[]; R
Univariate Polynomial Ring in x over Rational Field
sage: f = x^3 + x + 1
sage: K.<a> = NumberField(f)
sage: M.<b> = K.galois_closure(); M
Number Field in b with defining polynomial
    x^6 + 14*x^4 - 20*x^3 + 49*x^2 - 140*x + 379
sage: E.change_ring(M)
sage: T = F.torsion_subgroup(); T
Torsion Subgroup isomorphic to Z/2 + Z/2 associated ...
sage: T.gens()
```

$$\left(\frac{20}{9661}b^5 + \frac{147}{9661}b^4 + \frac{700}{28983}b^3 + \frac{1315}{9661}b^2 + \frac{5368}{28983}b + \frac{4004}{28983} : 0 : 1\right),$$

$$\left(\frac{10}{9661}b^5 + \frac{147}{19322}b^4 + \frac{350}{28983}b^3 + \frac{1315}{19322}b^2 - \frac{23615}{57966}b + \frac{2002}{28983} : 0 : 1\right)$$

We continue to assume that E is an elliptic curve over a number field K. For any positive integer n, the group $E[n]$ is isomorphic as an abstract abelian group to $(\mathbf{Z}/n\mathbf{Z})^2$. There are various related ways to see why this is true. One is to use the Weierstrass $\wp$-theory to parametrize $E(\mathbf{C})$ by the the complex numbers, i.e., to find an isomorphism $\mathbf{C}/\Lambda \cong E(\mathbf{C})$, where Λ is a lattice in $\mathbf{C}$ and the isomorphism is given by $z \mapsto (\wp(z), \wp'(z))$ with respect to an appropriate choice of coordinates on $E(\mathbf{C})$. It is then an easy exercise to verify that $(\mathbf{C}/\Lambda)[n] \cong (\mathbf{Z}/n\mathbf{Z})^2$.

Another way to understand $E[n]$ is to use that $E(\mathbf{C})_{\mathrm{tor}}$ is isomorphic to the quotient

$$\mathrm{H}_1(E(\mathbf{C}),\mathbf{Q})/\,\mathrm{H}_1(E(\mathbf{C}),\mathbf{Z})$$

of homology groups and that the homology of a curve of genus g is isomorphic to $\mathbf{Z}^{2g}$. Then

$$E[n] \cong (\mathbf{Q}/\mathbf{Z})^2[n] = (\mathbf{Z}/n\mathbf{Z})^2.$$

If $n = p$ is a prime, then upon chosing a basis for the two-dimensional $\mathbf{F}_p$-vector space $E[p]$, we obtain an isomorphism $\mathrm{Aut}(E[p]) \cong \mathrm{GL}_2(\mathbf{F}_p)$. We thus obtain a mod p Galois representation

$$\overline{\rho}_{E,p} : G_K \to \mathrm{GL}_2(\mathbf{F}_p).$$

This representation $\overline{\rho}_{E,p}$ is continuous if $\mathrm{GL}_2(\mathbf{F}_p)$ is endowed with the discrete topology, because the field

$$K(E[p]) = K(\{a, b : (a, b) \in E[p]\})$$

is a Galois extension of K of finite degree.

In order to attach an L-function to E, one could try to embed $\mathrm{GL}_2(\mathbf{F}_p)$ into $\mathrm{GL}_2(\mathbf{C})$ and use the construction of Artin L-functions from Section 9.5. Unfortunately, this approach is doomed in general, since $\mathrm{GL}_2(\mathbf{F}_p)$ frequently does not embed in $\mathrm{GL}_2(\mathbf{C})$. The following Sage session shows that for $p = 5, 7$, there are no 2-dimensional irreducible representations of $\mathrm{GL}_2(\mathbf{F}_p)$, so $\mathrm{GL}_2(\mathbf{F}_p)$ does not embed in $\mathrm{GL}_2(\mathbf{C})$. (The notation in the output below is `[degree of rep, number of times it occurs]`.)

```
sage: gap(GL(2,GF(2))).CharacterTable().CharacterDegrees()
[ [ 1, 2 ], [ 2, 1 ] ]
sage: gap(GL(2,GF(3))).CharacterTable().CharacterDegrees()
[ [ 1, 2 ], [ 2, 3 ], [ 3, 2 ], [ 4, 1 ] ]
sage: gap(GL(2,GF(5))).CharacterTable().CharacterDegrees()
[ [ 1, 4 ], [ 4, 10 ], [ 5, 4 ], [ 6, 6 ] ]
sage: gap(GL(2,GF(7))).CharacterTable().CharacterDegrees()
[ [ 1, 6 ], [ 6, 21 ], [ 7, 6 ], [ 8, 15 ] ]
```

Instead of using the complex numbers, we use the p-adic numbers, as follows. For each power p^m of p, we have defined a homomorphism

$$\overline{\rho}_{E,p^m} : G_K \to \mathrm{Aut}(E[p^m]) \approx \mathrm{GL}_2(\mathbf{Z}/p^m\mathbf{Z}).$$

We combine together all of these representations (for all $m \geq 1$) using the inverse limit. Recall that the p-adic numbers are

$$\mathbf{Z}_p = \varprojlim \mathbf{Z}/p^m\mathbf{Z},$$

which is the set of all compatible choices of integers modulo p^m for all m. We obtain a (continuous) homomorphism

$$\rho_{E,p} : G_K \to \mathrm{Aut}(\varprojlim E[p^m]) \cong \mathrm{GL}_2(\mathbf{Z}_p),$$

where $\mathbf{Z}_p$ is the ring of p-adic integers. The composition of this homomorphism with the reduction map $\mathrm{GL}_2(\mathbf{Z}_p) \to \mathrm{GL}_2(\mathbf{F}_p)$ is the representation $\overline{\rho}_{E,p}$, which we defined above, which is why we denoted it by $\overline{\rho}_{E,p}$. We next try to mimic the construction of $L(\rho, s)$ from Section 9.5 in the context of a p-adic Galois representation $\rho_{E,p}$.

Definition 10.2.1 (Tate module)**.** The p-adic Tate module of E is

$$T_p(E) = \varprojlim E[p^n].$$

Let M be the fixed field of $\ker(\rho_{E,p})$. The image of $\rho_{E,p}$ is infinite, so M is an infinite extension of K. Fortunately, one can prove that M is ramified at only finitely many primes (the primes of bad reduction for E and p—see [ST68]). If ℓ is a prime of K, let D_ℓ be a choice of decomposition group for some prime $\mathfrak{p}$ of M lying over ℓ, and let I_ℓ be the inertia group. We haven't defined inertia and decomposition groups for infinite Galois extensions, but the definitions are almost the same: choose a prime of $\mathcal{O}_M$ over ℓ, and let D_ℓ be the subgroup of $\mathrm{Gal}(M/K)$ that leaves $\mathfrak{p}$ invariant. Then the submodule $T_p(E)^{I_\ell}$ of inertia invariants is a module for D_ℓ and the characteristic polynomial $F_\ell(x)$ of Frob_ℓ on $T_p(E)^{I_\ell}$ is well defined (since inertia acts trivially). Let $R_\ell(x)$ be the polynomial obtained by reversing the coefficients of $F_\ell(x)$. One can prove that $R_\ell(x) \in \mathbf{Z}[x]$ and that $R_\ell(x)$, for $\ell \neq p$ does not depend on the choice of p. Define $R_\ell(x)$ for $\ell = p$ using a different prime $q \neq p$, so the definition of $R_\ell(x)$ does not depend on the choice of p.

Definition 10.2.2. The L-series of E is

$$L(E, s) = \prod_\ell \frac{1}{R_\ell(\ell^{-s})}.$$

A prime $\mathfrak{p}$ of $\mathcal{O}_K$ is a prime of *good reduction* for E if there is an equation for E such that $E \mod \mathfrak{p}$ is an elliptic curve over $\mathcal{O}_K/\mathfrak{p}$.

If $K = \mathbf{Q}$ and ℓ is a prime of good reduction for E, then one can show that that $R_\ell(\ell^{-s}) = 1 - a_\ell \ell^{-s} + \ell^{1-2s}$, where $a_\ell = \ell + 1 - \#\tilde{E}(\mathbf{F}_\ell)$ and $\tilde{E}$ is the reduction of a local minimal model for E modulo ℓ. (There is a similar statement for $K \neq \mathbf{Q}$.)

One can prove using fairly general techniques that the product expression for $L(E, s)$ defines a holomorphic function in some right half plane of $\mathbf{C}$, i.e., the product converges for all s with $\mathrm{Re}(s) > \alpha$, for some real number α.

