

Maths Finance  
Personal notes

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## Preface

This is my personal notes for maths finance. For organization :

- The first chapter represents basic concepts and knowledges in finance.
- The second chapter represents fundamentally what is probability and its related concepts. We find here the convergence of random variables, uniform integrability, Radon-Nikodym theorem and the three major theorems : monotone convergence theorem, dominated convergence theorem and Fatou's lemma. In section useful tools, important equalities and inequalities are found.
- The third chapter is about statistic which is for usual distributions, hypothesis tests, estimators such as MLE, OLS, GMM, IV and their properties.
- The fourth chapter is for stochastic processes and their properties (stationary, ergodicity). We find here popular processes such as Gaussian, Poisson, Markov, Brownian motion, Ornstein Uhlenbeck, martingale, cadlag. For stochastic integrals, one describes the indispensable Ito lemma and infinitesimal generator. For application in finance, Giranov's theorem is used to switch between physical measure and risk neutral measure.
- The fifth chapter is for optimization or optimal control. In deterministic case, we find Euler-Lagrange, Pontryagin's maximum principle, Hamilton-Jacobi-Bellman (HJB) and the well-known Ramsey–Cass–Koopmans model. In stochastic case, we find Kalman filter, Linear quadratic gaussian control. We also find here the finite difference method (FDM) and finite element method (FEM), which are powerful for solving partial differential equation (PDE) in HJB. Finally, we find the Merton's portfolio problem and the optimal stopping.
- The sixth chapter is about quantitative : classical decomposition ETS (error, trend, seasonality), ARIMA model, pairs trading (cointegration & correlation), modern portfolio theory, asset price simulation by GBM, option pricing (Black-Scholes model and binomial options pricing model), Greeks, delta hedging, Kelly criterion, utility, risk aversion and technical analysis by well-known indicators.

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<https://www.investopedia.com/>  
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[QuantJobInterviewQuestionsAndAnswers](https://QuantJobInterviewQuestionsAndAnswers)  
[wikipedia](https://fr.wikipedia.org/)  
...

# Chapter 1

## General finance

### 1 Abbreviations and financial words

- NAV : Net Asset Value.
- Holding (company) : Company which acquires financial holdings in other companies and which directs or controls their activity.
- Security : a certificate attesting credit, the ownership of stocks or bonds, or the right to ownership connected with tradable derivatives.
- A money manager : a person or financial firm that manages the securities portfolio of individual or institutional investors.

### 2 ETF

ETF (exchange traded fund) is a type of investment fund and **it is also** an exchange-traded product, i.e., it is traded on stock exchanges as other securities. ETFs are constituted of basket of funds, bonds, commodities,... Typically, people investing in ETFs are more interested in a diversified portfolio and want to keep their investment in an ETF for a longer period of time.

One of most common ETFs is SPDR (SPY) tracks S&P 500, which means SPY's evolution is exactly similar to S&P 500. Price of ETF is available in all trading time.

Their holdings (owner of ETF shares) are completely public and transparent. ETFs are bought and sold from other owners throughout the day on stock exchanges.

### 3 Mutual funds

A mutual fund is an investment vehicle that made up of a pool of funds collected from many investors. Mutual fund is operated by money manager,

who invests the **fund's capital** and attempt to produce gains and income for the **fund's investors**. Mutual fund's portfolio is structured and maintained to match the investment objectives stated in its prospectus (but not always reached).

Mutual funds disclose their holdings typically once a quarter (but can vary by fund). Shares of mutual fund are bought and sold from the issuer based on their price at day's end.

## 4 Hedge funds

Hedge funds are alternative investments using pooled funds that employ numerous different strategies to earn active return, called alpha, for their investors.

Hedge funds may be aggressively managed and they make use of derivatives and leverage in both domestic and international markets with the goal of generating much higher returns (compared to a metric, which can be specified market benchmark such as S&P 500,...)

It is important to note that hedge funds are generally only accessible to **accredited investors** as they require less SEC (Securities and Exchange Commission) regulations than other funds.

An accredited investor is an individual or a business entity that is allowed to trade securities that may not be registered with financial authorities. They are entitled to this privileged access by satisfying at least one requirement regarding their income, net worth, asset size, governance status, or professional experience. In the U.S., the term accredited investor is used by the Securities and Exchange Commission (SEC) under Regulation D to refer to investors who are financially sophisticated and have a reduced need for the protection provided by regulatory disclosure filings. Accredited investors include high-net-worth individuals (HNWIs), banks, insurance companies, brokers, and trusts.

Hedge funds do not need to disclose their investment activities, their holdings,

...

### 4.1 What hedge funds do ?

- Long/Short Equity: This strategy involves buying stocks that the hedge fund believes will increase in value (long positions) and selling stocks that are expected to decrease in value (short positions).
- Global Macro: This strategy involves making bets on global macroeconomic trends, such as interest rates, inflation, and currency fluctuations.
- Event-Driven: This strategy involves investing in companies that are going through major events, such as mergers, acquisitions, bankruptcies, or other corporate actions.
- Quantitative: This strategy involves using computer algorithms and mathematical models to analyze financial data and identify investment opportunities.

## 5 Funds comparison

ETFs are similar in many ways to mutual funds, except that ETFs are bought and sold from other owners throughout the day on stock exchanges whereas mutual funds are bought and sold between the issuer of fund and individuals based on their price at day's end.

<https://www.vietish.com/mutual-fund-vs-index-fund-vs-etf/#:~:text=ETF%20c%C5%A9ng%20nh%C6%B0%20mutual%20fund,v%E1%BB%9Bi%20nhau%20%E1%BB%9F%20secondary%20market>

### 5.1 Fee

- ETF : Expense ratio 0.01 - 1 %
- Mutual funds : Expense ratio 0.5 - 3 %
- Hedge funds : 2% of fund and 20% of profit (2 and 20 rule). Successful hedge funds can offer higher ratio, as they have no real rules to follows.

### 5.2 Liquidity

- ETF : as a stock
- Mutual funds : end of the day.
- Hedge funds : On agreement.

## 6 Margin and Leverage

Margin refers to the **amount of money** that a trader must put up to open and maintain a trading position.

$$\text{margin} = \frac{\text{your\_money}}{\text{total\_value}}$$

For example, if you have \$1000 and a broker requires a margin of 50% then you can borrow \$1000, which means \$2000 in total to buy stock. Your \$1000 is also your deposited money.

### 6.1 Margin call

Suppose that your stock decreases from \$2000 to \$1500 and the safe margin is 40%. So your actual margin is  $\frac{1000-(2000-1500)}{1500} = \frac{500}{1500} = 33.3\%$

Now, you need a margin call and you have two options to set margin at 40%:

- Put more amount  $x$  deposited money:

$$\frac{500+x}{1500+x} = 40\%$$

- Sell your stock of amount  $z$ , and this money is put directly in the deposit.

$$\frac{500 + z}{1500} = 40\%$$

## 6.2 Reverse margin call

As its name, it means that if your stock increases then you can have back your money.

## 6.3 Leverage

Leverage refers to the **amount of capital** that a trader can control with a smaller amount of money. Leverage notation :  $1 : X$ . For example, with leverage is  $1 : 100$  which means you have \$1 but you can control \$100.

## 6.4 Leverage in balance sheet

Many types:

- **Accounting leverage** is total assets divided by the total assets minus total liabilities.
- **Notional leverage** is total notional amount of assets plus total notional amount of liabilities divided by equity.
- Economic leverage is volatility of equity divided by volatility of an unlevered investment in the same assets.

## 7 Cash flow

Cash flow refers to the movement of cash in and out of a business or individual's finances. It is the **difference between the amount of cash received and spent** by a business or individual during a given time period. Positive cash flow means that more cash is coming in than going out, while negative cash flow means that more cash is going out than coming in.

Cash flow is an essential component of financial management because it provides insights into the financial health of a business or individual. It allows them to monitor their cash inflows and outflows, plan for future expenses and investments, and make informed decisions about borrowing, saving, and spending. Cash flow can also help a business or individual determine their ability to pay bills, meet financial obligations, and generate profits.

## 8 Drawdown

The drawdown is the measure of the decline from a historical peak in some variable (typically the cumulative profit or total open equity of a financial trading strategy) :

$$D(T) = \max_{t \in (0, T)} [X(t) - X(T), 0]$$

Average drawdown:

$$\frac{1}{T} \int_0^T D(t) dt$$

## 9 Difference between common stock and preferred stock

- Ownership and Voting Rights:
  - Common stockholders are the true owners of the company and have voting rights. They can participate in corporate decision-making by voting on important matters, such as the election of the board of directors and major company policy changes.
  - Preferred stockholders also have ownership in the company, but they typically **do not have voting rights** or **have limited voting rights**. Their ability to influence corporate decisions is generally restricted compared to common stockholders.
- Dividends
  - Common stockholders may receive dividends, but these dividends are typically not fixed and can vary based on the company's profitability and management decisions. Dividends for common stock are paid after all obligations to preferred stockholders have been met
  - Preferred stockholders have a higher claim on the company's assets and earnings compared to common stockholders. They usually receive fixed dividends at regular intervals, and these dividends are paid before any dividends are distributed to common stockholders. The fixed dividend rate may be expressed as a percentage of the stock's par value or as a fixed dollar amount
- Liquidation Preference
  - Common stockholders have the lowest priority in terms of receiving the remaining assets after all debts and obligations are settled. They are the last to be paid and may not receive anything if there are insufficient assets

- Preferred stockholders typically have a higher priority compared to common stockholders in receiving the company's assets during liquidation. They have a preferential claim and are entitled to receive their investment back, along with any accrued dividends, before any distribution is made to common stockholders
- Capital Appreciation
  - If the company performs well and its share price increases, common stockholders can benefit from the price appreciation and potentially sell their shares at a higher price.
  - Preferred stockholders generally do not benefit directly from capital appreciation. Their returns primarily come from the fixed dividends they receive.

## 10 Derivative

3 fields which work with derivatives:

- Hedging : Hedging is an investment strategy used to reduce the risk. It involves taking an opposite position to an existing to minimize the impact of adverse price movements.
- Speculating : perform high-risk investments with the intention of earning substantial profits. Speculators often engage in short-term trading.
- Arbitrage : refers to the practice of simultaneous buying and selling of assets in different markets to take advantage of price discrepancies and make **risk-free** profits. Arbitrageurs exploit temporary pricing inefficiencies that occur due to various factors like market imperfections, supply and demand imbalances, or differences in pricing across different markets.

Types of derivatives :

- Forwards : a simple contract between 2 parties at a moment in the future.
- Futures (highly standardized forward contract), having clearinghouse, counterparties do not know each other.
- Swaps : Two parties exchange cash flows based on an underlying asset at a future point of time. Underlying asset can be : Interest rate, stock price, bond price, commodity. Most of swap contract used are interest rate, e.g. exchange between a fixed interest rate and a variable one.
- Options : An option gives the holder the right, but not the obligation, to buy or sell the underlying asset at a specified price, called the strike price, within a specified time period. There are two main types of options: call options and put options.

The strike price, also known as the exercise price, is a key term in options trading. It refers to the price at which the underlying asset, such as a stock, can be bought or sold if the option is exercised.

For example, if an investor holds a *call* option with a strike price of \$50 and the stock price is currently trading at \$60, the investor has the right to buy the stock at \$50, even though it is trading at a higher price in the market. If the investor chooses to exercise this right, they would buy the stock at \$50 and sell it immediately for a profit of \$10 per share. **But the investor, have to pay for the option at the beginning** and it is calculated by Black–Scholes model, in 6.5.1.

Options has a expiration date !!!

In the US, options typically have an American-style exercise, which means that the option holder can exercise the option at any time before the expiration date. In contrast, options in the EU have a European-style exercise, which means that the option holder can only exercise the option on the expiration date.

Option contracts traded on futures exchanges are mainly American-style, whereas those traded over-the-counter are mainly European.

### 10.1 Who emits or creates options ?

**Traders** write an option by creating a new option contract that sells someone the right to buy or sell a stock at a specific price (strike price) on a specific date (expiration date). In other words, the writer of the option can be forced to buy or sell a stock at the strike price.

### 10.2 Covered option

A covered option is a financial transaction in which the holder of securities sells (or “writes”) a call or put option, against stock that they own or are shorting. The seller of a covered option receives compensation, or “premium”, for this transaction, which can limit losses; however, the act of selling a covered option also limits their profit potential to the upside.

### 10.3 Put option

Previously, we have an example for a call option, now we go into an example for *put* option.

Here are the details of the put option:

- Stock: XYZ Corp.
- Strike Price: \$50
- Expiration Date: 30 days from now
- Premium: \$3 per share

- Contract Size: 100 shares (**typically one option contract represents 100 shares**)

*Scenario 1: XYZ Corp.'s stock price decreases.* Suppose the price of XYZ Corp.'s stock falls to \$40 per share before the option's expiration date. In this case, you can exercise your put option, which can be understood as you buy (from market) 100 shares of XYZ Corp. at \$40 per share, and then you use the right of your put option, resell (to the trader that you buy this put option) at the strike price of \$50 per share. This allows you to profit from the difference between the strike price and the lower market price, which is \$10 per share. Considering the contract size of 100 shares, your profit would be \$10 per share multiplied by 100, minus the premium paid for the put option.

*Scenario 2: XYZ Corp.'s stock price increases or remains above the strike price.* If the stock price of XYZ Corp. remains above the strike price of \$50, or even increases, it would not be advantageous for you to exercise the put option. In this case, the put option may expire worthless, and your loss would be limited to the premium paid for the put option, which is \$3 per share multiplied by 100 shares.

## 10.4 Pay-off

The value of an option at maturity depends on the relationship between the delivery price ( $K$ ) and the underlying price ( $S_T$ ) at that time.

- For a long call this payoff is:

$$f_T = S_T - K$$

- For a long put :

$$f_T = K - S_T$$

## 10.5 Valuation of options

### 10.5.1 Intrinsic value

The intrinsic value is the difference between the underlying spot price and the strike price, in favor of the option holder.

For a call option, the option is in-the-money if the underlying spot price is higher than the strike price; then the intrinsic value is the underlying price minus the strike price. For a put option, the option is in-the-money if the strike price is higher than the underlying spot price; then the intrinsic value is the strike price minus the underlying spot price. Otherwise the intrinsic value is zero.

For example, when a call (bullish/long) option has the strike price \$18,000 and the underlying is priced at \$18,050 then there is a \$50 advantage even if the option were to expire today. This \$50 is the intrinsic value of the option.

In summary, intrinsic value :

- =  $\max(\text{current stock price} - \text{strike price}, 0)$  (for call option)
- =  $\max(\text{strike price} - \text{current stock price}, 0)$  (for put option)

### 10.5.2 Extrinsic value or time value

In finance, the extrinsic value or time value (TV) of an option is the premium that a rational investor would pay over its current exercise value (intrinsic value), based on the probability it will increase or decrease in favor of the buyer before expiry. Hence the longer the expiration time, the higher time value. Time value can be obtained by :

$$\text{Time value} = \text{Option price} - \text{Intrinsic value}$$

Time value *decays* to zero at expiration, with a general rule that it will lose  $\frac{1}{3}$  of its value during the first half of its life and  $\frac{2}{3}$  in the second half. Therefore, if an option moves closer to expiry, the intrinsic value is much more important than extrinsic value.

## 10.6 Volatility and implied volatility

Implied volatility and volatility are two related but distinct concepts used in options trading and financial analysis.

- Historical Volatility: Historical volatility is calculated based on **past price** data and represents the actual observed volatility over a given period. It provides insight into how much the price of the underlying asset has moved in the past.
- Implied Volatility : Implied volatility, on the other hand, is derived from the prices of options (about somewhat in the future). Therefore, it represents the forecast of future volatility rather than being based on historical data.

### 10.6.1 How to get implied volatility ?

In financial mathematics, the implied volatility of an option contract is that value of the volatility of the underlying instrument which, when input in an option pricing model (such as Black-Scholes), will return a theoretical value equal to the current market price of said option. more precisely, let  $C$  is option price,  $f$  is a function for calculating option price (e.g. Black-Scholes),  $\sigma$  volatility :

$$C = f(\sigma, \dots)$$

Then given the market price of option  $\bar{C}$ , the implied volatility  $\bar{\sigma}$  is calculated by :

$$\bar{\sigma} = f_{\sigma}^{-1}(\bar{C}, \dots)$$

## 10.7 Put–call parity

This defines a relationship between the price of a *European* call option and *European* put option, both with the *identical strike price and expiry*. If a portfolio consists of a long call option and a short put option, it is equivalent to (and hence has the same value as) a single forward contract at this strike price and expiry.

This is because if the price at expiry is above the strike price, the call will be exercised, while if it is below, the put will be exercised, and thus in either case one unit of the asset will be purchased for the strike price, exactly as in a forward contract.

Small reminder for forward contract : it is a non-standardized contract between two parties to buy or sell an asset at a specified future time (delivery and payment) while the price is agreed at the time that contract signed.

Note that, the validity of this relationship requires that certain assumptions be satisfied, e.g, transaction costs, financing costs must be negligible.

### 10.7.1 Statement

Put–call parity can be stated in the following equation:

$$C - P = D \cdot (F - K)$$

where  $C$  is the (current) value of a call,  $P$  is the (current) value of a put,  $D$  is the discount factor (to come back at present value),  $F$  is the forward price of the underlying asset, and  $K$  is the strike price.

Equivalently,

$$C - P = S - D \cdot K$$

where  $S$  is the spot price, which is obtained by discounting the forward price  $F$ .

## 10.8 Style of option

### 10.8.1 European option

The option is only exercised at the maturity, which means a fixed number of data after acquiring option.

### 10.8.2 American option

The option can be exercised at any moment before or at the maturity.

### 10.8.3 Bermudan option

A Bermudan option is an option where the buyer has the right to exercise at a **set of instants** (always discretely spaced). For example, if the maturity is 6 months, the buyer has the right to exercise at the end of each month.

This is intermediate between a European option and an American option. The name is jocular: Bermuda, a British overseas territory, is somewhat American and somewhat European.

#### 10.8.4 Asian option

An Asian option is a type of option where the payoff depends on the average price of the underlying asset over a predetermined period of time. This is different from a US option and a European option, where the payoff depends solely on the price of the underlying asset at a specific point in time. For example, a call option payoff :

$$\max(A(0, T) - K, 0)$$

where  $A(0, T)$  can be:

- In the continuous case, arithmetic average :

$$A(0, T) = \frac{1}{T} \int_0^T S(t) dt$$

- In the discrete case :

$$A(0, T) = \frac{1}{n} \sum_{i=1}^n S(i)$$

- Geometric average :

$$A(0, T) = \exp \left( \frac{1}{T} \int_0^T \log[S(t)] dt \right)$$

## 11 Spread trade

In finance, a spread trade (also known as relative value trade) is the simultaneous purchase of one security and sale of a related security. Each bought or sold security called leg. Spread trades are usually executed with *options* or *futures contracts* as the legs, but other securities are sometimes used.

These trades are executed to yield an overall net position whose value, called the spread, depends on the difference between the prices of the legs.

Common spreads are priced and traded as a unit on futures exchanges rather than as individual legs, thus ensuring simultaneous execution and eliminating the execution risk of one leg executing but the other failing.

Spread trades are executed to attempt to profit from the widening or narrowing of the spread, rather than from movement in the prices of the legs directly. Spreads are either “bought” or “sold” depending on whether the trade will profit from the widening or narrowing of the spread.

The volatility of the spread is typically *much lower* than the volatility of the individual legs, since a change in the market fundamentals of a commodity will tend to affect both legs similarly.

## 11.1 Example : Bull Call spread

Sometime called Call Debit Spread. Why named Bull Call spread ?

- Bull since the acquirer expect a bull trend
- Call since call option
- Spread, since it means strategy that use combination of two options

Theese two call options (one long and one short) are with different strike prices, but with the same expiration date, to bet on a rise of the underlying. The strike price in long position must be lower than one in short.

### 11.1.1 Example

- Long a call with strike \$53 expiring on day X for cost \$4.35
- Short a call with \$61 strike with the same expiration for \$1.39

So the total cost is  $4.35 - 1.39 = 2.96$ , which is equivalent to \$296. Note that, this is the **loss maximal** for the spread acquirer. This happens when the underlying price is lower then strike in the long (\$53)

**Gain maximal** : When the underlying price is surpass \$61. Then the gain is :  $6100 - 5300 - 296 = \$504$ . Hence, the Bull Call Spread is a strategy whose gains and losses maximal are known in advance.

So what happed if the underlying price is between the two strike values ? Let's call  $x$  the underlying price. Then the gain is :  $(x - 53 - 2.96) \times 100$ . We have a break even at underlying price of \$55.96, means that if  $x$  is greater than \$55.96 then we have gain and if  $x$  is lower than \$55.96 then we have loss.

**When used** ?By using a bull call spread, we are essentially betting that the underlying asset's price will increase moderately but not surpass the higher strike price.

We have in following several types of spread trades:

## 11.2 Intracommodity spreads

Also known as a calendar spread, horizontal spread or time spread, is an options trading strategy that involves simultaneously buying and selling options contracts with the same characteristics (underlying, strike price, free-rate...), the only difference is the expiration date. More precisely, we sell the nearby expiration and buy the further-out expiration (option). **Calendar spread has an expiration date and it is typically one of nearby (near-term) option**. If we perform inversly (buy the nearby expiration and sell the further-out expiration), it is known as a *reverse calendar spread*.

The key idea behind a calendar spread is to take advantage of time value which decays. As time passes, the value of the shorter-term option will decline at a **faster rate of decay**, potentially resulting in a profit. More precisely, let

$C_n$  and  $C_f$  be the price of near-term option and further-out option and suppose that the spread value has time value which decay exponentially :

$$(C_n - InVa)e^{r_n t} - (C_f - InVa)e^{r_f t}$$

*InVa* means intrinsic value, which is the same for both near-term and further-out option. We have the rate of decay  $r_n$  is greater than  $r_f$  since **the rate of decay is proportional to implied volatility and the closer expiration date the higher implied volatility**. The latter is because near-term options are more sensitive to near-term market events and uncertainties, leading to higher implied volatility.

### 11.2.1 Trading strategies

Pick expiration months : When selecting the expiration date of the long option, it is wise to go at least two to three months out. This will depend largely on your forecast. However, when selecting the short strike, it is a good practice to always sell the shortest dated option available.

### 11.2.2 Notes

- Risk (loss) is limited to the net debit, means the difference between the premium of two options.
- Benefits from an increase in implied volatility.

## 12 Strategies in trading

### 12.1 Pair trading

A pairs trade or pair trading is a market neutral trading strategy. The idea behind pair trading is to identify two assets that have a **strong historical correlation**, meaning that their prices tend to move together over time, but may diverge at several instants due to market fluctuations or other factors.

When one asset's price rises and the other's falls, traders can buy the falling asset and short sell the rising one, anticipating that the spread between them will eventually narrow and they will revert to their historical correlation. Traders may use statistical tools to help identify when the spread between two assets deviates from its average, and then execute the trades to take advantage of the expected convergence.

As with any trading strategy, pair trading involves risks and requires careful analysis and risk management.

#### 12.1.1 Intercommodity spreads

Intercommodity spreads are formed from two distinct but related commodities, reflecting the economic relationship between them.

Common examples are:

- The *crack* (cracking) spread between crude oil and one of its byproducts, reflecting the premium inherent in refining oil (from) into gasoline, gas oil, or heating oil.
- The *spark* spread between natural gas and electricity, for gas-fired power stations.
- The *crush* spread between soybeans and one of its byproducts, reflecting the premium inherent in processing soybeans into soy meal and soy oil

## 12.2 Mean reversion

Mean reversion is a financial term for the assumption that an asset's price will tend to converge to the average price over time.

When the current market price is less than the **average past price**, the security is considered attractive for purchase, with the expectation that the price will rise.

When the current market price is above the **average past price**, the market price is expected to fall. In other words, deviations from the average price are expected to revert to the average.

## 13 Long and short position

Key differences between long and short positions include:

- Directional Expectation: Long positions anticipate price appreciation (increasing), while short positions anticipate price depreciation (decreasing).
- Ownership: Long positions involve owning the asset, whereas short positions involve borrowing to sell (need to buy them back and return them to the lender)
- Profit/Loss Calculation: In a long position, the profit is realized if the asset's value increases, while the loss is incurred if the value decreases. In a short position, the profit is realized if the asset's value decreases, while the loss is incurred if the value increases.
- Risk Profile: Long positions typically have limited risk, as the maximum loss is generally limited to the amount invested. Short positions, especially when selling borrowed assets, have potentially unlimited risk if the asset's value rises significantly.

## 14 Straddle, Strangle, Butterfly

All these three concept is about strategies that we can play with different configuration of options : buy or sell, number of puts, number of calls, strike price,...

## 14.1 Straddle

In finance, a straddle strategy involves two transactions in options on the same underlying, with opposite positions (call and put). As a result, it involves the purchase or sale of particular option that allow the holder to profit based on how much the price of the underlying security moves, regardless of the direction of price movement.

### 14.1.1 Long straddle

We buy a call and put with same strike price and expiration date. There are two considered cases:

- If the stock price is close to the strike price at expiration of the options, the straddle leads to a loss.
- However, if there is a sufficiently large move in either direction, a significant profit will result (after compensating the premiums).

A straddle is appropriate when an investor is expecting a large move in a stock price but does not know in which direction the move will be. This strategy is a limited risk, since the most a purchaser may lose at maximum is the cost of both options. At the same time, there is theoretically unlimited profit potential (wow). See figure 1.1.

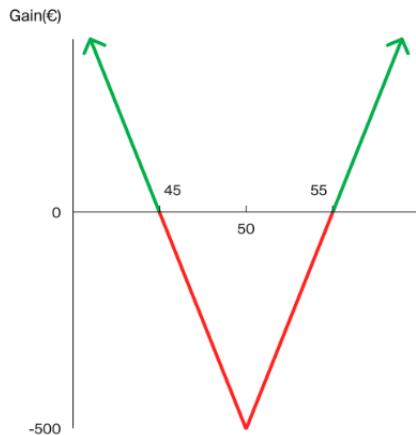


Figure 1.1: Long straddle, payoff in term of price evolution. Here suppose that we long straddle with a contract size of 100, with  $V = 5\text{€}$  is the cost of one straddle (equivalent to buy an option and sell an option). The strike price  $K = 50$ . We only have positive payoff if the spot price is less than  $K - V = 45$  or more than  $K + V = 55$ .

**When to use ?** For example, company XYZ is set to release its quarterly financial results in two weeks. A trader believes that the release of these results

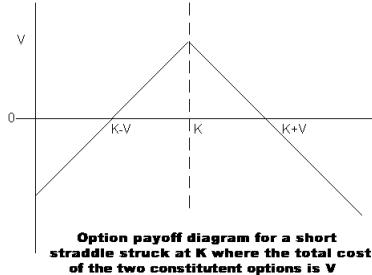


Figure 1.2: Short straddle, payoff in term of price evolution.

will cause a large movement in the price of XYZ's stock, but does not know whether the price will go up or down. He can enter into a long straddle, where he gets a profit no matter which way the price of XYZ stock moves, if the price changes enough either way.

If the options are American, the stock is sufficiently volatile, and option duration is long, the trader could profit from both options. This would require the stock to move both below the put option's strike price and above the call option's strike price at different times before the option expiration date.

#### 14.1.2 Short straddle

A short straddle is a non-directional options trading strategy that involves simultaneously selling a put and a call of the same underlying security, strike price and expiration date. The main objective here is to create a product that gives profit if at expiration, the the underlying security's price is closed to strike price, see figure 1.2. However, the risk is theoretically unlimited as large moves of the underlying security's price either up or down will cause losses proportional to the magnitude of the price move.

#### 14.1.3 Straps and strips

Strip and Strap are practically identical strategies to Straddle, but with one nuance : for Strip, we use 2 puts and 1 call while for Strap, we use 1 put and 2 calls. As with Straddle, we can be buy or sell Strap or Strip. The main objective is to give more favor on one side between upward and downward underlying's price, see figure 1.3. We buy an additional put if we think the decline will be stronger (Strip), or we will buy an additional call if we think the rise will be more violent (Strap).

### 14.2 Strangle

A strangle consists of one call of strike  $K_1$  and one put of strike  $K_2$ , with the same expiry and underlying. See figure 1.4 for long strangle.

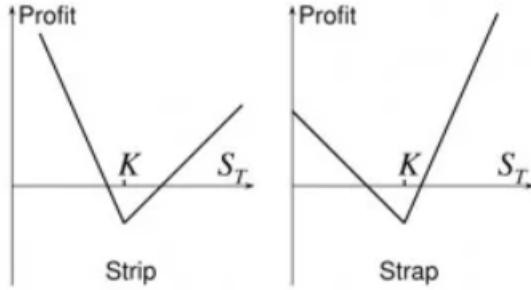


Figure 1.3: Long strap and long strip, payoff in term of price evolution.

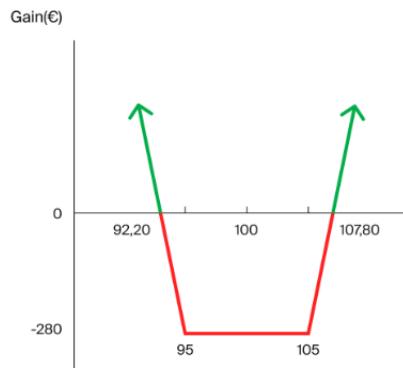


Figure 1.4: Long strangle. Here we buy a call contract with a strike price of 105 which costs 150€ and a put contract with a strike price of 95 which costs 130€. The total cost now amounts to 280€. Since a contract is of size 100, we have cost  $P = 2.80\text{€}$  for a strangle. We only have positive payoff if the spot price is less than  $K_1 - P = 92.20$  or more than  $K_2 + P = 107.80$ .

The objective of strangle is to freely set two bounds  $K_1$  and  $K_2$  instead of depending on  $V$  (cost of one straddle) as in long straddle (see figure 1.1).

For short strangle, we just sell the couple call and put instead of buying and consequently, the payoff representation is just a flip up down of one in figure 1.4.

### 14.3 Butterfly

A long butterfly options strategy consists of the following options:

- Long 1 call with a strike price of  $K_1 = X - a$
- Long 1 call with a strike price of  $K_2 = X + a$
- Short 2 calls with a strike price of  $X$

where  $X$  is the current market price of underlying (spot price) and  $a > 0$ . Let's see the figure 1.5. It is interesting to use a butterfly if we expect the underlying security to expire within a certain fluctuation range, parameterized by spot price  $X$  and  $a$ . As illustrated in figure 1.5, long butterfly has limited risk and it is non-directional. A butterfly is a spread (1.11) since we buy and sell simultaneously a related security.

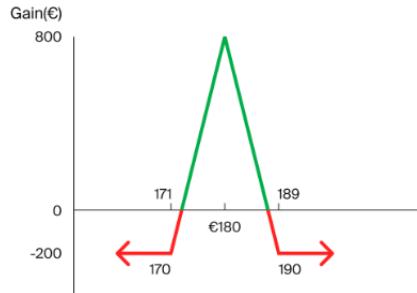


Figure 1.5: Long butterfly, payoff in term of price evolution. Here, we buy a contract of call options at strike  $K_1 = 170$  that cost 1000€, a contract of call options at strike  $K_2 = 190$  that cost 200€ and sell two contracts of call options at spot price  $X = 180$  for 1000€. Therefore, the equivalent cost for a contract of long butterfly is 200€. Since a contract is of size 100, we have cost  $P = 2\text{€}$  for a butterfly. We only have positive payoff if the spot price is less than  $K_1 + P = 172$  or more than  $K_2 - P = 178$ . The best payoff we can get is 800€ when  $X = 180$  at expiration.

We may note that the payoff representation of long butterfly is more or less similar to short straddle 1.14.1.2. However, the potential risk of a long butterfly is limited which is contrary to short straddle and it has a much lower profit potential than short straddle. Also, the cost for a butterfly is usually higher than for a short straddle.

For short butterfly, we just replace call by put and consequently, the payoff representation is just a flip up down of one in figure 1.5.

## 15 Contango et Backwardation

### 15.1 Contango

Contango is a situation where the futures price (or forward price) of a commodity is higher than the expected spot price of the contract at maturity (figure 1.6).

Here we note some informations about price :

- Spot price : The spot price refers to the current market price of a particular asset or commodity that is available for immediate delivery or settlement,

as opposed to a future price that reflects the expected price at a future date.

- Expected spot price at maturity : In general, there is no mathematical formula for expected spot price. It is more of an economic concept rather than a mathematical part. Keywords : Fundamental analysis, technical analysis, Monitor news and events, market sentiment, forecasting models, ...
- Future price : **In forward contracts or futures contracts**, where contract terms are agreed now but delivery and payment will occur at a future date.
- As definition of contango, **Future price and expected spot price are two thing differents**. The main difference is the two time instants : one at contract terms are agree and other at the payment. In the expected spot price, these two instants are the same in the future and in future price, one instant is now and other is in the future.

In a weaker definition, it's a situation where the cost of buying a commodity in the future is higher than the cost of buying it today.

Contango occurs when there is a higher demand for commodities in the future than there is currently, or when there is an expectation of a future shortage of the commodity. It can also occur due to the costs of storage and financing, as holding a commodity for delivery in the future involves additional costs such as storage fees and the cost of financing the purchase.

When a market is in contango, it creates an opportunity for traders to buy the commodity in the spot market and sell it in the futures market, with the expectation of making a profit when the futures contract expires. However, the potential profit from this strategy is limited by the costs of storage and financing, as well as any changes in the market conditions that may affect the future price of the commodity.

## 15.2 Backwardation

Backwardation is the opposition of contango. In other words, the spot price of a commodity is higher than the price of the futures contract for that same commodity.

This can occur when there is a shortage (thieu, pénurie) of the commodity, or when there is high demand for the commodity in the near term. In these situations, buyers are willing to pay more for the commodity right now, and sellers are willing to accept a lower price for delivery at a future date when they expect the supply to increase.

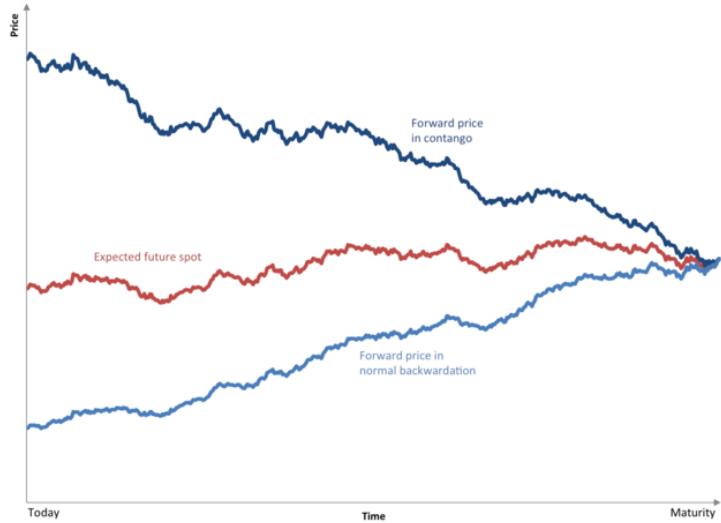


Figure 1.6: Contango and backwardation.

## 16 Making continuous price for future contracts

The expiration dates of the future contracts vary based on the underlying asset and range from monthly to quarterly, therefore there are no price of this asset type after these date. When expiration approaches, traders will roll their positions over to the next contract or let them expire. A basic approach to construct a continuous series would be to always use the front month contract's price and when the current front month contract expires, switch to the new front month contract.

However, there is one caveat, the price of the contracts when you rollover (reinvest) may not be the same, and in general, won't be the same. To properly evaluate the trading strategies with historical futures contract data, it is necessary to combine these contracts into a continuous price series. There are several methods:

### 16.1 Shift adjustment

Forward adjusting would shift the next contract to eliminate the gap by **subtracting** the adjustment factor from the next contract's price series. Backward adjusting would shift the previous contract to eliminate the gap by **adding** the adjustment factor to the previous contract's price series. Figure 1.7 below shows an example of these adjustments for an actual rollover. In this case, the size of this gap can be also called the adjustment factor.

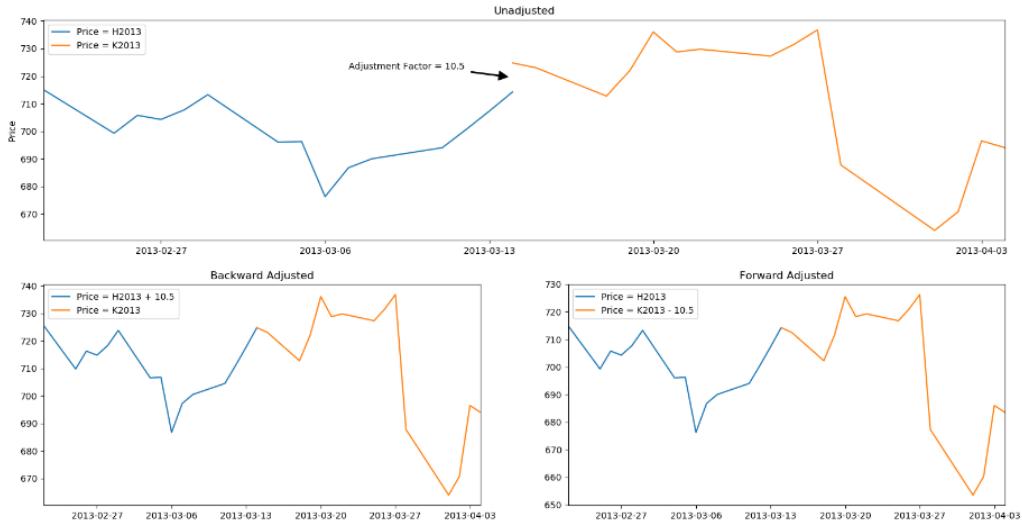


Figure 1.7: Shift adjustment, with backward and forward technical.

## 16.2 Proportional adjustment

In this case, the gap is removed by using **multiplication**. Then the adjustment factor is :

$$\frac{\text{PriceAfter}}{\text{PriceActual}}$$

This is calculated at the maturity of the price contract actual.

## 16.3 Perpetual method

Smooths the transitions between contracts by taking a weighted average of the contracts' prices during the transition period. The weight can be used based on :

- time left to expiration, with 5 rolling days:  $p_1 = 0.8x_1 + 0.2x_2, p_2 = 0.6x_1 + 0.4x_2, \dots, p_5 = 0x_1 + 1x_2 = x_2$
- volume  $w_1 = \frac{v_1}{v_1+v_2}, w_2 = \frac{v_2}{v_1+v_2}$
- open interest
- other properties of the contracts

## 16.4 Notes

- P&L : Preserverd in but not in Proportional adjustment

- Return rate : Preserverd in Proportional adjustment but not in Shift adjustment

The further we are from a contract's expiration; the more price speculation (dau co) is embedded into the price. The front or nearest month contract refers to the contract which has the soonest expiration date and thus has the least amount of speculation. Generally, front month contracts have the most trading activity.

## 17 Open interest

Open interest OI (Hop dong mo) is applicable for options contracts, also future contract in the derivatives market. It equals to the *number of (futures) contracts or positions (options) that are currently active*.

Examples:

Sharon, Cynthia and Kurt are trading on the same future contract. If Sharon buys a contract to enter a long-term trade, then OI will increase by 1. Cynthia also buys 6 contracts, then the OI will total 7. Now, if Kurt decides Going to short (ban khong) the market and sell three contracts, the open interest increased to 10 again. But if Sharon sell to Cynthia a contract, then the OI is still 10. (but the volume now is 11).

Rising open interest is often indicative of a strong developing price trend (bull market). This means that traders are supporting the trend by stepping up into the market. And vice versa, it will be a signal of a trend decline when traders lose confidence in the trend they leave the market (bear market).

Accordingly, Open Interest is an indicator that many traders in the futures market use in conjunction with other analysis to aid in trading decisions.

## 18 Price-earnings ratio

Called simply *P/E* :

$$\frac{\text{Shareprice}}{\text{Earning per share}}$$

## 19 Market to book

Market-to-book (M/B or P/B) is a financial ratio is calculated by dividing the market capitalization of a company by its book value. The book value is the value of a company's assets, minus its liabilities, as recorded on its balance sheet. M/B ratio is used to assess a company's financial performance and it provides insight into the growth potential and future prospects of a company.

## **20 Year-to-date**

Year-to-date (YTD) return is a financial metric that measures the performance of an investment or portfolio from the beginning of the current calendar year up to the present date. It provides a snapshot of the investment's performance over the specified period.

Here's an example to illustrate the calculation:

Let's say you invested \$10,000 in a stock on January 1st (which is a tradable day) of the current year, and the current value of your investment is \$12,000. The YTD return would be:

$$\text{YTD Return} = (\$12,000 - \$10,000) / \$10,000 * 100 = 20\%$$

## **21 Value stocks**

Value stocks are stocks that are believed to be undervalued by the market, meaning that their price is lower than their intrinsic value. In other words, investors believe that the stock's current price does not reflect the company's actual worth, and that the stock is therefore a good investment opportunity. Value stocks are often found in industries that are currently out of favor or have low growth prospects. Examples of value stocks may include companies that have a strong balance sheet, generate steady cash flows, pay dividends, and have a low price-to-earnings (P/E) ratio.

## **22 Growth stocks**

Growth stocks, on the other hand, are stocks of companies that are expected to grow at a faster rate than the overall market or their peers. These companies may be in industries that are experiencing high growth, have innovative products or services, or have a competitive advantage. Growth stocks are often characterized by a high P/E ratio, as investors are willing to pay a premium for the potential future growth of the company. Examples of growth stocks may include technology companies, biotech companies, and companies in emerging markets.

## **23 Cost of goods sold**

The cost of goods sold (GOGS, gia von hang ban) is the sum of all direct cost associated with making a product. It appears on an income statement and typically includes money spent on raw materials and labour. It does not include costs associated with marketing, sales or distribution.

## 24 Operating expense

An operating expense, operating expenditure, operational expense, operational expenditure or opex (chi phi hoat dong) is an ongoing cost for running a product, business, or system.

Its counterpart, a capital expenditure (capex, chi phi von), is the cost of developing or providing non-consumable parts for the product or system.

For example, *the purchase of a photocopier involves capex*, on the other hand, *the annual paper, toner, power and maintenance costs represent opex*.

For larger systems like businesses, opex may also include the cost of workers and facility expenses such as rent and utilities.

## 25 Underlying instrument

In finance, the underlying instrument refers to the financial asset or security that serves as the basis for a derivative contract or other financial product. The value of the derivative product is derived from the underlying instrument, and its price movements are closely tied to the performance of the underlying asset.

*Examples of underlying instruments include stocks, bonds, currencies, commodities, and indices.* For example, a stock option is a derivative contract that gives the holder the right to buy or sell shares of a particular stock (the underlying instrument) at a predetermined price and time.

## 26 Income and outcome

Income refers to the **money** or other **financial gains** received by an individual or organization, typically from employment, business activities, investments, or other sources. Income can be earned in the form of wages, salaries, commissions, bonuses, rental income, investment income, or other forms of compensation.

Outcome, on the other hand, refers to the **results** or **consequences**, in a broad sense, of a particular action or decision.

Example : in business, we want launch a new product. The *ressources* is money for materials, the *action or activity* is the developpement... Then the *output* is the result (product). The *outcome* is the consequence of the output, e.g. the user of this product get better. Finally, the gain of selling this product is *income*.

### 26.1 Difference between output and outcome

To illustrate the difference between output and outcome, let's take an example of a non-profit organization providing food assistance to a community. The number of food packages distributed to the community would be an output, while the improvement in the nutritional status or food security of the recipients would be an outcome.

## 27 Arbitrage and Speculation

### 27.1 Arbitrage

It is the practice of taking advantage of a difference (discrepancy) in prices in two or more markets. It strikes (makes) a combination of matching deals to capitalise on this difference. Then the profit being the difference between the market prices at which the unit is traded.

In simple terms, it is the possibility of a risk-free profit after transaction costs. Arbitrageurs buy an asset in one market where it is priced lower and sell it in another market where it is priced higher, profiting from the price difference. Arbitrageurs do not take on risk because they are guaranteed to make a profit as long as the price difference exists and they can execute the trades fast enough to capture the opportunity.

When used by academics, an arbitrage is a transaction that involves :

- No negative cash flow at any probabilistic or temporal state.
- At least one state, a positive cash flow.

In academic use, an arbitrage is risk-free

#### 27.1.1 Examples

- Geographic arbitrage: Taking advantage of price discrepancies in different locations.
- Currency arbitrage
- Risk arbitrage: Taking advantage of price discrepancies caused by an impending event, such as a merger or acquisition.
- Statistical arbitrage: Using complex quantitative models to identify price discrepancies between related financial instruments and taking positions to profit from these discrepancies.
- Convergence arbitrage: Traders simultaneously buy the cheaper instrument and sell the more expensive one, expecting them to converge in price over time.
- Dividend arbitrage: Capitalizing on differences in dividends paid by different stocks. Traders can buy a stock just before the **ex-dividend date** to receive the dividend payment and then sell the stock afterward. However, the price of stock can decrease in selling.
- Energy arbitrage: Exploiting price differences in different energy markets, such as buying electricity during low-demand periods and selling it during high-demand periods.

### **27.1.2 Arbitrage-free**

Arbitrage-free or no arbitrage or arbitrage equilibrium refers to a condition in financial markets where there is no opportunity for riskless profit by exploiting price differences between similar financial instruments. An arbitrage equilibrium is a precondition for a general economic equilibrium.

### **27.1.3 Statistical arbitrage**

We have seen that arbitrage is to take advantage of a price difference between **two or more markets**. Statistical arbitrage, on the other hand, is a specific type of arbitrage that relies on quantitative analysis and statistical methods to identify mispricings in financial instruments. It seeks to profit from temporary price discrepancies between related financial instruments **within the same market**.

## **27.2 Speculation**

Speculation involves buying or selling assets in the hope of making a profit from changes in their prices. Speculators take on risk and are exposed to the possibility of losing money if their predictions do not come true. Speculators may use a variety of methods to try to predict future price movements, such as technical analysis, fundamental analysis, or market sentiment.

In summary, speculation involves taking risks in the hope of making a profit, while arbitrage involves exploiting price discrepancies in different markets to make a guaranteed profit with no risk.

## **28 Risk neutral**

Risk neutral is a term commonly used in finance and economics to describe a situation or perspective in which an individual or organization does not consider risk when making decisions. In a risk neutral environment, the value of an investment or financial asset is determined solely by its expected return, without taking into account any risks associated with that investment. This means that a risk neutral person would be indifferent between two investments with the same expected return, even if one is riskier than the other.

### **28.1 Difference between risk-free and Risk neutral**

Risk-free investments are those with no risk of loss, meaning that the return on the investment is guaranteed. Typically, these investments have a low rate of return, such as savings accounts or government bonds. Investors choose risk-free investments when they prioritize capital preservation over capital growth and want to minimize the risk of losing money.

On the other hand, Risk neutral investments do have some level of risk and investors are compensated for taking on risk. For example, stocks, mutual funds,

and corporate bonds are considered Risk neutral investments because they offer the potential for higher returns but also come with the risk of losing money.

The key difference between the two is that risk-free investments have no risk of loss, while Risk neutral investments have some level of risk associated with them.

## 28.2 P and Q measure

$\mathbb{P}$  measure and  $\mathbb{Q}$  measure are probability measures (2.1.7.3.2) used to model the behavior of financial instruments and markets.

$\mathbb{P}$  measure, also known as the **physical measure**, is a probability measure that describes the real-world probabilities of events. It evaluates risk based on actual probabilities and is used to assess real-world scenarios.

$\mathbb{Q}$  measure, also known as the **risk neutral measure** (1.28.3), is a probability measure that allows for a simpler pricing of derivatives and other financial instruments by removing the effects of risk aversion. The  $\mathbb{Q}$  measure is constructed by adjusting the probabilities under the  $\mathbb{P}$  measure so that the expected return of the derivative is equal to the risk-free rate. Under  $\mathbb{Q}$  measure, the *risk premiums* (1.28.2.1) is absent.

### 28.2.1 Risk premium

The risk premium is the extra return that an investor expects to receive for taking on additional risk compared to a “risk-free” investment such as a government bond. In finance, risk is typically defined as the variability of returns, and investors are generally risk-averse, meaning they prefer less risky investments.

The risk premium is calculated as the *difference between the expected return on a risky investment and the expected return on a risk-free investment with a similar maturity*. For example, if the expected return on a stock is 10%, and the risk-free rate is 3%, the risk premium would be 7%.

Investors demand a risk premium to compensate them for the potential losses that may arise from investing in a risky asset. The size of the risk premium will depend on the perceived risk of the investment, the investor’s risk tolerance, and the prevailing market conditions.

## 28.3 Risk neutral measure

Also called an equilibrium measure, or equivalent martingale measure. This means a **probability (set)** such that *each share price is exactly equal to the discounted expectation of the share price* under this measure or equivalently, an asset should be worth the present value of the expected future returns on that asset.

**Note that measure here means probability measure** 2.1.7.3.2, not a method to measure. The risk neutral and risk neutral measure are two different concepts, one is the nature of investor and one is probability measure.

If there is only present value and no future value, which means  $t = 0$  and there is no impact of discounted rate, the risk neutral measure is the **probability (set)** for which the price of any trade is equal to the expectation of the trade's winnings and losses.

Risk neutral measure is heavily used in the pricing of financial derivatives due to the fundamental theorem of asset pricing, which implies that in a complete market (negligible transaction costs and every asset in every possible state of the world has a price), a derivative's price is the discounted expected value of the future payoff under the unique risk neutral measure.

### 28.3.1 Example

- A player hands over some money, say  $X\$$  to play.
- The host tosses an unbiased coin.
- If it comes up heads ( $\omega = \text{head}$ ) then the player is given  $2\$$ .
- If it comes up tails ( $\omega = \text{tail}$ ) then nothing is given back.

The question is how much does it cost  $X$  ? Theoretically, under the physical measure  $\mathbb{P}$  (see 1.28.2) :

$$\begin{cases} \mathbb{P}(\omega = \text{head}) = 0.5 \\ \mathbb{P}(\omega = \text{tail}) = 0.5 \end{cases}$$

the value of  $X$  is  $1\$$ . But let's consider two following persons :

- Person A that has  $1.50\$$  in their pocket but is under pressure from a traffic warden to pay  $2\$$  for a parking ticket (and nothing less than  $2\$$  will do).
- Person B that has  $10\$$  in their pocket and doesn't really need anything more than that.

Indeed, person A is more willing to pay more than person B since he really need money for parking ticket. Person B might be a harder sell, but perhaps he would come around if we charged only  $X = 50$  cents.

Let's consider now only person A. If he accepts to pay  $X = 1.50\$$  in order to get an expected payoff of  $1\$$  under measure  $\mathbb{P}$ , this is not fair. To be fair, we need to adjust measure  $\mathbb{P}$  to have risk neutral measure  $\mathbb{Q}$ , such that :

$$\begin{cases} \mathbb{Q}(\omega = \text{head}) = 0.75 \\ \mathbb{Q}(\omega = \text{tail}) = 0.25 \end{cases}$$

Then the expected winning value would then be equal to the price paid (here  $t = 0$ , which means there is no impact of discounted rate):

$$0.75 \times 2\$ + 0.25 \times 0\$ = 1.50\$$$

In conclusion, the risk neutral measure  $\mathbb{Q}$  for  $X = 1.50\$$  is probability set  $(0.75, 0.25)$ , which means there is no risk premium neither for the banker (bookie) nor person A.

If  $X = 0.50\$$  for person B, then the risk neutral measure  $\mathbb{Q}$  is probability set  $(0.25, 0.75)$

## 28.4 Risk neutral pricing

Risk neutral pricing is a pricing method that assumes that investors are indifferent to risk when valuing financial assets. This means that the expected return on an asset is adjusted for the risk-free rate of return (see 1.28.3.1 for example), and not for the risk associated with the asset. This method is commonly used in derivatives pricing, where the value of the derivative is based on the expected future value of the underlying asset, discounted at the risk-free rate.

## 29 Discounted rate

The discounted rate is used to determine the present value of future cash flows or investments. It takes into account the time value of money, which means that a dollar received or paid in the future is worth less than a dollar received or paid today.

### 29.1 Example

Suppose you have an investment opportunity that promises to pay you \$1,000 per year for the next five years. However, you want to determine the present value of these cash flows today, considering a discount rate of 8%.

To calculate the present value for each year's cash flow, you can use the formula:

$$\text{Present Value} = \text{Future Value} / (1 + \text{Discount Rate})^n$$

where n is the number of years from the present.

Let's calculate the present value for each year's cash flow, in table 1.1.

Year	1	2	3	4	5
$(1 + \text{Discount Rate})^n$	$\frac{1}{(1+0.08)}$	$\frac{1}{(1+0.08)^2}$	$\frac{1}{(1+0.08)^3}$	$\frac{1}{(1+0.08)^4}$	$\frac{1}{(1+0.08)^5}$
Discounted value	925.93	857.34	793.83	735.03	680.58

Table 1.1: Discounted value per each year with payoff of \$1000.

Total Present Value =  $925.93 + 857.34 + 793.83 + 735.03 + 680.58 = 3,992.71$ . This means this opportunity costs 3,992.71 at the present.

## 29.2 Estimation discounted rate

Usually, a discount rate value is the same as the rate of return, hence it can be estimated by :

- By using the risk premium, then added by risk free rate.
- By using return rate estimated by CAPM in section 6.3.10.
- By using return rate estimated by DDM in section 6.3.13.
- By using required return rate  $R_e$  inferred from WACC in section 6.3.16, given  $WACC, R_d, E, D, t$ .

## 30 Moneyness

In finance, moneyness is the relative position of the current price (or future price) of an underlying asset (e.g., a stock) with respect to the strike price of a derivative. Moneyness is firstly a three-fold classification:

- If the derivative would have positive intrinsic value if it were to expire today, it is said to be **in the money**.
- If the derivative would be worthless if expiring with the underlying at its current price, it is said to be **out of the money**.
- And if the current underlying price and strike price are equal, the derivative is said to be **at the money**.

There are two slightly different definitions, according to whether one uses the current price (spot) or future price (forward), specified as “at the money spot” or “at the money forward”.

## 31 Order book and high frequency trading

### 31.1 Order book

An order book is a record of all buy and sell orders for a particular financial asset, such as stocks, bonds, or derivatives, that are currently open and waiting to be executed. It is essentially a list of all the current bids to buy and asks to sell a particular asset at various prices.

The order book provides traders and investors with important information about market activity, including the current supply and demand for a particular asset, as well as the prices that buyers and sellers are willing to accept. By looking at the order book, traders can see where the market is headed and make informed decisions about when to buy or sell an asset.

In addition to displaying current buy and sell orders, the order book may also include information about recent trades that have taken place. This can

provide valuable insights into the current state of the market and the behavior of other traders and investors.

Order books can be accessed by traders and investors through various platforms, including online trading platforms and trading terminals provided by financial institutions. They are a critical tool for anyone looking to trade in financial markets, as they provide valuable information about the supply and demand dynamics of the market.

### 31.2 High frequency trading

High frequency trading (HFT) is a type of automated trading that uses sophisticated algorithms and computer programs to execute large volumes of trades at extremely high speeds. HFT strategies typically involve analyzing market data in real-time and making split-second decisions to buy or sell securities.

The main goal of HFT is to profit from small price discrepancies that exist for only a fraction of a second. HFT firms use advanced technology and ultra-fast networks to ensure that they receive market data and execute trades before their competitors. This can result in HFT firms earning profits in a matter of microseconds.

While HFT can be highly profitable, it is also controversial. Some critics argue that HFT creates instability in the financial markets, and that it can lead to increased volatility and market crashes. Others argue that HFT provides liquidity to the market and helps to reduce bid-ask spreads, which can benefit all market participants.

Note that, HFT is only important if we trade a big volume of stock.

### 31.3 Bid-ask spread

The bid-ask spread is the difference between the highest price a buyer is willing to pay for a financial asset (the bid price) and the lowest price a seller is willing to accept for that same asset (the ask price). In other words, it is the difference between the highest price a buyer is willing to pay for an asset and the lowest price a seller is willing to accept.

For example, if the bid price for a stock is \$100 and the ask price is \$101, then the bid-ask spread is \$1. The bid-ask spread represents the transaction costs of trading a particular asset and is an important factor to consider when buying or selling financial instruments.

The size of the bid-ask spread can vary depending on a number of factors, including the liquidity of the market, the volume of trading, and the volatility of the asset. In general, assets with higher trading volumes and greater liquidity tend to have smaller bid-ask spreads, while assets with lower trading volumes and less liquidity tend to have larger bid-ask spreads.

Traders and investors must consider the bid-ask spread when buying or selling an asset, as it represents a cost that must be paid to complete the transac-

tion. The larger the spread, the more expensive it is to trade an asset, which can impact overall profitability.

## 32 Survivorship bias

This is about the survivorship bias applied in finance. Let's consider before and after dotcom bubble, the stocks in S&P 500 have changed a lot. Then algo made before this moment **may not be compatible** with things after this moment.

## 33 Efficient market hypothesis

The efficient-market hypothesis (EMH) is a hypothesis in financial economics that states that asset prices reflect **immediately** from all information, equivalently, asset prices are always correctly valued. A direct implication is that it is impossible to “beat the market”.

Hedge funds work with condition that this hypothesis is not always true. E.g, in dotcom period, stocks are overvalued.

## 34 Network effect

The network effect in economics refers to the phenomenon where the value of a good or service increases as the number of users or participants in its network grows. This effect arises when a product or service becomes more useful or valuable the more people use it, leading to a positive feedback loop that reinforces its dominance in the market.

The adoption of a product by an additional user can be broken into two effects:

- Total effect : An increase in the value to all other users.
- Marginal effect : Enhancement of other non-users' motivation for using the product

Network effects can be direct or indirect:

- *Direct network effects* arise when a given user's utility increases with the number of other users of the same product or technology, meaning that adoption of a product by different users is complementary. Direct network effects can be seen with social networking services, including Twitter, Facebook, Airbnb, Uber, and LinkedIn; telecommunications devices like the telephone. E.g., utility of an user Airbnb increases when more people use this app, which means (maybe) having more reviews about an accommodation, or having more chooses if users are property...

- *Indirect network effects* (or cross-group) arise when there are “at least two different customer groups that are (inter)dependent, and the utility of one group grows as the other group grow”. For example, the product is OS (Windows, Ubuntu), hence, hardware consumers may become more valuable with the growth of software consumers. Other example, Airbnb, one customer group is landlord and other one is tenant.

Network effects are commonly mistaken for economies of scale, which describe *decreasing average production costs* in relation to *the total volume of units produced*. Economies of scale are a common phenomenon in *traditional* industries such as manufacturing, whereas network effects are most prevalent in *new economy* industries, particularly information and communication technologies. Network effects is the counterpart of economies of scale, as they function by increasing a customer’s willingness to pay by the prevalence rather than by decreasing cost.

Upon reaching certain number of users (critical point), a bandwagon effect can result. As the network continues to become more valuable with each new adopter, more people are incentivised to adopt, resulting in a positive feedback loop. **Multiple equilibria** and **market monopoly** are two main states in markets that exhibit network effects. Consumer expectation is a key factor to determine which state will result.

### 34.1 Two-sided market

A two-sided market, also called a two-sided network, is an **intermediary economic platform** having **two distinct user groups** that provide each other with network benefits.

Two-sided networks can be found in many industries, sharing the space with traditional product and service offerings :

- Credit cards : Cardholders and Merchants
- Health maintenance organizations (patients and doctors)
- Operating systems (end-users and developers)
- Yellow pages (advertisers and consumers)
- Video-game consoles (gamers and game developers)
- Recruitment sites (job seekers and recruiters)
- Search engines (advertisers and users)

## 35 Maths in finance

**Mathematical finance**, also called quantitative finance. It focuses on *developing and implementing* methods for derivatives pricing on the one hand,

and portfolio, risk management (including hedging) on the other. The level of mathematics in this field is very rigorous.

**Financial engineering**, on the other hand, is the *application* or *practice* of mathematical models and quantitative methods to design and create financial instruments and strategies. The level of mathematics in this field is less rigorous than mathematical finance.

**Computational finance** is the mix between mathematical finance and numerical methods. It benefits the power of computer to solve a financial problem. This field encompasses a wide range of techniques, including

- Mathematical modeling
- Monte Carlo simulation
- Probability and statistics
- Optimization (operations research, deterministic, stochastic, ...)
- Machine learning (history matching then forecast)
- ...

This field require a high skill in both maths and numerical analysis.

### 35.1 Implemtation package on quant

- Quantopian
- Pyfolio : Providing performance and risk analysis of financial portfolio.
- Alphalens : Predive stock package (alpha)
- Numerai
- Quantiacs
- Zipline (Backtesting engine)

# Chapter 2

# Probability

## 1 Random variable

We discover first some concepts that is needed for the definition of random variable, notably the  $\sigma$ -algebra and the measure space.

### 1.1 $\sigma$ -algebra

In mathematical analysis and in probability theory, a  $\sigma$ -algebra (also  $\sigma$ -field) on a set  $X$ , noted  $\Sigma$ , is a collection of subsets of  $X$  that satisfies :

1.  $\emptyset \in \Sigma$ .
2.  $A \in \Sigma \Rightarrow A^C$  ( in  $X$ )  $\in \Sigma$ . (*closed under complementation*)
3.  $A_1, A_2, \dots, A_\infty \in \Sigma \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$ . (*closed under countable unions*)

From the second and the third property, we can infer the useful property :

$$A_1, A_2, \dots \in \Sigma \Rightarrow A_1^C, A_2^C, \dots \in \Sigma \Rightarrow \bigcup_i A_i^C \in \Sigma \Rightarrow \left( \bigcup_i A_i^C \right)^C \in \Sigma \text{ or } \bigcap_i A_i \in \Sigma$$

This is also called countable intersection.

If  $X = \{a, b, c, d\}$  then one possible  $\sigma$ -algebra on  $X$  is

$$\Sigma = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$$

$\sigma$ -algebra can be applied for subset or random variable:

- $\sigma(A)$ , where  $A$  is a non-empty collection of subset in  $\Omega$ , is the unique smallest  $\sigma$ -algebra that contains every elements of  $A$ .
- $\sigma(X)$ , where  $X$  is a random variable (we will see this concept after in sec 2.1.6.4).

### 1.1.1 Example

What is the  $\sigma(R, S)$  if  $\Omega = \{1, 2, 3, 4, 5\}$ ,  $R = \{1, 2, 3\}$  and  $S = \{2, 4\}$  ?  
We first find the  $\sigma(R), \sigma(S)$ :

- $\sigma(R) = \{\emptyset, \Omega, R, R^C\}$
- $\sigma(S) = \{\emptyset, \Omega, S, S^C\}$

Then  $\sigma(R, S)$  must contain:

- $R, R^C, S, S^C$
- $R \cap S = \{2\}$
- $R \cap S^C = \{1, 3\}$
- $R^C \cap S = \{4\}$
- $R^C \cap S^C = \{5\}$

Since  $\{2\}, \{1, 3\}, \{4\}, \{5\}$  generate  $R, R^C, S, S^C$ ; then :

$$\sigma(R, S) = \sigma(\{\{2\}, \{1, 3\}, \{4\}, \{5\}\})$$

Note that  $\{2\}, \{1, 3\}, \{4\}, \{5\}$  are disjoint sets that their union forms  $\Omega$  then each union between two sets, three sets, four sets, themself and the empty set form  $\sigma(R, S)$ , which has  $2^4 = 16$  elements.

### 1.1.2 Cartesian product of two $\sigma$ -algebras

Let  $\sigma(A)$  and  $\sigma(B)$  be two  $\sigma$ -algebras, then the Cartesian product  $\sigma(A) \times \sigma(B)$  is in general not a  $\sigma$ -algebra. Thus

$$\sigma(A \times B) = \sigma(\sigma(A) \times \sigma(B))$$

### 1.1.3 Algebra

In some lectures, we can find the concept named algebra. The definition of an algebra is the same as  $\sigma$ -algebra except the third condition :

1.  $\emptyset \in \Sigma$ .
2.  $A \in \Sigma \Rightarrow A^C$  ( in  $X$ )  $\in \Sigma$ . (*closed under complementation*)
3.  $A_1, A_2, \dots, A_n \in \Sigma \Rightarrow \bigcup_i^n A_i \in \Sigma$ . (*closed under finite unions*)

Here is a simple example for the difference on the third condition between an algebra and a  $\sigma$ -algebra. Let's consider closed intervals  $[\frac{1}{n}, 1 - \frac{1}{n}]$ , then a **finite** union of these intervals is also a closed interval. However, a **countable** union can be an infinity union, then the result is  $(0, 1)$ , which is a open interval.

$\sigma$ -algebra is well an algebra, thus it is an extension of algebra. Then the converse is not true, let's see the following example.

### 1.1.3.1 Example

Consider a set  $X$  of natural numbers, i.e.  $X = \mathbb{N}$  and the algebra  $\Sigma$  on it that consists of all subsets of  $X$  with the property that either the set or its complement has finitely many elements :

$$\Sigma = \{A \in \mathbb{N}: |A| \text{ or } |A^C| \text{ is finite}\}$$

$\Sigma$  is well an algebra, but it is not a  $\sigma$ -algebra. This is because if we take  $A_i = \{2i\}, \forall i \geq 0$ , thus  $A_i$  is a singleton set that contains a even number, then the *countable* union condition of  $A_i$  infers that set  $E$  contains all even numbers must be in  $\Sigma$ . From here, by relative complement, set  $O = E^C$  contains all odd numbers must be in  $\Sigma$ . However,  $|E| = \infty$  and  $|O| = \infty$ , then  $E$  can not be in  $\Sigma$ .

### 1.1.4 Borel set

Given a space  $X$ , a Borel set is any set that can be formed from open sets (or, equivalently, from closed sets) through the operations of *countable* union, *countable* intersection, and relative complement.

The collection of all Borel sets in this space  $X$  forms a  $\sigma$ -algebra, known as the Borel algebra, denoted as  $B(X)$ . This Borel algebra on  $X$  is the smallest  $\sigma$ -algebra containing all open sets of  $X$ .

### 1.1.4.1 Example

Given the real number line ( $X = \mathbb{R}$ ), then the collection of all Borel sets (or intervals), which are formed from open intervals in  $\mathbb{R}$ , is a Borel algebra on  $\mathbb{R}$ , denoted  $B(\mathbb{R})$ .

## 1.2 $\pi$ -system

In mathematics, a  $\pi$ -system (or pi-system) on a set  $\Omega$  is a collection  $P$  of certain subsets  $\Omega$ , such that :

- $P$  is non-empty
- $A, B \in P$  then  $A \cap B \in P$

$\sigma$ -algebra is  $\pi$ -system : since  $A \cap B = (A^C \cup B^C)^C$  is an element in  $\sigma$ -algebra. However,  $\pi$ -system is not  $\sigma$ -algebra.

If  $\Sigma$  is a non-empty collection of subsets in  $\Omega$  then  $\mathcal{I}(\Sigma)$  or  $\mathcal{I}_\Sigma$  is the unique smallest  $\pi$ -system that contains every elements of  $\Sigma$ . We have explicitly its formulae :

$$\mathcal{I}_\Sigma = \{E_1 \cap \dots \cap E_n : 1 \leq n \in \mathbb{N} \text{ and } E_1, \dots, E_n \in \Sigma\}$$

## 1.3 Dynkin system

### 1.3.1 Definition

Let  $\Omega$  be a non-empty set, and let  $D$  be a collection of subsets of  $\Omega$ . Then  $D$  is a Dynkin system if

- $\Omega \in D$
- If  $A \in D$  then  $\Omega \setminus A \in D$
- If  $A_1, A_2, A_3, \dots$  is a sequence of pairwise disjoint sets in  $D$  ( $A_i \cap A_j = \emptyset, \forall i \neq j$ ) then  $\bigcup_{i=1}^n A_i \in D, \forall n = 1, \dots, \infty$ .

Dynkin systems are sometimes referred to as  $\lambda$ -systems or d-systems. Interesting properties:

- $\sigma$ -algebra is  $\lambda$ -system
- If a set is both  $\lambda$ -system and  $\pi$ -system, then it is a  $\sigma$ -algebra. In general,  $\lambda$ -system is not  $\pi$ -system and vice versa.

### 1.3.2 Dynkin's $\pi$ - $\lambda$ theorem

$$\begin{cases} P \text{ is a } \pi\text{-system} \\ D \text{ is a } \lambda\text{-system} \\ P \subseteq D \end{cases} \Rightarrow \sigma(P) \subseteq D$$

**Proof** [http://theanalysisofdata.com/probability/E\\_3.html](http://theanalysisofdata.com/probability/E_3.html)

## 1.4 Measurable space

Consider a set  $X$  and a  $\sigma$ -algebra  $\Sigma$  on  $X$ . Then the tuple  $(X, \Sigma)$  is called a measurable space. No measure (or empty subset  $\emptyset$ ) and  $X$  must be in the  $\sigma$ -algebra  $\Sigma$ .

**Examples** Given the set:  $X = \{1, 2\}$ :

- $\Sigma_1 = \{X, \emptyset\}$ . Then  $(X, \Sigma_1)$  is a measurable space.
- $\Sigma_2 = 2^X = \{X, \emptyset, \{1\}, \{2\}\}$  (powerset of  $X$ ). Then  $(X, \Sigma_2)$  is also a measurable space.

DO NOT confuse **measurable space** with **measure space** in sec 2.1.8.

## 1.5 Measurable function

### 1.5.1 Definition

Let  $(\Omega, \Sigma)$  and  $(Y, T)$  be measurable spaces. A function  $f : \Omega \rightarrow Y$  is said to be measurable on  $(\Sigma, T)$  if for every  $E \in T$ , the pre-image of  $E$  under  $f$  is in  $\Sigma$  :

$$f^{-1}(E) := \{\omega \in \Omega \mid f(\omega) \in E\} \in \Sigma, \quad \forall E \in T.$$

Note that  $f^{-1}$  here take input value of a  $\sigma$ -algebra, which is a set of elements of  $\Omega$ , which means :  $f^{-1} : \sigma\text{-algebra of } Y \rightarrow \sigma\text{-algebra of } \omega$ .

### 1.5.2 Role of measurable function

A measurable function has the ability to “transport” a measure defined for the domain’s measurable space  $(X, \Sigma)$  to the codomain’s measurable space  $(Y, T)$ .

### 1.5.3 $\mathcal{F}$ -measurability

Let  $\mathcal{F}$  be an arbitrary  $\sigma$ -algebra of  $\Omega$ .  $\mathcal{B}$  is Borel set (sec 2.1.1.4) on  $\mathbb{R}$ . A random variable or a map  $X : \Omega \rightarrow \mathbb{R}$  is said to be  **$\mathcal{F}$ -measurable** for short or  $(\mathcal{F}, \mathcal{B})$ -measurable for long if for all  $S \in \mathcal{B}$ , the pre-image (i.e. the inverse image) under  $X$  belongs to  $\mathcal{F}$ , that is :

$$X^{-1}(S) \in \mathcal{F}, \quad \forall S \in \mathcal{B}$$

or

$$\{\omega : X(\omega) \in S\} \in \mathcal{F}, \quad \forall S \in \mathcal{B}$$

From the two above definitions ( $\mathcal{F}$ -measurability and  $\sigma$ -algebra generated),  $X$  is  $\mathcal{F}$ -measurable if and only if  $\sigma(X) \subset \mathcal{F}$ .

In short,  $X$  is  $\mathcal{F}$ -measurable if all pre-images of  $X$  are in  $\mathcal{F}$ .

#### 1.5.3.1 Example 1

We use the example 2.1.6.4.2, if  $\mathcal{F} = \{\Omega, \emptyset, \{1, 2, 3\}, \{4, 5, 6\}, \{1, 4\}\} \supset \sigma(X)$ , then  $X$  is  $\mathcal{F}$ -measurable.

#### 1.5.3.2 Example 2

Let  $\Omega = \{-1, 0, 1\}$  and  $\mathcal{F} = \{\emptyset, \Omega, \{-1, 1\}, \{0\}\}$ :

- $X_1(\omega) := \omega$  is NOT  $\mathcal{F}$ -measurable. Since  $X^{-1}(\{1\}) = \{1\} \notin \mathcal{F}$ .
- $X_2(\omega) := \omega^2$  is  $\mathcal{F}$ -measurable.

### 1.5.3.3 Lebesgue measurable function

It means a measurable function  $f: (\mathbb{R}, \mathcal{L}) \rightarrow (\mathbb{R}, B(\mathbb{R}))$ , where  $\mathcal{L}$  is the  $\sigma$ -algebra of **Lebesgue measurable sets** (sec 2.1.7.2.1 and 2.1.7.3.3) and  $B(\mathbb{R})$  is the Borel algebra (2.1.1.4) on  $\mathbb{R}$ .

In the general case, the domain of  $f$  is not necessary  $\mathbb{R}$  and we call  $X$  this domain, which means  $f: X \rightarrow \mathbb{R}$ . Then  $f$  is Lebesgue measurable function if and only if the set

$$\{f > \alpha\} = \{x \in X : f(x) > \alpha\}$$

is Lebesgue measurable for all  $\alpha \in \mathbb{R}$ . This is also equivalent to any of  $\{f \geq \alpha\}$ ,  $\{f < \alpha\}$ ,  $\{f \leq \alpha\}$  being Lebesgue measurable for all  $\alpha \in \mathbb{R}$ , or the preimage of any open set being Lebesgue measurable. Continuous functions, monotone functions, step functions, semicontinuous functions, Riemann-integrable functions, and functions of bounded variation (2.2.9) are all Lebesgue measurable.

Lebesgue measurable functions are of interest in mathematical analysis because they can be integrated, which means with Lebesgue measure  $L$ , then

$$\int_{[a,b]} f(x) dL(x) = \int_{[a,b]} f(x) dx$$

## 1.6 Random variable

A random variable is a **measurable function** that maps from the measurable space of sample space and event space  $(\Omega, \mathcal{F})$  to a measurable space  $(Y, T)$ .

### 1.6.1 Why measurable function ?

It is because we want that random variables are available to transport the **probability measure**.

### 1.6.2 Examples

**Example 1 :** Let's  $\Omega = \{\text{Head}, \text{Tail}\}$ ,  $Y = \{0, 1\}$  and a function  $X: \Omega \rightarrow Y$  that:

$$\begin{cases} X(\text{Head}) = 0 \\ X(\text{Tail}) = 1 \end{cases}$$

In order to  $X$  be a random variable, we need to define two measurable spaces (domain and codomain) that  $X$  is measurable function :

- Domain :  $(\Omega, \mathcal{F})$  where  $\mathcal{F} = \{\emptyset, \Omega, \{\text{Head}\}, \{\text{Tail}\}\}$ .
- Codomain :  $(Y, T)$  where  $T = \{\emptyset, Y, \{0\}, \{1\}\}$ .

Note that we precise  $X^{-1}(\emptyset) = \emptyset$  and  $X^{-1}(Y) = \{\omega \in \Omega \mid X(\omega) \in Y\} = \{\text{Head}, \text{Tail}\} = \Omega$ .

Moreover, the two following measurable spaces can be used :

- Domain :  $(\Omega, \mathcal{F})$  where  $\mathcal{F} = \{\emptyset, \Omega\}$ .

- Codomain :  $(Y, T)$  where  $T = \{\emptyset, Y\}$ .

It means that, given a function, it is not unique a couple of (domain, codomain) that makes this function a random variable.

**Example 2** Using the above example, but with two measurable spaces :

- Domain  $(\Omega, \mathcal{F})$  where  $\mathcal{F} = \{\emptyset, \Omega\}$ .
- Codomain :  $(Y, T)$  where  $T = \{\emptyset, Y, \{0\}, \{1\}\}$ .

This is non measurable function because  $\{1\} \in T$  and  $\{f^{-1}(\{1\})\} = \{Tail\} \notin \Sigma$ .

**Example 3** Uniforme distribution transform. Let's consider two measurable spaces :  $([0, 1], \sigma([0, 1]))$  and  $([c, d], \sigma([c, d]))$ .  $Z$  is a function  $[0, 1] \rightarrow [c, d]$  such that :

$$Z(x) = (d - c)x + c, \forall x \in [0, 1]$$

Then  $X$  is a random variable.

**Example 4** Box–Muller transform. Let's consider two measurable spaces :  $[0, 1] \times [0, 1], \sigma([0, 1] \times [0, 1])$  and  $\mathbb{R}, \sigma(\mathbb{R})$ .  $Y_1 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  and  $Y_2 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  such that :

$$Y_1 = R \cos(\Theta) = \sqrt{-2 \ln U_1} \cos(2\pi U_2), \forall U_1 \in [0, 1], U_2 \in [0, 1]$$

$$Y_2 = R \sin(\Theta) = \sqrt{-2 \ln U_1} \sin(2\pi U_2), \forall U_1 \in [0, 1], U_2 \in [0, 1]$$

Then  $Y_1, Y_2$  are random variables.

### 1.6.3 Notes

- A random variable is just a measurable function and **it does not require** a probability measure. See figure 2.1.
- A measurable space consists of sample space and event space  $(\Omega, \mathcal{F})$  and is equiped with a probability measure is a probability space.

### 1.6.4 $\sigma$ -algebra generated by random variable

Suppose  $\Omega$  is a sample space. Let  $X : \Omega \rightarrow \mathbb{R}$  is a random variable and  $\mathcal{B}(\mathbb{R})$  is the Borel  $\sigma$ -algebra on  $\mathbb{R}$ . Indeed  $\mathcal{B}(\mathbb{R})$  contains all possible open interval in  $\mathbb{R}$ . The  $\sigma$ -algebra generated by  $X$ , noted  $\sigma(X)$  is:

$$\sigma(X) = \{X^{-1}(S) : \forall S \in \mathcal{B}(\mathbb{R})\}$$

or

$$\sigma(X) = \{\{\omega \in \Omega | X(\omega) \in S\} : \forall S \in \mathcal{B}(\mathbb{R})\}$$

since  $X^{-1}(S) := \{\omega \in \Omega | X(\omega) \in S\}$

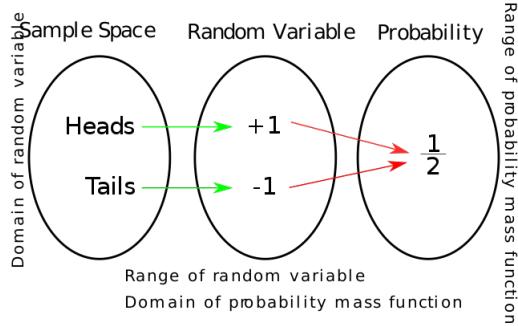


Figure 2.1: Relation between sample space, random variable and probability measure (mass function).

#### 1.6.4.1 Example 1

Let consider a random variable that always returns a constant,  $X(\omega) = c$ . Then :

- if  $c \in S$  then  $X^{-1}(S) = \Omega$
- if  $c \notin S$  then  $X^{-1}(S) = \emptyset$

Hence  $\sigma(X) = \{\Omega, \emptyset\}$ .

#### 1.6.4.2 Example 2

Let consider a random variable  $X$  that  $X(\omega) = -5$  if  $\omega \in \{1, 2, 3\}$  and  $X(\omega) = 5$  if  $\omega \in \{4, 5, 6\}$ .  $\Omega = \{1, 2, 3, 4, 5, 6\}$ . Then :

- If  $-5 \notin S$  and  $5 \notin S$  then  $X^{-1}(S) = \emptyset$
- If  $-5 \in S$  and  $5 \notin S$  then  $X^{-1}(S) = \{1, 2, 3\}$
- If  $-5 \notin S$  and  $5 \in S$  then  $X^{-1}(S) = \{4, 5, 6\}$
- If  $-5 \in S$  and  $5 \in S$  then  $X^{-1}(S) = \Omega$

Hence  $\sigma(X) = \{\Omega, \emptyset, \{1, 2, 3\}, \{4, 5, 6\}\}$ .

## 1.7 Measure

In mathematics, the concept of a **measure** can informally be :

- Geometrical measures (length, area, volume)
- Magnitude, mass
- Probability of events

### 1.7.1 Formal definition of measure

Let  $(X, \Sigma)$  be a **measurable space**. A *set function*  $\mu$  from  $\Sigma$  to the *extended real number line* ( $\mathbb{R}$  with two infinity elements  $\pm\infty$ ) is called a **measure** if the following conditions hold:

1. *Non-negativity*: For all  $E \in \Sigma$ ,  $\mu(E) \geq 0$ .
2. *Null empty set*:  $\mu(\emptyset) = 0$ .
3. *Countable additivity* : For all countable collections  $\{E_k\}_{k=1}^{\infty}$  of pairwise **disjoint** sets in  $\Sigma$ , we have

$$\mu \left( \bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \mu(E_k)$$

Note that, the three above conditions do not require the existance of all  $\mu(x)$  for  $x \in X$ , only  $\mu(E)$  where  $E \in \Sigma$  is required. The role of  $X$  here is just to say that  $\Sigma$  is a  $\sigma$ -algebra of  $X$ .

#### 1.7.1.1 Signed measure

This is a flexible concept of a measure.  $\mu$  is called signed measure if it satisfies only the *Countable additivity* : For all countable collections  $\{E_k\}_{k=1}^{\infty}$  of pairwise **disjoint** sets in  $\Sigma$ , we have

$$\mu \left( \bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \mu(E_k)$$

We call  $\mu$  finite signed measure if  $\mu(E) < \infty, \forall E \in \Sigma$ .

#### 1.7.1.2 $\sigma$ -finite measure

- $\mu(E)$  is called finite measure (w.r.t set  $E$ ) if  $\mu(E) < \infty, \forall E \in \Sigma$ .
- A measure  $\mu$  is called  $\sigma$ -finite measure if sample space  $X$  is a countable union of measurable subsets  $E_i$ , which means  $X = \bigcup_{i=1}^n E_i$  and each  $E_i$  has finite measure.

#### 1.7.1.3 $\mu$ -almost everywhere

Given two functions  $f$  and  $g$  from  $X$  to  $Y$  ( $f$  and  $g$  are not necessary measurable function) and a measure  $\mu$ . We say that  $f = g$   $\mu$ -almost everywhere if

$$\forall E \in 2^X, \text{ if } \mu(E) = 0 \text{ then } f(x) = g(x), \forall x \in X/E$$

### 1.7.2 Outer measure

#### Definition

Given a set  $X$ , let's  $2^X$  denote the collection of all subsets of  $X$ , including the empty set  $\emptyset$ . An *outer measure* on  $X$  is a set function  $\mu^*$  such that :

1. *Non-negativity*: For all  $E \in 2^X$ ,  $\mu^*(E) \geq 0$ .
2. *Null empty set*:  $\mu^*(\emptyset) = 0$ .
3. *Countable subadditivity* :  $\forall A, B_1, B_2, \dots, B_\infty \in 2^X$ ,

$$\text{if } A \subseteq \bigcup_{i=1}^{\infty} B_i \text{ then } \mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(B_i)$$

#### Equivalent definition

1. *Non-negativity*: For all  $E \in 2^X$ ,  $\mu^*(E) \geq 0$ .
2. *Null empty set*:  $\mu^*(\emptyset) = 0$ .
3. *Monotone*:  $\forall A, B \in 2^X$  if  $A \subseteq B$  then  $\mu^*(A) \leq \mu^*(B)$
4. *Outer measure of countable sum is less than countable sum of outer measure*:

$$\mu^* \left( \sum_{i=1}^{\infty} B_i \right) \leq \sum_{i=1}^{\infty} \mu^*(B_i), \quad \forall B_1, B_2, \dots, B_\infty \in 2^X$$

#### 1.7.2.1 Measurability of sets relative to an outer measure

Let  $X$  be a set with an outer measure  $\mu^*$ . One says that a set  $E \in 2^X$  is  $\mu^*$ -measurable (or Carathéodory-measurable relative to  $\mu^*$ ) if and only if it satisfies the **Carathéodory's criterion** :

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap E) + \mu^*(A \setminus E) \\ &= \mu^*(A \cap E) + \mu^*(A \cap E^C), \quad \forall A \in 2^X \end{aligned}$$

Informally, this says that a  $\mu^*$ -measurable set  $E$  is one which may be used as a building block (or basic block) such that any other set  $A$  can be decomposed into two parts, one is inside of the measurable set and the other is outside of the measurable set.

#### Properties

1. *Relative complement*: If set  $E \in 2^X$  is a  $\mu^*$ -measurable, then  $E^C$  (in  $X$ ) is also  $\mu^*$ -measurable.
2. *Countable additivity*: If  $E_1, E_2, \dots, E_\infty \in 2^X$  are  $\mu^*$ -measurable pairwise **disjoint**, then

$$\mu^* \left( \bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \mu^*(E_k)$$

3. *Closed under countable unions*: If  $E_1, E_2, \dots, E_\infty \in 2^X$  are  $\mu^*$ -measurable then  $\bigcup_{i=1}^{\infty} E_i$  is also  $\mu^*$ -measurable.

4. *Completeness*: If  $E \subseteq X$  such that  $\mu^*(E) = 0$  then  $E$  is  $\mu^*$ -measurable.

Because of complexity, we show only the proof for the last property :

- From the third characteristic (*monotone*) in the equivalent of outer measure,  $\forall B \in 2^X$ , if  $B \subseteq E$  then  $\mu^*(B) \leq \mu^*(E) = 0$ . However, from the first characteristic,  $\mu^*(B) \geq 0$ , then  $\mu^*(B) = 0$ .
- $\forall A \in 2^X$ , then  $A = (A \cap E) \cup (A \cap E^C)$ .
- On one hand, from the third characteristic (*countable subadditivity*) in the definition of outer measure, we have :

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^C) \leq \mu^*(A \cap E^C),$$

since  $A \cap E \subseteq E$ , then  $\mu^*(A \cap E) = 0$ .

- On the one hand,  $A \cap E^C \subseteq A$ , then from the *monotone* characteristic, we have  $\mu^*(A \cap E^C) \leq \mu^*(A)$ .
- Finally,  $\mu^*(A) = \mu^*(A \cap E^C)$  or  $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C)$ . We conclude that  $E$  is  $\mu^*$ -measurable by Carathéodory's criterion.

#### 1.7.2.2 Purpose of outer measure

Given a set  $X$ , the purpose of constructing an outer measure on all subsets of  $X$  is to **pick out** a class (or set) of subsets, which is  $\sigma$ -algebra and the outer measure on this class is a measure.

##### Proof

The class that we mentioned above, denoted  $\Sigma$ , is thus the set of all  $\mu^*$ -measurable subsets of  $X$ . We need to show that :

1.  $\Sigma$  is a  $\sigma$ -algebra. This is because :
  - $\emptyset \in \Sigma$  : since from the definition of outer space  $\mu^*(\emptyset) = 0$  and the *completeness*, property 4 says that  $\emptyset \in \Sigma$ .
  - *Relative complement*, property 1.
  - *Closed under countable unions*, property 3.
2. Outer measure  $\mu^*$  on  $\Sigma$  is a measure. This is because:
  - *Non-negativity* and *null empty set* are herited from outer measure.
  - *Countable additivity* is the property 2.

Note that, a measure is outer measure but the converse is not true.

### 1.7.2.3 Valuation

Given a class  $\mathcal{T}$  of open subsets of a topological space (which can be a set), a valuation is a set function such that :

1. *Non-negativity*: For all  $E \in \mathcal{T}$ ,  $v(E) \geq 0$ .
2. *Null empty set*:  $v(\emptyset) = 0$ .
3. *Monotone*:  $\forall A, B \in \mathcal{T}$  if  $A \subseteq B$  then  $v(A) \leq v(B)$ .
4. *Modularity*:

$$v(A \cup B) + v(A \cap B) = v(A) + v(B), \quad \forall A, B \in \mathcal{T}$$

At a first look, a valuation is similar to a outer space, the only thing that we can distinguish between them is the last condition.

The purpose of valuation is a raw or basic set function to construct a outer measure, in the following text.

### 1.7.2.4 Construction of outer measure

We discover a manner to construct an outer space from a given set  $X$  and a given valuation  $v$ .

$$\mu^*(E) = \inf \left\{ \sum_{i=0}^{\infty} p(A_i) \mid E \subseteq \bigcup_{i=0}^{\infty} A_i, \forall i \in \mathbb{N}, A_i \in 2^X \right\}$$

We have a convention that the infimum is infinite if no such sequence of  $A_i$  exists to cover  $E$ .

## 1.7.3 Examples of measure

### 1.7.3.1 Counting measure

Given measurable space  $(X, \Sigma)$ . The counting measure on  $\Sigma$  is defined by  $\mu(E) = \text{number of elements in } E$ , for all  $E \in \Sigma$ . The three conditions *non-negativity*, *null empty set* and *countable additivity* are trivially proved

### 1.7.3.2 Probability measure

$P$  on measurable space  $(\Omega, \mathcal{F})$  is measure since :

- *Non-negativity*:  $P(E) \geq 0, \forall E \in \mathcal{F}$ .
- *Null empty set*:  $P(\emptyset) = 0$ , in addition  $P(\Omega) = 1$ .
- *Countable additivity*:  $P\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} P(E_k)$  where  $\{E_k\}_{k=1, \dots, \infty}$  are pairwise disjoint.

### 1.7.3.3 Lebesgue measure

The Lebesgue measure is the standard way of assigning a measure to subsets of Euclidean  $n$ -dimensional space. It coincides with the standard measure of length, area, or volume for  $n = 1, 2, 3$ . For  $n > 3$ , it is also called  $n$ -dimensional volume,  $n$ -volume, hypervolume, or simply volume. Let's start with the case  $X = \mathbb{R}$  then we generalize for the case  $X = \mathbb{R}^n$

#### Definition for $X = \mathbb{R}$

Given  $X = \mathbb{R}$ , we denote  $2^{\mathbb{R}}$  a set that contains all open intervals in  $\mathbb{R}$ . Also, given a valuation (sec 2.1.7.2.3)  $l$  on  $2^{\mathbb{R}}$  such that  $l(I) = b - a$  with  $I = (a, b)$ . Then we construct an outer measure  $\lambda^*(E)$ , called **Lebesgue outer measure** as in sec 2.1.7.2.4, by :

$$\lambda^*(E) = \inf \left\{ \sum_{i=0}^{\infty} l(A_i) \mid E \subseteq \bigcup_{i=0}^{\infty} A_i, \forall i \in \mathbb{N}, A_i \in 2^{\mathbb{R}} \right\}$$

#### Definition for $X = \mathbb{R}^n$

Given  $X = \mathbb{R}^n$ , we denote  $2^{\mathbb{R}^n}$  a set that contains all open rectangular cuboids in  $\mathbb{R}^n$ ,  $C = I_1 \times I_2 \times \dots \times I_n$ . The valuation  $v$  on  $2^{\mathbb{R}^n}$  is given  $v(C) = l(I_1) \times l(I_2) \times \dots \times l(I_n)$ . Then Lebesgue outer measure is constructed by :

$$\lambda^*(E) = \inf \left\{ \sum_{i=0}^{\infty} v(A_i) \mid E \subseteq \bigcup_{i=0}^{\infty} A_i, \forall i \in \mathbb{N}, A_i \in 2^{\mathbb{R}^n} \right\}$$

Finally, we just select all sets  $E \in 2^{\mathbb{R}^n}$  such that each  $E$  is  $\lambda^*$ -measurable as in sec 2.1.7.2.1 and create a  $\sigma$ -algebra  $\Sigma$  on these  $E$ . In  $\Sigma$ , the Lebesgue outer measure  $\lambda^*$  is **Lebesgue measure**  $\lambda$ .

