Journal of Applied Engineering Mathematics

Volume 12, December 2025

Comparison of Manning and Navier-Stokes in Rectangular Channels

Kenneth T. Quintana

Civil and Construction Engineering Department
Brigham Young University
Provo, Utah 84602
ktquint@byu.edu

Abstract

Manning's equation is widely used for estimating flow in open channels due to its simplicity and empirical basis. However, its accuracy can be limited in complex flow conditions. This study compares the performance of Manning's equation with the Navier-Stokes equations in rectangular channels under varying Reynold's Numbers. Assuming laminar flow (Re < 500), we calculate values of Manning's n and compare them to traditional values. Our results indicate that for laminar flow, Manning's equation provides reasonable estimates for uniform flow. Under turbulent and transitional flows, the analytical solution to the laminar Navier-Stokes equations is no longer valid, so we made no comparison. The findings highlight the importance of selecting appropriate modeling techniques based on flow conditions for accurate hydraulic predictions.

Nomenclature

Navier-Stokes Variables:

$u_{i/j}$ = Velocity vector (m/s)

P = Pressure (Pa)

μ = Dynamic viscosity (Pa·s)

g_i = Gravitational acceleration vector (m/s^2)

t = Time (s)

$x_{i/j}$ = Spatial coordinate (m)

ρ = Fluid density (kg/m^3)

ν = Kinematic viscosity (m^2/s)

Manning's Equation Variables:

Q = Volumetric Flowrate (m^3/s)

A = Cross-sectional area of flow (m^2)

n = Manning's roughness coefficient

R_H = Hydraulic radius (m)

P_w = Wetted perimeter (m)

S = Channel slope (m/m)

V = Average velocity (m/s)

Shared Variables:

b = Channel bottom width (m)

h = Flow depth (m)

Re = Reynolds number (dimensionless)

Introduction

Measuring and predicting flow in open channels is a fundamental aspect of both hydraulic engineering and engineering hydrology. Several methods exist for estimating flow characteristics, the most import of which being volumetric flow rate (Q).

The most commonly used method for estimating flow in open channels is Manning's equation:

$$Q = \frac{1}{n} A R_H^{2/3} S^{1/2} \tag{1}$$

Where hydraulic radius R_H is defined as the ratio of the cross-sectional area of flow to the wetted perimeter:

$$R_H = \frac{A}{P_w} \tag{2}$$

Manning's equation is applied to all open channels, from culverts, streams, and rivers to large canals. It's popularity stems from its simplicity and empirical basis, allowing engineers to quickly estimate flow rates with limited data. However, Manning's equations has several limitations, including its empirical nature, assumptions of uniform flow, and sensitivity to the roughness coefficient n. These limitations can lead to inaccuracies in complex flow conditions, such as varying channel geometries, unsteady flows, and turbulent regimes.

For more complex flow conditions, the Navier-Stokes equations provide a comprehensive framework for modeling fluid dynamics. The Navier-Stokes equations describe the motion of viscous fluid substances and are derived from the principles of conservation of mass, momentum, and energy. They can capture a wide range of flow phenomena, including turbulence, boundary layer effects, and non-uniform flow profiles.

In this paper, we will not solve the full Navier-Stokes equations, but rather simplify them to matching conditions necessary to apply Manning's equation, except for turbulence. Our solutions to the Navier-Stokes equations will yield an analytical expression for volumetric flowrate under laminar flow conditions in rectangular channels. We will then compare the analytically derived flowrates to the empirically derived flowrates from Manning's equation to evaluate its accuracy under laminar flow conditions.

We will determine the flow regime using the Reynolds number (Re), defined as:

$$Re = \frac{V R_H}{\nu} \tag{3}$$

where V is the characteristic velocity (m/s), R_H is the hydraulic radius (m), and ν is the kinematic viscosity (m^2/s). For a rectangular channel, the hydraulic diameter can be expressed as:

We will consider laminar flow conditions to be such that $Re < 500$.

Methodology

Governing Equations

We start our derivation with the Navier-Stokes equations in three dimensions:

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \rho g_i \tag{4}$$

where ρ is the fluid density (kg/m^3), u_i is the velocity vector (m/s), P is the pressure (Pa), μ is the dynamic viscosity (Pa·s), and g_i is the gravitational acceleration vector (m/s^2).

Assumptions

The following assumptions were made to simplify the Navier-Stokes to match the conditions necessary to apply Manning's equation:

- The fluid is water $\rightarrow \rho = C, \mu = C$.
- Flow is steady $\rightarrow \frac{\partial}{\partial t} = 0$.
- Flow is uniform $\rightarrow \frac{\partial P}{\partial x} = 0$.
- Channel is infinite in the flow direction $\rightarrow \frac{\partial}{\partial x} = 0$.
- Flow is irrotational $\longrightarrow \mathbf{u} \cdot \nabla \mathbf{u} = 0$.

