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Manning's and Navier-Stokes in Rectangular Channels

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Abstract

Manning's equation is widely used for estimating flow in open channels due to its simplicity and empirical basis. However, its accuracy can be limited in complex flow conditions. This study compares the performance of Manning's equation with the Navier-Stokes equations in rectangular channels under varying Reynold's Numbers. Assuming laminar flow (Re < 500), we calculate values of Manning's n and compare them to traditional values. Our results indicate

Nomenclature

Navier-Stokes Variables:

$u_{i/j}$ = Velocity vector (m/s)

P = Pressure (Pa)

μ = Dynamic viscosity (Pa·s)

g_i = Gravitational acceleration vector (m/s^2)

t = Time (s)

$x_{i/j}$ = Spatial coordinate (m)

ρ = Fluid density (kg/m^3)

ν = Kinematic viscosity (m^2/s)

Manning's Equation Variables:

Q = Volumetric Flowrate (m^3/s)

A = Cross-sectional area of flow (m^2)

n = Manning's roughness coefficient

R_H = Hydraulic radius (m)

P_w = Wetted perimeter (m)

S = Channel slope (m/m)

V = Average velocity (m/s)

Shared Variables:

b = Channel bottom width (m)

h = Flow depth (m)

Re = Reynolds number (dimensionless)

Introduction

Measuring and predicting flow in open channels is a fundamental aspect of both hydraulic engineering and engineering hydrology. Several methods exist for estimating flow characteristics, the most import of which being volumetric flow rate (Q).

The most commonly used method for estimating flow in open channels is Manning's equation:

$$Q = \frac{1}{n} A R_H^{2/3} S^{1/2} \tag{1}$$

Where hydraulic radius R_H is defined as the ratio of the cross-sectional area of flow to the wetted perimeter:

$$R_H = \frac{A}{P_w} \tag{2}$$

Manning's equation is applied to all open channels, from culverts, streams, and rivers to large canals. It's popularity stems from its simplicity and empirical basis, allowing engineers to quickly estimate flow rates with limited data. However, Manning's equations has several limitations, including its empirical nature, assumptions of uniform flow, and sensitivity to the roughness coefficient n. These limitations can lead to inaccuracies in complex flow conditions, such as varying channel geometries, unsteady flows, and turbulent regimes.

For more complex flow conditions, the Navier-Stokes equations provide a comprehensive framework for modeling fluid dynamics. The Navier-Stokes equations describe the motion of viscous fluid substances and are derived from the principles of conservation of mass, momentum, and energy. They can capture a wide range of flow phenomena, including turbulence, boundary layer effects, and non-uniform flow profiles.

In this paper, we will not solve the full Navier-Stokes equations, but rather simplify them to matching conditions necessary to apply Manning's equation, except for turbulence. Our solutions to the Navier-Stokes equations will yield an analytical expression for volumetric flowrate under laminar flow conditions in rectangular channels. We will then compare the analytically derived flowrates to the empirically derived flowrates from Manning's equation to evaluate its accuracy under laminar flow conditions.

We will determine the flow regime using the Reynolds number (Re), defined as:

$$Re = \frac{V R_H}{\nu} \tag{3}$$

where V is the characteristic velocity (m/s), R_H is the hydraulic radius (m), and ν is the kinematic viscosity (m^2/s).

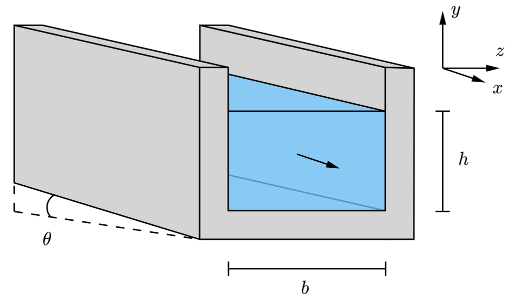

Figure 1: Rectangular channel and coordinate system.

We will consider laminar flow conditions to be such that $Re < 500$.

Methodology

Governing Equations

We start our derivation with the Navier-Stokes equations in three dimensions:

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \rho g_i \tag{4}$$

where ρ is the fluid density (kg/m^3), u_i is the velocity vector (m/s), P is the pressure (Pa), μ is the dynamic viscosity (Pa·s), and g_i is the gravitational acceleration vector (m/s^2).

Assumptions

The following assumptions were made to simplify the Navier-Stokes to match the conditions necessary to apply Manning's equation:

- The fluid is water $\rightarrow \rho = C, \mu = C$.
- Flow is steady $\rightarrow \frac{\partial}{\partial t} = 0$.
- Flow is uniform $\rightarrow \frac{\partial P}{\partial x} = 0$.
- Channel is infinite in the flow direction $\rightarrow \frac{\partial}{\partial x} = 0$.
- Flow is irrotational $\longrightarrow \mathbf{u} \cdot \nabla \mathbf{u} = 0$.

