Journal of Applied Engineering Mathematics

Volume 12, December 2025

Comparison of Manning and Navier-Stokes in Rectangular Channels

Kenneth T. Quintana

Civil and Construction Engineering Department
Brigham Young University
Provo, Utah 84602
ktquint@byu.edu

Abstract

Manning's equation is widely used for estimating flow in open channels due to its simplicity and empirical basis. However, its accuracy can be limited in complex flow conditions. This study compares the performance of Manning's equation with the Navier-Stokes equations in rectangular channels under varying Reynold's Numbers. Assuming laminar flow (Re < 500), we calculate values of Manning's n and compare them to traditional values. Our results indicate that for laminar flow, Manning's equation provides reasonable estimates for uniform flow. Under turbulent and transitional flows, the analytical solution to the laminar Navier-Stokes equations is no longer valid, so we made no comparison. The findings highlight the importance of selecting appropriate modeling techniques based on flow conditions for accurate hydraulic predictions.

Nomenclature

Q = Volumetric Flowrate (m^3/s)
A = Cross-sectional area of flow (m^2)
n = Manning's roughness coefficient
R_H = Hydraulic radius (m)
P_w = Wetted perimeter (m)
S = Channel slope (m/m)
V = Velocity (m/s)
b = Channel bottom width (m)
h = Flow depth (m)
ν = Kinematic viscosity (m^2/s)
g = Gravitational acceleration (m/s^2)
ρ = Fluid density (kg/m^3)
Re = Reynolds number (dimensionless)

Introduction

Manning's equation is an empirical formula used to estimate the flow of water in open channels. It is expressed as:

$$Q = \frac{1}{n} A R_H^{2/3} S^{1/2} \tag{1}$$

where Q is the discharge (m^3/s), A is the cross-sectional area of flow (m^2), R_H is the hydraulic radius (m), S is the channel slope (m/m), and n is Manning's roughness coefficient, which accounts for the channel's surface roughness. The hydraulic radius R_H is defined as the ratio of the cross-sectional area of flow to the wetted perimeter:

$$R_H = \frac{A}{P_w} \tag{2}$$

For a rectangular channel, the cross-sectional area A and wetted perimeter P_w can be expressed as:

$$A = b \cdot h \tag{3}$$

$$P_w = b + 2h \tag{4}$$

where b is the channel bottom width (m) and h is the flow depth (m).

The Navier-Stokes equations describe the motion of fluid substances and are fundamental to fluid dynamics. For incompressible flow, the Navier-Stokes equations can be written as:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g} \tag{5}$$

where ρ is the fluid density (kg/m^3), $\mathbf{u}$ is the velocity vector (m/s), p is the pressure (Pa), μ is the dynamic viscosity (Pa·s), and $\mathbf{g}$ is the gravitational acceleration vector (m/s^2). The Reynolds number (Re) is a dimensionless quantity used to predict flow patterns in different fluid flow situations. It is defined as:

$$Re = \frac{VR_H}{\nu} \tag{6}$$

where V is the characteristic velocity (m/s), R_H is the hydraulic radius (m), and ν is the kinematic viscosity (m^2/s). For a rectangular channel, the hydraulic diameter can be expressed as:

$$R_H = \frac{A}{P_w} \tag{7}$$

In this study, we compare the performance of Manning's equation with the Navier-Stokes equations in rectangular channels under varying Reynolds numbers. We focus on laminar flow conditions (Re < 500) to evaluate the accuracy of Manning's equation in predicting flow characteristics.

Methodology

Assumptions

The following assumptions were made to simplify the Navier-Stokes to match the conditions necessary to apply Manning's equation:

- The fluid is water $\rightarrow \rho = C, \mu = C$.
- Flow is steady $\rightarrow \frac{\partial}{\partial t} = 0$.
- Flow is uniform $\rightarrow \frac{\partial P}{\partial x} = 0$.
- Channel is infinite in the flow direction $\rightarrow \frac{\partial}{\partial x} = 0$.
- Flow is irrotational $\longrightarrow \mathbf{u}\cdot\nabla\mathbf{u} = 0$.

Applying our assumptions to the Navier-Stokes equations in the x-direction, we simplify it to:

$$-\rho g_x = \mu[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}] \tag{8}$$

where g_x is the component of gravitational acceleration in the x-direction (m/s^2), and u is the velocity component in the x-direction (m/s). With boundary conditions:

- No slip at the bottom and sides: $u = 0$ at $y = 0$, $z = 0$, and $z = b$.
- Symmetry at the free surface: $\frac{\partial u}{\partial y} = 0$ at $y = h$.