Conjecture 10.2.3. *The function $L(E, s)$ extends to a holomorphic function on all $\mathbf{C}$.*

10.2.1 Modularity of Elliptic Curves over Q

Fix an elliptic curve E over $\mathbf{Q}$. In this section we will explain what it means for E to be modular, and note the connection with Conjecture 10.2.3 from the previous section.

First, we give the general definition of modular form (of weight 2). The complex *upper half plane* is $\mathfrak{h} = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$. A *cuspidal modular form f* of level N

(of weight 2) is a holomorphic function $f : \mathfrak{h} \to \mathbf{C}$ such that $\lim_{z \to i\infty} f(z) = 0$ and for every integer matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant 1 and $c \equiv 0 \pmod{N}$, we have

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{-2} f(z).$$

For each prime number ℓ of good reduction, let $a_\ell = \ell + 1 - \#\tilde{E}(\mathbf{F}_\ell)$. If ℓ is a prime of bad reduction let $a_\ell = 0, 1, -1$, depending on how singular the reduction $\tilde{E}$ of E is over $\mathbf{F}_\ell$. If $\tilde{E}$ has a cusp, then $a_\ell = 0$, and $a_\ell = 1$ or -1 if $\tilde{E}$ has a node; in particular, let $a_\ell = 1$ if and only if the tangents at the cusp are defined over $\mathbf{F}_\ell$.

Extend the definition of the a_ℓ to a_n for all positive integers n as follows. If $\gcd(n, m) = 1$ let $a_{nm} = a_n \cdot a_m$. If p^r is a power of a prime p of good reduction, let

$$a_{p^r} = a_{p^{r-1}} \cdot a_p \ - \ p \cdot a_{p^{r-2}}.$$

If p is a prime of bad reduction let $a_{p^r} = (a_p)^r$.

Attach to E the function

$$f_E(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i z}.$$

It is an extremely deep theorem that $f_E(z)$ is actually a cuspidal modular form, and not just some random function.

The following theorem is called the modularity theorem for elliptic curves over $\mathbf{Q}$. Before it was proved it was known as the Taniyama-Shimura-Weil conjecture.

Theorem 10.2.4 (Wiles, Brueil, Conrad, Diamond, Taylor)**.** *Every elliptic curve over $\mathbf{Q}$ is modular, i.e, the function $f_E(z)$ is a cuspidal modular form.*

Corollary 10.2.5 (Hecke)**.** *If E is an elliptic curve over $\mathbf{Q}$, then the L-function $L(E, s)$ has an analytic continuous to the whole complex plane.*

Chapter 11

Galois Cohomology

11.1 Group Cohomology

11.1.1 Group Rings

Let G be a finite group. The group ring $\mathbf{Z}[G]$ of G is the free abelian group on the elements of G equipped with multiplication given by the group structure on G. Note that $\mathbf{Z}[G]$ is a commutative ring if and only if G is commutative.

For example, the group ring of the cyclic group $C_n = \langle a \rangle$ of order n is the free $\mathbf{Z}$-module on $1, a, \ldots, a^{n-1}$, and the multiplication is induced by $a^i a^j = a^{i+j} = a^{i+j \pmod{n}}$ extended linearly. For example, in $\mathbf{Z}[C_3]$ we have

$$(1 + 2a)(1 - a^2) = 1 - a^2 + 2a - 2a^3 = 1 + 2a - a^2 - 2 = -1 + 2a - a^2.$$

You might think that $\mathbf{Z}[C_3]$ is isomorphic to the ring $\mathbf{Z}[\zeta_3]$ of integers of $\mathbf{Q}(\zeta_3)$, but you would be wrong, since the ring of integers is isomorphic to $\mathbf{Z}^2$ as abelian group, but $\mathbf{Z}[C_3]$ is isomorphic to $\mathbf{Z}^3$ as abelian group. (Note that $\mathbf{Q}(\zeta_3)$ is a quadratic extension of $\mathbf{Q}$.)

11.2 Modules and Group Cohomology

Let A be a G module. This means that A is an abelian group equipped with a left action of G, i.e., a group homomorphism $G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ denotes the group of bijections $A \to A$ that preserve the group structure on A. Alternatively, A is a module over the ring $\mathbf{Z}[G]$ in the usual sense of module. For example, $\mathbf{Z}$ with the trivial action is a module over any group G, as is $\mathbf{Z}/m\mathbf{Z}$ for any positive integer m. Another example is $G = (\mathbf{Z}/n\mathbf{Z})^*$, which acts via multiplication on $\mathbf{Z}/n\mathbf{Z}$.

For each integer $n \geq 0$ there is an abelian group $\mathrm{H}^n(G, A)$ called the *nth cohomology group of G acting on A*. The general definition is somewhat complicated, but the definition for $n \leq 1$ is fairly concrete. For example, the *0th cohomology group*

$$\mathrm{H}^0(G, A) = \{x \in A : \sigma x = x \text{ for all } \sigma \in G\} = G^A$$

is the subgroup of elements of A that are fixed by every element of G.

The *first cohomology group*

$$\mathrm{H}^1(G, A) = C^1(G, A)/B^1(G, A)$$

is the group of 1-cocycles modulo 1-coboundaries, where

$$C^1(G, A) = \{f : G \to A \text{ such that } f(\sigma\tau) = f(\sigma) + \sigma f(\tau)\}$$

and if we let $f_a : G \to A$ denote the set-theoretic map $f_a(\sigma) = \sigma(a) - a$, then

$$B^1(G, A) = \{f_a : a \in A\}.$$

There are also explicit, and increasingly complicated, definitions of $\mathrm{H}^n(G, A)$ for each $n \geq 2$ in terms of certain maps $G \times \cdots \times G \to A$ modulo a subgroup, but we will not need this.

For example, if A has the trivial action, then $B^1(G, A) = 0$, since $\sigma a - a = a - a = 0$ for any $a \in A$. Also, $C^1(G, A) = \mathrm{Hom}(G, A)$. If $A = \mathbf{Z}$, then since G is finite there are no nonzero homomorphisms $G \to \mathbf{Z}$, so $\mathrm{H}^1(G, \mathbf{Z}) = 0$.

If X is any abelian group, then

$$A = \mathrm{Hom}(\mathbf{Z}[G], X)$$

is a G-module. We call a module constructed in this way *co-induced*.

The following theorem gives three properties of group cohomology, which uniquely determine group cohomology.

Theorem 11.2.1. *Suppose G is a finite group. Then*

1. *We have $\mathrm{H}^0(G, A) = A^G$.*

2. *If A is a co-induced G-module, then $\mathrm{H}^n(G, A) = 0$ for all $n \geq 1$.*

3. *If $0 \to A \to B \to C \to 0$ is any exact sequence of G-modules, then there is a long exact sequence*

$$0 \to \mathrm{H}^0(G, A) \to \mathrm{H}^0(G, B) \to \mathrm{H}^0(G, C) \to \mathrm{H}^1(G, A) \to \cdots$$

$$\cdots \to \mathrm{H}^n(G, A) \to \mathrm{H}^n(G, B) \to \mathrm{H}^n(G, C) \to \mathrm{H}^{n+1}(G, A) \to \cdots$$

Moreover, the functor $\mathrm{H}^n(G, -)$ is uniquely determined by these three properties.

We will not prove this theorem. For proofs see [Cp86, Atiyah-Wall] and [Ser79, Ch. 7]. The properties of the theorem uniquely determine group cohomology, so one should in theory be able to use them to deduce anything that can be deduced about cohomology groups. Indeed, in practice one frequently proves results about higher cohomology groups $\mathrm{H}^n(G, A)$ by writing down appropriate exact sequences, using explicit knowledge of H^0, and chasing diagrams.

Remark 11.2.2. Alternatively, we could view the defining properties of the theorem as the definition of group cohomology, and could state a theorem that asserts that group cohomology exists.

Remark 11.2.3. For those familiar with commutative and homological algebra, we have

$$\mathrm{H}^n(G, A) = \mathrm{Ext}^n_{\mathbf{Z}[G]}(\mathbf{Z}, A),$$

where $\mathbf{Z}$ is the trivial G-module.

Remark 11.2.4. One can interpret $\mathrm{H}^2(G, A)$ as the group of equivalence classes of extensions of G by A, where an extension is an exact sequence

$$0 \to A \to M \to G \to 1$$

such that the induced conjugation action of G on A is the given action of G on A. (Note that G acts by conjugation, as A is a normal subgroup since it is the kernel of a homomorphism.)

