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Comparison of Manning and Navier-Stokes in Rectangular Channels

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Abstract

Manning's equation is widely used for estimating flow in open channels due to its simplicity and empirical basis. However, its accuracy can be limited in complex flow conditions. This study compares the performance of Manning's equation with the Navier-Stokes equations in rectangular channels under varying Reynold's Numbers. Assuming laminar flow (Re < 500), we calculate values of Manning's n and compare them to traditional values. Our results indicate that for laminar flow, Manning's equation provides reasonable estimates for uniform flow. Under turbulent and transitional flows, the Navier-Stokes equations do not hold, so we made no comparison. The findings highlight the importance of selecting appropriate modeling techniques based on flow conditions for accurate hydraulic predictions.

Nomenclature

Q = Discharge (m^3/s)
A = Cross-sectional area of flow (m^2)
n = Manning's roughness coefficient
R_H = Hydraulic radius (m)
P_w = Wetted perimeter (m)
S = Channel slope (m/m)
V = Velocity (m/s)
b = Channel bottom width (m)
h = Flow depth (m)
ν = Kinematic viscosity (m^2/s)
g = Gravitational acceleration (m/s^2)
ρ = Fluid density (kg/m^3)
Re = Reynolds number (dimensionless)

Introduction

Manning's equation is an empirical formula used to estimate the flow of water in open channels. It is expressed as:

$$Q = \frac{1}{n} A R_H^{2/3} S^{1/2} \tag{1}$$

where Q is the discharge (m^3/s), A is the cross-sectional area of flow (m^2), R_H is the hydraulic radius (m), S is the channel slope (m/m), and n is Manning's roughness coefficient, which accounts for the channel's surface roughness. The hydraulic radius R_H is defined as the ratio of the cross-sectional area of flow to the wetted perimeter:

$$R_H = \frac{A}{P_w} \tag{2}$$

For a rectangular channel, the cross-sectional area A and wetted perimeter P_w can be expressed as:

$$A = b \cdot h \tag{3}$$

$$P_w = b + 2h \tag{4}$$

where b is the channel bottom width (m) and h is the flow depth (m).

The Navier-Stokes equations describe the motion of fluid substances and are fundamental to fluid dynamics. For incompressible flow, the Navier-Stokes equations can be written as:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g} \tag{5}$$

where ρ is the fluid density (kg/m^3), $\mathbf{u}$ is the velocity vector (m/s), p is the pressure (Pa), μ is the dynamic viscosity (Pa·s), and $\mathbf{g}$ is the gravitational acceleration vector (m/s^2). The Reynolds number (Re) is a dimensionless quantity used to predict flow patterns in different fluid flow situations. It is defined as:

$$Re = \frac{VR_h}{\nu} \tag{6}$$

where V is the characteristic velocity (m/s), R_H is the hydraulic radius (m), and ν is the kinematic viscosity (m^2/s). For a rectangular channel, the hydraulic diameter can be expressed as:

$$R_H = \frac{A}{P_w} \tag{7}$$

In this study, we compare the performance of Manning's equation with the Navier-Stokes equations in rectangular channels under varying Reynolds numbers. We focus on laminar flow conditions (Re < 500) to evaluate the accuracy of Manning's equation in predicting flow characteristics.

Methodology

Assumptions

The following assumptions were made to simplify the Navier-Stokes to match the conditions necessary to apply Manning's equation:

- The fluid is water $\rightarrow \rho = C, \mu = C$.
- Flow is steady $\rightarrow \frac{\partial}{\partial t} = 0$.
- Flow is uniform $\rightarrow \frac{\partial P}{\partial x} = 0$.
- Channel is infinite in the flow direction $\rightarrow \frac{\partial}{\partial x} = 0$.
- Flow is irrotational $\longrightarrow \mathbf{u}\cdot\nabla\mathbf{u} = 0$.

Applying our assumptions to the Navier-Stokes equations in the x-direction, we simplify it to:

$$-\rho g_x = \mu[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}] \tag{8}$$

where g_x is the component of gravitational acceleration in the x-direction (m/s^2), and u is the velocity component in the x-direction (m/s). With boundary conditions:

- No slip at the bottom and sides: $u = 0$ at $y = 0$, $z = 0$, and $z = b$.
- Symmetry at the free surface: $\frac{\partial u}{\partial y} = 0$ at $y = h$.

Next we non-dimensionalize the equations using the following variables:

$$\hat{y} = \frac{y}{h}, \quad \hat{z} = \frac{z}{b}, \quad \hat{u} = \frac{u\nu}{h^2 g\sin\theta} \tag{9}$$

where θ is the angle of the channel slope. Substituting these into the simplified Navier-Stokes equation, we obtain the non-dimensional form:

$$-1 = \frac{\partial^2\hat{u}}{\partial\hat{y}^2} + \left(\frac{h^2}{b^2}\right)\frac{\partial^2\hat{u}}{\partial\hat{z}^2} \tag{10}$$

We solve this equation using separation of variables and apply the boundary conditions to find the velocity profile $\hat{u}(\hat{y},\hat{z})$. Once we have the velocity profile, we calculate the volumetric flow rate Q by integrating the velocity over the cross-sectional area:

$$Q = \int_0^b\int_0^h u(y,z)\,dy\,dz \tag{11}$$

Flow Rate Calculations

The final form of the volumetric flow rate Q derived from the Navier-Stokes solution is:

$$Q = \frac{g\sin\theta h^3 b}{\nu}\left(\frac{1}{3} + \sum_{n=1}^{\infty}\frac{E_n}{\lambda_n^2}\left(1 - \frac{\sinh(\lambda_n)}{\lambda_n\cosh(\lambda_n)}\right)\right) \tag{12}$$

where $\lambda_n = \frac{n\pi h}{b}$ and E_n are constants determined by the boundary conditions.