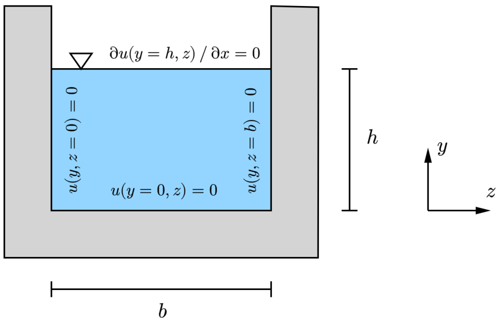

Figure 2: Boundary conditions for open channel flow.

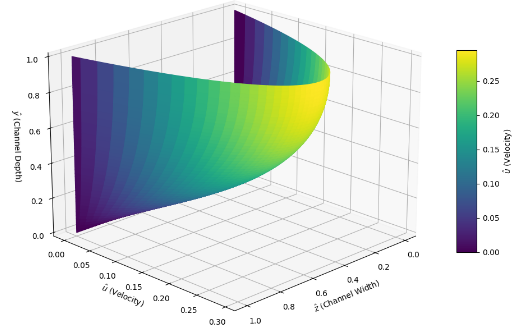

Figure 3: Non-dimensionalized velocity profile.

Applying our assumptions to the x-momentum equation, we simplify it to:

$$-\rho g_x = \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{5}$$

where g_x is the component of gravitational acceleration in the x-direction (m/s^2), and u is the velocity component in the x-direction (m/s).

With boundary conditions:

- No slip at the bottom and sides: $u = 0$ at $y = 0$, $z = 0$, and $z = b$.
- Symmetry at the free surface: $\frac{\partial u}{\partial y} = 0$ at $y = h$.

Velocity and Flowrate Calculations

Next we non-dimensionalize the equations using the following variables:

$$\hat{y} = \frac{y}{h}, \quad \hat{z} = \frac{z}{b}, \quad \hat{u} = \frac{u\nu}{h^2 g \sin\theta} \tag{6}$$

where θ is the angle of the channel slope. Substituting these into the simplified Navier-Stokes equation, we obtain the non-dimensional form:

$$-1 = \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \alpha^2 \frac{\partial^2 \hat{u}}{\partial \hat{z}^2} \tag{7}$$

where $\alpha = \frac{h}{b}$ is the aspect ratio of the channel.

We solve this equation using separation of variables and apply the boundary conditions to find the velocity profile

$$\hat{u}(\hat{y}, \hat{z}) = \hat{y} - \frac{\hat{y}^2}{2} + \sum_{n=1}^{\infty} A_n \sin(\lambda_n \hat{y}) \cosh\left[\frac{\lambda_n}{\alpha} \left(\hat{z} - \frac{1}{2} \right) \right] \tag{8}$$

where $A_n = \frac{-2}{\lambda_n^3 \cosh\left(\frac{\lambda_n}{2\alpha}\right)}$.

As shown in Figure 3, the flow distribution meets the boundary conditions defined previously.

The volumetric flow rate Q is calculated by integrating the velocity profile over the cross-sectional area:

$$Q = \int_0^b \int_0^h u(y, z)\, dy\, dz \tag{9}$$

We are left with a final expression for Q in terms of channel dimensions and fluid properties:

$$Q = \frac{h^3 b g \sin\theta}{\nu} \left[\frac{1}{3} - \frac{4h}{b} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^5} \tanh\left(\frac{\lambda_n b}{2h} \right) \right] \tag{10}$$

Manning's equation for volumetric flow rate in a rectangular channel is given by:

$$Q = \frac{1}{n} A R_H^{2/3} S^{1/2} \tag{11}$$

Substituting the expressions for A and R_H for a rectangular channel and rearranging to solve for n, we have:

$$n = \frac{1}{Q}(bh)\left(\frac{bh}{b+2h}\right)^{2/3} S^{1/2} \tag{12}$$

Reynold's Number Selection

To evaluate the performance of Manning's equation under laminar flow conditions, we selected a range of Reynolds numbers (Re) from 250 (laminar) to 12,500 (fully turbulent).

Re is defined as:

$$Re = \frac{VR_H}{\nu} \tag{13}$$

where V is the average velocity (m/s).

For each selected Re, we calculated the flowrate for a combination of channel widths (b = 0.01 to 5 m) and slopes (S = 0.01 to 0.1 m/m). From the Navier-Stokes derived flowrate, we calculated Manning's n using the rearranged Manning's equation. If Manning's equation were accurate, the calculated n values would fall within typical ranges for smooth rectangular channels (0.009 to 0.012).