To take away :

- A set  $E$  is said *Lebesgue measurable* if it is  $\lambda^*$ -measurable, where  $\lambda^*$  is Lebesgue outer measure.
- Non-measurable set  $E \in 2^{\mathbb{R}^n}$  do exist, an example is the Vitali set (2.1.7.3.4).
- Trivially, we see that the outer Lebesgue measure  $\lambda^*$  of a countable set is 0. From properties of *completeness*, we say that the countable set is  $\lambda^*$  measurable or Lebesgue measure  $\lambda$  of a countable set is 0. For example  $\mathbb{N}$  set of natural numbers and  $\mathbb{Q}$  set of rational numbers are both countable, then  $\lambda(\mathbb{N}) = 0$  and  $\lambda(\mathbb{Q}) = 0$

### 1.7.3.4 Vitali set

This is an example of non-measurable set.

First, we define an *equivalence* on two real numbers  $x, y \in \mathbb{R}$ , that  $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$ . For example,  $0.5 + \pi \sim \pi$  and  $\sqrt{3} \not\sim \pi, \dots$

Second, we put all real numbers that are equivalent (by above definition) to a group, we may have an infinite number of groups, no problem. Then by the axiom of choice, we can define a representative set  $A$  such that every real

number is equivalent to **exactly one element** of  $A$ . For example, in a group that contains  $\sqrt{2}, \sqrt{2} + 1, \sqrt{2} + 2, \dots$ , we just take  $\sqrt{2}$  and put it to  $A$ .

Third, instead of performing the second step in  $\mathbb{R}$ , we only do it in  $[0, 1]$  and this set  $A$  is called *Vitali set*.

Fourth, given a  $q \in \mathbb{Q} \cap [0, 1]$ , we define a translation :

$$A_q = \{a + q \pmod{1} \mid a \in A\}$$

Then we can see that

- $A_q$  are pairwise disjoint, ( $A = A_0$ ).
- $\bigcup A_q, \forall q \in \mathbb{Q} \cap [0, 1]$ , is a *countable* union and it is equal to  $[0, 1]$ .

Fifth, given a Lebesgue measure  $\mu$  (sec 2.1.7.3.3) and suppose that  $A$  is measurable set, then the *countable additivity* property of a measure say that:

$$\mu(\bigcup A_q) = \sum \mu(A_q) = \mu([0, 1])$$

Finally, by

- $\mu(A_q)$  is a constant since the Lebesgue measure is translation invariant
- There is infinit element  $\mu(A_q)$  in the sum.
- $\mu([0, 1])$  is also a fixed value that is strictly positive.

we have the contradiction.

## 1.8 Measure space

A measure space is generalized concept of probability space. A measure space is a triple  $(\Omega, \mathcal{F}, \mu)$  where :

- $\Omega$  is a set
- $\mathcal{F}$  is is a  $\sigma$ -algebra on the set  $\Omega$
- $\mu$  is a measure on  $(\Omega, \mathcal{F})$ .

In other words, a measure space consists of a measurable space  $(X, \mathcal{F})$  (see 2.1.4) together with a measure (see 2.1.7) on it.

If the measurable space consists of sample space and event space; and the measure  $\mu$  is the probability measure  $P$  (probability function), then the measure space is probability space.

### 1.8.1 Probability space

If the measurable space consists of sample space and event space  $(\Omega, \mathcal{F})$ ; and the measure  $\mu$  is the probability measure  $P$  (probability function), then the measure space is probability space. More precise :

- A sample space  $\Omega$ , which is the set of all possible **outcomes**.
- An event space  $\mathcal{F}$ , which is a set of events. An **event** is an subset of sample space.  $\mathcal{F}$  is a  $\sigma$ -algebra of  $\Omega$ .
- A probability measure  $P$  (2.1.7.3.2), which assigns each event in the event space a probability, which is a number between 0 and 1.

#### 1.8.1.1 Example 1

The throw of a standard die :

- $\Omega$  to be  $\{1, 2, 3, 4, 5, 6\}$ .
- For the event space  $\mathcal{F}$ , we could simply use the set of all subsets of the sample space, but  $\mathcal{F}$  must be a  $\sigma$ -algebra of  $\Omega$ . E.g., we want that  $\mathcal{F}$  contains events that die lands on  $\{5\}$  and die lands on even numbers  $\{2, 4, 6\}$ . Then naively,  $\mathcal{F} = \{\{5\}, \{2, 4, 6\}\}$ . However, the latter is not a  $\sigma$ -algebra of  $\Omega$ . What we need to do now is to make  $\mathcal{F}$  a smallest  $\sigma$ -algebra of  $\Omega$  that contains  $\{5\}$  and  $\{2, 4, 6\}$ , by taking  $\mathcal{F} = \{\Omega, \{5\}, \{2, 4, 6\}, \{1, 2, 3, 4, 6\}, \{1, 3, 5\}, \{2, 4, 5, 6\}, \{1, 3\}, \emptyset\}$ .
- The probability function, that map each event in  $\mathcal{F}$  to a probability, e.g event  $\{5\}$  corresponds to  $1/6$  and event  $\{2, 4, 6\}$  corresponds to  $1/2$ , ...

#### 1.8.1.2 Example 2

The fair coin is tossed three times.

- There are 8 possible outcomes:  $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ .
- We denote  $2^\Omega$  all possible subsets of  $\Omega$ . Any subset  $A \in 2^\Omega$  can make an event space by using  $\mathcal{F} = \sigma(A)$ , where  $\sigma(A)$  denotes the smallest  $\sigma$ -algebra of  $\Omega$  that contains  $A$ . Note that, two subsets can have the same  $\mathcal{F}$  such as  $A$  and  $A^C$ , since  $\sigma(A) = \sigma(A^C)$ . The number of subsets is  $2^{|\Omega|}$ . In this case, we have  $2^8 = 256$  subsets (or events). Let's take  $\mathcal{F} = \{\emptyset, \{HHH, HHT, HTH, THH\}, \{HTT, THT, TTH, TTT\}, \Omega\}$ , the event space with 2 events, the number of  $H$  is greater than one of  $T$  and vice-versa.
- The probability function : e.g.,  $P(\{HHH, HHT, HTH, THH\}) = 1/2$  and  $P(\{HTT, THT, TTH, TTT\}) = 1/2$

### 1.8.1.3 Example 3

As the same of above example, the fair coin is tossed three times but we know only the total number of tails (we do not know the result of three time tossing).

- The sample space  $\Omega = \{0, 1, 2, 3\}$ .
- The event space that number of tails is unpair :  $\mathcal{F} = \{\emptyset, \{1, 3\}, \{0, 2\}, \Omega\}$ .
- The probability function : e.g.  $P(\{1, 3\}) = P(\{HHT, HTH, THH, TTT\}) = 1/2$ .

### 1.8.1.4 Example 4

A number between 0 and 1 is chosen at random (**uniformly**).

- $\Omega = [0, 1]$
- E.g. the event that  $x < 0.1$  or  $x > 0.9$ .  $\mathcal{F} = \{\emptyset, \Omega, (0.9, 1], [0, 0.1], [0, 0.9], [0.1, 1]\}$ .
- The probability function :  $P((a, b)) = (b-a)$ , which generates the *Lebesgue measure* on  $[0,1]$ .

## 1.8.2 Equality of integrals then almost sure equality of random variables

This is a standard argument in measure theory. Given probability space  $(\Omega, \mathcal{F}, P)$ . Given two random variables  $X, Y$  with  $\mathbb{E}[|X|] < \infty, \mathbb{E}[|Y|] < \infty$  and

$$\int_F X \, dP = \int_F Y \, dP, \quad \forall F \in \mathcal{F}$$

Then  $X = Y$  almost surely.

#### Proof

The above problem is equivalent to, given :

$$\int_F X \, dP = 0, \quad \forall F \in \mathcal{F}$$

Then  $X = 0$  almost surely.

Let  $A_n := \{\omega : X(\omega) > \frac{1}{n}\}$ . If there is a  $n \in \mathbb{N}$  that the set  $A_n$  has nonzero probability, then

$$\int_{A_n} X \, dP \geq \frac{P(A_n)}{n} > 0$$

We obtain the contradiction, therefore,  $P(A_n) = 0, \forall n$ . This implies, by the *countable additivity* property of probability measure (2.1.7.3.2): With

$$A = \bigcup A_n = \{\omega : X(\omega) > 0\}$$

Then

$$P(A) = \sum_n P(A_n) = 0$$

In analogy, we can show that  $P(B) = 0$ , with  $B = \{\omega : X(\omega) < 0\}$ . Finally,  $X = 0$  almost surely.

## 1.9 Lebesgue integral

We first reminding the concept of Riemann integral

### 1.9.1 Riemann integral

Let's  $f : [a, b] \rightarrow \mathbb{R}$ . Then  $f$  is said *Riemann integrable* over  $[a, b]$  if

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^{\infty} (x_i - x_{i-1}) f(t_i) \text{ converges,}$$

where  $P$  is a partition over  $[a, b]$ ,  $\|P\| = \max_i (x_i - x_{i-1})$ ,  $t_i \in [x_i, x_{i-1}]$ . In this case, the Riemann integral is equal to the limit and denoted  $\int_a^b f(x) dx$ .

#### Properties

- If  $f$  is Riemann integrable, then it is bounded.
- If  $f$  is bounded and piecewise continuous, then it is Riemann integrable

### 1.9.2 Intuitive idea for Riemann integral and Lebesgue integral

We discover two main ideas for the intuitive of Lebesgue integral, one from sum of discontinuous horizontal rectangles and one from discontinuous vertical rectangles.

#### 1.9.2.1 Sum of discontinuous horizontal rectangles

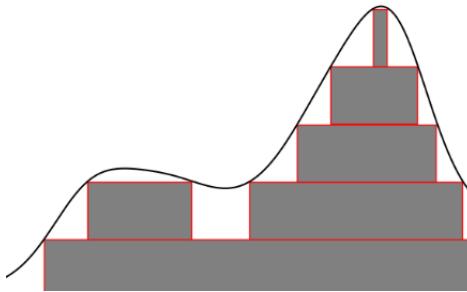


Figure 2.2: Lebesgue integral illustrated by summing up of discontinuous horizontal rectangles.

We see the idea in figure 2.2. A little bit formally, let  $(X, \Sigma, \mu)$  be a measure space and  $f$  is a non-negativity measurable function from  $(X, \Sigma)$  to  $(\mathbb{R}, B(\mathbb{R}))$ , then the Lebesgue integral can be written :

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^{\infty} (y_k - y_{k-1}) \mu(\{x \mid f(x) > y_k\})$$

where  $P$  is a partition on the ordinate over interval  $[0, \max_x(f(x))]$ .

#### 1.9.2.2 Sum of discontinuous vertical rectangles

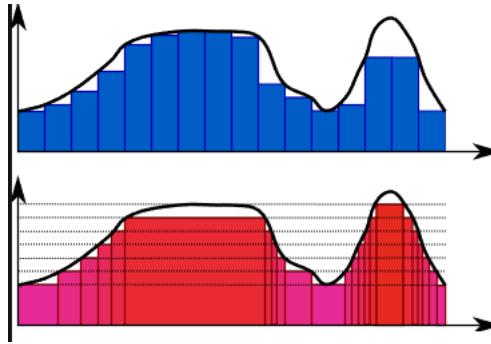


Figure 2.3: Intuitive idea between Riemann and Lebesgue integral by discontinuous vertical rectangles.

In figure 2.3, we see the difference between calculation of integral, proposed by Riemann and Lebesgue. Visually,

- In Riemann integral, we sum progressively small vertical rectangles.
- In Lebesgue integral, we group first small vertical rectangles that have the same height and let's call this group “*discontinuous vertical rectangle*”. Then we sum these discontinuous vertical rectangles. In figure 2.3, the color changes from rose to red for different discontinuous vertical rectangles, characterized by the height or output value of function.

Formally, the Lebesgue integral can be written:

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^{\infty} \xi_i \mu(\{x \mid y_{k-1} \leq f(x) < y_k\})$$

where  $\xi_i \in [y_{k-1}, y_k]$  and  $P$  is a partition on the ordinate over interval  $[0, \max_x(f(x))]$ .

This concept, which is considered as a generalization of Riemann integral, have a formal definition by using simple function in sec 2.1.9.3 that we discover right after.

### 1.9.2.3 Example

Riemann integral is invented at the mid-1800s and Lebesgue is invented at 1902 and it has several advantages compared to Riemann integral. The classical one is the following example. Let  $\mathbb{1}_{\mathbb{Q}}(x)$  be indicator function, which is equal to 1 if  $x \in \mathbb{Q}$ , 0 otherwise. Then it is shown that over interval  $[0, 1]$ ,  $\mathbb{1}_{\mathbb{Q}}(x)$  is not Riemann integrable, but we still calculate integral of this function by using Lebesgue integral. The idea is simple, we can write the integral by

$$1 \times l_{[0,1]}(\mathbb{Q}) + 0 \times l_{[0,1]}(\mathbb{R}/\mathbb{Q})$$

where  $l_{[0,1]}$  is Lebesgue measure on measurable space  $([0, 1], \mathcal{B}([0, 1]))$  or  $l$  means the length of interval. We see that in sec 2.1.7.3.3, Lebesgue measure of countable set  $\mathbb{Q}$  is 0 or  $l_{[0,1]}(\mathbb{Q}) = 0$ . Then by the countable additivity,  $l_{[0,1]}(\mathbb{R}/\mathbb{Q}) = 1$ .

### 1.9.3 Simple function

Given measure space  $(X, \Sigma, \mu)$  and non-negativity measurable function  $f$  is a non-negativity measurable function from  $(X, \Sigma)$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ .

An **indicator function**  $\mathbb{1}_A(x)$ , is defined equal to 1 if  $x \in A$  and 0 otherwise, where  $A \in \Sigma$ .

Then a **simple function**  $s$  is a finite linear combination of indicator functions of  $\mu$ -measurable sets  $A_k$  :

$$s(x) = \sum_{k=1}^{\infty} a_k \mathbb{1}_{A_k}(x),$$

where sets  $A_k$  are pairwise disjoint.

Given a simple function  $s(x)$ , among of its all possible decompositions (into indicator functions), it exists only one called canonical decomposition such that

- $a_k$  are pairwise disjoint and non null.
- $s(x) = 0, \forall x$  if and only if  $n = 0$ .

### 1.9.4 Definition of Lebesgue integral

Given a measure space  $(X, \Sigma, \mu)$  and a non-negative measurable function  $f$  from  $(X, \Sigma)$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , we define

$$\int_X f d\mu = \sup_s \left\{ \int_X s d\mu : 0 \leq s(x) \leq f(x) \text{ for all } x \in X, s \text{ is simple function} \right\}$$

It is proved that Lebesgue integral give always the same result by Riemann integral.

#### 1.9.4.1 Integral of simple function

To calculate  $\int_X s d\mu$ , we use the **canonical decomposition** of  $s$ :

$$\begin{aligned}\int_X s d\mu &= \int_X \sum_{k=1}^{\infty} a_k \mathbb{1}_{A_k} d\mu \\ &= \sum_{k=1}^{\infty} a_k \int_X \mathbb{1}_{A_k} d\mu\end{aligned}$$

To assign a value to the integral of the indicator function  $\mathbb{1}_A$  of a measurable set  $A$  consistent with the given measure  $\mu$ , the only reasonable choice is to set:

$$\int_X \mathbb{1}_A d\mu = \mu(A)$$

Finally,

$$\int_X s d\mu = \sum_{k=1}^{\infty} a_k \mu(A_k)$$

#### 1.9.4.2 Emphasizing

Note that, to emphasize that  $f$  and  $\mu$  are dependent on  $x \in X$ , we can write

$$\int_X f d\mu = \int_X f(x) \mu(dx) = \int_X f(x) d\mu(x)$$

In short  $d\mu$ ,  $\mu(dx)$  or  $d\mu(x)$  means the measure of infinitesimal set around  $x$ .

#### 1.9.4.3 In case of random variable

This is a particular case, where the measure space is probability space, the measurable function  $f$  is a random variable  $X$  and the measure  $\mu$  is probability measure  $P$ .

$$\begin{aligned}\int f d\mu &= \int X(\omega) dP(\omega) \\ &= \int X(\omega) p(\omega) d\omega \\ &= \mathbb{E}[X]\end{aligned}$$

where  $p$  is density function. **Do not confuse** the measure  $P$  with density function  $p$ . For example, if  $X \sim \mathcal{N}(0, 1)$ ,  $P(a \leq X \leq b) = \int_a^b p(\omega) d\omega$ .

### 1.9.5 When Lebesgue integral is defined and integrable

To handle any measurable function  $f$  that take value in  $\mathbb{R}$ , we use the concept of positive par and negative part :

$$f = f^+ - f^-$$

where  $f^+$  and  $f^-$  are positive part and negative part of  $f$ , they are always non-negativity. Moreover, since  $f$  is measurable function then both  $f^+$  and  $f^-$  are measurable functions and we can apply Lebesgue integral on them. Then

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

We say that the  $\int f d\mu$  is defined if and only if

$$\min(\int f^+ d\mu, \int f^- d\mu) < \infty$$

Note that if  $\int f d\mu = \pm\infty$ , it is also definded. The state of  $\int f d\mu$  is given by :

$\int f^- d\mu < \infty$	$\int f^+ d\mu < \infty$	$\int f^+ d\mu = \infty$
$\int f^- d\mu = \infty$	finite	$+\infty$

The measurable function  $f$  is said **Lebesgue integrable** if and only if

$$\int |f| d\mu < +\infty$$

#### 1.9.5.1 In case of random variable

We have seen that in sec 2.1.9.4.3,  $\int f d\mu$  is  $\mathbb{E}[X]$ , where  $X = f$ . We say that the  $\mathbb{E}[X]$  is defined if and only if

$$\min(\mathbb{E}[X^+], \mathbb{E}[X^-]) < \infty$$

and  $X$  is said integrable if and only if

- In continuous case :

$$\mathbb{E}[|X|] = \int |x| p(x) dx < +\infty$$

where  $p$  here is density function.

- In discret case :

$$\mathbb{E}[|X|] = \sum |x| P(X = x) < +\infty$$

### 1.9.6 Properties of Lebesgue integral

As Riemann integral, Lebesgue integral has the same properties for *linearity* and *monotonicity*

- If  $f$  and  $g$  are Lebesgue integrable functions, then  $af + bg$  is Lebesgue integrable and

$$\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu$$

- If  $f \leq g$ , then

$$\int f d\mu \leq \int g d\mu$$

## 1.10 Independence

Some time called independence  $\sigma$ -algebra. Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space, which means  $\mu(E)$  exists,  $\forall E \in \mathcal{F}$ .

Two events  $A, B \in \mathcal{F}$  are said independent, noted  $A \perp B$ , if

$$\mu(A \cap B) = \mu(A)\mu(B)$$

In general, the events  $A_1, \dots, A_n \in \mathcal{F}$  are independent, if :

$$\mu(A_1 \cap \dots \cap A_n) = \mu(A_1) \dots \mu(A_n)$$

### 1.10.1 Illustration example

#### 1.10.1.1 Non independence case

Suppose that we have one dice with probability space  $(\omega, \mathcal{F}, P)$  and two random variable  $A$  and  $B$  :

$\omega$	1	2	3	4	5	6
$A(\omega)$	0	-1	0	-1	0	-1
$B(\omega)$	0	-1	-1	0	0	-1

The sample space  $\Omega = \{1, 2, 3, 4, 5, 6\}$  and let's take  $\mathcal{F} = \sigma(\Omega)$ .

- $\{\omega | A(\omega) = -1\} = \{2, 4, 6\}$  then  $P(A = -1) = \frac{1}{2}$
- $\{\omega | B(\omega) = -1\} = \{2, 3, 6\}$  then  $P(B = -1) = \frac{1}{2}$
- $\{\omega | A(\omega) = -1 \cap B(\omega) = -1\} = \{2, 6\}$  then  $P(A = -1 \cap B = -1) = \frac{1}{3}$

Then  $P(A = -1)P(B = -1) = \frac{1}{4} \neq P(A = -1 \cap B = -1) = \frac{1}{3}$

$\omega_1$	1	2	3	4	5	6
$A(\omega_1)$	0	-1	0	-1	0	-1
$\omega_2$	a	b	c	d	e	f
$B(\omega_2)$	0	-1	-1	0	0	-1

#### 1.10.1.2 Independence case

Now suppose that we have two dices, one for  $A$  and the other for  $B$ . In order to distinguish better between two dices, we label six faces of one dice by  $a, b, c, d, e, f$ . Let's define two random variables  $A$  and  $B$  by :

The sample space  $\Omega = \{1, 2, 3, 4, 5, 6\} \times \{a, b, c, d, e, f\} = \{(1, a), \dots, (6, f)\}$  and let's take  $\mathcal{F} = \sigma(\Omega)$ . In this case each outcome, e.g.,  $(2, C)$  has probability  $\frac{1}{36}$ .

- $\{\omega | A(\omega) = -1\} = \{(2, .), (4, .), (6, .)\}$  then  $P(A = -1) = \frac{1}{2}$
- $\{\omega | B(\omega) = -1\} = \{(. , b), (. , c), (. , f)\}$  then  $P(B = -1) = \frac{1}{2}$
- $\{\omega | A(\omega) = -1 \cap B(\omega) = -1\} = \{2, 4, 6\} \times \{b, c, f\}$  then  $P(A = -1 \cap B = -1) = \frac{9}{36} = \frac{1}{4}$

Then  $P(A = -1)P(B = -1) = \frac{1}{4} = P(A = -1 \cap B = -1) = \frac{1}{4}$ . Then we can show that  $P(A)P(B) = P(AB)$ .

#### 1.10.2 Between sigma-algebra

Sub  $\sigma$ -algebra  $\mathcal{A}_1, \dots, \mathcal{A}_n$  of  $\mathcal{F}$  are independent if :

Events  $A_1, A_2, \dots, A_n$  are independent,  $\forall A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2, \dots, A_n \in \mathcal{A}_n$

Note that, if we say a random variable  $X$  is independent with sub  $\sigma$ -algebra  $\mathcal{H}$ , it means that  $\sigma(X)$  and  $\mathcal{H}$  are independent. Example, if we toss a coin twice,  $\mathcal{H}$  is the  $\sigma$ -algebra generated by the outcomes of the first toss and  $Y$  is random variable representing the outcome of the second toss.

#### 1.10.3 Independence of $\pi$ -systems implies independence of $\sigma$ -algebras

If  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$  are  $\pi$ -system are independent if and only if  $\sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2), \dots, \sigma(\mathcal{A}_n)$  are independent. (See here for demonstration [http://theanalysisofdata.com/probability/E\\_4.html](http://theanalysisofdata.com/probability/E_4.html))

#### 1.10.4 Conditional independence

Let  $A, B, C$  be events.  $A$  and  $B$  are said to be conditionally independent given  $C$  if and only if  $P(C) > 0$  and :

$$P(A | B, C) = P(A | C)$$

This property is often written:  $A \perp\!\!\!\perp B | C$  Equivalently, conditional independence may be stated as:

$$P(A, B|C) = P(A|C)P(B|C)$$

It demonstrates that  $(A \perp\!\!\!\perp B | C)$  is equivalent to  $(B \perp\!\!\!\perp A | C)$ .

#### 1.10.5 Uncorrelation and independence

*Uncorrelation does not necessarily mean independence :*

- Uncorrelated variables have a correlation coefficient of zero, which means there is **no linear relationship** between the variables. However, there may still have **a nonlinear relationship** between the variables, or they may be related in some other ways that is not captured by the correlation coefficient.
- Independent variables, on the other hand, are variables that have no relationship with each other, whether linear or nonlinear. So, all independent variables are uncorrelated, but not all uncorrelated variables are independent.

In summary, if two variables are independent, they are uncorrelated, but the inverse is not necessarily true.

#### 1.10.6 Note 1

Let's consider random variable  $X, Y, Z$ . If  $X$  is independent to  $Y$  and  $X$  is independent to  $Z$ , it **do not** imply that  $X$  is independent to  $aY + bZ$ .

**Example** Let  $Y$  and  $Z$  are two Bernoulli(0.5) variables and the variable  $X$  such that :

$Y$	0	1	0	1
$Z$	0	0	1	1
$X$	0	1	1	0

It means that if  $Y = 0$  and  $Z = 0$  then  $X = 0$  ... We see that :

- $P(X = 0|Y = 1) = P(X = 0|Y = 0) = P(X = 0) = 0.5$  (same as  $X = 1$ ), then  $P(X|Y) = P(X)$ , then  $X \perp Y$ .
- $P(X = 0|Z = 1) = P(X = 0|Z = 0) = P(X = 0) = 0.5$  (same as  $X = 1$ ), then  $P(X|Z) = P(X)$ , then  $X \perp Z$ .

But in case  $X = 0$ :

- $P(Y + Z = 0|X = 0) = 0.5 \neq P(Y + Z = 0) = 0.25$
- $P(Y + Z = 2|X = 0) = 0.5 \neq P(Y + Z = 2) = 0.25$
- $P(Y + Z = 1|X = 0) = 0 \neq P(Y + Z = 1) = 0.5$

and in case  $X = 1$

- $P(Y + Z = 0|X = 1) = 0 \neq P(Y + Z = 0) = 0.25$
- $P(Y + Z = 2|X = 1) = 0 \neq P(Y + Z = 2) = 0.25$
- $P(Y + Z = 1|X = 1) = 1 \neq P(Y + Z = 1) = 0.5$

Then  $X$  and  $Y + Z$  are not independent.

#### 1.10.7 Note 2

However, if  $X$  is independent to  $(Y, Z)$ , it **implies** that  $X$  is independent to  $aY + bZ$ .

Proof : Using the property that if  $X \perp (Y, Z)$  then  $f_1(X) \perp f_2(Y, Z), \forall f_1, f_2$ . Then taking  $f_1(X) = X$  and  $f_2(Y, Z) = Y + Z$ .

#### 1.10.8 Note 3

If  $X$  is independent to  $Y$  and  $Z$ , it **does not** imply that  $X$  is independent to  $YZ$ .

##### Example

Let's say we choose a number randomly from:  $\{1, 2, 3, 4, 6, 7, 8, 9\}$ . Let  $X$  be:

- $X = 1$  if (Chosen number is even AND less than five) OR (Chosen number is odd AND greater than 5)
- $X = 0$  otherwise

Let  $Y$  be:

- $Y = 1$  if chosen number is even
- $Y = 0$  otherwise

Let  $Z$  be:

- $Z = 1$  if your chosen number is greater than 5
- 0 otherwise

Now we know :

- $X = 1$  with probability 0.5 (2,4,7,9)
- $X = 0$  with probability 0.5 (1,3,6,8)

If we know :

- $Y = 1$ , then  $X$  is still 1 with  $p = 0.5$  (2,4) and 0 with  $p = 0.5$  (6,8)
- $Y = 0$ , then  $X$  is still 1 with  $p = 0.5$  (7,9) and 0 with  $p = 0.5$  (1,3)

If we know:

- $Z = 1$ , then  $X$  is still 1 with  $p = 0.5$  (7,9) and 0 with  $p = 0.5$  (6,8)
- $Z = 0$ , then  $X$  is still 1 with  $p = 0.5$  (2,4) and 0 with  $p = 0.5$  (1,3)

So  $X$  is independent of  $Y$  and  $X$  is independent of  $Z$ .

But knowing if a number is even AND knowing if it's greater than 5 ( $Y \& Z$ ), makes us know  $X$  with certainty. e.g.  $Y = 1, Z = 1$ , then  $YZ = 1$ ,  $X$  has to be 0 with probability 1 (as  $X$  is 0 if the number is an even number  $> 5$ )

In short :

- $P(X = 0|Y) = P(X = 0) = 0.5$  (same for  $X = 1$ ) then  $X \perp Y$
- $P(X = 0|Z) = P(X = 0) = 0.5$  (same for  $X = 1$ ) then  $X \perp Z$

But  $P(X = 0|YZ = 1) = 1 \neq P(X = 0) = 0.5$  (similar for  $X = 1$ ).

## 2 Useful tools

### 2.1 Notations

Usually used, but not conventionning :

- Capital  $X$  is for random variable.
- Lower case  $x$  is for a particular value of  $X$ .
- Capital  $P$  represents the probability of an event occurring. For example, if we toss a fair coin, the probability of getting heads is denoted by  $P(X = \text{heads}) = 0.5$ , and the probability of getting tails is denoted by  $P(X = \text{tails}) = 0.5$ .
- Lowercase  $p$  represents a probability density function (PDF) or probability mass function (PMF), depending on whether the variable is continuous or discrete. A PDF gives the probability of a continuous random variable taking on a particular value or falling within a certain range of values, while a PMF gives the probability of a discrete random variable taking on a particular value.

Example of using : In discrete case, for a given dice, we can write :

$$P(X = 1) = p(1) = \frac{1}{6}$$

where  $p$  is PMF.

But in continuous case, given  $X \sim \mathcal{N}(0, 1)$ ,

$$P(X = x) \approx P\left(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}\right) = p(x)dx$$

where  $p$  is PDF.

## 2.2 Law of the unconscious statistician

LOTUS, This is a name for a method that one (may) uses to calculate instinctly the expected value but do not know its name. Let  $X$  that one knows its probability distribution and a function  $g(X)$  that one does not know its distribution.

In discret case, let  $P_X$  is the probability mass associated to  $X$ :

$$\mathbb{E}[g(X)] = \sum_x g(x)P_X(X = x)$$

Let  $f_X(x)$  be the density function associated to  $X$ , then in continuous case we have:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$$

If we know the cumulative probability distribution function of  $X$ , means  $F_X$ :

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)dF_X(x)$$

## 2.3 Expectation with exponentiation

Let  $W$  and  $Z$  be non-negative random variables on probability space  $(\Omega, \mathcal{F}, P)$ . Then for any  $r > 0$ ,

$$\mathbb{E}[WZ^r] = r \int_0^{+\infty} t^{r-1} \mathbb{E}[W \mathbb{1}_{\{Z>t\}}] dt$$

**Proof**

$$\begin{aligned} & r \int_0^{+\infty} t^{r-1} \mathbb{E}[W \mathbb{1}_{\{Z>t\}}] dt \\ &= r \int_0^{+\infty} t^{r-1} \int_{Z>t} W dP dt \\ &= \int_0^{+\infty} \int_{Z>t} rt^{r-1} W dP dt \\ &= \int_{\Omega} \int_0^Z rt^{r-1} W dt dP \quad (\text{bound changing}) \\ &= \int_{\Omega} W \int_0^Z rt^{r-1} dt dP \\ &= \int_{\Omega} W Z^r dP \\ &= \mathbb{E}[WZ^r] \end{aligned}$$

## 2.4 Expected value of nonnegative random variable

Given  $X$  is a nonnegative random variable, then

$$\mathbb{E}[X] = \int_0^{+\infty} P(X \geq x) dx$$

**Proof**

Let  $f_X(x)$  be the density function of  $X$ , then

$$\begin{aligned} \int_0^{+\infty} P(X \geq x) dx &= \int_0^{+\infty} \int_x^{+\infty} f_X(z) dz dx \\ &= \int_0^{+\infty} \int_0^z f_X(z) dx dz \\ &= \int_0^{+\infty} f_X(z) \int_0^z dx dz \\ &= \int_0^{+\infty} f_X(z) z dz \\ &= \mathbb{E}[X] \end{aligned}$$

The second equality is that given  $0 \leq x \leq z \leq +\infty$ , if we fix  $x$  first and if we fix  $z$  first, we must have the same thing. The last equality is because  $X$  is nonnegative.

An other proof which is faster by using 2.2.3 with  $W = 1, r = 1$  and note that  $\mathbb{E}[\mathbb{1}_{\{Z>t\}}] = P(Z > t)$ .

### 2.4.1 Discrete case

Given  $X$  is a nonnegative random variable, then

$$\begin{aligned} \mathbb{E}[X] &= \int_0^{+\infty} P(X \geq x) dx &= \sum_{i=0}^{+\infty} \int_i^{i+1} P(X \geq x) dx \\ &\geq \sum_{i=0}^{+\infty} \int_i^{i+1} P(X \geq i+1) dx &= \sum_{i=0}^{+\infty} P(X \geq i+1) \int_i^{i+1} dx \\ &= \sum_{i=0}^{+\infty} P(X \geq i+1) &= \sum_{i=1}^{+\infty} P(X \geq i) \\ &= \sum_{i=0}^{+\infty} P(X > i) \end{aligned}$$

The greater or equal because if  $x$  in interval  $[i, i+1]$ ,  $P(X \geq x) \geq P(X \geq i+1)$ . The equality happens when  $X$  is a **integer valued** random variable.

### 2.4.2 More general case

Given  $X$  is a nonnegative random variable and  $a \geq 0$ , then

$$\mathbb{E}[(X - a)^+] = \int_a^{+\infty} P(X \geq x) dx$$

**Proof**

By using 2.2.3 with  $W = \mathbb{1}_{\{X \geq a\}}$ ,  $Z = X - a$ ,  $r = 1$  :

$$\begin{aligned}\mathbb{E}[(X - a)^+] &= \mathbb{E}[(X - a)\mathbb{1}_{\{X \geq a\}}] \\ &\leq \int_0^{+\infty} \mathbb{E}[\mathbb{1}_{\{X \geq a\}}\mathbb{1}_{\{X-a \geq x\}}] dx \\ &= \int_0^{+\infty} \mathbb{E}[\mathbb{1}_{\{X-a \geq x\}}] dx \\ &= \int_0^{+\infty} \mathbb{E}[\mathbb{1}_{\{X \geq a+x\}}] dx \\ &= \int_a^{+\infty} \mathbb{E}[\mathbb{1}_{\{X \geq x\}}] dx\end{aligned}$$

where the last equality is by variable change from  $x$  to  $x + a$ .

## 2.5 Wald's equation

### 2.5.1 Wald's First Identity

Let  $(X_n)_{n \in \mathcal{N}^*}$  be a sequence of real-valued, independent and identically distributed random variable,  $S_n = \sum_{i=1}^n X_n$  and  $T \leq 0$  be an integer-valued random variable. Then

$$\mathbb{E}[S_T] = \mathbb{E}[X_1]\mathbb{E}[T]$$

### 2.5.2 Wald's Second Identity

In addtion to Wald's First Identity, if  $\mathbb{E}[X_1] = 0$  and  $Var[X_1] = \sigma^2$ , then

$$\mathbb{E}[S_T^2] = \sigma^2\mathbb{E}[T]$$

## 2.6 Equality by expectation

Given  $X$  and  $Y$  two random variables in probability space  $(\Omega, \mathcal{F}, P)$ , if

$$\mathbb{E}[X.\mathbb{1}_A] = \mathbb{E}[Y.\mathbb{1}_A], \forall A \in \mathcal{F}$$

Then  $X = Y$  almost surely, which means  $P(A = B) = 1$

**Proof**

Let  $B = \omega: X > Y$ , then  $B \in \mathcal{F}$  and by the hypothese, we must have  $\mathbb{E}[(X - Y).\mathbb{1}_B] = 0$ . Note that  $X - Y$  is a strictly positive, then  $P(B) = 0$ . By analogy with  $C = \omega: X < Y$ , we must have  $P(C) = 0$ . Finally  $P(X = Y) = 1$ .

## 2.7 Layer cake representation

Given a measure space  $(\Omega, \mathcal{F}, \mu)$  and function  $f : \Omega \rightarrow \mathbb{R}^+$  which is measurable (2.1.5), from  $(\Omega, \mathcal{F})$  to  $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ . Then we have :

$$f(x) = \int_0^\infty \mathbb{1}_{L(f,t)}(x) dt \quad , \forall x \in \Omega$$

where  $L(f,t)$  denotes the super-level set

$$L(f,t) = \{\omega \in \Omega \mid f(\omega) \geq t\}$$

We call layer cake representation because when  $t$  increases, we have a new level layer.

Before proof, we consider the following lemma

### 2.7.1 Lemma

$$\mathbb{1}_{L(f,t)}(x) = \mathbb{1}_{[0,f(x)]}(t) \quad , \forall x \in \Omega$$

Thus :

- If  $x \in L(f,t)$ , it means  $f(x) \geq t$ , then  $\mathbb{1}_{L(f,t)}(x) = \mathbb{1}_{[0,f(x)]}(t) = 1$ .
- If  $x \notin L(f,t)$ , it means  $f(x) < t$ , then  $\mathbb{1}_{L(f,t)}(x) = \mathbb{1}_{[0,f(x)]}(t) = 0$

### 2.7.2 Proof

We use the above lemma, then :

$$\begin{aligned} \int_0^\infty \mathbb{1}_{L(f,t)}(x) dt &= \int_0^\infty \mathbb{1}_{[0,f(x)]}(t) dt \\ &= \int_0^{f(x)} dt \\ &= f(x) \end{aligned}$$

## 2.8 Inequalities on probability

### 2.8.1 Markov

Given  $X$  a non-negative random variable and  $a > 0$ :

$$P(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

### 2.8.2 Chebyshev

Given  $X$  a random variable admits a second moment, then  $\forall \varepsilon > 0$ :

$$P(|X - \mathbb{E}[X]| > \varepsilon) \leq \frac{\text{Var}(X)}{\varepsilon^2}$$

### 2.8.3 Jensen

Given  $X$  a real-valued random variable and  $\phi$  a convex function.

$$\phi(\mathbb{E}[X]) \leq \mathbb{E}[\phi(X)]$$

#### 2.8.3.1 Conditional Jensen's Inequality

Given  $X$  an real-value integrable random variable and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex function such that  $f(X)$  is also integrable, then

$$f(\mathbb{E}[X | \mathcal{G}]) \leq \mathbb{E}[f(X) | \mathcal{G}]$$

Note that, the inequality changes direction if  $f$  is concave. This is trivially proved by using  $-f$ , which is convex function.

### 2.8.4 Product with indicator function

If  $X$  and  $Y$  two random variables such that  $|X| < |Y|$ , then

$$|X| \cdot \mathbb{1}_{\{|X| > K\}} \leq |Y| \cdot \mathbb{1}_{\{|Y| > K\}}$$

since  $\{|X| > K\} \subseteq \{|Y| > K\}$ . Thus,  $\forall \omega, |X(\omega)| > K \rightarrow |Y(\omega)| > K$ .

### 2.8.5 Absolute sum decomposition

Let  $X$  and  $Y$  two random variables, then

$$|X + Y| \cdot \mathbb{1}_{\{|X+Y| \geq 2K\}} \leq 2|X| \cdot \mathbb{1}_{\{|X| \geq K\}} + 2|Y| \cdot \mathbb{1}_{\{|Y| \geq K\}}$$

#### Proof

- By inequality 2.2.8.4, We have :

$$|X + Y| \cdot \mathbb{1}_{\{|X+Y| \geq 2K\}} \leq (|X| + |Y|) \cdot \mathbb{1}_{\{|X| + |Y| \geq 2K\}}$$

- Now, for all  $\omega$  such that  $|X(\omega)| \leq |Y(\omega)|$ , then  $|X| + |Y| \leq 2|Y|$ , by inequality 2.2.8.4 :

$$(|X| + |Y|) \cdot \mathbb{1}_{\{|X| + |Y| \geq 2K\}} \leq 2|Y| \cdot \mathbb{1}_{\{2|Y| \geq 2K\}} = 2|Y| \cdot \mathbb{1}_{\{|Y| \geq K\}}$$

- By analogy, for all  $\omega$  such that  $|Y(\omega)| \geq |X(\omega)|$  then we have

$$(|X| + |Y|) \cdot \mathbb{1}_{\{|X| + |Y| \geq 2K\}} \leq 2|X| \cdot \mathbb{1}_{\{|X| \geq K\}}$$

Finally, for all  $\omega$ , we have :

$$(|X + Y|) \cdot \mathbb{1}_{\{|X+Y| \geq 2K\}} \leq 2|X| \cdot \mathbb{1}_{\{|X| \geq K\}} + 2|Y| \cdot \mathbb{1}_{\{|Y| \geq K\}}$$

### 2.8.6 Holder

Given  $p, q \in (1, \infty)$ , with  $1/p + 1/q = 1$ .

$$\mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{\frac{1}{p}} \mathbb{E}[|Y|^q]^{\frac{1}{q}}$$

In some case, Jensen inequality is used with  $X = |X|^r, Y = 1$  and  $p = \frac{s}{r}$ , with  $0 < r < s$  :

$$\mathbb{E}[|X|^r] \leq E[|X|^s]^{\frac{r}{s}}$$

### 2.8.7 Hoeffding's lemma

Let  $X$  be any real-valued random variable such that  $a \leq X \leq b$  almost surely. Then, for all  $s \in \mathbb{R}$  :

$$\mathbb{E} \left[ e^{s(X - \mathbb{E}[X])} \right] \leq \exp \left( \frac{s^2(b-a)^2}{8} \right)$$

### 2.8.8 Boole's inequality

Also called *union bound* for **countable sets**  $A_1, A_2, \dots$  we have

$$P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$$

Moreover, we have Bonferroni inequalities :

$$P(\bigcup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} P(A_i) - \sum_{i < j} P(A_i \cap A_j)$$

$$P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i) - \sum_{i < j} P(A_i \cap A_j) + \sum_{i < j < k} P(A_i \cap A_j \cap A_k)$$

and ect ...

### 2.8.9 Expected value with negative and positive part

Let's remind the definition for the negative and positive part of number  $x$  :

$$\begin{aligned} x^- &= -\min(x, 0) \\ x^+ &= \max(x, 0) \end{aligned}$$

Then given a random variable  $X$ , we have

$$\mathbb{E}[(X - a)^-] - \mathbb{E}[X^-] \leq a^+ \quad \forall a \in \mathbb{R}$$

**Proof**

$$\begin{aligned}
& \mathbb{E}[(X - a)^-] - \mathbb{E}[X^-] \\
&= \int_{-\infty}^a (a - x)p(x)dx + \int_a^{+\infty} 0p(x)dx \\
&\quad - \int_{-\infty}^0 -xp(x)dx - \int_0^{+\infty} 0p(x)dx \\
&= \int_{-\infty}^a ap(x)dx + \begin{cases} \int_0^a -xp(x)dx & \text{if } a \geq 0 \\ -\int_a^0 xp(x)dx & \text{if } a < 0 \end{cases} \\
&\leq \int_{-\infty}^a ap(x)dx \\
&\leq \max(a, 0) \\
&= a^+
\end{aligned}$$

In the same way, we have

$$\mathbb{E}[(X - a)^+] - \mathbb{E}[X^+] \leq a^- \quad \forall a \in \mathbb{R}$$

**Proof**

Let's remind that  $x = x^+ - x^-$ , we have :

$$\begin{aligned}
\mathbb{E}[(X - a)] &= \mathbb{E}[(X - a)^+] - \mathbb{E}[(X - a)^-] \\
\mathbb{E}[X] &= \mathbb{E}[X^+] - \mathbb{E}[X^-]
\end{aligned}$$

Then

$$\begin{aligned}
& \mathbb{E}[(X - a)^+] - \mathbb{E}[X^+] \\
&= \mathbb{E}[(X - a)] + \mathbb{E}[(X - a)^-] - \mathbb{E}[X] - \mathbb{E}[X^-] \\
&= \mathbb{E}[(X - a)^-] - \mathbb{E}[X^-] - a \\
&\leq a^+ - a \\
&= a^-
\end{aligned}$$

Finally, note that  $x^- = (-x)^+$  and  $x^+ = (-x)^-$ , then we can write :

$$\mathbb{E}[(a - X)^+] - \mathbb{E}[(-X)^+] \leq a^+ \quad \forall a \in \mathbb{R}$$

and

$$\mathbb{E}[(a - X)^-] - \mathbb{E}[(-X)^-] \leq a^- \quad \forall a \in \mathbb{R}$$

## 2.8.10 Divers

### 2.8.10.1 Inequality 1

Given random variable  $X$  and two functions  $f_1, f_2$ , with  $f_1(X) \leq f_2(X)$ , then :

$$P_X(f_1(X) \geq a) \leq P_X(f_2(X) \geq a)$$

### 2.8.10.2 Inequality 2

Given probability space  $(\Omega, \mathcal{F}, P)$ , then for all  $A \in \mathcal{F}$ , we have :

$$X \mathbb{1}_A \leq \max(X, 0)$$

#### Proof

Given  $\omega \in \Omega$ , then

- $\omega \in A$ , then we have  $X \leq \max(X, 0)$
- $\omega \notin A$ , then we have  $0 \leq \max(X, 0)$

## 2.9 Bounded variation

A function of bounded variation, also known as **BV function**, is a real-valued function whose total variation is bounded (finite).

### 2.9.1 Total variation

The total variation of a real-valued (or complex-valued) function  $f$ , defined on an interval  $[a, b] \subset \mathbb{R}$  is the quantity

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P-1} |f(x_{i+1}) - f(x_i)|$$

where the supremum runs over the set of all partitions  $\mathcal{P}$  where

$$\mathcal{P} = \{P = \{x_0, \dots, x_{n_P}\} \mid \text{of } [a, b]\}$$

## 2.10 Derivability and differentiability

- A function  $f$  is said **derivable** at  $x_0$  if there exists

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- A function  $f$  is said **differentiable** at  $x_0$  if there exists  $a$  such that

$$f(x_0 + \varepsilon) \approx f(x_0) + a\varepsilon, \quad \forall |\varepsilon| \ll 1$$

### 2.10.0.1 Properties

- A function is differentiable then it is derivable but the converse is not true, e.g.

$$f(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ x - 1 & \text{if } x < 0 \end{cases}$$

This function is derivable at  $x_0 = 0$ . But it is not differentiable since it is not continuous.

- A function  $f$  is differentiable but its derivative is does not have to be continuous, e.g.

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{otherwise} \end{cases}$$

is differentiable, but its derivative

$$f'(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{otherwise} \end{cases}$$

is not continuous at  $x = 0$ .

- A function  $f$  is differentiable but its derivative does not have to be Riemann integrable, e.g.

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin\left(\frac{1}{x^2}\right) & \text{otherwise} \end{cases}$$

is differentiable on  $[-1, 1]$  but its derivative is unbounded on  $[-1, 1]$ , then it is not Riemann integrable.

## 2.11 Radon–Nikodym theorem

The Radon–Nikodym theorem is a result in measure theory that shows under certain conditions, **there exists a relationship between two measures** defined on the same measurable space. Before going to a formal description, let's remind the definition for (ordinary) continuity, uniform continuity, absolute continuity for a function and absolute continuity for measure.

### 2.11.1 Ordinary continuity, uniform continuity and absolute continuity

For a function  $f: X \rightarrow Y$  with metric spaces  $(X, d_1)$  and  $(Y, d_2)$  :

- The function  $f$  is said to be **continuous** if  $\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in X : d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$ .
- The function  $f$  is said to be **uniformly continuous** if  $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X, \forall y \in X : d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$ .

At a quick look, these two definitions seem to be the same. However, let's focus on the position of  $(\forall x \in X)$ , which make the difference between these two concepts. In short, in the definition of continuity, we first fix  $x$  then  $\varepsilon$ , but in the definition of uniform continuity, we first fix  $\varepsilon$  then  $x$ .

### 2.11.1.1 Example

Given two metric space  $(\mathbb{R}, |.|)$  and  $(\mathbb{R}, |.|)$ , where  $|.|$  is for  $L_1$  distance. The function  $f(x) = \frac{1}{x}$  on  $(0, 1]$  is continuous but not uniformly continuous. This is because we fix first  $\varepsilon$ , then suppose that exists  $\delta$  such that for all  $x, y$  if  $|x - y| < \delta$  then  $\left| \frac{1}{x} - \frac{1}{y} \right| < \varepsilon$ . The latter is not true since :

$$\left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x - y|}{|xy|} > +\infty$$

if  $y = x + \frac{\delta}{2}$  and  $x, y \rightarrow 0^+$ .

By this example, we see that continuous functions can *fail* to be uniformly continuous if they are unbounded on a closed interval.

### 2.11.1.2 Absolute continuity

For a function  $f: X \rightarrow Y$  with metric spaces  $(X, d_1)$  and  $(Y, d_2)$ . The function  $f$  is **absolutely continuous** if  $\forall \varepsilon > 0$ ,  $\exists \delta > 0$  such that whenever a finite sequence of pairwise disjoint sub-intervals  $(x_k, y_k)$  in  $X$  satisfies

$$\sum_k d_1(y_k, x_k) < \delta$$

then

$$\sum_k d_2(f(y_k), f(x_k)) < \varepsilon$$

#### Equivalent definitions

Over interval  $[a, b]$ , the following statement are equivalent :

1.  $f$  is absolutely continuous.
2.  $f$  has a derivative  $f'$  almost everywhere (analogous to the notion of almost surely in probability theory) and the derivative  $f'$  is Lebesgue integrable (sec 2.1.9.5).
3. There exists a Lebesgue integrable function  $g$  on  $[a, b]$  such that

$$f(x) = f(a) + \int_a^x g(t) dt$$

for all  $x \in [a, b]$ .

### 2.11.2 Absolute continuity of measure

Given :

- A measure  $\mu$  (sec 2.1.7) on measurable space  $(\mathbb{R}, B(\mathbb{R}))$ , where  $B(\mathbb{R})$  denotes Borel algebra (in sec 2.1.1.4) on  $\mathbb{R}$ .

- A Lebesgue measure  $\lambda$  (in sec 2.1.7.3.3) on the same measurable space  $(\mathbb{R}, B(\mathbb{R}))$ .

Then the measure  $\mu$  is said absolutely continuous with respect to the Lebesgue measure  $\lambda$  or absolutely continuous (in short) if  $\mu$  is dominated by  $\lambda$  ( $\mu \ll \lambda$ ), which mean :

$$\lambda(E) = 0 \Rightarrow \mu(E) = 0, \quad \forall \lambda\text{-measurable } E \in B(\mathbb{R})$$

equivalently,

$$\mu(E) > 0 \Rightarrow \lambda(E) > 0, \quad \forall \lambda\text{-measurable } E \in B(\mathbb{R})$$

The same principle holds for measures on Borel subsets of  $\mathbb{R}^n, n \geq 2$ .

#### 2.11.2.1 Equivalent definitions

The following statements are equivalent :

1.  $\mu$  is absolutely continuous.
2.  $\forall \varepsilon > 0, \exists \delta > 0$  such that for all  $\lambda$ -measurable  $E$  if  $\lambda(E) < \delta$ , then

$$\mu(E) < \varepsilon$$

3. There exists a Lebesgue integrable function  $g$  on the real line such that

$$\mu(E) = \int_E g d\lambda$$

#### 2.11.3 Illustrative case

Given a measure  $\mu$  on a measurable space  $(X, \Sigma)$  and a density function  $f$ , then for each  $A \in \Sigma$ ,  $\nu(A)$  is defined by Lebesgue integral (sec 2.1.9) of  $f$  over  $A$  with measure  $\mu$ ,

$$\nu(A) = \int_A f d\mu,$$

is also a measure on  $(X, \Sigma)$ .

##### Example 1

If  $f$  represents the mass density and  $\mu$  is the Lebesgue measure in three-dimensional space  $\mathbb{R}^3$ , which means volume. Then  $\mu(A)$  is the volume of  $A$  and  $\nu(A)$  is a new measure, which is thus the mass of  $A$ .

##### Example 2

If  $f$  represents the probability density function and  $\mu$  is the Lebesgue measure in one-dimensional space  $\mathbb{R}$ , which means length. Then  $\mu(A)$  is the length of  $A$  and  $\nu(A)$  is a new measure, which is the probability that  $A$  happens.

#### 2.11.4 Statement

The Radon–Nikodym theorem involves a measurable space  $(X, \Sigma)$  on which two  $\sigma$ -finite measures (sec 2.1.7.1.2) are defined,  $\mu$  and  $\nu$ . It states that, if  $\nu \ll \mu$  (that is, if  $\nu$  is absolutely continuous with respect to  $\mu$ , sec 2.2.11.2), **then there exists** a  $\Sigma$ -measurable function  $f: X \rightarrow [0, \infty)$ , such that for any measurable set  $A \in \Sigma$ :

$$\nu(A) = \int_A f d\mu,$$

- The function  $f$  is  $\mu$ -almost everywhere unique, that is, if  $g$  is another function which satisfies the same property, then  $f = g$   $\mu$ -almost everywhere (in sec 2.1.7.1.3).
- The function  $f$  is commonly written  $\frac{d\nu}{d\mu}$  and is called the **Radon–Nikodym derivative**.

##### 2.11.4.1 Extension to signed or complex measures

The Radon–Nikodym theorem is still true if  $\mu$  is a nonnegative  $\sigma$ -finite measure, and  $\nu$  is a finite signed measure (sec 2.1.7.1.1).

##### 2.11.4.2 Corollary

If  $f$  and  $g$  are both  $\mathcal{H}$ -measurable, where  $\mathcal{H} \subseteq \Sigma$  and

$$\int_A f d\mu = \int_A g d\mu,$$

then  $f = g$   $\mu$ -almost everywhere.

If  $g$  and  $f$  are not both  $\mathcal{H}$ -measurable, a counter example is the Conditional expectation with respect to a sub sigma algebra (in sec 2.2.12)

## 2.12 Conditional expectation with respect to a (sub) $\sigma$ -algebra

Consider the following:

- $(\Omega, \mathcal{F}, P)$  is a probability space.
- $X: \Omega \rightarrow \mathbb{R}^n$  is a random variable on that probability space with **defined** expectation (sec 2.1.9.5.1).
- $\mathcal{H} \subseteq \mathcal{F}$ , which means  $\mathcal{H}$  is a sub  $\sigma$ -algebra of  $\mathcal{F}$ .

A conditional expectation of  $X$  given  $\mathcal{H}$ , denoted as  $\mathbb{E}[X | \mathcal{H}]$ , is any  $\mathcal{H}$ -measurable function or random variable  $H: \Omega \rightarrow \mathbb{R}^n$  which satisfies:

$$\int_A \mathbb{E}[X | \mathcal{H}] dP = \int_A H dP = \int_A X dP$$

for all events  $A \in \mathcal{H}$ . This implies

$$\mathbb{E}[X \mid \mathcal{H}] = \frac{\int_A X dP}{\int_A dP} = \frac{\int_A X dP}{P(A)}$$

since  $\mathbb{E}[X \mid \mathcal{H}]$  is a constant with given  $A$ .

### 2.12.1 Proof for existence

We can prove that  $\mathbb{E}[X \mid \mathcal{H}]$  is always existing. Since  $X$  has a defined expectation then  $\mathbb{E}[|X|] < \infty$  or  $[E] = \pm\infty$ . In case that  $\mathbb{E}[X] = \pm\infty$ , we just take  $\mathbb{E}[X \mid \mathcal{H}](A) = \pm\infty$  if  $\int_A X dP = \pm\infty$ . In case that  $\mathbb{E}[|X|] < \infty$ , or  $X$  has finite expectation, we show in the following the existence of  $\mathbb{E}[X \mid \mathcal{H}]$ .

Thus, let's  $\mu^X$  is defined by :

$$\mu^X(A) = \int_A X dP, \forall A \in \mathcal{F}$$

- Since  $X$  is not non-negativity then we can only say that  $\mu^X$  is signed measure (sec 2.1.7.1.1) and from that  $X$  has finite expectation then  $\mu^X$  is **finite signed measure**.
- If  $P(A) = 0$  then  $P(A_i) = 0$  for all  $\bigcup_{i=0}^{\infty} A_i = A$ , then  $\mu^X(A) = 0$  or  $\mu^X$  is absolutely continuous with respect to  $P$  or  $\mu^X \ll P$ .

Let  $P|_{\mathcal{H}}$  and  $\mu^X|_{\mathcal{H}}$  be respectively measure  $P$  and  $\mu^X$  restricted to  $\mathcal{H}$ , which is thus simply means  $P|_{\mathcal{H}}$  and  $\mu^X|_{\mathcal{H}}$  are measure on  $(\omega, \mathcal{H})$ . Since  $\mu^X \ll P$  in  $\mathcal{F}$  and  $\mathcal{H} \subseteq \mathcal{F}$  then  $\mu^X|_{\mathcal{H}} \ll P|_{\mathcal{H}}$ . By Radon–Nikodym theorem (sec 2.2.11), there exists

$$\mathbb{E}(X \mid \mathcal{H}) = \frac{d\mu^X|_{\mathcal{H}}}{dP|_{\mathcal{H}}}$$

### 2.12.2 Important note

- Do not confuse  $\mathbb{E}[X \mid \mathcal{H}]$  with  $X$ . Note that
  - $X$  is  $\mathcal{F}$ -measurable (2.1.5.3) and **is not**  $\mathcal{H}$ -measurable.
  - $\mathbb{E}[X \mid \mathcal{H}]$  is  $\mathcal{H}$ -measurable then it **is also**  $\mathcal{F}$ -measurable.
- For  $\mathbb{E}[X \mid \mathcal{H}]$ , its corresponding probability space is  $(\omega, \mathcal{H}, P|_{\mathcal{H}})$

### 2.12.3 Example

Consider the a probability space  $(\Omega, \mathcal{F}, P)$  related to an unfair dice and a random variable  $X$  on this space :

$\Omega$	$a$	$b$	$c$	$d$	$e$	$f$
$P(\omega)$	0.2	0.1	0.3	0.05	0.15	0.2
$X(\omega)$	1	2	3	4	5	6

Given the sub  $\sigma$ -algebra (of  $\mathcal{F}$ )  $\mathcal{H} = \{\Omega, \{a, b\}, \{c, d\}, \{e, f\}, \{c, d, e, f\}, \{a, b, e, f\}, \{a, b, c, d\}, \emptyset\}$ . Then  $\mathbb{E}[X | \mathcal{H}]$  is with probability space  $(\omega, \mathcal{H}, P|_{\mathcal{H}})$ .

- Since  $\mathbb{E}[X | \mathcal{H}]$  is also  $\mathcal{F}$ -measurable, then  $\mathbb{E}[X | \mathcal{H}](a)$  exists. We have  $\mathbb{E}[X | \mathcal{H}](a) = \mathbb{E}[X | \mathcal{H}](b) = \mathbb{E}[X | \mathcal{H}](\{a, b\})$  since  $\{a, b\}$  is the most primary element that contains  $a$ . Using the definition of conditional expectation in 2.2.12 :

$$\begin{aligned}\int_{\{a,b\}} \mathbb{E}[X | \mathcal{H}](a) dP &= \int_{\{a,b\}} X dP \\ \Rightarrow \mathbb{E}[X | \mathcal{H}](a)(0.2 + 0.1) &= (1 \times 0.2 + 2 \times 0.1) \\ \Rightarrow \mathbb{E}[X | \mathcal{H}](a) &= \frac{4}{3}\end{aligned}$$

- In analogy,  $\mathbb{E}[X | \mathcal{H}](c) = \mathbb{E}[X | \mathcal{H}](d)$  is inferred by :

$$\begin{aligned}\int_{\{c,d\}} \mathbb{E}[X | \mathcal{H}](c) dP &= \int_{\{c,d\}} X dP \\ \Rightarrow \mathbb{E}[X | \mathcal{H}](c)(0.3 + 0.05) &= (3 \times 0.3 + 4 \times 0.05) \\ \Rightarrow \mathbb{E}[X | \mathcal{H}](c) &= \frac{110}{35}\end{aligned}$$

- And  $\mathbb{E}[X | \mathcal{H}](e) = \mathbb{E}[X | \mathcal{H}](f)$  is inferred by :

$$\begin{aligned}\int_{\{e,f\}} \mathbb{E}[X | \mathcal{H}](e) dP &= \int_{\{e,f\}} X dP \\ \Rightarrow \mathbb{E}[X | \mathcal{H}](e)(0.15 + 0.2) &= (5 \times 0.15 + 6 \times 0.2) \\ \Rightarrow \mathbb{E}[X | \mathcal{H}](e) &= \frac{195}{35}\end{aligned}$$

Note that if we take  $\mathcal{H} = \{\Omega, \{a, b\}, \dots, \{c, d, e, f\}, \dots, \emptyset\}$ ,  $\mathcal{H}$  is exactly  $\mathcal{F}$  because the complement and the union properties of  $\sigma$ -algebra.

#### 2.12.4 Useful equalities

For all equalities in this section, suppose that we have a probability space  $(\Omega, \mathcal{F}, P)$ . Let  $\mathcal{H}$  be a  $\sigma$ -algebra and  $\mathcal{H} \subset \mathcal{F}$ .

##### 2.12.4.1 With indicator random variable

The conditional expectation can be usually used in form :

$$\mathbb{E}[X | A] = \frac{\mathbb{E}[X \cdot \mathbb{1}_A]}{P(A)}$$

#### 2.12.4.2 Independence with $\sigma$ -algebra

If  $X$  and  $\mathcal{H}$  are independent, then

$$\mathbb{E}[X | \mathcal{H}] = \mathbb{E}[X]$$

**Proof** Since  $X$  and  $\mathcal{H}$  are independent, then :

$$P(X = x | A) = P(X = x) \quad \forall A \in \mathcal{H}$$

Then

$$\begin{aligned} \mathbb{E}[X | \mathcal{H}] &= \sum_x x P(X = x | A) \\ &= \sum_x x P(X = x) \\ &= \mathbb{E}[X] \end{aligned}$$

**Proof 2** For all  $A \in \mathcal{H}$  :

$$\begin{aligned} \int_A \mathbb{E}[X | \mathcal{H}] dX &= \int_A X dX \\ &= \mathbb{E}[X \mathbf{1}_A] \\ &= \mathbb{E}[X] \mathbb{E}[\mathbf{1}_A] \quad (\text{independence}) \\ &= \mathbb{E}[X] P(A) \\ &= \int_A \mathbb{E}[X] dX \end{aligned}$$

Then by 2.1.8.2,  $\mathbb{E}[X | \mathcal{H}] = \mathbb{E}[X]$  almost surely.

#### 2.12.4.3 Random variable is $\mathcal{H}$ -measurable

If  $X$  is  $\mathcal{H}$ -measurable (do not confuse with  $\mathbb{E}[X | \mathcal{H}]$ , which must be  $\mathcal{H}$ -measurable by default) :

$$\mathbb{E}[X | \mathcal{H}] = X$$

**Proof**

$$\begin{aligned} \mathbb{E}[X | \mathcal{H}](A) &= \frac{\int_A X dP}{\int_A dP}, \quad \forall A \in \mathcal{H} \\ &= \frac{\int_A X(A) dP}{\int_A dP}, \quad (X \text{ is } \mathcal{H}\text{-measurable}) \\ &= X(A) \frac{\int_A dP}{\int_A dP} \\ &= X(A) \end{aligned}$$

#### 2.12.4.4 Expectation of converged sequence conditioned by $\sigma$ -algebra

Let  $\{X_n : n \geq 1\}$  be a sequence of random variables such that

- $X_n \geq 0$
- $X_n \nearrow X$
- $\mathbb{E}[X] < \infty$

then

$$\mathbb{E}[X_n | \mathcal{H}] \nearrow \mathbb{E}[X | \mathcal{H}] \quad \text{almost surely}$$

Thus since  $X_n \nearrow X$ , we just need to prove

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n | \mathcal{H}] = \mathbb{E}[X | \mathcal{H}] \quad \text{almost surely}$$

##### Proof 1

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[X_n | \mathcal{H}](H) &= \lim_{n \rightarrow \infty} \frac{\int_H X_n dP}{P(H)} \quad \forall H \in \mathcal{H} \\ &= \frac{\int_H \lim_{n \rightarrow \infty} X_n dP}{P(H)} \quad (\text{MCT, 2.2.15}) \\ &= \frac{\int_H X dP}{P(H)} \\ &= \mathbb{E}[X | \mathcal{H}](H) \end{aligned}$$

In this proof, we only have pointwise convergence, not almost surely, then we go to proof 2.

##### Proof 2

Set  $Y_n = X - X_n$ , then we are reduced to show  $Z_n = \mathbb{E}[Y_n | \mathcal{F}] \searrow 0$ .

1. We see that  $Y_n$  is decreasing and  $Y_n \geq 0$ , hence  $Z_n$  is also decreasing and  $Z_n \geq 0$ . This implies that  $Z_n$  admits a limit almost surely, denoted by  $Z_\infty$  and by Monotone Convergence Theorem (2.2.15) :

$$\lim_{n \rightarrow \infty} \mathbb{E}[Z_n] = \mathbb{E}[Z_\infty]$$

2. Since  $X_n$  is non-negative and  $X_n \nearrow X$  by Dominated convergence theorem 2.2.17.2 :

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X]$$

This implies  $\lim_{n \rightarrow \infty} \mathbb{E}[Y_n] = 0$

3. By law of iterated expectation 2.2.13:

$$\mathbb{E}[Z_n] = \mathbb{E}[\mathbb{E}[Y_n | \mathcal{F}]] = \mathbb{E}[Y_n]$$

Then  $\lim_{n \rightarrow \infty} \mathbb{E}[Z_n] = 0$ . This yields  $\mathbb{E}[Z_\infty] = 0$

4. Finally,  $Z_\infty \geq 0$  and  $\mathbb{E}[Z_\infty] = 0$ , then  $Z_\infty = 0$  almost surely.

#### 2.12.4.5 With a $\mathcal{H}$ -measurable random variable

If  $Y$  is bounded and  $\mathcal{H}$ -measurable , then

$$\mathbb{E}[YX | \mathcal{H}] = Y\mathbb{E}[X | \mathcal{H}]$$

##### Proof 1

$$\begin{aligned}\mathbb{E}[YX | \mathcal{H}](A) &= \frac{\int_A YX dP}{\int_A dP}, \quad \forall A \in \mathcal{H} \\ &= \frac{\int_A Y(A)X dP}{\int_A dP}, \quad (Y \text{ is } \mathcal{H}\text{-measurable}) \\ &= Y(A) \frac{\int_A X dP}{\int_A dP} \\ &= Y(A)\mathbb{E}[X | \mathcal{H}](A)\end{aligned}$$

##### Proof 2

1. Random variables  $X, Y$  can be assumed, without loss of generality, to be non-negative. We can have this assumption since we can write :

- $X = X^+ - X^-$  and  $Y = Y^+ - Y^-$ . where  $X^+, X^-$  means positive part and negative part, which are both non-negative.
- Expectation is in general linear, it means ,e.g.,

$$\mathbb{E}[XY] = \mathbb{E}[X^+Y^+] + \mathbb{E}[X^-Y^-] - \mathbb{E}[X^+Y^-] - \mathbb{E}[X^-Y^+]$$

2. Fix  $A \in \mathcal{H}$  and let  $Y = \mathbb{1}_A$ . Then for any  $H \in \mathcal{H}$  :

$$\begin{aligned}\int_H \mathbb{E}[\mathbb{1}_AX | \mathcal{H}] dP &= \int_H \mathbb{1}_AX dP \\ &= \int_{A \cap H} X dP \\ &= \int_{A \cap H} \mathbb{E}[X | \mathcal{H}] dP \\ &= \int_H \mathbb{1}_A \mathbb{E}[X | \mathcal{H}] dP\end{aligned}$$

Hence by 2.1.8.2,  $E(\mathbb{1}_AX | \mathcal{H}) = \mathbb{1}_A E(X | \mathcal{H})$  almost surely.

3. Any **simple function is a finite linear combination** of indicator functions. By linearity, the above property holds for simple functions: if  $Y_n$  is a simple function then  $\mathbb{E}(Y_nX|\mathcal{H}) = Y_n E(X|\mathcal{H})$ .

Since  $Y$  be  $\mathcal{H}$ -measurable and non-negative, then there exists a sequence of simple functions  $\{Y_n\}_{n \geq 1}$  converging monotonically (here meaning  $Y_n \leq Y_{n+1}$  and pointwise to  $Y$ . Consequently, with  $X \geq 0$  :

- The sequence  $\{Y_n X\}_{n \geq 1}$  converges monotonically and pointwise to  $XY$ .
  - $Y_n E(X|\mathcal{H})$  converges monotonically and pointwise to  $YE(X|\mathcal{H})$
4. Finally, it is to use Monotone convergence theorem (MCT, 2.2.15):

$$\begin{aligned}
\int_H YE(X|\mathcal{H}) dP &= \int_H \lim_{n \rightarrow \infty} Y_n E(X|\mathcal{H}) dP \\
&= \lim_{n \rightarrow \infty} \int_H Y_n E(X|\mathcal{H}) dP \quad (\text{MCT}) \\
&= \lim_{n \rightarrow \infty} \int_H Y_n E(X|\mathcal{H}) dP \\
&= \lim_{n \rightarrow \infty} \int_H E(Y_n X|\mathcal{H}) dP \quad (\text{simple function proved previously}) \\
&= \lim_{n \rightarrow \infty} \int_H Y_n X dP \\
&= \int_H \lim_{n \rightarrow \infty} Y_n X dP \quad (\text{MCT}) \\
&= \int_H YX dP \\
&= \int_H E(YX|\mathcal{H}) dP
\end{aligned}$$

Then by 2.1.8.2,  $\mathbb{E}[YX | \mathcal{H}] = Y\mathbb{E}[X | \mathcal{H}]$  almost surely.

#### 2.12.4.6 With binary operation $g(X, Y)$

Given

- $X, Y$  be real-valued random variables defined on this space
- $X$  is  $\mathcal{H}$ -measurable, which means  $\sigma(X) \subset \mathcal{H}$ .
- $Y$  is independent of  $\mathcal{H}$ , which implies that  $X$  and  $Y$  are independent.

Then we have :

$$\mathbb{E}[g(X, Y) | \mathcal{H}] = \mathbb{E}[g(X, Y) | X]$$

for any Borel-measurable ( $\mathbb{R}$ -measurable) function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

**Proof**

The proof is more or less similar to the proof in 2.2.12.4.5.

1. Random variables  $g(X, Y)$  can be assumed, without loss of generality, to be non-negative.

2. Let's define  $\mathcal{E} = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ . Then for all  $B \times C \in \mathcal{E}$  or for all  $B, C \in \mathcal{B}(\mathbb{R})$ , we show that we have the equality for  $g(X, Y) = \mathbb{1}_B(X)\mathbb{1}_C(Y)$  :

$$\begin{aligned}\mathbb{E}[\mathbb{1}_B(X)\mathbb{1}_C(Y) | \mathcal{H}] &= \mathbb{1}_B(X)\mathbb{E}[\mathbb{1}_C(Y) | \mathcal{H}] \quad (2.2.12.4.3) \\ &= \mathbb{1}_B(X)\mathbb{E}[\mathbb{1}_C(Y)] \quad (2.2.12.4.2) \\ &= \mathbb{1}_B(X)\mathbb{E}[\mathbb{1}_C(Y) | \sigma(X)] \quad (2.2.12.4.2) \\ &= \mathbb{E}[\mathbb{1}_B(X)\mathbb{1}_C(Y) | \sigma(X)] \quad (2.2.12.4.3) \\ &= \mathbb{E}[\mathbb{1}_B(X)\mathbb{1}_C(Y) | X]\end{aligned}$$

In addition,  $\mathcal{E}$  is a  $\pi$ -system (2.1.2) since if  $B_1 \times C_1$  and  $B_2 \times C_2$  in  $\mathcal{E}$ , then  $(B_1 \times C_1) \cap (B_2 \times C_2) = (B_1 \cap B_2) \times (C_1 \cap C_2)$  also in  $\mathcal{E}$ .

3. Let's define  $\mathcal{D}$  by:

$$\mathcal{D} := \{D \in \mathcal{B}(\mathbb{R}^2) : \mathbb{E}[g(X, Y) | \mathcal{H}] = \mathbb{E}[g(X, Y) | X] \text{ for } g(x, y) = \mathbb{1}_D(x, y)\}$$

By the definition, we have  $\mathcal{D} \subset \mathcal{B}(\mathbb{R}^2)$ . We show that  $\mathcal{B}(\mathbb{R}^2) \subset \mathcal{D}$ , then  $\mathcal{B}(\mathbb{R}^2) = \mathcal{D}$ .

First,  $\mathcal{D}$  is a Dynkin system (2.1.3) since :

- $\Omega \in \mathcal{D}$

- If  $D \in \mathcal{D}$ , then  $D^C \in \mathcal{D}$  since  $\mathbb{1}_{D^C} = 1 - \mathbb{1}_D$ , then

$$\mathbb{E}[\mathbb{1}_{D^C}(X, Y) | \mathcal{H}] = 1 - \mathbb{E}[\mathbb{1}_D(X, Y) | X] = 1 - \mathbb{E}[1 - \mathbb{1}_{D^C}(X, Y) | X] = \mathbb{E}[\mathbb{1}_{D^C}(X, Y) | X]$$

- If  $D_1$  and  $D_2$  are disjoint and

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{D_1}(X, Y) | \mathcal{H}] &= \mathbb{E}[\mathbb{1}_{D_1}(X, Y) | X] \\ \mathbb{E}[\mathbb{1}_{D_2}(X, Y) | \mathcal{H}] &= \mathbb{E}[\mathbb{1}_{D_2}(X, Y) | X]\end{aligned}$$

then

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{D_1}(X, Y) + \mathbb{1}_{D_2}(X, Y) | \mathcal{H}] &= \mathbb{E}[\mathbb{1}_{D_1}(X, Y) + \mathbb{1}_{D_2}(X, Y) | X] \\ \Leftrightarrow \mathbb{E}[\mathbb{1}_{D_1 \cup D_2}(X, Y) | \mathcal{H}] &= \mathbb{E}[\mathbb{1}_{D_1 \cup D_2}(X, Y) | X]\end{aligned}$$

since  $D_1$  and  $D_2$  are disjoint.

4. For all  $E = B \times C \in \mathcal{E}$ , we have  $\mathbb{1}_E(x, y) = \mathbb{1}_B(x)\mathbb{1}_C(y)$  or this means that  $E \in \mathcal{D}$ . This implies  $\mathcal{E} \subset \mathcal{D}$ . By Dynkin's theorem (2.1.3.2), we get  $\sigma(\mathcal{E}) \subset \mathcal{D}$  or  $\sigma(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})) \subset \mathcal{D}$  or  $\mathcal{B}(\mathbb{R}^2) \subset \mathcal{D}$  (see 2.1.1.2).
5. Finally, any **simple function is a finite linear combination** of indicator functions: if  $g_n(X, Y)$  is a simple function then  $\mathbb{E}[g_n(X, Y) | \mathcal{H}] = \mathbb{E}[g_n(X, Y) | X]$ . Moreover, there exists a sequence of simple functions  $\{g_n(X, Y)\}_{n \geq 1}$  converging monotonically (here meaning  $g_n(X, Y) \leq g_{n+1}(X, Y)$ ) and pointwise to  $g(X, Y)$ . By 2.2.12.4.4:

- $\mathbb{E}[g_n(X, Y) | \mathcal{H}] \nearrow \mathbb{E}[g(X, Y) | \mathcal{H}]$
- $\mathbb{E}[g_n(X, Y) | X] \nearrow \mathbb{E}[g(X, Y) | X]$

Then  $\mathbb{E}[g(X, Y) | \mathcal{H}] = \mathbb{E}[g(X, Y) | X]$  almost surely.

## 2.13 Law of iterated expectation

### 2.13.1 With one time conditioned

Also called law of total expectation, the tower rule or the smoothing theorem :

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$$

**Proof**

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X|Y]] &= \int_y \mathbb{E}[X|Y=y] P(Y=y) dy \\ &= \int_y \int_x x P(X=x|Y=y) dx P(Y=y) dy \\ &= \int_y \int_x x P(X=x, Y=y) dx dy \\ &= \int_x \int_y x P(X=x, Y=y) dy dx \quad (\text{by Fubini's theorem}) \\ &= \int_x x P(X=x) dx \\ &= \mathbb{E}[X]\end{aligned}$$

#### 2.13.1.1 Conditional by sigma algebra

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $\sigma$ -algebra  $\mathcal{H} \subseteq \mathcal{F}$ , then

$$\mathbb{E}[\mathbb{E}[X | \mathcal{H}]] = \mathbb{E}[X]$$

**Illustrative example**

Let's come back to example 2.2.12.3 :

$\Omega$	$a$	$b$	$c$	$d$	$e$	$f$
$P(\omega)$	0.2	0.1	0.3	0.05	0.15	0.2
$X(\omega)$	1	2	3	4	5	6

with  $\mathcal{H} = \{\Omega, \{a, b\}, \{c, d\}, \{e, f\}, \{c, d, e, f\}, \{a, b, e, f\}, \{a, b, c, d\}, \emptyset\}$ . On one hand,

$$\begin{aligned}\mathbb{E}[X] &= \sum_a X(a) P(a) \\ &= 1 \times 0.2 + 2 \times 0.1 + 3 \times 0.3 + 4 \times 0.05 + 5 \times 0.15 + 6 \times 0.2 \\ &= 3.45\end{aligned}$$

On the other hand :

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X \mid \mathcal{H}]] &= \sum_a \mathbb{E}[X \mid \mathcal{H}](a)P(a) \\
&= \frac{4}{3} \times 0.2 + \frac{4}{3} \times 0.1 + \frac{110}{35} \times 0.3 + \frac{110}{35} \times 0.05 + \frac{195}{35} \times 0.15 + \frac{195}{35} \times 0.2 \\
&= 3.45
\end{aligned}$$

### Proof

From sec 2.2.12,

$$\int_A \mathbb{E}[X \mid \mathcal{H}] = \int_A X \, dP, \quad \forall A \in \mathcal{H}$$

Then

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X \mid \mathcal{H}]] &= \int_{\Omega} \mathbb{E}[X \mid \mathcal{H}] \, dP \\
&= \sum_i^{\infty} \int_{A_i} \mathbb{E}[X \mid \mathcal{H}] \, dP \quad (A_i \text{ are pairwise disjoint and } \bigcup A_i = \Omega) \\
&= \sum_i^{\infty} \int_{A_i} X \, dP \\
&= \int_{\Omega} X \, dP \\
&= \mathbb{E}[X]
\end{aligned}$$

Note that to use  $\mathcal{H} = \sigma(Y)$ , then we refind  $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$ .

#### 2.13.2 With two time conditioned

$$\mathbb{E}[\mathbb{E}[X \mid Y, Z] \mid Z] = \mathbb{E}[X \mid Z]$$

This is a particular case of the general case law of iterated expectation (2.2.13.3).

##### 2.13.2.1 Illustrative example

Let's consider a fair dice with 6 faces labelled by  $a, \dots, f$  and three random variables  $X, Y, Z$  such that :

$\omega$	a	b	c	d	e	f
$X(\omega)$	1	2	3	4	5	6
$Y(\omega)$	0	-1	-1	0	0	-1
$Z(\omega)$	-1	0	-1	0	0	-1

In case  $Z = 0$

- On one hand

- $\mathbb{E}[X|Y=0, Z=0] = \frac{4+5}{2} = \frac{9}{2}$ .
- In the same manner,  $\mathbb{E}[X|Y=1, Z=0] = 2$ .
- Then  $\mathbb{E}[\mathbb{E}[X|Y, Z=0]|Z=0] = \mathbb{E}[X|Y=0, Z=0]P(Y=0, Z=0) + \mathbb{E}[X|Y=1, Z=0]P(Y=1, Z=0) = \frac{9}{2} \times \frac{2}{3} + 2 \times \frac{1}{3} = \frac{11}{3}$ .

- On the other hand  $\mathbb{E}[X|Z=0] = \frac{11}{3}$

### Formal proof

Note that  $\mathbb{E}[X | Y, Z]$  is random variable which can describe as  $g(Y, Z)$ . Then  $\mathbb{E}[\mathbb{E}[X | Y, Z] | Z]$  means the conditional expectation this function  $g(Y, Z)$ , given  $Z$ . We have :

$$\begin{aligned} & \mathbb{E}[\mathbb{E}[X | Y, Z] | Z] \\ &= \int_y \mathbb{E}[X | Y=y, Z=z] P(Y=y | Z=z) dy \\ &= \int_y \int_x X P(X=x | Y=y, Z=z) dx \frac{P(Y=y, Z=z)}{P(Z=z)} dy \\ &= \int_x \int_y \frac{P(X=x, Y=y, Z=z)}{P(Z=z)} dy dx \\ &= \int_x \frac{P(X=x, Z=z)}{P(Z=z)} dx \\ &= \int_x P(X=x | Z=z) dx \\ &= \mathbb{E}[X | Z] \end{aligned}$$

#### 2.13.3 General case

Let  $(\Omega, \mathcal{F}, P)$  be a probability space. Two sub  $\sigma$ -algebras  $\mathcal{H}_1$  and  $\mathcal{H}_2$  such that  $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \mathcal{F}$ . For a random variable  $X$  on  $(\Omega, \mathcal{F}, P)$ , if  $\mathbb{E}[X]$  is defined, which means  $\min(\mathbb{E}[X^-], \mathbb{E}[X^+]) < \infty$  (in sec 2.1.9.5.1), then

$$\mathbb{E}[\mathbb{E}[X | \mathcal{H}_2] | \mathcal{H}_1] = \mathbb{E}[X | \mathcal{H}_1]$$

##### 2.13.3.1 Proof

In general, we need to prove :

1. The existence of  $\mathbb{E}[\mathbb{E}[X | \mathcal{H}_2] | \mathcal{H}_1]$  and  $\mathbb{E}[X | \mathcal{H}_1]$ .
2.  $\mathbb{E}[\mathbb{E}[X | \mathcal{H}_2] | \mathcal{H}_1]$  and  $\mathbb{E}[X | \mathcal{H}_1]$  are both  $\mathcal{H}_1$  measurable or equivalently, they are both measurable function on  $(\Omega, \mathcal{H}_1)$ .
3.  $\int_A \mathbb{E}[\mathbb{E}[X | \mathcal{H}_2] | \mathcal{H}_1] dP = \int_A \mathbb{E}[X | \mathcal{H}_1] dP$  for all  $A \in \mathcal{H}_1$ .

The first requirement :

- Since  $X$  has a defined expectation then  $\mathbb{E}[X | \mathcal{H}_1]$  exists (sec 2.2.12).
- Let's  $Y = \mathbb{E}[X | \mathcal{H}_2]$ , since  $\mathbb{E}[Y] = \mathbb{E}[X]$  (sec 2.2.13.1.1) is defined, then  $\mathbb{E}[Y | \mathcal{H}_1]$  exists or  $\mathbb{E}[\mathbb{E}[X | \mathcal{H}_2] | \mathcal{H}_1]$  exists.

The second requirement is trivial by the definition of conditional expectation (sec 2.2.12).

The third requirement. From the definition of conditional expectation (sec 2.2.12) :

- $\int_A \mathbb{E}[X | \mathcal{H}_1] dP = \int_A X dP$  for all  $A \in \mathcal{H}_1$
- $\int_A \mathbb{E}[\mathbb{E}[X | \mathcal{H}_2] | \mathcal{H}_1] dP = \int_A \mathbb{E}[X | \mathcal{H}_2] dP$  for all  $A \in \mathcal{H}_1$ . Since  $\mathcal{H}_1 \subseteq \mathcal{H}_2$ , then  $A \in \mathcal{H}_2$ . This infers  $\int_A \mathbb{E}[X | \mathcal{H}_2] dP = \int_A X dP$  for all  $A \in \mathcal{H}_1$ .

We use the corollary of Radon–Nikodym theorem 2.2.11.4.2 for the second and the third to have q.e.d.

### 2.13.3.2 Corollary

- If  $\mathcal{H}_1 = \sigma(Z)$  and  $\mathcal{H}_2 = \sigma(Y, Z)$ , we refind  $\mathbb{E}[\mathbb{E}[X | Y, Z] | Z] = \mathbb{E}[X | Z]$ .
- If  $\sigma(Z) \subseteq \sigma(Y)$ , we can write  $\mathbb{E}[\mathbb{E}[X | Y] | Z] = \mathbb{E}[X | Z]$ .

## 2.14 Fubini–Tonelli theorem

Given measure space  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  with two measures  $\mu$  and  $\nu$  are  $\sigma$ -finite measure (2.1.7.1.2). Then the  $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$  is also a measure space with a product measure. If

$$f: X \times Y \rightarrow \mathbb{R}$$

is a function that is  $\mathcal{A} \otimes \mathcal{B}$ -measurable, then two functions

$$g_y(x) := x \mapsto \int_Y |f(x, y)| d\nu(y)$$

and

$$g_x(y) := y \mapsto \int_X |f(x, y)| d\mu(x)$$

are respectively  $\mathcal{A}$ -measurable and  $\mathcal{B}$ -measurable.

Besides, if

$$\int_X \int_Y |f(x, y)| d\nu(y) \mu(x)$$

is finite then

$$x \mapsto \int_Y f(x, y) d\nu(y)$$

$$y \mapsto \int_X f(x, y) d\nu(x)$$

are respectively  $\mathcal{A}$ -measurable and  $\mathcal{B}$ -measurable.

## 2.15 Monotone Convergence Theorem

If  $f_n : E \rightarrow [0, +\infty)$  is a sequence of measurable functions on a measurable set  $E$  such that  $f_n$  is monotone (increasing by default,  $f_1 \leq f_2 \leq \dots$ ), then :

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E \lim_{n \rightarrow \infty} f_n$$

If  $\lim_{n \rightarrow \infty} f_n = f$  pointwise almost everywhere (Beppo Levi theorem):

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$$

**In case of probability**, given probability space  $(\Omega, \mathcal{F}, P)$  and  $X_n$  is non-negative random variable and increasing ( $X_n \leq X_{n+1}$ ), with any  $H \in \mathcal{F}$ , then :

$$\lim_{n \rightarrow \infty} \int_H X_n = \int_H \lim_{n \rightarrow \infty} X_n$$

The Monotone Convergence Theorem (MCT), the Dominated Convergence Theorem (DCT 2.2.17), and Fatou's Lemma (2.2.16) are **three major** results in the theory of Lebesgue integration that answer the question : When we have that  $\lim_{n \rightarrow \infty}$  and  $\int$  can commute.

### 2.15.1 Example

Let  $X_n$  be a sequence of (finite expectation) random variables,  $T_n = \sum_{k=0}^n |X_k|$  and  $T = \lim_{n \rightarrow \infty} T_n = \sum_{k=0}^{\infty} |X_k|$ . Then  $T_n$  is increasing, by the Monotone Convergence Theorem :

$$\begin{aligned} \mathbb{E}[\lim_{n \rightarrow \infty} T_n] &= \lim_{n \rightarrow \infty} \mathbb{E}[T_n] \\ \Leftrightarrow \mathbb{E}[T] &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \mathbb{E}[|X_k|] \\ &= \sum_{k=0}^{\infty} \mathbb{E}[|X_k|] \end{aligned}$$

## 2.16 Fatou's lemma

Given a measure space  $(\Omega, \mathcal{F}, \mu)$  and a set  $X \in \mathcal{F}$ .  $\mathcal{B}_{\mathbb{R}_{\geq 0}}$  denotes the Borel set on real nonnegative number. Let  $f_n$  be measurable functions on  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\geq 0}})$ ,

$f_n : X \rightarrow \mathbb{R}_{\geq 0}$ . Define the function  $f : \Omega \rightarrow \mathbb{R}_{\geq 0}$  by setting

$$f(\omega) = \lim_{n \rightarrow +\infty} \inf_{k \geq n} f_k(\omega[0:k]) \quad \forall \omega \in X$$

Then  $f$  is also  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\geq 0}})$ -measurable and

$$\int_X f(\omega) d\mu(\omega) \leq \lim_{n \rightarrow +\infty} \inf_{k \geq n} \int_X f_k(\omega[0:k]) d\mu(\omega[0:k])$$

or

$$\int_X \lim_{n \rightarrow +\infty} \inf_{k \geq n} f_k \leq \lim_{n \rightarrow +\infty} \inf_{k \geq n} \int_X f_k$$

A literature explanation for this lemma is that integral of limit inferior is less than limit inferior of integral. The notation  $\lim_{n \rightarrow +\infty} \inf_{k \geq n}$  can be noted  $\liminf_{n \rightarrow +\infty}$  for short. If in case that  $X = \Omega$  then :

$$\mathbb{E} \left[ \lim_{n \rightarrow +\infty} \inf_{k \geq n} f_k \right] \leq \lim_{n \rightarrow +\infty} \inf_{k \geq n} \mathbb{E}[f_k]$$

### 2.16.1 Example

Let's consider  $X = [0, 2]$  and functions  $f_n$  defined by

$$f_n = \begin{cases} \mathbb{1}_{[0,1]} & \text{if } n \text{ is odd} \\ \mathbb{1}_{(1,2]} & \text{if } n \text{ is even} \end{cases}$$

Hence  $f_n$  are positive and

$$\lim_{n \rightarrow \infty} \inf_{k \geq n} f_k = \lim_{n \rightarrow \infty} \inf[f_{\text{odd}}, f_{\text{even}}] = \lim_{n \rightarrow \infty} 0 = 0 = f$$

By Fatou's lemma :

$$\begin{aligned} \int_0^2 f(x) dx &\leq \lim_{n \rightarrow \infty} \inf_{k \geq n} \int_0^2 f_k(x) dx \\ \Leftrightarrow \int_0^2 0 dx &\leq \lim_{n \rightarrow \infty} \inf_{k \geq n} 1 \\ \Leftrightarrow 0 &\leq 1 \quad (\text{ok}) \end{aligned}$$

### 2.16.2 Reverse Fatou lemma

If there exists a non-negative integrable function  $g$  on  $\Omega$  such that  $f_n(\omega) \leq g(\omega)$  for all  $\omega, n$  (statewise dominance 2.7.1), then

$$\int_X \limsup_{n \rightarrow +\infty} f_n d\mu \geq \limsup_{n \rightarrow +\infty} \int_X f_n d\mu$$

Here  $g$  integrable means that  $g$  is  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\geq 0}})$ -measurable and that  $\int_X g(\omega) d\mu(\omega) < \infty$ . The role of  $g$  here is to make an upper bound for  $f_n$  and make exist  $\int_X \limsup_{n \rightarrow +\infty} f_n d\mu$ .

## 2.17 Dominated convergence theorem

### 2.17.1 Statement

Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of measurable function (sec 2.1.5) on a measure space  $(S, \Sigma, \mu)$ .  $f_n$  is from  $(S, \Sigma)$  to  $(\mathbb{C}, B(\mathbb{C}))$ . Suppose that

1.  $f_n$  is pointwise convergent (sec 2.4.1.1) to a function  $f$ . In general case, it is *converge in measure*.
2.  $f_n$  dominated by an integrable function  $g$  in the sense that

$$|f_n(x)| \leq g(x), \forall x \in S \text{ and } \forall n \in \mathbb{N}$$

Then  $f$  is integrable and

$$\lim_{n \rightarrow \infty} \int |f_n - f| d\mu = 0,$$

which implies

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int \lim_{n \rightarrow \infty} f_n d\mu = \int f d\mu$$

### 2.17.2 Corollary

If in case of probability space, suppose that :

1.  $X_n \xrightarrow{P} X$  (converge in probability 2.4.3).
2.  $|X_n| \leq Y$ , where  $Y$  is integrable, which means  $\int |Y| dP < +\infty$ .