Flow Rate Calculations

Next we non-dimensionalize the equations using the following variables:

$$\hat{y} = \frac{y}{h}, \quad \hat{z} = \frac{z}{b}, \quad \hat{u} = \frac{u\nu}{h^2 g \sin\theta} \tag{9}$$

where θ is the angle of the channel slope. Substituting these into the simplified Navier-Stokes equation, we obtain the non-dimensional form:

$$-1 = \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \alpha^2 \frac{\partial^2 \hat{u}}{\partial \hat{z}^2} \tag{10}$$

where $\alpha = \frac{h}{b}$ is the aspect ratio of the channel.

We solve this equation using separation of variables and apply the boundary conditions to find the velocity profile

$$\hat{u}(\hat{y},\hat{z}) = \hat{y} - \frac{\hat{y}^2}{2} + \sum_{n=1}^{\infty} A_n \sin(\lambda_n \hat{y})\cosh(\frac{\lambda_n}{\alpha}(\hat{z} - \frac{1}{2})) \tag{11}$$

where $A_n = \frac{-2}{\lambda_n^3 \cosh(\frac{\lambda_n}{2\alpha})}$.

As shown in Figure 1, the flow distribution meets the boundary conditions defined previously.

The volumetric flow rate Q is calculated by integrating the velocity profile over the cross-sectional area:

$$Q = \int_0^b \int_0^h u(y,z)\, dy\, dz \tag{12}$$

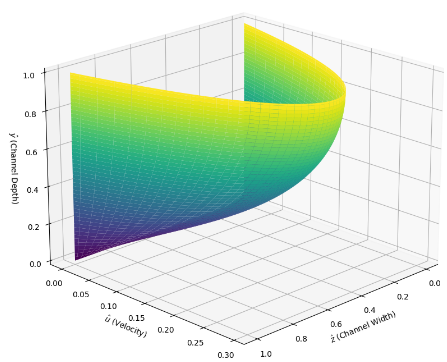

Figure 1: Rectangular channel geometry showing dimensions and coordinate system.

We are left with a final expression for Q in terms of channel dimensions and fluid properties:

$$Q = (\frac{h^3 bg\sin\theta}{\nu})\left(\frac{1}{3} - \frac{4h}{b}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^5}\tanh(\frac{\lambda_n b}{2h})\right) \quad (13)$$

Manning's equation for volumetric flow rate in a rectangular channel is given by:

$$Q = \frac{1}{n} A R_H^{2/3} S^{1/2} \quad (14)$$

Substituting the expressions for A and R_H for a rectangular channel, we have:

$$Q = \frac{1}{n}(bh)(\frac{bh}{b+2h})^{2/3} S^{1/2} \quad (15)$$

Solving for Manning's n

Rearranging for Manning's n, we get:

$$n = \frac{(bh)(\frac{bh}{b+2h})^{2/3} S^{1/2}}{Q} \quad (16)$$

Reynold's Number Selection

To evaluate the performance of Manning's equation under laminar flow conditions, we select a range of Reynolds numbers (Re) below 500. The Reynolds number is calculated using the formula:

$$Re = \frac{V R_H}{\nu} \quad (17)$$

where V is the average velocity (m/s), R_H is the hydraulic radius (m), and ν is the kinematic viscosity (m^2/s).

Simplifying the Reynold's number for a rectangular channel, we have:

$$Re = \frac{Q}{P_w \nu} \quad (18)$$

Results and Discussion

Conclusions

Put conclusions here.

Acknowledgements

The authors would like to acknowledge Dr. Vladimir P. Soloviev for his guidance and support in teaching the ME 505 course on Applied Engineering Mathematics. His patience and expertise were invaluable in guiding our understanding the fundamental concepts of differential equations and the solution methods we employed in this paper. The authors would also like to thank Dr. Julie Crockett for her assistance in understanding the fundamental principles of physics used to derive the Navier-Stokes equations. Her teachings inspired the initial research question that led to this study.

References

[1] Theodore L. Bergman and Adrienne S. Lavine. *Fundamentals of Heat and Mass Transfer*. 8th ed. 2019.

Appendix

Put appendix here.