11.2.1 Example Application of the Theorem

For example, let's see what we get from the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{m} \mathbf{Z} \to \mathbf{Z}/m\mathbf{Z} \to 0,$$

where m is a positive integer, and $\mathbf{Z}$ has the structure of trivial G module. By definition we have $\mathrm{H}^0(G, \mathbf{Z}) = \mathbf{Z}$ and $\mathrm{H}^0(G, \mathbf{Z}/m\mathbf{Z}) = \mathbf{Z}/m\mathbf{Z}$. The long exact sequence begins

$$0 \to \mathbf{Z} \xrightarrow{m} \mathbf{Z} \to \mathbf{Z}/m\mathbf{Z} \to \mathrm{H}^1(G, \mathbf{Z}) \xrightarrow{m} \mathrm{H}^1(G, \mathbf{Z}) \to \mathrm{H}^1(G, \mathbf{Z}/m\mathbf{Z}) \to \mathrm{H}^2(G, \mathbf{Z}) \xrightarrow{m} \mathrm{H}^2(G, \mathbf{Z}) \to \cdots$$

From the first few terms of the sequence and the fact that $\mathbf{Z}$ surjects onto $\mathbf{Z}/m\mathbf{Z}$, we see that $[m]$ on $\mathrm{H}^1(G, \mathbf{Z})$ is injective. This is consistent with our observation above that $\mathrm{H}^1(G, \mathbf{Z}) = 0$. Using this vanishing and the right side of the exact sequence we obtain an isomorphism

$$\mathrm{H}^1(G, \mathbf{Z}/m\mathbf{Z}) \cong \mathrm{H}^2(G, \mathbf{Z})[m].$$

As we observed above, when a group acts trivially the H^1 is Hom, so

$$\mathrm{H}^2(G, \mathbf{Z})[m] \cong \mathrm{Hom}(G, \mathbf{Z}/m\mathbf{Z}). \tag{11.2.1}$$

One can prove that for any $n > 0$ and any module A that the group $\mathrm{H}^n(G, A)$ has exponent dividing $\#G$ (see Remark 11.3.4). Thus (11.2.1) allows us to understand $\mathrm{H}^2(G, \mathbf{Z})$, and this comprehension arose naturally from the properties that determine H^n.

11.3 Inflation and Restriction

Suppose H is a subgroup of a finite group G and A is a G-module. For each $n \geq 0$, there is a natural map

$$\mathrm{res}_H : \mathrm{H}^n(G, A) \to \mathrm{H}^n(H, A)$$

called restriction. Elements of $\mathrm{H}^n(G, A)$ can be viewed as classes of n-cocycles, which are certain maps $G \times \cdots \times G \to A$, and the restriction maps restricts these cocycles to $H \times \cdots \times H$.

If H is a normal subgroup of G, there is also an inflation map

$$\mathrm{inf}_H : \mathrm{H}^n(G/H, A^H) \to \mathrm{H}^n(G, A),$$

given by taking a cocycle $f : G/H \times \cdots \times G/H \to A^H$ and precomposing with the quotient map $G \to G/H$ to obtain a cocycle for G.

Proposition 11.3.1. *Suppose H is a normal subgroup of G. Then there is an exact sequence*

$$0 \to \mathrm{H}^1(G/H, A^H) \xrightarrow{\mathrm{inf}_H} \mathrm{H}^1(G, A) \xrightarrow{\mathrm{res}_H} \mathrm{H}^1(H, A).$$

Proof. Our proof follows [Ser79, pg. 117] closely.

We see that $\mathrm{res} \circ \mathrm{inf} = 0$ by looking at cochains. It remains to prove that inf_H is injective and that the image of inf_H is the kernel of res_H.

1. *That* inf_H *is injective:* Suppose $f : G/H \to A^H$ is a cocycle whose image in $\mathrm{H}^1(G, A)$ is equivalent to 0 modulo coboundaries. Then there is an $a \in A$ such that $f(\sigma) = \sigma a - a$, where we identify f with the map $G \to A$ that is constant on the costs of H. But f depends only on the costs of σ modulo H, so $\sigma a - a = \sigma\tau a - a$ for all $\tau \in H$, i.e., $\tau a = a$ (as we see by adding a to both sides and multiplying by σ^{-1}).Thus $a \in A^H$, so f is equivalent to 0 in $\mathrm{H}^1(H, A^H)$.

2. *The image of* inf_H *contains the kernel of* res_H*:* Suppose $f : G \to A$ is a cocycle whose restriction to H is a coboundary, i.e., there is $a \in A$ such that $f(\tau) = \tau a - a$ for all $\tau \in H$. Subtracting the coboundary $g(\sigma) = \sigma a - a$ for $\sigma \in G$ from f, we may assume $f(\tau) = 0$ for all $\tau \in H$. Examing the equation $f(\sigma\tau) = f(\sigma) + \sigma f(\tau)$ with $\tau \in H$ shows that f is constant on the cosets of H. Again using this formula, but with $\sigma \in H$ and $\tau \in G$, we see that

$$f(\tau) = f(\sigma\tau) = f(\sigma) + \sigma f(\tau) = \sigma f(\tau),$$

   so the image of f is contained in A^H. Thus f defines a cocycle $G/H \to A^H$, i.e., is in the image of inf_H.

□

This proposition will be useful when proving the weak Mordell-Weil theorem.

Example 11.3.2. The sequence of Proposition 11.3.1 need not be surjective on the right. For example, suppose $H = A_3 \subset S_3$, and let S_3 act trivially on the cyclic group $C = \mathbf{Z}/3\mathbf{Z}$. Using the Hom interpretation of H^1, we see that $\mathrm{H}^1(S_3/A_3, C) = \mathrm{H}^1(S_3, C) = 0$, but $\mathrm{H}^1(A_3, C)$ has order 3.

Remark 11.3.3. On generalization of Proposition 11.3.1 is to a more complicated exact sequence involving the "transgression map" tr:

$$0 \to \mathrm{H}^1(G/H, A^H) \xrightarrow{\inf_H} \mathrm{H}^1(G, A) \xrightarrow{\mathrm{res}_H} \mathrm{H}^1(H, A)^{G/H} \xrightarrow{\mathrm{tr}} \mathrm{H}^2(G/H, A^H) \to \mathrm{H}^2(G, A).$$

Another generalization of Proposition 11.3.1 is that if $\mathrm{H}^m(H, A) = 0$ for $1 \leq m < n$, then there is an exact sequence

$$0 \to \mathrm{H}^n(G/H, A^H) \xrightarrow{\inf_H} \mathrm{H}^n(G, A) \xrightarrow{\mathrm{res}_H} \mathrm{H}^n(H, A).$$

Remark 11.3.4. If H is a not-necessarily-normal subgroup of G, there are also maps

$$\mathrm{cores}_H : \mathrm{H}^n(H, A) \to \mathrm{H}^n(G, A)$$

for each n. For $n = 0$ this is the trace map $a \mapsto \sum_{\sigma \in G/H} \sigma a$, but the definition for $n \geq 1$ is more involved. One has $\mathrm{cores}_H \circ \mathrm{res}_H = [\#(G/H)]$. Taking $H = 1$ we see that for each $n \geq 1$ the group $\mathrm{H}^n(G, A)$ is annihilated by $\#G$.

11.4 Galois Cohomology

Suppose L/K is a finite Galois extension of fields, and A is a module for $\mathrm{Gal}(L/K)$. Put

$$\mathrm{H}^n(L/K, A) = \mathrm{H}^n(\mathrm{Gal}(L/K), A).$$

Next suppose A is a module for the group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ and for any extension L of K, let

$$A(L) = \{x \in A : \sigma(x) = x \text{ all } \sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/L)\}.$$

We think of $A(L)$ as the group of elements of A that are "defined over L". For each $n \geq 0$, put

$$\mathrm{H}^n(L/K, A) = \mathrm{H}^n(\mathrm{Gal}(L/K), A(L)).$$

Also, put

$$\mathrm{H}^n(K, A) = \varinjlim_{L/K} \mathrm{H}^n(L/K, A(L)),$$

where L varies over all finite Galois extensions of K. (Recall: Galois means normal and separable.)

Example 11.4.1. The following are examples of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-modules:

$$\overline{\mathbf{Q}}, \quad \overline{\mathbf{Q}}^*, \quad \overline{\mathbf{Z}}, \quad \overline{\mathbf{Z}}^*, \quad E(\overline{\mathbf{Q}}), \quad E(\overline{\mathbf{Q}})[n], \quad \mathrm{Tate}_\ell(E),$$

where E is an elliptic curve over $\mathbf{Q}$.