Then  $X$  is integrable and

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X]$$

Note that this corollary is weaker than the lemma 2.4.5.4, says that converge in probability and uniformly integrable then converge in mean, since in proposition 2.3.3.5 :

- $|X_n| \leq Y$ , where  $Y$  is integrable  $\Rightarrow X_n$  is uniformly integrable.
- $X_n$  is uniformly integrable  $\not\Rightarrow$  there exists integrable  $Y$  such that  $|X_n| \leq Y$ .

### 2.17.3 Example

Finding

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$$

where  $f_n(x) = \frac{nx^n}{1+x}$ ,  $\forall x \in [0, 1]$ .

At a first look,  $f_n(x)$  is not be pointwise convergent on  $[0, 1]$ , in particular at  $x = 1$ . Let's use the integration by part :

$$\begin{aligned}\int_0^1 \frac{nx^n}{1+x} dx &= \frac{n}{2(n+1)} + \int_0^1 \frac{n}{(n+1)} \frac{x^{n+1}}{(1+x)^2} dx \\ &= \frac{n}{2(n+1)} + \int_0^1 g_n(x) dx\end{aligned}$$

Now use dominated convergence theorem to find :

$$\lim_{n \rightarrow \infty} \int_0^1 g_n(x) dx$$

1.  $g_n(x)$  is pointwise convergent to  $g(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1/4 & \text{if } x = 1 \end{cases}$

2.  $g_n(x)$  is dominated by the integrable function  $\frac{1}{(1+x)^2}$  on  $[0, 1]$ .

Then by the dominated convergence theorem :

$$\lim_{n \rightarrow \infty} \int_0^1 g_n(x) dx = \int_0^1 0 dx = 0$$

Finally,

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{nx^n}{1+x} dx = \frac{n}{2(n+1)} = \frac{1}{2}$$

### 3 Uniform integrability

In mathematics, uniform integrability is an important concept in measure theory. It plays a vital role in stochastic process, the theory of martingales (in sec 4.10.1),...

#### 3.1 Uniform integrability in measure theory

Let  $(\Omega, \mathcal{F}, \mu)$  be a positive measure space, which means  $\mu(\omega) \geq 0, \forall \omega \in \Omega$ .  $\mathcal{S}$  is a **family of measurable functions**  $f$  (in sec 2.1.5), from  $(\Omega, \mathcal{F})$  to  $(\mathbb{R}, B(\mathbb{R}))$ . We say that  $\mathcal{S}$  is uniformly integrable (abbreviated as UI) if the following conditions hold :

1.  $\sup_{f \in \mathcal{S}} \int |f| d\mu < +\infty$ .
2. For all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all  $A \in \mathcal{F}$  that satify  $\mu(A) < \delta$ , we have :

$$\int_{\omega \in A} |f(\omega)| d\mu(\omega) < \varepsilon, \quad \forall f \in \mathcal{S}$$

**Important note :** As mentionned in the definition, the uniformly integrability is eligible only for the **set (or family) of measurable functions** and **not for** a single measurable function.

### 3.2 Uniform integrability for set of random variable

Given probability space  $(\Omega, \mathcal{F}, P)$  and  $\mathcal{S}$  is a family of random variable  $X$ , from  $(\Omega, \mathcal{F})$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . We say that  $\mathcal{S}$  is uniformly integrable if the following conditions hold :

1.  $\sup_{X \in \mathcal{S}} \mathbb{E}[|X|] < +\infty$ .
2. Absolutely continuous : For all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all  $A \in \mathcal{F}$  that satify  $P(A) < \delta$ , we have :

$$\int_{\omega \in A} |X(\omega)| dP(\omega) < \varepsilon, \quad \forall X \in \mathcal{S}$$

or

$$\mathbb{E}[|X| \cdot \mathbb{1}_A] < \varepsilon, \quad \forall X \in \mathcal{S}$$

where  $\mathbb{1}_A(\omega) = 1$  if  $\omega \in A$ , 0 otherwise.

**Important note :** Again, the uniformly integrability is eligible only for the **set (or family) of random variables** and **not for** a single random variable.

#### 3.2.1 Conditional expectation and expectation of product with indicator function

**Do not confuse** between two following concepts :

- $\mathbb{E}[X | \mathcal{H}](A)$ , sometime shortly noted  $\mathbb{E}[X | A]$ , the conditional expectation (in sec 2.2.12).
- $\mathbb{E}[X \cdot \mathbb{1}_A]$ , the expectation of product between  $X$  and the indicator function  $\mathbb{1}_A$ .

Thus,

$$\mathbb{E}[X | A] = \frac{\int_A X dP}{P(A)} = \frac{\mathbb{E}[X \cdot \mathbb{1}_A]}{P(A)}$$

Note that,  $X \cdot \mathbb{1}_A$  is a new random variable whom distribution is modified from  $X$  (set  $X(\omega) = 0$  for all  $\omega \in A^C$ ).

### 3.2.2 Equivalent definition

The two previous conditions are equivalent to the following : For all  $\varepsilon > 0$ , there exists  $K > 0$  such that

$$\mathbb{E}[|X| \cdot \mathbb{1}_{\{|X|>K\}}] < \varepsilon, \quad \forall X \in \mathcal{S}$$

The set  $\{|X| > K\}$  is short notation for  $\{\omega : |X(\omega)| > K\}$ . Thus the above condition means that the integral of  $X$  at two extreme sides is very small. Then it is also equivalent to

$$\lim_{K \rightarrow \infty} \sup_{X \in \mathcal{S}} \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X|>K\}}] = 0$$

Since the uniform integrability is eligible for set of random variables, we have also the definition of uniform integrability for stochastic process (which is also set of random variables with indexation) in the following section.

## 3.3 Uniform integrability of stochastic process

From the uniform integrability of set of random variables (in sec 2.3.2), we can expand this concept for stochastic process. A SP  $X_n$  is called uniformly integrable if the following condition holds :

$$\lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n|>K\}}] = 0$$

### 3.3.1 Examples

- **Uniform integrable** stochastic process : Assume our probability space concerns a fair dice.  $X_n$  is the result of  $n^{th}$  rolling. Then

- with  $K = 0$  :  $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n|>0\}}] = 3.5$
- with  $K = 1$  :  $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n|>1\}}] = 4$
- ...
- with  $K = 6$  :  $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n|>1\}}] = 1$
- with  $K > 6$  :  $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n|>6\}}] = 0$

Then

$$\lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n|>K\}}] = 0$$

- **Non uniform integrable** stochastic process : Assume our probability space is  $([0, 1], \mathcal{F}, P)$  and outcome  $\omega$  is uniformly distributed in  $[0, 1]$ . Let's take

$$X_n(\omega) = \begin{cases} n & \text{if } \omega \in [0, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

Then  $\mathbb{E}[X_n] = 1$  or  $X_n$  is integrable with all  $n \in \mathbb{N}$ , even we have  $\sup_{n \in \mathbb{N}}(\mathbb{E}[|X_n|]) < \infty$ . However, the process  $(X_n)$  is not uniformly integrable since if  $n \leq K$ , then

$$\mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] = \begin{cases} 0 & \text{if } n \leq K \\ 1 & \text{if } n > K \end{cases}$$

This means

$$\lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] = 1 \neq 0$$

### 3.3.2 Equivalence between definitions

This is a proof for the equivalence between definition in 2.3.2 and in 2.3.2.2. The statement is the following.

$$\lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] = 0$$

if and only if we have two following conditions :

1.  $(X_n)$  is  $L^1$  bounded :

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|] < +\infty$$

2.  $(X_n)$  is absolutely continuous :

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_A] \rightarrow 0 \text{ as } P(A) \rightarrow 0$$

**In the direct sense**, let's  $h(K) = \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}]$ . First, we have

$$\begin{aligned} & \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|] \\ &= \sup_{n \in \mathbb{N}} (\mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] + \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| \leq K\}}]) \\ &\leq \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] + \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| \leq K\}}] \\ &\leq h(K) + K P(|X_n| \leq K) \\ &\leq h(K) + K, \quad \forall K > 0 \end{aligned}$$

Since  $\lim_{K \rightarrow \infty} h(K) = 0$ , then there exists  $K$  such that  $h(K) < \infty$ . Then  $X_n$  is  $L^1$  bounded. Second,

$$\begin{aligned} & \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_A] \\ &\leq \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{A \cup \{|X_n| > K\}}] \\ &= \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{(A \cap \{|X_n| \leq K\}) \cup \{|X_n| > K\}}] \end{aligned}$$

$$\begin{aligned}
&= \sup_{n \in \mathbb{N}} (\mathbb{E}[|X_n| \cdot \mathbb{1}_{A \cap \{|X_n| \leq K\}}] + \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}]) \\
&\leq \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{A \cap \{|X_n| \leq K\}}] + \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] \\
&\leq KP(A) + h(K)
\end{aligned}$$

Then if  $K \rightarrow \infty$ , we fix  $K$  first and then take  $P(A) \rightarrow 0$ , we have  $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_A] \rightarrow 0$ .

**In the converse sense**, for a given  $\varepsilon > 0$  suppose that there exists  $k$  such that

$$\mathbb{E}[|X_k| \cdot \mathbb{1}_{\{|X_k| > K\}}] > \varepsilon$$

By Markov's inequality 2.2.8.1

$$P(|X_k| > K) \leq \frac{\mathbb{E}[X_k]}{K}$$

Then if  $K \rightarrow \infty$ , then  $P(|X_k| > K) \rightarrow 0$ . By the property of absolutely continuous, if  $P(|X_k| > K) \rightarrow 0$  then  $\mathbb{E}[|X_k| \cdot \mathbb{1}_{\{|X_k| > K\}}] \rightarrow 0$  (contradiction). Then  $X_n$  is uniformly integrable.

### 3.3.3 Linear sum of two uniform integrable processes

The linear sum of two uniform integrable processes is also uniform integrable.

**Proof**

Let's consider  $X_n$  and  $Y_n$  are both uniform integrable. Then we show  $X_n + Y_n$  is uniformly integrable and  $\alpha X_n$  is uniformly integrable.

First, by Absolute sum decomposition inequality 2.2.8.5

$$\begin{aligned}
&\lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n + Y_n| \cdot \mathbb{1}_{\{|X_n + Y_n| > K\}}] \\
&\leq \lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \left( 2\mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > \frac{K}{2}\}}] + 2\mathbb{E}[|Y_n| \cdot \mathbb{1}_{\{|Y_n| > \frac{K}{2}\}}] \right) \\
&= 0
\end{aligned}$$

Second,

$$\begin{aligned}
&\lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|\alpha X_n| \cdot \mathbb{1}_{\{|\alpha X_n| > K\}}] \\
&= \lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \alpha \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > \frac{K}{|\alpha|}\}}] \\
&= 0
\end{aligned}$$

### 3.3.4 Lemma

Given process  $Y_n$  which is uniformly integrable. If the process  $X_n$  satifies that  $\exists K_0, \forall |X_n| \geq K_0$  :

$$|X_n| \leq |Y_n|$$

Then  $X_n$  is uniformly integrable.

**Proof**

For all  $K \geq K_0$  :

$$\begin{aligned} & \lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] \\ & \leq \lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|Y_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] \\ & \leq \lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|Y_n| \cdot \mathbb{1}_{\{|Y_n| > K\}}] \\ & = 0 \end{aligned}$$

where the last less or equal is by inequality 2.2.8.4.

### 3.3.5 Proposition

Let  $(X_n)_{n \in \mathbb{N}}$  be a stochastic process, and let  $Y$  be an integrable random variable, such that  $|X_n| \leq Y$  for all  $n \in \mathbb{N}$  (dominated). Prove that  $(X_n)$  is uniformly integrable.

Before the main proof, we prove the following lemma

#### 3.3.5.1 Lemma

If  $Y$  is integrable ( $\mathbb{E}[|Y|] < \infty$ ), then

$$\lim_{K \rightarrow \infty} \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|Y| > K\}}] = \lim_{K \rightarrow \infty} \int |Y| \cdot \mathbb{1}_{\{|Y| > K\}} dP = 0$$

**Proof**

Let's set the sequence  $(f_K)$  where  $f_K = |Y| \cdot \mathbb{1}_{\{|Y| > K\}}$ . Then :

1.  $(f_K)$  is pointwise convergent (2.4.1.1) to 0 (more precise, to  $f(\omega) = 0, \forall \omega$ ), as  $K \rightarrow +\infty$ .
2.  $(f_K)$  is dominated by  $|Y|$ , which means  $f_K(\omega) \leq |Y(\omega)|, \forall \omega$  and  $|Y|$  is integrable.

By the dominated convergence theorem in sec 2.2.17, we have

$$\lim_{K \rightarrow \infty} \int |Y| \cdot \mathbb{1}_{\{|Y| > K\}} dP = \int \lim_{K \rightarrow \infty} |Y| \cdot \mathbb{1}_{\{|Y| > K\}} dP = \int 0 dP = 0$$

#### 3.3.5.2 Proof

First, since  $|X_n| \leq Y$ , we have

$$\{|X_n| > K\} \subseteq \{Y > K\}.$$

Then

$$\int |X_n| \cdot \mathbb{1}_{\{|X_n| > K\}} dP = \int_{\{|X_n| > K\}} |X_n| dP \leq \int_{\{Y > K\}} Y dP = \int Y \cdot \mathbb{1}_{\{Y > K\}} dP$$

for all  $n \in \mathbb{N}$ , or

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] \leq \int Y \cdot \mathbb{1}_{\{Y > K\}} dP$$

Second, by lemma in sec 2.3.3.5.1,  $Y$  is integrable and  $Y = |Y|$  (since  $Y > |X_n| > 0$ ), we have :

$$\lim_{K \rightarrow \infty} \int Y \cdot \mathbb{1}_{\{Y > K\}} dP = 0$$

From the first and the second, we have :

$$\lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] = 0$$

### 3.3.5.3 Converse statement

However, conversely, if  $(X_n)_{n \in \mathbb{N}}$  is uniformly integrable, it does not guarantee that it exists a **integrable** random variable  $Y$  that  $|X_n| \leq Y$  for  $n \in \mathbb{N}$ .

#### Counter example

Let  $X_n = n$  with probability  $a_n$ , where  $a_n \in (0, 1)$  and  $X_n = 0$  otherwise. Assume all the  $X_n$  are independent. By definition of  $X_n$ , the expectation of the tail is

$$E[|X_n| \cdot \mathbb{1}_{\{|X_n| \geq K\}}] = \begin{cases} na_n & \text{if } n \geq K \\ 0 & \text{if } n < K \end{cases}$$

Set  $a_n = 1/(n \log n)$  for  $n \geq 2$ , then

$$E[|X_n| \cdot \mathbb{1}_{\{|X_n| \geq K\}}] \leq n \frac{1}{n \log n} = \frac{1}{\log n} = \frac{1}{\log K}$$

or

$$\lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > K\}}] = 0$$

Thus the family  $(X_n)$  is uniformly integrable. However, if there were to exist a random variable  $Z$  such that  $Z \geq |X_n|$  for all  $n$ , then we must have  $P(Z \geq n) \geq P(X_n \geq n) = a_n$ . From the inequality in sec 2.2.4.1, we have

$$\mathbb{E}[Z] \geq \sum_{i=1}^{+\infty} P(Z \geq i) \geq \sum_{i=2}^{+\infty} a_i = \sum_{i=2}^{+\infty} \frac{1}{i \log i}$$

However

$$\begin{aligned} \sum_{i=2}^{+\infty} \frac{1}{i \log i} &\geq \frac{1}{2 \log 2} + \frac{2}{4 \log 4} + \frac{4}{8 \log 8} + \dots + \frac{2^{k-1}}{2^k \log 2^k} \quad (k \rightarrow \infty) \\ &= \frac{1}{2 \log 2} \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} \right) \quad (k \rightarrow \infty) \\ &= +\infty \end{aligned}$$

Then we say that, it does not always exist an integrable random variable  $Z$  which dominate  $(X_n)$ .

### 3.3.6 Lemma

Random variable  $X$  is integrable if and only if  $X$  is absolutely continuous.  
**In the direct sense**, since  $X$  is integrable then by lemma 2.3.3.5.1 :

$$\lim_{K \rightarrow \infty} \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X| > K\}}] = 0$$

Then  $\forall \varepsilon > 0, \exists K_0, \forall K \geq K_0, \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X| > K_0\}}] < \varepsilon$ .

Then the same as in the proof 2.3.3.2:

$$\begin{aligned} & \mathbb{E}[|X| \cdot \mathbb{1}_A] \\ & \leq \mathbb{E}[|X| \cdot \mathbb{1}_{A \cup \{|X| > K_0\}}] \\ & = \mathbb{E}[|X| \cdot \mathbb{1}_{A \cap \{|X| \leq K_0\}}] + \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X| > K_0\}}] \\ & \leq K_0 P(A) + \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X| > K_0\}}] \end{aligned}$$

Then  $\forall 2\varepsilon > 0$ , let's  $\delta = \frac{\varepsilon}{K_0}$ . Then if  $P(A) < \delta$ , we have  $\mathbb{E}[|X| \cdot \mathbb{1}_A] < 2\varepsilon$  or  $X$  is absolutely continuous.

**In the converse sense**, since  $X$  is absolutely continuous, then given a  $\varepsilon > 0, \exists \delta$  such that if  $P(A) < \delta$ , we have  $\mathbb{E}[|X| \cdot \mathbb{1}_A] < \varepsilon$ . Moreover,  $P(|X| > K) \rightarrow 0$  as  $K \rightarrow \infty$  then  $\exists K_0, \forall K \geq K_0, P(|X| > K_0) < \delta$ . Finally,

$$\mathbb{E}[|X|] = \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X| \leq K_0\}}] + \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X| > K_0\}}] \leq K_0 + \varepsilon < \infty$$

or  $X$  is integrable.

## 4 Convergence of Random Variables

In this section, we want to see a sequence of random variables  $X_1, X_2, \dots, X_n$  converges to a random variable  $X$ . There are four types of convergences :

- Convergence in distribution
- Convergence in probability
- Convergence in mean
- Almost sure convergence.

All of them have the objective to prove that  $X_n \rightarrow X$ , but they are different in the method (by definition) and the intensity (from weak to strong) of convergence.

Before diving into the convergence of random variable, it may be necessary remind basic definitions on convergence of function.

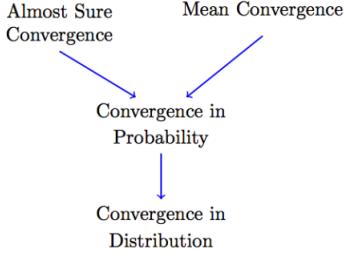


Figure 2.4: Relations between different types of convergence

## 4.1 Convergence of function

### 4.1.1 Pointwise convergence

Let  $\{f_n\}$  be a sequence of function,  $f_n : \mathbb{R} \rightarrow \mathbb{R}$ .  $\{f_n\}$  is called pointwise convergent to  $f$  if  $\forall x_0$  ( $x_0$  is given first),

$$|f_n(x_0) - f(x_0)| < \varepsilon, \forall \varepsilon > 0,$$

when  $n \rightarrow \infty$

### 4.1.2 Uniform convergence

Let  $\{f_n\}$  be a sequence of function,  $f_n : \mathbb{R} \rightarrow \mathbb{R}$ .  $\{f_n\}$  is called uniform convergence to  $f$  if  $\forall \varepsilon > 0$  ( $\varepsilon$  is given first),

$$|f_n(x) - f(x)| < \varepsilon, \forall x,$$

when  $n \rightarrow \infty$

#### 4.1.2.1 Interchange

If  $\{f_n\}$  converges uniformly to  $f$  then:

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$$

#### Proof

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx - \int_a^b f(x) dx &= \lim_{n \rightarrow \infty} \int_a^b (f_n(x) - f(x)) dx \\
 &\leq \lim_{n \rightarrow \infty} \int_a^b |f_n(x) - f(x)| dx \\
 &< \lim_{n \rightarrow \infty} (b-a)\varepsilon \\
 &= 0 \quad \text{by taking } \varepsilon \rightarrow 0
 \end{aligned}$$

### 4.1.3 Cauchy's criterion

The Cauchy's criterion is equivalent to uniform convergence. Let  $\{f_n\}$  be a sequence of function,  $f_n : \mathbb{R} \rightarrow \mathbb{R}$ .  $\{f_n\}$  is uniform convergent if  $\forall \varepsilon > 0$  ( $\varepsilon$  is given first),

$$\exists n_0, \forall p, q > n_0, |f_p(x) - f_q(x)| < \varepsilon$$

### 4.1.4 Notes and example

- In pointwise convergence,  $\varepsilon$  can depend on  $x_0$ , but in uniform convergence,  $\varepsilon$  is independent to  $x_0$ .
- The uniform convergence is stronger than pointwise convergence, it means that if we have uniform convergence then we have also pointwise convergence but conversely, it is not true.
- In pointwise and uniform uniform convergence,  $f$  must be given, but in Cauchy's criterion,  $f$  is no needed.

Let's take  $f_n(x) = x^n$  and consider two interval  $[0, r]$  where  $r < 1$  and  $[0, 1)$ .

- On  $[0, r]$ ,  $f_n$  is both poinwise convergent and uniform convergent to  $f = 0$ .
- On  $[0, 1)$ ,  $f_n$  is only poinwise convergent to  $f = 0$ , but not uniform convergent to  $f = 0$ . The problem for uniform convergence is if we fix first  $\varepsilon$  closed to 0, then  $|x^n - 0|$  can be greater  $\varepsilon$ , by taking  $x = 0.5^{1/n}$  then  $x^n = 0.5$  or by taking  $x = 1 - \frac{1}{n}$  then  $x^n = \frac{1}{e}$ .

## 4.2 Convergence in distribution

A sequence of random variables  $X_1, X_2, X_3, \dots, X_n$  converges in distribution (or converge weakly) to a random variable  $X$ , shown by  $X_n \xrightarrow{d} X$ , if :

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x)$$

for all  $x$  at which  $F_X(x)$  is continuous.  $F$  is cumulative distribution.

### 4.2.1 Equivalent definition

We have the convergence in distribution if any of the following statements are true:

- $\lim_{n \rightarrow \infty} P(X_n \leq x) \rightarrow P(X \leq x), \forall x.$
- $\lim_{n \rightarrow \infty} \mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X)]$  for all bounded and continuous functions  $f$ .

### 4.2.2 Example

Given a sequence of random variables  $X_1, X_2, X_3, \dots, X_n$  such that:

$$F_{X_n}(x) = \begin{cases} 1 - (1 - \frac{1}{n})^{nx} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that  $X_n$  converges in distribution to *Exponential*(1).

Remind that CDF of *Exponential*( $\lambda$ ) is :

$$F_X(x) = 1 - e^{-\lambda x}$$

Then

$$\begin{aligned} \lim_{n \rightarrow \infty} F_{X_n}(x) &= \lim_{n \rightarrow \infty} 1 - \left(1 - \frac{1}{n}\right)^{nx} \\ &= 1 - e^{-x}, \forall x > 0 \end{aligned}$$

Then  $X_n \xrightarrow{d} X$ , where  $X \sim \text{Exponential}(1)$ .

### 4.2.3 Integer-valued cases

Given sequence of random variables  $X_1, X_2, X_3, \dots, X_n$  and random variable  $X$ . Assume that  $X$  and  $X_n$  (for all  $n$ ) are non-negative and integer-valued, i.e.,

$$\begin{aligned} R_{X_i} &\subset \{0, 1, 2, \dots, i\}, i = 1, \dots, n \\ R_X &\subset \{0, 1, 2, \dots\} \end{aligned}$$

where  $R_X$  means the set of possible value of  $X$ . Then  $X_n \xrightarrow{d} X$  if and only if:

$$\lim_{n \rightarrow \infty} P_{X_n}(k) = P_X(k), \forall k \in R_X$$

#### 4.2.3.1 Application

Given a sequence of random variables  $X_1, X_2, X_3, \dots, X_n$  such that:

$$X_n \sim \text{Binomial}(n, \frac{\lambda}{n}), \quad \text{for } n \in \mathbb{N}, n > \lambda$$

where  $\lambda > 0$  is a constant. Show that  $X_n$  converges in distribution to *Poisson*( $\lambda$ ).

$$\lim_{n \rightarrow \infty} P_{X_n}(k) = \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$\begin{aligned}
&= \frac{n(n-1)\dots(n-k+1)}{k!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{\frac{n-\lambda(n-k)}{\lambda}} \\
&= \frac{\lambda^k}{k!} e^{-\lambda}
\end{aligned}$$

#### 4.2.4 Lemma

If :

- $X_n \xrightarrow{d} X$
- $X_n$  are uniformly integrable (2.3.3).
- $X$  is integrable.

Then  $\lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X]$  (convergence in mean 2.4.5).

**Proof**

Let  $f_K(x) = \begin{cases} x & \text{if } |x| \leq K \\ 0 & \text{otherwise} \end{cases}$ , then  $f_K$  is a bounded and continuous function. By the equivalent definition of convergence in distribution 2.4.2.1, we have  $\lim_{n \rightarrow \infty} \mathbb{E}[f_K(X_n)] \rightarrow \mathbb{E}[f_K(X)]$ .

With triangle inequality :

$$\begin{aligned}
&|\mathbb{E}[X_n] - \mathbb{E}[X]| \\
&\leq |\mathbb{E}[X_n] - \mathbb{E}[f_K(X_n)]| + |\mathbb{E}[f_K(X_n)] - \mathbb{E}[f_K(X)]| + |\mathbb{E}[f_K(X)] - \mathbb{E}[X]| \\
&= |\mathbb{E}[X_n \cdot \mathbb{1}_{\{|X_n|>K\}}]| + |\mathbb{E}[f_K(X_n)] - \mathbb{E}[f_K(X)]| + |\mathbb{E}[X \cdot \mathbb{1}_{\{|X|>K\}}]| \\
&\leq \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n|>K\}}] + |\mathbb{E}[f_K(X_n)] - \mathbb{E}[f_K(X)]| + \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X|>K\}}], \forall K
\end{aligned}$$

Then

$$\begin{aligned}
&\lim_{n \rightarrow \infty} |\mathbb{E}[X_n] - \mathbb{E}[X]| \\
&\leq \lim_{n \rightarrow \infty} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n|>K\}}] + \lim_{n \rightarrow \infty} |\mathbb{E}[f_K(X_n)] - \mathbb{E}[f_K(X)]| + \lim_{n \rightarrow \infty} \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X|>K\}}] \\
&\leq \lim_{K \rightarrow \infty} \sup_{n \rightarrow \infty} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n|>K\}}] + 0 + \lim_{K \rightarrow \infty} \sup_{n \rightarrow \infty} \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X|>K\}}] = 0
\end{aligned}$$

This is because  $X_n$  are uniformly integrable then  $\lim_{K \rightarrow \infty} \sup_{n \rightarrow \infty} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n|>K\}}] = 0$  (sec 2.3.3) and  $X$  is integrable then  $\lim_{K \rightarrow \infty} \sup_{n \rightarrow \infty} \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X|>K\}}] = 0$  (lemma 2.3.3.5.1).

### 4.3 Convergence in probability

A sequence of random variables  $X_1, X_2, X_3, \dots, X_n$  converges in probability to a random variable  $X$ , shown by  $X_n \xrightarrow{p} X$ , if :

$$\lim_{n \rightarrow \infty} P(|X_n - X| \geq \epsilon) = 0, \quad \text{for all } \epsilon > 0.$$

or in more details :

$$\lim_{n \rightarrow \infty} P(\{\omega : |X_n(\omega) - X(\omega)| \geq \epsilon\}) = 0, \quad \text{for all } \epsilon > 0.$$

#### 4.3.1 Law of large numbers

It states that if you repeat an experiment independently a large number of times and average the result, what you obtain should be close to the expected value. There are two main versions of the law of large numbers. They are called the **weak** (convergence in probability) and **strong** laws of the large numbers (Almost sure convergence, in sec 2.4.6.9).

Given i.i.d. random variables  $X_1, X_2, \dots, X_n$ . The sample mean, denoted by  $\bar{X}$ , is defined as:

$$\begin{aligned}\bar{X} &= \frac{X_1 + X_2 + \dots + X_n}{n} \\ \mathbb{E}[\bar{X}] &= \frac{\mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n]}{n} = \mathbb{E}[X_i] = \mu\end{aligned}$$

##### 4.3.1.1 Weak version

In the weak version, we assume that  $X_i$  admit a second moment, it means:

$$Var[\bar{X}] = \frac{1}{n^2} \sum_{i=1}^n Var[X_i] = \frac{Var[X]}{n}$$

Then,  $\bar{X} \xrightarrow{P} \mu$  for any  $\epsilon > 0$ :

$$\lim_{n \rightarrow \infty} P(|\bar{X} - \mu| \geq \epsilon) = 0$$

An simple proof can be done by using Chebyshev's inequality :

$$P(|\bar{X} - \mathbb{E}[\bar{X}]| \geq \epsilon) = P(|\bar{X} - \mu| \geq \epsilon) \leq \frac{Var[X]}{n\epsilon^2}$$

#### 4.3.2 Example

Let  $X_n \sim Exponential(n)$ , show that  $X_n \xrightarrow{P} 0$ .

$$\begin{aligned}\lim_{n \rightarrow \infty} P(|X_n - 0| \geq \epsilon) &= \lim_{n \rightarrow \infty} P(X_n \geq \epsilon) \quad \text{since } P(X_n < 0) = 0 \\ &= \lim_{n \rightarrow \infty} 1 - (1 - e^{-n\epsilon}) \\ &= \lim_{n \rightarrow \infty} e^{-n\epsilon} \\ &= 0\end{aligned}$$

### 4.3.3 Relation between convergence in probability and in distribution

As we mentioned previously, convergence in probability is stronger than convergence in distribution. That is, if  $X_n \xrightarrow{p} X$ , then  $X_n \xrightarrow{d} X$ . The converse is not necessarily true. For example, let  $X_1, X_2, X_3, \dots, X_n$  be a sequence of i.i.d.  $Bernoulli(\frac{1}{2})$ . Let also  $X \sim Bernoulli(\frac{1}{2})$  be independent from the  $X_i$ . Then,  $X_n \xrightarrow{d} X$ . However,  $X_n$  does not converge in probability to  $X$ , since  $|X_n - X|$  is in fact also a  $Bernoulli(\frac{1}{2})$  random variable and

$$P(|X_n - X| \geq \epsilon) = \frac{1}{2} \neq 0, \text{ for } 0 < \epsilon < 1.$$

### 4.4 Convergence in probability implies convergence in distribution

We prove first the lemma

$$P(Y \leq a) \leq P(X \leq a + \varepsilon) + P(|Y - X| > \varepsilon), \forall \varepsilon > 0$$

By using this lemma, we have  $\forall \varepsilon > 0$  :

$$P(X_n \leq a) \leq P(X \leq a + \varepsilon) + P(|X_n - X| > \varepsilon)$$

and

$$P(X \leq a - \varepsilon) \leq P(X_n \leq a) + P(|X_n - X| > \varepsilon)$$

This leads to

$$F_X(a - \varepsilon) \leq \lim_{n \rightarrow \infty} P(X_n \leq a) \leq F_X(a + \varepsilon), \text{ for all } \varepsilon > 0$$

Then we have Q.E.D.

### 4.5 Convergence in mean

Let  $r \geq 1$  be a fixed number. A sequence of random variables  $X_1, X_2, \dots, X_n$  converges in the  $r^{th}$  mean or in the  $L^r$  norm to a random variable  $X$ , shown by  $X_n \xrightarrow{L_r} X$ , if we have:

1.  $X$  is  $L^r$  integrable.
2.  $X_1, X_2, \dots, X_n$  are  $L^r$  integrable.
- 3.

$$\lim_{n \rightarrow \infty} \mathbb{E}[|X_n - X|^r] = 0.$$

- If  $r = 1$ , it is called by default convergence in mean or  $L^1$  convergence.
- If  $r = 2$ , it is called the mean-square convergence, and it is shown by  $X_n \xrightarrow{m.s.} X$

#### 4.5.1 Example

Let  $X_n \sim Uniform(0, \frac{1}{n})$ . Show that  $X_n \xrightarrow{L_r} 0$ .

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathbb{E}[|X_n - X|^r] &= \lim_{n \rightarrow \infty} \mathbb{E}[X_n^r] \\ &= \lim_{n \rightarrow \infty} \int_0^{\frac{1}{n}} x^r n dx \\ &= \lim_{n \rightarrow \infty} n \frac{1}{r+1} \left(\frac{1}{n}\right)^{r+1} \\ &= 0\end{aligned}$$

#### 4.5.2 Relation between convergence in mean and in probability

As we mentioned previously, convergence in mean is stronger than convergence in probability. That is, if  $X_n \xrightarrow{L_r} X$ , then  $X_n \xrightarrow{P} X$ . The converse is not necessarily true. For example, consider a sequence  $\{X_n, n = 1, 2, 3, \dots\}$  such that:

$$X_n = \begin{cases} n^2 & \text{with probability } \frac{1}{n} \\ 0 & \text{with probability } 1 - \frac{1}{n} \end{cases}$$

We show that  $X_n \xrightarrow{P} 0$ , thus :

$$\begin{aligned}\lim_{n \rightarrow \infty} P(|X_n - 0| \geq \epsilon) &= \lim_{n \rightarrow \infty} P(X_n \geq \epsilon) \\ &= \lim_{n \rightarrow \infty} P(X_n = n^2) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \\ &= 0\end{aligned}$$

But

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathbb{E}[|X_n - 0|^r] &= \lim_{n \rightarrow \infty} \mathbb{E}[X_n^r] \\ &= \lim_{n \rightarrow \infty} n^{2r} \frac{1}{n} + 0^r \left(1 - \frac{1}{n}\right) \\ &= \lim_{n \rightarrow \infty} n^{2r-1} \\ &= \infty\end{aligned}$$

where  $r \geq 1$ , then we do not have  $X_n \xrightarrow{L_r} 0$ .

### 4.5.3 Convergence in mean implies convergence in probability

The demonstration is trivial by using Markov's inequality (in sec 2.2.8.1)

$$P(|X_n - X| \geq \epsilon) \leq \frac{\mathbb{E}[|X_n - X|]}{\epsilon}, \forall n \in \mathbb{N}$$

We first fix  $\epsilon$  and take  $n \rightarrow \infty$ , then  $\mathbb{E}[|X_n - X|] \rightarrow 0$  or  $P(|X_n - X| \geq \epsilon) \rightarrow 0$ .

### 4.5.4 Lemma

$X_n \xrightarrow{L_1} X$  if and only if  $X_n \xrightarrow{p} X$  and  $X_n$  is uniformly integrable (sec 2.3.3).

**Proof**

In the direct sense, we see that the convergence in mean implies the convergence in probability in sec 2.4.5.3. On one hand, from the definition of convergence in mean (sec 2.4.5),  $\mathbb{E}[|X|] < +\infty$  and since  $\lim_{n \rightarrow \infty} \mathbb{E}[|X_n - X|] \rightarrow 0$ , with the triangle inequality, we have :

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|] < \mathbb{E}[|X|] + \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n - X|] < +\infty$$

On the other hand, always with the triangle inequality :

$$\begin{aligned} & \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_A] \\ & \leq \mathbb{E}[|X| \cdot \mathbb{1}_A] + \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n - X| \cdot \mathbb{1}_A] \\ & \leq KP(A) + \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X| > K\}}] + \max \left( \sup_{n \leq n_0} \mathbb{E}[|X_n - X| \cdot \mathbb{1}_A], \sup_{n > n_0} \mathbb{E}[|X_n - X| \cdot \mathbb{1}_A] \right) \\ & = KP(A) + \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X| > K\}}] + \max \left( KP(A) + \sup_{n \leq n_0} \mathbb{E}[|X_n - X| \cdot \mathbb{1}_{\{|X_n - X| > K\}}], \sup_{n > n_0} \mathbb{E}[|X_n - X| \cdot \mathbb{1}_A] \right) \end{aligned}$$

We have, given a  $\varepsilon > 0$  :

- $X$  is integrable and  $X_n$  are integrable (by def of converge in mean or by  $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|] < +\infty$  that we've just proved). Then  $\mathbb{E}[|X - X_n|] < \mathbb{E}[|X|] + \mathbb{E}[|X_n|] < +\infty$  or  $X - X_n$  are integrable.
- Since  $X$  and  $X_n - X, \forall n \leq n_0$  are integrable, set  $\{X, X_1 - X, \dots, X_{n_0} - X\}$  has a finite number of elements, thne we can choose  $K_0$  such that  $\forall K > K_0$ ,  $\mathbb{E}[|X| \cdot \mathbb{1}_{\{|X| > K\}}] < \varepsilon$  and  $\sup_{n \leq n_0} \mathbb{E}[|X_n - X| \cdot \mathbb{1}_{\{|X_n - X| > K\}}] < \varepsilon$  (sec 2.3.3.6).
- We can choose also  $n_0$  such that  $\forall n > n_0, \mathbb{E}[|X_n - X|] < \varepsilon$ , this infers  $\mathbb{E}[|X_n - X| \cdot \mathbb{1}_A] < \varepsilon$ .
- Take  $\delta = \frac{\varepsilon}{K_0}$ , then if  $P(A) < \delta$ , then  $K_0 P(A) < \varepsilon$ .

Finally,

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \cdot \mathbb{1}_A] \leq \epsilon + \epsilon + \max(\epsilon + \epsilon, \epsilon) = 4\epsilon, \text{ if } P(A) < \delta$$

or  $X_n$  are uniformly integrable.

**Conversely**, we show that if  $X_n \xrightarrow{p} X$  and  $X_n$  is uniformly integrable then  $X_n \xrightarrow{L_1} X$ .

First, we show that  $X$  is integrable. Since  $X_n \xrightarrow{p} X$  then there exists a sub-sequence  $(n_k)$  such that  $X_{n_k} \xrightarrow{a.s} X$  (sec 2.4.6.1). By Fatou's lemma (sec 2.2.16), we have :

$$\mathbb{E}[|X|] = \mathbb{E}[\liminf_{k \rightarrow \infty} |X_{n_k}|] \leq \liminf_{k \rightarrow \infty} \mathbb{E}[|X_{n_k}|] \leq \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|] < +\infty$$

Second, by triangle inequality:

$$\mathbb{E}[|X_n - X|] \leq \mathbb{E}[|X_n| + |X|] = \mathbb{E}[|X_n|] + \mathbb{E}[|X|] < +\infty$$

Then we have

- $Y = |X_n| + |X|$  is integrable.
- $|X_n - X| \leq Y = |X_n| + |X|$ .

By proposition 2.3.3.5,  $(X_n - X)$  is uniformly integrable.

Finally,

$$\begin{aligned} \mathbb{E}[|X_n - X|] \\ = & \mathbb{E}[|X_n - X| \cdot \mathbb{1}_{|X_n - X| > \varepsilon}] + \mathbb{E}[|X_n - X| \cdot \mathbb{1}_{|X_n - X| \leq \varepsilon}] \\ \leq & \mathbb{E}[|X_n - X| \cdot \mathbb{1}_{|X_n - X| > \varepsilon}] + \varepsilon, \quad \forall \varepsilon > 0 \end{aligned}$$

By two following statements :

- $X_n \xrightarrow{p} X$ , which means  $|X_n - X| \xrightarrow{p} 0$  or  $\lim_{n \rightarrow \infty} P(|X_n - X| > \varepsilon) = 0$
- The absolutely continuous property in the original definition of uniformly integrable (sec 2.3.2)

Then  $\lim_{n \rightarrow \infty} \mathbb{E}[|X_n - X| \cdot \mathbb{1}_{|X_n - X| > \varepsilon}] < \varepsilon_1, \forall \varepsilon_1 > 0$ , this implies :

$$\lim_{n \rightarrow \infty} \mathbb{E}[|X_n - X|] < \varepsilon + \varepsilon_1, \forall \varepsilon, \varepsilon_1 > 0$$

or  $X_n \xrightarrow{L_1} X$ .

## 4.6 Almost sure convergence

A sequence of random variables  $X_1, X_2, X_3, \dots, X_n$  converges almost surely to a random variable  $X$ , shown by  $X_n \xrightarrow{a.s.} X$  if:

$$A = \{\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)\}$$

$$P(A) = 1$$

where  $\Omega$  is the sample space, which is the set of all possible outcomes  $\omega$ .  $X_i(\omega)$  here is a mapping function from  $\Omega$  to the set of real numbers.

Each time  $\omega$  is fixed ( $= \omega_0$ ), it means while solving for A, we check if

$$X_1(\omega_0), X_2(\omega_0), X_3(\omega_0), \dots, X_n(\omega_0) \rightarrow X(\omega_0)$$

### 4.6.1 Properties

- If  $X_n \xrightarrow{a.s.} X$  then  $X_n \xrightarrow{p} X$ .
- If  $X_n \xrightarrow{p} X$  then there exists a sub-sequence  $(n_k)$  such that  $X_{n_k} \xrightarrow{a.s.} X$ .
- The dominated convergence theorem (sec 2.2.17.2) gives sufficient conditions for almost sure convergence to imply convergence in mean :

$$\left. \begin{array}{l} X_n \xrightarrow{a.s.} X (\Rightarrow X_n \xrightarrow{p} X) \\ |X_n| < Y \\ E[|Y|] < +\infty \end{array} \right\} \Rightarrow X_n \xrightarrow{L_1} X$$

### 4.6.2 Example

Consider the following random experiment: A fair coin is tossed once. Here, the sample space has only two elements  $S = \{H, T\}$ . We define a sequence of random variables  $X_1, X_2, X_3, \dots$  on this sample space as follows:

$$X_n(s) = \begin{cases} \frac{n}{n+1} & \text{if } \omega = H \\ (-1)^n & \text{if } \omega = T \end{cases}$$

We define also :

$$X(\omega) = 1, \forall \omega \in \{S, T\}$$

Is  $X_n \xrightarrow{a.s.} X$  ?

In the first step, we need to find  $A$ . Thus  $A = \{H\}$ .

Since  $P(A) = \frac{1}{2}$ . Then  $X_n$  is not almost sure convergence.

### 4.6.3 Example in a continuous case

Consider the sample space  $\Omega = [0, 1]$  with a probability measure that is uniform on this space, i.e.,

$$P([a, b]) = b - a, \forall a, b \text{ with } 0 \leq a \leq b \leq 1.$$

Define the sequence  $X_n, n = 1, 2, \dots$  as follows:

$$X_n(\omega) = \begin{cases} 1 & \text{if } 0 \leq \omega < \frac{n+1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

Also, define the random variable  $X$  on this sample space as follows

$$X(\omega) = \begin{cases} 1 & \text{if } 0 \leq \omega < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

In the first step, we need to find  $A$ . Thus  $A = [0, 1] - \{\frac{1}{2}\}$ . However, we still have  $P(A) = 1$ . Then  $X_n \xrightarrow{a.s.} X$ .

### 4.6.4 Remarks

We note that in **the continuous case** if there are single values (or infinite countable set)  $\omega$  such that  $X_n(\omega)$  do not converge, we still have the convergence (a.s.). The convergence is not true if there are intervals (even small) such that  $\forall \omega \in (a, b), X_n(\omega)$  do not converge.

Conversely, in **discrete case** (first example), if there is only one single values  $\omega$  such that  $X_n(\omega)$  do not converge, we do not have the a.s. convergence.

### 4.6.5 Relation between convergence in probability and almost sure convergence

Let's consider space sample  $\Omega = [0, 1], X(\omega) = 0, \forall \omega \in \Omega$  and  $\forall n \geq 1$ :

$$X_n(\omega) = \begin{cases} 1 & \text{if } \omega \in [\frac{k}{2^m}, \frac{k+1}{2^m}] \\ 0 & \text{otherwise.} \end{cases}$$

where  $m = \lfloor \log_2(n) \rfloor$  (floor part of  $\log_2(n)$ ) and  $k = n - 2^m$ . Here are some  $[\frac{k}{2^m}, \frac{k+1}{2^m}]$  for  $n = 1, 2, \dots$  :

$$\left[ \frac{0}{1}, \frac{1}{1} \right], \left[ \frac{0}{2}, \frac{1}{2} \right], \left[ \frac{1}{2}, \frac{2}{2} \right], \left[ \frac{0}{4}, \frac{1}{4} \right], \left[ \frac{1}{4}, \frac{2}{4} \right], \left[ \frac{2}{4}, \frac{3}{4} \right], \left[ \frac{3}{4}, \frac{4}{4} \right], \dots$$

First we show that  $X_n$  converges in probability :

$$\begin{aligned} \lim_{n \rightarrow \infty} P(|X_n - X| \geq \epsilon) &= \lim_{n \rightarrow \infty} P(|X_n - 0| \geq \epsilon) \\ &= \lim_{n \rightarrow \infty} P(X_n = 1) \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2^{\lfloor \log_2(n) \rfloor}} \\ = 0$$

Second, we show that  $X_n$  does converge a.s. Even, we can say

$$P(\{\omega \in \Omega: \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)\}) = P(\{\omega \in \Omega: \lim_{n \rightarrow \infty} X_n(\omega) = 0\}) = 0$$

since for each  $\omega \in \Omega$ ,  $X_n(\omega)$  outputs infinitely 1 as  $n \rightarrow \infty$ . This is because  $\omega$  will be found in a interval  $[\frac{k}{2^m}, \frac{k+1}{2^m}]$ .

#### Then convergence in probability do not imply convergence a.s.

Illustratively, we can see some first random variables of this example in figure 2.5.

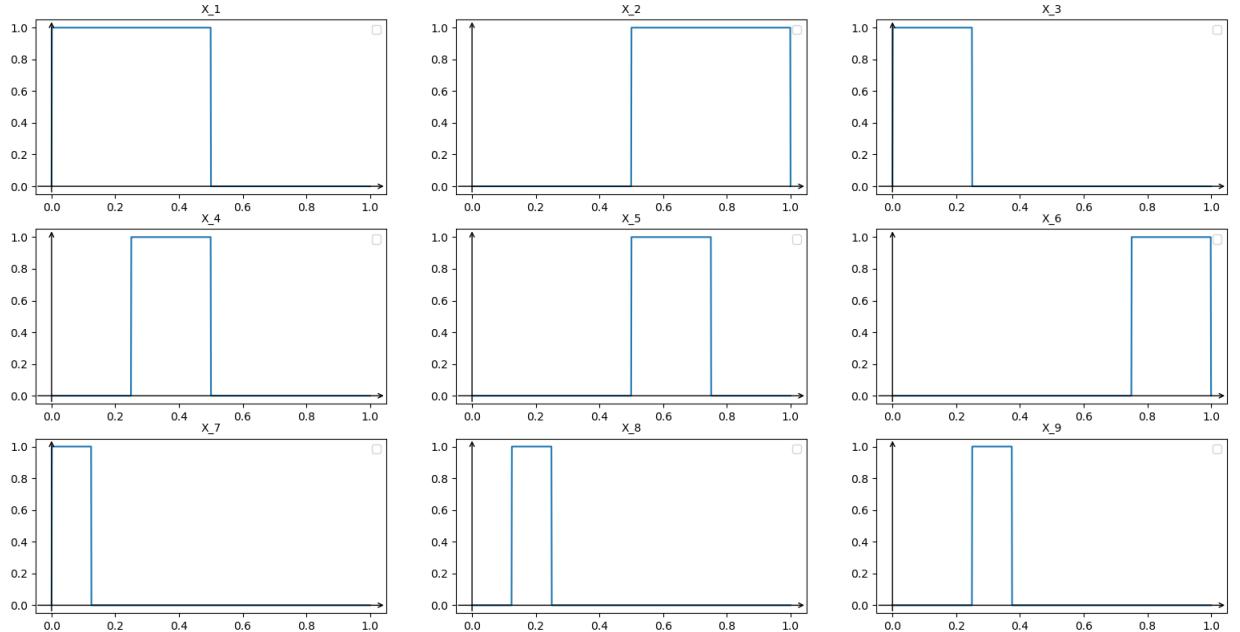


Figure 2.5: Distinction between convergence in probability and almost sure. Here  $\Omega = [0, 1]$  and  $\omega$  follows uniform distribution.  $\omega$  is represented by the horizontal axis. We present only the first 9 random variable  $X_1, \dots, X_9$  but it can be continued. The mainpoint is that with a fixed  $\omega$ , as  $n$  increases, we can refind 1.

##### 4.6.5.1 Note

- For convergence in probability :  $\omega$  is not necessarily fixed. For a given  $n$ , then we find set of  $\omega$  and move  $n \rightarrow \infty$  :

$$\lim_{n \rightarrow \infty} P(\{\omega: |X_n(\omega) - X(\omega)| \geq \epsilon\}) = 0, \quad \text{for all } \epsilon > 0.$$

In figure 2.5, given  $n$  first and we find  $\omega$ , then it is ok for convergence in probability.

- For convergence a.s. :  $\omega$  is fixed first and then move  $n \rightarrow \infty$  :

$$P(\{\omega \in \Omega: \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)\}) = 1$$

In figure 2.5, we fix  $\omega$  first then when  $n \rightarrow \infty$ , we refind always 1.

With a close attention, the position of  $\lim_{n \rightarrow \infty}$  is different in two above definitions.

#### 4.6.5.2 Convergence in mean and convergence almost surely

Moreover, we can show that  $X_n$  converges in mean :

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[|X_n - X|^r] &= \lim_{n \rightarrow \infty} \mathbb{E}[|X_n - 0|^r] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[X_n^r] \\ &= \lim_{n \rightarrow \infty} 1^r \frac{1}{2^{\lfloor \log_2(n) \rfloor}} + 0^r \left(1 - \frac{1}{2^{\lfloor \log_2(n) \rfloor}}\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2^{\lfloor \log_2(n) \rfloor}} \\ &= 0 \end{aligned}$$

**Then convergence in mean does not imply convergence a.s..** Note that convergence a.s. also so does not imply convergence in mean.

#### 4.6.6 Useful method in sufficient condition

In some problems, proving almost sure convergence directly can be difficult. Thus, it is desirable to know some **sufficient** conditions for almost sure convergence. Here is a result that is sometimes useful when we would like to prove almost sure convergence.

Consider the sequence  $X_1, X_2, X_3, \dots$ . If for all  $\epsilon > 0$ , we have :

$$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) < \infty,$$

then  $X_n \xrightarrow{a.s.} X$ .

**Example** Consider the sequence  $X_1, X_2, X_3, \dots$  such that:

$$X_n = \begin{cases} -\frac{1}{n} & \text{with probability } \frac{1}{2} \\ \frac{1}{n} & \text{with probability } \frac{1}{2} \end{cases}$$

Let  $X = 0$ , then

$$\begin{aligned}
\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) &= \sum_{n=1}^{\infty} P\left(\frac{1}{n} > \epsilon\right) \\
&= \sum_{n=1}^{\infty} P\left(\frac{1}{\epsilon} > n\right) \\
&\leq \sum_{n=1}^{\lfloor \frac{1}{\epsilon} \rfloor} P\left(\frac{1}{\epsilon} > n\right) \\
&= \lfloor \frac{1}{\epsilon} \rfloor \\
&< \infty
\end{aligned}$$

Then  $X_n \xrightarrow{a.s.} 0$

#### 4.6.7 Useful method in sufficient and necessary

The above condition provides only a sufficient direction for almost sure convergence. In particular, if we obtain

$$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) = \infty$$

then we still don't know whether the  $X_n$  converges to  $X$  almost surely or not. Here, we provide a condition that is both necessary and sufficient:

Consider the sequence  $X_1, X_2, X_3, \dots$ , for any  $\epsilon \rightarrow 0$ , define the set :

$$A_m = \{\omega \mid |X_n(\omega) - X(\omega)| < \epsilon, \text{ for all } n \geq m\}.$$

Then  $X_n \xrightarrow{a.s.} X$  if and only if for any  $\epsilon > 0$ , we have :

$$\lim_{n \rightarrow \infty} P(A_m) = 1$$

#### Example

Let  $X_1, X_2, X_3, \dots$  be independent random variables, where  $X_n \sim Bernoulli(\frac{1}{n})$  for  $n = 2, 3, \dots$ . The goal here is to check whether  $X_n \xrightarrow{a.s.} 0$ .

First,

$$\begin{aligned}
\sum_{n=1}^{\infty} P(|X_n - 0| > \epsilon) &= \sum_{n=1}^{\infty} P(X_n > \epsilon) \\
&= \sum_{n=1}^{\infty} P(X_n = 1) \\
&= \sum_{n=1}^{\infty} \frac{1}{n}
\end{aligned}$$

$$= \infty$$

Up to here, we are not sure that  $X_n$  converges. For any  $\epsilon > 0$ , we define  $A_m$  such that :

$$A_m = \{\omega \mid |X_n(\omega)| < \epsilon, \text{ for all } n \geq m\}.$$

Then

$$A_m = \{\omega \mid X_n(\omega) = 0, \text{ for all } n \geq m\}.$$

$$\begin{aligned} \lim_{m \rightarrow \infty} P(A_m) &= \lim_{m \rightarrow \infty} P(X_m = 0, X_{m+1} = 0, \dots, X_\infty = 0) \\ &\leq \lim_{m \rightarrow \infty} P(X_m = 0, X_{m+1} = 0, \dots, X_{2m} = 0) \\ &= \lim_{m \rightarrow \infty} P(X_m = 0)P(X_{m+1} = 0) \dots P(X_{2m} = 0) \quad (\text{independent}) \\ &= \lim_{m \rightarrow \infty} \frac{m-1}{m} \frac{m}{m+1} \dots \frac{2m-1}{2m} \\ &= \lim_{m \rightarrow \infty} \frac{m-1}{2m} \\ &= \frac{1}{2} < 1 \end{aligned}$$

Therefore, we conclude that  $X_n$  do not converge (a.s.) to 0. Note that  $X_n$  converges in probability. This example shows again convergence in probability does not imply convergence in a.s.

#### 4.6.8 Borel-Cantelli lemma

##### 4.6.8.1 Statement

Let  $E_1, E_2, \dots$  be a sequence of events (set of  $\omega$ ) in some probability space. The Borel–Cantelli lemma states that, if the sum of the probabilities of the events  $E_n$  is finite :

$$\sum_{n=1}^{\infty} P(E_n) < \infty$$

then the probability that infinitely many of them occur is 0, that is,

$$P\left(\limsup_{n \rightarrow \infty} E_n\right) = 0$$

where

$$\limsup_{n \rightarrow \infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

We call  $\limsup_{n \rightarrow \infty} E_n$  set-theoretic limit supremum, where we perform union first then intersection. For example of set-theoretic limit supremum, let  $E_n =$

$(-\frac{1}{n}, 1 - \frac{1}{n}]$ . Then

$$\limsup_{n \rightarrow \infty} E_n = \bigcap_n \bigcup_{j \geq n} \left( -\frac{1}{j}, 1 - \frac{1}{j} \right] = \bigcap_n \left( -\frac{1}{n}, 1 \right) = [0, 1)$$

### Example

Suppose  $X_n$  is a sequence of random variables with  $P(X_n = 0) = \frac{1}{n^2}$ . Let  $E_n = \{\omega: X_n(\omega) = 0\}$ . Then we have

$$\sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

where the sum of  $\frac{1}{n^2}$  is Basel problem.

$$\begin{aligned} \limsup_{n \rightarrow \infty} E_n &= \bigcap_n \bigcup_{k=n}^{\infty} E_k \\ &= \{\omega: \bigcap_n^{\infty} (X_n(\omega) = 0 \text{ or } X_{n+1}(\omega) = 0 \text{ or } \dots)\} \\ &= \{\omega: \bigcap_n^{\infty} \text{At least one of } X_n, \dots, X_{\infty} \text{ is 0}\} \\ &= \{\omega: \lim_{n \rightarrow \infty} \text{At least one of } X_n, \dots, X_{\infty} \text{ is 0}\} \end{aligned}$$

By Borel Cantelli lemma, then

$$P\left(\limsup_{n \rightarrow \infty} E_n\right) = 0$$

It means the probability that  $X_n = 0$  occurs for infinite set of  $n$  is 0.

#### 4.6.8.2 Converse result

Also called second Borel Cantelli lemma. If  $\sum_{n=1}^{\infty} P(E_n) = \infty$  and the events  $(E_n)_{n=1}^{\infty}$  are independent, then

$$P\left(\limsup_{n \rightarrow \infty} E_n\right) = 1$$

### Example

Let's reuse the example in sec 2.4.6.7. Let  $X_1, X_2, X_3, \dots$  be independent random variables, where  $X_n \sim \text{Bernoulli}(\frac{1}{n})$  for  $n = 2, 3, \dots$

First,

$$\sum_{n=1}^{\infty} P(X_n = 1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

and  $X_n$  are independent, then we have

$$P(\omega: \lim_{n \rightarrow \infty} \text{At least one of } X_n, \dots, X_\infty \text{ is 1}) = 1$$

It means the probability that  $X_n = 1$  occurs for infinite set of  $n$  is 1. Therefore,  $P(\omega: \lim_{n \rightarrow \infty} X_n \rightarrow 0) = 0$ , then  $X_n$  do not converge almost surely to  $X = 0$ .

#### 4.6.9 Strong version of law of large numbers

This is a strong version, in contrast with the one (weak) in subsection 2.4.3.1.1.

Let  $X_1, X_2, \dots, X_n$  be i.i.d. random variables with a finite expected value  $\mathbb{E}[X_i] = \mu < \infty$ . Then

$$\bar{X} \xrightarrow{a.s.} \mu$$

Note that in this strong version, we do not need to assume that  $X_i$  has the second moment.

## 5 Signatures of a distribution

### 5.1 Moment generating function

Given a random variable  $X$ , its Moment generating function (MGF) is defined by :

$$M_X(t) = \mathbb{E}[e^{tX}]$$

MGF of several popular distributions :

- Normal  $\mathcal{N}(\mu, \sigma^2)$ :  $\exp(t\mu + \frac{1}{2}\sigma^2 t^2)$
- Multivariate normal  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ :  $\exp(t^T \boldsymbol{\mu} + \frac{1}{2}t^T \boldsymbol{\Sigma} t)$

### 5.2 Characteristic function

Given a random variable  $X$ , its Characteristic function (CF) is defined by :

$$\varphi_X(t) = \mathbb{E}[e^{itX}]$$

CF of several popular distributions :

- Normal  $\mathcal{N}(\mu, \sigma^2)$ :  $\exp(it\mu - \frac{1}{2}\sigma^2 t^2)$
- Multivariate normal  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ :  $\exp(it^T \boldsymbol{\mu} - \frac{1}{2}t^T \boldsymbol{\Sigma} t)$

### 5.3 Remarks

While both MGF and CF are useful for describing the properties of random variables, they have some important differences :

- The MGF of a real-valued distribution does not always exist, indeed MGF only exists for random variables with finite moments. While the CF always exists.
- Additionally, the MGF is typically used for calculating moments and derivatives, while the CF is used for calculating convolutions and characterizing the shape of the distribution.

### 5.4 Probability generating function

Given  $X$  is a *discrete* random variable that takes value in  $\mathbb{N}$ . Then the probability generating function of  $X$ , noted  $G_X$  is defined by :

$$G_X(t) = \mathbb{E}(t^X) = \sum_{n=0}^{+\infty} P(X = n)t^n$$

We will show that  $G_X(t)$  converges with  $t \in [-1, 1]$ . Thus :

$$\begin{aligned} \sum_{n=0}^{+\infty} |P(X = n)t^n| &= \sum_{n=0}^{+\infty} |P(X = n)||t^n| \\ &\leq \sum_{n=0}^{+\infty} P(X = n) \\ &= 1 \end{aligned}$$

Remind the absolute convergence implies the (ordinary) convergence. The converse is not true. In short, if  $\sum_{n=0}^{+\infty} |a_n|$  converges then  $\sum_{n=0}^{+\infty} a_n$  converges. We infer from here that  $G_X(t) = \sum_{n=0}^{+\infty} P(X = n)t^n$  converges, since  $\sum_{n=0}^{+\infty} |P(X = n)t^n|$  converges.

Here, we note that the radius of convergence is surely the disk  $|t| \leq 1$  for all probability generating functions. Furthermore, in many examples, the radius of convergence is maybe larger than 1.

#### 5.4.1 Properties

- $G_X = G_Y$  (with  $t \in [-1, 1]$ ) if and only if  $X$  and  $Y$  are the same distribution.
- $G_{X+Y}(t) = G_X(t)G_Y(t)$  with  $X, Y$  are two independent random variables.
- $G'_X(1) = \mathbb{E}[X]$
- $E[X(X - 1)] = G''_X(1)$
- $Var[X] = G''_X(1) + G'_X(1) - G'_X(1)^2$

### 5.4.2 PGF of some usual distributions

- $X \sim \mathcal{U}([1, N])$  then  $G_X(t) = \frac{1}{N} \sum_{n=1}^N t^n$
- $X \sim \mathcal{B}(p)$  with  $p \in [0, 1]$  then  $G_X(t) = 1 - p + pt$
- $X \sim \mathcal{B}(n, p)$  with  $p \in [0, 1]$  then  $G_X(t) = (1 - p + pt)^n$
- $X \sim \mathcal{G}(p)$  with  $p \in [0, 1]$  then with  $t \in \left[-\frac{1}{1-p}, \frac{1}{1-p}\right]$ ,  $G_X(t) = \frac{pt}{1-(1-p)t}$
- $X \sim \mathcal{P}(\lambda)$  with  $\lambda > 0$ , then  $G_X(t) = e^{\lambda(t-1)}$

## 5.5 Expected value at any event

Given a probability space  $(\Omega, \mathcal{F}, P)$  and a random variable  $X$  the expect value of  $X$  at an event  $A \in \mathcal{F}$  is denoted by :

$$\mathbb{E}[X \mathbb{1}_A]$$

### 5.5.1 Lemma

Given two random variables  $X$  and  $Y$  on the same probability space  $(\Omega, \mathcal{F}, P)$ , then  $X = Y$  almost surely if

$$\mathbb{E}[X \mathbb{1}_A] = \mathbb{E}[Y \mathbb{1}_A] \quad , \forall A \in \mathcal{F}$$

#### Proof

We consider three types of  $A$  for any  $A \in \mathcal{F}$  :

1.  $A = \{X = Y\} = \{\omega | X(\omega) = Y(\omega)\}$
2.  $A = \{X < Y\} = \{\omega | X(\omega) < Y(\omega)\}$
3.  $A = \{X > Y\} = \{\omega | X(\omega) > Y(\omega)\}$

Then we need to show only for the second type and the third type that  $X = Y$ .

For the second type, consider the random variable  $X - Y$ , which the expected value is

$$\mathbb{E}[(X - Y) \mathbb{1}_A] = \mathbb{E}[X \mathbb{1}_A] - \mathbb{E}[Y \mathbb{1}_A] = 0$$

In this case,  $(X - Y) \mathbb{1}_A$  is strictly negative, then  $\mathbb{E}[(X - Y) \mathbb{1}_A] = 0$  can happen in two cases :

- There is no such as  $A$  as an interval.
- $A$  contains only a single outcome.

For the third type, we do the same thing. Finally, we can say that  $X = Y$  almost surely.

## 6 Multivariate normal distribution

### 6.1 The Method of Transformations

Starting example : Let  $X$  be a  $Uniform(0, 1)$  random variable, and let  $Y = e^X$ . What is the pdf of  $Y$  ?

In general, one calculates the cdf and then taking its derivative. As  $R_X = [0, 1]$ ,  $R_Y = [1, e]$ . The cdf of  $X$  is:

$$F_X(x) = x \quad (0 \leq x \leq 1)$$

Then the cdf of  $Y$  is:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(e^X \leq y) \\ &= P(X \leq \ln(y)) \\ &= F_X(\ln(y)) \\ &= \ln(y) \end{aligned}$$

Finally, the pdf of  $Y$ :

$$\begin{aligned} f_Y(y) &= F'_Y(y) \\ &= \frac{1}{y} \quad (1 \leq y \leq e) \end{aligned}$$

#### 6.1.1 Theorem 1

Suppose that  $X$  is a continuous random variable and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a strictly monotonic differentiable function. Let  $Y = g(X)$ . Then the PDF of  $Y$  is given by:

$$f_Y(y) = \begin{cases} \frac{f_X(x)}{|g'(x)|} &= f_X(x) \left| \frac{dx}{dy} \right| = f_X(x) \left| \frac{dx}{dg(x)} \right| \\ 0 &\text{if } y = g(x) \text{ does not have a solution.} \end{cases}$$

**Application** for the above example  $y = g(x) = e^x$ :

$$\begin{aligned} f_Y(y) &= f_X(x) \left| \frac{dx}{dg(x)} \right| \\ &= 1 \left| \frac{dx}{e^x dx} \right| \\ &= \frac{1}{y} \end{aligned}$$

#### 6.1.2 Theorem 2

Consider a continuous random variable  $X$  with domain  $R_X$ , and let  $Y = g(X)$ . Suppose that we can partition  $R_X$  into a finite number  $n$  of intervals

such that  $g(x)$  is strictly monotone and differentiable on each partition. Then the PDF of  $Y$  is given by:

$$f_Y(y) = \sum_i^n \frac{f_X(x_i)}{|g'(x_i)|} = \sum_i^n f_X(x_i) \left| \frac{dx_i}{dy} \right|$$

where,  $x_1, \dots, x_n$  are real solution for  $g(x) = y$

**Application :** Let  $X$  be a continuous random variable (standard normal distribution) with PDF :

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

and let  $Y = X^2$ . Find  $f_Y(y)$ .

We note that the function  $g(x) = x^2$  is strictly decreasing on the interval  $(-\infty, 0)$ , strictly increasing on the interval  $(0, \infty)$ , and differentiable on both intervals,  $g'(x) = 2x$ . Given an  $y > 0$ , we have two solutions for  $y = g(x)$ , in particular :

$$x_1 = \sqrt{y}, x_2 = -\sqrt{y}$$

Then :

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\ &= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}} \\ &= \frac{1}{2\sqrt{2\pi y}} e^{-\frac{1}{2}y} + \frac{1}{2\sqrt{2\pi y}} e^{-\frac{1}{2}y} \\ &= \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2}y} \end{aligned}$$

where  $y > 0$

At  $y = 0$ :

$$f_Y(0) = \frac{1}{\sqrt{2\pi}}$$

Then  $f_Y$  is not continue at 0.

### 6.1.3 Theorem 3, for bivariate case

Let  $X$  and  $Y$  be two jointly continuous random variables. Let  $(Z, W) = g(X, Y) = (g_1(X, Y), g_2(X, Y))$ , where  $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a continuous one-to-one (invertible) function with continuous partial derivatives. Let  $h = g^{-1}$ , i.e.,  $(X, Y) = h(Z, W) = (h_1(Z, W), h_2(Z, W))$ . Then  $Z$  and  $W$  are jointly continuous and their joint PDF,  $f_{ZW}(z, w)$ ,  $\forall (z, w) \in R_{ZW}$  is given by :

$$f_{ZW}(z, w) = f_{XY}(h_1(z, w), h_2(z, w)) |J|$$

where  $J$  is Jacobian matrix :

$$J = \det \begin{bmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix}$$

**Application :** Let  $X$  and  $Y$  be two independent standard normal random variables. Let also

$$\begin{cases} Z = 2X - Y \\ W = -X + Y \end{cases}$$

Find  $f_{ZW}(z, w)$

One needs to find  $h_1$  and  $h_2$  : From :

$$\begin{cases} X = Z + W \\ Y = Z + 2W \end{cases}$$

Then :

$$\begin{cases} h_1(z, w) = z + w \\ h_2(z, w) = z + 2w \end{cases}$$

$$J = \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1$$

Finally,

$$\begin{aligned} f_{ZW}(z, w) &= f_{XY}(h_1(z, w), h_2(z, w))|J| \\ &= f_{XY}(z + w, z + 2w)1 \\ &= \frac{1}{2\pi} e^{-\frac{(z+w)^2+(z+2w)^2}{2}} \end{aligned}$$

## 6.2 Bivariate case

In this case, the number of dimension  $k = 2$ .

Let  $X$  and  $Y$  two random variables, then

$$\Sigma = \begin{bmatrix} Var(X) & Cov(X, Y) \\ Cov(X, Y) & Var(Y) \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

where  $\rho$  is the correlation between  $X$  and  $Y$ . Remind the relation between covariance and correlation :

$$\rho(X, Y) = \frac{E[(X - E(X))(Y - E(Y))]}{\sqrt{Var(X)Var(Y)}} = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

Then  $Cov(X, Y) = \rho\sigma_X\sigma_Y$ . Remind that the density for multi-dimensions  $\mathcal{N}(\mu, \Sigma)$ :

$$\frac{1}{(2\pi)^{\frac{k}{2}} \det(\Sigma)^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

Then the expression in the exponentiel :

$$\begin{aligned}
& (x - \mu)^T \Sigma^{-1} (x - \mu) \\
&= [x - \mu_X \quad y - \mu_Y] \frac{1}{\sigma_X^2 \sigma_Y^2 (1 - \rho^2)} \begin{bmatrix} \sigma_Y^2 & -\rho \sigma_X \sigma_Y \\ -\rho \sigma_X \sigma_Y & \sigma_X^2 \end{bmatrix} \begin{bmatrix} x - \mu_X \\ y - \mu_Y \end{bmatrix} \\
&= \frac{\sigma_Y^2 (x - \mu_X)^2 - 2\rho \sigma_X \sigma_Y (x - \mu_X)(y - \mu_Y) + \sigma_X^2 (y - \mu_Y)^2}{\sigma_X^2 \sigma_Y^2 (1 - \rho^2)} \\
&= \frac{1}{1 - \rho^2} \left( \left( \frac{x - \mu_X}{\sigma_X} \right)^2 - \frac{2\rho(x - \mu_X)(y - \mu_Y)}{\sigma_X \sigma_Y} + \left( \frac{y - \mu_Y}{\sigma_Y} \right)^2 \right)
\end{aligned}$$

Then we have the first definition of two random variables are jointly normal distribution, with the density function :

### 6.2.1 Definition 1

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp \left( -\frac{1}{2(1-\rho^2)} \left( \left( \frac{x - \mu_X}{\sigma_X} \right)^2 - \frac{2\rho(x - \mu_X)(y - \mu_Y)}{\sigma_X \sigma_Y} + \left( \frac{y - \mu_Y}{\sigma_Y} \right)^2 \right) \right)$$

In some literature, we can have the second definition:

### 6.2.2 Definition 2

Two random variables are jointly normal distribution if any linear combination of them is univariate normal distribution.

### 6.2.3 Decomposition of bivariate normal distribution

Let  $X$  and  $Y$  be two bivariate normal random variables, i.e., their joint PDF is given by definition 1. Then there exist independent standard normal random variables  $Z_1$  and  $Z_2$  such that :

$$\begin{cases} X = \sigma_X Z_1 + \mu_X \\ Y = \sigma_Y (\rho Z_1 + \sqrt{1 - \rho^2} Z_2) + \mu_Y \end{cases}$$

where  $\rho$  is  $Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$

**Proof:**

Thus, the idea is that taking  $Z_1$  by standard normalizing  $X$  and then taking  $Z_2$  by removing correlated part with  $Z_1$ . Note that the denominator  $\sqrt{1 - \rho^2}$  help  $Z_2$  to be standard normal.

$$\begin{cases} Z_1 = \frac{X - \mu_X}{\sigma_X} \\ Z_2 = \frac{1}{\sqrt{1 - \rho^2}} \frac{Y - \mu_Y}{\sigma_Y} - \frac{\rho}{\sqrt{1 - \rho^2}} \frac{X - \mu_X}{\sigma_X} \end{cases}$$

We need to prove :

- 1,  $Z_1$  and  $Z_2$  are standard normal distribution.
- 2,  $Z_1$  and  $Z_2$  are independent.

First, one calculates the density function of joint distribution  $f_{Z_1 Z_2}(z_1, z_2)$ , using the theorem in sec 2.6.1.3 :

$$\begin{aligned} h_1(z_1, z_2) &= \sigma_X z_1 + \mu_X \\ h_2(z_1, z_2) &= \sigma_Y \rho z_1 + \sigma_Y \sqrt{1 - \rho^2} z_2 + \mu_y \\ \det(J) &= \det \begin{bmatrix} \sigma_X & 0 \\ \sigma_Y \rho z_1 & \sigma_Y \sqrt{1 - \rho^2} \end{bmatrix} = \sigma_X \sigma_Y \sqrt{1 - \rho^2} \end{aligned}$$

$$\begin{aligned} f_{Z_1 Z_2}(z_1, z_2) &= f_{XY}(\sigma_X z_1 + \mu_X, \sigma_Y \rho z_1 + \sigma_Y \sqrt{1 - \rho^2} z_2 + \mu_y) |J| \\ &= \frac{1}{2\pi} \exp \left( -\frac{1}{2(1-\rho^2)} (z_1^2 - 2\rho z_1 (\rho z_1 + \sqrt{1-\rho^2} z_2) + (\rho z_1 + \sqrt{1-\rho^2} z_2)^2) \right) \\ &= \frac{1}{2\pi} \exp \left( -\frac{z_1^2 + z_2^2}{2} \right) \end{aligned}$$

This mean that the two marginal probability density function  $f_{Z_1}(z_1), f_{Z_2}(z_2)$  are standard normal distribution :

$$\begin{aligned} f_{Z_1}(z_1) &= \int_{-\infty}^{+\infty} f_{z_1 z_2}(z_1, z_2) dz_2 \\ &= \frac{1}{2\pi} \exp \left( -\frac{z_1^2}{2} \right) \int_{-\infty}^{+\infty} \frac{1}{2\pi} \exp \left( -\frac{z_2^2}{2} \right) dz_2 \\ &= \frac{1}{2\pi} \exp \left( -\frac{z_1^2}{2} \right) \end{aligned}$$

$Z_1$  and  $Z_2$  are independent since :

$$f_{Z_1 Z_2}(z_1, z_2) = f_{Z_1}(z_1) f_{Z_2}(z_2)$$

In addition, recheck the covariance between  $Z_1$  and  $Z_2$  :

$$\begin{aligned} Cov(Z_1, Z_2) &= E[Z_1 Z_2] - E[Z_1] E[Z_2] \\ &= E \left[ \left( \frac{X - \sigma_X}{\sigma_X} \right) \left( -\frac{\rho}{\sqrt{1 - \rho^2}} \frac{X - \sigma_X}{\sigma_X} + \frac{1}{\sqrt{1 - \rho^2}} \frac{Y - \sigma_Y}{\sigma_Y} \right) \right] - 0 \\ &= E \left[ -\frac{\rho}{\sqrt{1 - \rho^2}} \left( \frac{X - \sigma_X}{\sigma_X} \right)^2 + \frac{1}{\sqrt{1 - \rho^2}} \left( \frac{X - \sigma_X}{\sigma_X} \right) \left( \frac{Y - \sigma_Y}{\sigma_Y} \right) \right] \\ &= -\frac{\rho}{\sqrt{1 - \rho^2}} 1 + \frac{1}{\sqrt{1 - \rho^2}} E \left[ \left( \frac{X - \sigma_X}{\sigma_X} \right) \left( \frac{Y - \sigma_Y}{\sigma_Y} \right) \right] \end{aligned}$$

$$\begin{aligned}
&= -\frac{\rho}{\sqrt{1-\rho^2}} + \frac{1}{\sqrt{1-\rho^2}} \text{Cov} \left[ \frac{X-\sigma_X}{\sigma_X}, \frac{Y-\sigma_Y}{\sigma_Y} \right] \\
&= -\frac{\rho}{\sqrt{1-\rho^2}} + \frac{1}{\sqrt{1-\rho^2}} \text{Cov} \left[ \frac{X}{\sigma_X}, \frac{Y}{\sigma_Y} \right] \\
&= -\frac{\rho}{\sqrt{1-\rho^2}} + \frac{1}{\sqrt{1-\rho^2}} \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y} \\
&= -\frac{\rho}{\sqrt{1-\rho^2}} + \frac{1}{\sqrt{1-\rho^2}} \rho \\
&= 0
\end{aligned}$$

#### 6.2.4 Proof for the equivalence between the two definitions

In the direction from definition 1 to definition 2 : Using the decomposition (sec 2.6.2.3).

In the direction from definition 2 to definition 1 : Given that  $aX + bY$  is normal distribution, for all  $a, b \in \mathbb{R}^2$ . Then prove that  $X$  and  $Y$  form a bivariate normal.

Let  $\rho$  the correlation of  $X$  and  $Y$  and let  $Z_1, Z_2$  be :

$$\begin{cases} Z_1 = \frac{X-\sigma_X}{\sigma_X} \\ Z_2 = -\frac{\rho}{\sqrt{1-\rho^2}} \frac{X-\sigma_X}{\sigma_X} + \frac{1}{\sqrt{1-\rho^2}} \frac{Y-\sigma_Y}{\sigma_Y} \end{cases}$$

Then  $Z_1, Z_2$  are both standard normal distribution. Now, we use MGF (Moment generating function, sec 2.5.1) to show that  $Z_1, Z_2$  follows bivariate (standard) normal distribution.

First, we remind the MGF for vector-valued random variables. Given a vector-valued random variable  $\mathbf{A}$  (dimension  $d > 1$ ), its MGF is :

$$M_{\mathbf{A}}(\mathbf{t}) = \mathbb{E} \left[ e^{\langle \mathbf{t}, \mathbf{A} \rangle} \right]$$

where  $\langle \cdot, \cdot \rangle$  is the dot product. We infer from this the MFG of  $[Z_1, Z_2]$ :

$$M_{Z_1, Z_2}(s, t) = \mathbb{E}[e^{sZ_1 + tZ_2}]$$

Since  $aX + bY$  is normal distribution then  $sZ_1 + tZ_2$  is also normal distribution, then  $e^{sZ_1 + tZ_2} \sim \text{Lognormal}(\mu_L, \sigma_L^2)$  (log-normal distribution), where  $\mu_L = \mathbb{E}[sZ_1 + tZ_2] = 0$  and  $\sigma_L^2 = \text{Var}(sZ_1 + tZ_2) = s^2 + t^2$ . Then the mean of this log-normal distribution is

$$\mathbb{E}[e^{sZ_1 + tZ_2}] = \mu_L + \frac{1}{2}\sigma_L^2 = \frac{1}{2}s^2 + \frac{1}{2}t^2$$

We found that  $M_{Z_1, Z_2}(s, t)$  has the same MGF as standard bivariate normal distribution. Remind that the MGF of standard bivariate normal distribution  $[A, B]$ :

$$M_{A, B}(s, t) = \mathbb{E} \left( \exp([s \quad t] \begin{bmatrix} A \\ B \end{bmatrix}) \right)$$

$$\begin{aligned}
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(a^2 + b^2)\right) \exp(sa + tb) da db \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(a^2 + b^2) + sa + sb - \frac{1}{2}(s^2 + t^2) + \frac{1}{2}(s^2 + t^2)\right) da db \\
&= \int_{-\infty}^{+\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(a - s)^2\right) da \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}(b - t)^2\right) db \exp\left(\frac{1}{2}(s^2 + t^2)\right) \\
&= \exp\left(\frac{1}{2}(s^2 + t^2)\right)
\end{aligned}$$

Then  $Z_1, Z_2$  form standard bivariate normal distribution, so  $X, Y$  has density function as in definition 1.

### 6.2.5 Conditional distribution in bivariate normal

Let  $X$  and  $Y$  be jointly normal random variables with parameters  $\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2$  and  $\rho$ . Find the conditional distribution  $p(Y|X = x)$ .

1, One way to solve this problem is by using the joint PDF formula :

$$f_{Y|X}(y|x) = \frac{f_{Y,X}(y,x)}{f_X(x)}$$

2, Or we can use the decomposition of bivariate normal distribution :

$$\begin{cases} X = \sigma_X Z_1 + \mu_X \\ Y = \sigma_Y (\rho Z_1 + \sqrt{1 - \rho^2} Z_2) + \mu_Y \end{cases}$$

With  $Z_1, Z_2$  are independent and standard normal distribution. Then :

$$f_{Y|X}(Y|X = x) = \sigma_Y (\rho \frac{x - \mu_X}{\sigma_X} + \sqrt{1 - \rho^2} Z_2) + \mu_Y$$

is also a normal distribution and :

$$E(Y|X = x) = \sigma_Y \rho \frac{x - \mu_X}{\sigma_X} + \mu_Y$$

$$Var(Y|X = x) = (1 - \rho^2) \sigma_Y^2$$

### 6.2.6 Remarks

The decomposition  $X, Y$  into two independent random variable  $Z_1$  and  $Z_2$  has three folds advantage :

- First, it is more convenient and insightful than the joint PDF formula.
- Second, sometimes the construction using  $Z_1$  and  $Z_2$  can be used to solve problems regarding bivariate normal distributions.
- Third, this method gives us a way to generate samples from the bivariate normal distribution, by using independent random variables.

### 6.2.7 Two non-independent normal distributions can form a non joint normal distribution

The fact that two random variables  $X$  and  $Y$  both have a normal distribution **does not imply** that the pair  $(X, Y)$  has a joint normal distribution.

Here is an famous counter example :

- $X$  is a standard normal distribution.
- $Y$  is defined by  $Y = X$  if  $|X| > a$  and  $Y = -X$  otherwise. As the symmetric of  $X$ , then  $Y$  is also normal distribution.

Then  $Z = X + Y = 2X$  if  $|X| > a$  and  $Z = X + Y = 0$  otherwise. One observes that  $Z$  is **not normal distribution**. The shape of  $Z$  is one peak at 0, then 0 extended to two sides until  $2a$  and  $-2a$ . After that  $f_Z(2x) = f_X(x)$ , where  $|x| > a$ .

**Remark :** By controlling  $a$ , we can set any value  $Cov(X, Y)$ .

By using the definition 2, then  $X, Y$  are not jointly normal distribution.

## 6.3 Multivariate case

For three or more random variables, the joint PDF, joint PMF, and joint CDF are defined in a similar way to what we have already seen for the case of two random variables (bivariate). Suppose that  $X_1, X_2, \dots, X_n$  are random variables. Then we have two equivalent definitions of Multivariate normal distribution :

### 6.3.1 Definition 1

If the joint distribution of  $\mathbf{x} = x_1, x_2, \dots, x_n$

$$f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)$$

we note  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  for multivariate normal distribution.

### 6.3.2 Definition 2

If any linear combination of  $k$  components is univariate normal distribution.

### 6.3.3 Generating a multivariate normal distribution

Given  $\boldsymbol{\mu}, \boldsymbol{\Sigma}$ , where  $\boldsymbol{\Sigma}$  is Hermitian (symmetric in real case), positive-definite matrix. We want to generate  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ :

- Generating  $\mathbf{z} = Z_1, Z_2, \dots, Z_n$  standard normal distribution and independent.
- With given  $\boldsymbol{\Sigma}$ , we can apply the Cholesky decomposition  $\boldsymbol{\Sigma} = LL^*$ , where  $L$  is a lower triangular matrix,  $L^*$  is conjugate transpose of  $L$ .
- Then  $L\mathbf{z} + \boldsymbol{\mu} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

### 6.3.4 Decomposition of multivariate normal distribution

Given  $\mathbf{x} = X_1, X_2, \dots, X_n$  are multivariate normal distribution  $\mathcal{N}(\mu, \Sigma)$ , then there exists the decomposition of  $\mathbf{x}$  into  $\mathbf{z} = Z_1, Z_2, \dots, Z_n$ :

$$\mathbf{x} = L\mathbf{z} + \mu$$

Or

$$\mathbf{z} = L^{-1}(\mathbf{x} - \mu)$$

where  $Z_1, Z_2, \dots, Z_n$  are all standard normal and independent.

Check in case of bivariate normal :

$$\Sigma = \begin{bmatrix} Var(X) & Cov(X, Y) \\ Cov(X, Y) & Var(Y) \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

Then :

$$L = \begin{bmatrix} \sigma_X & 0 \\ \rho\sigma_Y & \sqrt{1 - \rho^2}\sigma_Y \end{bmatrix}$$

### 6.3.5 Conditional distributions

Given that  $\mathbf{x} = X_1, X_2, \dots, X_n$  and  $\mathbf{x} \sim \mathcal{N}(\mu, \Sigma)$ . With a partition  $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2] = [[X_1, \dots, X_q], [X_{q+1}, \dots, X_n]]$ , we have

$$p(\mathbf{x}_2|\mathbf{x}_1) \sim \mathcal{N}(\mu_{2|1}, \Sigma_{2|1})$$

To prove that  $p(\mathbf{x}_2|\mathbf{x}_1)$  is normal, as in bivariate case, we can use the decomposition of  $\mathbf{x}$  into independent  $\mathbf{z}$  :  $[\mathbf{x}_1, \mathbf{x}_2] = L[\mathbf{z}_1, \mathbf{z}_2] + [\mu_1, \mu_2]$ , where  $L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$ . Suppose that we know  $\mathbf{x}_1$ , hence  $\mathbf{z}_1$ . So

$$\mathbf{x}_2|\mathbf{x}_1 = ([L_{21} \quad L_{22}] \mathbf{z} + \mu_2)|(\mathbf{z}_1, \mu_1)$$

Since  $\mathbf{z}_2$  contains components that are independent, hence any linear combination of them is univariate normal distribution, so  $\mathbf{x}_2|\mathbf{x}_1$  is multivariate normal.

**Expected value**

$$\begin{aligned} \mu_{2|1} &= E[[L_{21} \quad L_{22}] \mathbf{z} + \mu_2] \\ &= [L_{21} \quad L_{22}] E[\mathbf{z}] + \mu_2 \\ &= L_{21}\mathbf{z}_1 + \mu_2 \\ &= L_{21}L_{11}^{-1}(\mathbf{x}_1 - \mu_1) + \mu_2 \\ &= \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{x}_1 - \mu_1) + \mu_2 \end{aligned}$$

Note that :

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = LL^* = \begin{bmatrix} L_{11}L_{11} & L_{11}L_{21} \\ L_{21}L_{11} & L_{21}L_{21} + L_{22}L_{22} \end{bmatrix}$$

Then

$$\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} = L_{21}L_{11}(L_{11}L_{11})^{-1} = L_{21}L_{11}^{-1}$$

### Covariance

$$\begin{aligned} Cov(\mathbf{x}_2|\mathbf{x}_1, \mathbf{x}_2|\mathbf{x}_1) &= Cov\left[\begin{bmatrix} L_{21} & L_{22} \end{bmatrix} \mathbf{z} + \boldsymbol{\mu}_2, \begin{bmatrix} L_{21} & L_{22} \end{bmatrix} \mathbf{z} + \boldsymbol{\mu}_2\right] \\ &= Cov\left[\begin{bmatrix} L_{21} & L_{22} \end{bmatrix} \mathbf{z}, \begin{bmatrix} L_{21} & L_{22} \end{bmatrix} \mathbf{z}\right] \\ &= Cov[L_{21}\mathbf{z}_1 + L_{22}\mathbf{z}_2, L_{21}\mathbf{z}_1 + L_{22}\mathbf{z}_2] \\ &= Cov[L_{22}\mathbf{z}_2, L_{22}\mathbf{z}_2] \\ &= L_{22}L_{22}Cov(\mathbf{z}_2, \mathbf{z}_2) \\ &= L_{22}L_{22}I \\ &= L_{22}L_{22} \\ &= \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12} \end{aligned}$$

Note that :

$$\begin{aligned} \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12} &= L_{21}L_{21} + L_{22}L_{22} - L_{21}L_{11}(L_{11}L_{11})^{-1}L_{11}L_{21} \\ &= L_{22}L_{22} \end{aligned}$$

## 6.4 Elliptical distribution

An elliptical distribution is a type of multivariate probability distribution that generalizes the concept of the normal distribution, which means that elliptical distributions is a family that contains the normal distribution and its other related distributions.

It is called “elliptical” because the contours (lines) of constant probability density are elliptical in shape when plotted on a graph. Elliptical distributions are widely used in multivariate statistics and finance for modeling joint distributions of random variables.

### 6.4.1 Definition 1

Elliptical distributions are defined in terms of the *characteristic function* of probability theory. A random vector  $X$  on a Euclidean space has an elliptical distribution if its characteristic function  $\phi$  satisfies the following functional equation (for every column-vector).

$$\phi_{X-\mu}(t) = \psi(t'\boldsymbol{\Sigma}t)$$

for some location parameter  $\mu$ , some nonnegative-definite matrix  $\boldsymbol{\Sigma}$  and some scalar function (mapping n dimensions to 1 dimension)  $\psi$ .

#### 6.4.2 Definition 2

Some elliptical distributions are alternatively defined in terms of their density functions. An elliptical distribution with a density function  $f$  has the form:

$$f(x) = k \cdot g((x - \mu)' \Sigma^{-1} (x - \mu))$$

where  $k$  is the normalizing constant,  $x$  is an  $n$ -dimensional random vector with median vector  $\mu$  (which is also the mean vector if the latter exists), and  $\Sigma$  is the covariance matrix if the latter exists.

#### 6.4.3 Examples

Examples include the following multivariate probability distributions:

- Multivariate normal distribution
- Multivariate t-distribution (see its density function and the definition 2).

## 7 Stochastic dominance

Stochastic dominance is a concept to compare and rank probability distributions or random variables.

### 7.1 Statewise dominance

The simplest case of stochastic dominance is statewise dominance (also known as state-by-state dominance), defined as follows: Random variable  $A$  is statewise dominant over random variable  $B$  if  $A$  gives at least as good a result in every state (every possible set of outcomes), and a strictly better result in at least one state.

#### 7.1.0.1 Example

State	a	b	c	d
Player 1's reward	\$1	\$2	\$2	\$2
Player 2's reward	\$1	\$1	\$2	\$2

Table 2.1: Player 1 is statewise dominant player 2.

### 7.2 First-order stochastic dominance

Random variable  $A$  has first-order stochastic dominance over random variable  $B$  if for any value  $x$  (in codomain of  $A$  and  $B$ ),  $A$  gives at least as high a

probability of receiving at least  $x$  as does  $B$ , and for some  $x$ ,  $A$  gives a higher probability of receiving at least  $x$ . In notation form :

$$\begin{aligned} P[A \geq x] &\geq P[B \geq x], \quad \forall x, \\ P[A \geq x] &> P[B \geq x], \quad \text{and for some } x \end{aligned}$$

In terms of the *cumulative distribution functions* of the two random variables,  $A$  dominating  $B$  means that :

$$\begin{aligned} F_A(x) &\leq F_B(x), \quad \forall x, \\ F_A(x) &< F_B(x), \quad \text{and for some } x \end{aligned}$$

since  $F_A(x) = 1 - P[A \geq x]$  and  $F_B(x) = 1 - P[B \geq x]$

Note that **Statewise dominance implies first-order stochastic dominance**.

#### 7.2.0.1 Example

. Let continue with the above example, and now with an hypothesis that the probability of appearance of state  $a, b, c, d$  are the same.