Results and Discussion

For the range of Reynolds numbers we tested, we found that there were no values that yielded equivalent flowrates between the Navier-Stokes and Manning's equations when using typical values of Manning's n for smooth rectangular channels. Figure 4 shows the comparison of flowrates across all tested Reynolds numbers. The figure shows that there are regions where the two flowrates reach a perfect match. Table 1 summarizes the best matches found for each Reynolds number, along with the corresponding aspect ratio

Table 1: Best Match per Reynolds Number

Re	Optimal α	Error (%)
250	34.1	0.0544
500	24.7	0.0167
2,000	23.2	0.593
12,500	24.1	0.0509

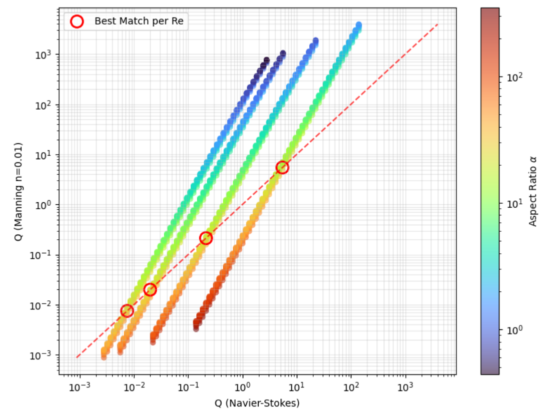

Figure 4: Comparison of Navier-Stokes and Manning's flowrates across Reynolds numbers.

Exploring further, we found that while exact matches were not achievable across the tested Reynolds numbers, there were aspect ratios where the percent error between the two flowrates was minimized. Figure 5 shows the error between the two flowrates across aspect ratios for all tested Reynolds numbers. The area highlighted in green indicates where the some comparisons have an error of less than 5%.

Our analysis revealed that for aspect ratios greater than about 45, the Navier-Stokes flowrate was greater than the Manning's flowrate. The opposite was true for aspect ratios less than about 17. Those values represent the bounds where Manning's equation and the Navier-Stokes equations yield similar flowrates. Table 2 summarizes the global aspect ratio range where the error between the two flowrates is less than 5%.

Table 2: Global α Range with $< 5.0\%$ Error

Statistic	Value
Min α	17.0
Max α	44.8
Mean α	26.7

The alpha values minimizing percent error suggest that agreement occurs when the depth of flow is much greater than the width of the channel. Considering the funda-

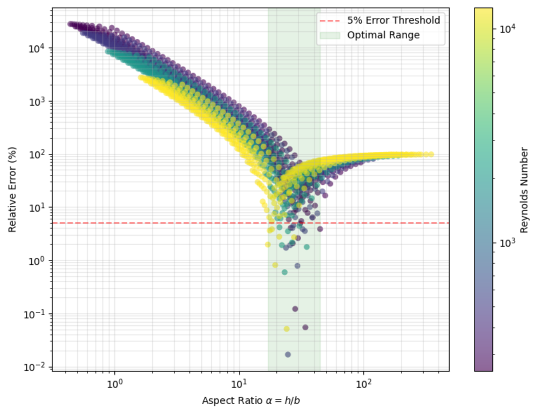

Figure 5: Error reduction of Navier-Stokes and Manning's Discharge.

mental physics governing the Navier-Stokes equations and the empirical nature of Manning's equation, we can see the cause for this agreement. Our Navier-Stokes account for the viscous forces acting across the entire cross-section of the flow, while Manning's equation simplifies these effects into a single roughness coefficient n. When the flow depth is significantly greater than the channel width, the influence of roughness on the channel walls is felt throughout the flow area. With roughness affecting the entire flow, the friction forces from Manning's equation behave more similarly to the viscous forces from the Navier-Stokes equations, leading to closer agreement in flowrates.

Conclusions

Our study compared the performance of Manning's equation with the Navier-Stokes equations in rectangular channels under laminar and turbulent flow conditions. We derived an analytical expression for volumetric flowrate using the Navier-Stokes equations and compared it to the flowrate calculated using Manning's equation. Our results indicated that exact matches between the two flowrates were not achievable across a range of Reynolds numbers when using typical values of Manning's n for smooth channels. However, we identified a range of aspect ratios (17 to 45) where the error between the two flowrates could be as low as 5%. This agreement occurred when the flow depth was significantly greater than the channel width, suggesting that under these conditions, the roughness assumptions of Manning's equation align more closely with the fundamental physics captured by the Navier-Stokes equations. These findings highlight the limitations of Manning's equation in laminar flow conditions and those of applying the simplified Navier-Stokes equations to turbulent flow. Future research could explore the performance of Manning's equation in more complex channel geometries and unsteady flow conditions, as well as investigate the effects of varying roughness coefficients on flow predictions.

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Appendix

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