Theorem 11.4.2 (Hilbert 90)**.** *We have* $\mathrm{H}^1(K, \overline{K}^*) = 0$.

Proof. See [Ser79]. The main input to the proof is linear independence of automorphism and a clever little calculation. □

Chapter 12

The Weak Mordell-Weil Theorem

12.1 Kummer Theory of Number Fields

Suppose K is a number field and fix a positive integer n. Consider the exact sequence

$$1 \to \mu_n \to \overline{K}^* \xrightarrow{n} \overline{K}^* \to 1.$$

The long exact sequence is

$$1 \to \mu_n(K) \to K^* \xrightarrow{n} K^* \to \mathrm{H}^1(K, \mu_n) \to \mathrm{H}^1(K, \overline{K}^*) = 0,$$

where $\mathrm{H}^1(K, \overline{K}^*) = 0$ by Theorem 11.4.2.

Assume now that the group μ_n of nth roots of unity is contained in K. Using Galois cohomology we obtain a relatively simple classification of all abelian extensions of K with Galois group cyclic of order dividing n. Moreover, since the action of $\mathrm{Gal}(\overline{K}/K)$ on μ_n is trivial, by our hypothesis that $\mu_n \subset K$, we see that

$$\mathrm{H}^1(K, \mu_n) = \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mu_n).$$

Thus we obtain an exact sequence

$$1 \to \mu_n \to K^* \xrightarrow{n} K^* \to \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mu_n) \to 1,$$

or equivalently, an isomorphism

$$K^*/(K^*)^n \cong \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mu_n),$$

By Galois theory, homomorphisms $\mathrm{Gal}(\overline{K}/K) \to \mu_n$ (up to automorphisms of μ_n) correspond to cyclic abelian extensions of K with Galois group a subgroup of the cyclic group μ_n of order n. Unwinding the definitions, what this says is that every cyclic abelian extension of K of degree dividing n is of the form $K(a^{1/n})$ for some element $a \in K$.

One can prove via calculations with discriminants, etc. that $K(a^{1/n})$ is unramified outside n and and the primes that divide $\mathrm{Norm}(a)$. Moreover, and this is a much bigger result, one can combine this with facts about class groups and unit groups to prove the following theorem:

Theorem 12.1.1. *Suppose K is a number field with $\mu_n \subset K$, where n is a positive integer. Then the maximal abelian exponent n extension L of K unramified outside a finite set S of primes is of finite degree.*

Sketch of Proof. We may enlarge S, because if an extension is unramified outside a set larger than S, then it is unramified outside S.

We first argue that we can enlarge S so that the ring

$$\mathcal{O}_{K,S} = \{a \in K^* : \mathrm{ord}_{\mathfrak{p}}(a\mathcal{O}_K) \geq 0 \text{ all } \mathfrak{p} \notin S\} \cup \{0\}$$

is a principal ideal domain. Note that for any S, the ring $\mathcal{O}_{K,S}$ is a Dedekind domain. Also, the condition $\mathrm{ord}_{\mathfrak{p}}(a\mathcal{O}_K) \geq 0$ means that in the prime ideal factorization of the fractional ideal $a\mathcal{O}_K$, we have that $\mathfrak{p}$ occurs to a nonnegative power. Thus we are allowing denominators at the primes in S. Since the class group of $\mathcal{O}_K$ is finite, there are primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ that generate the class group as a group (for example, take all primes with norm up to the Minkowski bound). Enlarge S to contain the primes $\mathfrak{p}_i$. Note that the ideal $\mathfrak{p}_i\mathcal{O}_{K,S}$ is the unit ideal (we have $\mathfrak{p}_i^m = (\alpha)$ for some $m \geq 1$; then $1/\alpha \in \mathcal{O}_{K,S}$, so $(\mathfrak{p}_i\mathcal{O}_{K,S})^m$ is the unit ideal, hence $\mathfrak{p}_i\mathcal{O}_{K,S}$ is the unit ideal by unique factorization in the Dedekind domain $\mathcal{O}_{K,S}$.) Then $\mathcal{O}_{K,S}$ is a principal ideal domain, since every ideal of $\mathcal{O}_{K,S}$ is equivalent modulo a principal ideal to a product of ideals $\mathfrak{p}_i\mathcal{O}_{K,S}$. Note that we have used that *the class group of $\mathcal{O}_K$ is finite.*

Next enlarge S so that all primes over $n\mathcal{O}_K$ are in S. Note that $\mathcal{O}_{K,S}$ is still a PID. Let

$$K(S,n) = \{a \in K^*/(K^*)^n : n \mid \mathrm{ord}_{\mathfrak{p}}(a) \text{ all } \mathfrak{p} \notin S\}.$$

Then a refinement of the arguments at the beginning of this section show that L is generated by all nth roots of the elements of $K(S,n)$. It thus sufficies to prove that $K(S,n)$ is finite.

There is a natural map

$$\phi : \mathcal{O}_{K,S}^* \to K(S,n).$$

Suppose $a \in K^*$ is a representative of an element in $K(S,n)$. The ideal $a\mathcal{O}_{K,S}$ has factorization which is a product of nth powers, so it is an nth power of an ideal. Since $\mathcal{O}_{K,S}$ is a PID, there is $b \in \mathcal{O}_{K,S}$ and $u \in \mathcal{O}_{K,S}^*$ such that

$$a = b^n \cdot u.$$

Thus $u \in \mathcal{O}_{K,S}^*$ maps to $[a] \in K(S,n)$. Thus ϕ is surjective.

Recall that we proved *Dirichlet's unit theorem* (see Theorem 8.1.2), which asserts that the group $\mathcal{O}_K^*$ is a finitely generated abelian group of rank $r+s-1$. More generally, we now show that $\mathcal{O}_{K,S}^*$ is a finitely generated abelian group of rank $r+s+\#S-1$. Once we have shown this, then since $K(S,n)$ is torsion group that is a quotient of a finitely generated group, we will conclude that $K(S,n)$ is finite, which will prove the theorem.

Thus it remains to prove that $\mathcal{O}_{K,S}^*$ has rank $r+s-1+\#S$. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ be the primes in S. Define a map $\phi : \mathcal{O}_{K,S}^* \to \mathbf{Z}^n$ by

$$\phi(u) = (\mathrm{ord}_{\mathfrak{p}_1}(u),\ldots,\mathrm{ord}_{\mathfrak{p}_n}(u)).$$

First we show that $\mathrm{Ker}(\phi) = \mathcal{O}_K^*$. We have that $u \in \mathrm{Ker}(\phi)$ if and only if $u \in \mathcal{O}_{K,S}^*$ and $\mathrm{ord}_{\mathfrak{p}_i}(u) = 0$ for all i; but the latter condition implies that u is a unit at each prime in S, so $u \in \mathcal{O}_K^*$. Thus we have an exact sequence

$$1 \to \mathcal{O}_K^* \to \mathcal{O}_{K,S}^* \xrightarrow{\phi} \mathbf{Z}^n.$$

Next we show that the image of ϕ has finite index in $\mathbf{Z}^n$. Let h be the class number of $\mathcal{O}_K$. For each i there exists $\alpha_i \in \mathcal{O}_K$ such that $\mathfrak{p}_i^h = (\alpha_i)$. But $\alpha_i \in \mathcal{O}_{K,S}^*$ since $\mathrm{ord}_{\mathfrak{p}}(\alpha_i) = 0$ for all $\mathfrak{p} \notin S$ (by unique factorization). Then

$$\phi(\alpha_i) = (0,\ldots,0,h,0,\ldots,0).$$

It follows that $(h\mathbf{Z})^n \subset \mathrm{Im}(\phi)$, so the image of ϕ has finite index in $\mathbf{Z}^n$. It follows that $\mathcal{O}_{K,S}^*$ has rank equal to $r+s-1+\#S$. □

12.2 Proof of the Weak Mordell-Weil Theorem

Suppose E is an elliptic curve over a number field K, and fix a positive integer n. Just as with number fields, we have an exact sequence

$$0 \to E[n] \to E \xrightarrow{n} E \to 0.$$

Then we have an exact sequence

$$0 \to E[n](K) \to E(K) \xrightarrow{n} E(K) \to \mathrm{H}^1(K, E[n]) \to \mathrm{H}^1(K, E)[n] \to 0.$$

From this we obtain a short exact sequence

$$0 \to E(K)/nE(K) \to \mathrm{H}^1(K, E[n]) \to \mathrm{H}^1(K, E)[n] \to 0. \qquad (12.2.1)$$

Now assume, in analogy with Section 12.1, that $E[n] \subset E(K)$, i.e., all n-torsion points are defined over K. Then

$$\mathrm{H}^1(K, E[n]) = \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), (\mathbf{Z}/n\mathbf{Z})^2),$$

and the sequence (12.2.1) induces an inclusion

$$E(K)/nE(K) \hookrightarrow \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), (\mathbf{Z}/n\mathbf{Z})^2). \qquad (12.2.2)$$

Explicitly, this homomorphism sends a point P to the homomorphism defined as follows: Choose $Q \in E(\overline{K})$ such that $nQ = P$; then send each $\sigma \in \mathrm{Gal}(\overline{K}/K)$ to $\sigma(Q) - Q \in E[n] \cong (\mathbf{Z}/n\mathbf{Z})^2$. Given a point $P \in E(K)$, we obtain a homomorphism $\varphi : \mathrm{Gal}(\overline{K}/K) \to (\mathbf{Z}/n\mathbf{Z})^2$, whose kernel defines an abelian extension L of K that has exponent n. The amazing fact is that L can be ramified at most at the primes of bad reduction for E and the primes that divide n. Thus we can apply theorem 12.1.1 to see that there are only finitely many such L.