State	a	b	c	d
Player 1's reward	\$1	\$2	\$2	\$2
Player 2's reward	\$1	\$1	\$2	\$2
Player 3's reward	\$3	\$3	\$1	\$1

- Player 1 statewise dominates player 2, then we can say that Player 1 first-order stochastic dominates player 2 (as remark above). Indeed,  $P(P1 \geq 1) = (P2 \geq 1) = 1$  and  $P(P1 \geq 2) = 3/4 > P(P2 \geq 2) = 1/2$ .
- Player 3 first-order stochastic dominates player 2 since  $P(P3 \geq 1) = (P2 \geq 1) = 1$ ,  $P(P3 \geq 2) = P(P2 \geq 2) = 1/2$  and  $P(P3 \geq 3) = 1/2 > P(P2 \geq 3) = 0$ .
- Player 1 and Player 3 cannot be ordered relative to each other.

### 7.3 Second-order

Given two distributions  $\rho$  and  $\nu$ , in terms of cumulative distribution functions  $F_\rho$  and  $F_\nu$ ,  $\rho$  is second-order stochastically dominant over  $\nu$  if and only if

$$\int_{-\infty}^x (F_\rho(t) - F_\nu(t)) dt \leq 0, \forall x$$

and with strict inequality at some  $x$ .

### 7.3.1 Equivalent definitions

There are two equivalent definition for the above one :

- $\rho$  dominates  $\nu$  in the second order if and only if  $\mathbb{E}_{X \sim \rho}[u(X)] \geq \mathbb{E}_{X \sim \nu}[u(X)]$  for nondecreasing and concave functions  $u(x)$ .
- **Decomposition** :  $\rho$  second-order stochastically dominates  $\nu$  if and only if there exist distributions  $y$  and  $z$  such that  $x_\rho \stackrel{d}{=} (x_\nu + y + z)$ , with  $y \leq 0$  and  $\mathbb{E}(z | x_\rho + y) = 0$ .

These are analogous with the equivalent definitions of first-order stochastic dominance, given above. Also, **first-order stochastic dominance implies for second-order dominance**, since  $F_\rho(t) \leq F_\nu(t)$  implies  $\int_{-\infty}^x (F_\rho(t) - F_\nu(t)) dt \leq 0$

### 7.3.2 Mean-preserving spread

Mean-preserving spread (MPS) is a change from one probability distribution A to another probability distribution B, where B is formed by spreading out one or more portions of A's probability density function or probability mass function while leaving the mean (the expected value) unchanged.

**Mathematical definitions** Let  $x_A$  and  $x_B$  be the random variables associated with gambles A and B. Then B is a mean-preserving spread of A if and only if  $x_B \stackrel{d}{=} (x_A + z)$  for some random variable  $z$  having  $E(z | x_A) = 0$  for all values of  $x_A$ . Here  $\stackrel{d}{=}$  means “is equal in distribution to” (that is, “has the same distribution as”).

**Example** This example shows MPS between A and B

Density	0.2	0.2	0.2	0.2	0.2
$x_A$	3	4	5	6	7
$x_B$	2	4	5	6	8

B is obtained by moving 20% of A from 3 to 2 and 20% of A from 7 to 8.

## 7.4 Third-order

Let  $F_\rho$  and  $F_\nu$  be the cumulative distribution functions.  $\rho$  dominates  $\nu$  in the third order if and only if both

$$\begin{aligned} \int_{-\infty}^x \left( \int_{-\infty}^z [F_\rho(t) - F_\nu(t)] dt \right) dz \leq 0, \forall x \\ \mathbb{E}_\rho(x) \geq \mathbb{E}_\nu(x) \end{aligned}$$

### 7.4.1 Properties

- If  $\rho$  dominates  $\nu$  in the third order, then  $\mathbb{E}_\rho(\log(x)) \geq \mathbb{E}_\nu(\log(x))$ . This means that the geometric mean of  $\rho$  must be greater than or equal to the geometric mean of  $\nu$ .

- If  $\rho$  dominates  $\nu$  in the third order, then  $\min_\rho(x) \geq \min_\nu(x)$ . This means that the left tail of  $F_\nu$  must be thicker than the left tail of  $F_\rho$ .
- As above observations, second-order dominance implies third-order dominance (trivial demonstration).

# Chapter 3

# Statistics

## 1 Theorems

### 1.1 Central limit theorem

If  $X_1, X_2, \dots, X_n \in \mathbb{R}$  are random samples drawn from a population with overall mean  $\mu$  and finite variance  $\sigma^2$ , and if  $\bar{X}_n$  is the sample mean of the first  $n$  samples, then the limiting form of the distribution,

$$Z = \lim_{n \rightarrow \infty} \left( \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \right)$$

is a standard normal distribution.

#### 1.1.1 Multidimensional

Now consider  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n \in \mathbb{R}^k$  are random vectors drawn from a population with mean  $\boldsymbol{\mu}$  and covariance matrix  $\boldsymbol{\Sigma}$ , then :

$$\sqrt{n}(\bar{\mathbf{X}}_n - \boldsymbol{\mu}) \xrightarrow{n \rightarrow \infty} \mathcal{N}_k(0, \boldsymbol{\Sigma})$$

## 1.2 Cochran's theorem

*Simplified version.* Let  $\mathbf{X} \in \mathbb{R}^n$  be a standard normal random vector, sampled from  $\mathcal{N}(0_{\mathbb{R}^n}, I_n)$  and  $F$  a subspace (that has dimension  $d$ ) of  $\mathbb{R}^n$ .  $F^\perp$  is its corresponding orthogonal subspace.  $P_F, P_{F^\perp}$  are matrix related to *orthogonal projections* on  $F, F^\perp$ ; then :

- Vectors  $P_F \mathbf{X}$  and  $P_{F^\perp} \mathbf{X}$  are independent, which means each component in  $P_F \mathbf{X}$  is independent with each component in  $P_{F^\perp} \mathbf{X}$ .
- $P_F \mathbf{X} \sim \mathcal{N}(0_{\mathbb{R}^n}, P_F)$  and  $P_{F^\perp} \mathbf{X} \sim \mathcal{N}(0_{\mathbb{R}^n}, P_{F^\perp})$ .
- $\sum_{i=1}^n P_F \mathbf{X}[i]^2 \sim \chi^2(d)$  and  $\sum_{i=1}^n P_{F^\perp} \mathbf{X}[i]^2 \sim \chi^2(n-d)$

### 1.2.1 Example

Let's  $F$  be a subspace represented by the plan  $x + y + z = 0$  with normalized normale vector  $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T$ . Then the matrix  $P_F$  to **project orthogonally** any point of  $\mathbb{R}^3$  into  $F$  is :

$$P_F = I - NN^T = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

and  $\dim(P_F) = 2$ . The matrix  $P_{F^\perp}$  to **project orthogonally** any point of  $\mathbb{R}^3$  into  $F^\perp$  is

$$P_{F^\perp} = NN^T = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and  $\dim(P_{F^\perp}) = 1$ .

Then given a vector  $X \sim \mathcal{N}(0_{\mathbb{R}^3}, I_3)$ ; vectors  $P_F X$  and  $P_{F^\perp} X$  are independent.

### 1.2.2 Note 1 : Linear transform of random variable vector

If  $\mathbf{X} \sim \mathcal{N}(0_{\mathbb{R}^n}, \Sigma)$ , then  $P\mathbf{X} \sim \mathcal{N}(0_{\mathbb{R}^n}, P\Sigma P^T)$ , where  $P$  is a matrix to represent a linear transform. This is because :

$$\text{Cov}(P\mathbf{X}) = E[(P\mathbf{X} - P0_{\mathbb{R}^n})(P\mathbf{X} - P0_{\mathbb{R}^n})^T] = P\Sigma P^T$$

Then we can demonstrate the second statement in Cochran's theorem :

$$P_F \mathbf{X} \sim \mathcal{N}(0_{\mathbb{R}^n}, P_F I_n P_F^T) = \mathcal{N}(0_{\mathbb{R}^n}, P_F),$$

since  $P_F$  is sysmetric ( $P_F = P_F^T$ ) and  $P_F$  is idempotent ( $P_F P_F = P_F$ ). To prove  $P_F$  is sysmetric, we can rely on :

$$P_F = A(A^T A)^{-1} A^T$$

where  $A$  is the matrix whose columns form an orthonormal basis for the subspace  $F$ .

### 1.2.3 Note 2 : Concatenation of two random variable vectors

If  $\mathbf{X} \sim \mathcal{N}(0_{\mathbb{R}^n}, \Sigma_X)$  and  $\mathbf{Y} \sim \mathcal{N}(0_{\mathbb{R}^m}, \Sigma_Y)$ , then :

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}(0_{\mathbb{R}^{m+n}}, \begin{bmatrix} \Sigma_X \\ \Sigma_Y \end{bmatrix} [\Sigma_X^T \quad \Sigma_Y^T])$$

Then we can demonstrate the first statement in Cochran's theorem :

$$\begin{aligned} \begin{bmatrix} P_F X \\ P_{F^\perp} X \end{bmatrix} &\sim \mathcal{N}(0_{\mathbb{R}^{2n}}, \begin{bmatrix} P_F \\ P_{F^\perp} \end{bmatrix} [P_F^T \quad P_{F^\perp}^T]) \\ &\sim \mathcal{N}(0_{\mathbb{R}^{2n}}, \begin{bmatrix} P_F & 0 \\ 0 & P_{F^\perp} \end{bmatrix}) \end{aligned}$$

Then  $P_F X$  and  $P_{F^\perp} X$  are independent.

#### 1.2.4 Note 3 : Why orthogonal projection ?

The orthogonal projection  $P_F$  and  $P_{F^\perp}$  help for the second statement and the third statement in Cochran's theorem, notably the fact that  $P_F$  and  $P_{F^\perp}$  are idempotent.

Note that, if  $P_F$  and  $P_{F^\perp}$  are not orthogonal projection to  $F$  and  $F^\perp$  respectively, e.g. :

$$P_F = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$P_{F^\perp} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

, the first statement in Cochran's theorem is still true.

### 1.3 Basu's theorem

Before diving into this theorem, we discover some necessary concepts : statistic, completeness, sufficiency and ancillary statistic.

#### 1.3.1 Statistic

Sample *statistic* is any quantity computed from values in a sample, e.g.

- Sample mean
- Sample median
- Sample variance
- ...

#### 1.3.2 Identifiability

Let  $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$  be a set of statistical models parameterized by  $\theta$  and  $\Theta$  be the space of  $\theta$ . We say that  $\mathcal{P}$  is identifiable if the mapping  $\theta \mapsto P_\theta$  is injective :

$$P_{\theta_1} = P_{\theta_2} \quad \Rightarrow \quad \theta_1 = \theta_2 \quad \forall \theta_1, \theta_2 \in \Theta$$

#### 1.3.3 Completeness

Completeness is a property of a *statistic* in relation to a set of observed data, which are sampled from a parameterized model (distribution).

### 1.3.3.1 Definiton

Consider :

- Let  $T$  be a statistic that is inferred from a set of observations  $X_1, \dots, X_n$ , which means  $T = h(X_1, \dots, X_n)$ , where  $h$  is a measurable function.
- We suppose that  $X_1, \dots, X_n$  are sampled from a distribution  $f_X(x; \theta)$  which is parametrized by  $\theta$ .
- Then the distribution of  $T$ , noted  $f_T(t; \theta)$  is also parametrized by  $\theta$ .

The statistic  $T$  is said to be complete for the distribution of  $X$  (parameterized by  $\theta$ ) if, for every measurable function  $g$ , such that :

If  $\mathbb{E}[g(T)](\theta) = 0, \forall \theta \in \Theta$ , then  $P(g(T) = 0) = 1, \forall \theta \in \Theta$

where we note that  $T$  is a function in term of  $\theta$ ,  $\Theta$  is the space of parameter  $\theta$ .

### 1.3.3.2 Example : Binomial model

Let  $(X_i)_{i=1, \dots, n}$  be  $n$  observations sampled from Bernoulli distribution with parameter  $p$ . Let's  $T = \sum_{i=1}^n X_i$  then  $T$  is a statistic of the set of observations. Since  $X_i$  is sampled from Bernoulli distribution then  $T$  follows the binomial distribution :  $T \sim B(n, p)$ .

We show that  $T$  is a complete statistic, with a parameter space  $(0, 1)$ .

$$\begin{aligned}\mathbb{E}[g(T)](p) &= \sum_{t=0}^n g(t)P(T=t) \\ &= \sum_{t=0}^n g(t) \binom{n}{t} p^t (1-p)^{n-t} \\ &= (1-p)^n \sum_{t=0}^n g(t) \binom{n}{t} \left(\frac{p}{1-p}\right)^t\end{aligned}$$

since  $p \in (0, 1)$  then if  $\mathbb{E}[g(T)](p) = 0, \forall p$ , then :

$$\sum_{t=0}^n g(t) \binom{n}{t} r^t = 0, \forall r$$

where  $r = \frac{p}{1-p}, r \in (0, +\infty)$ . Note that the above expression is a polynomial in  $r$  since  $t$  is integer, this infers that  $g(t) = 0, \forall t \in \{1, \dots, n\}$  or  $P(g(T) = 0) = 1$  since  $T$  only take values in  $\{1, \dots, n\}$ .

### 1.3.3.3 Continuation of above example

- If  $\Theta = \{0.5\}$  and  $n = 1$ , then  $\mathbb{E}[g(T)] = 0$  implies :

$$g(0)\frac{1}{2} + g(1)\frac{1}{2} = 0$$

All function  $g$  such that  $g(0) = \alpha$  and  $g(1) = -\alpha$  can be solution for the above equation. As  $\alpha$  can be different to 0, then  $P_T(g(T) = 0) \neq 1$  or  $T$  is not complete.

- If  $\Theta = \{0.25, 0.75\}$ , then  $\mathbb{E}[g(T)] = 0, p \in \Theta$  implies :

$$\begin{cases} g(0)\frac{3}{4} + g(1)\frac{1}{4} = 0 \\ g(0)\frac{1}{4} + g(1)\frac{3}{4} = 0 \end{cases}$$

Then  $g(0) = 0$  and  $g(1) = 0$  or  $P_T(g(T) = 0) = 1$ .  $T$  is now complete.

- If  $\Theta$  contains at least  $n + 1$  distinct values, then  $n + 1$  equations. Note that  $t = 0, \dots, n$  makes  $n + 1$  unknowns. The coefficient vectors given by  $\left[1, \left(\frac{p}{1-p}\right), \left(\frac{p}{1-p}\right)^2, \dots, \left(\frac{p}{1-p}\right)^n\right], \forall p \in \Theta$  are independent since  $\frac{p}{1-p}$  is monotone (ascending). In this case  $T$  is complete.

### 1.3.3.4 Example of not complete statistic

Let  $(X_i)_{i=1, \dots, n}$  be  $n$  observations sampled from  $\mathcal{N}(\theta, 1)$ . Let's  $T = X_1 - X_2$  is a statistic and let's take  $g(T) = T = X_1 - X_2$ . Then  $\mathbb{E}[g(T)](p) = \mathbb{E}[X_1] - \mathbb{E}[X_2] = \theta - \theta = 0, \forall \theta$ . However, this do not implies that  $P(X_1 = X_2) = 1$ .

### 1.3.3.5 Analogy to vector space

There is an analogy between completeness and vector space.

Suppose that we have a vector space of dimension  $n$ . Then a set of vectors  $v_1, \dots, v_n$  is complete if they can span the whole space. It means that any vector  $u \in \mathbb{R}^n$  can be written as a linear combination  $u = \sum_{i=1}^n a_i v_i$ . This is also equivalent to the fact that if any vector  $w \in \mathbb{R}^n$  is orthogonal to all  $v_i$ , then  $w = 0$ .

Now if we consider the discrete probability case, where  $f_T(t; \theta_i) = [f_T(t_1; \theta_i), \dots, f_T(t_m; \theta_i)]$  as a vector of dimension  $m$  and  $f_T(t; \theta_1), \dots, f_T(t; \theta_n)$  are  $n$  vectors. Then :

$$\begin{aligned} \mathbb{E}[g(T)](\theta) &= 0, \forall \theta = \theta_1, \dots, \theta_n \\ \Leftrightarrow \langle g(T) = 0, f_T(t; \theta) \rangle &\quad \forall \theta = \theta_1, \dots, \theta_n \end{aligned}$$

where  $g(T) = [g(t_1), \dots, g(t_m)]$ .

Then here we have the equivalent definition of completeness, which means that  $g(T)$  can expressed by a linear combination of  $f_T(t; \theta_1), \dots, f_T(t; \theta_n)$ .

### 1.3.4 Sufficient statistic

#### 1.3.4.1 Definition

A statistic  $t = T(X)$  is sufficient for underlying parameter  $\theta$  if the conditional probability distribution of the data  $X$ , given the statistic  $t = T(X)$ , does not depend on the parameter  $\theta$  :

$$P(X = x|T(X) = t, \theta) = P(X = x|T(X) = t)$$

In practice, we rarely use this formula to show that a statistic is sufficient. We generally prefer to use the Fisher-Neyman factorization theorem to show the statistic is sufficient.

#### 1.3.4.2 Fisher-Neyman factorization theorem

If the probability density function of  $X = [X_1, \dots, X_n]$  is  $f_X(x; \theta)$ , then  $T$  is sufficient for  $\theta$  if and only if there exists nonnegative functions  $g$  and  $h$  such that :

$$f_X(x; \theta) = h(x) g(T(x); \theta)$$

#### 1.3.4.3 Example

If  $X_1, \dots, X_n$  are independently sampled from a Poisson distribution of parameter  $\lambda$ , then the statistic  $T = \sum_{i=1}^n X_i$  is sufficient.

We have density function of  $X_i$ , denoted  $f_{X_i}(x_i; \lambda)$  :

$$f_{X_i}(x_i; \lambda) = \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}$$

Then density function of  $X$  is :

$$f_X([x_1, \dots, x_n]; \lambda) = \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = e^{-n\lambda} \lambda^{x_1 + \dots + x_n} \frac{1}{x_1! \dots x_n!}$$

Here we can take :

$$g(T(x); \lambda) = e^{-n\lambda} \lambda^{x_1 + \dots + x_n}$$

and

$$h(x) = \frac{1}{x_1! \dots x_n!}$$

### 1.3.5 Interpretation

<https://stats.stackexchange.com/questions/629902/sufficient-complete-statistic-leftrightarrow-injective-surjective>

### 1.3.6 Ancillary statistic

An ancillary statistic is a statistic whose distribution does not depend on the parameters of the model.

### 1.3.6.1 Example

Suppose  $X_1, \dots, X_n$  are independently sampled from  $\mathcal{N}(\mu, 1)$ . Then the following statistic is ancillary:

- $T = \max(X_1, \dots, X_n) - \min(X_1, \dots, X_n)$ , since  $T$  is does not depend on  $\mu$ , by soustraction.
- $T = \frac{\sum(X_i - \bar{X})^2}{n}$  estimation of variance, since  $T$  is does not depend on  $\mu$ , by soustration.

### 1.3.7 Basu's theorem

Let  $(P_\theta; \theta \in \Theta)$  be a family of distributions on a measurable space  $(X, \mathcal{A})$  (see 2.1.4) and statistics  $T$  and  $A$  that map from  $(X, \mathcal{A})$  to some measurable space  $(Y, \mathcal{B})$ . If :

- If  $T$  is a boundedly complete and sufficient statistic for  $\theta$ .
- $A$  is ancillary to  $\theta$ .

Then conditional on  $\theta$ ,  $T$  is independent of  $A$  :  $T \perp\!\!\!\perp A | \theta$ .

#### 1.3.7.1 Example : Independence of sample mean and sample variance of a normal distribution

Let  $(X_i)_{i=1, \dots, n}$  be iid sampled from  $\mathcal{N}(\mu, \sigma^2)$ . Then with respect to the parameter  $\mu$ , one can show that :

- Statistic (mean)  $\hat{\mu} = \frac{\sum X_i}{n}$  is a complete and sufficient statistic for  $\mu$ .
- Sample variance  $\hat{\sigma}^2 = \frac{\sum (X_i - \bar{X})^2}{n-1}$  is an ancillary statistic with respect to  $\mu$ .

Therefore, from Basu's theorem it follows that these statistics are independent, conditional on  $\mu$ .

## 2 Statistical inference : Classical method

### 2.1 Random sample

The collection of random variables  $X_1, X_2, X_3, \dots, X_n$  is said to be a random sample of size  $n$  if they are independent and identically distributed (i.i.d.), i.e. :

- $X_1, X_2, X_3, \dots, X_n$  are independent.
- $F_{X_1}(x) = F_{X_2}(x) = \dots = F_{X_n}(x)$ . The equality of cumulative distribution is equivalent to the equality of density distribution.

### 2.1.1 Sampling with replacement

If the population is large, then the probability of choosing one person twice is extremely low, and it can be shown that the results obtained from sampling with replacement are very close to the results obtained using sampling without replacement. The big advantage of sampling with replacement is that  $X_i$  will be independent and this makes the analysis much simpler.

### 2.1.2 Bootstrap

Bootstrap data is a statistical technique used to estimate the sampling distribution of a statistic (mean, median, variance,...) by resampling with replacement from a single data sample. In other words, bootstrap data involves creating multiple “bootstrapped” samples from a single dataset, with each bootstrapped sample being the same size as the original dataset.

$$\begin{aligned}(5, 4, 2, 1) &\rightarrow (5, 5, 2, 2) \\ &\rightarrow (5, 1, 1, 1) \\ &\rightarrow \dots\end{aligned}$$

Suppose we have a dataset of 50 observations on a variable of interest. We want to estimate the median of the population from which these observations were drawn, but we are uncertain about the sampling distribution of the median. To estimate the sampling distribution of the median, we can use bootstrap data as follows:

1. Randomly sample 50 observations with replacement from the original dataset to create a bootstrap sample. This bootstrap sample may contain some observations multiple times and may omit some observations altogether.
2. Calculate the median of the bootstrap sample.
3. Repeat steps 1 and 2 with a large number of times (e.g., 1000 times) to create a distribution of bootstrapped medians.
4. Use the bootstrapped median distribution to estimate the **standard error** of the median and construct a **confidence interval** for the population median.

## 2.2 Order Statistics

Given a random sample, we might be interested in quantities such as the largest, the smallest, or the middle value in the sample. Thus, we often order the observed data from the smallest to the largest. We call the resulting ordered random variables *order statistics*. More specifically, let  $X_1, X_2, X_3, \dots, X_n$  be a random sample from a *continuous* distribution with CDF  $F_X(x)$  and PDF

$f_X(x)$ . Let us order  $X_i$ 's from the smallest to the largest and denote the resulting sequence of random variables as:

$$X_{(1)}, X_{(2)}, \dots, X_{(n)}.$$

Thus, we have:

$$X_{(1)} = \min(X_1, X_2, \dots, X_n)$$

and

$$X_{(n)} = \max(X_1, X_2, \dots, X_n)$$

We call  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  the order statistics. Then the CDF and PDF of  $X_{(i)}$  are given by :

$$\begin{aligned} P(X_{(i)} = x) &= P(X = x)P(x \text{ is } i^{\text{th}} \text{ smallest}) \\ \Leftrightarrow f_{X_{(i)}}(x)dx &= f_X(x)dxP(x \text{ is } i^{\text{th}} \text{ smallest}) \\ \Leftrightarrow f_{X_{(i)}}(x) &= f_X(x)P(x \text{ is } i^{\text{th}} \text{ smallest}) \\ \Leftrightarrow f_{X_{(i)}}(x) &= f_X(x)[F_X(x)]^{i-1}[1-F_X(x)]^{n-i}n\binom{n-1}{i-1} \end{aligned}$$

$$\begin{aligned} F_{X_{(i)}}(x) &= \int f_{X_{(i)}}(x)dx \\ &= n\binom{n-1}{i-1} \int f_X(x)[F_X(x)]^{i-1}[1-F_X(x)]^{n-i}dx \\ &= n\binom{n-1}{i-1} \frac{1}{i} \int [1-F_X(x)]^{n-i} d[F_X(x)]^i \\ &= \binom{n}{i} \left[ [1-F_X(x)]^{n-i} [F_X(x)]^i \Big|_0^x - \int [F_X(x)]^i d[1-F_X(x)]^{n-i} \right] \\ &= \binom{n}{i} [1-F_X(x)]^{n-i} [F_X(x)]^i + \binom{n}{i}(n-i) \int [F_X(x)]^i [1-F_X(x)]^{n-i-1} f_X(x)dx \\ &= \binom{n}{i} [1-F_X(x)]^{n-i} [F_X(x)]^i + n\binom{n-1}{i} \int f_X(x) [F_X(x)]^i [1-F_X(x)]^{n-i-1} dx \\ &= \binom{n}{i} [1-F_X(x)]^{n-i} [F_X(x)]^i + f_{X_{(i+1)}}(x) \\ &= \sum_{k=i}^n \binom{n}{k} [1-F_X(x)]^{n-k} [F_X(x)]^k \end{aligned}$$

Also, the joint PDF of  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  is given by

$$f_{X_{(1)}, X_{(2)}, \dots, X_{(n)}}(x_1, x_2, \dots, x_n) = n! f_X(x_1) f_X(x_2) \dots f_X(x_n)$$

where  $x_1 \leq x_2 \leq \dots \leq x_n$ .

### 2.3 Estimator's bias

Let  $\hat{\Theta} = h(X_1, X_2, \dots, X_n)$  be a point estimator for  $\theta$  (true value). The **bias** of point estimator  $\hat{\Theta}$  is defined by:

$$B(\hat{\Theta}) = \mathbb{E}(\hat{\Theta}) - \theta$$

In case that  $B(\hat{\Theta}) = 0$ , it means that  $\hat{\Theta}$  is an unbiased estimator.

**Example.** Given  $X_i$  are iid. sample mean is unbiased and variance is biased  
:

$$\begin{aligned} B(\hat{\Theta}) &= B(\bar{X}) = B\left(\frac{X_1 + \dots + X_n}{n}\right) \\ &= \mathbb{E}\left(\frac{X_1 + \dots + X_n}{n}\right) - \mu \\ &= \frac{n\mathbb{E}(X_i)}{n} - \mu \\ &= 0 \end{aligned}$$

$$\begin{aligned} B(\hat{\Theta}) &= B(\hat{\sigma}^2) = B\left(\frac{1}{n} \sum_{k=1}^n (X_k - \bar{X})^2\right) \\ &= B\left(\frac{1}{n} \sum_{k=1}^n (X_k^2 - n\bar{X}^2)\right) \\ &= \mathbb{E}\left(\frac{1}{n} \sum_{k=1}^n (X_k^2 - n\bar{X}^2)\right) - \sigma^2 \\ &= \frac{1}{n} \sum_{k=1}^n \mathbb{E}(X_k^2) - \mathbb{E}(\bar{X}^2) - \sigma^2 \\ &= \frac{1}{n} \sum_{k=1}^n (Var(X_k) + \mathbb{E}(X_k)^2) - Var(\bar{X}) - \mathbb{E}(\bar{X})^2 - \sigma^2 \\ &= \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2 - \sigma^2 \\ &= -\frac{\sigma^2}{n} \end{aligned}$$

It is worth to note that  $\hat{\sigma}^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X})^2$  is an unbiased estimator for variance.

### 2.4 Mean squared error of estimator

This quantity is used to evaluate the performance of an estimator. The mean squared error (MSE) of a point estimator  $\hat{\Theta}$ , shown by  $MSE(\hat{\Theta})$ , is defined as:

$$MSE(\hat{\Theta}) = \mathbb{E}[(\hat{\Theta} - \theta)^2]$$

**Example** What is the better estimator for mean between  $X_1$  and  $\bar{X}$  ?

$$\begin{aligned} MSE(X_1) &= \mathbb{E}[(X_1 - \mu)^2] \\ &= Var(X_1) \\ &= \sigma^2 \end{aligned}$$

$$\begin{aligned} MSE(\bar{X}) &= \mathbb{E}[(\bar{X} - \mu)^2] \\ &= Var(\bar{X} - \mu) + \mathbb{E}[(\bar{X} - \mu)]^2 \\ &= \frac{\sigma^2}{n} + (0 - 0)^2 \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Hence, we take the estimator  $\bar{X}$  since it gives better error.

**Relation between MSE and Bias :**

$$\begin{aligned} MSE(\hat{\Theta}) &= \mathbb{E}[(\hat{\Theta} - \theta)^2] \\ &= Var(\hat{\Theta} - \theta) + \mathbb{E}[(\hat{\Theta} - \theta)]^2 \\ &= Var(\hat{\Theta}) + B(\hat{\Theta})^2 \end{aligned}$$

## 2.5 Asymptotic consistency

Loosely speaking, we say that an estimator is asymptotically consistent if as the sample size  $n$  gets larger,  $\hat{\Theta}$  converges (in probability, sec 2.4.3) to the real value of  $\theta$ . More precisely, we have the following definition:

Let  $\hat{\Theta}_1, \hat{\Theta}_2, \dots, \hat{\Theta}_n$  be a sequence of point estimators of  $\theta$ . We say that  $\hat{\Theta}^n$  is a consistent estimator of  $\theta$ , if

$$\lim_{n \rightarrow \infty} P(|\hat{\Theta}_n - \theta| \geq \epsilon) = 0, \quad \text{for all } \epsilon > 0.$$

Note that if we have  $\lim_{n \rightarrow \infty} MSE(\hat{\Theta}_n) = 0$ , then  $\hat{\Theta}_n$  is a consistent estimator.  
We note that :

$$\begin{aligned} P(|\hat{\Theta}_n - \theta| \geq \epsilon) &= P((\hat{\Theta}_n - \theta)^2 \geq \epsilon^2) \\ &\leq \frac{\mathbb{E}[(\hat{\Theta}_n - \theta)^2]}{\epsilon^2} \quad (\text{inequality Markov}) \\ &= \frac{MSE(\hat{\Theta}_n)}{\epsilon^2} \end{aligned}$$

An example for consistency is the sample mean estimator (sec 3.2.8)

## 2.6 Asymptotic efficiency

An estimator is said to be asymptotically efficient if it achieves the **smallest possible variance** among all consistent estimators as the sample size increases.

More formally, let  $\theta$  be an unknown parameter that we wish to estimate based on a random sample  $X_1, X_2, \dots, X_n$ . Let  $Y_n(\theta)$  be an estimator of  $\theta$ . The estimator  $Y_n$  is said to be asymptotically efficient if, as  $n$  approaches infinity, the variance of  $Y_n$  approaches the Cramer-Rao lower bound, which is the smallest possible variance among all unbiased estimators of  $\theta$ .

Intuitively, an asymptotically efficient estimator is the best possible estimator of  $\theta$  that we can construct using the available information from the sample. However, it is important to note that asymptotic efficiency only holds in the limit as the sample size approaches infinity, and may not hold for finite sample sizes.

## 2.7 Asymptotic normality

In simple terms, an estimator  $\hat{\Theta}$  is said to be asymptotically normal if its sampling distribution approaches a normal distribution as the sample size becomes large, which means :

$$\lim_{n \rightarrow \infty} \frac{\hat{\Theta} - \theta}{\sqrt{Var(\hat{\Theta})}} \sim \mathcal{N}(0, 1)$$

where  $Var(\hat{\Theta}) = g(\theta)$  is a function in term of  $\theta$ .

An example for asymptotic normality is the sample mean estimator (sec 3.2.8). Note that other assumptions and conditions may need to be met for asymptotic normality to hold.

## 2.8 Sample mean estimator

Let  $X_1, X_2, X_3, \dots, X_n$  be a random sample from a distribution with mean  $\mu$  and std  $\sigma$ .

Sample mean estimator :

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i$$

1.  $E(\hat{\mu}) = \mu$
2. Variance of sample mean estimator

$$Var[\hat{\mu}] = Var\left[\frac{1}{n} \sum_{i=1}^n x_i\right] = \frac{1}{n^2} \left( \sum_{i=1}^n Var[x_i] + \sum_i^n \sum_{i \neq j} Cov(x_i, x_j) \right) = \frac{\sigma^2}{n}$$

3. Consistency :  $\hat{\mu} \rightarrow \mu$ , by strong version of law of large numbers (sec 2.4.6.9)
4. Asymptotic normality : By the Central Limit Theorem (sec 3.1.1)

$$\lim_{n \rightarrow \infty} \left( \frac{\hat{\mu} - \mu}{\frac{\sigma}{\sqrt{n}}} \right)$$

or distribution of  $\hat{\mu}$  converges to  $\mathcal{N}(\mu, \frac{\sigma^2}{n})$ . Then  $\hat{\mu}$  is asymptotically normal.

## 2.9 Maximum likelihood estimation

In statistics, maximum likelihood estimation (MLE) is a method of estimating the parameters of an assumed probability distribution, given some observed data. This is achieved by maximizing a likelihood function so that, under the assumed statistical model, the observed data is most probable.

The point in the *parameter space* that maximizes the likelihood function is called the maximum likelihood estimate.

Formally, the problem for maximum likelihood estimation :

$$\hat{\theta}_{MLE}(\mathbf{X}) = \underset{\theta}{\operatorname{argmax}} \mathcal{L}(\theta; \mathbf{X})$$

where  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  a random sample.

### 2.9.1 Likelihood function

Likelihood function  $\mathcal{L}$  is defined as :

$$\mathcal{L}(\theta; \mathbf{X} = \mathbf{x}) = P_{X_1 X_2 \dots X_n}(x_1, x_2, \dots, x_n | \theta),$$

if  $X_i$  are discrete value and

$$\mathcal{L}(\theta; \mathbf{X} = \mathbf{x}) = f_{X_1 X_2 \dots X_n}(x_1, x_2, \dots, x_n | \theta)$$

if  $X_i$  are continuous value.  $P$  and  $f$  here are respectively mass probability function and density function.

**If  $X_i$  are independent**

$$\mathcal{L}(\theta; \mathbf{X} = \mathbf{x}) = \prod_{i=1}^n f_{X_i}(x_i | \theta)$$

**If  $X_i$  are independent and identically distributed**

$$\mathcal{L}(\theta; \mathbf{X} = \mathbf{x}) = \prod_{i=1}^n f(x_i | \theta)$$

In practice, it is often convenient to work with the natural logarithm of the likelihood function, called the Log-Likelihood  $l$  function :

$$l(\theta; \mathbf{X} = \mathbf{x}) = \ln(\mathcal{L}(\theta; \mathbf{X} = \mathbf{x})) = \sum_{i=1}^n \ln(f(x_i | \theta))$$

### 2.9.2 Asymptotic Properties

- Consistency (subsection 3.2.5). Thus we have the convergence in probability if  $n \rightarrow \infty$ :

$$\lim_{n \rightarrow \infty} P(|\hat{\theta}_{MLE} - \theta| > \epsilon) = 0$$

- $\hat{\theta}_{MLE}$  is asymptotically unbiased :

$$\lim_{n \rightarrow \infty} \mathbb{E}[\hat{\theta}_{MLE}] = \theta$$

- $\hat{\theta}_{MLE}$  is asymptotically normal (subsection 3.2.7)

$$\lim_{n \rightarrow \infty} \frac{\hat{\theta}_{MLE} - \theta}{\sqrt{Var(\hat{\theta}_{MLE})}} \rightarrow \mathcal{N}(0, 1).$$

- Asymptotically efficient (subsection 3.2.6), it reaches the Cramér-Rao bound. It means that  $Var(\hat{\theta}_{MLE})$  is smallest among all estimators.

### 2.9.3 Example

Given a coin that is unbiased, we toss this coin and have a random sample :

$$X_i = \begin{cases} 1 & \text{if face} \\ 0 & \text{if tail} \end{cases}$$

What is the parameter  $p$  that tossed coin is face ?

We know that density function of Bernoulli distribution with a parameter  $p$  need to be estimated is:

$$f(x|p) = p^x(1-p)^{1-x}$$

Log-Likelihood :

$$\begin{aligned} l(\theta; \mathbf{x}) &= \sum_{i=1}^n \ln(p^{x_i}(1-p)^{1-x_i}) \\ &= \sum_{i=1}^n x_i \ln(p) + \sum_{i=1}^n (1-x_i) \ln(1-p) \end{aligned}$$

$$\begin{aligned} \frac{\partial l(\theta; \mathbf{x})}{\partial \theta} &= \frac{\sum_{i=1}^n x_i}{p} - \frac{\sum_{i=1}^n (1-x_i)}{1-p} \\ &= \frac{\sum_{i=1}^n x_i - np}{p(1-p)} \end{aligned}$$

Hence :

$$\hat{p} = \frac{\sum_{i=1}^n x_i}{n}$$

However, in the more complicated model, finding and solving for the derivative of log likelihood is not always trivial. Let's take an example, that we want to estimate parameters of a GMM (Gaussian mixture model). This GMM is composed of two kernels, with  $\theta = (w_1, w_2, \mu_1, \sigma_1, \mu_2, \sigma_2)$ :

$$f(x|\theta) = \frac{w_1}{w_1 + w_2} \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{1}{2}(\frac{x - \mu_1}{\sigma_1})^2\right) + \frac{w_2}{w_1 + w_2} \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{1}{2}(\frac{x - \mu_2}{\sigma_2})^2\right)$$

Log-likelihood function :

$$l(\theta; \mathbf{x}) = \sum_{i=1}^n \ln f(x_i|\theta)$$

is hard to get the derivative for each parameter and also solve for them.

## 2.10 Confidence Intervals

Let  $X_1, X_2, \dots, X_n$  be a random sample (observation) from a distribution with a parameter  $\theta$  which is to be estimated. Suppose that we have observed  $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$  and the point estimator  $\hat{\theta}$  is a function of  $X_1, X_2, \dots, X_n$ . The problem is raised here, we do not know how close  $\hat{\theta}$  is to the real  $\theta$ . Hence, we have concept of **interval estimation**.

Instead of finding out only an estimation  $\hat{\theta}$ , we provide in addition :

- An *interval*  $[\hat{\theta}_l, \hat{\theta}_h]$ , with high and low estimate
- A *confidence level*  $(1 - \alpha)$ . We would like  $\alpha$  to be small. Common values for  $\alpha$  are 0.1, 0.05, 0.01, which correspond to confidence levels 90%, 95%, 99% respectively.

In summarizing, we need to find  $\hat{\theta}_l$  and  $\hat{\theta}_h$  such that the probability that the interval  $[\hat{\theta}_l, \hat{\theta}_h]$  includes  $\theta$  is larger than  $1 - \alpha$ :

$$P(\hat{\theta}_l \leq \theta \leq \hat{\theta}_h) \geq 1 - \alpha$$

### 2.10.1 Finding Interval Estimator

Let  $X$  be a continuous random variable with CDF  $F_X(x) = P(X \leq x)$ . Suppose that we are interested in finding two values  $x_h$  and  $x_l$  such that

$$P(x_l \leq X \leq x_h) = 1 - \alpha$$

**One way** to do this, is to chose  $x_l$  and  $x_h$  such that

$$P(X \leq x_l) = \frac{\alpha}{2} \quad \text{and} \quad P(X \geq x_h) = \frac{\alpha}{2}$$

Equivalently

$$F_X(x_l) = \frac{\alpha}{2} \quad \text{and} \quad F_X(x_h) = 1 - \frac{\alpha}{2}$$

Then

$$x_l = F_X^{-1}\left(\frac{\alpha}{2}\right) \quad \text{and} \quad x_h = F_X^{-1}\left(1 - \frac{\alpha}{2}\right)$$

### Example

Let  $X_1, X_2, \dots, X_n$  be a random sample from a normal distribution  $N(\theta, 1)$ . Find a 95% confidence interval for  $\theta$ . In subsection 3.2.8, we have that the sample mean is asymptotically normal :

$$\frac{\bar{X} - \theta}{\sqrt{Var(\bar{X})}} = \frac{\bar{X} - \theta}{\sqrt{\frac{1}{n}Var(X_i)}} = \frac{\bar{X} - \theta}{\sqrt{\frac{1}{n}}} \xrightarrow{n \rightarrow \infty} \mathcal{N}(0, 1)$$

$$P(-x_{\frac{\alpha}{2}} \leq \sqrt{n}(\bar{X} - \theta) \leq x_{\frac{\alpha}{2}}) = 1 - \alpha$$

where  $\alpha = 0.05$  and  $x_{\frac{\alpha}{2}} = 1.96$ . To find the interval :

$$P(\bar{X} + \frac{x_{\frac{\alpha}{2}}}{\sqrt{n}} \geq \theta \geq \bar{X} - \frac{x_{\frac{\alpha}{2}}}{\sqrt{n}}) = 1 - \alpha$$

Therefore, we can report the interval  $[\bar{X} - \frac{x_{\frac{\alpha}{2}}}{\sqrt{n}}, \bar{X} + \frac{x_{\frac{\alpha}{2}}}{\sqrt{n}}]$  as our  $(1 - \alpha)100\%$  confidence interval for  $\theta$ , which means that  $(1 - \alpha)100\%$  we have the real  $\theta$  in this interval.

### 2.10.2 Pivotal Quantity

Let  $X_1, X_2, X_3, \dots, X_n$  be a random sample from a distribution with a parameter  $\theta$  that is to be estimated. The random variable  $Q$  is said to be a pivot or a pivotal quantity, if it has the following properties:

- $Q$  depends only on  $X_1, X_2, X_3, \dots, X_n$  and parameter  $\theta$ .
- The probability distribution of  $Q$  does not depend on  $\theta$ .

#### 2.10.2.1 Example

Let  $X_1, X_2, X_3, X_4, X_5$  be a random sample from  $\mathcal{N}(\theta, 1)$ . Then

$$Q = \bar{X} - \theta$$

where  $\bar{X}$  is the sample mean, is a pivotal quantity since :

- $Q$  depends only on  $X_1, X_2, X_3, \dots, X_n$  and parameter  $\theta$ .
- The probability distribution of  $Q$  is  $\mathcal{N}(0, \frac{1}{5})$ .

## 2.11 Hypothesis testing

Often, we need to test whether a hypothesis is true or false :

- $H_0$  (the null hypothesis):  $\theta \in S_0$ .
- $H_1$  (the alternative hypothesis):  $\theta \in S_1$ .

We distinct between simple and composite:

- Simple hypothesis : only one hypothesis, e.g.  $\theta = \frac{1}{3}$
- Composite hypothesis : more than one hypothesis, e.g.  $\theta = \frac{1}{3}$  ou  $\theta = \frac{2}{3}$

**Example 1**, consider a radar system that uses radio waves to detect aircraft. The system receives a signal and, based on the received signal, it needs to decide whether an aircraft is present or not. Here, there are again two opposing hypotheses:

- $H_0$  : No aircraft is present. (null hypothesis)
- $H_1$ : An aircraft is present. (alternative hypothesis)

### Example 2

We have a coin and after a number of observations  $n$ , we would like to check whether it is fair or not. More specifically, let  $\theta$  be the probability of heads,  $\theta = P(H)$ , we have two hypotheses:

- $H_0$  : Coin is fair  $\theta = \frac{1}{2}$ . (null hypothesis)
- $H_1$ : The coin is not fair. (alternative hypothesis)

#### 2.11.1 Type I error

In order to find  $T$ , we need define the first notion called *Type I error* as the event that we reject  $H_0$  when  $H_0$  is true. :

$$P(H_0 \text{ is rejected} | H_0 \text{ is true})$$

equivalently

$$P(H_0 \text{ is rejected} | \theta), \quad \forall \theta \in S_0$$

If the probability of type I error satisfies

$$P(H_0 \text{ is rejected} | \theta) \leq \alpha, \quad \forall \theta \in S_0,$$

then we say the test has significance level  $\alpha$  or simply the test is a level  $\alpha$  test.

### 2.11.1.1 Example

Let  $X = X_1 + X_2 + \dots + X_n$ , where  $X_i$  is an observation. Logically, if we have threshold  $T$ , we can say that:

- If  $|X - \frac{n}{2}| \leq T : H_0$  is accepted
- If  $|X - \frac{n}{2}| > T : H_0$  is rejected

Then the Type I error is :

$$P\left(|X - \frac{n}{2}| > T | H_0 \text{ is true}\right)$$

As in the estimation of confidence intervals (sec 3.2.10), usually we take  $\alpha = 0.05$

Now, as  $X_i$  follows Bernoulli distribution then  $X$  follows Binomial distribution. If  $H_0$  is true,  $X \sim \text{Binomial}(n, \frac{1}{2})$ . Since  $\alpha = 0.05$  and Binomial distribution is symmetric (only in this case  $\theta = 0.5$ ) at  $\frac{n}{2}$ , then  $T$  can be found such as :

$$\sum_{k=50+T}^n 0.5^k (1-0.5)^{n-k} = \frac{\alpha}{2} = 0.025$$

If  $n$  is large enough.  $H_0$  is true means that  $X_i \sim \text{Bernoulli}(0.5)$  with mean 0.5 and variance 0.5(1-0.5). Applying CLT (sec 3.1.1)

$$Y = \frac{\bar{X} - 0.5}{\sqrt{\frac{0.5(1-0.5)}{n}}} = \frac{\sqrt{n}(\frac{\bar{X}}{n} - 0.5)}{0.5} \sim \mathcal{N}(0, 1)$$

$T$  can be found as:

$$\begin{aligned} \frac{\sqrt{n}(X - \frac{n}{2})}{0.5n} &= x_{\frac{\alpha}{2}} \\ T &= \frac{1}{2}\sqrt{nx_{\frac{\alpha}{2}}} = 9.8 \approx 10 \end{aligned}$$

### 2.11.1.2 Remarks

Suppose that we toss the coin 100 times and observe 55 heads. Based on the above discussion we should accept  $H_0$  (if  $\alpha = 0.05$ ). However, it is often recommended to say “we failed to reject  $H_0$ ” instead of saying “we are accepting  $H_0$ ”. The reason is that we have not really proved that  $H_0$  is true.

In fact, all we know is that the result of our experiment was not statistically contradictory to  $H_0$ .

### 2.11.2 Type II error

This is the second possible error that we can make is to accept  $H_0$  when  $H_0$  is false, equivalently when  $H_1$  is true.

$$P(H_0 \text{ is accepted} | H_1 \text{ is accepted})$$

Since the alternative hypothesis,  $H_1$ , is usually a composite hypothesis (so it includes more than one value of  $\theta$ ), the probability of type II error is usually a function of  $\theta$  :

$$\beta(\theta) = P(H_0 \text{ is accepted} | \theta)$$

or

$$\beta(\theta) = P(H_1 \text{ is rejected} | \theta)$$

for all  $\theta \in S_1$

#### 2.11.2.1 Example

Consider a radar system that uses radio waves to detect aircraft. Let  $X$  be the received signal. If  $X = 0$  there is no aircraft and if  $X = 1$ , there is an aircraft. Suppose that we know only noised signal, such that :

- $X = W$ , if no aircraft is present.
- $X = 1 + W$ , if an aircraft is present.

where  $W \sim \mathcal{N}(0, \frac{1}{9})$ .

Then the hypotheses :

- $H_0$  (null hypothesis): No aircraft is present.
- $H_1$  (alternative hypothesis): An aircraft is present.

What is the threshold  $c$  if we would like the probability of missing a present aircraft to be less than 1% ?

**Solution** In this problem, the type II error is considered ;

$$P(H_0 \text{ is accepted} | H_1 \text{ is accepted})$$

$H_1$  is accepceted means that  $X = 1 + W$  or  $X \sim \mathcal{N}(1, \frac{1}{9})$ . As  $\beta = 0.01$ , we must find  $c$  such that :

$$P(X < c) = 0.01$$

or

$$P(3(X - 1) < 3(c - 1)) = 0.01$$

As  $3(X - 1) \sim \mathcal{N}(0, 1)$ , then

$$3(c - 1) = \Phi^{-1}(0.01)$$

or

$$c = \frac{1}{3}\Phi^{-1}(0.01) + 1$$

where  $\Phi$  is cumulative distribution of the standard normal.

### 2.11.2.2 Note

We have the following explanation:

- Type I error : this is the probability of detecting (true) when there is no aircraft (false alarm, false positive). If given  $\alpha = 0.05$ , it means that a false alarm is less than 5%.
- Type II error : this is the probability of missing a present aircraft (false negative). If given  $\beta = 0.01$  it means that a false negative is less than 1%.

### 2.11.3 Hypothesis Testing for the Mean

We assume that we have a random sample  $X_1, X_2, \dots, X_n$  from a distribution and our goal is to make inference about **the mean of the distribution**  $\mu$ . We consider two tests : two-sided and one-sided. Hypotheses in two-sided test:

- $H_0 : \mu = \mu_0$
- $H_1 : \mu \neq \mu_0$

and hypotheses in one-sided test :

- $H_0 : \mu \leq \mu_0$
- $H_1 : \mu > \mu_0$

where  $\mu_0$  is given.

In both two tests, we use the sample mean :

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

Let's define :

$$Q(X_1, X_2, \dots, X_n) = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

if the true std  $\sigma$  of the distribution is given or

$$Q(X_1, X_2, \dots, X_n) = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

where  $S$  is sample standard deviation. Under hypothesis  $H_0$ , in the first case,  $Q$  follows standard normal distribution and in the second case,  $Q$  follows  $T(n-1)$  distribution (sec 3.4.2).

#### 2.11.3.1 Two-sided Tests for the Mean

- $H_0 : \mu = \mu_0$
- $H_1 : \mu \neq \mu_0$

In this case, the null hypothesis is a simple hypothesis and the alternative hypothesis is a **two-sided** hypothesis ( $\mu < \mu_0$  or  $\mu > \mu_0$ ). Then we need to find  $c$  such that :

$$\begin{aligned} P(\text{type I error}) &= P(H_0 \text{ is rejected } | H_0 \text{ is true}) \\ &= P(|Q| > c | \mu = \mu_0) \\ &\leq \alpha \end{aligned}$$

or

$$\begin{aligned} P(\text{type II error}) &= P(H_0 \text{ is accepted } | H_1 \text{ is true}) \\ &= P(|Q| < c | \mu \neq \mu_0) \\ &\leq \beta \end{aligned}$$

**Example** Let  $X_1, X_2, \dots, X_n$  be a random sample from a  $\mathcal{N}(\mu, \sigma^2)$  distribution, where  $\mu$  is unknown but  $\sigma$  is known. Design tests (type I error and type II error) to choose between the hypotheses :

- $H_0 : \mu = \mu_0$
- $H_1 : \mu \neq \mu_0$

For type I error, under the  $H_0$  hypothesis, the mean of the distribution  $\mu$  is equal to  $\mu_0$ , hence  $Q \sim \mathcal{N}(0, 1)$ . We need to choose threshold  $c$  such that :

$$P(|Q| > c) = \alpha$$

or since  $\mathcal{N}(0, 1)$  is symmetric :

$$P(Q > c) = \frac{\alpha}{2}$$

we infer that  $c = \Phi^{-1}(1 - \frac{\alpha}{2})$ , where  $\Phi$  is CDF of the standard normal. Then  $H_0$  is rejected if :

$$\left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \right| > \Phi^{-1}(1 - \frac{\alpha}{2})$$

For type II error, under the  $H_1$  hypothesis, the mean of the distribution  $\mu$  is not equal to  $\mu_0$ . Now  $\mu$  can take a range of value (composite hypothesis), therefore, we study for each case of  $\mu$ .

Since  $X_i \sim \mathcal{N}(\mu, \sigma^2)$ , then  $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$ , then  $Q = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim \mathcal{N}(\frac{\mu - \mu_0}{\sigma/\sqrt{n}}, 1)$ .

Let  $\beta$  is a acceptable false negative rate. **For each given value**  $\mu$  we find  $c(\mu) > 0$  such that :

$$\begin{aligned} P(|Q| < c(\mu) | \mu) &= \beta \\ P(-c(\mu) < Q < c(\mu) | \mu) &= \beta \end{aligned}$$

$$P\left(-c(\mu) - \frac{\mu - \mu_0}{\sigma/\sqrt{n}} < Q - \frac{\mu - \mu_0}{\sigma/\sqrt{n}} < c(\mu) - \frac{\mu - \mu_0}{\sigma/\sqrt{n}} | \mu\right) = \beta$$

$$\Phi\left(c(\mu) - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right) - \Phi\left(-c(\mu) - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right) = \beta$$

This is thus a hard problem to obtain  $c(\mu)$ . However, we can use this equation in the reverse direction, if  $c(\mu)$  is known, we can infer easily  $\beta$ , the false negative rate.

### 2.11.3.2 One-sided Tests for the Mean

- $H_0 : \mu \leq \mu_0$
- $H_1 : \mu > \mu_0$

In this case, we have one-sided test and note the following inequality:

$$\begin{aligned} P(\text{type I error}) &= P(H_0 \text{ is rejected } | H_0 \text{ is true}) \\ &= P(Q > c | \mu \leq \mu_0) \\ &= \alpha \end{aligned}$$

**Example** Let  $X_1, X_2, \dots, X_n$  be a random sample from a  $\mathcal{N}(\mu, \sigma^2)$  distribution, where  $\mu$  is unknown and  $\sigma$  is known. Design a level  $\alpha$  test to choose between

- $H_0 : \mu \leq \mu_0$
- $H_1 : \mu > \mu_0$

We study for each given value  $\mu \leq \mu_0$  :

$$\begin{aligned} P(Q > c | \mu \leq \mu_0) &= P\left(Q - \frac{\mu - \mu_0}{\sigma/\sqrt{n}} > c - \frac{\mu - \mu_0}{\sigma/\sqrt{n}} | \mu \leq \mu_0\right) \\ &\leq P\left(Q - \frac{\mu - \mu_0}{\sigma/\sqrt{n}} > c | \mu \leq \mu_0\right), \quad \text{since } \mu \leq \mu_0 \\ &= 1 - \Phi(c) \end{aligned}$$

Then we can choose  $c = \Phi^{-1}(1 - \alpha)$

### 2.11.4 P-Values

P-value is the lowest significance level  $\alpha$  that results in rejecting the null hypothesis.

**Example** Suppose that we have a two-sided test and the pivotal quantity  $Q \sim \mathcal{N}(0, 1)$ . The observation given that  $Q = 2$ . Then the p-value, which corresponds to  $c = 2$  (for a two-sided test) is:

$$2(1 - \Phi(2)) = 0.046$$

## 2.11.5 Likelihood Ratio Tests

### 2.11.5.1 Likelihood Ratio Test for Simple Hypotheses

Let  $X_1, X_2, \dots, X_n$  be a random sample (observation) from a distribution with a parameter  $\theta$ .

- $H_0 : \theta = \theta_0$
- $H_1 : \theta = \theta_1$

we define

$$\lambda(x_1, x_2, \dots, x_n) = \frac{\mathcal{L}(x_1, x_2, \dots, x_n | \theta = \theta_0)}{\mathcal{L}(x_1, x_2, \dots, x_n | \theta = \theta_1)}$$

To perform a likelihood ratio test (LRT), we choose a constant  $c$ . We reject  $H_0$  if  $\lambda < c$  and accept it if  $\lambda \geq c$ . The value of  $c$  can be chosen based on the desired  $\alpha$  (type I) or  $\beta$  (type II).

#### Example

Here, we look at the radar problem. More specifically, we observe the random variable  $X : X = \theta + W$ , where  $W \sim \mathcal{N}(0, \sigma^2 = \frac{1}{9})$ . We need to decide between

- $H_0 : \theta = \theta_0 = 0$
- $H_1 : \theta = \theta_1 = 1$

Let  $X = x$  an observation. Design a level  $\alpha = 0.05$  to decide between  $H_0$  and  $H_1$ .

$$\begin{aligned} L(x | \theta = \theta_0) &= \frac{1}{\sqrt{2\pi\frac{1}{3}}} \exp\left(-\frac{1}{2}9x^2\right) \\ L(x | \theta = \theta_1) &= \frac{1}{\sqrt{2\pi\frac{1}{3}}} \exp\left(-\frac{1}{2}9(x-1)^2\right) \\ \lambda(x) &= \exp\left(\frac{9(1-2x)}{2}\right) \end{aligned}$$

Equivalently, we accept  $H_0$  if

$$\lambda(x) \geq c \Leftrightarrow x \leq \frac{1}{2}(1 - \frac{2}{9}\ln c).$$

Let  $c' = \frac{1}{2}(1 - \frac{2}{9}\ln c)$ . Given  $\alpha$ , it means type I error, such that :

$$\begin{aligned} P(\text{type I error}) &= P(H_0 \text{ is rejected} | H_0 \text{ is true}) \\ &= P(X > c' | \theta = \theta_0) \leq \alpha \\ &= P(X > c') \leq \alpha \quad \left(X \sim \mathcal{N}(0, \frac{1}{9})\right) \end{aligned}$$

Then we can find the value  $c'$  and then  $c$ .

### 2.11.5.2 Likelihood Ratio Test for Composite Hypotheses

Let  $X_1, X_2, \dots, X_n$  be a random sample (observation) from a distribution with a parameter  $\theta$ .

- $H_0 : \theta \in S_0$
- $H_1 : \theta \in S_1$

we define

$$\lambda(x_1, x_2, \dots, x_n) = \frac{\max_{\theta} L(x_1, x_2, \dots, x_n | \theta \in S_0)}{\max_{\theta} L(x_1, x_2, \dots, x_n | \theta \in S)}$$

where  $S = S_0 \cup S_1$ . To perform a likelihood ratio test (LRT), we choose a constant  $c \in [0, 1]$ . We reject  $H_0$  if  $\lambda < c$  and accept it if  $\lambda \geq c$ .

## 2.12 Linear Regression

This model tries to find the linear relationship (finding  $\beta$ ) between variables between  $y$  and  $x = [x_1, x_2, \dots, x_p]$  :

$$y = \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p$$

With  $(i)$  denote a sampling, there are different types of linear regression such as :

- (Linear) least square :

$$\min_{\beta} \sum_k \left[ y^{(i)} - (\beta_1 x_1^{(i)} + \beta_2 x_2^{(i)} + \dots + \beta_p x_p^{(i)}) \right]^2$$

- Ridge (linear) regression :

$$\min_{\beta} \sum_k \left[ y^{(i)} - (\beta_1 x_1^{(i)} + \beta_2 x_2^{(i)} + \dots + \beta_p x_p^{(i)}) \right]^2 + \alpha \|\beta\|_2^2$$

- Lasso (linear) regression :

$$\min_{\beta} \sum_k \left[ y^{(i)} - (\beta_1 x_1^{(i)} + \beta_2 x_2^{(i)} + \dots + \beta_p x_p^{(i)}) \right]^2 + \alpha \|\beta\|_1$$

- Polynomial (linear) regression :

$$\min_{\beta} \left[ y^{(i)} - (\beta_0 + \beta_1 z_{(i)} + \beta_2 z_{(i)}^2 + \dots + \beta_n z_{(i)}^n) \right]^2$$

where  $z$  in this case is scalar.

Note that **Least Squares** and **Linear model** are completely two different concepts:

- **Least Squares** means minimizing the square distance between true value and its predicted value  $(y - \hat{y})^2$
- **Linear model** : only in case that  $\hat{y}$  is modeled by a linear relation.

### 2.12.1 Methods

We will discover several methods to resolve a *simple linear regression model* (finding  $\beta_0$  and  $\beta_1$ ). Our model is

$$Y = \beta_0 + \beta_1 X + \epsilon$$

with  $n$  observation  $(y_i, x_i), i = 1 \dots n$ .

#### 2.12.1.1 First method

In this first method, we must have the two following hypotheses :

- $\epsilon \sim \mathcal{N}(0, \sigma^2)$
- $\epsilon$  and  $X$  are independent.

First, we take expectation from both sides :

$$\mathbb{E}[Y] = \beta_0 + \beta_1 \mathbb{E}[X]$$

Second, we take the covariance between  $Y$  and  $X$ :

$$\begin{aligned} Cov(Y, X) &= Cov(\beta_0 + \beta_1 X + \epsilon, X) \\ &= \beta_1 Cov(X, X) + Cov(\epsilon, X) \\ &= \beta_1 Var(X) \end{aligned}$$

Finally,

$$\beta_1 = \frac{Cov(Y, X)}{Var(X)}$$

$$\beta_0 = \mathbb{E}[Y] - \frac{Cov(Y, X)}{Var(X)} \mathbb{E}[X]$$

where :  $\mathbb{E}[X] = \frac{\sum x_i}{n}$ ,  $\mathbb{E}[Y] = \frac{\sum y_i}{n}$ ,  $Var(X) = \frac{\sum (x_i - \mathbb{E}[X])^2}{n}$ ,  $Cov(X, Y) = \frac{\sum (x_i - \mathbb{E}[X])(y_i - \mathbb{E}[Y])}{n}$

#### 2.12.1.2 Second method

In this method, no hypothesis is needed. Let  $g$  be the sum of square error :

$$g(\beta_0, \beta_1) = \sum_i (y_i - \beta_0 - \beta_1 x_i)^2$$

First order condition :

$$\frac{\partial g}{\partial \beta_0} = 0$$

$$\Leftrightarrow -2 \sum (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\Leftrightarrow \beta_0 + \beta_1 \frac{\sum x_i}{n} = \frac{\sum y_i}{n}$$

$$\begin{aligned}\frac{\partial g}{\partial \beta_1} &= 0 \\ \Leftrightarrow \sum -2x_i(y_i - \beta_0 - \beta_1 x_i) &= 0 \\ \Leftrightarrow \beta_0 \sum x_i + \beta_1 \sum x_i^2 &= \sum x_i y_i\end{aligned}$$

Hence:

$$\beta_1 = \frac{(x_i - \bar{x})^2}{(x_i - \bar{x})(y_i - \bar{y})}$$

$$\beta_0 = \bar{y} - \beta_1 \bar{x}$$

#### 2.12.1.3 Third method

In this method by MLE (sec 3.2.9), we must have the the following hypothesis :

- $\epsilon \sim \mathcal{N}(0, \sigma^2)$

Hence  $Q = Y - \beta_0 - \beta_1 X$  follows standard normal distribution.

$$L(x_1, \dots, x_n, y_1, \dots, y_n | \beta_0, \beta_1) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} (y_i - \beta_0 - \beta_1 x_i)^2 \right)$$

Maximizing  $\ln(L)$  leads us to the second method.

#### 2.12.2 Ordinary Least Squares

Ordinary Least Squares (OLS) is a common statistical method used for estimating the relationship between output  $y$  (also called **dependent variable** or **reponse**) and  $p$  inputs  $(x_i)_{i=1, \dots, p}$  (also called **explanatory variables**, **regressors**, **independent variables** or **covariates**). Under the additional assumption that the noises (errors) *are normally distributed with zero mean*, OLS is the maximum likelihood estimator that outperforms any non-linear unbiased estimator. A remind about the difference between OLS and LLS, mainly by **noise** :

- Ordinary Least Squares (OLS) - Noises are white noise and i.i.d. from the normal distribution with zero mean and finite variance.

- Linear Least Squares (LLS) - Noises are white noise but with **different parameters** per sample or **correlated noise**. Note that LLS contains OLS and other LS methods such as Weighted Least Squares (WLS) and Generalized Least Squares (GLS).

Problem:

$$y = X\beta + \varepsilon$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix} \times \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

- $n$  observations  $\mathbf{x}_i, y_i, i = 1, \dots, n$ .
- $\mathbf{x}_i = [x_{i1}, \dots, x_{ip}]^T$ .
- row  $i$  of  $X$  is  $\mathbf{x}_i^T$  is  $i$ -th observations on all the explanatory variables.
- $X$  is called matrix of regressors and  $X[:, j]$  is called a regressor or all observations of explanatory variable  $j$ .
- Usually  $x_{i1} = 1, \forall i$  and  $\beta_1$  is called intercept.

Regressors do not have to be **independent**: there can be any desired relationship between the regressors (so long as it is not a **linear relationship**)

Solution :

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

- $X^T X$  is called Gram matrix
- $X^T y$  is called moment matrix
- $(X^T X)^{-1}$  is the cofactor matrix of  $\beta$ , closely related to its covariance matrix,  $C_\beta$ .
- $(X^T X)^{-1} X^T$  is Moore–Penrose pseudo-inverse of  $X$ .

We find that OLS (with linear model) is something equivalent linea MMSE in sec 3.3.5.1

### 2.12.2.1 Important notes

- In OLS, we impose the exogeneity assumption :  $\mathbb{E}[\varepsilon | X] = 0_p$ . We show that OLS estimator is unbiased :

$$\begin{aligned} \mathbb{E}[\hat{\beta}] &= \mathbb{E}[(X^T X)^{-1} X^T y] \\ &= \mathbb{E}[(X^T X)^{-1} X^T (X\beta + \varepsilon)] \end{aligned}$$

$$\begin{aligned}
&= \beta + \mathbb{E}[(X^T X)^{-1} X^T \varepsilon] \\
&= \beta + \mathbb{E}[\mathbb{E}[(X^T X)^{-1} X^T \varepsilon | X]] \\
&= \beta + \mathbb{E}[(X^T X)^{-1} X^T \mathbb{E}[\varepsilon | X]] \\
&= \beta
\end{aligned}$$

- Other consequences of the exogeneity assumption are :

$$\begin{aligned}
\mathbb{E}[\varepsilon] &= \mathbb{E}[\mathbb{E}[\varepsilon | X]] = \mathbb{E}[0_n] = 0_n \\
\mathbb{E}[X^T \varepsilon] &= \mathbb{E}[\mathbb{E}[X^T \varepsilon | X]] = \mathbb{E}[X^T \mathbb{E}[\varepsilon | X]] = 0_p
\end{aligned}$$

These consequences show that explanatory variables are **uncorrelated** with the errors :

$$\mathbb{E}[X^T \varepsilon] - \mathbb{E}[X^T] \mathbb{E}[\varepsilon] = 0_p - 0_p = 0_p$$

Here  $X$  is considered as a realization (contains  $n$  samplings or observations). Conversely, if  $\mathbb{E}[X^T \varepsilon] = 0$  and  $n \geq p$ , then we have  $\mathbb{E}[\varepsilon | X] = 0$  :

$$0_p = \mathbb{E}[X^T \varepsilon] = \mathbb{E}[\mathbb{E}[X^T \varepsilon | X]] = \mathbb{E}[X^T \mathbb{E}[\varepsilon | X]]$$

Thus  $0_p = \mathbb{E}[X^T \mathbb{E}[\varepsilon | X]]$  is a system of equation that has  $p$  equations and if we have  $n$  variables (in  $\mathbb{E}[\varepsilon | X]$ ),  $n \geq p$ , then the system has unique solution  $\mathbb{E}[\varepsilon | X] = 0$ .

- Orthogonality property, between theoretical residues to explanatory variables :

$$X^T(y - X\hat{\beta}) = 0_p$$

This is proved by the first order of  $\frac{1}{2} \|y - X\hat{\beta}\|_2^2$  with respect to  $\beta$ . Equivalently, we have :

$$\mathbb{E}[\mathbf{x}(y - \mathbf{x}^T \hat{\beta})] = 0_p$$

or :

$$\mathbb{E}[\mathbf{x}\hat{\varepsilon}] = 0_p$$

This means that OLS will simply pick the parameter  $\hat{\beta}$  that makes the resulting errors  $\hat{\varepsilon}$  appear uncorrelated with  $\mathbf{x}$ .

- After estimating  $\beta$  by  $\hat{\beta}$ , the fitted values (or predicted values) from the regression will be:

$$\hat{y} = X\hat{\beta} = Py = X(X^T X)^{-1} X^T y$$

$P$  is the projection matrix which maps  $y$  to  $\hat{y}$ . We can say  $P : R^n \rightarrow V$ , where  $V$  spanned by the columns of  $X$ .

- Another matrix, closely related to  $P$  is the *annihilator* matrix  $M = I_n - P$ ; this is a projection matrix onto the subspace orthogonal to  $V$ .

- Both matrices  $P$  and  $M$  are symmetric and idempotent (meaning that  $P^2 = P$  and  $M^2 = M$ ), and relate to the data matrix  $X$  via identities  $PX = X$  and  $MX = 0$ .
- Let  $\varepsilon = y - X\beta$  be the true noise (error, residual) then the estimated noise can be expressed as :

$$\hat{\varepsilon} = y - \hat{y} = y - X\hat{\beta} = (I_n - P)y = My = M(X\beta + \varepsilon) = M\varepsilon$$

### 2.12.2.2 Noise variance estimation

There are two estimators for the variance of estimated noise  $\hat{\varepsilon}$ , one is biased but has but has a smaller mean squared error and the other is unbiased.

- **Biased estimator**, by MLE :

$$\hat{\sigma}_1^2 = \frac{\hat{\varepsilon}^T \hat{\varepsilon}}{n}$$

- **Unbiased estimator**:

$$\hat{\sigma}_2^2 = \frac{\hat{\varepsilon}^T \hat{\varepsilon}}{n-p}$$

This is a minimum-variance unbiased estimator, which has lower variance than any other unbiased estimator. Also it is called the OLS estimator for noise variance.

Always with the unbiased estimator, let's set :

$$s^2 = \hat{\sigma}_2^2 = \frac{(y - X\hat{\beta})^T(y - X\hat{\beta})}{n-p} = \frac{(My)^T My}{n-p} = \frac{y^T M^T My}{n-p} = \frac{y^T My}{n-p}$$

where  $s$  (without square) is also called **standard error of regression**. Furthermore, we can show that  $s^2$  follows reduced chi-square statistic (sec 3.4.1.5).

### 2.12.3 Coefficient of determination

$R^2$  is a measure of the goodness of fit of a model. Given

- $y_1, y_2, \dots, y_n$  observations
- $f_1, f_2, \dots, f_n$  predicted value of  $y_i$
- $SS_{res} = \sum_i (f_i - y_i)^2$
- $SS_{tot} = \sum_i (y_i - \bar{y})^2$

Then  $R^2$  is defined by :

$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}} = 1 - \frac{\sum_i (f_i - y_i)^2}{\sum_i (y_i - \bar{y})^2}$$

$SS$  means sum of square. In a good fitting, we must have  $R^2 = 1$  or closed to 1. A trivial fitting by taking  $f_i = \bar{y}$  gives  $R^2 = 0$ . Note that a bad fitting can make  $R^2 < 0$ . In practice :  $R^2$  less than 10% is bad,  $R^2$  greater than 30% is good.

### 2.12.3.1 Other explanation

We continue the method described in subsection 3.2.12.1.1

$$\begin{aligned}
R^2 &= 1 - \frac{\sum_i (\hat{y}_i - y_i)^2}{\sum_i (y_i - \bar{y})^2} \\
&= 1 - \frac{\sum \epsilon_i^2}{nVar(Y)} \\
&= 1 - \frac{Var(\epsilon)}{Var(Y)} \quad \left( \frac{\sum \epsilon_i^2}{n} = \mathbb{E}[\epsilon^2] = Var(\epsilon) \right) \\
&= \frac{Var(Y) - Var(\epsilon)}{Var(Y)} \\
&= \frac{\beta_1^2 Var(X)}{Var(Y)} \quad (Var(Y) = \beta_1^2 Var(X) + Var(\epsilon)) \\
&= \frac{Cov(X, Y)^2}{Var(X)Var(Y)} \\
&= Cor(X, Y)^2
\end{aligned}$$

In this view,  $R^2$  is large if  $X$  and  $Y$  are highly correlated or highly uncorrelated.

### 2.12.3.2 R-squared in finance

R-squared is also a statistical measure that represents the proportion of the covariance between return (dependent variable,  $Y$ ) that's explained by an factors  $X$  (independent variable). Factors are described in in APT 6.3.11. So, if the  $R^2$  of a model is 0.5, then approximately half of the variation of return (or risk) can be explained by the variation of factors.

### 2.12.3.3 Coefficient of determination in OLS

$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}} = 1 - \frac{\sum_i (\hat{y}_i - y_i)^2}{\sum_i (y_i - \bar{y})^2} = 1 - \frac{y^T My}{y^T Ly} = \frac{y^T P^T L P y}{y^T Ly} = \frac{\sum_i (\hat{y}_i - \bar{y})^2}{\sum_i (y_i - \bar{y})^2}$$

where  $L$  is centering matrix  $L = I - \frac{1}{n} J_n$ .  $J_n$  is all one matrix.

### 2.12.3.4 Adjusted R-squared

$$1 - \frac{(1 - R^2)(N - 1)}{(N - p - 1)}$$

where  $p$  number of independent variables (explanatory variables or predictors) and  $N$  number of used observations.

Problematic : Mesuring the performance of model (fitting) in taking into account the number of independent variables. E.g, given 2 independent variables

(inputs)  $a$  and  $b$ ,  $y$  is output, then in model 1, we fit  $y$  with only  $a$  and in model 2, we fit  $y$  with  $a$  and  $b$ . Then the  $R^2$  of model 2 is equal or greater than model 1. But model 2 has advantage that it uses 2 independent variables, then in order to have a fair comparison,  $p$  is introduced in the adjusted R-squared.

## 2.13 Generalized method of moments

Let's start with some necessary concepts :

- **Non-parametric model** : Non-parametric model means that the number of parameters grows with the size of the observed dataset. **A non-parametric model does not imply that there are no parameters, but rather infinite parameters.** Conversely, parametric model has a fixed number of parameters. Finally, semi-parametric model is a type of model that combines both parametric and non-parametric components.
- **Population moments** : We use the term “population” moments to designate **theoretical moments** which are calculated based on the **entire population of the probability distribution**, as opposed to samples of moment.

The generalized method of moments (GMM) is a generic method for **estimating parameters in statistical models**. Usually it is applied in the context of semi-parametric models and the parameters of interest are finite-dimensional. In this situation, the full shape of the data's distribution function may not be known, and therefore maximum likelihood estimation (3.2.9) is not applicable.

The method involves choosing a set of moment conditions that are based on economic theory and then using these moment conditions to construct a set of equations that can be solved for the model parameters.

### 2.13.1 Method of Moments

Before diving into GMM, we start with the Method of Moments (MoM). This is one of the commonly-used methods to estimate parameters of interest. It expresses the **moments** as functions of the parameters of interest. The number of such equations is the same as the number of parameters to be estimated. Then we solve a system of equations for the parameters of interest.

#### 2.13.1.1 Method description

Suppose that the parameter  $\theta = (\theta_1, \theta_2, \dots, \theta_k)$  characterizes the distribution  $f_W(w; \theta)$  of the random variable  $W$ , which mean  $P(W = w_0) = f_W(w_0; \theta)$ .

**Theoretical part** : We calculate the first  $k$  the population moments as functions of  $\theta$  :

$$\begin{aligned}\mu_1 &\equiv \mathbb{E}[W] = g_1(\theta_1, \theta_2, \dots, \theta_k) \\ \mu_2 &\equiv \mathbb{E}[W^2] = g_2(\theta_1, \theta_2, \dots, \theta_k)\end{aligned}$$

$$\vdots \\ \mu_k \equiv \mathbb{E}[W^2] = g_k(\theta_1, \theta_2, \dots, \theta_k)$$

**Empirical estimation part :** A sample of size  $n$  is drawn, resulting in the values  $w_1, \dots, w_n$ . For  $j = 1, \dots, k$ , let

$$\hat{\mu}_j = \frac{1}{n} \sum_{i=1}^n w_i^j$$

be the  $j$ -th sample moment, an estimate of  $\mu_j$ . The MoM **estimator** for  $(\theta_1, \theta_2, \dots, \theta_k)$  denoted by  $(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k)$  is defined to be the solution (if one exists) to the system of equations :

$$\left\{ \begin{array}{l} \hat{\mu}_1 = g_1(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k) \\ \hat{\mu}_2 = g_2(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k) \\ \vdots \\ \hat{\mu}_k = g_k(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k) \end{array} \right.$$

#### 2.13.1.2 Advantages and disadvantages

- The MoM it is an alternative to MLE.
- The MoM is fairly simple and it has not explicit objective function to optimize while in MLE, we have the likelihood function to maximize and solving for MLE is more complicated.
- The MoM yields estimators which are asymptotically consistent 3.2.5 and biased. However, MLE is asymptotically consistent, unbiased, efficient and normal (see 3.2.9.2).
- In some cases the likelihood equations may be intractable without computers, whereas the MoM estimators can be computed much more quickly and easily. Due to easy computability, method-of-moments estimates may be used as the first approximation to the solutions of the likelihood equations, and successive improved approximations may then be found by the Newton method. In this way the method of moments can assist in finding maximum likelihood estimates.

#### 2.13.1.3 Example to estimate parameter of Gamma distribution

Suppose that  $X_1, \dots, X_n$  are independent and identically distributed random variables according to the Gamma distribution with density :

$$f_X(x; \alpha, \beta) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)}$$

where  $x > 0$  and we seek to estimate the vector of parameters  $\theta = (\alpha, \beta)$ .

We first determine the population moments (theoretical moments). The first moment is given by :

$$\mu_1 = \mathbb{E}[X] = \alpha\beta$$

and the second moment :

$$\mu_2 \equiv \mathbb{E}[X^2] = \beta^2\alpha(\alpha + 1)$$

Then  $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$  by MoM estimator is :

$$\begin{cases} \hat{\mu}_1 = \hat{\alpha}\hat{\beta} \\ \hat{\mu}_2 = \hat{\beta}^2\hat{\alpha}(\hat{\alpha} + 1) \end{cases}$$

By solving this system of equations, we get  $\alpha = \frac{\hat{\mu}_1^2}{\hat{\mu}_2 - \hat{\mu}_1^2}$  and  $\beta = \frac{\hat{\mu}_2 - \hat{\mu}_1^2}{\hat{\mu}_1}$ .

#### 2.13.1.4 Example for not to use MoM

Let's reuse the example in 3.2.12.1. We want to find  $\beta_0$  and  $\beta_1$  for a simple linear regression model :

$$Y = \beta_0 + \beta_1 X$$

with  $n$  observation  $(y_i, x_i), i = 1 \dots n$ .

We first determine the population moments (theoretical moments) :

$$\begin{cases} \mathbb{E}[Y] = \beta_0 + \beta_1 \mathbb{E}[X] \\ \mathbb{E}[Y^2] = \beta_0^2 + 2\beta_0\beta_1 \mathbb{E}[X] + \beta_1^2 \mathbb{E}[X^2] \end{cases}$$

By solving the corresponding system of equations, we get :

$$\begin{cases} \beta_1 = \frac{\hat{\sigma}_Y}{\hat{\sigma}_X} \\ \beta_0 = \mu_Y - \frac{\hat{\sigma}_Y}{\hat{\sigma}_X} \mu_X \end{cases}$$

where  $\mu, \hat{\sigma}$  mean estimated mean and estimated std.

The estimate for  $\beta_1$  in this case is  $\frac{\hat{\sigma}_Y}{\hat{\sigma}_X}$  and is completely different from the estimate  $\frac{Cov(X, Y)}{Var(X)}$  in 3.2.12.1.1. This is explained by the fact that latter estimate is equivalent to MLE (see 3.2.12.1.2 and 3.2.12.1.3).

#### 2.13.1.5 Example of transition

Given a normal distribution  $x \sim \mathcal{N}(\mu, \sigma^2)$ , suppose that we use the first two moments :

$$\mu = \mathbb{E}[x] \rightarrow \hat{\mu} = \frac{1}{N} \sum_{i=1}^N x_i$$

$$\sigma^2 = \mathbb{E}[(x - E[x])^2] \rightarrow \hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \hat{\mu})^2$$

Using the two above equations, we can estimate parameters  $\hat{\mu}, \hat{\sigma}^2$  and with  $\{x_1, \dots, x_N\}$ , which are observations of the random variable. By solving equations, we're able to reach an estimation to the parameters. However, if we consider the third and fourth moments of  $x$  as well, similarly, there are two more equations:

$$0 = \mathbb{E}[(x - E[x])^3] \rightarrow 0 = \frac{1}{N} \sum_{i=1}^N (x_i - \hat{\mu})^3$$

$$3\sigma^4 = \mathbb{E}[(x - E[x])^3] \rightarrow 3\hat{\sigma}^4 = \frac{1}{N} \sum_{i=1}^N (x_i - \hat{\mu})^4$$

Here arise the problem, there are four equations and only two unknowns. It is referred to as over-identification problem and normally there is no solution to the system. This is why we need generalized method of moments to estimate the parameters in the over-identified problem.

#### 2.13.1.6 Example

Given  $N$  samples from a uniform distribution  $[0, \theta]$  how would you estimate  $\theta$ ?

We can estimate  $d$  by two methods, MLE or method of moments. Let  $\mathbf{x} = x_1, \dots, x_N$  be a  $N$  sampling result.

By MLE :

$$\mathcal{L}(\theta, \mathbf{x}) = \prod_{i=1}^N l(\theta, x_i) = \begin{cases} \frac{1}{\theta^N} & \text{if } \max(x_1, \dots, x_N) \leq \theta \\ 0 & \text{if } \max(x_1, \dots, x_N) > \theta \end{cases}$$

Then we can see that the max of  $\mathcal{L}(\theta, \mathbf{x})$  happens if and only if  $\mathcal{L}(\theta, \mathbf{x}) > 0$  and  $\mathcal{L}(\theta, \mathbf{x}) = \frac{1}{\theta^N}$  where  $\max(x_1, \dots, x_N) \leq \theta$ . As  $\frac{1}{\theta^N}$  is decreasing function, then max of  $\mathcal{L}(\theta, \mathbf{x})$  is when  $\theta = \max(x_1, \dots, x_N)$

By method of moments :

$$\mathbb{E}[\mathbf{X}] = \frac{\theta}{2}$$

Then

$$\theta = 2 \frac{\sum_{i=1}^N x_i}{N}$$

#### 2.13.2 Description of GMM

Suppose the available data consists of  $T$  observations  $\{Y_t\}_{t=1, \dots, T}$ , where each observation  $Y_t$  is an  $n$ -dimensional multivariate random variable. We assume that the data come from a certain statistical model, defined up to an unknown parameter  $\theta \in \Theta$ . The goal of the estimation problem is to find the “true” value of this parameter  $\theta_0$  or at least a reasonably close estimate.

A general assumption of GMM is that the data  $Y_t$  be generated by a *weakly stationary* (4.2.1.1) and *ergodic* (4.2.2) stochastic process. Note that the case of independent and identically distributed (iid) is a special case of this condition.

In order to apply GMM, we need to have *moment conditions*, that is, we need to construct  $g(Y_t, \theta) \in \mathbb{R}^p$  ( $g$  is vector-valued function whose codomain is  $p$ -dimension) such that, if  $\theta = \theta_0$ , we have :

$$m(\theta_0) \equiv \mathbb{E}[g(Y_t, \theta_0)] = 0$$

**Important note :** Let  $l$  be the dimension of  $\theta$ . If  $p = l$ , we are in the case called **just-identified** and we solve directly the system of  $p$  equations and do not consider the following part. Next, we consider that  $p > l$  and we are in the case **over-identified**.

The function  $m(\theta)$  must differ from zero for  $\theta \neq \theta_0$  and the parameter  $\theta$  must be **identifiable point**. Here, it means there is unique  $\theta_0$  that make  $m(\theta_0) = 0$ . This is always the case since  $p > l$ .

The basic idea behind GMM is to replace the theoretical expected value  $\mathbb{E}[\cdot]$  with its empirical analog—sample average (this is similar to MoM):

$$\hat{m}(\theta) \equiv \frac{1}{T} \sum_{t=1}^T g(Y_t, \theta)$$

and then to minimize an objective function in term of  $\hat{m}(\theta)$ , where the solution is  $0^p$ . In general, we take the norm function  $\|\hat{m}(\theta)\|$  as the objective function to minimize to make  $\hat{m}(\theta)$  as close to zero as possible, hence  $\theta$  is close to  $\theta_0$ . For example, the norm associated with the weight matrix  $W$ , where  $W$  is a positive-definite weighting matrix :

$$\|\hat{m}(\theta)\|_W^2 = \hat{m}(\theta)^T W \hat{m}(\theta)$$

In practice, the weighting matrix  $W$  is computed based on the available data set, which will be denoted as  $\hat{W}$ . Thus, the GMM estimator can be written:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \left( \frac{1}{T} \sum_{t=1}^T g(Y_t, \theta) \right)^T \hat{W} \left( \frac{1}{T} \sum_{t=1}^T g(Y_t, \theta) \right)$$

Under suitable conditions this GMM estimator is :

- **Consistent :**

$$\lim_{T \rightarrow \infty} \hat{\theta} \xrightarrow{P} \theta_0$$

- **Asymptotically normal :**

$$(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}\left[0, \frac{1}{T}(G^T W G)^{-1} G^T W \Omega W^T G (G^T W^T G)^{-1}\right]$$

where  $G = \mathbb{E}[\nabla_{\theta} g(Y_t, \theta_0)]$  and  $\Omega = \mathbb{E}[g(Y_t, \theta_0)g(Y_t, \theta_0)^T]$

- **Asymptotically efficient** : It is showed that if we chose  $W$  proportional to  $\Omega^{-1}$  (denoted  $W \propto \Omega^{-1}$ ), then we have the most efficient estimator in the class of all (generalized) method of moment estimators and

$$(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}\left[0, \frac{1}{T}(G^\top \Omega^{-1} G)^{-1}\right]$$

### 2.13.3 Implementation

One difficulty with implementing the outlined method is that we cannot take  $W = \Omega^{-1}$  because, by the definition of matrix  $\Omega$ , we need to know the value of  $\theta_0$  in order to compute this matrix. In the case of  $Y_t$  being iid we can estimate  $W$  by :

$$\hat{W}_T(\hat{\theta})\hat{\Omega}^{-1} = \left( \frac{1}{T} \sum_{t=1}^T g(Y_t, \hat{\theta})g(Y_t, \hat{\theta})^T \right)^{-1}$$

where  $\hat{\Omega}$  is thus an estimate of covariance matrix and it is Hermitian positive definite matrix. To inverse  $\hat{\Omega}$  we can apply Cholesky decomposition (6.2.1).

There are several approaches exist to deal with this issue, the first one being the most popular:

- **Two-step feasible GMM:**

- Step 1: Take  $W = I$  (the identity matrix) or some other positive-definite matrix, and compute preliminary GMM estimate  $\hat{\theta}_{(1)}$ . This estimator is consistent for  $\theta_0$ , although not efficient.
- Step 2: Update  $W = \hat{W}_T(\hat{\theta}_{(1)})$  and reestimate  $\theta$ . **Note that there is no iteration.**

- **Iterated GMM:**

- The same procedure as 2-step GMM, but it is repeated several times. That is, the estimate obtained in step 2 is used to calculate the weighting matrix for step 3, and so on until some convergence criterion is met.

- **Continuously updating GMM:**

- Estimates  $\hat{\theta}$  simultaneously with estimating the weighting matrix  $W$ :

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \left( \frac{1}{T} \sum_{t=1}^T g(Y_t, \theta) \right)^T \hat{W}_T(\theta) \left( \frac{1}{T} \sum_{t=1}^T g(Y_t, \theta) \right)$$

#### 2.13.4 Sargan–Hansen test

When the number of moment conditions  $p$  is greater than  $l$ , the dimension of the parameter  $\theta$ , the model is said to be over-identified, it can happen that **there exists no**  $\theta_z$  such that  $m(\theta_z) \equiv \mathbb{E}[g(Y_t, \theta_z)] = 0$ , which means the chosen function  $g$  is not fit for estimating  $\theta_0$ .

Formally we consider two hypotheses:

- $H_0 : m(\theta_z) = 0$  (the null hypothesis that the model is “valid”).
- $H_1 : m(\theta) \neq 0, \forall \theta \in \Theta$  (the alternative hypothesis that model is “invalid”).

Under hypothesis  $H_0$ , the following so-called **J-statistic** is asymptotically chi-squared distributed with  $p - l$  degrees of freedom. Define  $J$  to be:

$$J \equiv T \cdot \left( \frac{1}{T} \sum_{t=1}^T g(Y_t, \hat{\theta}) \right)^T \hat{W}_T \left( \frac{1}{T} \sum_{t=1}^T g(Y_t, \hat{\theta}) \right) \xrightarrow{d} \chi_{p-l}^2$$

#### 2.13.5 Frequently used functions $g$

Many other popular estimation techniques can be cast in terms of GMM optimization:

- Ordinary least squares (3.2.12.2) is equivalent to GMM with moment conditions :

$$g(X, Y; \beta) = \mathbf{x} \cdot (y - \mathbf{x}^T \beta)$$

where  $(\cdot)$  emphasizes the multiplication between a vector and a scalar.

- Weighted least squares :

$$g(X, Y; \beta) = \text{diag}(\mathbf{w}) \mathbf{x} \cdot (y - \mathbf{x}^T \beta)$$

where  $\text{diag}(\mathbf{w})$  means the diagonal matrix forms by given weight  $\mathbf{w}$ .

- Instrumental variables (3.2.14) regression :

$$g(X, Y, Z; \beta) = \mathbf{z} \cdot (y - \mathbf{x}^T \beta)$$

- Non-linear least squares (NLLS):

$$g(X, Y; \beta) = \nabla_\beta f(\mathbf{x}, \beta) \cdot (y - f(\mathbf{x}, \beta))$$

- Maximum likelihood estimation (MLE):

$$g(X; \beta) = \nabla_\theta \ln f(\mathbf{x}, \theta)$$

**Remark :** For ordinary least squares case, it is inferred from the orthogonality property (3.2.12.2.1).

### 2.13.5.1 Example

We come back to example 3.2.13.1.4 and now we use ordinary least squares but casting as MoM to solve it. As a reminder, we want to find  $\beta_0$  and  $\beta_1$  for a simple linear regression model :

$$Y = \beta_0 + \beta_1 X = [1 \quad X] \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

From  $\mathbb{E}[\mathbf{x} \cdot (y - \mathbf{x}^T \beta)] = 0$  (3.2.13.5), we have :

$$\begin{aligned} & \mathbb{E} \left[ \begin{bmatrix} 1 \\ X \end{bmatrix} (Y - \beta_0 - \beta_1 X) \right] = 0 \\ \Leftrightarrow & \begin{cases} \mathbb{E}[Y] - \beta_0 - \beta_1 \mathbb{E}[X] = 0 \\ \mathbb{E}[XY] - \beta_0 \mathbb{E}[X] - \beta_1 \mathbb{E}[X^2] = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} \beta_1 = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\text{Var}(X)} = \frac{\text{Cov}(X,Y)}{\text{Var}(X)} \\ \beta_0 = \mathbb{E}[Y] - \frac{\text{Cov}(X,Y)}{\text{Var}(X)} \mathbb{E}[X] \end{cases} \end{aligned}$$

### 2.13.5.2 Example

We describe a OLS casted by MoM for estimate parameters in Fama–French three-factor model 6.3.14. Let's consider :

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 = [1 \quad X_1 \quad X_2 \quad X_3] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

From  $\mathbb{E}[\mathbf{x} \cdot (y - \mathbf{x}^T \beta)] = 0$  (3.2.13.5), we have :

$$\begin{aligned} & \mathbb{E} \left[ \begin{bmatrix} 1 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} (Y - \beta_0 - \beta_1 X_1 - \beta_2 X_2 - \beta_3 X_3) \right] = 0 \\ \Leftrightarrow & \begin{cases} \mathbb{E}[Y] - \beta_0 - \beta_1 \mathbb{E}[X_1] - \beta_2 \mathbb{E}[X_2] - \beta_3 \mathbb{E}[X_3] = 0 \\ \mathbb{E}[X_1 Y] - \beta_0 \mathbb{E}[X_1] - \beta_1 \mathbb{E}[X_1^2] - \beta_2 \mathbb{E}[X_1 X_2] - \beta_3 \mathbb{E}[X_1 X_3] = 0 \\ \mathbb{E}[X_2 Y] - \beta_0 \mathbb{E}[X_2] - \beta_1 \mathbb{E}[X_1 X_2] - \beta_2 \mathbb{E}[X_2^2] - \beta_3 \mathbb{E}[X_2 X_3] = 0 \\ \mathbb{E}[X_3 Y] - \beta_0 \mathbb{E}[X_3] - \beta_1 \mathbb{E}[X_1 X_3] - \beta_2 \mathbb{E}[X_2 X_3] - \beta_3 \mathbb{E}[X_3^2] = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} \beta_0 + \beta_1 \mathbb{E}[X_1] + \beta_2 \mathbb{E}[X_2] + \beta_3 \mathbb{E}[X_3] = \mathbb{E}[Y] \\ \beta_1 \text{Var}(X_1) + \beta_2 \text{Cov}(X_1, X_2) + \beta_3 \text{Cov}(X_1, X_3) = \text{Cov}(X_1, Y) \\ \beta_1 \text{Cov}(X_1, X_2) + \beta_2 \text{Var}(X_2) + \beta_3 \text{Cov}(X_2, X_3) = \text{Cov}(X_2, Y) \\ \beta_1 \text{Cov}(X_1, X_3) + \beta_2 \text{Cov}(X_2, X_3) + \beta_3 \text{Var}(X_3) = \text{Cov}(X_3, Y) \end{cases} \end{aligned}$$

$$\Leftrightarrow \begin{cases} \beta_0 + \beta_1 \mathbb{E}[X_1] + \beta_2 \mathbb{E}[X_2] + \beta_3 \mathbb{E}[X_3] = \mathbb{E}[Y] \\ \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) & Cov(X_1, X_3) \\ Cov(X_1, X_2) & Var(X_2) & Cov(X_2, X_3) \\ Cov(X_1, X_3) & Cov(X_2, X_3) & Var(X_3) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} Cov(X_1, Y) \\ Cov(X_2, Y) \\ Cov(X_3, Y) \end{bmatrix} \end{cases}$$

Here, we can inverse the covariance matrix and infer  $\beta_1, \beta_2, \beta_3$  then  $\beta_0$ .