Manning's equation for volumetric flow rate in a rectangular channel is given by:

$$Q = \frac{1}{n} A R_H^{2/3} S^{1/2} \tag{13}$$

Substituting the expressions for A and R_H for a rectangular channel, we have:

$$Q = \frac{1}{n}(bh)\left(\frac{bh}{b+2h}\right)^{2/3} S^{1/2} \tag{14}$$

We calculate the flow rate using both the Navier-Stokes derived equation and Manning's equation for various channel dimensions and slopes under laminar flow conditions.

Finally, we compare the flow rate obtained from the Navier-Stokes solution with that predicted by Manning's equation to evaluate its accuracy under laminar flow conditions.

Results and Discussion

Substituting the non-dimensionalized variables into the boundary conditions, they become:

- No slip at the bottom and sides: $\hat{u} = 0$ at $\hat{y} = 0$ and $\hat{z} = 0$, $\hat{z} = 1$.
- Symmetry at the free surface: $\frac{\partial \hat{u}}{\partial \hat{y}} = 0$ at $\hat{y} = 1$.

We can now solve for $\hat{u}$ as $\hat{u} = u_1(\hat{y}) + u_2(\hat{y}, \hat{z})$, where u_1 is the particular solution and u_2 is the homogeneous solution.

The equation for the particular solution, u_1, is:

$$\frac{d^2 u_1}{d\hat{y}^2} = -1 \tag{15}$$

Integrating twice and applying the boundary conditions, we find:

$$u_1(\hat{y}) = \hat{y} - \frac{\hat{y}^2}{2} \tag{16}$$

The homogeneous solution, u_2, satisfies:

$$\frac{\partial^2 u_2}{\partial \hat{y}^2} + \alpha^2 \frac{\partial^2 u_2}{\partial \hat{z}^2} = 0 \tag{17}$$

where $\alpha = \frac{h}{b}$. Using separation of variables, we assume $u_2(\hat{y}, \hat{z}) = Y(\hat{y})Z(\hat{z})$. Substituting into the homogeneous equation and separating variables, we obtain:

$$\frac{Y''}{Y} = -\lambda^2, \quad \frac{Z''}{Z} = \alpha^2 \lambda^2 \tag{18}$$

where λ^2 is the separation constant. Solving these ordinary differential equations, we find:

$$Y(\hat{y}) = A\cos(\lambda \hat{y}) + B\sin(\lambda \hat{y}) \tag{19}$$

$$Z(\hat{z}) = C\cosh(\frac{\lambda \hat{z}}{\alpha}) + D\sinh(\frac{\lambda \hat{z}}{\alpha}) \tag{20}$$

Applying the boundary conditions to u_2, we determine the constants A, B, C, and D. The no-slip conditions at the sides ($\hat{z} = 0$ and $\hat{z} = 1$) lead to:

$$C = 0, \quad \cos(\lambda) = 0 \implies \lambda_n = \frac{(2n-1)\pi}{2} \tag{21}$$

Thus, the homogeneous solution simplifies to:

$$u_2(\hat{y}, \hat{z}) = \sum_{n=1}^{\infty} E_n \sinh(\lambda_n \hat{y}) \sin\left(\lambda_n \frac{b}{h} \hat{z}\right) \tag{22}$$

where $\lambda_n = \frac{n\pi h}{b}$ and E_n are constants determined by the boundary conditions. Combining the particular and homogeneous solutions, the complete velocity profile is:

$$\hat{u}(\hat{y}, \hat{z}) = \hat{y} - \frac{\hat{y}^2}{2} + \sum_{n=1}^{\infty} E_n \sinh(\lambda_n \hat{y}) \sin\left(\lambda_n \frac{b}{h} \hat{z}\right) \tag{23}$$

To find the volumetric flow rate Q, we integrate the velocity profile over the cross-sectional area:

$$Q = \int_0^b \int_0^h u(y, z)\, dy\, dz \tag{24}$$

Substituting the dimensional form of $\hat{u}$ and performing the integration, we obtain:

$$Q = \frac{g \sin\theta h^3 b}{\nu}\left(\frac{1}{3} + \sum_{n=1}^{\infty} \frac{E_n}{\lambda_n^2}\left(1 - \frac{\sinh(\lambda_n)}{\lambda_n \cosh(\lambda_n)}\right)\right) \tag{25}$$

Finally, we compare this flow rate with that predicted by Manning's equation to evaluate its accuracy under laminar flow conditions.

Conclusions

Put conclusions here.

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Appendix

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