Theorem 12.2.1. *If $P \in E(K)$ is a point, then the field L obtained by adjoining to K all coordinates of all choices of $Q = \frac{1}{n}P$ is unramified outside n and the primes of bad reduction for E.*

Sketch of Proof. First one proves that if $\mathfrak{p} \nmid n$ is a prime of good reduction for E, then the natural reduction map $\pi : E(K)[n] \to \tilde{E}(\mathcal{O}_K/\mathfrak{p})$ is injective. The argument that π is injective uses "formal groups", whose development is outside the scope of this course. Next, as above, $\sigma(Q) - Q \in E(K)[n]$ for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$. Let $I_\mathfrak{p} \subset \mathrm{Gal}(L/K)$ be the inertia group at $\mathfrak{p}$. Then by definition of interia group, $I_\mathfrak{p}$ acts trivially on $\tilde{E}(\mathcal{O}_K/\mathfrak{p})$. Thus for each $\sigma \in I_\mathfrak{p}$ we have

$$\pi(\sigma(Q) - Q) = \sigma(\pi(Q)) - \pi(Q) = \pi(Q) - \pi(Q) = 0.$$

Since π is injective, it follows that $\sigma(Q) = Q$ for $\sigma \in I_\mathfrak{p}$, i.e., that Q is fixed under all $I_\mathfrak{p}$. This means that the subfield of L generated by the coordinates of Q is unramified at $\mathfrak{p}$. Repeating this argument with all choices of Q implies that L is unramified at $\mathfrak{p}$. □

Theorem 12.2.2 (Weak Mordell-Weil)**.** *Let E be an elliptic curve over a number field K, and let n be any positive integer. Then $E(K)/nE(K)$ is finitely generated.*

Proof. First suppose all elements of $E[n]$ have coordinates in K. Then the homomorphism (12.2.2) provides an injection of $E(K)/nE(K)$ into

$$\mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), (\mathbf{Z}/n\mathbf{Z})^2).$$

By Theorem 12.2.1, the image consists of homomorphisms whose kernels cut out an abelian extension of K unramified outside n and primes of bad reduction for E. Since this is a finite set of primes, Theorem 12.1.1 implies that the homomorphisms all factor through a finite quotient $\mathrm{Gal}(L/K)$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$. Thus there can be only finitely many such homomorphisms, so the image of $E(K)/nE(K)$ is finite. Thus $E(K)/nE(K)$ itself is finite, which proves the theorem in this case.

Next suppose E is an elliptic curve over a number field, but do *not* make the hypothesis that the elements of $E[n]$ have coordinates in K. Since the group $E[n](\mathbf{C})$ is finite and its elements are defined over $\overline{\mathbf{Q}}$, the extension L of K got by adjoining to K all coordinates of elements of $E[n](\mathbf{C})$ is a finite extension. It is also Galois, as we saw when constructing Galois representations attached to elliptic curves. By Proposition 11.3.1, we have an exact sequence

$$0 \to \mathrm{H}^1(L/K, E[n](L)) \to \mathrm{H}^1(K, E[n]) \to \mathrm{H}^1(L, E[n]).$$

The kernel of the restriction map $\mathrm{H}^1(K, E[n]) \to \mathrm{H}^1(L, E[n])$ is finite, since it is isomorphic to the finite group cohomology group $\mathrm{H}^1(L/K, E[n](L))$. By the argument of the previous paragraph, the image of $E(K)/nE(K)$ in $\mathrm{H}^1(L, E[n])$ under

$$E(K)/nE(K) \hookrightarrow \mathrm{H}^1(K, E[n]) \xrightarrow{\mathrm{res}} \mathrm{H}^1(L, E[n])$$

is finite, since it is contained in the image of $E(L)/nE(L)$. Thus $E(K)/nE(K)$ is finite, since we just proved the kernel of res is finite. □

Chapter 13

Exercises

1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

   (a) Find the Smith normal form of A.

   (b) Prove that the cokernel of the map $\mathbf{Z}^3 \to \mathbf{Z}^3$ given by multiplication by A is isomorphic to $\mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}$.

2. Show that the minimal polynomial of an algebraic number $\alpha \in \overline{\mathbf{Q}}$ is unique.

3. Which of the following rings have infinitely many prime ideals?

   (a) The integers $\mathbf{Z}$.

   (b) The ring $\mathbf{Z}[x]$ of polynomials over $\mathbf{Z}$.

   (c) The quotient ring $\mathbf{C}[x]/(x^{2005} - 1)$.

   (d) The ring $(\mathbf{Z}/6\mathbf{Z})[x]$ of polynomials over the ring $\mathbf{Z}/6\mathbf{Z}$.

   (e) The quotient ring $\mathbf{Z}/n\mathbf{Z}$, for a fixed positive integer n.

   (f) The rational numbers $\mathbf{Q}$.

   (g) The polynomial ring $\mathbf{Q}[x, y, z]$ in three variables.

4. Which of the following numbers are algebraic integers?

   (a) The number $(1 + \sqrt{5})/2$.

   (b) The number $(2 + \sqrt{5})/2$.

   (c) The value of the infinite sum $\sum_{n=1}^{\infty} 1/n^2$.

   (d) The number $\alpha/3$, where α is a root of $x^4 + 54x + 243$.

5. Prove that $\overline{\mathbf{Z}}$ is not noetherian.

6. Let $\alpha = \sqrt{2} + \frac{1+\sqrt{5}}{2}$.

(a) Is α an algebraic integer?

(b) Explicitly write down the minimal polynomial of α as an element of $\mathbf{Q}[x]$.

7. Which are the following rings are orders in the given number field.

   (a) The ring $R = \mathbf{Z}[i]$ in the number field $\mathbf{Q}(i)$.

   (b) The ring $R = \mathbf{Z}[i/2]$ in the number field $\mathbf{Q}(i)$.

   (c) The ring $R = \mathbf{Z}[17i]$ in the number field $\mathbf{Q}(i)$.

   (d) The ring $R = \mathbf{Z}[i]$ in the number field $\mathbf{Q}(\sqrt[4]{-1})$.

8. We showed in the text (see Proposition 3.1.3) that $\overline{\mathbf{Z}}$ is integrally closed in its field of fractions. Prove that and every nonzero prime ideal of $\overline{\mathbf{Z}}$ is maximal. Thus $\overline{\mathbf{Z}}$ is not a Dedekind domain only because it is not noetherian.

9. Let K be a field.

   (a) Prove that the polynomial ring $K[x]$ is a Dedekind domain.

   (b) Is $\mathbf{Z}[x]$ a Dedekind domain?

10. Prove that every finite integral domain is a field.

11. (a) Give an example of two ideals I, J in a commutative ring R whose product is *not* equal to the set $\{ab : a \in I, b \in J\}$.

    (b) Suppose R is a principal ideal domain. Is it always the case that

    $$IJ = \{ab : a \in I, b \in J\}$$

    for all ideals I, J in R?

12. Is the set $\mathbf{Z}[\frac{1}{2}]$ of rational numbers with denominator a power of 2 a fractional ideal?

13. Suppose you had the choice of the following two jobs[1]:

    Job 1 Starting with an annual salary of \$1000, and a \$200 increase every year.

    Job 2 Starting with a semiannual salary of \$500, and an increase of \$50 every 6 months.

    In all other respects, the two jobs are exactly alike. Which is the better offer (after the first year)? Write a Sage program that creates a table showing how much money you will receive at the end of each year for each job. (Of course you could easily do this by hand – the point is to get familiar with Sage.)