## 2.14 Instrumental variables estimation

### 2.14.1 Context

In statistics, econometrics, epidemiology and related disciplines, the method of instrumental variables (IV) is used to estimate causal relationships between dependent variable  $A$  and explanatory variables  $B$ .

We say that  $A$  (effect) and  $B$  (cause) have causal relationship if the cause  $B$  is partly (or totally) responsible for the effect  $A$ , and the effect  $A$  is partly (or totally) dependent on the cause  $B$ .

Intuitively, to know whether  $A$  and  $B$  have causal relationship, we can perform an independency test 3.4.1.4 or just measure the correlation between  $A$  and  $B$ . However, these methods **require** that the error term  $E$ , which is obtained by removing  $B$  from  $A$ , is **uncorrelated** with  $B$ . In the case that  $E$  is correlated with  $B$ , a solution is to use instrumental variable.

**Example :** For estimating the causal relationships, we can use OLS (3.2.12.2). However, in the case of no exogeneity assumption, then the explanatory variable of interest can be correlated with the error term (endogenous) and the solution given by OLS is **biased** (see 3.2.12.2.1). In this case, we use instrumental variable.

### 2.14.2 Mechanism of instrumental variable

A (valid) instrument or instrumental variable is a variable  $Z$  such that it is correlated with the explanatory variable  $B$  but is not correlated with error term  $E$ . In other words, a variation on the instrumental variable  $Z$  has an effect on the explanatory variable  $B$  but no effect on error term  $E$ . By studying the variations of dependent variable and the explanatory variable linked to the variations of instrumental variable, researcher can uncover the causal effect of the explanatory variable on the dependent variable.

In summary, we need to construct  $Z$  which is correlated with  $B$  and uncorrelated with  $E$ .

#### Causes of correlation

The correlation between  $B$  and  $E$  may occur when :

- There are omitted variables that affect both the dependent and explanatory variables. For example, the true model is expressed by  $A = \alpha B +$

$\beta C + F$ , where  $C$  is the second explanatory variable and  $F$  is the true error. However, we omit  $C$ , then  $B$  and  $E = \beta C + F$  can be correlated.

- The explanatory variables  $B$  are subject to non-random measurement error, which means when measuring  $B$ , we get  $B = B_{true} + \varepsilon_B$  and this  $\varepsilon_B$  is not random. If  $\varepsilon_B$  is correlated to  $E$ , then  $B$  and  $E$  is also correlated.
- Changes in the dependent variable  $A$  change value of at least one of the covariates  $B$  (reverse causation). This may be understood by if causal relation between  $A$  and  $B$  is **perfectly expressed**, then changes in  $A$  change only  $E$  but not  $B$ . If changes in  $A$  change  $B$ , then the causal relation may not be correct and this is caused by plural reasons, such as omitted variables as above.

### 2.14.3 Example illustrative

Suppose a researcher wishes to estimate the causal effect of smoking  $X$  on general health  $Y$ . Correlation between smoking  $X$  and health  $Y$  does not imply that smoking causes poor health because other variables, such as depression or food, may affect both health and smoking, or because health may affect smoking. It is not possible to conduct controlled experiments on smoking status in the general population.

The researcher may attempt to estimate the causal effect of smoking on health from observational data by using the tax rate for tobacco products  $Z$  as an instrument for smoking. The tax rate for tobacco products is a reasonable choice for an instrument because the researcher assumes that it can only be **correlated with health through its effect on smoking and but not through depression or food**. If the researcher then finds tobacco taxes and state of health to be correlated, this may be viewed as evidence that smoking causes changes in health.

### 2.14.4 Two-stage least squares

#### 2.14.4.1 Statement

Let's reuse the OLS problem (3.2.12.2)

$$y = X\beta + \varepsilon$$

but in this case we do not have exogeneity assumption  $\mathbb{E}[\varepsilon | X] = 0$ . Here,  $X \in \mathbb{R}^{n \times p}$  and  $y, \varepsilon \in \mathbb{R}^n$ .  $\mathbb{E}[\varepsilon] = 0$ , if not we use the intercept  $\beta_0$  to offset  $\varepsilon$  and get  $\mathbb{E}[\varepsilon] = 0$ .

Suppose that we can find  $Z$  that is highly correlated with each component of  $X$  but is not correlated with  $\varepsilon$  and that the relation between  $X$  and  $Z$  is linear :

$$X = Z\delta + v$$

where  $Z \in \mathbb{R}^{n \times q}$ ,  $\delta \in \mathbb{R}^{q \times p}$  and  $v \in \mathbb{R}^{n \times p}$ .

We formulate the problem as following : given data for  $Y, X, Z$ , we need to estimate  $\beta$  **without bias**.

#### 2.14.4.2 Solution description

Intuitively, as the name of method Two-stage least squares (TSLS), we apply the least square two times :

- In the first one :

$$\hat{\delta} = (Z^T Z)^{-1} Z^T X$$

Then :

$$\hat{X} = Z(Z^T Z)^{-1} Z^T X = P_Z X$$

- In the second one :

$$\begin{aligned}\hat{\beta} &= (\hat{X}^T \hat{X})^{-1} \hat{X}^T y \\ &= (X^T P_Z^T P_Z X)^{-1} X^T P_Z^T y \\ &= (X^T P_Z X)^{-1} X^T P_Z y\end{aligned}$$

since  $(Z^T Z)^{-1}$  is symmetric and  $P_Z$  is also symmetric.

Next, we show that  $\hat{\beta}$  is **unbiased**. Since  $Z$  and  $\varepsilon$  is not correlated, then  $\mathbb{E}[Z^T \varepsilon] - \mathbb{E}[Z^T] \mathbb{E}[\varepsilon] = 0_q$  or  $\mathbb{E}[Z^T \varepsilon] = 0_q$ . Usually,  $n \geq \max(q, p)$  to avoid under-identified problem, then  $0_q = \mathbb{E}[Z^T \varepsilon] = \mathbb{E}[\mathbb{E}[Z^T \varepsilon | X, Z]] = \mathbb{E}[Z^T \mathbb{E}[\varepsilon | X, Z]]$ . This is a system of  $q$  equations and  $n$  variables (in  $\mathbb{E}[\varepsilon | X, Z]$ ), then we must have  $\mathbb{E}[\varepsilon | X, Z] = 0$ .

Let's check the mean of estimator IV :

$$\begin{aligned}\mathbb{E}[\hat{\beta}] &= \mathbb{E}[(X^T P_Z X)^{-1} X^T P_Z y] \\ &= \mathbb{E}[(X^T P_Z X)^{-1} X^T P_Z (X\beta + \varepsilon)] \\ &= \beta + \mathbb{E}[(X^T P_Z X)^{-1} X^T P_Z \varepsilon] \\ &= \beta + \mathbb{E}[\mathbb{E}[(X^T P_Z X)^{-1} X^T P_Z \varepsilon | X, Z]] \\ &= \beta + \mathbb{E}[(X^T P_Z X)^{-1} X^T P_Z \mathbb{E}[\varepsilon | X, Z]] \\ &= \beta\end{aligned}$$

**In a particular case** that  $q = p$ , then  $X^T Z$  and  $Z^T X$  are of  $p \times p$  or square matrix, then  $(X^T Z)^{-1}$  and  $(Z^T X)^{-1}$  exist. In addition, note that the property of two square matrix  $L, K$  :  $(LK)^{-1} = K^{-1}L^{-1}$ . Hence,

$$\begin{aligned}\hat{\beta} &= (X^T P_Z X)^{-1} X^T P_Z y \\ &= (X^T Z (Z^T Z)^{-1} Z^T X)^{-1} X^T Z (Z^T Z)^{-1} Z^T y \\ &= (Z^T X)^{-1} Z^T Z (X^T Z)^{-1} X^T Z (Z^T Z)^{-1} Z^T y \\ &= (Z^T X)^{-1} Z^T y\end{aligned}$$

### 3 Statistical inference : Bayesian Inference

In classical or frequentist approach. The unknown quantity  $\theta$  is assumed to be a **fixed** (non-random) quantity that is to be estimated by the observed data.

In this section, we would like to discuss a different framework for inference, namely the Bayesian approach. In the Bayesian framework, we treat the unknown quantity,  $\theta$ , as a random variable.

#### 3.1 Bayes' theorem

With  $A, B$  are two random variables that have the same probability space  $(\Omega, \mathcal{F}, P)$ , then

$$P(A = a | B = b) = \frac{P(A = a \cap B = b)}{P(B = b)} = \frac{P(A = a | B = b)P(B = b)}{P(B = b)}$$

for all  $a \in \mathcal{F}_A$  and  $b \in \mathcal{F}_B$ .  $a \cap b \neq \emptyset$  indicates that two events  $a$  and  $b$  have common elements in sample space  $\Omega$ .

Note that do not confuse  $P(A \cap B)$  and  $P(A, B)$ :

- In  $P(A \cap B)$  :  $A, B$  take value from the same event space  $\mathcal{F}$  and the function probability  $P : \Omega \rightarrow [0, 1]$ .
- In  $P(A, B)$  :  $A, B$  take value respectively from  $\mathcal{F}_A$  and  $\mathcal{F}_B$ . The function probability  $P : \Omega_A \times \Omega_B \rightarrow [0, 1]$ .

Or in some probability literature, with  $z \rightarrow x$ :

- $z$  latent variable
- $x$  observation
- $p(x)$  marginal or evidence
- $p(z|x)$  posterior probability
- $p(x|z)$  likelihood
- $p(z)$  prior probability

Then :

$$p(z|x) = \frac{p(x|z)p(z)}{p(x)}$$

##### 3.1.1 Example 1

Let  $X \sim Uniform(0, 1)$ . Suppose that we know  $Y|X = x \sim Geometric(x)$ . Find the posterior density of  $X$  given  $Y = 2$ ,  $f_{X|Y}(x|2)$ .

**Solution:**

- $X$  (prior) is uniform with density function :

$$P(X = x) = 1$$

- $Y|X$  (likelihood) follows geometric distribution  $\mathcal{G}(x)$ , with density function :

$$P(Y|X = x) = (1 - x)^{y-1}x$$

Then posterior :

$$\begin{aligned} f_{X|Y}(x|2) &= P(X|Y = 2) \\ &= \frac{P(Y = 2|X = x)P(X = x)}{P(Y = 2)} \\ &= \frac{(1 - x)x}{\int_0^1 (1 - x)x dx} \\ &= \frac{(1 - x)x}{\frac{1}{6}} \\ &= 6(1 - x)x \end{aligned}$$

### 3.1.2 Example 2

You have a coin and your prior assumption is that its probability of heads  $\theta$  is chosen from a uniform distribution on  $[0, 1]$ . You toss the coin 10 times and get 6 heads. What is the estimate of  $\theta$ ?

**Solution:**

- $\theta$  (prior) is uniform with density function :

$$P(\Theta = \theta) = 1$$

- $X|\Theta$  (likelihood) follows binomial  $\mathcal{B}(10, \theta)$ , with density function :

$$P(X = 6|\Theta = \theta) = \binom{6}{10} \theta^6 (1 - \theta)^{10-6}$$

Then posterior :

$$P(\Theta|X = 6) = \frac{P(X = 6|\Theta)P(\Theta)}{P(X = 6)}$$

We find  $\theta$  in this case, which is the expected value :

$$\begin{aligned} \mathbb{E}_{\Theta|X}[\theta|6] &= \int_0^1 \theta P(\theta|6)d\theta \\ &= \frac{1}{P(X = 6)} \int_0^1 \binom{6}{10} \theta^7 (1 - \theta)^4 d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sum_{\theta} P(X = 6|\Theta = \theta)P(\Theta = \theta)} \int_0^1 \binom{6}{10} \theta^7(1-\theta)^4 d\theta \\
&= \frac{\int_0^1 \binom{6}{10} \theta^7(1-\theta)^4 d\theta}{\int_0^1 \binom{6}{10} \theta^6(1-\theta)^4 d\theta} \\
&= \frac{0.0530}{0.0909} \\
&= 0.58333
\end{aligned}$$

The integral can be calculated by a calculatrice or using e.g. python.

**Other solution:**

We remind that the density function of *beta*-distribution, depends on  $a$  and  $b$ . This density function is denoted by :

$$\beta(a, b)(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{(a-1)}(1-x)^{(b-1)}$$

where  $\Gamma$  is *gamma*-function,  $\Gamma(n) = (n-1)!$ . Then :

- $\theta$  (prior) is uniform and it is *beta*-distribution. Density function :

$$f(\theta) = \beta(1, 1)(\theta)$$

- $(X = 6|\Theta = \theta)$ . Its density function is also *beta*-distribution :

$$P(X = 6|\theta) = \binom{6}{10} \theta^6(1-\theta)^4 = \beta(7, 5)(\theta) \frac{\binom{6}{10}}{\frac{\Gamma(12)}{\Gamma(5)\Gamma(7)}}$$

- $P(X = 6)$  is a constant.

Then posterior distribution is (by Baye) :

$$f_{\theta|X}(\theta) = \frac{\binom{6}{10} \theta^6(1-\theta)^4}{P(X = 6)} = \beta(7, 5)(\theta) \frac{\binom{6}{10}}{P(X = 6) \frac{\Gamma(12)}{\Gamma(5)\Gamma(7)}}$$

Here we can say that posterior distributio  $\theta|X$  follows also posterior distribution  $\beta(7, 5)$ , since  $\frac{\Gamma(12)}{\Gamma(5)\Gamma(7)}$  and  $\frac{\binom{6}{10}}{P(X=6)}$  is just for normalization. Simple check if these two quantities are equal :

$$\frac{\Gamma(12)}{\Gamma(5)\Gamma(7)} = 2310$$

$$\frac{\binom{6}{10}}{P(X = 6)} = \frac{210}{0.0909} = 2310.23$$

As the expected value of  $\beta(a, b)$  distribution is  $\frac{a}{a+b}$ , hence  $\mathbb{E}(\theta|X) = \frac{7}{7+5} = \frac{7}{12} = 0.58333$ .

## 3.2 Conditional probability

The probability of  $Y$  conditional on  $X = x$  is denoted :

$$P(Y|X = x)$$

### 3.2.1 Example

Suppose that the signal  $X \sim \mathcal{N}(0, \sigma_X^2)$  is transmitted over a communication channel. Assume that the received signal is given by  $Y = X + W$ , where  $W \sim \mathcal{N}(0, \sigma_W^2)$  is independent of  $X$ . Show that  $X|Y$  is normal distribution.

**False reasoning:**

From  $Y = X + W$ , we have:

$$(X|Y = y) = y - W$$

This is wrong since  $X|Y$  depends on  $W$ , violated the fact that  $X$  and  $W$  are independent.

**Good reasoning:**

Since :

- $Y = X + W$ , then  $\alpha Y + \beta X = (\alpha + \beta)X + \alpha W$ .
- In addition,  $X$  and  $W$  are independent normal distribution, then  $(\alpha + \beta)X + \alpha W$  is also normal distribution for all  $\alpha, \beta$ .

. The two above observations infer that  $\alpha Y + \beta X$  is normal distribution for all  $\alpha, \beta$  or  $X, Y$  are jointly normal. Then from density function of bivariate normal  $f_{X,Y}$ , we infer that  $f_{X|Y}$  is normal distribution.

## 3.3 Maximum a posteriori estimation MAP

Maximum a posteriori (MAP) is a statistical method used to estimate the most likely value of a parameter in a probability distribution, given some observed data. In other words, it is a method for finding the point in the parameter space that maximizes the posterior probability, which is the probability of the parameter given the observed data and any prior knowledge we may have about the parameter.

Mathematically, the MAP estimate of a parameter  $\theta$  can be written as:

$$\hat{\theta}_{MAP}(\mathbf{X} = \mathbf{x}) = \underset{\theta}{\operatorname{argmax}} f(\theta|\mathbf{x})$$

By Baye's theorem, the MAP estimate takes into account both the observed data and any prior knowledge we may have about the parameter :

$$\hat{\theta}_{MAP}(\mathbf{X} = \mathbf{x}) = \underset{\theta}{\operatorname{argmax}} f(\mathbf{x}|\theta)f(\theta)$$

Compared to MLE in sec 3.2.9, we have in addition prior distribution  $f(\theta)$ .

### 3.3.1 Example

Let  $X$  be a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Also, suppose that

$$Y|X = x \sim Geometric(x)$$

Find the MAP estimate of  $X$  given  $Y = 3$ .

**Solution** We need to maximize  $g(x)$ :

$$g(x) = P_{Y|X}(3|x)f_X(x) = (1-x)^2x^2$$

Then MAP estimation is at  $x$  such that  $g'(x) = 0$ , equivalently  $x = \frac{1}{2}$ .

## 3.4 Minimum Mean Squared Estimator

### 3.4.1 Definition

Given a random variable  $X$  that follows a certain distribution. Then estimator MMSE of  $X$  is  $a$  such that:

$$\min_a \mathbb{E}[(X - a)^2]$$

Using first order condition w.r.t  $a$ , we have  $a = \mathbb{E}[X]$ . And the minimum of  $\mathbb{E}[(X - a)^2]$  is thus  $Var(X)$ .

### 3.4.2 Conditional expectation

Given a posterior distribution,  $f_{X|Y}(x|y)$  contains all the knowledge that we have about the unknown quantity  $X$ . Therefore, to find a point estimate of  $X$ , we can just choose a summary statistic of the posterior such as its mean, median, or mode. If we choose the mode (the value of  $x$  that maximizes  $f_{X|Y}(x|y)$ ), we obtain the MAP estimate of  $X$ . Another option would be to choose the expected value of posterior :

$$\hat{x} = \mathbb{E}[X|Y = y]$$

We also call  $\mathbb{E}[X|Y = y]$  is conditional expectation.

This is a particular case of conditional expectation with respect to a sub sigma algebra in sec 2.2.12, with

$$\mathbb{E}[X|Y] := \mathbb{E}[X|\sigma(Y)]$$

### 3.4.3 Example

Let  $X$  be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

We also know that

$$f_{Y|X}(y|x) = \begin{cases} 2xy - x + 1 & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the MMSE estimate of  $X$ , given  $Y = y$  is observed.

#### Solution

We need to find  $\mathbb{E}[X|Y]$  with the density function :

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)}$$

Finding  $f_Y(y)$  :

$$\begin{aligned} f_Y(y) &= \int_0^1 f_{Y|X}(y|x)f_X(x)dx \\ &= \int_0^1 (2xy - x + 1)2x dx \\ &= \int_0^1 (4x^2y - 2x^2 + 2x) dx \\ &= y \frac{4x^3}{3} - \frac{2x^3}{3} + x^2 \Big|_0^1 = \frac{4y}{3} + \frac{1}{3} \end{aligned}$$

Hence:

$$f_{X|Y}(x|y) = \frac{(2xy - x + 1)2x}{\frac{4y}{3} + \frac{1}{3}}$$

MMSE :

$$\begin{aligned} \mathbb{E}[X|Y = y] &= \int_0^1 \frac{(2xy - x + 1)2x}{\frac{4y}{3} + \frac{1}{3}} x dx \\ &= \frac{3y + \frac{1}{2}}{4y + 1} \end{aligned}$$

### 3.4.4 Mean of MMSE estimator

Let  $\hat{X}_M$  is MMSE estimator of posterior  $X|Y$ , which means:

$$\hat{X}_M = \mathbb{E}[X|Y]$$

Then by the law of iterated expectation (in sec 2.2.13), we have :

$$\mathbb{E}[\hat{X}_M] = \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$$

### 3.4.5 Lemma

Let  $\hat{X}_M = \mathbb{E}[X|Y]$  be the MMSE estimator of  $X$  given  $Y$  and let  $\tilde{X} = X - \hat{X}_M$  be the estimation error, then we have :

1.  $W = \mathbb{E}[\tilde{X}|Y] = 0$
2. For any function  $g(Y)$ , we have  $\mathbb{E}[\tilde{X}g(Y)] = 0$
3.  $Cov(\tilde{X}, \hat{X}_M) = 0$
4.  $Var(X) = Var(\hat{X}_M) + Var(\tilde{X})$

$$\begin{aligned}\mathbb{E}[\tilde{X}|Y] &= \mathbb{E}[X - \hat{X}_M|Y] \\ &= \mathbb{E}[X|Y] - \mathbb{E}[\hat{X}_M|Y] \\ &= \hat{X}_M - \mathbb{E}[\hat{X}_M] \\ &= \hat{X}_M - \hat{X}_M \\ &= 0\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\tilde{X}g(Y)] &= \mathbb{E}[(X - \hat{X}_M)g(Y)] \\ &= \mathbb{E}[Xg(Y)] - \mathbb{E}[\hat{X}_Mg(Y)] \\ &= g(Y)\mathbb{E}[X] - \mathbb{E}[\hat{X}_M]g(Y) \quad \text{E in term of } X \\ &= 0 \quad \text{Since } \mathbb{E}[\hat{X}_M] = \mathbb{E}[X]\end{aligned}$$

$$\begin{aligned}Cov(\tilde{X}, \hat{X}_M) &= \mathbb{E}[\tilde{X}\hat{X}_M] - \mathbb{E}[\tilde{X}]\mathbb{E}[\hat{X}_M] \\ &= \mathbb{E}[\tilde{X}\hat{X}_M] \quad (\mathbb{E}[\tilde{X}] = 0) \\ &= \mathbb{E}[\tilde{X}g(y)] \quad (\text{As } \hat{X}_M \text{ is function of } y) \\ &= 0\end{aligned}$$

$$\begin{aligned}Var(X) &= Var(\tilde{X} + \hat{X}_M) \\ &= Var(\tilde{X}) + Var(\hat{X}_M) + 2Cov(\tilde{X}, \hat{X}_M) \\ &= Var(\tilde{X}) + Var(\hat{X}_M)\end{aligned}$$

This infers that:

$$\begin{aligned}\mathbb{E}[X^2] - \mathbb{E}[X]^2 &= \mathbb{E}[\tilde{X}^2] - \mathbb{E}[\tilde{X}]^2 + \mathbb{E}[\hat{X}_M^2] - \mathbb{E}[\hat{X}_M]^2 \\ \Leftrightarrow \mathbb{E}[X^2] &= \mathbb{E}[\tilde{X}^2] + \mathbb{E}[\hat{X}_M^2] \quad \text{Since } \mathbb{E}[X]^2 = \mathbb{E}[\hat{X}_M]^2 \text{ and } \mathbb{E}[\tilde{X}] = 0\end{aligned}$$

### 3.5 Linear MMSE Estimation of Random Variables

We might face some difficulties if we want to use the MMSE in practice.

- $f_{X|Y}(x|y)$  might not be easy to find in some problems
- computing  $\mathbb{E}[X|Y = y]$  might not be easy
- the function  $g(y) = \mathbb{E}[X|Y = y]$  might have a complicated form

Hence, an alternative solution is to suppose that the estimator of  $X$  is a linear function of  $Y$ :

$$\hat{X}_L = aY + b$$

Then we find  $a, b$  such that :

$$a^*, b^* = \underset{a,b}{\operatorname{argmin}} h(a,b) = \underset{a,b}{\operatorname{argmin}} \mathbb{E}[(X - \hat{X}_L)^2]$$

The solution  $(a^*, b^*)$  satisfy these following things :

1.  $a^* = \frac{\text{Cov}(X,Y)}{\text{Var}(Y)}, b^* = \mathbb{E}[X] - a^*\mathbb{E}[Y]$
2.  $h(a^*, b^*) = (1 - \rho^2)\text{Var}(X)$
3.  $\mathbb{E}[(X - a^*Y - b)Y] = 0$  or  $\mathbb{E}[\tilde{X}Y] = 0$  (orthogonality principle)

The first thing is obtained by taking the derivative of  $h(a,b)$ , (as in section 3.2.12.1.2) :

$$\begin{aligned} h(a,b) &= \mathbb{E}[(X - aY - b)^2] \\ &= \mathbb{E}[(X^2 + a^2Y^2 + b^2 - 2aXY - 2bX + 2abY)] \\ &= \mathbb{E}[X^2] + a^2\mathbb{E}[Y^2] + b^2 - 2a\mathbb{E}[XY] - 2b\mathbb{E}[X] + 2ab\mathbb{E}[Y] \end{aligned}$$

$$\begin{cases} 2a\mathbb{E}[Y^2] - 2\mathbb{E}[XY] + 2b\mathbb{E}[Y] = 0 \\ 2b - 2\mathbb{E}[X] + 2a\mathbb{E}[Y] = 0 \end{cases} \quad (3.1)$$

$$\begin{cases} a = \frac{\text{Cov}(X,Y)}{\text{Var}(Y)} \\ b = \mathbb{E}[X] - a\mathbb{E}[Y] \end{cases}$$

Here, we find the same result as in sec 3.2.12.1.1.

The second thing:

$$\begin{aligned} h(a^*, b^*) &= \mathbb{E}[(X - a^*Y - b^*)^2] \\ &= \text{Var}(X - a^*Y - b^*) \quad \text{Since } \mathbb{E}[X - a^*Y - b^*] = 0 \\ &= \text{Var}(X) + (a^*)^2\text{Var}(Y) - 2a^*\text{Cov}(X, Y) \end{aligned}$$

$$\begin{aligned}
&= \text{Var}(X) + \frac{\text{Cov}(X, Y)^2}{\text{Var}(Y)^2} \text{Var}(Y) - 2 \frac{\text{Cov}(X, Y)}{\text{Var}(Y)} \text{Cov}(X, Y) \\
&= \text{Var}(X) - \frac{\text{Cov}(X, Y)^2}{\text{Var}(Y)} \\
&= (1 - \rho^2) \text{Var}(X)
\end{aligned}$$

The third thing :

$$\begin{aligned}
\mathbb{E}[(X - a^*Y - b^*)Y] &= \mathbb{E}[XY] - a^*\mathbb{E}[Y^2] - b^*\mathbb{E}[Y] \\
&= 0 \quad \text{as 3.1}
\end{aligned}$$

### 3.5.1 Random Vector

This is an extension for random vector instead of random variable. Suppose that we have a random vector  $\mathbf{X} = [X_1, \dots, X_m]^T$  that we want to model by a linear relation of  $\mathbf{Y} = [Y_1, \dots, Y_n]^T$  by :

$$\mathbf{X} = \mathbf{AY} + \mathbf{b}$$

where  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^n$ .

Remind that in the case of random variable  $X$  and  $Y$ , the solution is:

$$\hat{X}_L = \frac{\text{Cov}(X, Y)}{\text{Var}(Y)}(Y - \mathbb{E}[Y]) + \mathbb{E}[X]$$

By analogy, in the case of random vector:

$$\hat{\mathbf{X}}_L = \text{Cov}(\mathbf{X}, \mathbf{Y}) \text{Cov}(\mathbf{Y}, \mathbf{Y})^{-1}(\mathbf{Y} - \mathbb{E}[\mathbf{Y}]) + \mathbb{E}[\mathbf{X}]$$

and

$$\begin{aligned}
\mathbb{E}[\tilde{\mathbf{X}}] &= 0 \\
\text{Cov}(\tilde{\mathbf{X}}, Y_j) &= \mathbb{E}[\tilde{\mathbf{X}}Y_j] = 0
\end{aligned}$$

where  $\tilde{\mathbf{X}} = \mathbf{X} - \hat{\mathbf{X}}_L$ .

The first condition gives us  $m$  equations and the second give us  $m \times n$  equations, as the same as the number of variable in  $\mathbf{A}$  and  $\mathbf{b}$ , which is  $m \times (n+1)$ . Linear MMSE for random vector is equivalent to OLS in sec 3.2.12.2.

#### 3.5.1.1 Example

Let  $X$  be an unobserved random variable with  $\mathbb{E}[X] = 0$ ,  $\text{Var}(X) = 4$ . Assume that we have observed  $Y_1$  and  $Y_2$  given by

$$\begin{aligned}
Y_1 &= X + W_1 \\
Y_2 &= X + W_2
\end{aligned}$$

where  $\mathbb{E}[W_1] = \mathbb{E}[W_2] = 0$ ,  $Var(W_1) = 1$ , and  $Var(W_2) = 4$ . Assume that  $W_1, W_2$ , and  $X$  are independent random variables. Find the linear MMSE estimator of  $X$ , given  $Y_1$  and  $Y_2$ .

**Solution 1**

We will find directly  $\mathbf{A}$  and  $\mathbf{b}$

$$\begin{aligned}\mathbf{A} &= Cov(\mathbf{X}, \mathbf{Y})Cov(\mathbf{Y}, \mathbf{Y})^{-1} \\ &= [Cov(X, Y_1) \quad Cov(X, Y_2)] \begin{bmatrix} Cov(Y_1, Y_1) & Cov(Y_1, Y_2) \\ Cov(Y_2, Y_1) & Cov(Y_2, Y_2) \end{bmatrix}^{-1}\end{aligned}$$

$$Cov(X, Y_1) = Cov(X, X + W_1) = Cov(X, X) = Var(X) = 4$$

$$Cov(X, Y_2) = Cov(X, X + W_2) = Cov(X, X) = Var(X) = 4$$

$$Cov(Y_1, Y_1) = Cov(X + W_1, X + W_1) = Cov(X, X) + Cov(W_1, W_1) = Var(X) + Var(W_1) = 5$$

$$Cov(Y_1, Y_2) = Cov(X + W_1, X + W_2) = Cov(X, X) = Var(X) = 4$$

$$Cov(Y_2, Y_2) = Cov(X + W_2, X + W_2) = Cov(X, X) + Cov(W_2, W_2) = Var(X) + Var(W_2) = 8$$

$$\mathbf{A} = [4 \quad 4] \begin{bmatrix} 5 & 4 \\ 4 & 8 \end{bmatrix}^{-1} = [4 \quad 4] \frac{1}{24} \begin{bmatrix} 8 & -4 \\ -4 & 5 \end{bmatrix}^{-1} = \left[ \frac{2}{3} \quad \frac{1}{6} \right]$$

$$\mathbf{b} = \mathbb{E}[\mathbf{X}] - \mathbf{A}\mathbb{E}[\mathbf{Y}] = 0$$

**Solution 2**

By using orthogonality principle. Let  $\tilde{X} = aY_1 + bY_2 + c$ :  
The condition  $\mathbb{E}[\tilde{\mathbf{X}}] = 0$  gives us:

$$\begin{aligned}\mathbb{E}[X - (aY_1 + bY_2 + c)] &= 0 \\ \Leftrightarrow \mathbb{E}[X] - a\mathbb{E}[Y_1] - b\mathbb{E}[Y_2] - c &= 0 \\ \Leftrightarrow c &= 0\end{aligned}$$

The condition  $\mathbb{E}[\tilde{\mathbf{X}}Y_1] = 0$  gives us:

$$\begin{aligned}\mathbb{E}[(X - aY_1 - bY_2 - c)Y_1] &= 0 \\ \Leftrightarrow \mathbb{E}[(X - a(X + W_1) - b(X + W_2) - c)(X + W_1)] &= 0 \\ \Leftrightarrow \mathbb{E}[X^2] - a\mathbb{E}[X^2] - b\mathbb{E}[X^2] - c\mathbb{E}[X] - a\mathbb{E}[W_1^2] - c\mathbb{E}[W_1] &= 0 \\ \Leftrightarrow 4 - 4a - 4b - a &= 0\end{aligned}$$

$$\Leftrightarrow 5a + 4b = 4$$

The condition  $\mathbb{E}[\tilde{\mathbf{X}}Y_2] = 0$  gives us:

$$\begin{aligned} & \mathbb{E}[(X - aY_1 - bY_2 - c)Y_2] = 0 \\ & \Leftrightarrow \mathbb{E}[(X - a(X + W_1) - b(X + W_2) - c)(X + W_2)] = 0 \\ & \Leftrightarrow \mathbb{E}[X^2] - a\mathbb{E}[X^2] - b\mathbb{E}[X^2] - c\mathbb{E}[X] - b\mathbb{E}[W_2^2] - c\mathbb{E}[W_2] = 0 \\ & \Leftrightarrow 4 - 4a - 4b - 4b = 0 \\ & \Leftrightarrow 4a + 8b = 4 \end{aligned}$$

Then we can have  $a = \frac{2}{3}$  and  $b = \frac{1}{6}$

### 3.6 Bayesian Hypothesis Testing

Suppose that we need to decide between two hypotheses  $H_0$  and  $H_1$ . In the Bayesian setting, we assume that we know prior probabilities of  $H_0$  and  $H_1$ . That is, we know  $P(H_0)$  and  $P(H_1)$ . We observe the random variable (or the random vector)  $Y = y$ . Then we compare these two following posteriors  $P(H_0|Y = y)$  and  $P(H_1|Y = y)$  and select the hypothesis with higher posterior.

The average error probability for a hypothesis test can be written as :

$$P_e = P(\text{choose } H_1 | H_0)P(H_0) + P(\text{choose } H_0 | H_1)P(H_1).$$

#### 3.6.1 Example

Suppose that the random variable  $X$  is transmitted over a communication channel. Assume that the received signal is given by

$$Y = X + W$$

, where  $W \sim \mathcal{N}(0, \sigma^2)$  is independent of  $X$ . Suppose that  $X = 1$  with probability  $p$ , and  $X = -1$  with probability  $1 - p$ . The goal is to decide between  $X = 1$  and  $X = -1$  by observing the random variable  $Y$ , which means to find higher posterior for this problem.

**Solution**

$$\begin{aligned} P(H_0|Y = y) &= P(X = 1|Y = y) \\ &= \frac{P(Y = y|X = 1)P(X = 1)}{P(Y = y)} \\ &= \frac{\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)p}{\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)p + \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y+1)^2}{2\sigma^2}\right)(1-p)} \end{aligned}$$

$$P(H_1|Y = y) = P(X = -1|Y = y)$$

$$\begin{aligned}
&= \frac{P(Y = y|X = -1)P(X = -1)}{P(Y = y)} \\
&= \frac{\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y+1)^2}{2\sigma^2}\right)(1-p)}{\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)p + \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y+1)^2}{2\sigma^2}\right)(1-p)}
\end{aligned}$$

$$\begin{aligned}
P(H_0|Y = y) &> P(H_1|Y = y) \\
\Leftrightarrow \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)p &> \exp\left(-\frac{(y+1)^2}{2\sigma^2}\right)(1-p) \\
\Leftrightarrow \exp\left(\frac{2y}{\sigma^2}\right) &> \frac{1-p}{p} \\
\Leftrightarrow y &> \frac{\sigma^2}{2} \ln\left(\frac{1-p}{p}\right)
\end{aligned}$$

### 3.7 Bayesian Interval Estimation

Given a posterior, it's logic that we can have an interval estimation of  $X$ , given  $Y = y$ .

$$P(a \leq X \leq b|Y = y) = 1 - \alpha$$

where interval  $[a, b]$  is said to be a  $(1 - \alpha)100\%$  credible interval.

#### 3.7.1 Example

Let  $X$  and  $Y$  be jointly normal and  $X \sim \mathcal{N}(0, 1)$ ,  $Y \sim \mathcal{N}(1, 4)$ , and  $\rho(X, Y) = 0.5$ . Find a 95% credible interval for  $X$ , given  $Y = 2$  is observed.

##### Solution

From sec 2.6.2.1,

$$\begin{aligned}
&f_{XY}(x, y) \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left( (\frac{x-\mu_X}{\sigma_X})^2 - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + (\frac{y-\mu_Y}{\sigma_Y})^2 \right)\right) \\
&= \frac{1}{4\pi\sqrt{1-0.5^2}} \exp\left(-\frac{1}{2(1-0.5^2)} \left( x^2 - \frac{xy}{2} + \frac{y^2}{4} \right)\right)
\end{aligned}$$

$$\begin{aligned}
&f_{X|Y}(x|2) \\
&= \frac{f_{XY}(x, y)}{f_Y(2)} \\
&= \frac{\frac{1}{4\pi\sqrt{1-0.5^2}} \exp\left(-\frac{1}{2(1-0.5^2)} (x-1)^2\right)}{\frac{1}{\sqrt{2\pi}2} \exp\left(-\frac{1}{2} \frac{2^2}{2^2}\right)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}\sqrt{1-0.5^2}} \exp\left(-\frac{2}{3}(x-1)^2 + \frac{1}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}\sqrt{3/4}} \exp\left(-\frac{1}{2}\frac{(x-1/4)^2}{3/4}\right)
\end{aligned}$$

The above density function is of  $\mathcal{N}(\cdot, \frac{3}{4})$

Or a faster way is to use directly conditional distribution in sec 2.6.2.5. It means  $X|Y = y$  follows normal distribution with :

$$E(X|Y = y) = \sigma_X \rho \frac{y - \mu_Y}{\sigma_Y} + \mu_X = 0.5 \frac{2-1}{2} = 0.25$$

$$Var(X|Y = y) = (1 - \rho^2)\sigma_X^2 = \frac{3}{4}$$

Now the problem is just to find the credible (or confidence) interval , such as in sec 3.2.10.

## 4 Distribution and hypothesis testing

### 4.1 Chi-squared distribution

Chi-squared distribution (noted  $\chi^2$ ) with  $k$  degrees of freedom is the distribution of a sum of the squares of  $k$  independent standard normal random variables.

If  $Z_1, \dots, Z_k$  are independent, standard normal random variables, then the sum of their squares

$$Q = \sum_{i=1}^k Z_i^2$$

is distributed according to the chi-squared distribution with  $k$  degrees of freedom. This is usually denoted as

$$Q \sim \chi^2(k) \quad \text{or} \quad Q \sim \chi_k^2$$

It can be shown that the Chi-squared distribution, in fact, is a particular case of gamma distribution :

$$Q \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

#### 4.1.1 Adequacy test

The  $\chi^2$  adequacy test (also called the  $\chi^2$  conformity test or  $\chi^2$  adjustment test) makes it possible to check whether a sampling of a random variable  $Y$  follows to a distribution defined a priori.

#### 4.1.1.1 Adequacy test for a multinomial distribution

In this case, we test the adequacy for a **multinomial** distribution. Remind some properties of the multinomial distribution :

- $n$  number of trials
- $k$  number of events
- $p_1, p_2, \dots, p_k$  event probabilities and  $\sum_{i=1}^k p_i = 1$ .
- $X_i$  be the random variable that count the number of the event  $i$  which happens.
- Support :  $X_i \in \{0, 1, \dots, n\}$  with  $1 \leq i \leq k$
- PMF :  $\frac{n!}{x_1! \dots x_k!} p_1^{x_1} \dots p_k^{x_k}$ , where  $\sum_{i=1}^k x_i = n$ .
- Mean  $\mathbb{E}(X_i) = np_i$
- Variance  $Var(X_i) = np_i(1 - p_i)$
- Covariance  $Cov(X_i, X_j) = -np_i p_j$
- Let  $Y_i^l$  be a indicator random variable :

$$\begin{cases} Y_i^l = 1 & \text{if event } i \text{ happens at trial } l \\ Y_i^l = 0 & \text{if not} \end{cases}$$

$Y_i^l$  follows Bernoulli distribution. We can express  $X_i$  as:

$$X_i = Y_i^1 + Y_i^2 + \dots + Y_i^n$$

#### On the sidelines : Proof for covariance

$$\begin{aligned} Cov(X_i, X_j) &= \mathbb{E}(X_i X_j) - \mathbb{E}(X_i) \mathbb{E}(X_j) \\ &= \mathbb{E}\left(\sum_k Y_i^k \sum_l Y_j^l\right) - n^2 p_i p_j \\ &= \mathbb{E}\left(\sum_{l=h} Y_i^l Y_j^h + \sum_l \sum_{h \neq l} Y_i^l Y_j^h\right) - n^2 p_i p_j \\ &= \mathbb{E}\left(\sum_l \sum_{h \neq l} Y_i^l Y_j^h\right) - n^2 p_i p_j, \quad \text{since } i \neq j \text{ then } Y_i^l Y_j^l = 0 \\ &= \sum_l \mathbb{E}(Y_i^l) \sum_{h \neq l} \mathbb{E}(Y_j^h) - n^2 p_i p_j \\ &= np_i(n-1)p_j - n^2 p_i p_j \\ &= -np_i p_j \end{aligned}$$

Let's come back the main subject :

**Hypothesis null :**  $(X_i)_{i=1,\dots,k}$  follow the multinomial distribution. We show that under the hypothesis null, we have :

$$\sum_{i=1}^k \frac{(\hat{N}_i - np_i)^2}{np_i} \sim \chi^2(k-1)$$

by two methods.

### First method

From the above hypothesis, we have that  $Y_i^l$  is sampled from the iid distribution (Bernoulli) with mean  $\mu_{Y_i} = p_i$  and std  $\sigma_{Y_i} = p_i(1-p_i)$ . If  $n$  is sufficiently big, then the theorem central limite says:

$$Z_i = \frac{\frac{X_i}{n} - \mu_{Y_i}}{\frac{\sigma_{Y_i}}{\sqrt{n}}} = \frac{X_i - n\mu_{Y_i}}{\sqrt{n}\sigma_{Y_i}} = \frac{n\hat{p}_i - np_i}{\sqrt{np_i(1-p_i)}}$$

$Z_i$  is a standard normal distribution. Since  $\sum_{i=1}^k X_i = n$ , we take only  $k-1$  first  $Z_i$  to guarantee that  $(Z_i)_{i=1,\dots,k-1}$  are independent. Then

$$T = \sum_{i=1}^{k-1} Z_i^2 = \sum_{i=1}^{k-1} \frac{(n\hat{p}_i - np_i)^2}{np_i(1-p_i)} = \sum_{i=1}^{k-1} \frac{(\hat{N}_i - np_i)^2}{np_i(1-p_i)} \sim \chi^2(k-1)$$

Note that we have the following equality. With  $\sum_{i=1}^k \hat{N}_i = n = n(\sum_{i=1}^k p_i)$ , then we can rewrite  $T$  :

$$T = \sum_{i=1}^k \frac{(\hat{N}_i - np_i)^2}{np_i}$$

Demonstration for  $k=2$ :

$$\begin{aligned} T &= \frac{(\hat{N}_1 - np_1)^2}{np_1(1-p_1)} \\ &= \frac{(\hat{N}_1 - np_1)^2}{np_1} + \frac{(\hat{N}_1 - np_1)^2}{n(1-p_1)} \\ &= \frac{(\hat{N}_1 - np_1)^2}{np_1} + \frac{(\hat{N}_2 - np_2)^2}{np_2} \end{aligned}$$

Since  $\hat{N}_1 + \hat{N}_2 = np_1 + np_2$ .

### Second method

Let  $Y$  be a random variable such that  $P(Y = i) = p_i$  and consider the following random vector  $Z$ :

$$\begin{aligned} Z &= [Z_1, \dots, Z_k]^T \\ &= \left[ \frac{[Y=1]-p_1}{\sqrt{p_1}}, \frac{[Y=2]-p_2}{\sqrt{p_2}}, \dots, \frac{[Y=k]-p_k}{\sqrt{p_k}} \right]^T \end{aligned}$$

where:

$$[Y=j] = \begin{cases} 1 & \text{if } Y=j \\ 0 & \text{otherwise} \end{cases}$$

We have :

- $\mathbb{E}(Z_i) = 0$
- $\text{Var}(Z_i) = 1 - p_i$
- $\text{Cov}(Z_i, Z_j) = -\sqrt{p_i p_j}, i \neq j$
- Covariance matrix  $\Sigma = I_k - \begin{bmatrix} \sqrt{p_1} \\ \vdots \\ \sqrt{p_k} \end{bmatrix} [\sqrt{p_1}, \dots, \sqrt{p_k}]^T = I_k - \sqrt{p} \sqrt{p}^T$

Then by applying the central limit theorem (3.1.1) in the multidimensional case:

$$\tilde{Z} = \sqrt{n} \frac{Z^{(i)} + \dots + Z^{(n)}}{n} \rightarrow \mathcal{N}(0, \Sigma)$$

Here, we have two ways to show that  $\tilde{Z} \sim \chi^2(k-1)$

**First method :** Using Cochran theorem in section 3.1.2, let's suppose that random vector  $\mathbf{A} \sim \mathcal{N}(0, I_k)$ . Let  $F$  be the subspace that attaches to  $\sqrt{p}$ , then  $P_F = \sqrt{p} \sqrt{p}^T$ . Let  $F^\perp$  be the subspace that perpendicular to  $\sqrt{p}$ , then  $P_{F^\perp} = I_k - \sqrt{p} \sqrt{p}^T$ . This is because  $I_k$  corresponds to matrix for projecting from  $\mathbb{R}^k$  to itself  $\mathbb{R}^k$ ;  $F$  and  $F^\perp$  are orthogonal, then for getting  $P_{F^\perp}$  we just take the subtraction. From  $\dim(\sqrt{p} \sqrt{p}^T) = 1$ , then  $\dim(P_F) = 1$ , this infers  $\dim(P_{F^\perp}) = k-1$ . Hence, we get  $P_{F^\perp} \mathbf{A} \sim \mathcal{N}(0, I_k - \sqrt{p} \sqrt{p}^T)$  and  $\sum_i P_{F^\perp} \mathbf{A}[i] \sim \chi^2(k-1)$ .

Remind that  $\tilde{Z}$  has the same distribution as  $P_{F^\perp} \mathbf{A}$ , then we must have  $\sum_i \tilde{Z}[i] \sim \chi^2(k-1)$ .

**Second method :**  $\Sigma = I_k - \sqrt{p} \sqrt{p}^T$  has eigenvalues 0 (with eigenvector  $V_1 = \sqrt{p}$ ) and 1 (with  $k-1$  eigenvectors  $V_2, \dots, V_k$ ). We see that  $V_1$  corresponds to subspace  $F$  and  $V_2, \dots, V_k$  corresponds to subspace  $F^\perp$ . Also, since  $\Sigma$  is symmetric, we have  $V_1 \perp [V_2, \dots, V_k]$ .

Reminding a short proof, given  $A$  is symmetric and two different couples (eigenvalue, eigenvector)  $(\lambda, x)$  and  $(\mu, y)$ :

$$\lambda \langle x, y \rangle = \langle \lambda x, y \rangle = \langle Ax, y \rangle = \langle x, A^T y \rangle$$

$$= \langle x, Ay \rangle = \langle x, \mu y \rangle = \mu \langle x, y \rangle$$

therefore  $\lambda \langle x, y \rangle = \mu \langle x, y \rangle$ , so  $\langle x, y \rangle = 0$

Suppose that  $Q = [V_2, \dots, V_k]$  are orthonormalized (if not can use e.g. Gram–Schmidt). Let's set  $\mathbf{B} = Q^T \tilde{Z}$ , hence after this linear transformation  $\mathbf{B} \sim \mathcal{N}(0, Q^T \Sigma Q)$ . This linear transformation of normal distribution can be proved by Moment generating function (sec 2.5.1). Note that :

$$Q^T \Sigma Q = Q^T (I_k - \sqrt{p} \sqrt{p}^T) Q = Q^T Q - Q^T \sqrt{p} \sqrt{p}^T Q = I_{k-1}$$

Then  $\mathbf{B} \sim \mathcal{N}(0, I_{k-1})$ . Also,

$$\sum_i \tilde{Z}[i]^2 = \tilde{Z}^T \tilde{Z} = \mathbf{B}^T Q^T Q \mathbf{B} = \sum_i \mathbf{B}[i]^2 \sim \chi^2(k-1)$$

**Let's resume to the main**, after the analyse covariance for matrix, from  $\sum_i \tilde{Z}[i]^2 \sim \chi^2(k-1)$ , it means:

$$\begin{aligned} \sum_{i=1}^k \tilde{Z}[i]^2 &= \sum_{i=1}^k \left( \frac{1}{\sqrt{n}} \sum_{j=1}^n \frac{[Y^{(j)} = i] - p_i}{\sqrt{p_i}} \right)^2 \\ &= \sum_{i=1}^k \frac{(n\hat{p}_i - np_i)^2}{np_i} \\ &= \sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i} \\ &\sim \chi^2(k-1) \end{aligned}$$

where  $O_i$  means the observation for event  $i$  and  $E_i$  means the true expected value for event  $i$ . Note that  $E_i$  is not always available, and we maybe need to estimate these values.

#### 4.1.2 Degree of freedom

We need to determine the degree of freedom, in order to use correctly Chi squared distribution. Indeed, it is not necessary to analyse the covariance matrix as before, just using the following :

- Number of normal distributions :  $J$ .
- With  $N_c$  number of constraints, (e.g.  $\sum_i O_i = n$ ), the degree of freedom :  $k = J - N_c$
- In addition, if there are  $s$  number of unknown parameters (of a distribution to test) need to be estimated :  $k = J - N_c - s$  (see 3.4.1.2.2).

#### 4.1.2.1 Example 1

Check the balance of a dice with  $n = 600$  throws (adequacy test for uniform distribution):

number	1	2	3	4	5	6
effectifs	88	109	107	94	105	97

$$\begin{aligned}
T &= \sum_{i=1}^6 \frac{(O_i - E_i)^2}{E_i} \\
&= \frac{(88 - 100)^2}{100} + \frac{(109 - 100)^2}{100} + \frac{(107 - 100)^2}{100} + \frac{(94 - 100)^2}{100} + \frac{(105 - 100)^2}{100} + \frac{(97 - 100)^2}{100} \\
&= 3.44
\end{aligned}$$

$T \sim \chi^2(6 - 1)$  (since there is only one constraint  $\sum_i O_i = n$ ), risk  $\alpha = 0.05$  :  $P(T < 11.07) = 0.95$ . Since  $3.44 < 11.07$  then we can say the dice is unbiased.

#### 4.1.2.2 Example 2

We consider a random variable  $Y$  taking positive or zero integer values. A sampling of  $n = 100$  values of this variable is distributed as follows:

number	0	1	2	3	4
effectifs	31	45	16	7	1

We wish to test the hypothesis according to which  $Y$  follows Poisson distribution (adequacy test), with a risk  $\alpha = 0.05$ .

In hypothesis according to Poisson distribution, we get :

$$\lambda = 0.45 + 2 \times 0.16 + 3 \times 0.07 + 4.01 = 1.02$$

With is  $\lambda = 1.02$ , we infer the expected effectifs by  $n \times \frac{\lambda^k}{k!} e^{-\lambda}$ :

number	0	1	2	3	4
effectifs	36.06	36.78	18.76	6.48	1.92

Since this parameter  $\lambda$  is **estimated**, we will reduce the number of degrees of freedom by one unit, we have also a condition  $\sum_i O_i = n$ , then the degree of freedom is  $5 - 1 - 1 = 3$ .

$$\begin{aligned}
T &= \sum_{i=0}^4 \frac{(O_i - E_i)^2}{E_i} \\
&= \frac{(36.06 - 31)^2}{36.06} + \frac{(36.78 - 45)^2}{36.78} + \frac{(18.76 - 16)^2}{18.76} + \frac{(6.48 - 7)^2}{6.48} + \frac{(1.92 - 1)^2}{1.92} \\
&= 3.4357
\end{aligned}$$

The  $\chi^2$  law with three degrees of freedom gives  $P(T < 7.815) = 0.95$ . Therefore, we do not reject the hypothesis that the random variable  $Y$  follows a Poisson distribution, with a risk of error of 5%.

### 4.1.3 Homogeneity test

The question here is to ask whether two lists of numbers with the same total effectif  $n$  can derive from the same law of probability.

The procedure is more or less similar to adequacy test, except that :

$$T = \sum_{i=1}^J \frac{(O_i - Q_i)^2}{Q_i}$$

where  $O, Q$  means the first and second observation.

### 4.1.4 Independency test

This test makes it possible to verify the absence of a statistical link between two variables X and Y. The two are said to be independent when there is no statistical link between them, in other words, knowledge of X does not in any way make it possible to pronounce on Y.

Reminding the independence :

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

Need to check the property above.

Example : check the independency between X and Y :

X / Y	1	2	3	4	Total
A	50	70	110	60	290
B	60	75	100	50	285
Total	110	145	210	110	575

- Observation  $O_{ij} = nP(X = x_i, Y = y_j)$
- Estimated value  $E_{ij} = nP(X = x_i)P(Y = y_j)$

$$T = \sum_{ij} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

Therefore  $T$  follows  $\chi^2$ , we need to find out the degree of freedom  $q$  for  $T$ . X and Y take a fini number of values, I pour X, J pour Y.

- Number of values :  $IJ$
- One constraint :  $\sum_{i,j} O_{ij} = 1$ .
- For X, there are  $(I - 1)$  probabilities  $P(X = x_i)$  need to be estimated and the same for Y, there are  $(J - 1)$  probabilities  $P(Y = y_i)$  need to be estimated.
- Hence,  $q = IJ - 1 - (I - 1) - (J - 1) = (I - 1)(J - 1) = 3$ .

$$\begin{aligned}
T &= \sum_{ij} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \\
&= \sum \frac{(50 - 575 \frac{290}{575} \frac{110}{575})^2}{575 \frac{290}{575} \frac{110}{575}} \\
&= 2.4235
\end{aligned}$$

#### 4.1.5 Reduced chi-squared statistic

In statistics, the reduced chi-square statistic is used extensively in goodness of fit testing. The reduced chi-squared  $\chi_\nu^2$  is defined as chi-squared  $\chi^2$  per degree of freedom:

$$\chi_\nu^2 = \frac{\chi^2}{\nu},$$

where chi-squared is a weighted sum of squared deviations:

$$\chi^2 = \sum_i \frac{(O_i - C_i)^2}{\sigma_i^2}$$

with  $O_i$  is observation,  $C_i$  is reference or mean (true value),  $\sigma_i$  is std of distribution  $i$ . The degree of freedom  $\nu = n - m$ , equals the number of observations  $n$  minus the number of fitted parameters  $m$ .

Goodness of Fit:

- A reduced chi-squared value close to 1 indicates a good fit.
- A reduced chi-squared value significantly greater than 1 suggests an underfitting model
- A reduced chi-squared value significantly smaller than 1 suggests an overfitting model.

In ordinary least squares, the definition simplifies to:

$$\chi_\nu^2 = \frac{RSS}{\nu}$$

where  $RSS = \sum r^2$  is residual sum of squares.

## 4.2 Student distribution

Let  $Z$  random variable of normal distribution, centered and reduced.  $U$  is an independent variable compared to  $Z$  and follows Chi-squared distribution of  $k$  degree of freedom.

$$T = \frac{Z}{\sqrt{U/k}}$$

follows Student distribution with degree of freedom  $k$ .

#### 4.2.1 Student distribution in sampling

Let  $X_1, \dots, X_n$   $n$  random variables mutually independent and distributed from normal distribution  $\mathcal{N}(\mu, \sigma^2)$ .

Empirical mean:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

Variance estimated without bias :

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

Consider  $\bar{X}$  as a random variable, hence  $\bar{X}$  is a normal distribution,  $mean(\bar{X}) = \mu$ . Then by CTL,

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

is a normal standard distribution.

Replacing  $\sigma$  (true value) by  $S$  (estimated):

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\sigma}{S} \frac{(\bar{X} - \mu)}{\sigma/\sqrt{n}}$$

$T$  follows Student distribution of  $(n - 1)$  degree liberty, because we set :

$$\frac{U}{n-1} = \frac{S^2}{\sigma^2}$$

where

$$U = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2}$$

Then

$$T = \frac{Z}{\sqrt{U/(n-1)}}$$

We need to prove :

- $Z$  (**normalized** sample mean) and  $U$  (**normalized** sample variance) are independent. One can use Basu's theorem (sec 3.1.3.7). Note that here we have **normalized** case, if not, sample mean and sample variance are independent just for  $\mu$  (sec 3.1.3.7.1).

- $U \sim \chi^2(n-1)$ . This can be done by using theorem of Cochran (sec 3.1.2).

#### 4.2.1.1 Application

To get the bilateral confidence interval for  $\mu$ , in case of unknown variance. Then with risk  $\alpha$  or with confidance  $(1 - \alpha)$ , this bilateral confidence interval is given by :

$$\left[ \bar{X} - t_{\alpha/2}^{n-1} \frac{S}{\sqrt{n}}; \bar{X} + t_{\alpha/2}^{n-1} \frac{S}{\sqrt{n}} \right]$$

where  $t^{n-1}$  follow Student distribution with degree of freedom  $n - 1$ .

Ex: Given the table of height for 8 persons, estimate the confident interval lied to mean of a normal distribution which has unknown variance.

$i$	1	2	3	4	5	6	7	8
$x_i$	155	160	161	167	171	177	180	181

Mean estimated:

$$\bar{X} = \frac{1}{8} \sum_{i=1}^8 X_i = 169$$

Variance estimated without bias :

$$S = \frac{1}{7} \sum_{i=1}^8 (X_i - \bar{X})^2 = 96.85$$

Then:

$$T = \frac{\bar{X} - \mu}{S/\sqrt{8}}$$

follows Student distribution, of degree 7.

#### 4.2.2 Standard error

The standard error (SE) of a statistic (usually an estimate of a parameter) is the standard deviation (std) of its sampling distribution or an estimate of that standard deviation.

##### 4.2.2.1 Example

Let  $X_1, \dots, X_n$  be a random sample from a distribution that has mean  $\mu$  and std  $\sigma$ . Then sample mean is used to estimate  $\mu$  by :

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n X_i$$

Then the standard error of  $\hat{\mu}$ , denoted  $s.e.(\hat{\mu})$  is  $\frac{\sigma}{\sqrt{n}}$ .

### 4.2.3 *t*-statistic

Let  $\hat{\beta}$  be an estimator of parameter  $\beta$  in some statistical model. Then a *t*-statistic for this parameter is any quantity of the form :

$$t = \frac{\hat{\beta} - \beta_0}{s.e.(\hat{\beta})}$$

where :

- $\beta_0$  is a constant, using in hypothesis  $H_0 : \beta = \beta_0$ .
- where *s.e.* means standard error (sec 3.4.2.2).

If  $\hat{\beta}$  is an estimator by using ordinary least squares with the classical linear regression model, and if the true value of the parameter  $\beta$  is equal to  $\beta_0$ . Then *t* follows the Student distribution with  $(n - k)$  degrees of freedom, where  $n$  is the number of observations, and  $k$  is the number of regressors (including the intercept).

In the majority of models, the estimator  $\hat{\beta}$  is consistent (sec 3.2.5) for  $\beta$  and is distributed asymptotically normally (sec 3.2.7). If the true value of the parameter  $\beta$  is equal to  $\beta_0$ , and the quantity *s.e.*( $\hat{\beta}$ ) correctly estimates the asymptotic variance of this estimator, then the *t*-statistic will asymptotically have the standard normal distribution.

*t*-statistic is very similar to the *z*-score ( $z = \frac{x-\mu}{\sigma}$ ) but with the difference that *t*-statistic is used when the sample size is small or the population standard deviation is unknown. It is also used along with *p*-value when running hypothesis tests where the *p*-value tells us what the odds are of the results to have happened.

#### 4.2.3.1 Tests for correlation

Theoretically, we say that  $A$  and  $B$  are uncorrelated if  $r = \text{corr}(A, B) = 0$ , otherwise  $A$  and  $B$  are correlated. However, in practice, in simple way, we use a relaxed threshold :

$$\frac{-1.96}{\sqrt{n}} \geq r \leq \frac{1.96}{\sqrt{n}}$$

where  $n$  is the number of observations for  $A$  and  $B$ . For example,  $A = [0.3, 0.6, -0.2]$ ,  $B = [0.2, -0.7, 0.1]$ , then  $n = 3$ .

A more sophisticated way is to use *t*-statistic, called test de Spearman. Let's :

$$t = r \sqrt{\frac{n-2}{1-r^2}}$$

Then *t* is distributed approximately as Student's distribution with  $n - 2$  degrees of freedom under the null hypothesis.

### 4.3 Box-Cox transformation

In statistics and econometrics, the Box-Cox transformation is often used to eventually convert a distribution into normal distribution.

$$B(x, \lambda) = \begin{cases} \frac{x^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln(x) & \text{if } \lambda = 0 \end{cases}$$

In figure 3.1, we see the advantage of Box-Cox transformation. A difficulty is to estimate  $\lambda$  to for applying Box-Cox transformation.

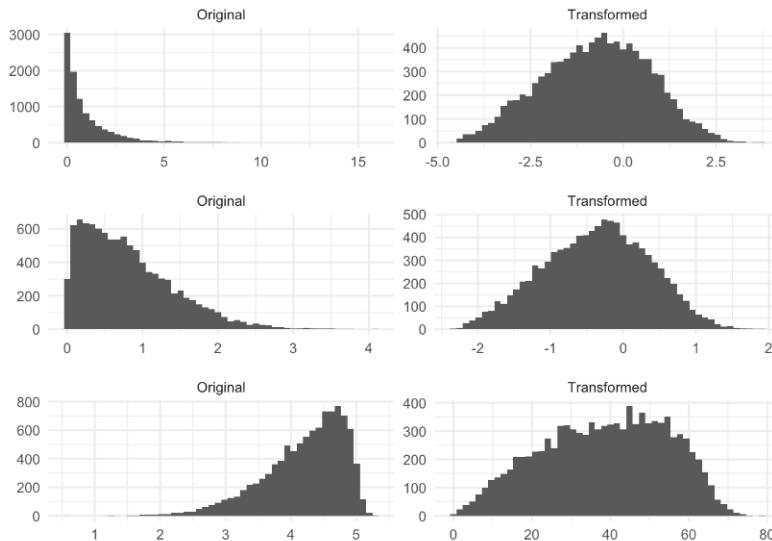


Figure 3.1: Original distributions and corresponding transformed distributions.

### 4.4 Prediction interval

In statistical, specifically predictive inference, a prediction interval is an estimate of an interval in which a **future observation** will fall, with a certain probability, given what has already been observed. Prediction intervals are often used in regression analysis.

#### 4.4.1 Unknown mean, known variance

- Given a normal distribution with unknown mean  $\mu$  but known variance (normalized to 1). Then the sample mean  $\bar{X}$  of the observations  $X_1, X_2, \dots, X_n$  has distribution  $\mathcal{N}(\mu, 1/n)$  (Central limit theorem).
- While the future observation or observation to test  $X_t$  has distribution  $\mathcal{N}(\mu, 1)$

- We remind that sum of two independent normal distribution follows  $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ . Then taking the difference of these ( $\bar{X}$  and  $X_t$ ) cancels the  $\mu$  and yields a normal distribution of variance  $1 + 1/n$ .

In short :

$$X_t - \bar{X} \sim \mathcal{N}(0, 1 + 1/n)$$

Then the prediction interval for  $X_t$  can be inferred from  $\mathcal{N}(\bar{X}, 1 + 1/n)$ .

Notice that this prediction distribution is more conservative (careful) than using the estimated mean  $\bar{X}$  and known variance 1, as this uses variance  $1 + \frac{1}{n}$ , hence yields wider intervals. This is necessary for the desired confidence interval property to hold.

#### 4.4.2 Known mean, unknown variance

- Given a normal distribution with known mean 0 but unknown variance  $\sigma^2$ .
- $\bar{X}$  is the mean of the observations  $X_1, X_2, \dots, X_n$  follows normal distribution  $\mathcal{N}(0, \sigma^2/n)$ .
- Future observation or observation to test  $X_t$  that follows  $\mathcal{N}(0, \sigma^2)$ .
- Since  $X_1, \dots, X_n$  are i.i.d., then  $\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \sim \chi_{n-1}^2$ .
- Then  $\frac{X_t/\sigma}{\sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2(n-1)}}} \sim T^{(n-1)}$  or  $\frac{X_t}{s_n} \sim T^{(n-1)}$ , where  $s_n^2 = \sum_{i=1}^n \frac{(X_i - \bar{X})^2}{n-1}$  (sample variance without bias).

Then the prediction interval for  $X_t$  can be inferred from  $s_n T^{(n-1)}$ .

#### 4.4.3 Unknown mean, unknown variance

From previous results :

- $(X_t - \bar{X}) \sim \mathcal{N}(0, \sigma^2(1 + \frac{1}{n}))$ .
- $\sum_{i=1}^n \left( \frac{X_i - \mu}{\sigma} - \frac{\bar{X} - \mu}{\sigma} \right)^2 \sim \chi_{n-1}^2$  or  $\frac{(n-1)s_n^2}{\sigma^2} \sim \chi_{n-1}^2$

Then :

$$\frac{\frac{X_t - \bar{X}_n}{\sigma \sqrt{1 + \frac{1}{n}}}}{\sqrt{\frac{(n-1)s_n^2}{\sigma^2} / (n-1)}} \sim T^{(n-1)}$$

Or :

$$\frac{X_t - \bar{X}_n}{s_n \sqrt{1 + \frac{1}{n}}} \sim T^{(n-1)}$$

Then the prediction interval for  $X_t$  can be inferred from  $\bar{X}_n + s_n \sqrt{1 + \frac{1}{n}} T^{(n-1)}$ .

## 4.5 *F*-distribution

Let  $S_1$  and  $S_2$  be two independent random variables with chi-square distributions with respective degrees of freedom  $d_1$  and  $d_2$ . Then

$$X = \frac{S_1/d_1}{S_2/d_2}$$

follows F-distribution (Fisher–Snedecor distribution), noted  $F(d_1, d_2)$ .

It can be shown to follow that the probability density function for  $X$  is given by :

$$f(x; d_1, d_2) = \frac{\sqrt{\frac{(d_1 x)^{d_1} d_2^{d_2}}{(d_1 x + d_2)^{d_1 + d_2}}}}{x B(\frac{d_1}{2}, \frac{d_2}{2})}$$

where B is the beta function.

## 4.6 Analysis of variance (ANOVA)

ANOVA, or Analysis of Variance, is a statistical method used to analyze the differences between group means. It is commonly used to determine if there are significant differences between the means of three or more groups.

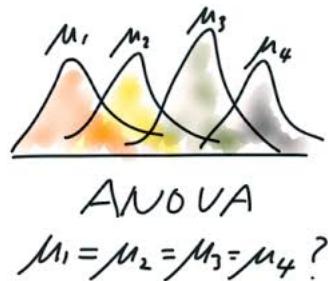


Figure 3.2: Illustration for ANOVA

### 4.6.1 Example

Let's take an example to illustrate the method. Imagine a breeder who wants to buy new cows for his milk production. He has three different breeds of cows and therefore wonders whether the breed is important for his choice. He has as information the breed of each of his animals (this is the discrete explanatory variable or variability factor, which can take 3 different values A,B,C) and their daily milk production (this is the continuous variable to be explained, which corresponds to the volume of milk in liters).

In this example, the null hypothesis is that all the cows produce the same daily quantity of milk (to the nearest random factor) whatever the breed. The

alternative hypothesis is that one of the breeds produces significantly more or less milk than the others.

A	20.1	19.8	21.3	20.7			
B	22.6	24.1	23.8	22.5	23.4	24.5	22.9
C	31.2	31.6	31.0	32.1	31.4		

$$\mu_A = 20.475, \mu_B = 23.4, \mu_C = 31.46, \mu = 25.1875$$

The sum of square for inter-classes:

$$SS_{inter} = 4 \times (20.475 - 25.1875)^2 + 7 \times (23.4 - 25.1875)^2 + 5 \times (31.46 - 25.1875)^2 = 307.918$$

Here the degree of freedom is 3-1, since we have the condition  $4 \times \mu_A + 7 \times \mu_B + 5 \times \mu_C = 16 \times \mu$ , see 3.4.1.2.

The sum of square for intra-class:

$$SS_{intra} = \sum_A (20.1 - 20.475)^2 + \sum_B (22.6 - 23.4)^2 + \sum_C (31.2 - 31.46)^2 = 5.6$$

Here the degree of freedom is 4+7+5-3, since we have the one condition for each case A, B, C.

Then :

$$f = \frac{\frac{SS_{inter}}{3-1}}{\frac{SS_{intra}}{4+7+5-3}} = 357.44$$

$f \sim F(d_1, d_2)$  (3.4.5). This  $f$  corresponds to  $p$ -value  $4.338 \times 10^{-12}$ , which means the null hypothesis “milk production does not vary from one breed to another” has probability  $4.338 \times 10^{-12}$ . It is extremely low ( $\ll 0.05$ ), we can therefore reject this hypothesis. We can conclude: the breeds do not produce the same quantity of milk.

## 4.7 Dickey–Fuller test

### 4.7.1 Unit root

Given the time serie signal (stochastic process)  $y_t$  :

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_p y_{t-p} + \varepsilon_t.$$

Then the carateristic polynomial of  $y_t$  is given by :

$$x^p - x^{p-1} a_1 - x^{p-2} a_2 - \cdots - a_p = 0$$

We say that this signal has an unit root if  $x = 1$  is a root of the carateristic polynomial. A unit root can have a multiplicity  $r > 1$ .

#### 4.7.1.1 Example

Given :

$$y_t = \alpha y_{t-1} + \varepsilon_t$$

There cases need to be considered:

- $\alpha > 1$  : explosive, can check visually by observing  $t$  tends to infini. Non-stationary
- $\alpha = 1$  : Non-stationary
- $\alpha < 1$  : Stationary from an instant  $t_0$ .
- $\alpha < 0$  possible but unrealistic in real scenario.

The most subtle case is to check if there is an unit root in the AR model.

#### 4.7.2 Order of integration

The order of integration of a time series is a summary statistic, which reports the minimum number of differences required to obtain a covariance-stationary series.

A time series is integrated of order  $d$ , denoted  $I(d)$ , if

$$(1 - L)^d X_t$$

is a stationary process, where  $L$  is the lag operator and  $1 - L$  is the difference operator, i.e.

$$(1 - L)X_t = X_t - X_{t-1} = \Delta X$$

In other words, a process is integrated to order  $d$  if taking  $d$  times difference operator yields a stationary process.

Price, rate, and yield data can be assumed as  $I(1)$  series, while returns (obtained by differencing the price) can be assumed as  $I(0)$  series.

#### 4.7.3 Dickey–Fuller test

In statistics, the Dickey–Fuller test tests the null hypothesis that a unit root (sec 3.4.7.1) is present in an autoregressive time series model. The alternative hypothesis is different depending on which version of the test is used, but is usually stationarity or trend-stationarity. The test is named after the statisticians David Dickey and Wayne Fuller, who developed it in 1979.

The first model is the Test for an unit root :

$$y_t = \rho y_{t-1} + u_t$$

To find  $\rho$ , the regression model can be written as :

$$\Delta y_t = (\rho - 1)y_{t-1} + u_t = \delta y_{t-1} + u_t$$

Test if  $\delta = 0$  or  $\rho = 1$  (unit root). We can use OLS (Ordinary Least Squares, 3.2.12.2), for each time instant  $t$  to get  $\delta$ , then check if mean of  $\delta$  is 0. Using OLS because it works with hypothesis that  $u_t$  is white noise.

To check if mean of  $\delta$  is 0, we use  $t$ -statistic (see 3.4.2.3). Since the test is done over the residual term  $(y_t - y_{t-1})$  rather than raw data, it is not possible to use standard  $t$ -distribution to provide critical values, since  $(y_t - y_{t-1})$  and  $(y_{t-1} - y_{t-2})$  are not independent. Therefore, this  $t$ -statistic has a specific distribution simply known as the Dickey–Fuller table.

The two other models in Dickey–Fuller test :

Model 2, Test for an unit root with drift :

$$\Delta y_t = \alpha + (\rho - 1)y_{t-1} + u_t = \alpha + \delta y_{t-1} + u_t$$

Model 3, Test for an unit root with drift and deterministic time trend :

$$\Delta y_t = \alpha + \beta t + (\rho - 1)y_{t-1} + u_t = \alpha + \beta t + \delta y_{t-1} + u_t$$

## 4.8 Augmented Dickey–Fuller test

Augmented Dickey–Fuller test (ADF) is an extended version of the Dickey–Fuller test for a larger and more complicated set of time series models :

$$\Delta y_t = \alpha + \beta t + \delta y_{t-1} + u_t + \delta_1 \Delta y_{t-1} + \dots + \delta_{p-1} \Delta y_{t-p+1}$$

where we have extended the Model 3 in Dickey–Fuller test by adding an  $p$  autoregressive model (6.1.5) :  $\delta_1 \Delta y_{t-1} + \dots + \delta_{p-1} \Delta y_{t-p+1}$ , with  $\Delta y_i = y_i - y_{i-1}$ .

Here the lag length  $p$  has to be determined **when applying** the test. One possible approach is to *test down* from high orders and examine the  $t$ -statistic on coefficients.

## 4.9 Kolmogorov–Smirnov test

The Kolmogorov-Smirnov (K-S) test is a statistical test used to determine whether a sample or dataset follows a specific distribution. Its primary role is to assess the goodness of fit between the observed data and a theoretical distribution.

### 4.9.1 Between dataset and known distribution

Suppose that :

- $F$  is the CDF of the known distribution.
- $F_n$  is the EDF (empirical distribution function), commonly also called an empirical Cumulative Distribution Function (eCDF) from the dataset. Given  $n$  is the number of observations of dataset, the EDF is calculated by :

$$F_n = \frac{\text{number of observations less or equal to } x}{n}$$

- The K-S test statistic:

$$D_n = \sup_x |F(x) - F_n(x)|$$

Then with given  $\alpha$  and  $n$ , we look in K-S test table to get the value  $K_{n,\alpha}$  to compare with  $D_n$ . If  $D_n < K_{n,\alpha}$ , we accept the hypothesis  $H_0$  that the dataset follows the known distribution, if not, we reject  $H_0$ .

**Example** Test if the following dataset follows the normality.

68, 72, 75, 77, 79, 81, 82, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 122, 124, 126, 128, 130

From the dataset and the requirement, we infer that the known distribution is normal distribution with  $\mu = 100.76$  and  $\sigma = 15.39$ . Then we calculate  $F_n$ ,  $D_n$  and ....

#### 4.9.2 Between two dataset

In this case the K-S statistic is :

$$D_{n,m} = \sup_x |F_{1,n}(x) - F_{2,m}(x)|$$

and the critic value  $K_{m,n,\alpha}$

$$K_{m,n,\alpha} = c(\alpha) \sqrt{\frac{n+m}{mn}}$$

where  $c(\alpha)$  is obtained from table :

$\alpha$	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001
$c(\alpha)$	1.073	1.138	1.224	1.358	1.48	1.628	1.731	1.949

#### 4.10 Jarque–Bera test

In statistics, the Jarque–Bera test is a goodness-of-fit test of whether sample data have the skewness and kurtosis matching a normal distribution.

The test statistic JB is defined as

$$JB = \frac{n}{6} \left( S^2 + \frac{1}{4}(K - 3)^2 \right)$$

where n is the number of observations (or degrees of freedom in general); S is the sample skewness, K is the sample kurtosis :

$$S = \frac{\hat{\mu}_3}{\hat{\sigma}^3} = \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^3}{\left( \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \right)^{3/2}}$$

$$K = \frac{\hat{\mu}_4}{\hat{\sigma}^4} = \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^4}{\left( \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \right)^2}$$

If the data comes from a normal distribution, the JB statistic asymptotically has a chi-squared ( $\chi_2$ ) distribution with **two** degrees of freedom, so the statistic can be used to test the hypothesis that the data are from a normal distribution.

# Chapter 4

## Stochastic process

### 1 Stochastic process

#### 1.1 Definition

A stochastic process is defined as a collection (family) of random variables defined on a common probability space  $(\Omega, \mathcal{F}, P)$ , indexed par  $T$ , output values in state space  $S$ , where  $S$  is the sample space of measurable space  $(S, \Sigma)$ . Reminding that the definition of random variable in sec 2.1.6, random variable is a measurable function that maps from a measurable space to an other measurable space. Here, it means from  $(\Omega, \mathcal{F})$  to  $(S, \Sigma)$ .

In other words, for a given probability space  $(\Omega, \mathcal{F}, P)$  and a measurable space  $(S, \Sigma)$ , a stochastic process is a collection of  $S$ -valued random variables, which can be written as

$$\{X(t) : t \in T\}$$

A stochastic process can also be written as a *function*  $X$  in term of  $t \in T$  and  $\omega \in \Omega : X(t, \omega)$ .

#### 1.2 Examples

##### 1.2.1 Bernoulli process

One of the simplest stochastic processes, which is a sequence of **independent and identically distributed** random variables  $(X_t)_{t=1,\dots,n}$  that follow Bernoulli distribution  $Bernoulli(p)$ .

Since random variables  $X_i$  are independent, we can take the probability space  $(\Omega, \mathcal{F}, P)$  where  $\Omega = \{H, T\}$ ,  $\mathcal{F} = \{\emptyset, H, T, \Omega\}$ ,  $P(\omega = H) = p$  and  $P(\omega = T) = 1 - p$ . Here we assume that the domain of  $X$  is measurable space  $(\Omega, \mathcal{F})$  which is related to outcomes of a coin. The state space  $S = \{0, 1\}$ .

### 1.2.2 Coin and dice

Let's suppose that we have a coin and a dice :

- Coin  $C(\omega) = \begin{cases} 0 & \text{if } \omega = H \\ 1 & \text{if } \omega = T \end{cases}$ , where  $\Omega_C = \{H, T\}$ ,  $P_C(H) = P_C(T) = 0.5$
- Dice  $D(\omega) = \begin{cases} 1 & \text{if } \omega = a \\ 2 & \text{if } \omega = b \\ 3 & \text{if } \omega = c \\ 4 & \text{if } \omega = d \\ 5 & \text{if } \omega = e \\ 6 & \text{if } \omega = f \end{cases}$ , where  $\Omega_D = \{a, b, c, d, e, f\}$ ,  $P_D(a) = \dots = P_D(f) = 1/6$ .

Let  $(C_t)_{t=1,\dots,n}$  be sequence of iid random variables (of coin) or stochastic process of coin and let  $(D_t)_{t=1,\dots,n}$  be sequence of iid random variables (of dice) or stochastic process of dice. We set up a stochastic process  $(X_t)_{t=1,\dots,n}$  :

$$X_t = C_t + D_t$$

The main thing here is that  $C_t$  and  $D_t$  **do not have the same probability space**. In order to make sense for  $X_t$ , it requires that, for a given  $\omega$ ,  $C_t(\omega) + D_t(\omega)$  can give a output value. We can deal with this by adapting probability space for  $C_t$  and  $D_t$ , with a new probability space :

- $\Omega = \Omega_C \times \Omega_D = \{(H, a), \dots, (T, f)\}$
- $\mathcal{F} = \sigma(\Omega)$ , which is thus a smallest  $\sigma$ -algebra on  $\Omega$  that contains all element in  $\Omega$ . In this case, this is also the biggest  $\sigma$ -algebra on  $\Omega$ .
- $P((H, a)) = P_C(H) \times P_D(a) \dots$

Note that, this is also the probability space for  $X_t$ . The new random variable  $C$  and  $D$  are :

- Coin  $C(\omega) = \begin{cases} 0 & \text{if } \omega = (H, .) \\ 1 & \text{if } \omega = (T, .) \end{cases}$
- Dice  $D(\omega) = \begin{cases} 1 & \text{if } \omega = (., a) \\ 2 & \text{if } \omega = (., b) \\ 3 & \text{if } \omega = (., c) \\ 4 & \text{if } \omega = (., d) \\ 5 & \text{if } \omega = (., e) \\ 6 & \text{if } \omega = (., f) \end{cases}$

where '.' means that it is not independent to value.

The state space of  $X_t$  is  $\{1, 2, 3, 4, 5, 6, 7\}$ .

## 1.3 Filtration

### 1.3.1 Starting example

Again, let's suppose that we have a coin and a dice as in sec 4.1.2.2 :

- Coin  $C(\omega) = \begin{cases} 0 & \text{if } \omega = H \\ 1 & \text{if } \omega = T \end{cases}$ , where  $\Omega_C = \{H, T\}$ ,  $P_C(H) = P_C(T) = 0.5$
- Dice  $D(\omega) = \begin{cases} 1 & \text{if } \omega = a \\ 2 & \text{if } \omega = b \\ 3 & \text{if } \omega = c \\ 4 & \text{if } \omega = d \\ 5 & \text{if } \omega = e \\ 6 & \text{if } \omega = f \end{cases}$ , where  $\Omega_D = \{a, b, c, d, e, f\}$ ,  $P_D(a) = \dots = P_D(f) = 1/6$ .

Now, we want to set up a stochastic process  $(X_t)_{t=1,2}$  (only two time index 1 and 2) :

$$\begin{cases} X_1 = C \\ X_2 = X_1 + D \end{cases}$$

This example differs from one in sec 4.1.2.2 as we must toss the coin before throwing the dice, where in sec 4.1.2.2, the coin and the dice are executed at the same time. In other words,  $X_2$  depends on  $X_1$  and in sec 4.1.2.2,  $X_t$  are independent.

From the definition of stochastic (sec 4.1.1),  $X_1$  and  $X_2$  must be defined on the common probability space. Then the probability space  $(\Omega, \mathcal{F}, P)$  for  $X_1$  and  $X_2$  is the same as example in sec 4.1.2.2.

Let  $\omega = (\omega_C, \omega_D)$  be an outcome. The keypoint here is that **at the moment we toss the coin**, we know that the first component of outcome  $\omega_C$  can be only  $H$  or  $T$ . For the second component  $\omega_D$ , we still have no information about it. In other words, for the coin, we can split (event)  $\{H, T\}$  into  $H$  and  $T$  but still not for the dice. Hence for  $X_1$ , its event space can only be :

$$\begin{aligned} \mathcal{F}_1 = & \{\emptyset, \{(H, a), (H, b), (H, c), (H, d), (H, e), (H, f)\}, \\ & \{(T, a), (T, b), (T, c), (T, d), (T, e), (T, f)\}, \Omega\} \end{aligned}$$

Note that,  $\mathcal{F}_1$  is a  $\sigma$ -algebra and is truly a subset of  $\mathcal{F}$ .

**At the moment we throw the dice**, now we can split  $\{a, b, c, d, e, f\}$  into each each single element. Hence  $\mathcal{F}_2$  is just  $\mathcal{F}$  that is equal to  $\sigma(\Omega)$ .

We call  $(\mathcal{F}_t)_{t=1,2}$  is a filtration. In some litterature, we can find that the probability space of a stochastic process is expressed by using filtration instead of event space, such as  $(\Omega, \mathcal{F}_n, P)$ .

### 1.3.2 Definition

A family  $(\mathcal{F}_t)_{0 \leq t \leq T}$  of  $\sigma$ -algebra on  $\Omega$  is called a filtration if  $\mathcal{F}_s \subset \mathcal{F}_t$  whenever  $s \leq t$ . For brevity, we denote  $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ .

The name “filtration” is related to the fact that the greater  $t$ , the more refined event space  $\mathcal{F}_t$ .

### 1.3.3 Construction of filtration

Let  $(X_n)_{n \in \mathbb{N}}$  be a stochastic process on the probability space  $(\Omega, \mathcal{A}, P)$ . Then  $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$  is a filtration, where :

$$F_n := \sigma(X_k \mid k \leq n) = \sigma(X_1, X_2, \dots, X_n),$$

which denotes the  $\sigma$ -algebra generated by the **sequence** of random variables  $X_1, X_2, \dots, X_n$ . More precisely,

$$\sigma(X_1, X_2, \dots, X_n) = \bigvee_{i=1}^n \sigma(X_i) = \sigma\left(\bigcup_{i=1}^n \sigma(X_i)\right),$$

where  $a \vee b$  ( $\vee$  pronounced vee) is an operator outputs the smallest  $\sigma$ -algebra that contains  $a$  and  $b$ . In this document  $a \vee b = \sigma(a \cup b)$ .

The function  $\sigma(X_i)$  is the  $\sigma$ -algebra generated by random variable  $X_i$  (sec 2.1.6.4). Generally,  $\bigcup_{i=1}^n \sigma(X_i)$  is not a  $\sigma$ -algebra. We will see a method to get  $\sigma(\bigcup_{i=1}^n \sigma(X_i))$ , let's define  $\Pi$  ( $\pi$ -system in sec 2.1.2), where :

$$\Pi = \left\{ \bigcap_{i=1}^k A_i : A_i \in \sigma(X_{j_i}); j_i \in \{1, \dots, n\}; \forall 1 \leq k \leq n \right\},$$

this notation for  $\Pi$  is may quite complicated, let's explain it (from right to left):

1. Take any number  $k$  between 1 and  $n$ .
2. Take any  $k$   $\sigma$ -algebra  $\sigma(X_{j_1}), \dots, \sigma(X_{j_k})$  from  $\sigma(X_1), \dots, \sigma(X_n)$ .
3. From each chosen  $\sigma$ -algebra  $\sigma(X_{j_i})$ , select  $A_i$ .
4. An element of  $\Pi$  is  $\bigcap_{i=1}^k A_i$ .

Then we have that:

$$\sigma\left(\bigcup_{i=1}^n \sigma(X_i)\right) = \sigma(\Pi)$$

#### Proof

*On one hand*, take  $k = 1$ , then  $\forall A \in \sigma(X_i)$ , we have  $A \in \Pi$ . This implies  $\sigma(X_i) \subset \Pi, \forall i$ . The latter implies:

$$\bigcup_{i=1}^n \sigma(X_i) \subset \Pi$$

This implies:

$$\sigma \left( \bigcup_{i=1}^n \sigma(X_i) \right) \subseteq \sigma(\Pi)$$

*On the other hand*, given  $k$   $\sigma$ -algebra  $\sigma(X_{j_1}), \dots, \sigma(X_{j_k})$ . Let's consider an element in  $\Pi : \bigcap_{i=1}^k A_i$ , where  $A_i \in \sigma(X_{j_i})$ . We show that this is also an element of  $\sigma(\bigcup_{i=1}^n \sigma(X_i))$ .

Before going into details, let's see some common mistakes that we usually fail into. Given that  $A_1 \in \Sigma_1$  and  $A_2 \in \Sigma_2$ , then :

- $(A_1 \cap A_2) \subset (\Sigma_1 \cup \Sigma_2) : \text{This is wrong}$
- $(A_1 \cup A_2) \subset (\Sigma_1 \cup \Sigma_2) : \text{This is wrong}$

For example : Given  $\Sigma_1 = \{\{a, c\}, \{b, d\}\}$  and  $\Sigma_2 = \{\{a, d\}, \{b, c\}\}$ . Then,  $\Sigma_1 \cup \Sigma_2 = \{\{a, c\}, \{b, d\}, \{a, d\}, \{b, c\}\}$ . In addition,  $\Sigma_1 \cap \Sigma_2 = \emptyset$ . Let's take  $A_1 = \{a, c\}$  and  $A_2 = \{b, c\}$ , then  $A_1 \cap A_2 = \{c\} \not\subset \Sigma_1 \cup \Sigma_2$ . Also,  $A_1 \cup A_2 = \{a, b, c\} \not\subset \Sigma_1 \cup \Sigma_2$ .

Let's return to the proof. With the properties of  $\sigma$ -algebra in sec 2.1.1 :

$$\begin{aligned} A_i &\in \sigma(X_{j_i}) \\ A_i^C &\in \sigma(X_{j_i}) \\ A_i^C &\in \bigcup_{l=1}^n \sigma(X_l) \\ A_i^C &\in \sigma \left( \bigcup_{l=1}^n \sigma(X_l) \right) \\ \bigcup_{i=1}^k A_i^C &\in \sigma \left( \bigcup_{l=1}^n \sigma(X_l) \right) \\ \left( \bigcup_{i=1}^k A_i^C \right)^C &\in \sigma \left( \bigcup_{l=1}^n \sigma(X_l) \right) \\ \bigcap_{i=1}^k A_i &\in \sigma \left( \bigcup_{l=1}^n \sigma(X_l) \right) \end{aligned}$$

This means :

$$\Pi \subseteq \sigma \left( \bigcup_{i=1}^n \sigma(X_i) \right)$$

This implies:

$$\sigma(\Pi) \subset \sigma \left( \sigma \left( \bigcup_{i=1}^n \sigma(X_i) \right) \right) = \sigma \left( \bigcup_{i=1}^n \sigma(X_i) \right)$$

Finally,

$$\sigma\left(\bigcup_{i=1}^n \sigma(X_i)\right) = \sigma(\Pi)$$

#### 1.3.4 Example : coin, cards, dice

This is an extension of example in sec 4.1.3.1

Let consider the case that we first toss a coin, then we draw a card from a hand  $\{J, Q, K\}$ , finally we throw a dice. Let

- $X_1$  be the outcome of the tossed coin
- $X_2$  be the outcome of the drawed card
- $X_3$  be the outcome of the threw dice

For particular sample space :

- For the coin  $\Omega_C = \{H, T\}$
- For the hand  $\Omega_H = \{J, Q, K\}$
- For the dice  $\Omega_D = \{1, 2, 3, 4, 5, 6\}$

The sample space  $\Omega = \Omega_C \times \Omega_H \times \Omega_D = \{(H, J, 1), \dots, (H, J, 6), \dots, (H, K, 6), \dots, (T, K, 6)\}$  which has 36 elements. **Note that the sample space  $\Omega$  must be the same for  $X_1, X_2, X_3$ .** With  $\omega \in \Omega$ , where  $\omega = (x, y, z)$ , we can define :

- $X_1(\omega) = X_1(x, y, z) = x, x \in \{H, T\}$
- $X_2(\omega) = X_2(x, y, z) = y, y \in \{J, Q, K\}$
- $X_3(\omega) = X_3(x, y, z) = z, z \in \{1, 2, 3, 4, 5, 6\}$

Here we use a kind of identify mapping, but it's not mandatory, we can use other mappings. Then the state space  $S = \{H, T, J, Q, K, 1, 2, 3, 4, 5, 6\} = \Omega_C \cup \Omega_H \cup \Omega_D$ .