Applying our assumptions to the x-momentum equation, we simplify it to:

$$-\rho g_x = \mu [\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}] \tag{5}$$

where g_x is the component of gravitational acceleration in the x-direction (m/s^2), and u is the velocity component in the x-direction (m/s).

With boundary conditions:

- No slip at the bottom and sides: $u = 0$ at $y = 0$, $z = 0$, and $z = b$.
- Symmetry at the free surface: $\frac{\partial u}{\partial y} = 0$ at $y = h$.

Velocity and Flowrate Calculations

Next we non-dimensionalize the equations using the following variables:

$$\hat{y} = \frac{y}{h}, \quad \hat{z} = \frac{z}{b}, \quad \hat{u} = \frac{u\nu}{h^2 g \sin\theta} \tag{6}$$

where θ is the angle of the channel slope. Substituting these into the simplified Navier-Stokes equation, we obtain the non-dimensional form:

$$-1 = \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \alpha^2 \frac{\partial^2 \hat{u}}{\partial \hat{z}^2} \tag{7}$$

where $\alpha = \frac{h}{b}$ is the aspect ratio of the channel.

We solve this equation using separation of variables and apply the boundary conditions to find the velocity profile

$$\hat{u}(\hat{y}, \hat{z}) = \hat{y} - \frac{\hat{y}^2}{2} + \sum_{n=1}^{\infty} A_n \sin(\lambda_n \hat{y}) \cosh(\frac{\lambda_n}{\alpha}(\hat{z} - \frac{1}{2})) \tag{8}$$

where $A_n = \frac{-2}{\lambda_n^3 \cosh(\frac{\lambda_n}{2\alpha})}$.

As shown in Figure 1, the flow distribution meets the boundary conditions defined previously.

The volumetric flow rate Q is calculated by integrating the velocity profile over the cross-sectional area:

$$Q = \int_0^b \int_0^h u(y, z)\, dy\, dz \tag{9}$$

We are left with a final expression for Q in terms of channel dimensions and fluid properties:

$$Q = (\frac{h^3 b g \sin\theta}{\nu}) \left(\frac{1}{3} - \frac{4h}{b} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^5} \tanh(\frac{\lambda_n b}{2h}) \right) \tag{10}$$

Manning's equation for volumetric flow rate in a rectangular channel is given by:

$$Q = \frac{1}{n} A R_H^{2/3} S^{1/2} \tag{11}$$

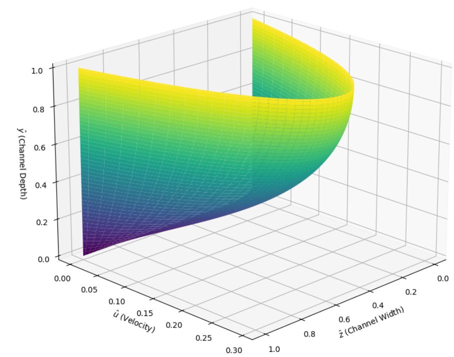

Figure 1: Rectangular channel geometry showing dimensions and coordinate system.

Substituting the expressions for A and R_H for a rectangular channel, we have:

$$Q = \frac{1}{n}(bh)(\frac{bh}{b + 2h})^{2/3} S^{1/2} \tag{12}$$

Solving for Manning's n

Rearranging for Manning's n, we get:

$$n = \frac{(bh)(\frac{bh}{b+2h})^{2/3} S^{1/2}}{Q} \tag{13}$$

Reynold's Number Selection

To evaluate the performance of Manning's equation under laminar flow conditions, we select a range of Reynolds numbers (Re) below 500. The Reynolds number is calculated using the formula:

$$Re = \frac{V R_H}{\nu} \tag{14}$$

where V is the average velocity (m/s), R_H is the hydraulic radius (m), and ν is the kinematic viscosity (m^2/s).

Simplifying the Reynold's number for a rectangular channel, we have:

$$Re = \frac{Q}{P_w \nu} \tag{15}$$

Results and Discussion

Conclusions

Put conclusions here.

Acknowledgements

The authors would like to acknowledge Dr. Vladimir P. Soloviev for his guidance and support in teaching the ME 505 course on Applied Engineering Mathematics. His patience and expertise were invaluable in guiding our understanding the fundamental concepts of differential equations and the solution methods we employed in this paper. The authors would also like to thank Dr. Julie Crockett for her assistance in understanding the fundamental principles of physics used to derive the Navier-Stokes equations. Her teachings inspired the initial research question that led to this study.

Appendix

Put appendix here.