14. Let $\mathcal{O}_K$ be the ring of integers of a number field. Let F_K denote the abelian group of fractional ideals of $\mathcal{O}_K$.

[1] From *The Education of T.C. MITS* (1942).

(a) Prove that F_K is torsion free.

(b) Prove that F_K is not finitely generated.

(c) Prove that F_K is countable.

(d) Conclude that if K and L are number fields, then there exists some (non-canonical) isomorphism of groups $F_K \approx F_L$.

15. From basic definitions, find the rings of integers of the fields $\mathbf{Q}(\sqrt{11})$ and $\mathbf{Q}(\sqrt{-6})$.

16. In this problem, you will give an example to illustrate the failure of unique factorization in the ring $\mathcal{O}_K$ of integers of $\mathbf{Q}(\sqrt{-6})$.

    (a) Give an element $\alpha \in \mathcal{O}_K$ that factors in two distinct ways into irreducible elements.

    (b) Observe explicitly that the (α) factors uniquely, i.e., the two distinct factorization in the previous part of this problem do not lead to two distinct factorization of the ideal (α) into prime ideals.

17. Factor the ideal (10) as a product of primes in the ring of integers of $\mathbf{Q}(\sqrt{11})$. You're allowed to use a computer, as long as you show the commands you use.

18. Let $\mathcal{O}_K$ be the ring of integers of a number field K, and let $p \in \mathbf{Z}$ be a prime number. What is the cardinality of $\mathcal{O}_K/(p)$ in terms of p and $[K : \mathbf{Q}]$, where (p) is the ideal of $\mathcal{O}_K$ generated by p?

19. Give an example of each of the following, with proof:

    (a) A non-principal ideal in a ring.

    (b) A module that is not finitely generated.

    (c) The ring of integers of a number field of degree 3.

    (d) An order in the ring of integers of a number field of degree 5.

    (e) The matrix on K of left multiplication by an element of K, where K is a degree 3 number field.

    (f) An integral domain that is not integrally closed in its field of fractions.

    (g) A Dedekind domain with finite cardinality.

    (h) A fractional ideal of the ring of integers of a number field that is not an integral ideal.

20. Let $\varphi : R \to S$ be a homomorphism of (commutative) rings.

    (a) Prove that if $I \subset S$ is an ideal, then $\varphi^{-1}(I)$ is an ideal of R.

    (b) Prove moreover that if I is prime, then $\varphi^{-1}(I)$ is also prime.

21. Let $\mathcal{O}_K$ be the ring of integers of a number field. The Zariski topology on the set $X = \mathrm{Spec}(\mathcal{O}_K)$ of all prime ideals of $\mathcal{O}_K$ has closed sets the sets of the form
$$V(I) = \{\mathfrak{p} \in X : \mathfrak{p} \mid I\},$$
where I varies through *all* ideals of $\mathcal{O}_K$, and $\mathfrak{p} \mid I$ means that $I \subset \mathfrak{p}$.

    (a) Prove that the collection of closed sets of the form $V(I)$ is a topology on X.

    (b) Let Y be the subset of nonzero prime ideals of $\mathcal{O}_K$, with the induced topology. Use unique factorization of ideals to prove that the closed subsets of Y are exactly the finite subsets of Y along with the set Y.

    (c) Prove that the conclusion of (a) is still true if $\mathcal{O}_K$ is replaced by an order in $\mathcal{O}_K$, i.e., a subring that has finite index in $\mathcal{O}_K$ as a $\mathbf{Z}$-module.

22. Explicitly factor the ideals generated by each of 2, 3, and 5 in the ring of integers of $\mathbf{Q}(\sqrt[3]{2})$. (Thus you'll factor 3 separate ideals as products of prime ideals.) You may assume that the ring of integers of $\mathbf{Q}(\sqrt[3]{2})$ is $\mathbf{Z}[\sqrt[3]{2}]$, but do *not* simply use a computer command to do the factorizations.

23. Let $K = \mathbf{Q}(\zeta_{13})$,where ζ_{13} is a primitive 13th root of unity. Note that K has ring of integers $\mathcal{O}_K = \mathbf{Z}[\zeta_{13}]$.

    (a) Factor 2, 3, 5, 7, 11, and 13 in the ring of integers $\mathcal{O}_K$. You may use a computer.

    (b) For $p \neq 13$, find a conjectural relationship between the number of prime ideal factors of $p\mathcal{O}_K$ and the order of the reduction of p in $(\mathbf{Z}/13\mathbf{Z})^*$.

    (c) Compute the minimal polynomial $f(x) \in \mathbf{Z}[x]$ of ζ_{13}. Reinterpret your conjecture as a conjecture that relates the degrees of the irreducible factors of $f(x) \pmod{p}$ to the order of p modulo 13. Does your conjecture remind you of quadratic reciprocity?

24. (a) Find by hand and with proof the ring of integers of each of the following two fields: $\mathbf{Q}(\sqrt{5})$, $\mathbf{Q}(i)$.

    (b) Find the ring of integers of $\mathbf{Q}(a)$, where $a^5+7a+1 = 0$ using a computer.

25. Let p be a prime. Let $\mathcal{O}_K$ be the ring of integers of a number field K, and suppose $a \in \mathcal{O}_K$ is such that $[\mathcal{O}_K : \mathbf{Z}[a]]$ is finite and coprime to p. Let $f(x)$ be the minimal polynomial of a. We proved in class that if the reduction $\overline{f} \in \mathbf{F}_p[x]$ of f factors as
$$\overline{f} = \prod g_i^{e_i},$$
where the g_i are distinct irreducible polynomials in $\mathbf{F}_p[x]$, then the primes appearing in the factorization of $p\mathcal{O}_K$ are the ideals $(p, g_i(a))$. In class, we did not prove that the exponents of these primes in the factorization of $p\mathcal{O}_K$ are the e_i. Prove this.

26. Let $a_1 = 1 + i$, $a_2 = 3 + 2i$, and $a_3 = 3 + 4i$ as elements of $\mathbf{Z}[i]$.

    (a) Prove that the ideals $I_1 = (a_1)$, $I_2 = (a_2)$, and $I_3 = (a_3)$ are coprime in pairs.

    (b) Compute $\#\mathbf{Z}[i]/(I_1 I_2 I_3)$.

    (c) Find a single element in $\mathbf{Z}[i]$ that is congruent to n modulo I_n, for each $n \leq 3$.

27. Find an example of a field K of degree at least 4 such that the ring $\mathcal{O}_K$ of integers of K is not of the form $\mathbf{Z}[a]$ for any $a \in \mathcal{O}_K$.

28. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$, and suppose that $\mathcal{O}_K/\mathfrak{p}$ is a finite field of characteristic $p \in \mathbf{Z}$. Prove that there is an element $\alpha \in \mathcal{O}_K$ such that $\mathfrak{p} = (p, \alpha)$. This justifies why we can represent prime ideals of $\mathcal{O}_K$ as pairs (p, α), as is done in SAGE. (More generally, if I is an ideal of $\mathcal{O}_K$, we can choose one of the elements of I to be *any* nonzero element of I.)

29. (*) Give an example of an order $\mathcal{O}$ in the ring of integers of a number field and an ideal I such that I cannot be generated by 2 elements as an ideal. Does the Chinese Remainder Theorem hold in $\mathcal{O}$? [The (*) means that this problem is more difficult than usual.]

30. For each of the following three fields, determining if there is an order of discriminant 20 contained in its ring of integers:

    $$K = \mathbf{Q}(\sqrt{5}), \quad K = \mathbf{Q}(\sqrt[3]{2}), \quad \text{and}\ldots$$

    K any extension of $\mathbf{Q}$ of degree 2005. [Hint: for the last one, apply the exact form of our theorem about finiteness of class groups to the unit ideal to show that the discriminant of a degree 2005 field must be large.]