In the next step, we discover  $\sigma(X_1), \sigma(X_2), \sigma(X_3)$  :

$$\begin{aligned}\sigma(X_1) &= \{A \times \Omega_H \times \Omega_D \mid A \in \mathcal{P}(\Omega_C)\} \\ &= \{\emptyset \times \{J, Q, K\} \times \{1, 2, 3, 4, 5, 6\}, \\ &\quad \{H\} \times \{J, Q, K\} \times \{1, 2, 3, 4, 5, 6\}, \\ &\quad \{T\} \times \{J, Q, K\} \times \{1, 2, 3, 4, 5, 6\}, \\ &\quad \{H, T\} \times \{J, Q, K\} \times \{1, 2, 3, 4, 5, 6\}\}\end{aligned}$$

which has  $2^2$  elements.  $\mathcal{P}$  denotes the powerset. More precisely:

- $\emptyset \times \{J, Q, K\} \times \{1, 2, 3, 4, 5, 6\} = \emptyset$ . The Cartesian product of the empty set is also the empty set.

- $\{H, T\} \times \{J, Q, K\} \times \{1, 2, 3, 4, 5, 6\} = \Omega$
- $\{H\} \times \{J, Q, K\} \times \{1, 2, 3, 4, 5, 6\}$  is the complement of  $\{T\} \times \{J, Q, K\} \times \{1, 2, 3, 4, 5, 6\}$  in  $\Omega$  and vice-versa.

$$\begin{aligned}
\sigma(X_2) &= \{\Omega_C \times A \times \Omega_D | A \in \mathcal{P}(\Omega_H)\} \\
&= \{\{H, T\} \times \emptyset \times \{1, 2, 3, 4, 5, 6\}, \\
&\quad \{H, T\} \times \{J\} \times \{1, 2, 3, 4, 5, 6\}, \\
&\quad \dots \\
&\quad \{H, T\} \times \{J, Q\} \times \{1, 2, 3, 4, 5, 6\}, \\
&\quad \dots \\
&\quad \{H, T\} \times \{J, Q, K\} \times \{1, 2, 3, 4, 5, 6\}\}
\end{aligned}$$

which has  $2^3$  elements.

$$\begin{aligned}
\sigma(X_3) &= \{\Omega_C \times \Omega_H \times A | A \in \mathcal{P}(\Omega_D)\} \\
&= \{\{H, T\} \times \{J, Q, K\} \times \emptyset, \\
&\quad \{H, T\} \times \{J, Q, K\} \times \{1\}, \\
&\quad \dots \\
&\quad \{H, T\} \times \{J, Q, K\} \times \{1, 2\}, \\
&\quad \dots \\
&\quad \{H, T\} \times \{J, Q, K\} \times \{1, 2, 3\}, \\
&\quad \dots \\
&\quad \{H, T\} \times \{J, Q, K\} \times \{1, 2, 3, 4, 5, 6\}\}
\end{aligned}$$

which has  $2^6$  elements.

**Before throwing the dice.**

$$F_0 = \{\emptyset, \Omega\}$$

which has **2** elements (events). Here we only have two possibilities, the event  $\Omega$  definitely occurs ( $P(\Omega) = 1$ ) or  $\emptyset$  definitely does not occur ( $P(\emptyset) = 0$ ). For any other event, we can not measure it.

**After tossing the coin and before drawing a card**

$$F_1 = \sigma(X_1)$$

We see  $F_0 \subset F_1$ .

**After drawing a card and before throwing the dice**

$$F_2 = \sigma(X_1, X_2)$$

$$\begin{aligned}
&= \sigma\{\sigma(X_1) \cup \sigma(X_2)\} \\
&= \sigma\{\{A \times \Omega_H \times \Omega_D | A \in \mathcal{P}(\Omega_C)\} \cup \{\Omega_C \times B \times \Omega_D | B \in \mathcal{P}(\Omega_H)\}\} \\
&= \sigma\{\{A \times \Omega_H | A \in \mathcal{P}(\Omega_C)\} \cup \{\Omega_C \times B | B \in \mathcal{P}(\Omega_H)\}\} \times \Omega_D
\end{aligned}$$

In the first method, let's extract  $U = \{A \times \Omega_H | A \in \mathcal{P}(\Omega_C)\} \cup \{\Omega_C \times B | B \in \mathcal{P}(\Omega_H)\} =$

$$\begin{aligned}
&\{\emptyset, \{(H, J), (H, Q), (H, K)\}, \{(T, J), (T, Q), (T, K)\}, \Omega_C \times \Omega_H\} \\
&\cup \{\emptyset, \{(H, J), (T, J)\}, \{(H, Q), (T, Q)\}, \{(H, K), (T, K)\}, \{(H, J), (T, J), (H, Q), (T, Q)\}, \\
&\quad \{(H, J), (T, J), (H, K), (T, K)\}, \{(H, Q), (T, Q), (H, K), (T, K)\}, \Omega_C \times \Omega_H\} \\
&= \{\emptyset, \{(H, J), (H, Q), (H, K)\}, \{(T, J), (T, Q), (T, K)\}, \\
&\quad \{(H, J), (T, J)\}, \{(H, Q), (T, Q)\}, \{(H, K), (T, K)\}, \{(H, J), (T, J), (H, Q), (T, Q)\}, \\
&\quad \{(H, J), (T, J), (H, K), (T, K)\}, \{(H, Q), (T, Q), (H, K), (T, K)\}, \Omega_C \times \Omega_H\}
\end{aligned}$$

Now we need to find  $\sigma(U)$ , where  $U$  is collection of subsets. It also means that find the smallest  $\sigma$ -algebra that contains all events in  $U$  as in general,  $U$  itself is not a  $\sigma$ -algebra. Then, to get  $\sigma(U)$ , for each  $A, B \in U$ , we add  $A \cup B$  and  $(A \cup B)^C$  to  $U$  until having (smallest)  $\sigma$ -algebra, after that we can remove the symbol  $\sigma$ .

$$\begin{aligned}
F_2 &= \sigma(X_1, X_2) \\
&= \sigma\left(\{\emptyset, \{(H, J), (H, Q), (H, K)\}, \{(T, J), (T, Q), (T, K)\}, \right. \\
&\quad \{(H, J), (T, J)\}, \{(H, Q), (T, Q)\}, \{(H, K), (T, K)\}, \{(H, J), (T, J), (H, Q), (T, Q)\}, \\
&\quad \{(H, J), (T, J), (H, K), (T, K)\}, \{(H, Q), (T, Q), (H, K), (T, K)\}, \Omega_C \times \Omega_H, \quad (\text{new events start from here}) \\
&\quad \{(H, J), (H, Q), (H, K), (T, J)\}, \{(T, Q), (T, K)\}, \quad (\text{tuple (4,2)}) \\
&\quad \{(H, J), (T, J), (T, Q), (T, K)\}, \{(H, Q), (H, K)\}, \\
&\quad \{(H, J), (H, Q), (H, K), (T, Q)\}, \{(T, J), (T, K)\}, \\
&\quad \{(H, Q), (T, J), (T, Q), (T, K)\}, \{(H, J), (H, K)\} \\
&\quad \{(H, J), (H, Q), (H, K), (T, K)\}, \{(T, J), (T, Q)\}, \\
&\quad \{(H, K), (T, J), (T, Q), (T, K)\}, \{(H, J), (H, Q)\}, \\
&\quad \{(H, J), (H, Q), (H, K), (T, J), (T, Q)\}, \{(T, K)\} \quad (\text{tuple (5,1)}) \\
&\quad \{(H, J), (H, Q), (H, K), (T, J), (T, K)\}, \{(T, Q)\} \\
&\quad \{(H, J), (H, Q), (H, K), (T, Q), (T, K)\}, \{(T, J)\} \\
&\quad \{(H, Q), (H, K), (T, J), (T, Q), (T, K)\}, \{(H, K)\} \\
&\quad \{(H, J), (H, K), (T, J), (T, Q), (T, K)\}, \{(H, Q)\} \\
&\quad \{(H, J), (H, Q), (T, J), (T, Q), (T, K)\}, \{(H, J)\}\Big) \times \Omega_D
\end{aligned}$$

Here we see that there is the elementary set  $\{(T, K)\}, \{(T, Q)\}, \{(T, J)\}, \{(H, K)\}, \{(H, Q)\}, \{(H, J)\}$  is a subset of above set. Then  $\sigma(U)$  is just  $\sigma(\Omega_C \times \Omega_H)$  (which has  $2^6$  elements)

In the second method, we use  $\sigma(\Pi)$  (in sec 4.1.3.3), where:

$$\Pi = \{A \cap B | A \in \sigma(X_1), B \in \sigma(X_2)\}$$

Also, we see that there are all elementary set  $\{(T, K)\}, \{(T, Q)\}, \{(T, J)\}, \{(H, K)\}, \{(H, Q)\}, \{(H, J)\}$  in  $\Pi$ , then  $\sigma(\Pi)$  must be also  $\sigma(\Omega_C \times \Omega_H)$ . The latter is :

$$\begin{aligned} F_2 &= \sigma(\Omega_C \times \Omega_H) \\ &= \{A \times \Omega_D | A \in \mathcal{P}(\Omega_C \times \Omega_H)\} \\ &= \{\emptyset \times \{1, 2, 3, 4, 5, 6\}, \\ &\quad \{(H, J)\} \times \{1, 2, 3, 4, 5, 6\}, \\ &\quad \dots \\ &\quad \{(H, J), (H, Q)\} \times \{1, 2, 3, 4, 5, 6\}, \\ &\quad \dots \\ &\quad \{(H, J), (H, Q), \dots, (T, K)\} \times \{1, 2, 3, 4, 5, 6\}\} \end{aligned}$$

We see that  $F_1 \subset F_2$ .

**After throwing the dice**

$$F_3 = \sigma(X_1, X_2, X_3) = \sigma(\sigma(X_1) \cup \sigma(X_2) \cup \sigma(X_3))$$

which is the same as

$$F_3 = \sigma(\omega) = \mathcal{P}(\Omega) = \mathcal{P}(\Omega_C \times \Omega_H \times \Omega_D)$$

which has  $2^{36} = 68719476736$  elements (events). We see that  $F_2 \subset F_3$ .

### 1.3.5 Random walk

In case of 1-dimensional, we start at 0 at the beginning. At each time  $i$ , we toss a fair coin, if head ( $H$ ), we go left one unit and if tail ( $T$ ), we go right one unit. Let's note  $(X_i)_{i=1, \dots, n}$  this stochastic process and we will discover its probability space and state space. By setting  $Z_i$  is random variable for tossing coin at time  $i$ , then :

$$X_i = Z_1 + \dots + Z_i$$

Note that  $i$  varies from 0 to fixed value  $n$ .

- Probability space  $\Omega = \{H, T\} \times \{H, T\} \times \dots \times \{H, T\} = \{H, T\}^n$
- Event space  $\mathcal{F} = \mathcal{P}(\Omega)$  ( $\mathcal{P}$  is power set).
- Filtration  $F_i = \mathcal{P}(\{H, T\}^i) \times \{H, T\}^{(n-i)}$
- Probability measure, given  $\omega \in \Omega$ ,  $P(\omega) = \frac{1}{2^n}$

State space is  $\{-n, -(n - 1), \dots, -1, 0, 1, \dots, n - 1, n\}$

A random walk is a particular case of Markov process 4.6.

## 1.4 Adapted stochastic process

### 1.4.1 Stochastic process adapted to a filtration

A stochastic process  $(X_t)_{\{0 \leq t \leq T\}}$  is said to be  $\mathbb{F}$  – *adapted* if the random variable  $X_t$  is  $\mathcal{F}_t$ -measurable (see 2.1.5.3), with  $0 \leq t \leq T$ .

### 1.4.2 Generated Filtration

Generated filtration or natural filtration associated to a stochastic process is a filtration which records the “past behaviour” of this stochastic process at each time.

More formally, let  $\Omega$  be the sample space and let  $(S, T)$  be a measurable space for the codomain of stochastic process  $W$ . The generated filtration associated to stochastic process  $W$  up to instant  $n$  is  $(\mathcal{F}_i^W)_{0 \leq i \leq n}$ , where :

$$\mathcal{F}_i^W = \sigma(W_j^{-1}(A), \forall j \leq i, \forall A \in T)$$

It also means the smallest  $\sigma$ -algebra on domain  $\Omega$  that contains all pre-images of  $T$ . Sometime, we use the notation :

$$\mathcal{F}_i^W = \sigma(W_0, \dots, W_i)$$

### 1.4.3 Stochastic process adapted to a generated filtration

Let  $\mathcal{F}_t^W$  the generated filtration associated to stochastic process  $W_t$  at time  $t$ . Then a stochastic process  $X_t$  is adapted to  $\mathcal{F}_t^W$  if  $X_l$  is  $\mathcal{F}_l^W$ -measurable ( $\sigma(X_l) \subset \mathcal{F}_l^W$ ), for  $0 \leq l \leq t$ . This point is crucial for Ito calculus or stochastic integrals 4.16.

## 1.5 Continuity of stochastic process

As in the convergence of random variables (in sec 2.4), we have also several types of continuity for stochastic processes.

### 1.5.1 Stochastically continuous

A continuous time stochastic process  $(X_t)_{\{t \geq 0\}}$  is said to be stochastically continuous if  $\forall t \geq 0, \forall \varepsilon > 0$  :

$$P(\lim_{s \rightarrow t} |X_t - X_s| > \varepsilon) = 0$$

Then stochastically continuous is also call continuous **in probability**.

### 1.5.2 Almost surely

Let  $X(t)$  be a random process. We say that  $X(t)$  is **almost surely continuous** or *continuous with probability one* at time  $t$  if :

$$\mathbf{P} \left( \left\{ \omega \in \Omega \mid \lim_{s \rightarrow t} |X_s(\omega) - X_t(\omega)| = 0 \right\} \right) = 1$$

We can say there is a **finite** number of  $\omega$  that  $X$  is not continuous at  $t$ .

### 1.5.3 Mean-square

Let  $X(t)$  be a random process. We say that  $X(t)$  is **mean-square continuous** at time  $t$  if

$$\lim_{\delta \rightarrow 0} \mathbb{E} [|X(t + \delta) - X(t)|^2] = 0.$$

It is important to note that the mean-square continuity does not imply that every possible realization of  $X(t)$  is a continuous function. It roughly means that  $\mathbb{E}[X(t + \delta) - X(t)]$  is small.

### 1.5.4 Continuous stochastic process

A continuous time stochastic process  $(X_t)_{\{t \geq 0\}}$  is said to be *continuous* or to have *continuous sample paths* if for every given  $\omega \in \mathcal{F}_\infty$ ,  $X(t, \omega)$  is continuous with respect to  $t$ .

The we have the same thing for *right-continuous* and *left-continuous*.

## 1.6 The sameness between two stochastic process

Given a probability space  $(\Omega, \mathcal{F}, P)$ . Let  $X_\alpha$  and  $Y_\alpha$  ( $\alpha \in I$ ) be two stochastic processes in this space. We use notation  $\alpha$  instead of  $i$  or  $t$ , in order to not be biased at first glance to discret or continuous case.

### 1.6.1 Modification

We say that  $Y$  is a *modification*, a *version* of  $X$  or is  $M$ -equivalent to  $X$  if for every  $\alpha \in I$ , we have

$$P(X_\alpha = Y_\alpha) = P(\{\omega \in \Omega : X_\alpha(\omega) = Y_\alpha(\omega)\}) = 1$$

### 1.6.2 Indistinguishable

We say that  $X$  and  $Y$  are indistinguishable or  $D$ -equivalent if

$$P(X_\alpha = Y_\alpha, \forall \alpha \in I) = P(\{\omega \in \Omega : X_\alpha(\omega) = Y_\alpha(\omega), \forall \alpha \in I\}) = 1$$

### 1.6.3 Remark

If  $X_\alpha$  and  $Y_\alpha$  are  $D$ -equivalent, then they are  $M$ -equivalent. This is trivial by using the above definitions. However, the converse statement is not true. Let's discover by the following classical example.

Suppose that  $\Omega = [0, 1]$ ,  $\mathcal{F} = \mathcal{B}([0, 1])$ , which means the Borel sets (sec 2.1.1.4) of  $[0, 1]$ ,  $P$  is the uniform probability (i.e., Lebesgue measure, sec 2.1.7.3.3) on  $[0, 1]$ .  $I$  here is also  $[0, 1]$ , we define two continuous stochastic processes  $X_t$  and  $Y_t$  by :

$$X(t, \omega) = 0$$

$$Y(t, \omega) = \mathbb{1}_{t=\omega}$$

These two processes are  $M$ -equivalent but not  $D$ -equivalent, in interval  $I$

**Proof**

First, these two processes are  $M$ -equivalent because given  $t \in [0, 1]$  then :

$$\begin{cases} X(t, \omega) = Y(t, \omega) = 0 & \text{if } \omega \neq t \\ X(t, \omega) \neq Y(t, \omega) & \text{if } \omega = t \end{cases}$$

Therefore on interval  $[0, 1]$  there is only one point  $\omega = t$  that  $X(t, \omega) \neq Y(t, \omega)$ . This means  $P(X(t, \cdot) = Y(t, \cdot)) = 1, \forall t \in [0, 1]$ .

Second, we show that in general, these two processes are not  $D$ -equivalent on any interval  $I = [a, b]$ , where  $0 \leq a < b \leq 1$ . Thus

$$P(X(t, \cdot) \neq Y(t, \cdot), \forall t \in I) = P(\{\omega : \omega \in [a, b]\}) = b - a > 0$$

This means

$$P(X(t, \cdot) = Y(t, \cdot), \forall t \in I) < 1$$

### 1.6.4 Lemma

If  $X_\alpha$  and  $Y_\alpha$  are  $M$ -equivalent and  $I$  is countable, then  $X$  and  $Y$  are  $D$ -equivalent.

**Proof**

By using the *countable additivity* property of probability measure.

### 1.6.5 Lemma

Let  $X_t$  and  $Y_t$  be right-continuous processes (resp. left-continuous processes) and they are  $M$ -equivalent, at each time  $t \geq 0$ . Then, they are  $D$ -equivalent.

**Proof**

First, by lemma 4.1.6.4,  $X_t$  and  $Y_t$  are  $D$ -equivalent with  $t \in I = Q$ , the set of rationals, which is countable.

Then using a partition on  $R$  such as  $\{0, \frac{1}{n}, \dots, \frac{k}{n}, \dots, \frac{\infty}{n}\}$ . We see that all elements are rational. Suppose that  $X_t$  and  $Y_t$  are right-continuous, then they can be approximated by :

$$X(t, \omega) = \lim_{n \rightarrow \infty} X^n(t, \omega) = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} X\left(\frac{k}{n}, \omega\right) \mathbb{1}_{[\frac{k}{n}, \frac{k+1}{n})}(t)$$

$$Y(t, \omega) = \lim_{n \rightarrow \infty} Y^n(t, \omega) = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} Y\left(\frac{k}{n}, \omega\right) \mathbb{1}_{[\frac{k}{n}, \frac{k+1}{n})}(t)$$

Since

- $\frac{k+1}{n} \rightarrow \frac{k}{n}$  as  $n \rightarrow \infty$
- each  $t > 0$  can be squeezed by a couple  $\frac{k}{n}$  and  $\frac{k+1}{n}$
- $X_t$  and  $Y_t$  are right-continuous

then  $X^n(t, \omega) = Y^n(t, \omega)$  as  $n \rightarrow \infty$  or  $X(t, \omega) = Y(t, \omega)$  for all  $\omega \in \Omega$  or they are  $D$ -equivalent.

## 1.7 Jointly measurable process

Given probability space  $(\Omega, \mathcal{F}, P)$ . The stochastic process  $X(\alpha, \omega)$  where  $\alpha \in I$ , defined by  $X: I \times \Omega \rightarrow \mathbb{R}$ . Then  $X_\alpha$  is said to be **jointly measurable** if for each  $\omega \in \Omega$ ,  $X(., \omega): I \rightarrow \mathbb{R}$  is a measurable function (sec 2.1.5).

In other words, a stochastic process is measurable if the function

$$X: (I \times \Omega, B(I) \otimes \mathcal{F}) \rightarrow (\mathbb{R}, B(\mathbb{R}))$$

is measurable.  $\otimes$  means the Kronecker product. Thus, we call *jointly measurable* to emphasize that  $X$  is with two dependants  $\alpha$  and  $\omega$ .

### 1.7.1 Progressively measurable process

This is the case where the event space  $\mathcal{F}$  is a filtration  $F_t, t \geq 0$  (sec 4.1.3). The process  $X_t$  is said to be progressively measurable if at each time  $t$  the function

$$X(s, \omega): [0, t] \times \Omega \rightarrow \mathbb{R}$$

is  $B([0, t]) \otimes F_t$ -measurable. This implies that  $X$  is  $F_t$ -adapted.

### 1.7.2 Example of non measurable stochastic process

Suppose that for each  $t \in [0, 1]$ , the random variable  $X_t$  is uniformly distributed on  $[-1, 1]$ . Suppose further that the collection of random variables  $(X_t)_{\{t \geq 0\}}$  is pairwise independent. Then  $(X_t)_{\{t \geq 0\}}$  is not jointly measurable process.

#### Proof

The proof is conducted by contradiction. Formally,  $X$  is represented by

$$X: [0, 1] \times \Omega \rightarrow [-1, 1]$$

Here we define two measure spaces :

- $(\Omega, \mathcal{F}, P)$ , we can take, e.g,  $\Omega = [0, 2], \mathcal{F} = B([0, 2])$  and probability measure  $P$  corresponds an uniform distribution on  $[0, 2]$ . In this case  $X(., \omega) = \omega - 1$ .
- $([0, t], B([0, t]), L)$  is thus a measure space with Lebesgue measure  $L$  (sec 2.1.7.3.3).

Suppose that  $X$  is a jointly measurable process, which mean  $X$  is  $B([0, 1]) \otimes \mathcal{F}$ -measurable. Then  $X$  is also  $B([0, t]) \otimes \mathcal{F}$ -measurable since  $B([0, t]) \subseteq B([0, 1]), \forall t \in [0, 1]$ . By Fubini–Tonelli theorem (sec 2.2.14), and note that  $X(., .) \in [-1, 1]$  then

$$\int_{[0, t]} \int_{\Omega} |X(s, \omega)| dP(\omega) dL(s)$$

is finite, then

$$\begin{aligned} Y_t(\omega) &= \int_{[0, t]} X_s(\omega) dL(s) \\ &= \int_{[0, t]} X_s(\omega) ds \quad (\text{Lebesgue measurable function 2.1.5.3.3}) \end{aligned}$$

is  $\mathcal{F}$ -measurable or  $Y_t, \forall t \in [0, 1]$  is also a random variable on  $(\Omega, \mathcal{F}, P)$ . In addition,  $Y(t, \omega)$  is continuous with respect to  $t$  for a given  $\omega$ .

For a given  $t$ , we have :

$$\begin{aligned} \mathbb{E}[Y_t^2] &= \mathbb{E}\left[\left(\int_0^t X(s, \omega) ds\right)^2\right] \\ &= \mathbb{E}\left[\int_0^t X(s, \omega) ds \int_0^t X(r, \omega) dr\right] \\ &= \mathbb{E}\left[\int_0^t \int_0^t X(s, \omega) X(r, \omega) ds dr\right] \\ &= \int_0^t \int_0^t \mathbb{E}[X(s, \omega) X(r, \omega)] ds dr \\ &= \int_0^t \int_0^t \mathbb{E}[X(s, \omega) X(r = s, \omega)] ds dr + \int_0^t \int_0^t \mathbb{E}[X(s, \omega)] \mathbb{E}[X(r \neq s, \omega)] ds dr \\ &= \int_0^t \int_0^t \mathbb{E}[X_s^2] \mathbb{1}_{r=s} ds dr \quad (\text{since i.i.d of } X_s) \\ &= \mathbb{E}[X_0^2] \int_0^t \int_0^t \mathbb{1}_{r=s} ds dr \quad (\text{since } \mathbb{E}[X_s^2] \text{ is a constant}) \\ &= 0 \quad (\text{since area of diagonal line}) \end{aligned}$$

From  $\mathbb{E}[Y_t^2] = 0$ , we infer that  $P(Y_t = 0) = 1$  or process  $Y_t$  and process constant 0 are  $M$ -equivalent. In addition,  $Y_t$  is continuous, then by lemma

4.1.6.5,  $Y_t$  and process constant 0 are  $D$ -equivalent or  $Y_t = 0, \forall t \in [0, 1]$  almost surely. Remind that  $Y_t(\omega) = \int_0^t X(s, \omega) ds$ , then  $X_t = Y_{t+\Delta} - Y_t = 0$  almost surely. This is contradictory with the fact that  $X_t$  is uniformly in  $[-1, 1]$ .

### 1.7.3 Lemma

If process  $X$  is stochastically continuous (in 4.1.5.1), then  $X$  has a jointly measurable version.

### 1.7.4 Lemma

All right-continuous (left-continuous, 4.1.5.4) processes are jointly measurable.

Example 4.1.7.2 show that the role of continuity.

## 1.8 Independent increments

Let  $X = (X_t)_{t \in T}$  be a stochastic process with  $X_0 = 0$ .  $(F_t^X)_{t \in T}$  is the natural filtration of  $X$ . Then the two following statements are equivalent :

1.  $X$  has independent increments, i.e., for all  $0 \leq s_1 < t_1 \leq s_2 < t_2$  we have  $(X_{t_1} - X_{s_1}), \dots, (X_{t_2} - X_{s_2})$  are independent.
2. For every  $s < t \in T$ , we have  $X_t - X_s$  is independent of  $F_s^X$ .

**From 1  $\Rightarrow$  2 :** We need to prove that  $\sigma(X_{t_2} - X_{s_2}) \perp F_{s_2}^X, \forall t_2 > s_2$ . We have

$$\sigma(X_{t_2} - X_{s_2}) \perp \sigma(X_{t_1} - X_{s_1}), \forall 0 \leq s_1 < t_1 \leq s_2 < t_2$$

Let's take  $s_1 = 0, t_1 = t$  and note that  $X_0 = 0$ , hence :

$$\sigma(X_{t_2} - X_{s_2}) \perp \sigma(X_t), \forall 0 < t \leq s_2$$

In addtion, we have also  $\sigma(X_{t_2} - X_{s_2}) \perp \sigma(X_0)$  since  $X_0$  is a constant. Then

$$\sigma(X_{t_2} - X_{s_2}) \perp \sigma(X_t), \forall 0 \leq t \leq s_2$$

Then  $\sigma(X_{t_2} - X_{s_2}) \perp \sigma(\cup_i^n \sigma(X_{a_i}))$ ,  $n \rightarrow \infty$  and  $0 = a_1 < a_2 < \dots < a_n = s_2$ . Note that  $\sigma(X_{t_2} - X_{s_2})$  is a  $\sigma$ -algebra then it is a  $\pi$ -system. Also,  $\sigma(\cup_i^n \sigma(X_{a_i}))$  is a  $\pi$ -system. From the property that if  $\pi$ -systems are independent, then their corresponding  $\sigma$ -algebras are also independent (see 2.1.10.3), we have :

$$\sigma(\sigma(X_{t_2} - X_{s_2})) \perp \sigma(\cup_i^n \sigma(X_{a_i}))$$

or

$$\sigma(X_{t_2} - X_{s_2}) \perp \sigma(\cup_i^n \sigma(X_{a_i})) = \sigma((X_t)_{0 \leq t \leq s_2}) = F_{s_2}^X$$

**From 2  $\Rightarrow$  1 :** We need to prove that  $\sigma(X_{t_2} - X_{s_2}) \perp \sigma(X_{t_1} - X_{s_1})$ . We have :

$$\sigma(X_{t_2} - X_{s_2}) \perp F_{s_2}^X$$

$$\begin{aligned}
&\Leftrightarrow \sigma(X_{t_2} - X_{s_2}) \perp \sigma((X_t)_{0 \leq t \leq s_2}) \\
&\Rightarrow \sigma(X_{t_2} - X_{s_2}) \perp \sigma(X_{t_1}, X_{s_1}), \forall 0 \leq s_1 < t_1 \leq s_2 \quad (\text{Since } \sigma(X_{t_1}, X_{s_1}) \subset \sigma((X_t)_{0 \leq t \leq s_2})) \\
&\Rightarrow \sigma(X_{t_2} - X_{s_2}) \perp \sigma(X_{t_1} - X_{s_1}), \forall 0 \leq s_1 < t_1 \leq s_2 \quad (\text{Since } \sigma(X_{t_1} - X_{s_1}) \subset \sigma(X_{t_1}, X_{s_1}))
\end{aligned}$$

This means  $(X_{t_2} - X_{s_2})$  and  $(X_{t_1} - X_{s_1})$  are independent, for all  $0 \leq s_1 < t_1 \leq s_2$ .

## 1.9 Stopping time

A stopping time is a *random variable* whose value is interpreted as the time at which a given stochastic process stops. A stopping time is often defined by a stopping rule, a mechanism for deciding whether to continue or stop a process on the basis of the *present and past events*.

### 1.9.1 Definition

#### 1.9.1.1 Discrete time

Let  $\tau$  be a random variable, which is defined on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}}, P)$  with values in  $\mathbb{N} \cup \{+\infty\}$ . Then  $\tau$  is called a stopping time if the following condition holds:

$$\{\omega | \tau(\omega) \leq n\} \in \mathcal{F}_n, \forall n$$

Intuitively, this condition means that the “decision” of whether to stop at time  $n$  must be based only on the information present at time  $n$ , not on any future information.

#### 1.9.1.2 General case

Let  $\tau$  be a random variable, which is defined on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, P)$  with values in  $T$ . In most cases,  $T = [0, +\infty)$ . Then  $\tau$  is called a stopping time if the following condition holds:

$$\{\omega | \tau(\omega) \leq t\} \in \mathcal{F}_t^X, \forall t \in T$$

### 1.9.2 Example

Let's consider you are in a casino and play a game:

- Playing exactly five games corresponds to the stopping time  $\tau = 5$ , and *it is* a stopping rule.
- Playing until you have the maximum amount ahead you will ever be *is not* a stopping rule and does not provide a stopping time, as it requires information about the future as well as the present and past.
- Playing until you double your money (borrowing if necessary) *is not* a stopping rule, as there is a positive probability that you will never double your money.

- Playing until you either double your money or run out of money *is* a stopping rule, even though there is potentially no limit to the number of games you play, since the probability that you stop in a finite time is 1.

### 1.9.3 Example

Let's SP  $X$  in the above definition be a Brownian motion  $B$ :

- Let  $a \in \mathbb{R}$ . Then  $\tau := \inf\{t \geq 0 \mid B_t = a\}$  is a stopping time for Brownian motion, corresponding to the stopping rule: "stop as soon as the Brownian motion hits the value  $a$ ."
- Another stopping time is given by  $\tau := \inf\{t \geq 1 \mid B_s > 0, \forall s \in [t-1, t]\}$ . It corresponds to the stopping rule "stop as soon as the Brownian motion has been positive over a contiguous stretch of length 1 time unit."
- In general, if  $\tau_1$  and  $\tau_2$  are stopping times on the same filtration, then their minimum  $\tau_1 \wedge \tau_2$ , their maximum  $\tau_1 \vee \tau_2$ , and their sum  $\tau_1 + \tau_2$  are also stopping times. (This is not true for differences and products, because these may require "looking into the future" to determine when to stop.)

### 1.9.4 Stopped process

A stopped process is a stochastic process that is forced to assume the same value after a prescribed time.

#### 1.9.4.1 Definition

Let

- $(\Omega, \mathcal{F}, P)$  be a probability space.
- $(S, \Sigma)$  be a measurable space.
- $X : T \times \Omega \rightarrow S$  be a stochastic process, where  $T$  can be discrete ( $T = \{1, 2, \dots, +\infty\}$ ) or continuous ( $T = [0, +\infty)$ ).
- $\tau : \Omega \rightarrow T$  be a stopping time, which is adapted to filtration  $\{\mathcal{F}_t | t \geq 0\}$  of  $\mathcal{F}$ .

Then the stopped process  $X^\tau$  is defined for  $t \geq 0$  and  $\omega \in \Omega$  by

$$X_t^\tau(\omega) := X_{\min(t, \tau(\omega))}(\omega)$$

Note that, after the stopping time, the process continues but it is now a constant.

#### 1.9.4.2 Example

Let's resume the example in sec 4.1.3.4. We have that  $(\Omega, \mathcal{F}, F_t, P), X_t, T$ , and  $S$  are defined or determined. Remind that  $T = \{1, 2, 3\}$ . We define the stopping time  $\tau$

$$\begin{cases} \tau(\omega) = 2 \text{ if } \omega = (H, Q, .) \\ \tau(\omega) = 3 \text{ otherwise} \end{cases}$$

Suppose that an outcome is  $\omega = (H, Q, 3)$ , then  $X_1 = H, X_2 = Q, X_3 = Q$ , where  $X_3 = X_2$ .

## 2 Properties of stochastic process

### 2.1 Stationary

In probability theory, a stochastic process is said to be (strict-sense) stationary if its statistical properties, such as its mean, variance, and autocorrelation, remain constant over a time length. In other words, a stationary process has the same statistical properties at any point in time, regardless of when those properties are measured.

More formally, a stochastic process  $X(t)$  is said to be stationary if, for any set of time instants  $\{t_1, t_2, \dots, t_n\}$  (length  $n$ ) and any integer  $k$ , the joint probability distribution of  $X(t_1), X(t_2), \dots, X(t_n)$  is the same as the joint probability distribution of  $X(t_1 + k), X(t_2 + k), \dots, X(t_n + k)$ . This means that the statistical properties of the process do not change over time, and are invariant under time-shifts.

Stationarity is an important concept in probability theory, as it simplifies the analysis of stochastic processes by reducing the number of parameters that need to be estimated.

#### 2.1.1 Weakly stationary

A stochastic process is said to be weakly stationary (also known as weak-sense stationary or wide-sense stationary or WSS) if its mean and autocorrelation are stationary :

$$\mathbb{E}[X(t_1)] = \mathbb{E}[X(t_2)]$$

$$R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] = \mathbb{E}[X(t_1 + \Delta)X(t_2 + \Delta)]$$

Sometimes we write :

$$\mathbb{E}[X(t)X(t + \tau)] = R_X(\tau) = \mathbb{E}[X(t + \tau)X(t)] = R_X(-\tau)$$

Hence  $R_X$  is an even function.

**Example**

Consider the random process  $\{X(t), t \in \mathbb{R}\}$  defined as  $X(t) = \cos(t + U)$ , where  $U \sim \text{Uniform}(0, 2\pi)$ . Show that  $X(t)$  is a weak-sense stationary process.

### 2.1.2 Jointly Wide-Sense Stationary

Two random processes  $\{X(t), t \in \mathbb{R}\}$  and  $\{Y(t), t \in \mathbb{R}\}$  are said to be jointly wide-sense stationary if

- $X(t)$  and  $Y(t)$  are each weak-sense stationary.
- $R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2)$

$R_{XY}$  is not an even function as  $R_X(t)$  or  $R_Y(t)$

### 2.1.3 Cyclostationary processes

A continuous-time random process  $\{X(t), t \in \mathbb{R}\}$  is cyclostationary if there exists a positive real number  $T$  such that, for all  $t_1, t_2, \dots, t_r \in \mathbb{R}$ , the joint CDF of

$$X(t_1), X(t_2), \dots, X(t_r)$$

is the same as the joint CDF of

$$X(t_1 + T), X(t_2 + T), \dots, X(t_r + T).$$

#### Example

The random process  $\{X(t), t \in \mathbb{R}\}$  defined as  $X(t) = A \cos(\omega t)$ , where  $A$  is a random variable, is cyclostationary.

### 2.1.4 Weak-sense cyclostationary

A continuous-time random process  $\{X(t), t \in \mathbb{R}\}$  is weak-sense cyclostationary or wide-sense cyclostationary if there exists a positive real number  $T$  such that :

- $\mu_X(t + T) = \mu_X(t)$ , for all  $t \in \mathbb{R}$
- $R_X(t_1 + T, t_2 + T) = R_X(t_1, t_2)$ , for all  $t_1, t_2 \in \mathbb{R}$

## 2.2 Ergodicity

In physics, statistics, econometrics and signal processing, a stochastic process is said to be in an ergodic regime if **the time-averages converge to the ensemble averages**. In this regime, any collection of random samples from a process must represent the average statistical properties of the entire regime. Conversely, a process that is not in ergodic regime is said to be in non-ergodic regime.

In ergodicity, we may need to consider the concepts of *statistics along time* and *statistics along realizations*. In the figure 4.1, related with a continuous SP, the time variable is  $t$  (horizontal axis) and the realizations are indexed with

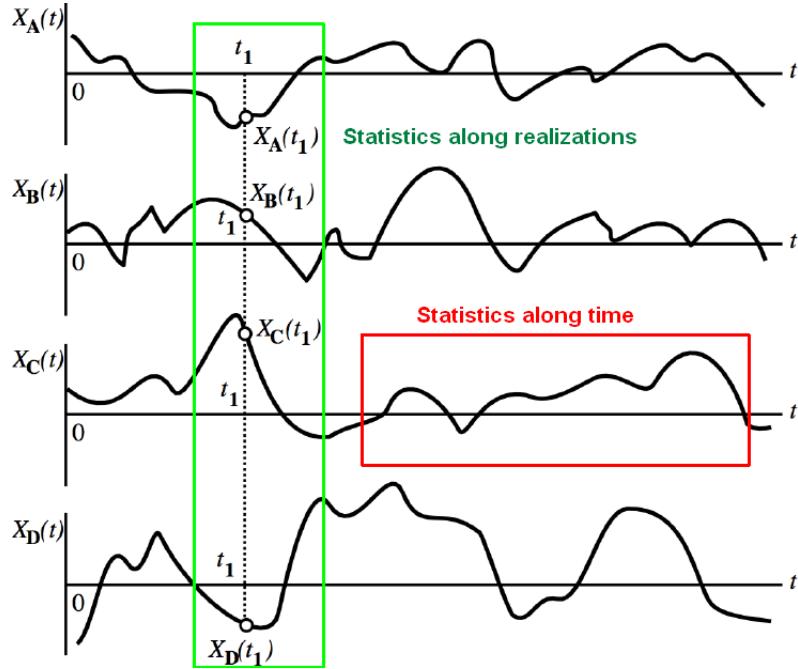


Figure 4.1: Statistics along realization and along time.

$A, B, C, D$  (vertical axis) and  $X$  is denoted for SP. This illustrates the meaning of the two axes associated with a SP.

Then **the time-averages converge to the ensemble averages** can be written as:

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T X_A(s) ds = E[X(t)]$$

where  $A$  is indexed for a realization.

### 2.2.1 Difference with stationary

There are processes that are stationary but not ergodic. Let consider these two following examples:

1, Suppose that we have two coins: one coin has head and tail and the other has two heads. We choose (at random) one of the coins first, and then perform a sequence of independent tosses of our selected coin, with output 1 for heads and 0 for tails. Here are some realizations:

- (Fair) : 1,1,0,1,0,0...
- (Unfair) : 1,1,1,1,1,1,...
- (Fair) : 0,1,1,1,0,0...

On one hand, the long term time-average is either  $\frac{1}{2}$  if fair or 1 if unfair. On the other hand, the ensemble average is  $\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 = \frac{3}{4}$ . Hence, this random process is not ergodic in mean, but it is stationary.

2, A little bit the same as the example before, but with 2 dices, one has 6 faces from 1 to 6 and the other has 4 faces from 1 to 4. We choose (at random) one of two dices first, and then perform a sequence of independent throw of our selected dice.

Here are some realizations:

- (D4) : 1,2,4,3,2,1,4...
- (D6) : 2,5,6,3,4,1,...
- ...

On one hand, the long term time-average is either  $\frac{5}{2}$  if (D4) or  $\frac{7}{2}$  if (D6). On the other hand, the ensemble average is  $\frac{1}{2} \times \frac{5}{2} + \frac{1}{2} \times \frac{7}{2} = 3$ . In the same manner, this random process is not ergodic in mean, but it is stationary.

### 2.2.2 Ergodicity implies mean stationarity

From the definition of ergodicity and given two time instant  $t_1, t_2$  and a realization  $A$ , we have :

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T X_A(s) ds = E[X(t_1)]$$

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T X_A(s) ds = E[X(t_2)]$$

This implies:

$$E[X(t_1)] = E[X(t_2)]$$

## 3 Processing on random variable

### 3.1 Power Spectral Density

Consider a wide-sense stationary (WSS) random process  $X(t)$ . Since  $E[X(t)]$  is constant for all instants  $t$ , then we exploit rather the autocorrelation  $R_X(\tau)$  to know the properties of random process. We define the *Power Spectral Density* (PSD) by the Fourier transform of  $R_X(\tau)$ :

$$S_X(f) = \mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau) e^{-2j\pi f\tau} d\tau$$

May a question is raised here is that why we use the Fourier transform ? One of possible answer is that we want to capture the periodic of a random process. Measuring directly the periodic (if existing) on one or more realizations of

random process may be a hard task, specially in case of low number of realizations. An alternative solution is to use the autocorrelation. Then the Fourier transform of the autocorrelation is for getting the periodic or other processings.

If  $X(t)$  is a real-valued random process, then  $R_X(\tau)$  is an even, real-valued function of  $\tau$ , hence:

- $S_X(f) = S_X(-f)$
- $S_X(f) \geq 0$ , which means to reconstruct  $R_X(\tau)$ , there are only positive coefficient in the sum. (Proof in 4.3.3)

### Expected power

$$E[X(t)^2] = R_X(0) = \int_{-\infty}^{\infty} S_X(f) e^{2j\pi f 0} df = \int_{-\infty}^{\infty} S_X(f) df$$

**Cross Spectral Density** (between two SP  $X$  and  $Y$ :

$$S_{XY}(f) = \mathcal{F}\{R_{XY}(\tau)\} = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-2j\pi f \tau} d\tau$$

## 3.2 Linear Time-Invariant (LTI) Systems

We remind the impulse response in signal processing. Denote  $X(t), h(t), Y(t)$  respectively input, filter and output :

$$Y(t) = \int_{-\infty}^{\infty} h(\alpha) X(t - \alpha) d\alpha = \int_{-\infty}^{\infty} h(t - \alpha) X(\alpha) d\alpha$$

Image that  $h(t)$  is a blackbox and we do not know it. A simple method to  $h(t)$  is to apply at input a Dirac pulse :

$$X(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then at output we have  $Y(t) = h(t)$ , or simply the impulse response.

Now in case of random process, consider an LTI system with impulse response  $h(t)$ . Let  $X(t)$  be a WSS random process. If  $X(t)$  is the input of the system, then the output,  $Y(t)$ , is also a random process. More specifically, we can write:

$$Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(\alpha) X(t - \alpha) d\alpha.$$

Then we have the following properties:

- $Y(t)$  is also WSS.
- $X(t)$  and  $Y(t)$  are jointly WSS.

- $\mu_Y(t) = \mu_Y = \mu_X \int_{-\infty}^{\infty} h(\alpha) d\alpha$
- $R_{XY}(\tau) = h(-\tau) * R_X(\tau) = \int_{-\infty}^{\infty} h(-\alpha) R_X(t - \alpha) d\alpha$
- $R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$

(\*)  $Y(t)$  is WSS :

First,

$$\begin{aligned}
\mathbb{E}[Y(t)] &= \mathbb{E} \int_{-\infty}^{\infty} h(\alpha) X(t - \alpha) d\alpha \\
&= \int_{-\infty}^{\infty} \mathbb{E}[h(\alpha) X(t - \alpha)] d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha) \mathbb{E}[X(t - \alpha)] d\alpha \\
&= \mu_X \int_{-\infty}^{\infty} h(\alpha) d\alpha \\
&= \mu_X H(0), \quad H \text{ denotes Fourier tf} \\
&= \mu_Y
\end{aligned}$$

Second,

$$\begin{aligned}
R_Y(\tau) &= \mathbb{E}[Y(t)Y(t + \tau)] \\
&= Cov[Y(t), Y(t + \tau)] + \mathbb{E}[Y(t)]\mathbb{E}[Y(t + \tau)] \\
&= Cov \left[ \int_{-\infty}^{\infty} h(\alpha) X(t - \alpha) d\alpha, \int_{-\infty}^{\infty} h(\alpha) X(t + \tau - \alpha) d\alpha \right] + \mu_Y^2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) Cov[X(t - \alpha), X(t + \tau - \beta)] d\alpha d\beta + \mu_Y^2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) (\mathbb{E}[X(t - \alpha), X(t + \tau - \beta)] - \mathbb{E}[X(t - \alpha)]\mathbb{E}[X(t + \tau - \beta)]) d\alpha d\beta + \mu_Y^2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) (R_X(\tau + \alpha - \beta) - \mu_X^2) d\alpha d\beta + \mu_Y^2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) R_X(\tau + \alpha - \beta) d\alpha d\beta - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) \mu_X^2 d\alpha d\beta + \mu_Y^2 \\
&= \int_{-\infty}^{\infty} h(\alpha) h(\tau + \alpha) * R_X(\tau + \alpha) d\alpha - \int_{-\infty}^{\infty} h(\alpha) d\alpha \mu_X \int_{-\infty}^{\infty} h(\beta) d\beta \mu_X + \mu_Y^2 \\
&= \int_{-\infty}^{\infty} h(\alpha) h(\tau + \alpha) * R_X(\tau + \alpha) d\alpha \\
&= h(-\tau) * [h(\tau) * R_X(\tau)] \quad \text{Since } h(-t) * g(t) = \int_{-\infty}^{\infty} h(\alpha) g(t + \alpha) d\alpha
\end{aligned}$$

Hence  $Y(t)$  is WSS, Moreover:

$$S_Y(f) = \mathcal{F}\{R_Y(\tau)\}$$

$$\begin{aligned}
&= \mathcal{F}\{h(-\tau) * h(\tau) * R_X(\tau)\} \\
&= H(-f)H(f)S_X(f) \\
&= \bar{H}(f)H(f)S_X(f) \\
&= |H(f)|^2S_X(f)
\end{aligned}$$

$H(-f) = \bar{H}(f)$  since  $h(t)$  is assumed to be a real signal.  
 (\*\*\*)  $X(t)$  and  $Y(t)$  are jointly WSS

$$\begin{aligned}
R_{XY}(\tau) &= \mathbb{E}[X(t)Y(t-\tau)] \quad (\text{by definition}) \\
&= \mathbb{E}[X(t) \int_{-\infty}^{\infty} h(\alpha)X(t-\tau-\alpha)d\alpha] \\
&= \mathbb{E}\left[\int_{-\infty}^{\infty} h(\alpha)X(t)X(t-\tau-\alpha)d\alpha\right] \\
&= \int_{-\infty}^{\infty} h(\alpha)\mathbb{E}[X(t)X(t-\tau-\alpha)]d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)R_X(-\tau-\alpha)d\alpha \\
&= h(\tau) * R_X(-\tau) \\
&= h(\tau) * R_X(\tau)
\end{aligned}$$

Hence  $X(t)$  and  $Y(t)$  are jointly WSS. Moreover:

$$\begin{aligned}
S_{XY}(f) &= \mathcal{F}\{R_{XY}(\tau)\} \\
&= \mathcal{F}\{h(\tau) * R_X(\tau)\} \\
&= H(f)S_X(f)
\end{aligned}$$

### 3.3 Power in a Frequency Band

Consider the following transfer function  $H$ , which is the Fourier transform of a filter  $h$  (in sec 4.3.2) :

$$H(f) = \begin{cases} 1 & f_1 \leq |f| \leq f_2 \\ 0 & \text{otherwise} \end{cases}$$

Then the power spectral density of output is:

$$S_Y(f) = |H(f)|^2S_X(f) = \begin{cases} S_X(f) & f_1 \leq |f| \leq f_2 \\ 0 & \text{otherwise} \end{cases}$$

An the expected power :

$$\mathbb{E}[Y(t)^2] = \int_{-\infty}^{\infty} S_Y(f)$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} S_X(f) \\
&= 2 \int_{f_1}^{f_2} S_X(f) \quad \text{Since } S_X(-f) = S_X(f)
\end{aligned}$$

Equivalently,  $\mathbb{E}[Y(t)^2]$  here corresponds to the expected power of  $X(t)$ , taking into account frequencies over the range  $f_1 \leq |f| \leq f_2$ .

Here note that if  $f_1 \rightarrow f_2$ , then we must have  $S_X(f)$  non-negative since  $\mathbb{E}[Y(t)^2] \geq 0$ .

### 3.4 White Noise

The random process  $X(t)$  is called a white noise process if its Power Spectral Density function is constant (or flat), denoted by  $\frac{N_0}{2}$  by convention :

$$S_X(f) = \frac{N_0}{2}$$

The autocorrelation of white noise is :

$$R_X(\tau) = \mathcal{F}^{-1}\left\{\frac{N_0}{2}\right\} = \frac{N_0}{2}\delta(\tau)$$

where  $\delta$  is a Dirac function.

#### 3.4.1 White Gaussian noise

The random process  $X(t)$  is called a **white Gaussian noise** if :

- $X(t)$  is white noise.
- $X(t)$  is Gaussian process (in sec 4.4), with zero mean  $\mu_X = 0$ .
- $X(t)$  is a stationary.

#### 3.4.2 Difference between white noise and white Gaussian noise

In case of white noise, we have  $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$ . This means that  $X(t_1)$  and  $X(t_2)$  are uncorrelated or white noise is uncorrelated. In case of white Gaussian noise, we have the stronger thing that  $X(t_1)$  and  $X(t_2)$  are independent (if two gaussian distributions are uncorrelated then they are independent).

## 4 Gaussian Process

In probability theory and statistics, a Gaussian process  $X_t$  is a stochastic process if every finite collection of  $k$  random variables

$$X_{t_1}, X_{t_2}, \dots, X_{t_k}$$

has a multivariate normal distribution, i.e. every finite linear combination of them is normally distributed (sec 2.6.3).

**Note that do not confuse** between *stochastic process* and *multivariate random variable* :

- First, in stochastic process, variables must be indexed (ordered).
- Second, in stochastic process, the number of index can tend to infinity, which differs from multivariate random variable that has the fixed number of random variable.

## 4.1 LTI of a Gaussian Process

Let  $X(t)$  be a Gaussian process. If  $X(t)$  is the input to an LTI system (in sec 4.3.2), then the output random process,  $Y(t)$ , is also a Gaussian process.

$$Y(t) = \int_{-\infty}^{\infty} h(\alpha)X(t - \alpha)d\alpha$$

### 4.1.1 Illustration

Let's :

$$h(\alpha) = \begin{cases} a & \text{if } \alpha = 0 \\ b & \text{if } \alpha = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$Y(t) = aX(t) + bX(t - 1)$$

Consider  $[Y(t_1), \dots, Y(t_k)]$  that equals to  $aX(t_1) + bX(t_1 - 1), \dots, aX(t_k) + bX(t_k - 1)$ . Then any linear combination :

$$\sum_{i=1}^k c_i Y(t_i) = \sum_{i=1}^k c_i aX(t_i) + \sum_{i=1}^k c_i bX(t_i - 1)$$

On the left hand side, we have a normal distribution since it is a linear combination of  $2k$  jointly normal random variable. This implies that  $Y(t)$  is Gaussian process

### 4.1.2 LTI of stationary Gaussian process

Moreover, if  $X(t)$  is *stationary* Gaussian process, then  $Y(t)$  is also a stationary Gaussian process. Note that, for Gaussian processes, wide-sense stationarity implies (strict) stationarity.

## 4.2 Gaussian Process Regression

Given fixed  $n$ , suppose that  $X_{t_i}, t_1 < \dots < t_n$  is a Gaussian process or  $(X_{t_1}, \dots, X_{t_n})$  is standard multivariate normal and :

- $X = (X_{i_1}, \dots, X_{i_q})$ , known variables, where  $i_1, \dots, i_q \in \{t_1, \dots, t_n\}$ . This means we know  $[X_{i_1}, \dots, X_{i_q}] = [y_{i_1}, \dots, y_{i_q}]$ .
- $X^*$  : resting  $n - q$  variables need to be predicted (or inference).
- $[X, X^*] \sim \mathcal{N}(\mathbf{0}, \Sigma)$
- Covariance matrix  $\Sigma = \begin{bmatrix} \mathbf{K} & \mathbf{K}_* \\ \mathbf{K}_*^T & \mathbf{K}_{**} \end{bmatrix}$  is unknown and need to be estimated before doing inference. In real case, if we need taking into account for noise, which means :

$$y = X + \varepsilon$$

$$\text{where } \varepsilon \sim \mathcal{N}(\mathbf{0}, \sigma_n^2 I_q). \text{ Then } [X, X^*] \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \mathbf{K} + \sigma_n^2 I_q & \mathbf{K}_* \\ \mathbf{K}_*^T & \mathbf{K}_{**} \end{bmatrix}\right)$$

### 4.2.1 Hyper-parameter estimation

**Kernel** or covariance functions, usually, we use the Squared exponential, combined with constant  $C$  :

$$K_{SE}(x, x') = C \exp\left(-\frac{(x - x')^2}{2l^2}\right)$$

where  $C, l$  are two hyper-parameters need to be estimated. We can suppose that the covariance between two variables  $X_{t_a}$  and  $X_{t_b}$  is:

$$\Sigma(X_{t_a}, X_{t_b}) = K_{SE}(t_a, t_b)$$

Considering now only known variable  $X$ , as  $X$  follows multivariate normal distribution  $\mathcal{N}(\mathbf{0}, \mathbf{K} + \sigma_n^2 I_q)$ :

$$f_{\mathbf{X}|\Theta}(\mathbf{y}|\theta) = \frac{1}{\sqrt{(2\pi)^q |\mathbf{K} + \sigma_n^2 I_q|}} \exp\left(-\frac{1}{2} \mathbf{y}^T (\mathbf{K} + \sigma_n^2 I_q)^{-1} \mathbf{y}\right)$$

where  $\Theta$  contains hyper-parameters of kernel  $C, l$  in  $\mathbf{K}$  and also  $\sigma_n$ .  $\mathbf{y}$  and  $\theta$  are explicite values in the support of  $\mathbf{X}$  and  $\Theta$ , which means  $f_{\mathbf{X}|\Theta}(\mathbf{y}|\theta)$  is the value where  $\mathbf{X} = \mathbf{y}$  and  $\Theta = \theta$ . Then the log likelihood :

$$\log f_{\mathbf{X}|\Theta}(\mathbf{y}|\theta) = -\frac{1}{2} \mathbf{y}^T (\mathbf{K} + \sigma_n^2 I_q)^{-1} \mathbf{y} - \frac{1}{2} \log(|\mathbf{K} + \sigma_n^2 I_q|) - \frac{q}{2} \log(2\pi)$$

As  $\log f_{\mathbf{X}|\Theta}(\mathbf{y}|\theta)$  is differential, we can use for example the bfgs method to minimize  $\log f_{\mathbf{X}|\Theta}(\mathbf{y}|\theta)$  and get  $\theta$ .

### 4.2.2 Inference

After estimateing  $\theta$ , we have the explicit form of  $\mathcal{N}(\mathbf{0}, \Sigma)$ . In the inference, we use the **conditional distribution** as described in the sec 2.6.3.5, then *with each test point or prediction  $X_i^*$* :

$$f_{X_i^*|\mathbf{X}}(x_i^*|\mathbf{x}) \sim \mathcal{N}(\mu_i^*, \sigma_i^{*2})$$

where :

$$\mu_i^* = \mathbf{k}_i^{*T} [\mathbf{K} + \sigma_n^2 I_q]^{-1} \mathbf{y}$$

$$\sigma_i^{*2} = Cov(X_i^*, X_i^*) - \mathbf{k}_i^{*T} [\mathbf{K} + \sigma_n^2 I_q]^{-1} \mathbf{k}_i^*$$

with  $\mathbf{k}_i^* = Cov(X_i^*, \mathbf{X})$ .

As previously mentioned, the inference is performed for each point  $X_i^*$  instead of all points  $X^*$  because we want to obtain the variance  $\sigma_i^{*2}$  for each point. The latter value is important for giving a vision on confidence interval of estimated value.

#### 4.2.2.1 Cholesky decomposition trick

Since the complexity for matrix inversion  $[\mathbf{K} + \sigma_n^2 I_q]^{-1}$  is  $\mathcal{O}(q^3)$ . Then in practice, we use the Cholesky decomposition which have  $\mathcal{O}(q^2)$  for matrix inversion. Remind that given  $A$  symmetric semi positive (Then  $A = LL^*$  or  $LL^T$  in real case):

$$A \setminus b = A^{-1}b = L^T \setminus (L \setminus b)$$

where the notation  $A \setminus b$  denote the vector  $x$  that solves  $Ax = b$ . Then we get two new formulas for mean  $\mu_i^*$  and variance  $\sigma_i^*$ :

$$\mu_i^* = \mathbf{k}_i^{*T} \alpha$$

$$\sigma_i^* = Cov(X_i^*, X_i^*) - \mathbf{v}^T \mathbf{v}$$

where  $\mathbf{K} + \sigma_n^2 I_q = LL^T$ ,  $\alpha = L^T \setminus (L \setminus b)$  and  $\mathbf{v} = L \setminus \mathbf{k}_i^*$ .

### 4.2.3 Multi-features

Now in case that  $X_{t_i} = [a_1, \dots, a_d]$ , where  $d$  is number of features and  $a_i$  is an features. Then we only need to use the new kernel for multi dimension, such as :

$$K_{SE}(x, x') = C \exp \left( -\frac{\|x - x'\|_2^2}{2l^2} \right)$$

#### 4.2.4 Standardization

It is desirable to standardize the data while learning Gaussian processes regression. There are a number of reasons:

- In common Gaussian processes regression model we suppose that output  $y$  has zero mean, so we should mean shift  $y$  to match our assumption.
- In case of multi-features, we have different  $std$  (scale). So, we should standardize features to be in the same scale, then better for estimation  $\theta$ , if not, for each feature, we can put its scale as a new hyper-parameter.
- Gaussian processes regression is prone to numerical problems as we have to inverse ill-conditioned covariance matrix. To make this problem less severe, you should standardize your data.

### 4.3 Bayesian optimization

Bayesian optimization (BO) is a powerful method for optimizing hyperparameters in machine learning models. It is particularly useful when the objective function that we are trying to optimize is expensive to evaluate, meaning that each evaluation takes a significant amount of time or computational resources.

The basic idea behind Bayesian optimization is to build a probabilistic model of the objective function, and use this model to decide which hyperparameters to try next. The model is updated as new evaluations are made, so that it becomes increasingly accurate over time.

*Here, we can use Gaussian Processes Regression as model, with input is hyperparameters and output is the score of performance.*

At each step of the optimization process, Bayesian optimization chooses the next set of hyperparameters to evaluate based on a trade-off between exploration and exploitation. On one hand, it tries to explore new regions of the hyperparameter space where the objective function might be better. On the other hand, it tries to exploit regions that are known to be good based on previous evaluations.

Paper : “A Tutorial on Bayesian Optimization” by Peter I. Frazier

#### 4.3.1 Acquisition function

It means a mathematical function used to guide the search for the optimal solution in an efficient manner. The primary goal of an acquisition function is to balance the trade-off between exploration and exploitation in the optimization process. In other words, given expected value (mean) and uncertainty (variance) at different points in the search space, the acquisition function estimate a potential point that have best performance.

There are several common acquisition functions used in Bayesian optimization:

- Upper confidence bound

- Expected Improvement
- Probability of Improvement

#### 4.3.1.1 Upper confidence bound

Assuming that you want to solve a maximization problem, the upper confidence bound (UCB) can be written as :

$$a(x) = \mu(x) + \kappa\sigma(x)$$

with  $\kappa$  is a constant for the trade-off. Comparing two points  $x_1$  and  $x_2$ :

- If their means  $\mu$  are the same, then BO will pick the one that has larger  $\sigma$ . This is called **exploration**.
- If their std  $\sigma$  are the same, then BO will pick the one that has larger  $\mu$ . This is called **exploitation**.

#### 4.3.1.2 Expected improvement

Expected improvement (EI) is a short for expected value of improvement.

Let's suppose that after  $n$  evaluations at  $x_1, \dots, x_n$ , we have  $f(x_1) = y_1, \dots, f(x_n) = y_n$ . Let  $y^* = \max(y_1, \dots, y_n) = f^*$ . Then the objective is to find  $x_{n+1}$ , such as:

$$x_{n+1} = \operatorname{argmax}_x \mathbb{E} [[f(x) - f^*]^+],$$

where  $[x]^+ = \max(0, x)$ . This means for a given  $f(x)$  with mean and uncertainty, we consider **only the part that is greater  $f^*$** .

Since the set of  $f(x)$  and  $f(x_1), \dots, f(x_n)$  form a multivariate normal distribution, then the conditional distribution

$$f(x)|f(x_1), \dots, f(x_n) \sim \mathcal{N}(\mu(x), \sigma(x)^2)$$

where mean  $\mu(x)$  and std  $\sigma(x)$  are calculated as in sec 2.6.3.5. For each  $x$ , let  $z_0 = \frac{y^* - \mu(x)}{\sigma(x)}$  be the separated point (in the following integral), we have :

$$\begin{aligned} & \mathbb{E} [f(x) - f^*]^+ \\ &= \int_{z_0}^{\infty} (\mu(x) + \sigma(x)z - y^*)\phi(z)dz \\ &= (\mu(x) - y^*)\Phi\left(\frac{\mu(x) - y^*}{\sigma(x)}\right) + \sigma(x)\phi\left(\frac{\mu(x) - y^*}{\sigma(x)}\right) \end{aligned}$$

where  $\Phi(\cdot)$  and  $\phi(\cdot)$  are the CDF and PDF of standard normal distribution  $\mathcal{N}(0, 1)$ . This calculation is more or less similar to the one in Black-Scholes model (sec 6.5.1). Then we need to maximize the above function to find  $x_{n+1}$ .

## 5 Poisson Process

### 5.1 Preliminaries

#### 5.1.1 Counting processes

A random process  $\{N(t), t \in [0, \infty)\}$  is said to be a counting process if  $N(t)$  is the number of events occurred from time 0 up to and including time  $t$ . For a counting process, we assume

1.  $N(0) = 0$
2.  $N(t) \in \{0, 1, 2, \dots\}$
3. for  $0 \leq s < t$ ,  $N(t) - N(s)$  shows the number of events that occur in the interval  $(s, t]$

#### 5.1.2 Independent increments

Let  $\{X(t), t \in [0, \infty)\}$  be a continuous-time random process. We say that  $X(t)$  has independent increments if, for all  $0 \leq t_1 < t_2 < t_3 < \dots < t_n$ , the random variables

$$X(t_2) - X(t_1), X(t_3) - X(t_2), \dots, X(t_n) - X(t_{n-1})$$

are independent. We have seen the independent increments in sec 4.1.8.

A counting process has independent increments if the numbers of arrivals in non-overlapping (disjoint) intervals are independent.

#### 5.1.3 Stationary increments

We say that  $X(t)$  has stationary increments if, for all  $t_2 > t_1 \geq 0$ , and all  $r > 0$ , the two random variables

$$X(t_2) - X(t_1) \text{ and } X(t_2 + r) - X(t_1 + r)$$

have the same distributions. Equivalently,  $N(t_2) - N(t_1)$  has the same distribution as  $N(t_2 - t_1) - N(0) = N(t_2 - t_1)$ .

In other words, the distribution of the difference depends only on the length of the interval  $(t_1, t_2]$ , and not on the exact location of the interval on the real line.

#### 5.1.4 Poisson distribution

$X \sim \text{Poisson}(\mu)$  if its range is  $R_X = \{0, 1, 2, 3, \dots\}$  and its PMF is given by:

$$p(X = k) = e^{-\mu} \frac{\mu^k}{k!}, k \in R_X$$

Here are some useful facts:

1. If  $X \sim Poisson(\mu)$ , then  $\mathbb{E}[X] = \mu$  and  $Var[X] = \mu$
  2. If  $X_i \sim Poisson(\mu_i)$  and  $X_i$  are independent, then
- $$X_1 + X_2 + \dots + X_n \sim Poisson(\mu_1 + \mu_2 + \dots + \mu_n)$$
3. The Poisson distribution can be viewed as the limit of binomial distribution : Let  $\mu > 0$  be a fixed real number,  $Y_n \sim Binomial(n, p = \frac{\mu}{n})$  then the PMF of  $Y_n$  converges to a Poisson PMF when  $n \rightarrow \infty$ .

### 5.1.5 First arrival and Interarrival Times

The first arrival time, noted  $X_1$ , is the time from 0 to the first occurrence (arrival).

The interarrival times, noted  $X_i, (i > 1)$  is the time between the  $(i - 1)^{th}$  occurrence and  $i^{th}$  occurrence.

Let  $N(t)$  be a Poisson process with rate  $\lambda$ ,  $\forall i$  :

$$\begin{aligned} P(X_i > t) &= P(N(t) = 0) \\ &= e^{-\lambda t} \frac{(\lambda t)^0}{0!} \\ &= e^{-\lambda t} \end{aligned}$$

We try to find the distribution of  $X_i$ , thus the CDF :

$$\begin{aligned} F_{X_i}(t) &= P(X_i \leq t) \\ &= 1 - P(X_i > t) \\ &= 1 - e^{-\lambda t} \end{aligned}$$

Hence  $X_i \sim Exponential(\lambda)$ .

### 5.1.6 Arrival's distribution

Now that we know the distribution of the interarrival times, we can find the distribution of arrival times:

- $T_1 = X_1$
- $T_2 = X_1 + X_2$
- $T_3 = X_1 + X_2 + X_3$
- $\vdots$

More specifically,  $T_n$  is the sum of  $n$  independent  $Exponential(\lambda)$  random variables. If  $T_n = X_1 + X_2 + \dots + X_n$  where the  $X_i$  are independent  $Exponential(\lambda)$  random variables, then  $T_n \sim Gamma(n, \lambda)$  or Erlang distribution. The PDF of  $T_n$  :

$$f_{T_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$$

## 5.2 Definition of the Poisson Process

### 5.2.1 First definition

Let  $\lambda > 0$  be fixed. The counting process  $\{N(t), t \in [0, \infty)\}$  is called a Poisson process with rates  $\lambda$  if all the following conditions hold :

1.  $N(0) = 0$ .
2.  $N(t)$  has independent increments.
3.  $N(\tau) \sim \text{Poisson}(\lambda\tau)$  distribution, hence  $N(t)$  has stationary increments.

### 5.2.2 Second definition

Let  $\lambda > 0$  be fixed. The counting process  $\{N(t), t \in [0, \infty)\}$  is called a Poisson process with rate  $\lambda$  if all the following conditions hold:

1.  $N(0) = 0$
2.  $N(t)$  has independent and stationary increments
3. We have if  $\Delta \rightarrow 0$  that
  - $P(N(\Delta) = 0) = 1 - \lambda\Delta + o(\Delta)$
  - $P(N(\Delta) = 1) = \lambda\Delta + o(\Delta)$
  - $P(N(\Delta) \geq 2) = o(\Delta)$

### 5.2.3 Equivalence between these two definitions

**From the first definition to the second one**, using Taylor Series. For example, with *exponential distribution* that :

- $P(N(\Delta) = 0) = P(X_1 > \Delta) = e^{-\lambda\Delta} = 1 - \lambda\Delta + o(\Delta)$
- $P(N(\Delta) = 1) = P(X_1 < \Delta) = 1 - e^{-\lambda\Delta} = \lambda\Delta + o(\Delta)$
- $P(N(\Delta) \geq 2) = 1 - P(N(\Delta) = 0) - P(N(\Delta) = 1) = o(\Delta)$

**Inversely, from the second definition to the first one,**  
We define:

$$g_k(t) = P(N(t) = k)$$

Then using the property *independent increment* and with  $\Delta \rightarrow 0$

$$\begin{aligned} & g_k(t + \Delta) \\ &= P(N(t + \Delta) = k) \\ &= P(N(t) = k)P(N(\Delta) = 0) + P(N(t) = k - 1)P(N(\Delta) = 1) + \sum_{i=2}^k P(N(t) = k - i)P(N(\Delta) = i) \end{aligned}$$

$$\begin{aligned}
&= g_k(t)(1 - \lambda\Delta + o(\Delta)) + g_{k-1}(t)(\lambda\Delta + o(\Delta)) + \sum_{i=2}^k g_{k-i}(t)o(\Delta) \\
&= g_k(t)(1 - \lambda\Delta) + g_{k-1}(t)\lambda\Delta + o(\Delta)
\end{aligned}$$

Or,

$$g_k(t + \Delta) - g_k(t) = -\lambda\Delta g_k(t) + \lambda\Delta g_{k-1}(t) + o(\Delta)$$

Or,

$$\begin{aligned}
g'_k(t) &= -\lambda g_k(t) + \lambda g_{k-1}(t) \\
e^{\lambda t} g'_k(t) + e^{\lambda t} \lambda g_k(t) &= e^{\lambda t} \lambda g_{k-1}(t) \\
\frac{d}{dt}[e^{\lambda t} g_k(t)] &= \lambda [e^{\lambda t} g_{k-1}(t)]
\end{aligned}$$

Or,

$$\frac{d^k}{dt^k}[e^{\lambda t} g_k(t)] = \lambda^k [e^{\lambda t} g_0(t)] \quad (k \geq 1)$$

Now we try to find out  $g_0(t)$ , indeed :

$$\begin{aligned}
g_0(t) &= P(N(t) = 0) \\
&= \lim_{n \rightarrow \infty} \prod_{i=1}^n P\left(N\left(\frac{t}{n}\right) = 0\right) \quad (\text{independent increments}) \\
&= \lim_{n \rightarrow \infty} \prod_{i=1}^n \left(1 - \lambda \frac{t}{n} + o\left(\frac{t}{n}\right)\right) \quad (\text{Third property of } 2^{nd} \text{ definition}) \\
&= \lim_{n \rightarrow \infty} \left(1 - \lambda \frac{t}{n}\right)^n \\
&= e^{-\lambda t} \quad (\text{Definition of } e)
\end{aligned}$$

Hence,

$$\frac{d^k}{dt^k}[e^{\lambda t} g_k(t)] = \lambda^k \quad (k \geq 1)$$

This implies that :

$$e^{\lambda t} g_k(t) = \lambda^k \left( \frac{t^k}{k!} + \sum_{h=0}^{k-1} c_h \frac{t^h}{h!} \right) \quad (k \geq 1)$$

Note that we have  $g_i(0) = P(N(0) = i) = 0$  for  $i = 1, \dots, k$ , these  $k$  conditions imply that  $c_h = 0$  for  $h = 0, \dots, k-1$ . Finally,

$$g_k(t) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

or  $N(t) \sim Poisson(\lambda\tau)$

## 5.3 Merging and splitting

### 5.3.1 Merging Independent Poisson Processes

Let  $N_1(t), N_2(t), \dots, N_m(t)$  be  $m$  independent Poisson processes with rates  $\lambda_1, \lambda_2, \dots, \lambda_m$ . Let's also

$$N(t) = N_1(t) + N_2(t) + \dots + N_m(t)$$

Then,  $N(t)$  is a Poisson process with rate  $\lambda_1 + \lambda_2 + \dots + \lambda_m$ .

### 5.3.2 Splitting (Thinning) of Poisson Processes

Let  $N(t)$  be a Poisson process with rate  $\lambda$ . Here, we divide  $N(t)$  to two processes  $N_1(t)$  and  $N_2(t)$  in the following way: For each arrival, a coin with  $P(H) = p$  is tossed. If the coin lands heads up, the arrival is counted for the first process ( $N_1(t)$ ), otherwise it is counted for the second process. The coin tosses are independent and are independent of  $N(t)$ . Then,

- $N_1(t)$  is a Poisson process with rate  $\lambda p$
- $N_2(t)$  is a Poisson process with rate  $\lambda(1 - p)$
- $N_1(t)$  and  $N_2(t)$  are independent.

## 5.4 Nonhomogeneous Poisson Processes

Let  $N(t)$  be the number of customers arriving at a fast food restaurant by time  $t$ . We think that the customers arrive somewhat randomly, so we might want to model  $N(t)$  as a Poisson process. However, we notice that this process does not have *stationary increments*. For example, we note that the arrival rate of customers is larger during lunch time compared to, say, 4 p.m. In such scenarios, we might model  $N(t)$  as a nonhomogeneous Poisson process. Such a process has all the properties of a Poisson process, except for the fact that **its rate is a function of time**, i.e.,  $\lambda = \lambda(t)$ . We have the following definition :

Let  $\lambda(t) : [0, \infty) \rightarrow [0, \infty)$  be an integrable function. The counting process  $\{N(t), t \in [0, \infty)\}$  is called a nonhomogeneous Poisson process with rate  $\lambda(t)$  if all the following conditions hold.

- $N(0) = 0$
- $N(t)$  has independent increments
- for any  $t \in [0, \infty)$ :
  - $P(N(t + \Delta) - N(t) = 0) = 1 - \lambda(t)\Delta + o(\Delta)$
  - $P(N(t + \Delta) - N(t) = 1) = \lambda(t)\Delta + o(\Delta)$
  - $P(N(t + \Delta) - N(t) \geq 2) = o(\Delta)$

More specifically, we can write

$$N(t + s) - N(t) \sim \text{Poisson} \left( \int_t^{s+t} \lambda(\alpha) d\alpha \right)$$

## 6 Discrete-Time Markov Chains

Also called Discrete Markov process.

### 6.1 Introduction

Consider :

- The discrete stochastic process  $\{X_n, n = 0, 1, 2, \dots\}$
- State space (countable)  $S \subset \{1, 2, \dots\}$  and  $R_{X_i} \in S$

We say that this process is a Markov chain if

$$P(X_{m+1} = j | X_m = i, X_{m-1} = i_{m-1}, \dots, X_0 = i_0) = P(X_{m+1} = j | X_m = i) = p_{ij}$$

which means the  $X_{m+1}$  depends only on  $X_m$  and not other previous random variables. If the number of states is finite, e.g.,  $|S| = r$ , we call it a finite Markov chain. From here, we consider that  $S = \{1, 2, \dots, r\}$ .

#### 6.1.1 Transition probability matrix

The probability to pass from state  $i$  to  $j$  is noted as  $p_{ij}$  and they forms a transition probability matrix :

$$P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1r} \\ p_{21} & p_{22} & \dots & p_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ p_{r1} & p_{r2} & \dots & p_{rr} \end{bmatrix}$$

Given that we are in state  $i$ , then the next state must be one of the possible states, hence :

$$\sum_{j=1}^r p_{ij} = 1$$

or, the sum of each row in the transition probability matrix must be 1.

#### 6.1.2 Probability distributions

Suppose that we know the probability distribution of  $X_0$  :

$$\pi^{(0)} = [P(X_0 = 1) P(X_0 = 2) \dots P(X_0 = r)]$$

Then the distribution of  $X_n$  is :

$$\pi^{(n)} = [P(X_n = 1) P(X_n = 2) \dots P(X_n = r)] = \pi^{(0)} P^n$$

Note that it is  $\pi^{(0)} P^n$  and **not**  $P^n \pi^{(0)T}$ .

### 6.1.3 n-Step Transition Probabilities

It means the probability that after  $n$  steps, we get the state  $j$  from  $i$ .

$$p_{ij}^{(n)} = P(X_{n_0+n} = j | X_{n_0} = i)$$

The Chapman-Kolmogorov equation can be written as

$$p_{ij}^{(m+n)} = P(X_{m+n} = j | X_0 = i) = \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}.$$

## 6.2 Clustering of state

- We say that state  $j$  is **accessible** from state  $i$ , written as  $i \rightarrow j$ , if  $\exists n, p_{ij}^{(n)} > 0$ . We assume every state is accessible from itself since  $p_{ii}^{(0)} = 1$ .
- Two states  $i$  and  $j$  are said to **communicate**, written as  $i \leftrightarrow j$ , if they are accessible from each other.
- Two states  $i$  and  $j$  **belong to the same class** (or group) if and only if  $i \leftrightarrow j$ .
- Given a Markov chain, it is said to be **irreducible** if all states communicate with each other.

Given a state  $i$  in a Markov chain, it is said to be :

- **recurrent** if  $f_{ii} = P(\exists n_i \geq 1, X_{n_i} = i | X_0 = i) = 1$ . It means just that if we go start from state  $i$ , then there is at least one path to return back to  $i$ .
- **transient** if  $f_{ii} = P(\forall n \geq 1, X_n \neq i | X_0 = i) < 1$ . It means just that, from a certain moment, if we go start from state  $i$ , we have no path to return to  $i$ .

To bring away :

- Every state is either recurrent or transient.
- If two states are in the same class, either both of them are recurrent, or both of them are transient.
- A class is said to be recurrent (transient) if the states in that class are recurrent (transient).
- If we can show there is a state recurrent (transient) in class, then this class is recurrent (transient)

### 6.3 Periodicity

The period of a state  $i$  is the largest integer  $d$  satisfying the following property:

$$d = \begin{cases} p_{ii}^{(n)} > 0 & \text{if } n \equiv 0 \pmod{d} \\ p_{ii}^{(n)} = 0 & \text{if } n \not\equiv 0 \pmod{d} \end{cases}$$

The period of state  $i$  is shown by  $d(i)$ .

- If  $d(i) > 1$ , we say that state  $i$  is periodic.
- If  $d(i) = 1$ , we say that state  $i$  is not periodic or aperiodic.
- If  $i \leftrightarrow j$ , then  $d(i) = d(j)$

Consider a finite irreducible Markov chain  $X_n$ , it is said to be aperiodic if:

- There is a self-transition state in the chain,  $p_{ii} > 0$  for some  $i$ .
- We can start from state  $i$  and come back to state  $i$  by two paths, with length respectively  $m$  and  $l$  and  $\gcd(m, l) = 1$  ( $\gcd$  means greatest common divisor).
- There exists a positive integer  $n$  such that all elements of the matrix  $P^n$  are strictly positive, which means  $p_{ij}^{(n)} > 0, \forall i, j$ .

### 6.4 Using the law of Total Probability with Recursion

A very useful technique in the analysis of Markov chains is using law of total probability. Here, we will use this technique to find absorption probabilities, mean hitting times, and mean return times.

#### 6.4.1 Absorption Probabilities

The *absorbing* states means that once we enter those states, we never leave them :  $p_{ii} = 1$ . Note that absorbing state is recurrent state but the inverse is not true. Consider a finite Markov chain  $\{X_n, n = 0, 1, 2, \dots\}$  with state space  $S = \{1, 2, \dots, r\}$ . Suppose that all states are either absorbing or transient. Let  $l \in S$  be **an absorbing state**. Define :

$$a_{il} = P(\text{absorption in } l | X_0 = i), \quad \text{for all } i \in S$$

By the above definition, we have  $a_{ll} = 1$ , and  $a_{lj} = 0$  if  $j$  is any other absorbing state.

To find the unknown values of  $a_{il}$ , we can use the following equations (or **law of total probability with recursion**):

$$a_{il} = \sum_k p_{ik} a_{kl}, \quad \text{for } i \in S$$

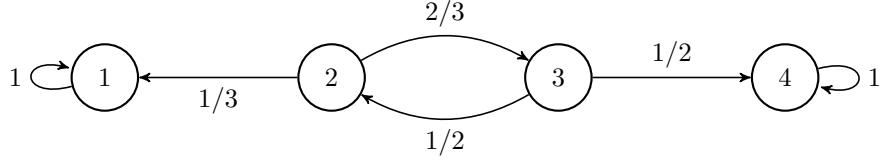


Figure 4.2: A state transition diagram.

#### 6.4.1.1 Example

Given the diagram in figure 4.2. At a first look, we can say that  $a_{11} = 1$ ,  $a_{41} = 0$ . We will find  $a_{21}$  and  $a_{31}$  by law of total probability with recursion :

$$\begin{cases} a_{21} = p_{21}a_{11} + p_{22}a_{21} + p_{23}a_{31} + p_{24}a_{41} = \frac{1}{3} + \frac{2}{3}a_{31} \\ a_{31} = p_{31}a_{11} + p_{32}a_{21} + p_{33}a_{31} + p_{34}a_{41} = \frac{1}{2}a_{21} \end{cases}$$

Then  $a_{21} = \frac{1}{2}$  and  $a_{31} = \frac{1}{4}$ .

#### 6.4.1.2 Note

If the PDF of  $X_0$ ,  $\pi^{(0)} = [P(X_0 = 1)P(X_0 = 2)\dots P(X_0 = r)]$  is given, then absorption probabilities of state  $l$  compared to other states  $j$ , without knowing state at  $X_0$  is:

$$\frac{\sum_i a_{il}P(X_0 = i)}{\sum_j \sum_i a_{ij}P(X_0 = i)}$$

#### 6.4.2 Mean First Hitting Times

Here, we would like to study the expected time until the process hits a state or a set of state for the first time.

Consider a finite Markov chain  $\{X_n, n = 0, 1, 2, \dots\}$  with state space  $S = \{1, 2, \dots, r\}$ . Given a state  $l$  in  $S$ , random variable  $T_{il}$  denotes the time that first hits state  $l$  or  $T_{iA}$  denotes the time that first hits set of state  $A$ , both from state  $i$ , then the mean first hitting times :

$$t_{il} = \mathbb{E}[T_{il}] \text{ and } t_{iA} = \mathbb{E}[T_{iA}]$$

By the above definition, we have  $t_{ll} = 0$  or  $t_{lA} = 0$  if  $l \in A$ . To find the unknown values of  $t_{il}$  or  $t_{iA}$ , we can use the law of total probability with recursion :

$$t_{il} = 1 + \sum_k p_{ik}t_{kl}, \quad \text{for } i \neq l$$

and

$$t_{iA} = 1 + \sum_k p_{ik}t_{kA}, \quad \text{for } i \notin A,$$

here the unit of time is 1.

#### 6.4.2.1 Example

Let's resume the figure 4.2, we want to the mean first hitting time for the absorbing class  $A = \{1, 4\}$ . At a first look, we can say that  $t_{1A} = 0, t_{4A} = 0$ . We will find  $t_{2A}$  and  $t_{3A}$  by law of total probability with recursion :

$$\begin{cases} t_{2A} = 1 + p_{21}t_{1A} + p_{22}t_{2A} + p_{23}t_{3A} + p_{24}t_{4A} = 1 + \frac{2}{3}t_{3A} \\ t_{3A} = 1 + p_{31}t_{1A} + p_{32}t_{2A} + p_{33}t_{3A} + p_{34}t_{4A} = 1 + \frac{1}{2}t_{2A} \end{cases}$$

Then  $t_{2A} = \frac{5}{2}$  and  $t_{3A} = \frac{9}{4}$

#### 6.4.2.2 Note

If the PDF of  $X_0, \pi^{(0)} = [P(X_0 = 1)P(X_0 = 2)\dots P(X_0 = r)]$  is given. Then the mean first hitting times  $A$  without knowing state at  $X_0$  is:

$$\sum_i t_{iA} P(X_0 = i)$$

#### 6.4.3 Mean First Return Times

Another interesting thing is the first return time. In particular, assuming the chain is in state  $l$ , we consider the expected time (number of steps) needed until the chain returns to state  $l$ .

Consider a finite Markov chain  $\{X_n, n = 0, 1, 2, \dots\}$  with state space  $S = \{1, 2, \dots, r\}$ . Let  $r_l$  be the mean return time to state  $l$ , then

$$r_l = 1 + \sum_k p_{lk} t_{kl}$$

where  $t_{kl}$  is the expected time until the chain hits state  $l$  from state  $k$ , calculated as in section 4.6.4.2. Note that  $r_l \geq 1, \forall l$  and to have the first return time, usually, we need to find first the first hitting time.

#### 6.4.3.1 Example

If resuming the figure 4.2 :

$$r_2 = 1 + p_{21}t_{12} + p_{22}t_{22} + p_{23}t_{32} + p_{24}t_{42} = 1 + p_{21}t_{12} + p_{23}t_{32} = +\infty$$

since  $p_{21} = \frac{1}{3}$  and  $t_{12} = +\infty$ .

### 6.5 Limiting and stationary distributions

#### 6.5.1 Limiting Distributions

We would like to discuss long-term behavior of Markov chains. In particular, we would like to know the distribution of  $X_n$  as  $n$  becomes large. Let  $P$  be the transition matrix. More specifically :

$$\lim_{n \rightarrow +\infty} [P(X_n = 1)P(X_n = 2) \dots P(X_n = r)] = \lim_{n \rightarrow +\infty} \pi^{(n)} = \lim_{n \rightarrow +\infty} \pi^{(0)} P^n$$

When the limiting distribution exists, it does not depend on the initial state  $X_0$ .

#### 6.5.1.1 Example

Consider a Markov chain  $X_n$  with two possible states,  $S = \{0, 1\}$ . In particular, suppose that the transition matrix is given by :

$$P = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix},$$

where  $0 < a, b < 1$ . Find the limiting distribution of  $X$ .

##### Solution

We find first  $P^n$ . The matrix  $P$  have two eigenvalues, the first  $\lambda = 1$  with eigenvector  $[1, 1]^T$  and  $\lambda = 1 - a - b$  with eigenvector  $[a, -b]^T$ . Then  $P$  can be decomposed by :

$$P = \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-a-b \end{bmatrix} \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix}^{-1}$$

Then

$$\begin{aligned} P^n &= \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1-a-b)^n \end{bmatrix} \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1-a-b)^n \end{bmatrix} \begin{bmatrix} b & a \\ 1 & -1 \end{bmatrix} \frac{1}{a+b} \\ &= \frac{1}{a+b} \begin{bmatrix} b+a(1-a-b)^n & a-a(1-a-b)^n \\ b-b(1-a-b)^n & a+b(1-a-b)^n \end{bmatrix} \end{aligned}$$

$$\text{As } \lim_{n \rightarrow +\infty} (1-a-b)^n = 0, \text{ then } \lim_{n \rightarrow +\infty} P^n = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix}$$

Finally,

$$\lim_{n \rightarrow +\infty} \pi^{(0)} P^n = [\alpha \quad 1-\alpha] \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix} = \left[ \frac{b}{a+b} \quad \frac{a}{a+b} \right]$$

#### 6.5.2 Stationary distribution

We consider Markov chains with a finite number of states. In general, a finite Markov chain can consist of several transient as well as recurrent states. As  $n$  becomes large the chain will enter a recurrent class and it will stay there forever. Therefore, when studying long-run behaviors we focus only on the recurrent classes. If a finite Markov chain has more than one recurrent class, then the chain will get absorbed in one of the recurrent classes.

Thus, we can limit our attention to the case where our Markov chain consists of one recurrent class. Consider a finite Markov chain  $\{X_n, n = 0, 1, 2, \dots\}$  with state space  $S = \{1, 2, \dots, r\}$ . Assume that the chain is irreducible and aperiodic. Then,

- The set of equations

$$\begin{aligned} \pi &= \pi P \\ s.t. \quad \sum_{j \in S} \pi_j &= 1 \end{aligned}$$

has a unique solution, that we called **stationary distribution**, which equals to limiting distribution  $\lim_{n \rightarrow \infty} \pi^{(n)}$  in sec 4.6.5.1.

- $r_j$  is the mean return time to state  $j$  and it equals to

$$r_j = \frac{1}{\pi_j}, \quad \text{for all } j \in S$$

## 6.6 Countably Infinite Markov Chains

When a Markov chain has an infinite (but countable) number of states, we need to distinguish between two types of recurrent states: *positive* recurrent and *null* recurrent states.

Let  $i$  be a recurrent state. Assuming  $X_0 = i$ , let  $R_i$  be the number of transitions needed to return to state  $i$  **in a given path**, i.e.,

$$\begin{aligned} R_i &= \min\{n \geq 1 : X_n = i\} \\ r_i &= \mathbb{E}[R_i] \end{aligned}$$

State  $i$  is said to be :

- positive recurrent if  $r_i < \infty$
- null recurrent if  $r_i = \infty$

### 6.6.1 Properties

Consider a *infinite* Markov chain  $\{X_n, n = 0, 1, 2, \dots\}$  with state space  $S = \{1, 2, \dots, \infty\}$ . Assume that the chain is irreducible and aperiodic. Then, one of the following cases can occur:

1. All states are transient, and

$$\lim_{n \rightarrow \infty} P(X_n = j | X_0 = i) = 0, \quad \forall i, j$$

2. All states are null recurrent, and

$$\lim_{n \rightarrow \infty} P(X_n = j | X_0 = i) = 0, \quad \forall i, j$$

3. All states are positive recurrent. In this case, there exists a limiting distribution,  $\pi = [\pi_1, \dots, \pi_n]$ , where

$$\pi_j = \lim_{n \rightarrow \infty} P(X_n = j | X_0 = i) > 0, \quad \forall i, j$$

$\pi$  is calculated as in 4.6.5.2.

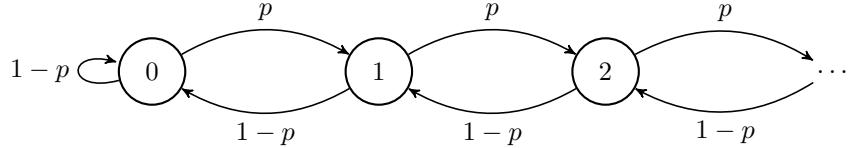
Note that, prove that all states are positive recurrent can be a hard task, then how do we use the above theorem ? We can try to find a stationary distribution  $\pi$  by solving the equations :

$$\begin{aligned} \pi &= \pi P \\ \text{s.t. } \sum_{j \in S} \pi_j &= 1 \end{aligned}$$

If the above equations have a unique solution, we conclude that the chain is positive recurrent and the stationary distribution is the limiting distribution of this chain. On the other hand, if no stationary solution exists, we conclude that the chain is either transient or null recurrent.

### 6.6.2 Example

Consider the following Markov chain :



Assume that  $0 < p < \frac{1}{2}$ . Does this chain have a limiting distribution?

#### Solution

This chain is irreducible since all states communicate with each other. It is also aperiodic since it includes a self-transition,  $P_{00} > 0$ . Let's write the equations for a stationary distribution. For state 0, we can write :

$$\pi_0 = (1 - p)\pi_0 + (1 - p)\pi_1,$$

or

$$\pi_1 = \frac{p}{1 - p}\pi_0,$$

For state 1, we can write

$$\begin{aligned} \pi_1 &= p\pi_0 + (1 - p)\pi_2 \\ &= (1 - p)\pi_1 + (1 - p)\pi_2, \end{aligned}$$

which results in

$$\pi_2 = \frac{p}{1 - p}\pi_1$$

Similarly, we have

$$\pi_i = \alpha\pi_{i-1} = \dots = \alpha^i\pi_0$$

where  $\alpha = \frac{p}{1-p}$ . Since  $0 < p < \frac{1}{2}$  then  $0 < \alpha < 1$ .

Finally, we must have

$$1 = \sum_{i=0}^{+\infty} \pi_i = \sum_{i=0}^{+\infty} \alpha^i \pi_0 = \frac{1}{1-\alpha} \pi_0$$

This implies:

$$\begin{cases} \pi_0 = 1 - \alpha \\ \pi_i = \alpha^i \pi_0 \end{cases}$$

Since we have found a stationary distribution, we conclude that all states are positive recurrent.