31. Prove that the quantity $C_{r,s}$ in our theorem about finiteness of the class group can be taken to be $\left(\frac{4}{\pi}\right)^s \frac{n!}{n^n}$, as follows (adapted from [SD01, pg. 19]): Let S be the set of elements $(x_1, \ldots, x_n) \in \mathbf{R}^n$ such that

    $$|x_1| + \cdots |x_r| + 2 \sum_{v=r+1}^{r+s} \sqrt{x_v^2 + x_{v+s}^2} \leq 1.$$

    (a) Prove that S is convex and that $M = n^{-n}$, where

    $$M = \max\{|x_1 \cdots x_r \cdot (x_{r+1}^2 + x_{(r+1)+s}^2) \cdots (x_{r+s}^2 + x_n^2)| : (x_1, \ldots, x_n) \in S\}.$$

    [Hint: For convexity, use the triangle inequality and that for $0 \leq \lambda \leq 1$, we have

    $$\lambda\sqrt{x_1^2 + y_1^2} + (1 - \lambda)\sqrt{x_2^2 + y_2^2}$$
    $$\geq \sqrt{(\lambda x_1 + (1 - \lambda)x_2)^2 + (\lambda y_1 + (1 - \lambda)y_2)^2}$$

for $0 \leq \lambda \leq 1$. In polar coordinates this last inequality is

$$\lambda r_1 + (1-\lambda) r_2 \geq \sqrt{\lambda^2 r_1^2 + 2\lambda(1-\lambda) r_1 r_2 \cos(\theta_1 - \theta_2) + (1-\lambda)^2 r_2^2},$$

which is trivial. That $M \leq n^{-n}$ follows from the inequality between the arithmetic and geometric means.

(b) Transforming pairs x_v, x_{v+s} from Cartesian to polar coordinates, show also that $v = 2^r (2\pi)^s D_{r,s}(1)$, where

$$D_{\ell,m}(t) = \int \cdots \int_{\mathcal{R}_{\ell,m}(t)} y_1 \cdots y_m dx_1 \cdots dx_\ell dy_1 \cdots dy_m$$

and $\mathcal{R}_{\ell,\Updownarrow}(t)$ is given by $x_\rho \geq 0$ $(1 \leq \rho \leq \ell)$, $y_\rho \geq 0$ $(1 \leq \rho \leq m)$ and

$$x_1 + \cdots + x_\ell + 2(y_1 + \cdots + y_m) \leq t.$$

(c) Prove that

$$D_{\ell,m}(t) = \int_0^t D_{\ell-1,m}(t-x)dx = \int_0^{t/2} D_{\ell,m-1}(t-2y)ydy$$

and deduce by induction that

$$D_{\ell,m}(t) = \frac{4^{-m} t^{\ell+2m}}{(\ell+2m)!}$$

32. Let K vary through all number fields. What torsion subgroups $(U_K)_{\mathrm{tor}}$ actually occur?

33. If $U_K \approx \mathbf{Z}^n \times (U_K)_{\mathrm{tor}}$, we say that U_K has rank n. Let K vary through all number fields. What ranks actually occur?

34. Let K vary through all number fields such that the group U_K of units of K is a finite group. What finite groups U_K actually occur?

35. Let $K = \mathbf{Q}(\zeta_5)$.

    (a) Show that $r = 0$ and $s = 2$.

    (b) Find explicit generators for the group of units U_K.

    (c) Draw an illustration of the log map $\varphi : U_K \to \mathbf{R}^2$, including the hyperplane $x_1 + x_2 = 0$ and the lattice in the hyperplane spanned by the image of U_K.

36. Let K be a number field. Prove that $p \mid d_K$ if and only if p ramifies in K. (Note: This fact is proved in many books.)

37. Using Zorn's lemma, show that there are homomorphisms $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \{\pm 1\}$ with finite image that are not continuous, since they do not factor through the Galois group of any finite Galois extension. [Hint: The extension $\mathbf{Q}(\sqrt{d}, d \in \mathbf{Q}^*/(\mathbf{Q}^*)^2)$ is an extension of $\mathbf{Q}$ with Galois group $X \approx \prod \mathbf{F}_2$. The index-two open subgroups of X correspond to the quadratic extensions of $\mathbf{Q}$. However, Zorn's lemma implies that X contains many index-two subgroups that do not correspond to quadratic extensions of $\mathbf{Q}$.]

38. (a) Give an example of a finite nontrivial Galois extension K of $\mathbf{Q}$ and a prime ideal $\mathfrak{p}$ such that $D_{\mathfrak{p}} = \mathrm{Gal}(K/\mathbf{Q})$.

    (b) Give an example of a finite nontrivial Galois extension K of $\mathbf{Q}$ and a prime ideal $\mathfrak{p}$ such that $D_{\mathfrak{p}}$ has order 1.

    (c) Give an example of a finite Galois extension K of $\mathbf{Q}$ and a prime ideal $\mathfrak{p}$ such that $D_{\mathfrak{p}}$ is not a normal subgroup of $\mathrm{Gal}(K/\mathbf{Q})$.

    (d) Give an example of a finite Galois extension K of $\mathbf{Q}$ and a prime ideal $\mathfrak{p}$ such that $I_{\mathfrak{p}}$ is not a normal subgroup of $\mathrm{Gal}(K/\mathbf{Q})$.

39. Let S_3 by the symmetric group on three symbols, which has order 6.

    (a) Observe that $S_3 \cong D_3$, where D_3 is the dihedral group of order 6, which is the group of symmetries of an equilateral triangle.

    (b) Use (39a) to write down an explicit embedding $S_3 \hookrightarrow \mathrm{GL}_2(\mathbf{C})$.

    (c) Let K be the number field $\mathbf{Q}(\sqrt[3]{2}, \omega)$, where $\omega^3 = 1$ is a nontrivial cube root of unity. Show that K is a Galois extension with Galois group isomorphic to S_3.

    (d) We thus obtain a 2-dimensional irreducible complex Galois representation

    $$\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Gal}(K/\mathbf{Q}) \cong S_3 \subset \mathrm{GL}_2(\mathbf{C}).$$

    Compute a representative matrix of Frob_p and the characteristic polynomial of Frob_p for $p = 5, 7, 11, 13$.

40. Look up the Riemann-Roch theorem in a book on algebraic curves.

    (a) Write it down in your own words.

    (b) Let E be an elliptic curve over a field K. Use the Riemann-Roch theorem to deduce that the natural map

    $$E(K) \to \mathrm{Pic}^0(E/K)$$

    is an isomorphism.

41. Suppose G is a finite group and A is a finite G-module. Prove that for any q, the group $\mathrm{H}^q(G, A)$ is a torsion abelian group of exponent dividing the order $\#A$ of A.

42. Let $K = \mathbf{Q}(\sqrt{5})$ and let $A = U_K$ be the group of units of K, which is a module over the group $G = \mathrm{Gal}(K/\mathbf{Q})$. Compute the cohomology groups $\mathrm{H}^0(G, A)$ and $\mathrm{H}^1(G, A)$. (You shouldn't use a computer, except maybe to determine U_K.)

43. Let $K = \mathbf{Q}(\sqrt{-23})$ and let C be the class group of $\mathbf{Q}(\sqrt{-23})$, which is a module over the Galois group $G = \mathrm{Gal}(K/\mathbf{Q})$. Determine $\mathrm{H}^0(G, C)$ and $\mathrm{H}^1(G, C)$.

44. Let E be the elliptic curve $y^2 = x^3 + x + 1$. Let $E[2]$ be the group of points of order dividing 2 on E. Let

$$\overline{\rho}_{E,2} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}(E[2])$$

be the mod 2 Galois representation associated to E.

(a) Find the fixed field K of $\ker(\overline{\rho}_{E,2})$.

(b) Is $\overline{\rho}_{E,2}$ surjective?

(c) Find the group $\mathrm{Gal}(K/\mathbf{Q})$.

(d) Which primes are ramified in K?

(e) Let I be an inertia group above 2, which is one of the ramified primes. Determine $E[2]^I$ explicitly for your choice of I. What is the characteristic polynomial of Frob_2 acting on $E[2]^I$.

(f) What is the characteristic polynomial of Frob_3 acting on $E[2]$?

(g) Let K be a number field. Prove that there is a finite set S of primes of K such that

$$\mathcal{O}_{K,S} = \{a \in K^* : \mathrm{ord}_{\mathfrak{p}}(a\mathcal{O}_K) \geq 0 \text{ all } \mathfrak{p} \notin S\} \cup \{0\}$$

is a prinicipal ideal domain. The condition $\mathrm{ord}_{\mathfrak{p}}(a\mathcal{O}_K) \geq 0$ means that in the prime ideal factorization of the fractional ideal $a\mathcal{O}_K$, we have that $\mathfrak{p}$ occurs to a nonnegative power.

(h) Let $a \in K$ and n a positive integer. Prove that $L = K(a^{1/n})$ is unramified outside the primes that divide n and the norm of a. This means that if $\mathfrak{p}$ is a prime of $\mathcal{O}_K$, and $\mathfrak{p}$ is coprime to $n\,\mathrm{Norm}_{L/K}(a)\mathcal{O}_K$, then the prime factorization of $\mathfrak{p}\mathcal{O}_L$ involves no primes with exponent bigger than 1.