## 7 Continuous-Time Markov Chains

### 7.1 Preliminary question

In Discrete-Time Markov Chains, after a unit time  $\Delta = 1$  ( $X_{n\Delta}$  to  $X_{(n+1)\Delta}$ ), we stay in the same state or jump (move) to other state. Consider  $T_i$  the time that we stay in the same state  $i$ :

- If  $p_{ii} = 1$ , then  $T_i = \infty$
- If  $p_{ii} = 0$ , then  $T_i = \Delta$
- If  $1 > p_{ii} > 0$ , then probability of success for a jump is  $(1 - p_{ii})$ . Hence,  $T_i = k\Delta$ , where  $k \sim \text{Geometric}(1 - p_{ii})$ .

**Remarks important in In Discrete-Time Markov Chains :**

- After a constant time  $\Delta$ , we have the right to jump eventually to other state.
- $T_i$  the time that we stay in the same state, is inferred from  $p_{ii}$  or is dependent of  $p_{ii}$ .

Hence raised a question, what if we want to model  $T_i$  that takes a continuous value instead of discrete values ( $k\Delta$ ) ? Then Continuous-Time Markov Chains comes for this issue.

## 7.2 Definition

More specifically, we will consider a random process  $\{X(t), t \in [0, \infty)\}$ . Again, we assume that we have a countable state space  $S \subset \{1, 2, \dots\}$ . If  $X(0) = i$ , then  $X$  stays in state  $i$  for a random amount of time, say  $T_1$ , where  $T_1$  is a continuous random variable. At time  $T_1$ , the process jumps to a new state  $j$  and will spend a random amount of time  $T_2$  in that state and so on. We have the following definition:

A continuous-time Markov chain  $X(t)$  is defined by two components:

- A *jump chain* that consists transition probabilities  $p_{ij}$ .
- A set of *holding time* parameters  $\lambda_i$  of **exponential distribution**, where  $i \in S$ . **Do not confuse** this parameter with the mean time that we stay in state  $i$ , which is thus  $\frac{1}{\lambda_i}$ .

We assume  $p_{ii} = 0$  and there is no absorbing states  $i \in S$ . Then the mechanism for jumping from a state to other state is :

- If  $X(t) = i$ , the time until the state changes has  $Exponential(\lambda_i)$  distribution.
- If  $X(t) = i$ , the next state will be  $j \neq i$  with probability  $p_{ij}$ .

The process satisfies the Markov property. That is, for all times where state changes  $0 < t_1 < t_2 < \dots < t_n < t_{n+1}$ , we have :

$$\begin{aligned} P(X(t_{n+1}) = j | X(t_n) = i_n, X(t_{n-1}) = i_{n-1}, \dots, X(t_1) = i_1) \\ = P(X(t_{n+1}) = j | X(t_n) = i_n), \end{aligned}$$

where  $j \neq i_n$  and  $i_k \neq i_{k-1}, \forall k = n-1, \dots, 2$ .

### 7.2.1 Second definition

A continuous Markov chain can be sufficiently defined by the *continuous transition matrix*  $P(t)$  :

$$P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1r}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2r}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{r1}(t) & p_{r2}(t) & \dots & p_{rr}(t) \end{bmatrix}$$

where  $p_{ij}(t) = P(X(t+s) = j | X(s) = i)$ .

We must distinguish the jump chain (as transition matrix  $P$  in discrete Markov Chain case) and this transition matrix  $P(t)$  in the continuous case, which is dependent on  $t$ .

### 7.2.2 Why exponential distribution ?

Here raised a question, why exponential distribution is taken for modeling time until changing state ?

It is because its property “Memorylessness” :

$$P(X > t + s \mid X > t) = P(X > s),$$

where  $X$  means the time that state changes or event occurs. This means that the probability of an event occurring in the future is independent of how much time has already passed.

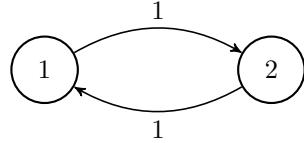
In probability and statistic domain, we have only the Geometric distribution (discret case) and Exponential distribution (continuous case) that have the Memorylessness property.

*And why Memorylessness property ?*

This is because Markov property implies the memorylessness. To make any prediction about the future, it should not matter how long the process has been in a state. Thus, the time that the process spends in each state must have a “memoryless” property. Note that, it’s the same as First arrival and Interarrivals in Poisson process in sec 4.5.1.5.

### 7.2.3 Example

Given continuous-time Markov chain  $X(t)$  with the jump chain:



Assume the holding time parameters are given by  $\lambda_0 = \lambda_1 = \lambda > 0$ , and the jump chain is  $p_{12} = p_{21} = 1$ . Find the transition matrix  $P(t)$ .

**Solution**

$$\begin{aligned} p_{11}(t) &= P(X(t) = 1 \mid X(0) = 1) \\ &= P(\text{even number of change for state}) \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{2n}}{(2n)!} \\ &= e^{-\lambda t} \sum_{n=0}^{\infty} \left[ \frac{e^{\lambda t} + e^{-\lambda t}}{2} \right]^n \quad \text{Taylor serie} \\ &= \frac{1}{2} + \frac{1}{2} e^{-2\lambda t} \end{aligned}$$

$$p_{12}(t) = 1 - p_{11}(t) = \frac{1}{2} - \frac{1}{2} e^{-2\lambda t}$$

Finally,

$$P(t) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} \end{bmatrix}$$

As mentionned in sec 4.7.2.1, it is sufficiently to use this transition matrix  $P(t)$  to describe the continuous-time Markov chain  $X(t)$ .

### 7.3 Stationary and limiting distributions

#### 7.3.1 Stationary distribution

As in the case of discrete-time Markov chains, for “nice” chains, a unique stationary distribution exists and it is equal to the limiting distribution.

Let  $X(t)$  be a continuous-time Markov chain with transition matrix  $P(t)$  and state space  $S = \{1, 2, \dots\}$ . A probability distribution  $\pi$  on  $S$ , i.e, a vector  $\pi = [\pi_1, \pi_2, \dots]$ , where  $\pi_i \in [0, 1]$  and  $\sum_{i \in S} \pi_i = 1$  is said to be a **stationary distribution** for  $X(t)$  if

$$\pi = \pi P(t), \quad \text{for all } t \geq 0.$$

This implies that if the probability distribution of  $X(0)$  is  $\pi$ , then the distribution of  $X(t)$  is also given by  $\pi$ , for any  $t \geq 0$ .

##### 7.3.1.1 Example

Let's come back at the example in 4.7.2.3. We try to solve for :

$$\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} \end{bmatrix}$$

s.t.  $\pi_0 + \pi_1 = 1$

Then we get  $\pi_0 = \pi_1 = \frac{1}{2}$

#### 7.3.2 Limiting distributions

The probability distribution  $\pi = [\pi_1, \pi_2, \dots]$  is called the limiting distribution of the continuous-time Markov chain  $X(t)$  if

$$\pi_i = \lim_{t \rightarrow \infty} P(X(t) = i)$$

and

$$\sum_{i \in S} \pi_i = 1$$

Note that, the limiting distribution equals to the stationary distribution. The latter can be checked by using the example 4.7.2.3.

### 7.3.3 A method for stationary distribution

In theory, we can find the stationary (and limiting) distribution by solving  $\pi = \pi P(t)$  or by finding  $\lim_{t \rightarrow \infty} P(X(t) = i)$ . However, in practice, finding  $P(t)$  itself is usually very difficult.

Let  $X(t), t \geq 0$  be a continuous-time Markov chain with an irreducible positive recurrent jump chain. Suppose that the unique stationary distribution of the *jump chain* (only discret transition probability  $p_{ij}$ ) is given by

$$\tilde{\pi} = [\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \dots]$$

and  $\lambda_i$  is the parameter in Exponential distribution for the time until the state changes from state  $i$ . Then

$$\pi_i = \lim_{t \rightarrow \infty} P(X(t) = i) = \frac{\frac{\tilde{\pi}_i}{\lambda_i}}{\sum_{k \in S} \frac{\tilde{\pi}_k}{\lambda_k}},$$

$$\text{if } 0 < \sum_{k \in S} \frac{\tilde{\pi}_k}{\lambda_k} < \infty.$$

An intuitive for this formula is that, one we have  $\tilde{\pi}_i$ , we go further to get  $\pi_i$  by multiplying with the average time  $\frac{1}{\lambda_i}$  that we stay in state  $i$ .

## 7.4 The generator matrix

The generator matrix, usually shown by  $G$ , gives us a way of analyzing continuous-time Markov chains, specially for getting *stationary distribution* or *limiting distribution*. It can be considered as an **operator** to get the derivative of continuous transition matrix  $P(t)$ .

### 7.4.1 Definition

Let's define matrix  $G$  as :

$$\begin{cases} g_{ii} = -\lambda_i \\ g_{ij} = \lambda_i p_{ij} \text{ with } j \neq i \end{cases}$$

where  $p_{ij}$  are discret transition probabilities in jump chaine. Note that in  $p_{ii} = 0$  by assumption in defintion of continuous Markov process. From  $\sum_{j \neq i} p_{ij} = 1$  then

$$\sum_j g_{ij} = 0.$$

Then we have that  $P'(t)$  is expressed by

- “forward equations” :

$$P'(t) = P(t)G$$

- “backward equations”:

$$P'(t) = GP(t)$$

**Proof**

$$\begin{aligned}
P'_{ij}(t) &= \frac{\partial P_{ij}(t)}{\partial t} \\
&= \lim_{\Delta \rightarrow 0} \frac{P_{ij}(t + \Delta) - P_{ij}(t)}{\Delta} \\
&= \lim_{\Delta \rightarrow 0} \frac{\sum_k P_{ik}(t)P_{kj}(\Delta) - P_{ij}(t)}{\Delta} \quad (\text{Chapman-Kolmogorov}) \\
&= \lim_{\Delta \rightarrow 0} \frac{\sum_{k \neq j} P_{ik}(t)P_{kj}(\Delta) + P_{ij}(t)P_{jj}(\Delta) - P_{ij}(t)}{\Delta} \\
&= \frac{\sum_{k \neq j} P_{ik}(t)[\lambda_k \Delta p_{kj}] + P_{ij}(t)[1 - \lambda_j \Delta] - P_{ij}(t)}{\Delta} \quad (\text{showed in next paragraph}) \\
&= \sum_{k \neq j} P_{ik}(t)\lambda_k p_{kj} - P_{ij}(t)\lambda_j \\
&= \sum_{k \neq j} P_{ik}(t)g_{kj} + P_{ij}(t)g_{jj} \\
&= \sum_k P_{ik}(t)g_{kj}
\end{aligned}$$

We need to prove:

- $\lim_{\Delta \rightarrow 0} P_{kj}(\Delta) = \lambda_k \Delta p_{kj}, \forall k \neq j$ , which means that state changes from  $k$  to  $j$  after exactly time length  $\Delta$ .
- $\lim_{\Delta \rightarrow 0} P_{jj}(\Delta) = 1 - \lambda_j \Delta$ , which means that state rests in state  $j$  after exactly time length  $\Delta$ .

$$\begin{aligned}
\lim_{\Delta \rightarrow 0} P_{kj}(\Delta) &= \lim_{\Delta \rightarrow 0} P(X(\Delta) = j | X(0) = k) \\
&= \lim_{\Delta \rightarrow 0} P(X(\Delta) \neq k | X(0) = k) p_{kj} \\
&= \lim_{\Delta \rightarrow 0} P(T < \Delta) p_{kj} \\
&= \lim_{\Delta \rightarrow 0} (1 - e^{-\lambda_k \Delta}) p_{kj} \\
&= \lambda_k \Delta p_{kj}
\end{aligned}$$

$$\begin{aligned}
\lim_{\Delta \rightarrow 0} P_{jj}(\Delta) &= \lim_{\Delta \rightarrow 0} P(X(\Delta) = j | X(0) = j) \\
&= \lim_{\Delta \rightarrow 0} [1 - P(X(\Delta) \neq j | X(0) = j)] \\
&= 1 - \lambda_j \Delta
\end{aligned}$$

### 7.4.2 Example

Let's check the property  $P'(t) = P(t)G = GP(t)$  in example 4.7.2.3. The generator matrix  $G$  is

$$G = \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}$$

Then

$$P'(t) = \begin{bmatrix} -\lambda e^{-2\lambda t} & \lambda e^{-2\lambda t} \\ \lambda e^{-2\lambda t} & -\lambda e^{-2\lambda t} \end{bmatrix}$$

$$\begin{aligned} P(t)G &= \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix} \\ &= \begin{bmatrix} -\lambda e^{-2\lambda t} & \lambda e^{-2\lambda t} \\ \lambda e^{-2\lambda t} & -\lambda e^{-2\lambda t} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} GP(t) &= \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix} \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} \end{bmatrix} \\ &= \begin{bmatrix} -\lambda e^{-2\lambda t} & \lambda e^{-2\lambda t} \\ \lambda e^{-2\lambda t} & -\lambda e^{-2\lambda t} \end{bmatrix} \end{aligned}$$

### 7.4.3 Corollary

Consider a continuous Markov chain  $X(t)$  with the state space  $S$  and the generator Matrix  $G$ . The probability distribution  $\pi$  on  $S$  is a *stationary distribution* for  $X(t)$  if and only if it satisfies:

$$\pi G = 0$$

Suppose that we have :  $\pi = \pi P(t), \forall t$

$$\begin{aligned} \frac{d}{dt}[\pi] &= \frac{d}{dt}[\pi P(t)], \quad \forall t \\ \Leftrightarrow 0 &= \pi P'(t), \quad \forall t \\ \Leftrightarrow 0 &= \pi GP(t), \quad \forall t \end{aligned}$$

As  $P(0) = P(t=0) = I$  the identity matrix, this implies  $\pi G = 0$

Inversely, suppose that  $\pi G = 0$ , then

$$\pi P'(t) = \pi GP(t) = 0, \forall t$$

Then  $\pi P(t) = C \in \mathbb{R}^r, \forall t$ . With  $P(0) = I$ , this implies  $C = \pi$ . Finally,  $\pi P(t) = \pi$

### 7.4.3.1 Example

Let's resum in example 4.7.2.3. The generator matrix  $G$  is

$$G = \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}$$

Then to find the stationary distribution, we need to solve :

$$\pi G = 0 = [\pi_0 \quad \pi_1] \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix} = 0$$

s.t  $\pi_0 + \pi_1 = 1$

Then  $\pi_0 = \pi_1 = \frac{1}{2}$ .

### 7.4.4 Why named generator matrix ?

Consider a continuous-time Markov chain  $X(t)$ . Assume  $X(t_0) = i$ . The chain will jump to the next state at time  $t_0 + T$ , where  $T \sim \text{Exponential}(\lambda_i)$ . We have (in Exponential distribution) that

$$\lim_{\Delta \rightarrow 0} P(T < \Delta) = \lambda_i \Delta$$

Thus, in a short interval of length  $\Delta$ , the probability of leaving state  $i$  is approximately  $\lambda_i \Delta$ . For this reason,  $\lambda_i$  is often called the transition rate out of state  $i$ . Furthermore, since we go from state  $i$  to state  $j$  with probability  $p_{ij}$ , then  $g_{ij} = \lambda_i p_{ij}$  is the transition rate from state  $i$  to state  $j$ .

Therefore, we call "generator matrix"  $G$  because it is something characterizing the dynamics for a continuous Markov process  $X_t$ , which means they contain all sufficient information to describe  $X_t$  or to **generate realizations**, as the continuous transition matrix  $P(t)$  does. There is an equivalence between two concepts, which means once we know one of them, we can infer the other :

- If we know  $P(t)$ , then  $G = P(t)^{-1}P'(t)$ . Since the latter is trur for all  $t$ , we take  $t = 0$ . Then  $G = P(0)^{-1}P'(0) = P'(0)$  (since  $P(0) = I$ ).
- If we know  $G$ , then we get  $P(t)$  by solving the system of differential equations :

$$P'(t) = GP(t),$$

where  $P(0) = I$ .

This generator matrix  $G$  is also known as the "infinitesimal generator" (4.18) or the "Q-matrix."

## 8 Brownian Motion

In mathematics, this is also described by the *Wiener process*, a continuous-time stochastic process named in honor of Norbert Wiener. The Wiener process  $W_t$  is characterized by four facts :

1.  $W_0 = 0$  almost surely.
2.  $W_t$  is almost surely continuous (sec 4.1.5.2),  $P(\lim_{t \rightarrow t_0} W_t = W_{t_0}) = 1$ .
3.  $W_t$  has independent increments : for all  $0 \leq s_1 < t_1 \leq s_2 < t_2$ , increments  $W_{t_1} - W_{s_1}$  and  $W_{t_2} - W_{s_2}$  are independent. By 4.1.8, the alternative of this condition is  $W_t - W_s$  and  $W_s$  are independent,  $\forall s < t$ .
4.  $W_t - W_s \sim \mathcal{N}(0, |t - s|)$

## 8.1 Notes

Given a fixed  $\Delta_t$ , let's call  $\Delta W = W_t - W_{t+\Delta_t}$ , then :

1.  $\mathbb{E}[\Delta W] = 0$
2.  $\mathbb{E}[(\Delta W)^2] = \text{Var}[\Delta W] = \Delta_t$
3.  $\text{Var}[(\Delta W)^2] = \mathbb{E}[(\Delta W)^4] - \mathbb{E}[(\Delta W)^2]^2 = 3\Delta_t^2 - \Delta_t^2 = 2\Delta_t^2$ . Remind that  $\mathbb{E}[(\Delta W)^{2n}] = (\Delta_t)^n (2n-1)!!$ , where  $!!$  means double factorial (e.g.  $5!! = 1.3.5$ ) and here  $n = 2$ .
4. If  $\Delta_t \rightarrow 0$ , from the third observation, then we can consider that  $\Delta_t^2 = 0$ , hence  $\text{Var}[(\Delta W)^2] = 0$ . Note that if variance of a random variable  $X$  is 0, this implies that  $X$  is constant. Then  $(\Delta W)^2$  is constant. From second observation,  $\mathbb{E}[(\Delta W)^2] = \Delta_t$ . Finally, we have  $(\Delta W)^2 = \mathbb{E}[(\Delta W)^2] = \Delta_t$ .
5. A Brownian motion is known that *everywhere continuous but nowhere differentiable*. This is because :
  - A differentiable curve means that when zooming in, the locality of curve is more and more as a straight line. But this is not the case for a trajectory of Brownian motion.
  - Given a point, the slope at its left side and its right side are stochastic. Therefore they are not the same and this is not differentiable.

Then  $dW_t$  exists only in a stochastic definition but not deterministic definition.

Sometime, we use the notation  $B_t$  instead of  $W_t$ .

## 8.2 Quadratic variation

In mathematics, quadratic variation is used in the analysis of stochastic processes such as Brownian motion and other martingales. Quadratic variation is something like the *variation* of a process.

Suppose that  $X_t$  is a real-valued stochastic process defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and with time index  $t$  ranging over the non-negative real numbers. Its quadratic variation is a **stochastic process**, written as  $[X]_t$  or  $\langle X \rangle_t$ , defined as :

$$[X]_t = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (X_{t_k} - X_{t_{k-1}})^2$$

where  $P$  ranges over partitions of the interval  $[0, t]$  and the norm of the partition  $P$  is the mesh, which is the  $\max_k |t_k - t_{k-1}|$ .

### 8.2.1 Covariation of two processes

Given two processes  $X_t$  and  $Y_t$ , this is also called cross-variance and denotes :

$$[X, Y]_t = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (X_{t_k} - X_{t_{k-1}}) (Y_{t_k} - Y_{t_{k-1}}).$$

We can say that this covariation is something that corresponds to covariance of two random variables.

#### 8.2.1.1 Properties

- In the case  $X = Y$ , we have that covariation  $[X, X]_t$  is equal to quadratic variation  $[X]$ .
- $[X, Y]_t = \int_0^t dX_s dY_s$
- $[X, Y]_t = \frac{1}{2}([X + Y]_t - [X]_t - [Y]_t)$
- If  $X_t$  and  $Y_t$  are independent Brownian motion, then  $[X, Y]_t = 0$ . The demonstration is by showing that  $\mathbb{E}[[X, Y]_t^2 - 0] = 0$  as in exercice 4.8.2.2. This infers that  $d[X, Y]_s = dX_s dY_s = 0$
- If  $Z_1 = a_1 X_1 + b_1 Y_1$  and  $Z_2 = a_2 X_2 + b_2 Y_2$ , it is trivial to prove (by definition) that :

$$[Z_1, Z_2]_t = a_1 a_2 [X_1, X_2]_t + a_1 b_2 [X_1, Y_2]_t + b_1 a_2 [Y_1, X_2]_t + b_1 b_2 [Y_1, Y_2]_t$$

If  $Z_1 = Z_2 = Z = aX + bY$ , then :

$$[Z]_t = a^2 [X]_t + 2ab [X, Y]_t + b^2 [Y]_t$$

### 8.2.2 Exercice

Show that with a partition  $\Pi$ , given  $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq \dots \leq t_n = T$ , then the quadratic variation of Wiener process  $W_t$  is:

$$[W]_T = \lim_{\|\Pi\| \rightarrow 0} \sum_{k=1}^n (W_{t_k} - W_{t_{k-1}})^2 = T$$

**Solution.** Observation, if  $(t_k - t_{k-1}) = \frac{T}{n}, \forall k$ , then  $[W]_T = \sum \frac{T}{n} \varepsilon^2 = T \mathbb{E}[\varepsilon^2] = T$ , where  $\varepsilon \leftarrow \mathcal{N}(0, 1)$

We prove that:

$$\mathbb{E} \left[ ([W]_T - T)^2 \right] = 0$$

$$\begin{aligned}
\mathbb{E} \left[ ([W]_T - T)^2 \right] &= \text{Var}([W]_T - T) + \mathbb{E}^2 \left[ ([W]_T - T) \right] \\
&= \text{Var}([W]_T) + \left( \sum_{k=1}^n \mathbb{E} \left[ (W_{t_k} - W_{t_{k-1}})^2 \right] - T \right)^2 \\
&= \sum_{k=1}^n \text{Var} \left[ (W_{t_k} - W_{t_{k-1}})^2 \right] + \left( \sum_{k=1}^n (t_k - t_{k-1}) - T \right)^2 \\
&= \sum_{k=1}^n \mathbb{E} \left[ (W_{t_k} - W_{t_{k-1}})^4 \right] - \sum_{k=1}^n \mathbb{E}^2 \left[ (W_{t_k} - W_{t_{k-1}})^2 \right] + (t_n - t_0 - T)^2 \\
&= \sum_{k=1}^n 3(t_k - t_{k-1})^2 - \sum_{k=1}^n (t_k - t_{k-1})^2 + (T - t_0)^2 \\
&= \sum_{k=1}^n 2(t_k - t_{k-1})^2 \\
&\leq 2 \max_k (t_k - t_{k-1}) \left( \sum_{k=1}^n (t_k - t_{k-1}) \right) \\
&= 2 \max_k (t_k - t_{k-1}) T
\end{aligned}$$

Note that, if  $X \sim \mathcal{N}(0, \sigma^2)$ , then  $\mathbb{E}(X^{2n}) = (\sigma^2)^n (n2 - 1)!!$ , where double factorial is, e.g.  $7!! = 7 \times 5 \times 3 \times 1$ . This can be shown by using partial integral on  $\int x^{2n} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \frac{x^2}{\sigma^2}} dx$

We observe that when  $\max_k (t_k - t_{k-1}) \rightarrow 0$  then  $\mathbb{E} \left[ ([W]_T - T)^2 \right] \rightarrow 0$ , hence  $[W]_T = T$ .

### 8.3 Multidimensional Brownian motion

We discover here the multidimensional version of Brownian motion, which is very commonly used in continuous-time market models. When modelling simultaneously several risky asset prices, there may exist dependences between assets. That is why we would like to build a multidimensional Brownian motion, where components can be correlated. Let's

1.  $\Gamma = (\gamma_{ij})_{i,j \in \{1,2,\dots,n\}}$  is a matrix of constants.
2.  $\mu \in \mathbb{R}^n$

3.  $\mathbf{W}_t = (W_t^{(1)}, \dots, W_t^{(n)})^T$  is a vector made of independent standard Brownian motions. It is also called multidimensional standard Brownian motion.
4. For all  $t$ , let's set  $\mathbf{B}_t = \Gamma \mathbf{W}_t$ .

Then  $B_t$  is a multidimensional Brownian motion. This construction is similar as the construction of multivariate normal distribution from standard normal and independent distribution (2.6.3.4). Moreover, let's  $\Sigma = \Gamma \Gamma^T$  :

$$Cov(B_t^{(i)}, B_t^{(j)}) = t \sum_{k=1}^n \gamma_{ik} \gamma_{jk} = t \Sigma_{ij} \quad (\text{covariation not covariance})$$

$$Corr(B_t^{(i)}, B_t^{(j)}) = \frac{\sum_{k=1}^n \gamma_{ik} \gamma_{jk}}{\sqrt{\sum_{k=1}^n \gamma_{ik}^2} \sqrt{\sum_{k=1}^n \gamma_{jk}^2}} = \frac{Cov(B_t^{(i)}, B_t^{(j)})}{\sqrt{Var(B_t^{(i)})} \sqrt{Var(B_t^{(j)})}}$$

In the above definition,  $\mu$  and  $\Gamma$  are two parameters that characterize multidimensional Brownian motion. In practice, if we have data of  $\mathbf{B}_t$  until certain instant  $t_0$ , then we can use this data to infer  $\mu$  and  $\Gamma$ , then use these estimated parameters for prediction.

#### 8.4 Integral of Brownian motion w.r.t. time

It is recommended to read section Stochastic Integrals (4.16) first in order to get some necessary concepts.

Let  $B_t$  be a Brownian motion, we define :

$$X_t = \int_0^t B_s ds$$

Some question may be raised here :

1. Expectation ?
2. Variance ?
3. Is it a martingale ?
4. Is it an Ito process or a Riemann integral ?

Suppose that we work in probability space  $(\Omega, \mathcal{F}_t, P)$ . The integral of Brownian motion is basically that, for each outcome  $\omega \in \Omega$ , we can treat  $\int_0^t B_s(\omega) ds$  as a Riemann integral. Moreover, note that

$$d(tB_t) = B_t dt + t dB_t$$

Therefore,

$$\begin{aligned}
\int_0^t B_s ds &= tB_t - \int_0^t s dB_s \\
&= t \int_0^t dB_s - \int_0^t s dB_s \\
&= \int_0^t (t-s) dB_s
\end{aligned}$$

Here we can see that  $X_t$  is an Ito process. Next, for expectation :

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}\left[\int_0^t B_s ds\right] \\
&= \mathbb{E}\left[\int_0^t (t-s) dB_s\right] \\
&= \int_0^t (t-s) \mathbb{E}[dB_s] \\
&= 0
\end{aligned}$$

For variance :

$$\begin{aligned}
Var(X_t) &= Var\left(\int_0^t B_s ds\right) \\
&= Var\left(\int_0^t (t-s) dB_s\right) \\
&= \int_0^t (t-s)^2 Var(dB_s) \\
&= \int_0^t (t-s)^2 ds \\
&= \frac{t^3}{3}
\end{aligned}$$

Regarding the martingality, note that

$$\begin{aligned}
\int_0^{t_2} B_s ds - \int_0^{t_1} B_s ds &= t_2 B_{t_2} - \int_0^{t_2} s dB_s - t_1 B_{t_1} + \int_0^{t_1} s dB_s \\
&= t_2 B_{t_2} - t_1 B_{t_1} - \int_{t_1}^{t_2} s dB_s \\
&= t_2 (B_{t_2} - B_{t_1}) + (t_2 - t_1) B_{t_1} - \int_{t_1}^{t_2} s dB_s
\end{aligned}$$

$$\begin{aligned}
&= t_2 \int_{t_1}^{t_2} dB_s + (t_2 - t_1)B_{t_1} - \int_{t_1}^{t_2} s dB_s \\
&= (t_2 - t_1)B_{t_1} + \int_{t_1}^{t_2} (t_2 - s) dB_s
\end{aligned}$$

For  $t_2 > t_1 > 0$ . Therefore,

$$\begin{aligned}
\mathbb{E}[X_{t_2} | \mathcal{F}_{t_1}] &= \mathbb{E}\left[\int_0^{t_2} B_s ds | \mathcal{F}_{t_1}\right] \\
&= \mathbb{E}\left[\int_0^{t_1} B_s ds + (t_2 - t_1)B_{t_1} + \int_{t_1}^{t_2} (t_2 - s) dB_s | \mathcal{F}_{t_1}\right] \\
&= \int_0^{t_1} B_s ds + (t_2 - t_1)B_{t_1} + \int_{t_1}^{t_2} (t_2 - s)\mathbb{E}[dB_s | \mathcal{F}_{t_1}] \\
&= \int_0^{t_1} B_s ds + (t_2 - t_1)B_{t_1} \\
&= X_1 + (t_2 - t_1)B_{t_1}
\end{aligned}$$

We see that  $X_t$  is not martingale since  $(t_2 - t_1)B_{t_1} \neq 0$ .

## 9 Lévy process

A Lévy process is a stochastic process  $X_{t(t \geq 0)}$  that satisfies the following properties:

1.  $X_0 = 0$  almost surely.
2. Continuity in probability: For any  $\varepsilon > 0$ , it holds that  $\lim_{h \rightarrow 0} P(|X_{t+h} - X_t| > \varepsilon) = 0$ .
3. Independence of increments: For any  $0 \leq t_1 < t_2 < \dots < t_n < \infty$ , we have  $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$  are mutually independent.
4. Stationary increments : For any  $s < t$ ,  $X_t - X_s$  is equal in distribution to  $X_{t-s}$ .

The last condition is noted by  $X \stackrel{d}{=} Y$ , which means  $P(X \leq z) = P(Y \leq z), \forall z$ . To be equal in distribution, random variables *need not be defined on the same probability space*. Two random variables having equal moment generating functions (2.5.1) have the same distribution.

## 9.1 Properties

Levy process is the general case for many well-known process, such as :

- If  $X_t - X_s \sim \mathcal{N}(0, |t-s|)$  and the continuity in probability becomes continuity almost surely, we have Weiner process (4.8).
- If  $X_t - X_s$  follows Poisson distribution with expected value  $\lambda(t-s)$ , we have Poisson process 4.5.

## 9.2 Moments

If Levy process has finite moments at every instant  $t$ , the  $n^{th}$  moment at instant  $t$   $\mu_n(t) = \mathbb{E}[X_t^n]$

- is a polynomial function of  $t$  :

$$\mu_n(t) = a_m t^m + \dots + a_1 t + a_0$$

- satisfies a binomial relation :

$$\mu_n(t+s) = \sum_{k=0}^n \binom{n}{k} \mu_k(t) \mu_{n-k}(s)$$

# 10 Martingale

## 10.1 Martingale

A martingale is a stochastic process which stays the same, on average. That is, the expected future value conditional on the present is equal to the current value.

### 10.1.1 Definition

Given probability space  $(\Omega, \mathcal{F}_t, P)$ , where  $\mathcal{F}_t$  is filtration, a martingale is that a stochastic process that satisfies,

- in discret case:
  - $\mathbb{E}(|X_n|) < \infty$  (integrable)
  - $\mathbb{E}(X_{n+1} | X_n, \dots, X_1) = X_n$
- in continuous case:
  - $\mathbb{E}(|X_t|) < \infty$
  - $\mathbb{E}(X_t | \{X_\tau, \tau \leq s\}) = X_s$  or  $\mathbb{E}(X_t | \mathcal{F}_s) = X_s \quad \forall s < t$

where the second condition (conditional expectation 2.2.12) is equivalent to,  
 $\forall s < t$ ,

$$\begin{aligned} \mathbb{E}[X_t | \mathcal{F}_s] &= \mathbb{E}[X_s | \mathcal{F}_s] \\ \Leftrightarrow \mathbb{E}[X_t - X_s | \mathcal{F}_s] &= 0 \\ \Leftrightarrow \mathbb{E}[X_t - X_s | A] &= 0, \quad \forall A \in \mathcal{F}_s \\ \Leftrightarrow \frac{\mathbb{E}[(X_t - X_s) \mathbb{1}_A]}{P(A)} &= 0, \quad \forall A \in \mathcal{F}_s \text{ ( see 2.2.12.4.1)} \\ \Leftrightarrow \mathbb{E}[(X_t - X_s) \mathbb{1}_A] &= 0, \quad \forall A \in \mathcal{F}_s \end{aligned}$$

### 10.1.2 Properties

#### 10.1.2.1 Constant expectation

Given  $X_t$  is a martingale, then

$$\mathbb{E}[X_t] = \mathbb{E}[X_0]$$

##### Proof

By using law of iterated expectations (2.2.13.1.1) :

$$\mathbb{E}_s[\mathbb{E}_t[X_t | \mathcal{F}_s]] = \mathbb{E}_t[X_t]$$

Then from the definition of martingale :

$$\begin{aligned} \mathbb{E}_t[X_t | \mathcal{F}_s] &= X_s \\ \Rightarrow \mathbb{E}_s[\mathbb{E}_t[X_t | \mathcal{F}_s]] &= \mathbb{E}_s[X_s] \\ \Rightarrow \mathbb{E}_t[X_t] &= \mathbb{E}_s[X_s] \quad s \leq t \end{aligned}$$

By analogy, in discret case:

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(X_n) = \dots = \mathbb{E}(X_0)$$

#### 10.1.2.2 Linear combination of two martingales

Given two martingales  $X_t$  and  $Y_t$  are both adapted to filtration  $\mathcal{F}_t$ , then

$$aX_t + bY_t$$

is also a martingale,  $\forall a, b \in \mathbb{R}$ .

##### Proof

$$\begin{aligned} \mathbb{E}[aX_t + bY_t | \mathcal{F}_s] &= a\mathbb{E}[X_t | \mathcal{F}_s] + b\mathbb{E}[Y_t | \mathcal{F}_s] \\ &= aX_s + bY_s \end{aligned}$$

In addition  $\mathbb{E}[|aX_t + bY_t|] < \infty$  since  $X_t$  and  $Y_t$  are both integrable. Then we have Q.E.D

### 10.1.3 Examples

An example of martingale is the wealth of a gambler as a function of time, assuming that he is playing a fair game. Furthermore, a Brownian motion (4.8) is continuous time martingale and a symmetric random walk is discrete time martingale.

- A gambler's fortune (capital) is a martingale if all the betting games which the gambler plays are fair. To be more specific: suppose  $X_n$  is a gambler's fortune after  $n$  tosses of a fair coin, where the gambler wins \$1 if the coin comes up heads and loses \$1 if it comes up tails. The gambler's conditional expected fortune after the next trial, given the history, is equal to their present fortune. This sequence is thus a martingale.
- $X_n^2 - n$ , where  $X_n$  in the above exemple is also martingale. In general, if  $X_{n+1} = X_n + \epsilon$ ,  $\mathbb{E}[\epsilon] = 0$ ,  $\text{Var}(\epsilon) = 1$ ,  $\epsilon$  and  $X_n$  are independent, hence  $X_n^2 - n$  is martingale :

$$\begin{aligned}\mathbb{E}(X_{n+1}^2 - n - 1) &= \mathbb{E}(X_n^2 + 2X_n\epsilon + \epsilon^2) - n - 1 \\ &= \mathbb{E}(X_n^2) - 2\mathbb{E}(X_n)\mathbb{E}(\epsilon) + \mathbb{E}(\epsilon^2) - n - 1 \\ &= \mathbb{E}(X_n^2) - n\end{aligned}$$

#### 10.1.3.1 Example of not martingale

Let's consider an biased random walk. We start at position 0. At each time step, flip a biased coin: If it's heads with probability 0.75, take a step to the right (increase by 1). If it's tails with probability 0.25, take a step to the left (decrease by 1).

Let  $X_i$  be our position at time instant  $i$ . We check if  $X_i$  is a martingale:

- First,  $\mathbb{E}[X_1] = 0.75 - 0.25 = 0.5$  and  $\mathbb{E}[X_i] = \frac{1}{2}i$ , so  $\mathbb{E}[X_i] < \infty$  (ok).
- Second,  $\mathbb{E}(X_{n+1}|X_n, \dots, X_1) = X_n + 0.5$ , which is not equals to  $X_n$  (not ok).

Hence,  $X_i$  is not martingale.

### 10.1.4 Martingales and stopping times

If  $X_t$  is a martingale and  $\tau$  is a stopping time (a random variable, 4.1.9), both with filtration  $\mathcal{F}_t$ , then  $X_t^\tau$  is a martingale.

**Proof** (for discret case)

1. Let's consider the discret case with martingale  $X_n$  :

$$X_n^\tau = X_{n \wedge \tau}$$

$$\begin{aligned}
&= \mathbb{1}_{\{\tau > n\}} X_n + \mathbb{1}_{\{\tau \leq n\}} X_\tau \\
&= \mathbb{1}_{\{\tau > n\}} X_n + \sum_{k=1}^n \mathbb{1}_{\{\tau = k\}} X_k
\end{aligned}$$

Since  $\mathbb{1}_{\{\tau > n\}}$ ,  $X_n$ ,  $\mathbb{1}_{\{\tau = k\}}$  and  $X_k$  are all  $\mathcal{F}_n$ -measurable, then  $X_n^\tau$  is also  $\mathcal{F}_n$ -measurable.

2. At instant  $n$ ,

$$\begin{aligned}
\mathbb{E}[|X_n^\tau|] &= \mathbb{E}[|\mathbb{1}_{\{\tau > n\}} X_n + \sum_{k=1}^n \mathbb{1}_{\{\tau = k\}} X_k|] \\
&\leq \mathbb{E}[|\mathbb{1}_{\{\tau > n\}} X_n|] + \sum_{k=1}^n \mathbb{E}[|\mathbb{1}_{\{\tau = k\}} X_k|] \\
&< \mathbb{E}[|X_n|] + \sum_{k=1}^n \mathbb{E}[|X_k|] \\
&< \infty
\end{aligned}$$

Then  $X_n^\tau$  is integrable (has finite expectation) with all  $n$ .

3. Finally,

$$\begin{aligned}
\mathbb{E}[X_{n+1}^\tau \mid \mathcal{F}_n] &= \mathbb{E}[X_n^\tau + \mathbb{1}_{\{\tau \geq n+1\}}(X_{n+1} - X_n) \mid \mathcal{F}_n] \\
&= \mathbb{E}[X_n^\tau \mid \mathcal{F}_n] + \mathbb{E}[\mathbb{1}_{\{\tau \geq n+1\}}(X_{n+1} - X_n) \mid \mathcal{F}_n] \\
&= X_n^\tau + \begin{cases} \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] - X_n & \text{if } \tau \geq n+1 \\ 0 & \text{otherwise} \end{cases} \\
&= X_n^\tau + \begin{cases} 0 & \text{if } \tau \geq n+1 \quad (\text{since } X_n \text{ is martingale)} \\ 0 & \text{otherwise} \end{cases} \\
&= X_n^\tau
\end{aligned}$$

In continuous case, it is more difficult since we need to prove that  $\mathbb{E}[X_\tau]$  is finite.

## 10.2 Submartingale

Submartingale is a stochastic process that is integrable and

$$\mathbb{E}[X_t \mid \mathcal{F}_s] \geq X_s, \quad \forall s < t,$$

which means that the process increases on average.

By analogy to martingale, with law of iterated expectation, a property of submartingale is :

$$\mathbb{E}[X_t] \geq \mathbb{E}[X_s], \quad \forall s < t$$

### 10.2.1 Lemma

If  $X$  is a martingale and  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a convex function such that  $f(X)$  is integrable, then,  $f(X)$  is a submartingale. This is a direct consequence of conditional Jensen's inequality (sec 2.2.8.3.1) :

$$\mathbb{E}[f(X_t) | \mathcal{F}_s] \geq f(\mathbb{E}[X_t | \mathcal{F}_s]) = f(X_s)$$

### 10.2.2 Lemma

In the case that  $X$  is a submartingale, beside the condition that  $f$  is convex, we need a supplement condition that  $f$  is **increasing** function, which implies:

$$\mathbb{E}[f(X_t) | \mathcal{F}_s] \geq f(\mathbb{E}[X_t | \mathcal{F}_s]) \geq f(X_s)$$

then  $f(X)$  is submartigale.

### 10.2.3 Lemma

If  $X$  is a submartingale and  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a **decreasing** and **concave** function such that  $f(X)$  is integrable, then,  $f(X)$  is a **supermartingale**. By Conditional Jensen's Inequality 2.2.8.3.1 in case of concave function and  $f$  is decreasing function, we have:

$$\mathbb{E}[f(X_t) | \mathcal{F}_s] \leq f(\mathbb{E}[X_t | \mathcal{F}_s]) \leq f(X_s)$$

### 10.2.4 Inequality

If  $X$  is a submartingale and  $\mathbb{1}_A$  is an indicator function ( $A \in \mathcal{F}_s$ ), then

$$\mathbb{E}[X_s \mathbb{1}_A] \leq \mathbb{E}[X_t \mathbb{1}_A], \quad s < t$$

This corresponds to the equality in sec 4.10.1.1.

**Proof**

$$\begin{aligned} \frac{\mathbb{E}[X_s \mathbb{1}_A]}{P(A)} &= \mathbb{E}[X_s | A] \\ &\leq \mathbb{E}[\mathbb{E}[X_t | \mathcal{F}_s] | A] \\ &= \mathbb{E}[X_t | A] \\ &= \frac{\mathbb{E}[X_t \mathbb{1}_A]}{P(A)} \end{aligned}$$

where the one before last equality is by the general case law of iterated expectation (2.2.13.3). Here  $\mathcal{H}_2 = \mathcal{F}_s$  and  $\mathcal{H}_1 = \{\Omega, \emptyset, A, A^C\}$ . We see that  $\mathcal{H}_1 \subseteq \mathcal{H}_2$  since  $A \in \mathcal{F}_n$ .

### 10.3 Supermartingale

Supermartingale is a stochastic process that is integrable and

$$\mathbb{E}[X_t \mid \mathcal{F}_s] \leq X_s, \quad \forall s < t,$$

which means that the process decreases on average. As in case of submartingale in 4.10.2, we have also a respective property and lemma for supermartingale.

### 10.4 Doob decomposition theorem

Before diving into the statement of Doob decomposition theorem, we discover first the concept of predictable process.

#### 10.4.1 Predictable process

<https://math.stackexchange.com/questions/2190739/what-exactly-is-a-predictable-process>

Let's take example comes from quantitative finance. Suppose we have a certain amount of money  $V_n$  at a certain (discrete) time  $t_n$ . We decide to invest a certain percentage  $\alpha_n$  of this money in a risky title which has value  $S_n$  at time  $t_n$ .

$S_n$  can be modeled as a random variable and also  $\alpha_n$ . We conceptualize the sigma algebra  $\mathbb{F}_n$  generated by  $S_n$  as the information from the values of  $S$  in the first  $n$  periods.

What makes  $S_n$  and  $\alpha_n$  different is that we **have control** of the amount of money  $\alpha_{n+1}$  that we want to invest at the time  $t_n$  in the title  $S$ , because this decision must be made before  $t_{n+1}$ . In other words, the value of  $\alpha_{n+1}$  must depend exclusively on a measurable function of the first  $n$  values of  $S$ .

We translate this concept to math as :

$$\alpha_{n+1} \in \mathcal{F}_n$$

or equivalently,

$$\mathbb{E}[\alpha_{n+1} \mid \mathcal{F}_n] = \alpha_{n+1}$$

Note that :

- Every deterministic process is a predictable process.
- Every continuous time adapted process that is **left continuous** is obviously a predictable process. This infer that Brownian motion (4.8) is predictable since it is almost surely continuous then left continuous.

#### 10.4.2 Statement

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $I = \{0, 1, 2, \dots, N\}$ ,  $(\mathcal{F}_n)_{n \in I}$  a filtration and  $(X_n)_{n \in I}$  an adapted stochastic process with  $\mathbb{E}[|X_n|] < \infty$  for all  $n \in I$ . Then

there exists a martingale  $M = (M_n)_{n \in I}$  and an integrable predictable process  $A = (A_n)_{n \in I}$  starting with  $A_0 = 0$  such that  $X_n = M_n + A_n$  for every  $n \in I$ .

Here predictable means that  $A_n$  is  $\mathcal{F}_{n-1}$ -measurable for every  $n \in I \setminus \{0\}$  or informally,  $A_n(\omega)$  is known with  $\omega \in \mathcal{F}_{n-1}$  or  $\mathbb{E}[A_n | \mathcal{F}_{n-1}] = A_n$ .

This decomposition is almost surely unique.

#### 10.4.3 Proof

##### Existence

Using conditional expectations (sec 2.2.13.1.1), let's define the processes  $A$  and  $M$ , explicitly by

$$A_n = \begin{cases} \sum_{k=1}^n (\mathbb{E}[X_k | \mathcal{F}_{k-1}] - X_{k-1}) & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

$$M_n = \begin{cases} X_0 + \sum_{k=1}^n (X_k - \mathbb{E}[X_k | \mathcal{F}_{k-1}]) & \text{if } n \geq 1 \\ X_0 & \text{if } n = 0 \end{cases}$$

Then we clearly see that  $X_n = M_n + A_n$ . By definition of  $A_n$ , we see that it is  $\mathcal{F}_{n-1}$ -measurable. First,  $\mathbb{E}[|A_n|] < \infty$  and  $\mathbb{E}[|M_n|] < \infty$  since  $\mathbb{E}[|X_n|] < \infty, \forall n$ . Second,

$$\begin{aligned} & \mathbb{E}[M_n | \mathcal{F}_{n-1}] \\ &= \mathbb{E} \left[ X_0 + \sum_{k=1}^n (X_k - \mathbb{E}[X_k | \mathcal{F}_{k-1}]) \mid \mathcal{F}_{n-1} \right] \\ &= \mathbb{E}[X_n | \mathcal{F}_{n-1}] - \mathbb{E}[\mathbb{E}[X_n | \mathcal{F}_{n-1}] | \mathcal{F}_{n-1}] + \mathbb{E}[X_0 | \mathcal{F}_{n-1}] + \sum_{k=1}^{n-1} (\mathbb{E}[X_k | \mathcal{F}_{n-1}] - \mathbb{E}[\mathbb{E}[X_k | \mathcal{F}_{k-1}] | \mathcal{F}_{n-1}]) \\ &= \mathbb{E}[X_0 | \mathcal{F}_{n-1}] + \sum_{k=1}^{n-1} (\mathbb{E}[X_k | \mathcal{F}_{n-1}] - \mathbb{E}[\mathbb{E}[X_k | \mathcal{F}_{k-1}] | \mathcal{F}_{n-1}]) \\ &= X_0 + \sum_{k=1}^n (X_k - \mathbb{E}[X_k | \mathcal{F}_{k-1}]) \\ &= M_{n-1} \end{aligned}$$

where  $\mathbb{E}[X_n | \mathcal{F}_{n-1}] - \mathbb{E}[\mathbb{E}[X_n | \mathcal{F}_{n-1}] | \mathcal{F}_{n-1}] = 0$  by general law of total expectation in sec 2.2.13.3 and  $\mathbb{E}[X_k | \mathcal{F}_{n-1}]$  with  $k \leq n-1$  is thus  $X_k$  since we know value of process until instant  $n-1$ . Note that, the law of total expectation is not applicable for  $\mathbb{E}[\mathbb{E}[X_k | \mathcal{F}_{k-1}] | \mathcal{F}_{n-1}]$  since  $\mathbb{E}[X_k | \mathcal{F}_{k-1}]$  is already known if given  $\mathcal{F}_{n-1}$ .

##### Uniqueness

Now suppose that we can decompose  $X_n$  by  $X_n = A_n + M_n$  and  $X_n = A'_n + M'_n$ , let's consider  $Y_n = M_n - M'_n = A'_n - A_n$ . Since  $M_n$  and  $M'_n$  are martingale adapted to  $\mathcal{F}_n$  then  $M_n - M'_n$  is also martingale (sec 4.10.1.2.2). In

addition,  $A_n$  and  $A'_n$  are predictable process then  $A'_n - A_n$  is also predictable process (trivial demonstration). Then  $Y_n$  is martingale :

$$\mathbb{E}[Y_n | \mathcal{F}_{n-1}] = Y_{n-1}$$

and predictable process :

$$\mathbb{E}[Y_n | \mathcal{F}_{n-1}] = Y_n$$

This means  $Y_n = Y_{n-1} = \dots = Y_0 = 0$ , since we have  $Y_0 = A'_0 - A_0 = 0$  by definition of  $A$ . This means the decomposition is almost surely unique.

#### 10.4.4 Corollary

$X$  is a (real-valued) submartingale if and only if it has a Doob decomposition into a martingale  $M$  and an integrable predictable process  $A$  that is almost surely increasing. In case of supermartingale,  $A$  is almost surely decreasing.

### 10.5 Lemma

If  $X_n$  is a stochastic process adaptad to  $\mathcal{F}_n$  and there exists an integrable random variable  $Y$  such that  $X_n = \mathbb{E}[Y | \mathcal{F}_n], \forall n \in I$  where  $I$  is index set. Then the  $X_n$  is called **right-closable** and we have also  $X_n$  is martingale. This right-closable property implies that  $X_n$  is uniformly integrable (2.3.3).

**Proof**

$$\begin{aligned} \frac{\mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > M\}}]}{P(|X_n| > M)} &= \mathbb{E}[|X_n| | \{|X_n| > M\}] \\ &= \mathbb{E}[|\mathbb{E}[Y | \mathcal{F}_n]| | \{|X_n| > M\}] \\ &\leq \mathbb{E}[\mathbb{E}[|Y| | \mathcal{F}_n] | \{|X_n| > M\}] \\ &= \mathbb{E}[|Y| | \{|X_n| > M\}] \\ &= \frac{\mathbb{E}[|Y| \cdot \mathbb{1}_{\{|X_n| > M\}}]}{P(|X_n| > M)}, \forall n \in I \\ \Leftrightarrow \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > M\}}] &\leq \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|X_n| > M\}}] \end{aligned}$$

where

- The first equality is by 2.2.12.4.1.
- The inequality is by Conditional Jensen's Inequality 2.2.8.3.1 with convex function  $f(x) = |x|$ .
- The one before last equality is by the general case law of iterated expectation 2.2.13.3, here  $\mathcal{H}_2 = \mathcal{F}_n$  and  $\mathcal{H}_1 = \{\Omega, \emptyset, \{|X_n| > K\}, \{|X_n| \leq K\}\}$ . We see that  $\mathcal{H}_1 \subseteq \mathcal{H}_2$  since  $\{|X_n| > K\} \in \mathcal{F}_n$

The above equations are written by a lazy manner, for the correct notation, we have to write  $\mathbb{E}[\mathbb{E}[Y | \mathcal{F}_n] | \{|X_n| > K\}]$  by  $\mathbb{E}[\mathbb{E}[Y | \mathcal{F}_n] | \mathcal{H}_1](\{|X_n| > K\})$ . By the same manner with ignoring conditional Jensen's inequality, we have  $\mathbb{E}[X_n \cdot \mathbb{1}_{\{|X_n| > M\}}] = \mathbb{E}[Y \cdot \mathbb{1}_{\{|X_n| > M\}}]$ .

Next, we use the similar method as in 2.3.3.2.

$$\begin{aligned}
& \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|X_n| > M\}}] \\
& \leq \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|X_n| > M\} \cup \{|Y| > K\}}] \\
& = \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|Y| \leq K\} \cap \{|X_n| > M\}}] + \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|Y| > K\}}] \\
& \leq KP(|X_n| > M) + \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|Y| > K\}}] \\
& \leq K \frac{\mathbb{E}[|X_n|]}{M} + \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|Y| > K\}}] \quad (\text{Markov inequality 2.2.8.1}) \\
& \leq K \frac{\mathbb{E}[|Y|]}{M} + \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|Y| > K\}}]
\end{aligned}$$

where the last inequality is by applying Conditional Jensen's Inequality 2.2.8.3.1 with convex function  $f(x) = |x|$ :

$$|X_n| = |\mathbb{E}[Y | \mathcal{F}_n]| \leq \mathbb{E}[|Y| | \mathcal{F}_n]$$

and this infers  $\mathbb{E}[|X_n|] \leq \mathbb{E}[|Y|]$  by law of iterated expectation.

Finally,

$$\lim_{M \rightarrow \infty} \sup_{n \in I} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{|X_n| > M\}}] \leq \lim_{M \rightarrow \infty} \sup_{n \in I} \left( \lim_{K \rightarrow \infty} K \frac{\mathbb{E}[|Y|]}{M} + \lim_{K \rightarrow \infty} \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|Y| > K\}}] \right) = 0$$

Here we note that  $K$  converges first to infinity and  $M$  can be chosen after  $K$ . In addition,  $\lim_{K \rightarrow \infty} \mathbb{E}[|Y| \cdot \mathbb{1}_{\{|Y| > K\}}] \rightarrow 0$  by 2.3.3.5.1. Hence  $X_n$  is uniformly integrable.

## 10.6 Uniform integrability of a backward submartingale

Let  $\{X_t\}_{t \in T}$  be a submartingale with respect to a filtered probability space  $(\Omega, \mathcal{F}_t, P)$ . Then  $X_{t_n}$  is uniformly integrable for any decreasing sequence  $t_n$  (backward) which is lower bounded in  $[0, T]$ .

### 10.6.1 Proof

Thus,  $t_n$  decreases and converges to  $t_\infty = a$ . By the corollary of Doob decomposition theorem (sec 4.10.4.4), we can decompose  $X_{t_n}$  into a martingale  $M_{t_n}$  and an integrable predictable process  $A_{t_n}$ . Note that,  $t_n$  is decreasing sequence, then  $A_{t_n}$  is decreasing instead of increasing (backward). By definition, we have  $A_0 = 0$  and since  $t_n \geq 0, \forall n$  then  $A_{t_n} \geq A_0 = 0$ .

First,  $0 \leq A_{t_n} \leq A_{t_0}$  or  $|A_{t_n}| \leq A_{t_0}$ , which is integrable then  $A_{t_n}$  is uniformly integrable by proposition 2.3.3.5.

Second, we show that  $M_{t_n}$  is also uniformly integrable. **Note that a martingale is by default not uniformly integrable**, in this case  $M_{t_n}$  is right-closable (sec 4.10.5) because

$$M_{t_n} = \mathbb{E}[M_{t_0} \mid \mathcal{F}_{t_n}], \forall n$$

Hence,  $M_{t_n}$  is uniformly integrable.

Finally,  $X_{t_n}$  is uniformly integrable by lemma 2.3.3.3.

### 10.6.2 Corollary

Let  $\{X_t\}_{t \in T}$  be a submartingale with respect to a filtered probability space  $(\Omega, \mathcal{F}_t, P)$ . Then given  $a \in \mathbb{R}$ , we define  $X_{t_n}^{+a} = \max(X_{t_n}, a)$ . Then  $X_{t_n}^{+a}$  is uniformly integrable for any decreasing sequence  $t_n$  which is lower bounded in  $[0, T]$ .

#### Proof

By lemma 4.10.6, which say that  $X_{t_n}$  is uniformly integrable.

$$0 < |X_{t_n}^{+a}| = |\max(X_{t_n}, a)| \leq \max(|X_{t_n}|, |a|) = Y_n$$

Thus  $Y_n$  is integrable and by lemma 2.3.3.4,  $X_{t_n}^{+a}$  is uniformly integrable.

## 10.7 Elementary process

Given interval  $[0, L]$  and a partition of  $T$  with  $0 = t_0 < t_1 < \dots < t_N = L$ . Let  $(\Omega, \mathcal{F}_{t_k}, P)$  be probability space. Let  $(Z_k)$  be a stochastic process adapted to  $\mathcal{F}_{t_k}$ . An **elementary process**  $\xi_t$  (of  $Z$ ) is a stochastic process defined by:

$$\xi(t, \omega) := Z_0(\omega) \mathbb{1}_{\{t=0\}} + \sum_{k=1}^N Z_k(\omega) \mathbb{1}_{(t_{k-1}, t_k]}(t), \quad \forall (t, \omega) \in T \times \Omega$$

Thus  $\xi(t, \omega)$  is a step function for a given  $\omega$ . The stochastic integral (more details in 4.16) for a given stochastic process  $Z(t, \omega)$  as integrand and stochastic process  $X_t$  as integrator, can be equivalently expressed in general by elementary process  $\xi(t, \omega)$ :

$$\begin{aligned} \int_0^L Z(t, \omega) dX(t, \omega) &= \sum_{k=1}^N Z_k(\omega) (X(t_k, \omega) - X(t_{k-1}, \omega)) \\ &= \int_0^L \xi(t, \omega) dX(t, \omega) \end{aligned}$$

Note that, for this expression to make sense,  $X(t_k)$  must be also  $\mathcal{F}_{t_k}$ -measurable function. The result of a stochastic integral is a **random variable**.

In the case that the upper integral bound is  $l \in [0, L]$ , then

$$Y_l = \int_0^l \xi(t, \omega) dX(t, \omega) = \sum_{k=1}^N Z_k(\omega) (X(t_k \wedge l, \omega) - X(t_{k-1} \wedge l, \omega)),$$

is a **stochastic process**, also called **integral process**. The notation  $t_k \wedge t$  is just  $\min(t_k, t)$ . If bounds of integral are not presiced, we can denote  $Y = \int \xi dX$  for short. In the differential form, the equivalent equation can be written :

$$dY = \xi dX$$

#### 10.7.1 Theorem

An adapted (to  $\mathcal{F}_{t_k}$ ) integrable process  $X$  is

1. a submartingale if and only if

$$\mathbb{E} \left[ \int_0^{+\infty} \xi dX \right] \geq 0$$

for all nonnegative bounded elementary processes  $\xi$ .

2. a supermartingale if and only if

$$\mathbb{E} \left[ \int_0^{+\infty} \xi dX \right] \leq 0$$

for all nonnegative bounded elementary processes  $\xi$ .

3. a martingale if and only if

$$\mathbb{E} \left[ \int_0^{+\infty} \xi dX \right] = 0$$

for all bounded elementary processes  $\xi$ .

Before going details for each statement, we show a common result :

$$\begin{aligned} \mathbb{E} \left[ \int_0^{+\infty} \xi dX \right] &= \mathbb{E} \left[ \sum_{k=1}^n Z_k(\omega)(X(t_k, \omega) - X(t_{k-1}, \omega)) \right] \quad (t_n = +\infty) \\ &= \sum_{k=1}^n (\mathbb{E}_{t_k} [Z_k X(t_k)] - \mathbb{E}_{t_{k-1}} [Z_k X(t_{k-1})]) \\ &= \sum_{k=1}^n (\mathbb{E}_{t_{k-1}} [\mathbb{E}_{t_k} [Z_k X(t_k) | \mathcal{F}_{t_{k-1}}]] - \mathbb{E}_{t_{k-1}} [Z_k X(t_{k-1})]) \\ &= \sum_{k=1}^n (\mathbb{E}_{t_{k-1}} [\mathbb{E}_{t_k} [Z_k X(t_k) | \mathcal{F}_{t_{k-1}}]] - Z_k X(t_{k-1})) \end{aligned}$$

#### Proof for 1

(\*) We show that  $X$  is submartingale  $\Rightarrow \mathbb{E} \left[ \int_0^{+\infty} \xi dX \right] \geq 0$ , for all nonnegative bounded elementary processes  $\xi$ .

We remind that with two random variable  $A$  and  $B$ , if  $\mathbb{E}[A] \geq B$  and  $\alpha$  is a non-negative constant, then  $\mathbb{E}[\alpha A] \geq \alpha B$ .

Since  $X$  is a submartingale, then  $\mathbb{E}_{t_k} [X(t_k) | \mathcal{F}_{t_{k-1}}] \geq X(t_{k-1})$ . Apply the above reminding, with  $Z_k$  is a non-negative random variable, then  $\mathbb{E}_{t_k} [Z_k X(t_k) | \mathcal{F}_{t_{k-1}}] \geq Z_k X(t_{k-1})$ . So we have the Q.E.D.

(\*\*) Conversely, we show that  $\mathbb{E} \left[ \int_0^{+\infty} \xi dX \right] \geq 0$  for all nonnegative bounded elementary processes  $\xi \Rightarrow X$  is submartingale.

We show that  $\mathbb{E}[X(t)|\mathcal{F}_s] \geq X(s), \forall s < t$  by contradiction.

On one hand, suppose that there exists  $s < t$  that  $\mathbb{E}[X(t)|\mathcal{F}_s] < X(s)$ . This implies  $\mathbb{E}_s[\mathbb{E}_t[X(t)|\mathcal{F}_s]] < \mathbb{E}_s[X(s)]$  or  $\mathbb{E}[X(t)] < \mathbb{E}[X(s)]$ .

On other hand, we take the elementary processes  $\xi = \mathbb{1}_{(s,t]}$ , which is thus 1 in  $(s, t]$  and 0 otherwise. From  $\mathbb{E} \left[ \int_0^{+\infty} \xi dX \right] \geq 0$ , we have  $\mathbb{E}[X(t)] - \mathbb{E}[X(s)] \geq 0$  or  $\mathbb{E}[X(t)] \geq \mathbb{E}[X(s)]$ .

Then we have the contradiction. This contradiction show that  $\mathbb{E}[X(t)|\mathcal{F}_s] \geq X(s), \forall s < t$  or  $X$  is a submartingale.

### Proof for 2

The same way as proof for 1.

### Proof for 3

Using 1 and 2, with the fact that martingale is also submartingale and supermartingale.

#### 10.7.2 Lemma

Let  $X$  be a process and  $\xi$  be a bounded elementary process. Define  $Y_t = \int_0^t \xi dX$ . Then

1. If  $X$  is a submartingale then

$$\begin{cases} Y \text{ is a submartingale if } \xi \text{ is non-negative.} \\ Y \text{ is a supermartingale if } \xi \text{ is non-positive.} \end{cases}$$

2. If  $X$  is a supermartingale then

$$\begin{cases} Y \text{ is a supermartingale if } \xi \text{ is non-negative.} \\ Y \text{ is a submartingale if } \xi \text{ is non-positive.} \end{cases}$$

3. If  $X$  is a martingale then so is  $Y$ .

### Proof for 1

Given  $s < t$ , on one hand :

$$\begin{aligned} \mathbb{E}_s[\mathbb{E}_t[Y_t | \mathcal{F}_s]] &= \mathbb{E}[Y_t] \\ &= \mathbb{E} \left[ \int_0^t \xi dX \right] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left[ \sum_{k=1}^n Z_k (X(t_k \wedge t) - X(t_{k-1} \wedge t)) \right] \\
&= \sum_{k=1}^n \mathbb{E} [Z_k (X(t_k \wedge t) - X(t_{k-1} \wedge t))]
\end{aligned}$$

On the other hand :

$$\begin{aligned}
\mathbb{E}[Y_s] &= \mathbb{E} \left[ \int_0^s \xi \, dX \right] \\
&= \sum_{k=1}^n \mathbb{E} [Z_k (X(t_k \wedge s) - X(t_{k-1} \wedge s))]
\end{aligned}$$

(\*) In case that  $\xi$  is non-negative :

Suppose that  $\mathbb{E}_t[Y_t | \mathcal{F}_s] < Y_s$ , then  $\mathbb{E}_s[\mathbb{E}_t[Y_t | \mathcal{F}_s]] < \mathbb{E}[Y_s]$  or

$$\sum_{k=1}^n \mathbb{E} [Z_k (X(t_k \wedge t) - X(t_{k-1} \wedge t))] < \sum_{k=1}^n \mathbb{E} [Z_k (X(t_k \wedge s) - X(t_{k-1} \wedge s))]$$

This is equivalent to

$$\mathbb{E}[Z_l (X(s) - X(t_l))] + \sum_{k=l+1}^{h-1} \mathbb{E} [Z_k (X(t_k) - X(t_{k-1})) + \mathbb{E}[Z_h (X(t) - X(t_{h-1}))] < 0$$

where  $s \in (t_{l-1}, t_l]$  and  $t \in (t_{h-1}, t_h]$ . By analogy to the proof for theorem 4.10.7.1, we have if  $X$  is a submartingale and  $\xi$  is non-negative, then  $\mathbb{E}[Z_k (X(a) - X(b))] \geq 0, \forall a > b$  and  $\forall k$ . Then

$$\mathbb{E}[Z_l (X(s) - X(t_{l-1})) + \sum_{k=l+1}^{h-1} \mathbb{E} [Z_k (X(t_k) - X(t_{k-1})) + \mathbb{E}[Z_h (X(t) - X(t_{h-1}))]] \geq 0$$

This latter is equivalent to  $\mathbb{E}_s[\mathbb{E}_t[Y_t | \mathcal{F}_s]] \geq \mathbb{E}_s[Y_s]$  (contradiction). Then we must have  $\mathbb{E}_t[Y_t | \mathcal{F}_s] \geq Y_s$  or  $Y_t$  is submartingale.

(\*\*) In case that  $\xi$  is non-positive :

By the same manner in case that  $\xi$  is non-negative (contradiction), with remark that if  $X$  is a submartingale and  $\xi$  is non-positive, then  $\mathbb{E}[Z_k (X(a) - X(b))] \leq 0, \forall a > b$ .

### Proof for 2

By the same manner as 1.

### Proof for 3

Given  $s < t$ , on one hand :

$$\mathbb{E}[Y_t | \mathcal{F}_s] = \mathbb{E} \left[ \int_0^t \xi \, dX \right]$$

$$\begin{aligned}
&= \mathbb{E}\left[\sum_{k=1}^n Z_k(X(t_k \wedge t) - X(t_{k-1} \wedge t) | \mathcal{F}_s)\right] \\
&= \sum_{k=1}^n (\mathbb{E}[Z_k X(t_k \wedge t) | \mathcal{F}_s] - \mathbb{E}[X(t_{k-1} \wedge t) | \mathcal{F}_s]) \\
&= \sum_{k=1}^{l-1} (Z_k X(t_k) - Z_k X(t_{k-1})) + \mathbb{E}[Z_l X(t_l) | \mathcal{F}_s] - Z_l X(t_{l-1}) \\
&\quad + \sum_{k=l+1}^{h-1} (\mathbb{E}[Z_k X(t_k) | \mathcal{F}_s] - \mathbb{E}[X(t_{k-1}) | \mathcal{F}_s]) + \mathbb{E}[Z_h X(t) | \mathcal{F}_s] - \mathbb{E}[Z_h X(t_{h-1}) | \mathcal{F}_s]
\end{aligned}$$

On the other hand :

$$\begin{aligned}
Y_s &= \int_0^s \xi \, dX \\
&= \sum_{k=1}^n Z_k(X(t_k \wedge s) - X(t_{k-1} \wedge s)) \\
&= \sum_{k=1}^{l-1} (Z_k X(t_k) - Z_k X(t_{k-1})) + Z_l X(s) - Z_l X(t_{l-1})
\end{aligned}$$

where  $s \in (t_{l-1}, t_l]$  and  $t \in (t_{h-1}, t_h]$

$$\begin{aligned}
&\mathbb{E}[Y_t | \mathcal{F}_s] - Y_s \\
&= \mathbb{E}[Z_h X(t) | \mathcal{F}_s] - \mathbb{E}[Z_h X(t_{h-1}) | \mathcal{F}_s] + \sum_{k=l+1}^{h-1} (\mathbb{E}[Z_k X(t_k) | \mathcal{F}_s] - \mathbb{E}[X(t_{k-1}) | \mathcal{F}_s]) \\
&\quad + \mathbb{E}[Z_l X(t_l) | \mathcal{F}_s] - Z_l X(s)
\end{aligned}$$

Since  $X$  is a martingale then  $\mathbb{E}[Z_k X(a) | \mathcal{F}_s] = X(s)$  with  $\forall a \geq s$ , then  $\mathbb{E}[Y_t | \mathcal{F}_s] - Y_s = 0$  or  $Y_t$  is a martingale.

## 10.8 Upcrossings, downcrossings and martingale convergence

### 10.8.1 Upcrossings and downcrossings

Figure 4.3 show an illustration for upcrossings and downcrossings of a process. We denote  $U[a, b]$  number of upcrossings and  $D[a, b]$  number of downcrossings.

Note that between any two upcrossings there is a downcrossing and similarly, between any two downcrossings there is an upcrossing. It follows that  $U[a, b]$  and  $D[a, b]$  can differ by at most 1, and they are either both finite or both infinite.

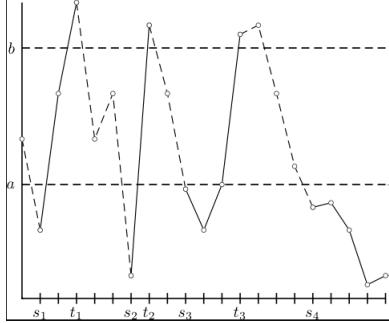


Figure 4.3: Upcrossing and downcrossing for given interval  $[a, b]$ . Thus  $a, b$  can be consider as thresholding. This process has 3 upcrossings and 4 downcrossings.

### 10.8.2 Lemma

Let's consider a probability space  $(\Omega, \mathcal{F}, P)$  and a filtration of  $\sigma$ -algebras  $\mathcal{F}_t$ , where  $t \in I$  and  $I$  is a set that contains indexes. Let  $X_t$  be a stochastic process adapted to  $\mathcal{F}_t$ . Given real numbers  $a < b$  and  $t_0 = 0$ . Let's define a sequence (can be infinite)  $s_k, t_k$  by

$$\begin{cases} s_k = \inf\{m \geq t_{k-1} : x_m \leq a\} \\ t_k = \inf\{m \geq s_k : x_m \geq b\} \end{cases}$$

Thus  $s_k$  is a sequence of index that  $X_t$  passes below  $a$  first after reaching  $b$  and  $t_k$  is a sequence of index that  $X_t$  passes above  $b$  first after touching  $a$ . Let the elementary process  $\xi_A$  be defined as :

$$\xi_A(t, \omega) = \sum_{k=1}^n \mathbb{1}_{(s_k, t_k]}(t)$$

where  $s_k, t_k$  depends on  $\omega$ . Then :

$$\int_0^T \xi_A dX = \sum_{k=1}^{\infty} (X_{t_k \wedge T} - X_{s_k \wedge T})$$

Since  $I$  is a finite set, then there exists  $t_N$  the last upcrossing instant or  $U[a, b] = N$ . There are two cases need to be considered:

- After  $t_N$ ,  $X_t$  is always strictly greater than  $a$ . In this case :

$$\int_0^T \xi_A dX = \sum_{k=1}^N (X_{t_k \wedge T} - X_{s_k \wedge T}) \geq (b - a)U[a, b]$$

- After  $t_N$ ,  $X_t$  has one more downcrossing at  $s_{N+1}$  ( $X_{s_{N+1}} \leq a$ ) and then  $X$  is always strictly less than  $b$ . We have :

$$X_T - X_{s_{N+1}} \geq -\max(a - X_T, 0)$$

since :

- If  $X_T \geq X_{s_{N+1}}$ , then  $X_T - X_{s_{N+1}} \geq 0$ . As  $-\max(a - X_T, 0)$  is something negative, then  $X_T - X_{s_{N+1}} \geq -\max(a - X_T, 0)$ .
- If  $X_T < X_{s_{N+1}}$ , then  $X_T - X_{s_{N+1}} > X_T - a = -\max(a - X_T, 0) = X_T - a$ , since  $X_{s_{N+1}} \leq a$ .

Then :

$$\int_0^T \xi_A dX = \sum_{k=1}^N (X_{t_k \wedge T} - X_{s_k \wedge T}) + X_T - X_{s_{N+1}} \geq (b-a)U[a, b] - \max(a - X_T, 0),$$

In both cases, we have

$$\int_0^T \xi_A dX \geq (b-a)U[a, b] - \max(a - X_T, 0)$$

### 10.8.3 Doob's upcrossing lemma

It is also called Doob's upcrossing inequality. Let  $X_t$  be a supermartingale with time  $t$  running through a **countable index set**  $I$ . Then with any  $a < b$ , we have :

$$(b-a)\mathbb{E}[U[a, b]] \leq \mathbb{E}[(a - X_T) \vee 0]$$

#### Proof

Let's denote  $J_n$  an increasing sequence ( $J_{n-1} \subseteq J_n$ ) that contains **finite index set** and converges to **countable index set**  $I$  ( $\lim_{n \rightarrow \infty} J_n \rightarrow I$ ). Given  $a$  and  $b$ , let  $f_n : J_n \rightarrow \mathbb{R}^+$  be measurable function that

$$f_n(x) = \begin{cases} 1 & \text{if } x \text{ is a upcrossing point } t_k \\ 0 & \text{otherwise} \end{cases}$$

Then  $f_n$  converges pointwise to  $f : I \rightarrow \mathbb{R}^+$ . By convergence monotone theorem 2.2.15, we have

$$(U[a, b], J_n) = \lim_{n \rightarrow \infty} \int_T f_n = \int_T \lim_{n \rightarrow \infty} f_n = \int_T f = (U[a, b], I)$$

where  $(U[a, b], J)$  is the number of upcrossing in set  $J$ .

Let's come back to Doob's upcrossing lemma, it is enough to prove this inequality for any **finite index set**  $J$  since  $I$  is countable. From the lemma 4.10.8.2, we have :

$$\begin{aligned} (b-a)\mathbb{E}[U[a, b]] &\leq \mathbb{E}\left[\int \xi_A dX\right] + \mathbb{E}[(a - X_T) \vee 0] \\ &\leq \mathbb{E}[(a - X_T) \vee 0] \\ &= \mathbb{E}[(a - X_T)^+] = \mathbb{E}[(X_T - a)^-] \end{aligned}$$

We have  $\mathbb{E} [\int \xi_A dX] \leq 0$  since  $\xi_A$  is nonnegative bound evementary process and  $X$  is supermartingale (theorem 4.10.7.1). The notation  $x^+ = \max(x, 0)$  means the positive part of  $x$  and  $x^- = -\min(x, 0)$  means the negative part of  $x$ . Both positive part et negative part are nonnegative.

#### 10.8.4 Lemma

Given a stochastic process  $X$  and let  $t_n$  be a monotone sequence. Note that by indexing with  $n$ , then the  $\{t_n : n \in \mathbb{N}\}$  is countable set. If  $U[a, b](\omega)$  of  $X_{t_n}(\omega)$  is finite for all  $a, b \in \mathbb{R}$ , then  $X_{t_n}$  almost surely converges. That means  $\lim_{n \rightarrow \infty} X_{t_n} \xrightarrow{a.s.} X_{t_\infty}$ , where  $X_{t_\infty}$  is a random variable.

##### Proof

Let's consider

$$\begin{aligned} \Lambda &= \{\omega : \lim_{n \rightarrow \infty} X_{t_n}(\omega) \text{ does not converge}\} \\ &= \{\omega : \liminf_{n \rightarrow \infty} X_{t_n}(\omega) < \limsup_{n \rightarrow \infty} X_{t_n}(\omega)\} \\ &= \bigcup_{a < b; a, b \in \mathbb{Q}} \{\omega : \liminf_{n \rightarrow \infty} X_{t_n}(\omega) < a < b < \limsup_{n \rightarrow \infty} X_{t_n}(\omega)\} \\ &= \bigcup_{a < b; a, b \in \mathbb{Q}} \{\omega : U[a, b](\omega) = +\infty\} \end{aligned}$$

We use  $a, b \in \mathbb{Q}$  instead of  $a, b \in \mathbb{R}$  to benefit the countable additivity of a measure (2.1.7.1). As  $\mathbb{Q}$  is countable set, then  $\mathbb{Q}^2$  is also countable set then we can apply the countable additivity:

$$P(\Lambda) = \sum_{a < b; a, b \in \mathbb{Q}} P(\{\omega : U[a, b](\omega) = +\infty\}) = 0$$

Then  $X_{t_n}$  converges almost surely (2.4.6).

### 10.9 Doob's Forward Convergence Theorem

Also called Doob's martingale convergence theorem.

#### 10.9.1 Statement

Let  $(X_n)_{n=1,2,\dots}$  be a supermartingale with the filtration  $\mathcal{F}_n$  such that

$$\sup_{n \in \mathbb{N}} \mathbb{E}[X_n^-] < +\infty$$

Then, with probability one, the limit

$$\lim_{n \rightarrow +\infty} X_n(\omega[: n]) \xrightarrow{a.s.} X_\infty(\omega), \quad \forall \omega \in \mathcal{F}_\infty$$

exists (where  $X_\infty$  is a random variable) and  $\mathbb{E}[X_\infty]$  is finite (or  $X_\infty$  is integrable). In other words, the random variables  $X_n$  converge **almost surely** (sec 2.4.6) to a random variable named  $X_\infty$ .

For continuous case, see 4.12.

### 10.9.2 Important note

Given the process  $X_n$ , there are two concepts that must be distinguished :

1.  $\mathbb{E}[|X_n|] < +\infty, \forall n \in N$ , which means that  $X_n$  is bounded in  $L_1$  for each  $n \in \mathbb{N}$ .
2.  $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|] < +\infty$ , which means the process is (bounded) in  $L_1$ .

The second is stronger than the first. If we have the first, it does not mean we have the second. For example, if  $\mathbb{E}[|X_n|] = n$ , then each  $X_n$  is bounded in  $L_1$  for all  $n \in N$ , but the process  $(X_n)$  is not bounded in  $L_1$ .

### 10.9.3 Proof

1. Given an (fixed)  $\omega$ , then for all  $a < b$ , we have :

$$\begin{aligned} (b-a)\mathbb{E}[U[a,b]] &\leq \mathbb{E}\left[\left(X_n - a\right)^-\right] \\ &\leq \mathbb{E}\left[X_n^-\right] + a^+ \end{aligned}$$

where

- The first inequality is from Doob's upcrossing lemma (sec 4.10.8.3) with  $X_n$  is a supermartingale and index  $n \in \mathbb{N}$ , which is a countable index set.
- The second inequality is from sec 2.2.8.9.

In addition, we have  $\sup_{n \in \mathbb{N}} \mathbb{E}[X_n^-] < +\infty$  then  $\exists M$  that  $\mathbb{E}[X_n^-] < M, \forall n$  then :

$$\mathbb{E}[U[a,b]] \leq \frac{M+a^+}{b-a} < +\infty$$

2. By lemma 4.10.8.4, as  $n \in \mathbb{N}$  is a countable set and  $U[a,b]$  of  $X_n$  is finite, then  $X_n$  converges almost surely to  $X_\infty$ .
3. We show that  $\mathbb{E}[X_\infty]$  is finite. Since  $X_n$  is a supermartingale then from definition in sec 4.10.3, we have

$$\mathbb{E}[X_n] < X_1$$

Then  $\mathbb{E}[X_\infty] < +\infty$ . Now suppose that  $\mathbb{E}[X_\infty] = -\infty$ . From  $\mathbb{E}[X_\infty] = \mathbb{E}[X_\infty^\pm] - \mathbb{E}[X_\infty^-]$  and  $\mathbb{E}[X_\infty^\pm]$  is something positive, we infer that  $\mathbb{E}[X_\infty^-] = +\infty$ . But we have that  $\mathbb{E}[X_n^-] < M, \forall n$ , then we have the contradiction. This infers that  $\mathbb{E}[X_\infty] < -\infty$  then  $\mathbb{E}[X_\infty]$  is finite.

An other way to show that  $\mathbb{E}[X_\infty]$  is finite is to use Fatou's lemma (2.2.16). As  $X_n \xrightarrow{a.s} X_\infty$ , then  $\liminf_{n \rightarrow \infty} |X_n| = |X_\infty|$ .

$$\mathbb{E}[|X_\infty|] = \mathbb{E}[\liminf_{n \rightarrow \infty} |X_n|]$$

$$\begin{aligned}
&\leq \liminf_{n \rightarrow \infty} \mathbb{E}[|X_n|] \quad (\text{Fatou's lemma}) \\
&\leq \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|] \\
&= \sup_{n \in \mathbb{N}} \mathbb{E}[|X_n^+ - X_n^-|] \\
&\leq \sup_{n \in \mathbb{N}} \mathbb{E}[X_n^+ + X_n^-] \quad \text{note that } X_n^+, X_n^- \geq 0 \\
&= \sup_{n \in \mathbb{N}} \mathbb{E}[X_n^+ - X_n^- + 2X_n^-] \\
&\leq \sup_{n \in \mathbb{N}} \mathbb{E}[X_n] + 2 \sup_{n \in \mathbb{N}} \mathbb{E}[X_n^-] \\
&\leq \mathbb{E}[X_1] + M < +\infty
\end{aligned}$$

A document for Martingale Convergence Theorem can be found at : [https://ocw.mit.edu/courses/15-070j-advanced-stochastic-processes-fall-2013/66b6c8fdb52304e3777ce8286beaf7d/MIT15\\_070JF13\\_Lec11Add.pdf](https://ocw.mit.edu/courses/15-070j-advanced-stochastic-processes-fall-2013/66b6c8fdb52304e3777ce8286beaf7d/MIT15_070JF13_Lec11Add.pdf)

#### 10.9.4 Analogy for submartingale

If  $(X_n)_{n=1,2,\dots}$  is a submartingale such that  $\sup_{n \in \mathbb{N}} \mathbb{E}[X_n^+]$  is (upper) bounded ( $< +\infty$ ), then  $\lim_{n \rightarrow +\infty} X_n \xleftarrow{a.s} X_\infty$  and  $\mathbb{E}[X_\infty]$  is finite.

#### 10.9.5 Analogy for martingale

If  $(X_n)_{n=1,2,\dots}$  is a martingale such that  $\sup_{n \in \mathbb{N}} \mathbb{E}[X_n^+]$  and  $\sup_{n \in \mathbb{N}} \mathbb{E}[X_n^-]$  are both (upper) bounded or equivalently  $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|]$  is bounded, then  $\lim_{n \rightarrow +\infty} X_n \xleftarrow{a.s} X_\infty$  and  $\mathbb{E}[X_\infty]$  is finite.

The condition that the martingale  $(X_n)$  is bounded is essentialn for example, an unbiased  $\pm 1$  random walk is a martingale but does not converge.

### 10.10 Extended of Doob's Forward Convergence Theorem

Let  $(X_n)_{n=1,2,\dots}$  be a martingale with filtration  $\mathcal{F}_n$ , where  $X_n$  is uniformly integrable (in sec 2.3.3). Then it exists unique random variable  $X_\infty$ , which is  $\mathcal{F}_\infty$ -measurable such that :

$$X_n = \mathbb{E}[X_\infty | \mathcal{F}_n]$$

#### Proof

- Since  $X_n$  is uniformly integrable, this implies that  $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|] < \infty$  (sec 2.3.3.2). Then Doob's Forward Convergence Theorem in sec 4.10.9 says that  $X_n$  converges almost surely to  $X_\infty$
- With  $k > n$ :

- Since  $X$  is a martingale, then  $\mathbb{E}[X_k | \mathcal{F}_n] = X_n$
- Since  $\lim_{k \rightarrow \infty} X_k \xrightarrow{a.s.} X_\infty$ , then  $\lim_{k \rightarrow \infty} X_k \xrightarrow{p} X_\infty$ . In addition,  $X_k$  is uniformly integrable, then by 2.4.5.4,  $X_k \xrightarrow{L_1} X_\infty$  or  $\lim_{k \rightarrow \infty} \mathbb{E}[|X_k - X_\infty|] = 0$  or we can write  $\lim_{s \rightarrow \infty} \mathbb{E}[X_k | \mathcal{F}_n] = \mathbb{E}[X_\infty | \mathcal{F}_n]$ .

From these two points, we get  $X_n = \mathbb{E}[X_\infty | \mathcal{F}_n]$ .

- Uniqueness : if there is  $Z$  such that  $\mathbb{E}[Z | \mathcal{F}_n] = X_n, \forall n$ . Then

$$\begin{aligned}\mathbb{E}[\mathbf{1}_A(X_\infty - Z)] &= \mathbb{E}[X_\infty \cdot \mathbf{1}_A] - \mathbb{E}[Z \cdot \mathbf{1}_A] \\ &= \mathbb{E}[X_\infty | \mathcal{F}_t](A)P(A) - \mathbb{E}[Z | \mathcal{F}_t](A)P(A) \quad (\text{By 2.2.12.4.1}) \\ &= X_t(A)P(A) - X_t(A)P(A) \\ &= 0 \quad \forall A \in \mathcal{F}_n\end{aligned}$$

Then by the equality by expectation 2.2.6, we have  $X_\infty = Z$ .

In continuous case, we have theorem 4.12.1.3

## 10.11 Exercice

Let  $(X_n)_{n \in \mathbb{N}}$  be a standard random walk. That is,  $X_1 = 0$  and

$$P(X_{n+1} = X_n + 1 | \mathcal{F}_n) = P(X_{n+1} = X_n - 1 | \mathcal{F}_n) = \frac{1}{2}$$

Then, for every integer  $a$ , we have  $X_n = a$  for some  $n$  with probability one.

### Solution

We aim to solve this exercice by contradiction. Without loss of generality, let's take  $a < 0$ . Let  $T$  be the first time  $n$  that  $X_n = a$ . If we never have  $n$  such that  $X_n = a$ , then  $T = +\infty$  and  $X_n > a, \forall n$ . The latter means that  $\sup_{n \in \mathbb{N}} \mathbb{E}[X_n^-] < +\infty$ . Since  $X_n$  is martingale then  $X_n$  is also supermartingale. Then Doob's Forward Convergence Theorem (in sec 4.10.9) says that  $X_n$  converge almost surely to  $X_\infty$ , or for all  $\varepsilon > 0$ , it exists  $n_0$  such that  $\forall m, n > n_0, |X_m - X_n| < \varepsilon$ . However, the property of random walk is that  $|X_{n+1} - X_n| = 1$ . Here, we have the contradiction.

## 10.12 Levy's Upwards and Downwards Theorems

A very useful application of martingale convergence is the following results concerning conditional expectations by  $\sigma$ -algebra (sec 2.2.13.1.1), for a limit of increasing or decreasing  $\sigma$ -algebras.

### 10.12.1 Upward

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  be an increasing sequence of sub  $\sigma$ -algebras of  $\mathcal{F}$ , which means  $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}, \forall n$ . Set  $\mathcal{F}_\infty \equiv \sigma(\bigcup_n \mathcal{F}_n)$ . For any random variable  $X$  in  $L^1$  (integrable). Then

$$\lim_{n \rightarrow \infty} \mathbb{E}[X | \mathcal{F}_n] \xrightarrow{a.s.} \mathbb{E}[X | \mathcal{F}_\infty]$$

Note that  $X(\omega)$  that take  $\omega \in \Omega$ , not in filtration  $\mathcal{F}_n$ .

**Proof**

Let's  $Y_n = \mathbb{E}[X | \mathcal{F}_n]$ . First, by law of iterated expectation in sec 2.2.13.3 :

$$\mathbb{E}[Y_{n+1} | \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[X | \mathcal{F}_{n+1}] | \mathcal{F}_n] = \mathbb{E}[X | \mathcal{F}_n] = Y_n,$$

then  $Y_n$  is a martingale.

Second, we show that  $\sup_{n \in \mathbb{N}} \mathbb{E}[|Y_n|] < +\infty$ . Thus

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|Y_n|] = \sup_{n \in \mathbb{N}} \mathbb{E}[|\mathbb{E}[X | \mathcal{F}_n]|] \leq \sup_{n \in \mathbb{N}} \mathbb{E}[\mathbb{E}[|X| | \mathcal{F}_n]] = \mathbb{E}[|X|] < +\infty$$

where the inequality is by the conditional Jensen's Inequality in 2.2.8.3.1, with  $f(x) = |x|$  is convex.

Finally, from the first and the second, with Doob's martingale convergence theorem (sec 4.10.9.5), we have Q.E.D

### 10.12.2 Downward

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  be an decreasing sequence of sub  $\sigma$ -algebras of  $\mathcal{F}$ , which means  $\mathcal{F}_k \supseteq \mathcal{F}_{k+1}, \forall k$ . Set  $\mathcal{F}_\infty \equiv \bigcap_n \mathcal{F}_n$ . For any random variable  $X$  in  $L^1$  (integrable). Then

$$\lim_{n \rightarrow \infty} \mathbb{E}[X | \mathcal{F}_n] \xrightarrow{a.s.} \mathbb{E}[X | \mathcal{F}_\infty]$$

## 10.13 Optional stopping theorem

The optional stopping theorem, also called Doob's optional sampling theorem says that **under certain conditions**, the expected value of a martingale at a stopping time is equal to its initial expected value.

### 10.13.1 Statement

Consider a discrete-time version (hence by analogy for continuous version). Let  $(X_t)_{t \in \mathbb{N}_0}$  be a discrete-time martingale and  $\tau$  a stopping time, both with respect to a filtration  $(\mathcal{F}_t)_{t \in \mathbb{N}_0}$ . Assume that **one of the following three** conditions holds:

1. The stopping time  $\tau$  is **almost surely bounded**, i.e., there exists a constant  $c \in \mathbb{N}$  such that  $\tau \leq c$  almost surely.

2. The stopping time  $\tau$  has **finite expectation** and the conditional expectations of the absolute value of the **martingale increments** are almost surely bounded, more precisely:

- $\mathbb{E}[\tau] < \infty$ ,
- $\exists c, \mathbb{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \leq c$

3.  $\exists c, |X_t^\tau| = |X_{t \wedge \tau}| \leq c$  almost surely for all  $t$ .

**Then**

- $X_\tau$  is an almost surely *well defined* random variable.
- $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$

### 10.13.2 Examples

Consider a random walk  $X$  that starts at 0. Let's  $Y_n = X_n^2 - n$ , by the second example in 4.10.1.3,  $Y_n$  is a martingale.  $\tau$  is the time at which  $X$  first reaches  $\pm m$ . Since  $Y_n$  satisfies the second condition (see proof in 4.10.13.2.1), by optional stopping theorem :

$$\mathbb{E}[Y_0] = \mathbb{E}[Y_\tau]$$

This gives

$$\begin{aligned}\mathbb{E}[Y_0] &= 0 = \mathbb{E}[Y_\tau] \\ &= \mathbb{E}[X_\tau^2] - \mathbb{E}[\tau] \\ &= m^2 - \mathbb{E}[\tau]\end{aligned}$$

Then  $\mathbb{E}[\tau] = m^2$

#### 10.13.2.1 Proof

We show firstly  $\mathbb{E}[\tau] < \infty$  and secondly  $\exists c, \mathbb{E}[|Y_{n+1} - Y_n| \mid \mathcal{F}_n] \leq c$ .  
In the first one,

- By 2.2.4.1 and note that  $\tau$  is a integer valued random variable, then :

$$\mathbb{E}[\tau] = \sum_{k=0}^{+\infty} P(\tau > k)$$

- For a given instant  $k$ , if  $\tau > k$  (until instant  $k$ ,  $X_t$  does not touch  $\pm m$ ). If we go to right (or left)  $2m$  consecutive times, we are sure to touch  $m$ , hence :

$$P(\tau \leq k + 2m \mid \tau > k) \geq \frac{1}{2^{2m}}$$

Equivalently,

$$P(\tau > k + 2m \mid \tau > k) \leq 1 - \frac{1}{2^{2m}}$$

By Bayes :

$$P(\tau > k + 2m \mid \tau > k) = \frac{P((\tau > k + 2m) \cap (\tau > k))}{P(\tau > k)} = \frac{P(\tau > k + 2m)}{P(\tau > k)}$$

or

$$\begin{aligned} P(\tau > k + 2m) &= P(\tau > k + 2m \mid \tau > k)P(\tau > k) \\ P(\tau > k + 2m) &\leq \left(1 - \frac{1}{2^{2m}}\right)P(\tau > k) \end{aligned}$$

Replacing  $k = (k - 1)2m$ , we have :

$$\begin{aligned} P(\tau > k2m) &\leq \left(1 - \frac{1}{2^{2m}}\right)P(\tau > (k - 1)2m) \\ &\leq \left(1 - \frac{1}{2^{2m}}\right)^k P(\tau > 0) \\ &= \left(1 - \frac{1}{2^{2m}}\right)^k \end{aligned}$$

- If  $h < g$ , then  $P(\tau > h) \geq P(\tau > g)$

Finally,

$$\begin{aligned} \mathbb{E}[\tau] &= \sum_{k=0}^{+\infty} P(\tau > k) \\ &= \sum_{k=0}^{+\infty} \sum_{l=0}^{2m-1} P(\tau > k2m + l) \\ &\leq \sum_{k=0}^{+\infty} 2mP(\tau > k2m) \\ &= \sum_{k=0}^{+\infty} 2m \left(1 - \frac{1}{2^{2m}}\right)^k \\ &= 2m2^{2m} < \infty \end{aligned}$$

For the second one,

$$\begin{aligned} |Y_{n+1} - Y_n| &= |X_{n+1}^2 - (n+1) - X_n^2 + n| \\ &= |(X_{n+1} \pm 1)^2 - X_n^2 - 1| \\ &= |\pm 2X_n + 1 - 1| \\ &= 2|X_n| \\ &= 2 \end{aligned}$$

### 10.13.3 Proof for optional stopping theorem (in discret case)

First, we need to distinguish between  $X_\tau$  and  $X_t^\tau$

- $X_\tau$  is a **random variable** evaluated at stoppting time  $\tau$ .
- $X_t^\tau$  is a **stochastic process** that is stopped at time  $\tau$ .