(i) Write down a proof of Hilbert's Theorem 90, formulated as the statement that for any number field K, we have

$$\mathrm{H}^1(K, \overline{K}^*) = 0.$$

1. Let k be any field. Prove that the only nontrivial valuations on $k(t)$ which are trivial on k are equivalent to the valuation (**??**) or (**??**) of page **??**.

2. A field with the topology induced by a valuation is a topological field, i.e., the operations sum, product, and reciprocal are continuous.

3. Give an example of a non-archimedean valuation on a field that is not discrete.

4. Prove that the field $\mathbf{Q}_p$ of p-adic numbers is uncountable.

5. Prove that the polynomial $f(x) = x^3 - 3x^2 + 2x + 5$ has all its roots in $\mathbf{Q}_5$, and find the 5-adic valuations of each of these roots. (You might need to use Hensel's lemma, which we don't discuss in detail in this book. See [Cas67, App. C].)

6. In this problem you will compute an example of weak approximation, like I did in the Example **??**. Let $K = \mathbf{Q}$, let $|\cdot|_7$ be the 7-adic absolute value, let $|\cdot|_{11}$ be the 11-adic absolute value, and let $|\cdot|_\infty$ be the usual archimedean absolute value. Find an element $b \in \mathbf{Q}$ such that $|b - a_i|_i < \frac{1}{10}$, where $a_7 = 1$, $a_{11} = 2$, and $a_\infty = -2004$.

7. Prove that -9 has a cube root in $\mathbf{Q}_{10}$ using the following strategy (this is a special case of Hensel's Lemma, which you can read about in an appendix to Cassel's article).

   (a) Show that there is an element $\alpha \in \mathbf{Z}$ such that $\alpha^3 \equiv 9 \pmod{10^3}$.

   (b) Suppose $n \geq 3$. Use induction to show that if $\alpha_1 \in \mathbf{Z}$ and $\alpha^3 \equiv 9 \pmod{10^n}$, then there exists $\alpha_2 \in \mathbf{Z}$ such that $\alpha_2^3 \equiv 9 \pmod{10^{n+1}}$. (Hint: Show that there is an integer b such that $(\alpha_1 + b \cdot 10^n)^3 \equiv 9 \pmod{10^{n+1}}$.)

   (c) Conclude that 9 has a cube root in $\mathbf{Q}_{10}$.

8. Compute the first 5 digits of the 10-adic expansions of the following rational numbers:

   $$\frac{13}{2}, \quad \frac{1}{389}, \quad \frac{17}{19}, \quad \text{the 4 square roots of 41.}$$

9. Let $N > 1$ be an integer. Prove that the series

   $$\sum_{n=1}^{\infty} (-1)^{n+1} n! = 1! - 2! + 3! - 4! + 5! - 6! + \cdots .$$

   converges in $\mathbf{Q}_N$.

10. Prove that -9 has a cube root in $\mathbf{Q}_{10}$ using the following strategy (this is a special case of "Hensel's Lemma").

(a) Show that there is $\alpha \in \mathbf{Z}$ such that $\alpha^3 \equiv 9 \pmod{10^3}$.

(b) Suppose $n \geq 3$. Use induction to show that if $\alpha_1 \in \mathbf{Z}$ and $\alpha^3 \equiv 9 \pmod{10^n}$, then there exists $\alpha_2 \in \mathbf{Z}$ such that $\alpha_2^3 \equiv 9 \pmod{10^{n+1}}$. (Hint: Show that there is an integer b such that $(\alpha_1 + b10^n)^3 \equiv 9 \pmod{10^{n+1}}$.)

(c) Conclude that 9 has a cube root in $\mathbf{Q}_{10}$.

11. Let $N > 1$ be an integer.

(a) Prove that $\mathbf{Q}_N$ is equipped with a natural ring structure.

(b) If N is prime, prove that $\mathbf{Q}_N$ is a field.

12. (a) Let p and q be distinct primes. Prove that $\mathbf{Q}_{pq} \cong \mathbf{Q}_p \times \mathbf{Q}_q$.

(b) Is $\mathbf{Q}_{p^2}$ isomorphic to either of $\mathbf{Q}_p \times \mathbf{Q}_p$ or $\mathbf{Q}_p$?

13. Prove that every finite extension of $\mathbf{Q}_p$ "comes from" an extension of $\mathbf{Q}$, in the following sense. Given an irreducible polynomial $f \in \mathbf{Q}_p[x]$ there exists an irreducible polynomial $g \in \mathbf{Q}[x]$ such that the fields $\mathbf{Q}_p[x]/(f)$ and $\mathbf{Q}_p[x]/(g)$ are isomorphic. [Hint: Choose each coefficient of g to be sufficiently close to the corresponding coefficient of f, then use Hensel's lemma to show that g has a root in $\mathbf{Q}_p[x]/(f)$.]

14. Find the 3-adic expansion to precision 4 of each root of the following polynomial over $\mathbf{Q}_3$:

$$f = x^3 - 3x^2 + 2x + 3 \in \mathbf{Q}_3[x].$$

Your solution should conclude with three expressions of the form

$$a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3 + O(3^4).$$

15. (a) Find the normalized Haar measure of the following subset of $\mathbf{Q}_7^+$:

$$U = B\left(28, \frac{1}{50}\right) = \left\{x \in \mathbf{Q}_7 : |x - 28| < \frac{1}{50}\right\}.$$

(b) Find the normalized Haar measure of the subset $\mathbf{Z}_7^*$ of $\mathbf{Q}_7^*$.

16. Suppose that K is a finite extension of $\mathbf{Q}_p$ and L is a finite extension of $\mathbf{Q}_q$, with $p \neq q$ and assume that K and L have the same degree. Prove that there is a polynomial $g \in \mathbf{Q}[x]$ such that $\mathbf{Q}_p[x]/(g) \cong K$ and $\mathbf{Q}_q[x]/(g) \cong L$. [Hint: Combine your solution to 13 with the weak approximation theorem.]

17. Prove that the ring C defined in Section 9 really is the tensor product of A and B, i.e., that it satisfies the defining universal mapping property for tensor products. Part of this problem is for you to look up a functorial definition of tensor product.

18. Find a zero divisor pair in $\mathbf{Q}(\sqrt{5}) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{5})$.

19. (a) Is $\mathbf{Q}(\sqrt{5}) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{-5})$ a field?

    (b) Is $\mathbf{Q}(\sqrt[4]{5}) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt[4]{-5}) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{-1})$ a field?

20. Suppose ζ_5 denotes a primitive 5th root of unity. For any prime p, consider the tensor product $\mathbf{Q}_p \otimes_{\mathbf{Q}} \mathbf{Q}(\zeta_5) = K_1 \oplus \cdots \oplus K_{n(p)}$. Find a simple formula for the number $n(p)$ of fields appearing in the decomposition of the tensor product $\mathbf{Q}_p \otimes_{\mathbf{Q}} \mathbf{Q}(\zeta_5)$. To get full credit on this problem your formula must be correct, but you do *not* have to prove that it is correct.

21. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a finite-dimensional vector space V over a field K (with valuation $|\cdot|$). Carefully prove that the topology induced by $\|\cdot\|_1$ is the same as that induced by $\|\cdot\|_2$.

22. Suppose K and L are number fields (i.e., finite extensions of $\mathbf{Q}$). Is it possible for the tensor product $K \otimes_{\mathbf{Q}} L$ to contain a nilpotent element? (A nonzero element a in a ring R is *nilpotent* if there exists $n > 1$ such that $a^n = 0$.)

23. Let K be the number field $\mathbf{Q}(\sqrt[5]{2})$.

    (a) In how many ways does the 2-adic valuation $|\cdot|_2$ on $\mathbf{Q}$ extend to a valuation on K?

    (b) Let $v = |\cdot|$ be a valuation on K that extends $|\cdot|_2$. Let K_v be the completion of K with respect to v. What is the residue class field $\mathbf{F}$ of K_v?

24. Prove that the product formula holds for $\mathbf{F}(t)$ similar to the proof we gave in class using Ostrowski's theorem for $\mathbf{Q}$. You may use the analogue of Ostrowski's theorem for $\mathbf{F}(t)$, which you had on a previous homework assignment. (Don't give a measure-theoretic proof.)

25. Prove Theorem **??**, that "The global field K is discrete in $\mathbb{A}_K$ and the quotient $\mathbb{A}_K^+/K^+$ of additive groups is compact in the quotient topology." in the case when K is a finite extension of $\mathbf{F}(t)$, where $\mathbf{F}$ is a finite field.

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