We show that  $X_\tau$  is almost sure well-defnied :

- In the first condition, random variable  $\tau$  is a.s. bounded by  $c$  or  $P(|\tau| < c) = 1$ . This means stopping time  $\tau$  is a.s. finite. Therefore, there exists a random variable  $X_\tau$  that represents the value of  $X_t$  at the stopping time  $\tau$ .
- In the second condition,  $\mathbb{E}[\tau] < \infty$  implies that the stopping time  $\tau$  is almost surely finite, then the same as above case.
- In third condition, as  $X_t$  is martingale then by 4.10.1.4,  $X_t^\tau$  is also martingale. In addition,  $X_t^\tau$  is bounded, then by 4.10.9,  $X_t^\tau$  converges a.s. to a random variable. Moreover,  $\lim_{t \rightarrow \infty} X_t^\tau = X_\tau$ . Finally,  $X_\tau$  is a.s. well defined.

We show that  $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ . First, we show that the stopped process  $X_t^\tau$  is bounded by an integrable random variable. This is obvious with the third condition, then we need to show it with the first and the second.

- With first condition,  $\tau < c$  a.s., then :

$$\begin{aligned} |X_t^\tau| &\leq \sum_{s=0}^c |X_s| \\ &:= H \end{aligned}$$

$\mathbb{E}[H]$  is trivially integrable since  $\mathbb{E}[X_s] < \infty, \forall s$ .

- With the second condition,

$$\begin{aligned} |X_t^\tau| &= |X_0 + \sum_{s=0}^{t-1 \wedge \tau - 1} (X_{s+1} - X_s)| \\ &\leq |X_0| + \sum_{s=0}^{t-1 \wedge \tau - 1} |(X_{s+1} - X_s)| \\ &\leq |X_0| + \sum_{s=0}^{\tau-1} |X_{s+1} - X_s| \\ &\leq |X_0| + \sum_{s=0}^{\infty} |X_{s+1} - X_s| \mathbb{1}_{\{s < \tau\}} \\ &:= G \end{aligned}$$

$\mathbb{E}[G]$  is integrable since :

$$\begin{aligned}
\mathbb{E}[G] &= \mathbb{E}[|X_0|] + \sum_{s=0}^{\infty} \mathbb{E}[|X_{s+1} - X_s| \mathbf{1}_{\{s < \tau\}}] \quad \text{as in 2.2.15.1} \\
&= \mathbb{E}[|X_0|] + \sum_{s=0}^{\infty} \mathbb{E}[|X_{s+1} - X_s| | s < \tau] P(s < \tau) \quad \text{by 2.2.12.4.1} \\
&\leq \mathbb{E}[|X_0|] + \sum_{s=0}^{\infty} c P(s < \tau) \\
&= \mathbb{E}[|X_0|] + c \sum_{s=0}^{\infty} P(\tau > s) \\
&= \mathbb{E}[|X_0|] + c \mathbb{E}[\tau] \quad \text{by 2.2.4.1} \\
&< \infty
\end{aligned}$$

Now, since

- $X_t^\tau$  converges a.s to  $X_\tau$  (since  $X_\tau$  is well-defined).
- $X_t^\tau$  is bounded by an integrable random variable.

Then by the Dominated Convergence Theorem 2.2.17.2,

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_t^\tau] = \mathbb{E}[X_\tau]$$

Since  $X_t^\tau$  is also martingale by 4.10.1.4, then

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_t^\tau] = \mathbb{E}[X_0^\tau] = \mathbb{E}[X_0] \quad (\text{since } \tau \leq 0)$$

Finally,

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$$

## 11 cadlag

### 11.1 Definition

It means *continue à droite, limite à gauche* in french, which means *right continuous with left limits*. Figure 4.4 show an example of cadlag function. The reference for left ou right is an abscisse point  $x$ , **not to confuse with the left or the right of function**. In this figure :

- **Continue à droite** :  $\lim_{x \rightarrow a^+} f(x) = f(a)$ .
- **Limite à gauche** :  $\lim_{x \rightarrow a^-} f(x)$  has a limite.

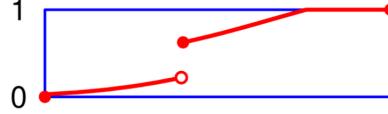


Figure 4.4: cadlag function.

### 11.1.1 Remark

Only right continuous does not imply left limit, e.g. the following function

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

is right continuous but has not left limit at  $x = 0$ .

## 11.2 cadlag process

A cadlag process  $X_t$  is a type of stochastic process that is, for all  $\omega \in \Omega$ , then  $X(t, \omega)$  or  $X_t$  is a cadlag function. cadlag processes allow understanding and modeling real-world phenomena subject to random fluctuations, since we have a mix between continuities and sudden jumps when  $t$  increases.

## 11.3 Theorem

Let  $X$  be an adapted stochastic process with index  $t \in \mathbb{R}^+$ . If either of the following conditions holds :

- For all  $t$ ,  $\mathbb{E}[|X_t|] < \infty$  ( $X_t$  is integrable) and  $\mathbb{E}[\int_0^t \mathbb{1}_A dX]$  is bounded, for all  $A \subseteq \mathbb{R}^+$ .
- For all  $t$ ,  $\int_0^t \mathbb{1}_A dX$  is bounded in probability, for all  $A \subseteq \mathbb{R}^+$ .

Then,  $X$  has a version  $Y$  (ou modification, sec 4.1.6.1) which has left and right limit for all  $t$ , and there is a countable subset  $S \subset \mathbb{R}_+$  for which  $Y_t$  is right-continuous at every  $t \notin S$ .

### 11.3.1 Proof

- $X$  has a version  $Y$  which has left and right limit for all  $t \in \mathbb{R}^+$ .

Let  $t_n \in \mathbb{R}^+$  be sequence which converges monotonically to  $t$ , we show that  $X_{t_n}$  almost surely converges.

By lemma 4.10.8.2 :

$$(b - a)U_{[a,b]}^{X_{t_n}} \leq \int \mathbb{1}_B dX + (a - X_{t_n})^+$$

where  $B$  is indicator set, which equals to 1 between a downcrossing and its nextby upcrossing, 0 otherwise.

- If for all  $t$ ,  $\mathbb{E}[|X_t|] < \infty$  ( $X$  is integrable) and  $\mathbb{E}[\int_0^t \mathbb{1}_A dX]$  is bounded, for all  $A \subseteq \mathbb{R}^+$ . Then

$$\begin{aligned}\mathbb{E}[U_{[a,b]}^{X_{t_n}}] &\leq \frac{\mathbb{E}[\int \mathbb{1}_B dX] + \mathbb{E}[(a - X_{t_n})^+]}{(b-a)} \\ &< \frac{M_1 + a^- + \mathbb{E}[X_{t_n}^+]}{(b-a)} \\ &< \frac{M_1 + a^- + M_2}{(b-a)}\end{aligned}$$

where  $M_1$  is a bound for  $\mathbb{E}[\int \mathbb{1}_B dX]$  (since  $B \subset \mathbb{R}^+$ ), inequality  $\mathbb{E}[(a - X_{t_n})^+] \leq a^- + \mathbb{E}[X_{t_n}^+]$  by 2.2.8.9 and  $M_2$  is a bound for  $\mathbb{E}[X_{t_n}^+]$ . Therefore,  $U_{[a,b]}^{X_{t_n}}$  is finite for all  $n$ .

- If for all  $t$ ,  $\int_0^t \mathbb{1}_A dX$  is bounded in probability, for all  $A \subseteq \mathbb{R}^+$ . Then there exists a function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $P(|\int_0^t \mathbb{1}_A dX| > K) < f(K)$  and  $\lim_{K \rightarrow \infty} f(K) \rightarrow 0$ . Hence

$$\begin{aligned}P((b-a)U_{[a,b]}^{X_{t_n}} > K) &\leq P(\int \mathbb{1}_B dX + (a - X_{t_n})^+ > K) \\ &= P(|\int \mathbb{1}_B dX| + (a - X_{t_n})^+ > K) \\ &\leq P(|\int \mathbb{1}_B dX| > \frac{K}{2}) + P((a - X_{t_n})^+ > \frac{K}{2}) \\ &\leq f(\frac{K}{2}) + P((a - X_{t_n})^+ > \frac{K}{2})\end{aligned}$$

Note that  $\int \mathbb{1}_B dX \leq 0$  and **for a given  $t_n$** , we move  $K \rightarrow \infty$  after, then  $\lim_{K \rightarrow \infty} P((b-a)U_{[a,b]}^{X_{t_n}} > K) = 0$  or  $P(U_{[a,b]}^{X_{t_n}} = \infty) = 0$  or  $U_{[a,b]}^{X_{t_n}}$  is finite for all  $n$ .

Finally, from lemma 4.10.8.4,  $X_{t_n}$  almost surely converges.

- There is a countable subset  $S \subset \mathbb{R}_+$  for which  $Y_t$  is right-continuous at every  $t \notin S$ . <https://almostsuremath.com/2009/12/18/cadlag-modifications/>

### 11.3.2 Corollary

If now  $X$  is (in addition) right continuous in probability, then  $X$  has a version  $Y_t$  which is cadlag for all  $t$ .

## 11.4 Cadlag martingales

### 11.4.1 Lemma

Let  $X$  be a submartingale (4.10.2). Then, it has a version  $Y$  which has left and right limit for all  $t$ , and there is a countable subset  $S \subset \mathbb{R}_+$  for which  $Y_t$  is right-continuous at every  $t \notin S$ .

### Proof

We see that  $X$  is integrable since  $X$  is submartingale, moreover a set  $A \subseteq [0, t]$  can be expressed by  $A = \bigcup_{k=1}^N [l_k, u_k]$ ,  $N$  can be infinity. Then on one hand,  $\forall A :$

$$\mathbb{E}\left[\int_0^t \mathbb{1}_A dX\right] = \mathbb{E}\left[\sum_{k=1}^N (X_{u_k} - X_{l_k})\right] = \sum_{k=1}^N (\mathbb{E}[X_{u_k}] - \mathbb{E}[X_{l_k}]) \geq 0$$

since  $X_t$  is submartingale which implies  $\mathbb{E}[X_{u_k}] \geq \mathbb{E}[X_{l_k}]$ . From this we infer that  $\mathbb{E}\left[\int_0^t \mathbb{1}_{A^C} dX\right]$  is positive since  $A^C$  is also a subset of  $[0, t]$ .

On the other hand,

$$\begin{aligned} \mathbb{E}\left[\int_0^t \mathbb{1}_A dX\right] &= \mathbb{E}\left[\int_0^t \mathbb{1}_{[0,t]} dX\right] - \mathbb{E}\left[\int_0^t \mathbb{1}_{A^C} dX\right] \\ &\leq \mathbb{E}[X_t] - \mathbb{E}[X_0] \quad \text{since } \mathbb{E}\left[\int_0^t \mathbb{1}_{A^C} dX\right] \geq 0 \\ &< \infty \quad \text{since } X \text{ is integrable} \end{aligned}$$

This shows that  $\mathbb{E}\left[\int_0^t \mathbb{1}_A dX\right]$  is bounded, for all  $A \subseteq [0, t]$ . Then by theorem 4.11.3, we have Q.E.D.

**Remark** This result is also valable for supermartingale (just replacing  $X$  by  $-X$ ) and martingale.

#### 11.4.2 Lemma

Suppose that a submartingale  $X$  is adapted to a right-continuous filtration :  $\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$  or  $\mathcal{F}_t = \mathcal{F}_{t^+}$ , which means that no information is added with infinitesimal step.

Then  $X$  has a cadlag version if and only if  $t \mapsto \mathbb{E}[X_t]$  is right-continuous.

### Proof

By lemma 4.11.4.1, since  $X_t$  is a submartingale, it has a version  $Y_t$  which has left and right limit for all  $t$ , such that there is a countable subset  $S \subset \mathbb{R}_+$  for which  $Y_t$  is right-continuous at every  $t \notin S$ .

**In direct sense**, we show that if  $t \mapsto \mathbb{E}[X_t]$  is right-continuous then  $X$  has a cadlag version. Thus we need to prove that :

$$\lim_{t \rightarrow a^+} Y_t = Y_a, \quad \forall a \in S$$

or equivalently, if  $t_n$  is a decreasing sequence to  $a$ :

$$\lim_{n \rightarrow \infty} Y_{t_n} = Y_a, \quad \forall a \in S$$

Here is a sketch of proof

1.  $Y_{t_n} \xrightarrow[n \rightarrow \infty]{a.s.} Y_a$  since  $Y_t$  has right limit.

2.  $Y_{t_n}$  is uniformly integrable, by lemma 4.10.6.
3.  $\lim_{n \rightarrow \infty} \mathbb{E}[Y_{t_n}] = \mathbb{E}[Y_{a^+}]$ . This is because  $\lim_{n \rightarrow \infty} \mathbb{E}[|Y_{t_n} - Y_{a^+}|] = 0$  (convergence in mean) by lemma 2.4.5.4, with the first two points.
4.  $\lim_{n \rightarrow \infty} \mathbb{E}[Y_{t_n}] = \mathbb{E}[Y_a]$ . This is because  $\mathbb{E}[X_t]$  is right-continuous, then  $\mathbb{E}[Y_t]$  is also right continuous.
5.  $\mathbb{E}[Y_a] = \mathbb{E}[Y_{a^+}]$ , then  $\mathbb{E}[Y_a | \mathcal{F}_a] = \mathbb{E}[Y_{a^+} | \mathcal{F}_a]$ , then  $Y_a = \mathbb{E}[Y_{a^+} | \mathcal{F}_a]$  (since  $Y_a = \mathbb{E}[Y_a | \mathcal{F}_a]$ ), then  $Y_a = \mathbb{E}[Y_{a^+} | \mathcal{F}_{a^+}]$  (since  $\mathcal{F}_a = \mathcal{F}_{a^+}$ ), then  $Y_a = Y_{a^+}$  (since  $Y_{a^+} = \mathbb{E}[Y_{a^+} | \mathcal{F}_{a^+}]$ ). We have Q.E.D.

**In converse sense**, it is trivial by using the two last points in above proof.

## 12 Martingale convergence with continuous index

In sec 4.10.9, we've seen the convergence for martingale with discret index. In this section, we work with continuous index. We start first with stochastic processes which are only right-continuous (e.g. cadlag), then continuous martingale stochastic processes.

### 12.1 Only right-continuous

$$\lim_{t \rightarrow a^+} X_t = X_a$$

#### 12.1.1 Doob's first martingale convergence theorem

Let  $X_t$  be a right continuous submartingale that satisfies following conditions:

$$\sup_{t \geq 0} \mathbb{E}[X_t^+] < +\infty$$

Then, the limit  $\lim_{t \rightarrow \infty} X_t = X_\infty$  exists with probability one (a.s. convergence) and  $\mathbb{E}[X_\infty]$  is finite.

**Proof**

Let's  $\mathcal{F}_t$  the filtration of  $X$ . We define the stochastic process  $Y$  by

$$Y_t = \begin{cases} X_{\frac{t}{1-t}} & \text{if } t < 1 \\ 0 & \text{if } t \geq 1 \end{cases}$$

with  $G_t = \mathcal{F}_{\frac{t}{1-t}}$  when  $t < 1$  and  $G_t = \mathcal{F}_\infty$  otherwise. By consequence,  $Y_t$  is right continuous with  $t \geq 0$ , submartingale with  $t < 1$  and  $\sup_{0 \leq t < 1} \mathbb{E}[Y_t^+] < +\infty$ .

Now the main problem is to show that  $\lim_{t \rightarrow 1^-} Y_t$  converges almost surely. We will show that  $Y_t$  is a cadlag by 4.11.3 (with the first condition). Note that  $Y_t$  is  $L_1$ -bounded for  $t \geq 1$  since  $Y_t = 0$ . If  $t < 1$  :

$$\begin{aligned}\mathbb{E}[|Y_t|] &= \mathbb{E}[|Y_t^+|] + \mathbb{E}[|Y_t^-|] \\ &= \mathbb{E}[Y_t^+] + \mathbb{E}[Y_t^-] \\ &\leq 2\mathbb{E}[Y_t^+] - \mathbb{E}[Y_t] \\ &< +\infty - \mathbb{E}[Y_0] \\ &< +\infty\end{aligned}$$

Then we only need to show that,  $\forall t, \mathbb{E}[\int_0^t \mathbb{1}_A dY]$  is bounded for all  $A \subseteq \mathbb{R}^+$ . Let's  $A = \bigcup_{k=1}^N [l_k, u_k]$ ,  $N$  which can be infinity. Suppose that  $l_m$  is the first time moment that is greater than 1. We have :

$$\begin{aligned}\mathbb{E}\left[\int_0^t \mathbb{1}_A dY\right] &= \mathbb{E}\left[\sum_{k=1}^{m-1} (Y_{u_k} - Y_{l_k})\right] \\ &= \mathbb{E}[Y_{u_{m-1}}] - \mathbb{E}[Y_{l_{m-1}}] + \sum_{k=1}^{m-2} (\mathbb{E}[Y_{u_k}] - \mathbb{E}[Y_{l_k}]) \\ &\geq \mathbb{E}[Y_{u_{m-1}}] - \mathbb{E}[Y_{l_{m-1}}] + 0 \quad (\text{since } Y_t \text{ is submartingale with } t < 1) \\ &= \begin{cases} -\mathbb{E}[Y_{l_{m-1}}] & \text{if } u_{m-1} < 1 \\ 0 & \text{if } u_{m-1} \geq 1 \end{cases} \\ &= \begin{cases} -\mathbb{E}[Y_{l_{m-1}}^+] + \mathbb{E}[Y_{l_{m-1}}^-] & \text{if } u_{m-1} < 1 \\ 0 & \text{if } u_{m-1} \geq 1 \end{cases} \\ &= \begin{cases} > -\infty + 0 & \text{if } u_{m-1} < 1 \quad \text{since } Y_{l_{m-1}}^- \leq 0 \\ = 0 & \text{if } u_{m-1} \geq 1 \end{cases}\end{aligned}$$

Then  $\mathbb{E}[\int_0^t \mathbb{1}_A dY]$  is lower bounded. Next,

$$\begin{aligned}\mathbb{E}\left[\int_0^t \mathbb{1}_A dY\right] &= \mathbb{E}\left[\sum_{k=1}^{m-1} (Y_{u_k} - Y_{l_k})\right] \\ &= \mathbb{E}[Y_{u_{m-1}}] - \sum_{k=m-1}^2 (\mathbb{E}[Y_{l_k}] - \mathbb{E}[Y_{u_{k-1}}]) - \mathbb{E}[Y_{l_1}] \\ &\leq \mathbb{E}[Y_{u_{m-1}}] - \mathbb{E}[Y_{l_1}] \quad (\text{since } Y_t \text{ is submartingale with } t < 1) \\ &\leq \mathbb{E}[|Y_{u_{m-1}}|] + \mathbb{E}[|Y_{l_1}|] \\ &< +\infty\end{aligned}$$

Then  $\mathbb{E}[\int_0^t \mathbb{1}_A dY]$  is bounded. Therefore, we have  $Y$  is cadlag, it means it has left limit or  $\lim_{t \rightarrow 1^-} Y_t$  converges a.s. or  $\lim_{t \rightarrow \infty} X_t \rightarrow X_\infty$ .

In case of supermartingale, we need  $\sup_{t \geq 0} \mathbb{E}[X_t^-] < +\infty$  and in case of martingale we need  $\sup_{t \geq 0} \mathbb{E}[|X_t|] < +\infty$ .

### 12.1.2 Corollary

If  $X$  is a nonnegative right-continuous supermartingale then the limit  $X_\infty = \lim_{t \rightarrow \infty} X_t$  exists and is finite, with probability one.

#### Proof

Since  $X_t$  is nonnegative then  $\mathbb{E}[X_t^-] = 0 < \infty$ , so by the previous theorem 4.12.1.1, we have Q.E.D.

### 12.1.3 Theorem

Let  $X$  be right-continuous martingale that is uniformly integrable, then there exists unique random variable  $X_\infty$  which is  $\mathcal{F}_\infty$ -measurable such that :

$$X_t = \mathbb{E}[X_\infty | \mathcal{F}_t], \quad \forall t \in \mathbb{R}_+$$

#### Proof

- As uniformly integrable process  $X$  infers that  $X$  is  $L_1$ -bounded, then Doob's first martingale convergence theorem (4.12.1.1) shows that the limit  $X_\infty = \lim_{t \rightarrow \infty} X_t$  exists and is almost surely finite.
- With  $(s > t)$  :
  - Since  $X$  is a martingale, then  $\mathbb{E}[X_s | \mathcal{F}_t] = X_t$ .
  - Since  $\lim_{s \rightarrow \infty} X_s \xrightarrow{a.s.} X_\infty$ , then  $\lim_{s \rightarrow \infty} X_s \xrightarrow{p} X_\infty$ . In addition,  $X_s$  is uniformly integrable, then by 2.4.5.4,  $X_s \xrightarrow{L_1} X_\infty$  or  $\lim_{s \rightarrow \infty} \mathbb{E}[|X_s - X_\infty|] = 0$  or we can write  $\lim_{s \rightarrow \infty} \mathbb{E}[X_s | \mathcal{F}_t] = \mathbb{E}[X_\infty | \mathcal{F}_t]$ .

From these two points, we get  $X_t = \mathbb{E}[X_\infty | \mathcal{F}_t]$ .

- Uniqueness : if there is  $Z$  such that  $\mathbb{E}[Z | \mathcal{F}_t] = X_t, \forall t$ . Then

$$\begin{aligned} \mathbb{E}[\mathbb{1}_A(X_\infty - Z)] &= \mathbb{E}[X_\infty \cdot \mathbb{1}_A] - \mathbb{E}[Z \cdot \mathbb{1}_A] \\ &= \mathbb{E}[X_\infty | \mathcal{F}_t](A)P(A) - \mathbb{E}[Z | \mathcal{F}_t](A)P(A) \quad (\text{By 2.2.12.4.1}) \\ &= X_t(A)P(A) - X_t(A)P(A) \\ &= 0 \quad \forall A \in \mathcal{F}_t \end{aligned}$$

Then by the equality by expectation 2.2.6, we have  $X_\infty = Z$ .

This is the continuous version of 4.10.10

## 12.2 Continuous

A stochastic process  $X_t$  is continuous if and only if :

$$\lim_{t \rightarrow a} X_t = X_a$$

### 12.2.1 Continuous martingale convergence

Let  $X$  be a **continuous martingale**. Then, almost surely, one of the following is satisfied :

- $X_\infty = \lim_{t \rightarrow \infty} X_t$  exists and is finite.
- $\limsup_{t \rightarrow \infty} X_t = +\infty$  and  $\liminf_{t \rightarrow \infty} X_t = -\infty$ . In this case, the process hits every value in  $\mathbb{R}$  at arbitrarily large times.

We can separate into two cases :

- $\sup_{t \geq 0} \mathbb{E}[|X_t|] < C$ . We use the theorem 4.12.1.1 and we have the first statement.
- $\sup_{t \geq 0} \mathbb{E}[|X_t|] = \infty$ . In this case, we can consider the continuous Brownian motion, which is a continuous martingale. Thus continuous Brownian motion is not  $L_1$ -bounded and  $X_t$  can reach  $+\infty$  or  $-\infty$  if  $t \rightarrow \infty$ .

### 12.2.2 Continuous submartingale convergence

Let  $X$  be a **continuous submartingale**. Then, almost surely, one of the following is satisfied.

- $X_\infty = \lim_{t \rightarrow \infty} X_t$  exists and is finite.
- $\limsup_{t \rightarrow \infty} X_t = +\infty$ .

The same as in 4.12.2.1, except that this is submartingale, then  $\liminf_{t \rightarrow \infty} X_t = -\infty$  is not possible.

## 13 Doob's martingale inequality

### 13.1 Statement of the inequality

#### 13.1.1 Discret case

Let  $X_1, \dots, X_n$  be a discrete-time submartingale relative to a filtration  $\mathcal{F}_1, \dots, \mathcal{F}_n$  and  $C > 0$ . Then

$$P\left(\max_{1 \leq i \leq n} X_i \geq C\right) \leq \frac{\mathbb{E}[X_n \mathbb{1}_{\{\max_{1 \leq i \leq n} X_i \geq C\}}]}{C} \leq \frac{\mathbb{E}[\max(X_n, 0)]}{C} = \frac{\mathbb{E}[X_n^+]}{C}$$

**Proof**

Let define sets:

$$W_i = \{\omega: X_i \geq C \text{ and } X_j < C, \forall j < i\}$$

therefore, we have that  $W_i$  are pairwise disjoint and

$$P(\max_{1 \leq i \leq n} X_i \geq C) = P\left(\bigcup_{i=1}^n W_i\right) = P(W)$$

$$\text{where } W = \bigcup_{i=1}^n W_i = \{\omega: \max_{1 \leq i \leq n} X_i(\omega) \geq C\}.$$

$$\begin{aligned} CP(W_i) &= C \int_{W_i} dP \\ &= \int_{W_i} C dP \\ &\leq \int_{W_i} X_i dP \\ &\leq \int_{W_i} \mathbb{E}[X_n | \mathcal{F}_i](W_i) dP \\ &= \int_{W_i} X_n dP \end{aligned}$$

where the last equality is by 2.2.12. Since  $W_i$  are pairwise disjoint sets, then

$$\begin{aligned} CP(W) &= CP\left(\bigcup_{i=1}^n W_i\right) \\ &= \sum_{i=1}^n CP(W_i) \\ &\leq \sum_{i=1}^n \int_{W_i} X_n dP \\ &= \int_W X_n dP \quad (\text{by disjoint sets}) \\ &= \int_{\Omega} X_n \mathbb{1}_W dP \\ &= \mathbb{E}[X_n \mathbb{1}_W] = \mathbb{E}[X_n \mathbb{1}_{\{\max_{1 \leq i \leq n} X_i \geq C\}}] \\ &\leq \mathbb{E}[\max(X_n, 0)] \end{aligned}$$

where we have the last inequality because 2.2.8.10.2

Note that, if the submartingale is restricted to a random variable and non-negative then we have Markov inequality 2.2.8.1.

### 13.1.2 Continuous case

Let  $X_t$  be a right-continuous submartingale then

$$P\left(\sup_{0 \leq t \leq T} X_t \geq C\right) \leq \frac{\mathbb{E}[X_T \mathbb{1}_{\{\sup_{0 \leq t \leq T} X_t \geq C\}}]}{C} \leq \frac{\mathbb{E}[\max(X_T, 0)]}{C} = \frac{\mathbb{E}[X_T^+]}{C}$$

#### Proof

The proof can be inspired from discret case 4.13.1.1. Given a fixed  $\omega$ , we consider the first time at which the process reaches the value  $C$  (first hitting):

$$\tau(\omega) = \inf_{0 \leq t \leq T} X_t(\omega) \geq C$$

where  $\tau$  now is also a random variable. From this defintion, we can trivially show that :

$$\{\omega : \sup_{0 \leq t \leq T} X_t \geq C\} = \{\omega : \tau(\omega) \leq T\} = W$$

Hence,

$$\begin{aligned} C \mathbb{1}_W &\leq X_\tau \mathbb{1}_W \\ \Leftrightarrow C \mathbb{E}[\mathbb{1}_W] &\leq \mathbb{E}[X_\tau \mathbb{1}_W] \\ \Leftrightarrow CP(W) &\leq \mathbb{E}[X_\tau \mathbb{1}_W] \\ &\leq \mathbb{E}[X_T \mathbb{1}_W] \\ &\leq \mathbb{E}[X_T^+] \end{aligned}$$

where the before last inequality is by 4.10.2.4 and the last inequality is by 2.2.8.10.2.

## 13.2 Further inequalities

### 13.2.1 Inequality I

Let  $X_t$  be a right-continuous martingale or right-continuous positive submartingale. Then with  $p \geq 1$  :

$$P\left(\sup_{0 \leq t \leq T} |X_t| \geq C\right) \leq \frac{\mathbb{E}[|X_T|^p]}{C^p}$$

#### Proof

- In case that  $X_t$  is martingale, then with  $f(x) = |x|^p$  where  $p \geq 1$  is a convex function and by lemma 4.10.2.1, we have  $|X_t|^p$  is submartingale.
- In case that  $X_t$  is positive submartingale, then with  $f(x) = |x|^p$  where  $p \geq 1$  is a convex function and  $f$  is increasing for  $x \geq 0$ , by lemma 4.10.2.2, we have  $|X_t|^p$  is submartingale.

Therefore, in both cases,  $|X_t|^p$  is submartingale, then by inequality 4.13.1.2, we have :

$$P\left(\sup_{0 \leq t \leq T} |X_t|^p \geq C\right) \leq \frac{\mathbb{E}[|X_T|^p \mathbb{1}_{\{\sup_{0 \leq t \leq T} |X_t|^p \geq C\}}]}{C} \leq \frac{\mathbb{E}[\max(|X_T|^p, 0)]}{C} = \frac{\mathbb{E}[|X_T|^p]}{C}$$

Now we replace the constant  $C$  by  $C^p$ , then

$$\begin{aligned} P\left(\sup_{0 \leq t \leq T} |X_t|^p \geq C^p\right) &\leq \frac{\mathbb{E}[|X_T|^p \mathbb{1}_{\{\sup_{0 \leq t \leq T} |X_t|^p \geq C^p\}}]}{C^p} \leq \frac{\mathbb{E}[|X_T|^p]}{C^p} \\ \Leftrightarrow P\left(\sup_{0 \leq t \leq T} |X_t| \geq C\right) &\leq \frac{\mathbb{E}[|X_T|^p \mathbb{1}_{\{\sup_{0 \leq t \leq T} |X_t| \geq C\}}]}{C^p} \leq \frac{\mathbb{E}[|X_T|^p]}{C^p} \end{aligned}$$

### 13.2.2 Inequality II

Let  $X_t$  is a non-negative submartingale and  $p > 1$ , then

$$\mathbb{E}[|X_T|^p] \leq \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_T|^p]$$

Sometimes known as Doob's maximal inequality. The first inequality is evident since  $\sup_{0 \leq t \leq T} |X_t|^p \geq |X_T|^p$ . For short, we denote  $X_T^* = \sup_{0 \leq t \leq T} X_t$ .

#### Proof

The non-negative is because we are in case that  $x^p$  and it requires that  $x \geq 0$ . By using (twice) the expectation with exponentiation 2.2.3 :

$$\begin{aligned} \mathbb{E}[(X_T^*)^p] &= p \int_0^\infty t^{p-1} \mathbb{E}[\mathbb{1}_{\{X_T^* > t\}}] dt \quad (\text{first use}) \\ &= p \int_0^\infty t^{p-1} P(X_T^* > t) dt \\ &= p \int_0^\infty t^{p-2} t P(X_T^* > t) dt \\ &\leq p \int_0^\infty t^{p-2} \mathbb{E}[X_T \mathbb{1}_{\{X_T^* > t\}}] dt \quad (\text{inequality 4.13.1.2}) \\ &= \frac{p}{p-1} \mathbb{E}[X_T (X_T^*)^{p-1}] \quad (\text{second use}) \\ &= \frac{p}{p-1} \mathbb{E}[(X_T)^p]^{\frac{1}{p}} \mathbb{E}[(X_T^*)^{(p-1)q}]^{\frac{1}{q}} \quad (\text{Holder inequality 2.2.8.6}) \\ &= \frac{p}{p-1} \mathbb{E}[(X_T)^p]^{\frac{1}{p}} \mathbb{E}[(X_T^*)^p]^{\frac{p-1}{p}} \quad (\text{by taking } q = \frac{p}{p-1}) \end{aligned}$$

We infer that :

$$\mathbb{E}[(X_T^*)^p]^{\frac{1}{p}} \leq \frac{p}{p-1} \mathbb{E}[(X_T)^p]^{\frac{1}{p}}$$

$$\Leftrightarrow \mathbb{E}[(X_T^*)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[(X_T)^p]$$

### 13.2.3 Inequality III

Let  $X_t$  is a non-negative submartingale, then

$$\mathbb{E}[X_T^*] \leq \frac{e}{e-1} (1 + \mathbb{E}[X_T \ln X_T])$$

#### 13.2.3.1 Lemma

For all  $x > 0$  and  $y$  :

$$x \ln(\max(1, y)) + \min(1, y) \leq x \ln x + e^{-1}y + 1$$

Let's consider :

$$f(x) = x \ln x - x \ln(\max(1, y)) + e^{-1}y + 1 - \min(1, y)$$

and its derivative :

$$f'(x) = \ln x + 1 - \ln(\max(1, y))$$

Then the first-order optimality condition gives :

$$x_0 = \exp(\ln(\max(1, y)) - 1) = e^{-1}(\max(1, y))$$

Replace in  $f$  :

$$\begin{aligned} f(x_0) &= e^{-1}(\max(1, y))(\ln(\max(1, y)) - 1) - e^{-1}(\max(1, y)) \ln(\max(1, y)) + e^{-1}y + 1 - \min(1, y) \\ &= -e^{-1}(\max(1, y)) + e^{-1}y + 1 - \min(1, y) \\ &= e^{-1}(y - \max(1, y)) + 1 - \min(1, y) \\ &= e^{-1}(\min(1, y) - 1) + 1 - \min(1, y) \\ &= (1 - \min(1, y))(1 - e^{-1}) \\ &\geq 0 \end{aligned}$$

#### 13.2.3.2 Proof

Since  $X_T^*$  is non-negative, then by 2.2.4.2 :

$$\begin{aligned} \mathbb{E}[(X_T^* - 1)^+] &= \int_1^\infty P(X_T^* \geq x) dx \\ &\leq \int_1^\infty \frac{\mathbb{E}[X_T \mathbb{1}_{\{X_T^* \geq x\}}]}{x} dx \quad (\text{by 4.13.1.2}) \\ &= \int_1^\infty \frac{1}{x} \int_\Omega X_T \mathbb{1}_{\{X_T^* \geq x\}} dP dx \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} X_T \int_1^{X_T^*} \frac{1}{x} dx dP \\
&= \int_{\Omega} X_T \ln(\max(1, X_T^*)) dP \\
&= \mathbb{E}[X_T \ln(\max(1, X_T^*))]
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathbb{E}[X_T^*] &= \mathbb{E}[(X_T^* - 1)^+] + \mathbb{E}[(X_T^* - 1)^-] \\
&\leq \mathbb{E}[X_T \ln(\max(1, X_T^*))] + \mathbb{E}[(X_T^* - 1)^-] \\
&= \mathbb{E}[X_T \ln(\max(1, X_T^*))] + \mathbb{E}[\min(1, X_T^*)] \\
&\leq \mathbb{E}[X_T \ln X_T + e^{-1} X_T^* + 1] \quad (\text{by lemma 4.13.2.3.1}) \\
&= \mathbb{E}[X_T \ln X_T] + 1 + e^{-1} \mathbb{E}[X_T^*]
\end{aligned}$$

then

$$\mathbb{E}[X_T^*] \leq \frac{e}{e-1} (1 + \mathbb{E}[X_T \ln X_T])$$

## 14 Local martingale

In a simple intuition, a local martingale is a stochastic process which is locally a martingale.

### 14.1 Formal definition

A process  $X_t$  is a local martingale if there exists a sequence of stopping times  $\tau_k : \Omega \rightarrow [0, +\infty)$  such that :

- $\tau_k$  are almost surely increasing :  $P(\tau_k < \tau_{k+1}) = 1$ .
- $\tau_k$  tends toward infinity almost surely :  $P\left(\lim_{k \rightarrow \infty} \tau_k = \infty\right) = 1$
- The stopped process  $X_t^{\tau_k} := X_{\min\{t, \tau_k\}} = X_{t \wedge \tau_k}$  is a martingale for every  $k$ .

### 14.2 Example

Consider the following process :

$$X_t = \begin{cases} W_{\frac{t}{1-t}}^T & \text{for } 0 \leq t < 1 \\ -1 & \text{for } 1 \leq t < \infty \end{cases}$$

where  $W$  is Brownian motion and  $T = \min\{l : W_l = -1\}$ , which means the first time that  $W$  hits  $-1$ . Some simple remarks :

- $X_t$  is continuous almost surely.
- $X_t$  is not martingale since  $\mathbb{E}[X_t]$  is not constant for all  $t$  since before that  $W_l$  hits -1,  $\mathbb{E}[X_t] = 0$  and after that  $W_l$  hits -1,  $\mathbb{E}[X_t] = -1$ .

Prove that  $X_t$  is local martingale.

### Proof

Let's consider the sequence :

$$\tau_k = \frac{k}{k+1} \mathbb{1}_{\{T \geq k\}} + \left( \frac{T}{T+1} + k \right) \mathbb{1}_{\{T < k\}}$$

From the definition of sequence  $\tau_k$  :

- $\tau_k$  is almost surely increasing, since

$$\begin{cases} \frac{T}{T+1} + k < \frac{T}{T+1} + (k+1) & \text{if } T < k \\ \frac{k}{k+1} < \frac{T}{T+1} + (k+1) & \text{if } k \leq T < k+1 \\ \frac{k}{k+1} < \frac{k+1}{k+2} & \text{if } T \geq k+1 \end{cases}$$

- $\tau_k$  tends toward infinity almost surely, since if  $k$  is large enough then  $\lim_{k \rightarrow +\infty} \frac{T}{T+1} + k = +\infty$ .
- The stopped process  $X_t^{\tau_k} := X_{\min\{t, \tau_k\}}$  is a martingale for every  $k$ . For some remarks:

$$\begin{cases} \tau_k = \frac{k}{k+1} < 1 & \text{if } T \geq k \\ \tau_k = \frac{T}{T+1} + k > 1 & \text{if } T < k \end{cases}$$

Let's rewrite  $X_t$  by :

$$X_t = W_{\frac{t}{1-t}}^T \mathbb{1}_{\{0 \leq t < 1\}} + (-1) \mathbb{1}_{\{t \geq 1\}}$$

Then consider  $X_t^{\tau_k} = X_{t \wedge \tau_k}$  with 4 following configurations :

– If  $\tau_k < 1$  or  $T \geq k$  :

- \* In the case  $t < 1$ . Then  $t \wedge \tau_k = t \wedge \frac{k}{k+1}$ . In corresponding to scale of  $W$  with transformation  $l = \frac{t}{1-t}$ , it means the stopping time (w.r.t.  $l$ ) is  $\frac{k}{1-\frac{k}{k+1}} = k$ . Hence, in this case the stopping time is  $T \wedge k = k$  **with time instant**  $l = \frac{t}{1-t}$  :

$$X_t^{\tau_k} = W_{\frac{t}{1-t}}^k$$

- \* In the case  $t > 1$  :

$$X_t^{\tau_k} = W_k$$

- If  $\tau_k > 1$  or  $T < k$  :
  - \* In the case that  $t < 1$  :

$$X_t^{\tau_k} = W_{\frac{t}{1-t}}^T$$

- \* In the case that  $t > 1$  :

$$X_t^{\tau_k} = W_T$$

Finally,

$$X_t^{\tau_k} = W_{\frac{t}{1-t}}^{T \wedge k} \mathbb{1}_{\{0 \leq t < 1\}} + W_{T \wedge k} \mathbb{1}_{\{t \geq 1\}}$$

We now show that  $X_t^{\tau_k}$  is a martingale. First, as  $W_l$  is a martingale and with 4.10.1.4, the stopped process  $W_l^{T \wedge k}$  is also a martingale, i.e.

$$\mathbb{E}[W_v^{T \wedge k} | \tilde{\mathcal{F}}_u] = W_u^{T \wedge k}, \quad \forall v \geq u \quad (*)$$

Therefore,

- Case  $1 \geq t \geq s$ . On one hand, if  $1 > t$ , by (\*) :

$$\mathbb{E}[X_t^{\tau_k} | \mathcal{F}_s] = X_s^{\tau_k}, \quad \forall 1 > t \geq s$$

On the other hand, if  $t = 1$ , always by (\*), with  $v = T \wedge k$ , then

$$\begin{aligned} \mathbb{E}[W_{T \wedge k} | \tilde{\mathcal{F}}_u] &= W_u^{T \wedge k}, \quad \forall u \\ \Leftrightarrow \mathbb{E}[X_1^{\tau_k} | \mathcal{F}_s] &= X_s^{\tau_k}, \quad \forall 1 \geq s \end{aligned}$$

Hence, in both hands:

$$\mathbb{E}[X_t^{\tau_k} | \mathcal{F}_s] = X_s^{\tau_k}, \quad \forall 1 \geq t \geq s$$

- Case  $t \geq s \geq 1$ .

$$\begin{aligned} \mathbb{E}[X_t^{\tau_k} | \mathcal{F}_s] &= \mathbb{E}[W_{T \wedge k} | \mathcal{F}_s] \quad (\text{since } t \geq 1) \\ &= W_{T \wedge k} \quad (\text{since } s \geq 1) \\ &= X_s^{\tau_k} \end{aligned}$$

- Case  $t \geq 1 \geq s$ .

$$\begin{aligned} \mathbb{E}[X_t^{\tau_k} | \mathcal{F}_s] &= \mathbb{E}[W_{T \wedge k} | \mathcal{F}_s] \quad (\text{since } t \geq 1) \\ &= \mathbb{E}[W_{T \wedge k}^{T \wedge k} | \mathcal{F}_s] \\ &= \mathbb{E}[W_{T \wedge k}^{T \wedge k} | \tilde{\mathcal{F}}_{\frac{s}{1-s}}] \quad (\text{since } s \leq 1) \\ &= W_{\frac{s}{1-s}}^{T \wedge k} \quad (\text{since } W_l^{T \wedge k} \text{ is martingale}) \\ &= X_s^{\tau_k} \end{aligned}$$

### 14.3 Example

Given a standard 2-dimensional Brownian motion  $\mathbf{B}_t = (X_t, Y_t)$ , see 4.8.3, we define the process  $M_t$  as

$$M_t = e^{X_t} \cos(Y_t)$$

Show that the process  $M_t$  is a local martingale and

$$[M]_t = \int_0^t e^{2X_s} ds$$

#### Proof

Hence by taking derivative of  $M_t$  as showed in example 4.17.1.3:

$$dM_t = e^{X_t} \cos(Y_t) dX_t - e^{X_t} \sin(Y_t) dY_t$$

Since

- $X_t$  and  $Y_t$  are local martingale.
- $e^{X_t} \cos(Y_t)$  and  $e^{X_t} \sin(Y_t)$  are predictable since they are left continuous process.

Then by a property of stochastic integral (4.16.5), we have :

$$\int e^{X_s} \cos(Y_s) dX_s - e^{X_s} \sin(Y_s) dY_s = \int dM_t = M_t$$

is local martingale.

Using property of quadratic variation (4.8.2),  $[M]_t = [M, M]_t = \int_0^t dM_s dM_s$ , then

$$\begin{aligned} [M]_t &= \int_0^t (e^{X_s} \cos(Y_s) dX_s - e^{X_s} \sin(Y_s) dY_s)(e^{X_s} \cos(Y_s) dX_s - e^{X_s} \sin(Y_s) dY_s) \\ &= \int_0^t e^{2X_s} \cos(Y_s)^2 (dX_s)^2 + e^{2X_s} \sin(Y_s)^2 (dY_s)^2 - 2e^{2X_s} \cos(Y_s) \sin(Y_s) dX_s dY_s \\ &= \int_0^t e^{2X_s} \cos(Y_s)^2 (dX_s)^2 + e^{2X_s} \sin(Y_s)^2 (dY_s)^2 \quad (dX_s dY_s = 0 \text{ since } X_s, Y_s \text{ are independent}) \\ &= \int_0^t e^{2X_s} ds \quad (\text{since } (dX_s)^2 = (dY_s)^2 = ds) \end{aligned}$$

### 14.4 Properties

- Every martingale is a local martingale but the inverse is not true.
- Every bounded local martingale is a martingale
- Local martingale that is bounded from below is a supermartingale
- Local martingale that is bounded from above is a submartingale

## 15 Semimartingale

A real valued stochastic process  $X$  defined on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  is called a semimartingale if it can be decomposed as :

$$X_t = M_t + A_t$$

where  $M_t$  is a local martingale (4.14) and  $A_t$  is a cadlag adapted process of locally bounded variation (or finite-variation, 2.2.9). (cadlag means continue à droite, limite à gauche in 4.11)

### 15.0.1 Properties

1. The semimartingales **form the largest class of stochastic processes** for which the Ito integral can be defined. This means that if a stochastic process is semimartingale, then its Ito integral exists.
2. Linear combinations of semimartingales are semimartingales.
3. Products of semimartingales are semimartingales, which is a consequence of the integration by parts formula for the Ito integral.
4. The quadratic variation (4.8.2) exists for every semimartingale.

## 16 Stochastic Integrals

This means to calculate the following integral with integrand  $X_t$  and integrator  $B_t$  are both stochastic process :

$$Y_t - Y_0 = \int_0^t X_s dB_s,$$

The result of the integration if exists, here  $Y_t$ , is then another stochastic process. Stochastic integral (SI) is very important in mathematical finance and stochastic differential equations. For an example in mathematical finance, the integrand  $X_s$  represents how much stock we hold at  $s$ , the integrator  $B_s$  represents the price at  $s$ , and the integral is how much money we have (or lose) from instant 0 to instant  $t$ .

The relation between  $Y, X$  and  $B$  can be equivalently represented by following ways :

- $Y = X \cdot B$
- $dY_t = X_t dB_t$

A necessary condition for the existing of integration is that  $B$  must be a semimartingale (4.15). Usually, we work with the case that  $B$  is Brownian motion. As in ordinary integral, a fundamental method to calculate SI is to construct a partition and let's this partition going to zero, then use elementary process (4.10.7).

## 16.1 Integrability of SI with Brownian motion integrator

We discover here a condition in that a stochastic integral is defined or integrable. Let  $\mathcal{L}^2 = \mathcal{L}^2(S, T)$  ( $S, T$  are two bounds) be the class that contains all SPs  $g$  such that :

- $g(t, \omega) : [0, \infty] \times \Omega \rightarrow \mathbb{R}$
- $g(t, \omega)$  is  $\mathcal{F}_t^B$ -adapted, where  $B$  is Wiener process (see 4.1.4.3)
- $\mathbb{E} \left[ \int_S^T g(t, \omega)^2 dt \right] < \infty$  (square-integrable process)

If a SP  $h \in \mathcal{L}(S, T)^2$ , then  $h$  is integrable on  $[S, T]$ , or

$$\int_S^T h(t, \omega) dB_t(\omega) = \lim_{n \rightarrow \infty} \int_S^T \phi_n(t, \omega) dB_t(\omega)$$

is defined or integrable.  $\phi_n(t, \omega)$  is call the elementary function of  $h(t, \omega)$  and is definded by :

$$\phi_n(t, \omega) = \sum_{i=0}^n h(t_i, \omega) \mathbb{1}_{[t_i, t_{i+1})}(t)$$

### 16.1.1 Notes

The second condition,  $g(t, \omega)$  is  $\mathcal{F}_t^B$ -adapted,  $\mathcal{F}_t^g \subseteq \mathcal{F}_t^B$  is for the objective to benefit the properties of Weiner process  $B$ :

- $g(., \omega)$  and  $B(., \omega)$  have the same sample space. Furthermore, note that, when evaluating,  $\omega$  is the same for  $g(., \omega)$  and  $B(., \omega)$ .
- $\mathcal{F}_t^B$  represents only the information available up to time  $t$  based on the observed behavior of the Wiener process  $B$  up to that point.  $g$  is adapted to  $\mathcal{F}_t^B$  guarantees that  $g$  only uses information from the past and present, not the future.
- $g(t_i, \omega)$  and  $B(t_i) - B(t_{i+1})$  are independent, since  $\mathcal{F}_{t_i}^g \subset \mathcal{F}_{t_i}^B$  and  $\mathcal{F}_{t_i}^B$ ,  $B(t_i) - B(t_{i+1})$  are independent (see subsec 4.1.8).

### 16.1.2 Properties of integrable SI

If  $g \in \mathcal{L}^2(S, T)$ :

- $\mathbb{E} \left[ \int_S^T g_t dW_t \right] = 0$
- $\mathbb{E} \left[ \left( \int_S^T g_t dW_t \right)^2 \right] = \int_S^T E[g_t^2] dW_t$
- $\int_S^T g_t dW_t$  is  $\mathcal{F}_T^B$ -measurable.

## 16.2 Properties

- *Associativity* : Let  $J, K$  be predictable processes 4.10.4.1, and  $K$  be  $X$ -integrable ( $\int K_s dX_s$  exists). Then,  $J$  is  $K \cdot X$ -integrable if and only if  $JK$  is  $X$ -integrable, in which case :

$$J \cdot (K \cdot X) = (JK) \cdot X$$

- *Dominated convergence* : Suppose that  $H_n \rightarrow H$  and  $|H_n| \leq J$ , where  $J$  is an  $X$ -integrable process. Then  $H_n \cdot X \xrightarrow{P} H \cdot X$ , means convergence in probability at each time  $t$ . In this case, the Dominated convergence theorem (2.2.17) can be extended to  $dX$  instead of  $d\mu$ .

- *The stochastic integral commutes with the operation of taking quadratic covariations* (in 4.8.2) : If  $X$  and  $Y$  are semimartingales then any process  $H$  which is  $X$ -integrable process will also be  $[X, Y]$ -integrable. The commutation is represented by

$$[H \cdot X, Y] = H \cdot [X, Y]$$

A consequence :

$$\begin{aligned} [H \cdot X] &= [H \cdot X, H \cdot X] \\ &= H[X, H \cdot X] \quad (\text{by taking } Y = H \cdot X) \\ &= H[H \cdot X, X] \quad (\text{communatation of quadratic covariation}) \\ &= H^2 \cdot [X, X] \quad (\text{by taking } Y = X) \\ &= H^2 \cdot [X] \end{aligned}$$

## 16.3 Ito lemma

### 16.3.1 Ito process

An Itô process is defined to be an **adapted stochastic process** that can be expressed as the sum of an integral with respect to time and an integral with respect to Brownian motion  $B$ :

$$X_t = X_0 + \int_0^t \mu(s) ds + \int_0^t \sigma(s) dB_s$$

or  $X(t)$  is called Ito process if  $X(t)$  satisfies the following stochastic differential equation :

$$dX_t = U_t dt + V_t dB_t$$

where  $U_t, V_t \in \mathcal{L}^2$  are two stochastic process.

In a most simlpe case, Ito process can be written by

$$dX_t = \mu(t) dt + \sigma(t) dB_t$$

where  $\mu(t), \sigma(t)$  are deterministic.

### 16.3.2 Motivation

Suppose  $X_t$  is an Ito process, which means :

$$dX_t = \mu(t)dt + \sigma(t)dB_t$$

where  $B_t$  is a Wiener process (*stochastic*),  $\mu(t)$  and  $\sigma(t)$  are *deterministic* function of  $t$ , independent with  $B_t$ . In general, it's not possible to write a solution  $X_t$  directly in terms of  $B_t$ . However, we can have a stochastic solution. First, formally write an integral solution :

$$X_t = X_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dB_s$$

It is proved that  $X_t$  has the same moments as the normal distribution  $\mathcal{N}\left(\int_0^t \mu(s)ds, \int_0^t \sigma(s)^2 ds\right)$ . Here we show that it is true until order 4.

$$\begin{aligned} \mathbb{E}(X_t) &= \mathbb{E}\left[\int_0^t \mu(s)ds + \int_0^t \sigma(s)dB_s\right] \\ &= \int_0^t \mathbb{E}[\mu(s)ds] + \int_0^t \mathbb{E}[\sigma(s)dB_s] \\ &= \int_0^t \mu(s)ds + \int_0^t \mathbb{E}[\sigma(s)]\mathbb{E}[dB_s] \quad \text{since } \sigma(s), dB_s \text{ are independent} \\ &= \int_0^t \mu(s)ds + 0 \end{aligned}$$

$$\begin{aligned} \text{Var}(X_t) &= \text{Var}\left[\int_0^t \mu(s)ds + \int_0^t \sigma(s)dB_s\right] \\ &= \text{Var}\left[\int_0^t \sigma(s)dB_s\right], \quad \text{since } \int_0^t \mu(s)ds \text{ is deterministic} \\ &= \mathbb{E}\left[\left(\int_0^t \sigma(s)dB_s\right)^2\right] - \mathbb{E}\left[\int_0^t \sigma(s)dB_s\right]^2 \\ &= \mathbb{E}\left[\left(\sum_i e_i \Delta B_i\right)^2\right] - \mathbb{E}\left[\sum_i e_i \Delta B_i\right]^2 \\ &= \mathbb{E}\left[\sum_i \sum_j e_i e_j \Delta B_i \Delta B_j\right] - \left(\sum_i e_i \mathbb{E}[\Delta B_i]\right)^2 \\ &= \sum_i \sum_j e_i e_j \mathbb{E}[\Delta B_i \Delta B_j] - 0 \\ &= \left(\sum_{i=j} + 2 \sum_{i \neq j}\right) e_i e_j \mathbb{E}[\Delta B_i \Delta B_j] \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=j} e_i^2 \mathbb{E}[\Delta B_i^2] \\
&= \int_0^t \sigma(s)^2 ds
\end{aligned}$$

where  $e_i = \sigma(s_i)$ ,  $\Delta B_i = B_{s_i} - B_{s_{i+1}}$ .  $\mathbb{E}[(\Delta B_i)^{2n+1}] = 0$ .  $\Delta B_i, \Delta B_j, (i \neq j)$  are independent.

$$\begin{aligned}
\mathbb{E} \left[ \left( \frac{X_t - E(X_t)}{\sqrt{Var(X_t)}} \right)^3 \right] &= \frac{1}{\sqrt{Var(X_t)^3}} \mathbb{E} \left[ \left( \int_0^t \sigma(s) dB_s \right)^3 \right] \\
&= \frac{1}{\sqrt{Var(X_t)^3}} \mathbb{E} \left[ \left( \sum_i e_i \Delta B_i \right)^3 \right] \\
&= \frac{1}{\sqrt{Var(X_t)^3}} \mathbb{E} \left[ \sum_i \sum_j \sum_k e_i e_j e_k \Delta B_i \Delta B_j \Delta B_k \right] \\
&= \frac{1}{\sqrt{Var(X_t)^3}} \sum_i \sum_j \sum_k e_i e_j e_k \mathbb{E} [\Delta B_i \Delta B_j \Delta B_k] \\
&= \frac{1}{\sqrt{Var(X_t)^3}} \left( \sum_{i=j=k} +3 \sum_{i=j \neq k} +6 \sum_{i \neq j \neq k} \right) e_i e_j e_k \mathbb{E} [\Delta B_i \Delta B_j \Delta B_k] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{E} \left[ \left( \frac{X_t - E(X_t)}{\sqrt{Var(X_t)}} \right)^4 \right] &= \frac{1}{\sqrt{Var(X_t)^4}} \mathbb{E} \left[ \left( \int_0^t \sigma(s) dB_s \right)^4 \right] \\
&= \frac{1}{\sqrt{Var(X_t)^4}} \mathbb{E} \left[ \left( \sum_i e_i \Delta B_i \right)^4 \right] \\
&= \frac{1}{\sqrt{Var(X_t)^4}} \mathbb{E} \left[ \sum_i \sum_j \sum_k \sum_l e_i e_j e_k e_l \Delta B_i \Delta B_j \Delta B_k \Delta B_l \right] \\
&= \frac{1}{\sqrt{Var(X_t)^4}} \sum_i \sum_j \sum_k \sum_l e_i e_j e_k e_l \mathbb{E} [\Delta B_i \Delta B_j \Delta B_k \Delta B_l] \\
&= \frac{1}{\sqrt{Var(X_t)^4}} \left( \sum_{i=j=k=l} +4 \sum_{i=j=k \neq l} +6 \sum_{i=j \neq k=l} +12 \sum_{i=j \neq k \neq l} +24 \sum_{i \neq j \neq k \neq l} \right) \times \\
&\quad e_i e_j e_k e_l \mathbb{E} [\Delta B_i \Delta B_j \Delta B_k \Delta B_l]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{Var(X_t)}^4} \left( \sum_{i=j=k=l} + 6 \sum_{i=j \neq k=l} \right) e_i e_j e_k e_l \mathbb{E} [\Delta B_i \Delta B_j \Delta B_k \Delta B_l] \\
&= \frac{1}{\sqrt{Var(X_t)}^4} \left( \sum_i e_i^4 \mathbb{E} [\Delta B_i^4] + 6 \sum_i \sum_{j \neq i} e_i^2 e_j^2 \mathbb{E} [\Delta B_i^2] \mathbb{E} [\Delta B_j^2] \right) \\
&= \frac{1}{\sqrt{Var(X_t)}^4} \left( \sum_i e_i^4 3 \mathbb{E} [\Delta B_i^2]^2 + 6 \sum_i \sum_{j \neq i} e_i^2 e_j^2 \mathbb{E} [\Delta B_i^2] \mathbb{E} [\Delta B_j^2] \right) \\
&= \frac{1}{\sqrt{Var(X_t)}^4} 3 \int_0^t \sigma(s)^2 ds \int_0^t \sigma(s)^2 ds \\
&= \frac{1}{\sqrt{Var(X_t)}} 3 Var(X_t) Var(X_t) \\
&= 3
\end{aligned}$$

Since  $X_t$  follows the normal distribution, then we can sample :

$$X_t \sim \mathcal{N} \left( \int_0^t \mu(s) ds, \int_0^t \sigma(s)^2 ds \right)$$

or

$$X_t = \int_0^t \mu(s) ds + \epsilon \leftarrow \mathcal{N}(0, \int_0^t \sigma(s)^2 ds)$$

However, sometimes we are faced with a more complicated stochastic differential equation, in which the process  $Y_t$  appears on both  $dt$  and  $dB_t$  parts, say:

$$dY_t = a_1(Y_t, t) dt + a_2(Y_t, t) dB_t$$

for some functions  $a_1$  and  $a_2$ . In this case, we cannot immediately write a formal solution as we did in the simpler case above. Instead, we hope to write the process  $Y_t$  as a function of a simpler process  $X_t$  taking the form above. That is, we want to identify three functions  $f(t, x)$ ,  $\mu_t$  and  $\sigma_t$  such that :

$$\begin{cases} Y_t = f(t, X_t) \\ dX_t = \mu_t dt + \sigma_t dB_t \end{cases}$$

In practice, Ito's lemma is used in order to find this transformation.

### 16.3.3 Informal derivation

Supposons we have already  $X_t$  is a Ito process, which means  $X_t$  satisfies:

$$dX_t = \mu_t dt + \sigma_t dB_t$$

now  $\mu_t$  and  $\sigma_t$  now can be stochastic process.

If  $f(t, X_t)$  is at least twice-differentiable scalar function, its expansion in a Taylor series is :

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} dX_t^2 + \dots$$

Substituting  $dX_t = \mu_t dt + \sigma_t dB_t$  :

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t} (\mu_t dt + \sigma_t dB_t) + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (\mu_t dt + \sigma_t dB_t)^2 + \dots$$

Because in the limit  $dt \rightarrow 0$ , the terms  $dt^2$  and  $dtdB_t$  tend to zero faster than  $dB_t^2$ , which is thus  $dt$ . Setting the  $dt^2$  and  $dtdB_t$  terms to zero, substituting  $dt$  for  $dB_t^2$  (see subsection 4.8.1 of Brownian motion), and collecting the  $dt$  and  $dB_t$  terms, we obtain **Ito's formula**:

$$df = \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 \right) dt + \left( \frac{\partial f}{\partial X_t} \sigma_t \right) dB_t$$

By identifying :

$$\begin{cases} a_1(f(t, X_t), t) &= \frac{\partial f(t, X_t)}{\partial t} + \frac{\partial f(t, X_t)}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} \sigma_t^2 \\ a_2(f(t, X_t), t) &= \frac{\partial f(t, X_t)}{\partial X_t} \sigma_t \end{cases}$$

Then we need to solve the above system of DE for getting  $f$  hence  $Y_t$ .

#### 16.3.4 Ito's formula

We remind what found in the previous section 4.16.3.3, if  $X_t$  is a Ito process

:

$$dX_t = \mu_t dt + \sigma_t dB_t$$

where  $\mu_t, \sigma_t \in \mathcal{L}^2$  are stochastic process.

Then for  $Y_t = f(t, X_t)$ , we have :

$$dY_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} dX_t^2$$

or

$$dY_t = \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 \right) dt + \left( \frac{\partial f}{\partial X_t} \sigma_t \right) dB_t$$

#### 16.3.5 Ito's formula for multidimensional Brownian motion

Given  $\mathbf{B}_t$  a  $d$ -multidimensional standard Brownian motion, which means that  $\mathbf{B}_t$  contains independent standard Brownian motions and  $\mathbf{X}_t$  is a multidimensional Ito process, which means

$$d\mathbf{X}_t = \mu_t dt + \Gamma_t d\mathbf{B}_t$$

where  $\mu_t = (\mu_t^{(1)}, \dots, \mu_t^{(d)})$  and  $\Gamma_t[i, j] = \sigma_t^{(ij)}$  are both stochastic process.

Let's  $Y_t = f(\mathbf{X}_t, t)$ , then Ito's formula for multidimensional Brownian motion says that :

$$dY_t = \left( \frac{\partial f}{\partial t} + \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T \mu_t + \frac{1}{2} \text{tr} \left( \frac{\partial^2 f}{\partial \mathbf{X}_t^2} \Gamma \Gamma^T \right) \right) dt + \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T \Gamma_t d\mathbf{B}_t$$

### Proof

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial t} dt + \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T d\mathbf{X}_t + \sum_{i,j} \frac{1}{2} \frac{\partial^2 f}{\partial X_t^{(i)} \partial X_t^{(j)}} dX_t^{(i)} dX_t^{(j)} + o(dt) \quad (\text{Taylor}) \\ &= \frac{\partial f}{\partial t} dt + \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T d\mathbf{X}_t + \sum_{i,j} \frac{1}{2} \frac{\partial^2 f}{\partial X_t^{(i)} \partial X_t^{(j)}} d\text{Cov}(X_t^{(i)}, X_t^{(j)}) \\ &= \frac{\partial f}{\partial t} dt + \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T d\mathbf{X}_t + \sum_{i,j} \frac{1}{2} \frac{\partial^2 f}{\partial X_t^{(i)} \partial X_t^{(j)}} \Gamma \Gamma^T[i, j] dt \\ &= \left( \frac{\partial f}{\partial t} + \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T \mu_t + \frac{1}{2} \sum_{i,j} \frac{1}{2} \frac{\partial^2 f}{\partial X_t^{(i)} \partial X_t^{(j)}} \Gamma \Gamma^T[i, j] \right) dt + \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T \Gamma_t d\mathbf{B}_t \\ &= \left( \frac{\partial f}{\partial t} + \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T \mu_t + \frac{1}{2} \text{tr} \left( \frac{\partial^2 f}{\partial \mathbf{X}_t^2} \Gamma \Gamma^T \right) \right) dt + \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T \Gamma_t d\mathbf{B}_t \end{aligned}$$

#### 16.3.6 Ito's formula reduced

If we the function  $Y_t$  is only depends on an Ito process  $X_t$ , which means  $Y_t = f(X_t)$  and :

$$dX_t = \mu_t dt + \sigma_t dB_t$$

This means  $f = f(X_t)$  and from section 4.16.3.4, then

$$dY_t = \frac{\partial f}{\partial X_t} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} dX_t^2$$

or

$$\begin{aligned} dY_t &= \left( \frac{\partial f}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 \right) dt + \left( \frac{\partial f}{\partial X_t} \sigma_t \right) dB_t \\ &= \left( \frac{\partial f}{\partial X_t} \mu_t dt + \frac{\partial f}{\partial X_t} \sigma_t dB_t \right) + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 dt \\ &= \frac{\partial f}{\partial X_t} (\mu_t dt + \sigma_t dB_t) + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 dt \\ &= \frac{\partial f}{\partial X_t} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 dt \end{aligned}$$

In case of multidimensional :

$$dY_t = \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T d\mathbf{X}_t + \sum_{i,j} \frac{1}{2} \frac{\partial^2 f}{\partial X_t^{(i)} \partial X_t^{(j)}} dX_t^{(i)} dX_t^{(j)}$$

or

$$dY_t = \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T d\mathbf{X}_t + \frac{1}{2} \text{tr} \left( \frac{\partial^2 f}{\partial \mathbf{X}_t^2} \Gamma \Gamma^T \right) dt$$

#### 16.3.6.1 Useful equality

By Ito's formula reduced 4.16.3.6, if  $Y_t = f(W_t)$  where  $W_t$  is a Brownian motion, then :

$$\begin{aligned} df(W_t) &= \frac{\partial f}{\partial W_t} dW_t + \frac{1}{2} \frac{\partial^2 f}{\partial W_t^2} dt \\ &= f'(W_t) dW_t + \frac{1}{2} f''(W_t) dt \end{aligned}$$

This implies :

$$f(W_s)|_0^t = \int_0^t f'(W_s) dW_s + \frac{1}{2} \int_0^t f''(W_s) ds$$

#### 16.3.7 Example for solving SDE

Solve the following Linear SDE :

$$\begin{aligned} dY_t &= a_1(Y_t, t) dt + a_2(Y_t, t) dB_t \\ &= (c(t)Y_t + g(t)) dt + (p(t)Y_t + q(t)) dB_t \end{aligned}$$

From the above, we get the system of DE :

$$\begin{cases} c(t)f(t, X_t) + g(t) &= \frac{\partial f(t, X_t)}{\partial t} + \frac{\partial f(t, X_t)}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} \sigma_t^2 \\ p(t)f(t, X_t) + q(t) &= \frac{\partial f(t, X_t)}{\partial X_t} \sigma_t \end{cases}$$

Then we need to solve for  $\mu_t$ ,  $\sigma_t$  and  $f$ , hence  $X_t = X_0 + \int_0^t \mu(s) ds + \int_0^t \sigma(s) dB_s$ . Finally  $Y_t = f(t, X_t)$

We discover next the Geometric Brownian Motion, which is a special case of Linear SDE.

#### 16.3.8 Notes

Not every  $a_1(Y_t, t)$  and  $a_2(Y_t, t)$  can make the stochastic differential equation

$$dY_t = a_1(Y_t, t) dt + a_2(Y_t, t) dB_t$$

has a solution. To check that (if there are solutions), we can see if (By Ito lemma)

$$\begin{cases} a_1(f(t, X_t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 \\ a_2(f(t, X_t), t) = \frac{\partial f}{\partial X_t} \sigma_t \end{cases}$$

has solution. An example of no solution can be found in 4.16.6.1.

### 16.3.9 Geometric Brownian Motion

A stochastic process  $Y_t$  is said to follow a GBM if it satisfies the following stochastic differential equation (see 6.4.2):

$$dY_t = aY_t dt + bY_t dB_t$$

Let  $Y_t = f(t, x)$ , apply Ito's lemma by identifying coeffs of  $dt$  and  $dB_t$ , the objective then to find  $f(t, x), \mu_t, \sigma_t$  satisfy:

$$\begin{cases} \frac{\partial f}{\partial x} \sigma_t = bf(t, x) \\ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma_t^2 = af(t, x) \end{cases}$$

From the first sde (stochastic differential equation), we have :

$$f(t, x) = C(t) \exp\left(\frac{b}{\sigma_t} x\right)$$

Then replace this into the second sde :

$$\begin{aligned} C'(t) \exp\left(\frac{b}{\sigma_t} x\right) - C(t) \exp\left(\frac{b}{\sigma_t} x\right) \frac{bx}{\sigma_t^2} \sigma'_t + C(t) \exp\left(\frac{b}{\sigma_t} x\right) \frac{b}{\sigma_t} \mu_t + \frac{1}{2} C(t) \exp\left(\frac{b}{\sigma_t} x\right) \left(\frac{b}{\sigma_t}\right)^2 \sigma_t^2 \\ = aC(t) \exp\left(\frac{b}{\sigma_t} x\right) \end{aligned}$$

or :

$$C'(t) - C(t) \frac{bx}{\sigma_t^2} \sigma'_t + C(t) \frac{b}{\sigma_t} \mu_t + \frac{1}{2} C(t) b^2 = aC(t)$$

where  $C(t), \mu_t, \sigma_t$  need to be identified. First, we see that there are only  $x$  in the left side, this makes  $\sigma_t$  be a constant  $A$  to get that  $\sigma'_t = 0$ .

$$C'(t) = \left(a - \frac{b}{A} \mu_t - \frac{1}{2} b^2\right) C(t)$$

Here we can simply take  $\mu_t = a - \frac{1}{2} b^2$  and  $\sigma_t = A = b, C(t) = B$ . Then

$$\begin{aligned} Y_t &= Bf(t, X_t) \\ &= B \exp(X_t) \\ &= B \exp\left(\int_0^t (\mu_s) ds + \epsilon \leftarrow \mathcal{N}(0, \int_0^t \sigma_s^2 ds)\right) \end{aligned}$$

$$\begin{aligned}
&= B \exp \left( \int_0^t (a - \frac{1}{2}b^2)dt + \epsilon \leftarrow \mathcal{N}(0, \int_0^t b^2 ds) \right) \\
&= B \exp \left( (a - \frac{1}{2}b^2)t + \epsilon \leftarrow \mathcal{N}(0, b^2 t) \right) \\
&= B \exp \left( (a - \frac{1}{2}b^2)t + bB_t \right)
\end{aligned}$$

But if we want go a little further (for more solution of  $C(t), \mu_t, \sigma_t$ ) :

$$C(t) = B \exp(D(t)) = B \exp \left( \int_0^t (a - \frac{b}{A}\mu_s - \frac{1}{2}b^2)ds \right)$$

then

$$\begin{aligned}
Y_t &= f(t, X_t) \\
&= B \exp(D(t)) \exp \left( \frac{b}{A}X_t \right) \\
&= B \exp \left( D(t) + \frac{b}{A}X_t \right) \\
&= B \exp \left( \int_0^t (a - \frac{b}{A}\mu_s - \frac{1}{2}b^2)ds + \frac{b}{A} \left( \int_0^t (\mu_s)dt + \mathcal{N}(0, \int_0^t A^2 ds) \right) \right) \\
&= B \exp \left( \int_0^t (a - \frac{1}{2}b^2)dt + \frac{b}{A} \mathcal{N}(0, A^2 t) \right) \\
&= B \exp \left( \int_0^t (a - \frac{1}{2}b^2)dt + \mathcal{N}(0, b^2 t) \right) \\
&= B \exp \left( (a - \frac{1}{2}b^2)t + bB_t \right)
\end{aligned}$$

We found the same solution as before.

Finally, the SP  $Y_t$  is well determined (defined) if we have the initial condition, which means  $Y_0 = \beta$ . Then in this case, we have  $B = \beta$ .

## 16.4 Integration by parts

As with ordinary calculus, integration by parts is an important result in stochastic calculus. Let  $X_s$  and  $Y_s$  be semimartingales :

$$\int_0^t Y_s dX_s = Y_s X_s \Big|_0^t - \int_0^t X_s dY_s - \int_0^t dY_s dX_s$$

$$= Y_s X_s|_0^t - \int_0^t X_s dY_s - [Y, X]_t$$

The integration by parts formula for the stochastic process differs from the standard result due to the inclusion of a *quadratic covariation* term (4.8.2)  $[Y, X]_t$ . Equivalently, we have the differential format :

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

If and case with  $[a, b]$ :

$$\int_a^b Y_s dX_s = Y_s X_s|_a^b - \int_a^b X_s dY_s - [Y, X]_b + [Y, X]_a$$

#### 16.4.1 Corollary

Suppose that  $X$  or  $Y$  has finite total variation (2.2.9.1), then the quadratic covariation term  $[X, Y]_t = 0$ . In this case, the stochastic integration by parts become :

$$\int_0^t Y_s dX_s = Y_s X_s|_0^t - \int_0^t X_s dY_s$$

#### 16.4.2 Example

[https://www.ceremade.dauphine.fr/~salez/stoc\\_TD.pdf](https://www.ceremade.dauphine.fr/~salez/stoc_TD.pdf) Let  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous (deterministic) function and  $B_t$  be Brownian motion, we define :

$$X_t := \int_0^t e^{\int_s^t \alpha(u) du} dB_s$$

Show that  $X_t$  satisfies the following SDE :

$$dX_t = \alpha(t) X_t dt + dB_t$$

Let's rewrite :

$$\begin{aligned} X_t &= \int_0^t e^{\int_0^t \alpha(u) du - \int_0^s \alpha(u) du} dB_s \\ &= \int_0^t e^{\int_0^t \alpha(u) du} e^{\int_0^s -\alpha(u) du} dB_s \\ &= e^{\int_0^t \alpha(u) du} \int_0^t e^{\int_0^s -\alpha(u) du} dB_s \\ &= G_t H_t \end{aligned}$$

where  $G_t = e^{\int_0^t \alpha(u) du}$  and  $H_t = \int_0^t e^{\int_0^s -\alpha(u) du} dB_s$ .

By integration by parts :

$$dX_t = d(G_t H_t)$$

$$\begin{aligned}
&= G_t dH_t + H_t dG_t + dG_t dH_t \\
&= e^{\int_0^t \alpha(u) du} e^{\int_0^t -\alpha(u) du} dB_t + \int_0^t e^{\int_0^s -\alpha(u) du} dB_s e^{\int_0^t \alpha(u) du} \alpha(t) dt + e^{\int_0^t \alpha(u) du} \alpha(t) dt e^{\int_0^t -\alpha(u) du} dB_t \\
&= dB_t + X_t \alpha(t) dt + \alpha(t) dt dB_t \\
&= dB_t + X_t \alpha(t) dt \quad (\text{since } dt dB_t \ll dt \text{ and } dt dB_t \ll dB_t)
\end{aligned}$$

## 16.5 Local martingale preservation

- If  $X$  is a local martingale and  $H$  is a locally bounded predictable process, then  $H \cdot X$  is also a local martingale.
- If  $X$  is a local martingale and  $H$  is not locally bounded, there are examples where  $H \cdot X$  is not a local martingale. However, this can only occur when  $X$  is not continuous.
- If  $X$  is a continuous local martingale, then the two following statements are equivalent :
  - Predictable process  $H$  is  $X$ -integrable.
  - $\int_0^t H_s^2 d[X]_s < \infty, \forall t$  and  $H \cdot X$  is local martingale.
- If  $X$  is a discontinuous local martingale and  $\sqrt{H^2 \cdot [X]}$  is locally integrable, then  $H \cdot X$  exists and is a local martingale.

## 16.6 Ito isometry

Let  $W : [0, T] \times \Omega \rightarrow \mathbb{R}$  denote the canonical real-valued Wiener process defined up to time  $T > 0$ , and let  $X, Y$  be stochastic processes that are adapted to the natural filtration  $\mathcal{F}^W$  of the Wiener process. Then :

$$\mathbb{E} \left[ \int_0^t X_s Y_s ds \right] = \mathbb{E} \left[ \left( \int_0^t X_s dW_s \right) \left( \int_0^t Y_s dW_s \right) \right]$$

*Reminder* : an isometry is an operator that **preserves distance**. For example, a reflection is an isometry or two consecutive reflections is equivalent to a translation, which is also an isometry.

*Why isometry in the above equality ?* Let's consider an operator and two spaces. In each space, we define its own distance :

- Operator :  $X_t \rightarrow f(X_t) := \int_0^t X_s dW_s$ , thus a SP to its corresponding SP by SI.
- Domain space :  $\mathcal{L}_{\text{adapted}}^2([0, T] \times \Omega)$  that contains all square-integrable adapted processes. The distance here is mean of inner products. Then given  $X_t$  and  $Y_t$ , the distance  $d(X_t, Y_t) = \mathbb{E} \left[ \int_0^t X_s Y_s ds \right]$

- Codomain space :  $\mathcal{L}_{(\Omega)}$  that contains all random variables on sample space  $\Omega$ . The distance here is the mean of product. Then given  $A$  and  $B$ , the distance  $d(A, B) = \mathbb{E}[AB]$ .

*Application :* Ito isometry is used to compute variance for random variable which is thus underg form of stochastic process, if we take  $X = Y$ , then :

$$\mathbb{E} \left[ \int_0^t X_s^2 ds \right] = \mathbb{E} \left[ \left( \int_0^t X_s dW_s \right)^2 \right]$$

#### 16.6.1 Example

Given  $W_t$  Brownian motion, calculate :

$$A_t = \mathbb{E} \left[ \left( \int_0^t (1 + s^2 + W_s) dW_s \right)^2 \right]$$

By Ito isometry, then

$$\begin{aligned} A_t &= \mathbb{E} \left[ \int_0^t (1 + s^2 + W_s)^2 ds \right] \\ &= \mathbb{E} \left[ \int_0^t ((1 + s^2)^2 + 2(1 + s^2)W_s + W_s^2) ds \right] \\ &= \mathbb{E} \left[ \int_0^t (1 + 2s^2 + s^4 + 2W_s + 2s^2W_s + W_s^2) ds \right] \\ &= t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + \frac{1}{2}t^2 \end{aligned}$$

since :

- $\mathbb{E}[\int_0^t W_s ds] = \int_0^t \mathbb{E}[W_s] ds = 0$
- $\mathbb{E}[\int_0^t s^2 W_s ds] = \int_0^t s^2 \mathbb{E}[W_s] ds = 0$
- $\mathbb{E}[\int_0^t W_s^2 ds] = \int_0^t \mathbb{E}[W_s^2] ds = \int_0^t s ds = \frac{1}{2}t^2$

Generally speaking, the expected value of an integral is an iterated integral, and so the normal mathematical rules for interchange of integrals apply. Do not confuse with the big three theorems (MCT, Fatou, DCT) in which there is a limit.

Note that, the Ito isometry is useful in this case because if we think to use Ito's lemma (4.16.3.3):

- In the case  $Y_t = \int_0^t (2(1 + s^2)W_s + W_s^2) ds$ . we need to find  $dY_t = a_1((t, W_t), t)dt + a_2((t, W_t), t)dW_t$ , where  $a_1((s, W_s), s) = 2(1 + s^2)W_s + W_s^2$  by solving :

$$\begin{cases} a_1(f(s, W_s), s) = \frac{\partial f(s, W_s)}{\partial s} + \frac{\partial f(s, W_s)}{\partial W_s} \mu_s + \frac{1}{2} \frac{\partial^2 f(s, W_s)}{\partial W_s^2} \sigma_s^2 \\ a_2(f(s, W_s), s) = \frac{\partial f(s, W_s)}{\partial W_s} \sigma_s \end{cases}$$

with  $a_2((s, W_s), s) = 0, \mu_s = 0, \sigma_s = 1$ , then :

$$\begin{cases} 2(1 + s^2)W_s + W_s^2 = \frac{\partial f(s, W_s)}{\partial s} + \frac{1}{2} \frac{\partial^2 f(s, W_s)}{\partial W_s^2} \\ 0 = \frac{\partial f(s, W_s)}{\partial W_s} \end{cases}$$

or

$$\begin{cases} 2(1 + s^2)W_s + W_s^2 = \frac{\partial f(s, W_s)}{\partial s} \\ 0 = \frac{\partial f(s, W_s)}{\partial W_s} \end{cases}$$

But we can not go further ...

- In the case  $Y_t = \int_0^t (1 + s^2 + W_s) dW_s$ , we need to solve:

$$\begin{cases} 0 = \frac{\partial f(s, W_s)}{\partial s} + \frac{1}{2} \frac{\partial^2 f(s, W_s)}{\partial W_s^2} \\ 1 + s^2 + W_s = \frac{\partial f(s, W_s)}{\partial W_s} \end{cases}$$

or

$$\begin{cases} 0 = 2sW_s + g'(s) + \frac{1}{2} &= \frac{\partial f(s, W_s)}{\partial s} + \frac{1}{2} \frac{\partial^2 f(s, W_s)}{\partial W_s^2} \\ (1 + s^2)W_s + \frac{1}{2}W_s^2 + g(s) &= f(s, W_s) \end{cases}$$

Here, we can not go further ...

### 16.6.2 In multidimensional case

Let  $\mathbf{W}_t$  be the standard  $p$ -dimensional Brownian motion and  $\mathbf{X}_t$  be  $p \times p$  matrix whose each element is a stochastic process adapted to the natural filtration of the Brownian motion. Then :

$$\mathbb{E} \left[ \left\| \int_0^t \mathbf{X}_s d\mathbf{B}_s \right\|_2^2 \right] = \mathbb{E} \left[ \int_0^t \|\mathbf{X}_s\|_F^2 d\mathbf{B}_s \right]$$

In the LHS, inner of norm is a vector and in the RHS, inner of norm is a matrix, therefore we have norm 2 and Frobenius norm.

## 17 Stochastic differentiation

### 17.1 Examples

#### 17.1.1 Example for using Ito's formula

Let  $X_t$  denote a standard Brownian Motion, compute the differential of the following stochastic process :

$$Y_t = e^{\frac{t}{2}} \cos(X_t)$$

Here  $Y_t = f(t, X_t) = e^{\frac{t}{2}} \cos(X_t)$ , and  $dX_t = \mu(t)dt + \sigma(t)dB_t$ , with  $\mu(t) = 0$  and  $\sigma(t) = 1$ , by Ito's formula 4.16.3.4:

$$\begin{aligned} dY_t &= \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 \right) dt + \left( \frac{\partial f}{\partial X_t} \sigma_t \right) dX_t \\ &= \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \right) dt + \left( \frac{\partial f}{\partial X_t} \right) dX_t \\ &= \left( \frac{1}{2} e^{\frac{t}{2}} \cos(X_t) - \frac{1}{2} e^{\frac{t}{2}} \cos(X_t) \right) dt - e^{\frac{t}{2}} \sin(X_t) dX_t \\ &= -e^{\frac{t}{2}} \sin(X_t) dX_t \end{aligned}$$

### 17.1.2 Example for using Ito's formula

<https://quant.stackexchange.com/questions/14102/worked-examples-of-applying-itos-lemma>  
Show that for  $\alpha, \sigma \in \mathbb{R}$  the process  $X$  defined by :

$$X_t := e^{-\alpha t} \left( X_0 + \sigma \int_0^t e^{\alpha s} dW_s \right)$$

satisfies the following SDE:

$$X_t - X_0 = \sigma W_t - \alpha \int_0^t X_s ds$$

Let's

$$\begin{aligned} H_t &= X_0 + \sigma \int_0^t e^{\alpha s} dW_s \\ &= X_0 + \int_0^t 0 dt + \int_0^t \sigma e^{\alpha s} dW_s \end{aligned}$$

or

$$dH_t = 0 dt + \sigma e^{\alpha t} dW_t$$

then  $H_t$  is a Ito process. Let's  $X_t = f(t, H_t) = e^{-\alpha t} H_t$ , then use Ito's formula 4.16.3.4, we have :

$$\begin{aligned} dX_t &= \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial H_t} 0 + \frac{1}{2} \frac{\partial^2 f}{\partial H_t^2} (\sigma e^{\alpha t})^2 \right) dt + \left( \frac{\partial f}{\partial H_t} (\sigma e^{\alpha t}) \right) dW_t \\ &= -\alpha e^{-\alpha t} H_t dt + e^{-\alpha t} \sigma e^{\alpha t} dW_t \quad (\text{since } \frac{\partial^2 f}{\partial H_t^2} = 0) \\ &= -\alpha X_t dt + \sigma dW_t \end{aligned}$$

or

$$X_t - X_0 = -\alpha \int_0^t X_s ds + \sigma W_t$$

### 17.1.3 Example for using Ito's formula with multidimension

We have a standard 2-dimensional Brownian motion  $\mathbf{X}_t = (X_t^{(1)}, X_t^{(2)})$  (in section 4.8.3), compute the differential of the following stochastic process :

$$Y_t = e^{X_t^{(1)}} \cos(X_t^{(2)})$$

$$\begin{aligned} dY_t &= \left[ \frac{\partial f}{\partial \mathbf{X}_t} \right]^T d\mathbf{X}_t + \frac{1}{2} \text{tr} \left( \frac{\partial^2 f}{\partial \mathbf{X}_t^2} \Gamma \Gamma^T \right) dt \\ &= \begin{bmatrix} e^{X_t^{(1)}} \cos(X_t^{(2)}) & -e^{X_t^{(1)}} \sin(X_t^{(2)}) \end{bmatrix} \begin{bmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{bmatrix} + \frac{1}{2} \text{tr} \left( \begin{bmatrix} e^{X_t^{(1)}} \cos(X_t^{(2)}) & -e^{X_t^{(1)}} \sin(X_t^{(2)}) \\ -e^{X_t^{(1)}} \sin(X_t^{(2)}) & -e^{X_t^{(1)}} \cos(X_t^{(2)}) \end{bmatrix} \Gamma \Gamma^T \right) dt \\ &= e^{X_t^{(1)}} \cos(X_t^{(2)}) dX_t^{(1)} - e^{X_t^{(1)}} \sin(X_t^{(2)}) dX_t^{(2)} + 0 \quad (\text{since } \Gamma = I) \\ &= e^{X_t^{(1)}} \cos(X_t^{(2)}) dX_t^{(1)} - e^{X_t^{(1)}} \sin(X_t^{(2)}) dX_t^{(2)} \end{aligned}$$

## 18 Infinitesimal generator

The infinitesimal generator is something characterizing a stochastic process and it encodes a great deal of information about this process. **Not every** stochastic process that can have infinitesimal generator, **only** stochastic processes which are Feller process that can have infinitesimal generator. Feller process is a continuous-time Markov process 4.7, which satisfies certain regularity conditions.

In continuous Markov process, infinitesimal generator is the generator matrix 4.7.4.

As infinitesimal generator is usually applied to Brownian motion, then it is necessary show that Brownian motion is a Feller process.

### 18.1 Markov property

A stochastic process  $X_t$  is said to have Markov property if and only if :

$$P(X_t \in A \mid \mathcal{F}_s) = P(X_t \in A \mid X_s), \quad \text{with } s \leq t$$

Note that,  $\mid \mathcal{F}_s$  is a filtration that describes the all informations from 0 to  $s$  and  $\mid X_s$  here is what happens at only  $s$ .  $A$  is just a subset of state space  $S$ .

Alternatively, we have an equivalent formulation :

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid X_s], \quad \text{with } s \leq t$$

for all bounded measurable functions  $f$ .

### 18.1.1 Intuition for equivalent between two formulations

An intuition to prove the equivalent between two formulations is to note in discret case :

$$\begin{aligned}\mathbb{E}[f(X_{n+1}) \mid \mathcal{F}_n] &= \mathbb{E}[f(X_{n+1}) \mid X_n] \\ \Leftrightarrow \sum_x f(X_{n+1} = x) P(X_{n+1} = x \mid \mathcal{F}_n) &= \sum_x f(X_{n+1} = x) P(X_{n+1} = x \mid X_n)\end{aligned}$$

It is trivial to show the first formulation implies the second. For inverse, we think to a system of equations, where each equation correspond to a function  $f$ . The bounded property is to guarantee that  $f(X_{n+1} = x) \neq \infty$  and measurable property to guarantee that  $f(X_{n+1})$  is also a random variable.

## 18.2 Brownian motion has Markov property

We prove here that Brownian motion (4.8) has Markov property or Brownian motion is a particular cas of Markov process. Thus any real-valued stochastic process with *stationary and independent increments* has Markov property. Note that, for a Brownian motion, increments are independent and stationary since it follows  $\mathcal{N}(0, \Delta_t)$ .

### 18.2.1 Proof

Given  $X_t$  is Brownian motion, then  $\forall t \geq s$  and for any bounded-measurable function  $f$ , we have

$$\begin{aligned}\mathbb{E}[f(X_t) \mid \mathcal{F}_s] &= \mathbb{E}[f(X_t - X_s + X_s) \mid \mathcal{F}_s] \\ &= \mathbb{E}[g(X_t - X_s, X_s) \mid \mathcal{F}_s] \quad (\text{we define } g(x, y) := f(x + y)) \\ &= \mathbb{E}[g(X_t - X_s, X_s) \mid X_s] \quad (\text{by lemma 2.2.12.4.6}) \\ &= \mathbb{E}[f(X_t), X_s] \mid X_s\end{aligned}$$

Then by the second formulation,  $X_t$  has Markov property.

## 18.3 Definition

For a homogenous Feller process  $(X_t)_{t \geq 0}$ , Feller semigroup  $T = (T_t)_{t \geq 0}$  and state space  $E$  we define the generator  $(\mathcal{A}, D(\mathcal{A}))$  by :

$$\left\{ \begin{array}{l} D(\mathcal{A}) = \left\{ f \in C_0(E) : \lim_{t \downarrow 0} \frac{T_t f - f}{t} \text{ exists as uniform convergence (2.4.1.2)} \right\}, \\ \mathcal{A}f = \lim_{t \downarrow 0} \frac{T_t f - f}{t}, \quad \text{for any } f \in D(\mathcal{A}) \end{array} \right.$$

where :

- $\lim_{t \downarrow 0}$  is the same as  $\lim_{t \rightarrow 0^+}$

- $C_0(E)$  denotes Banach space of continuous functions on  $E$  vanishing at infinity, equipped with the supremum norm.
- $T_t$  is an operator and the family  $T = (T_t)_{t \geq 0}$  forms a Feller semigroup.  $T_t$  is also called Fellerian.

A semigroup is a set of *algebraic structures* (numbers, functions, operators,...), which is associative under a **provided operation**. In other words, it's a couple  $S, \star$  composed of a set  $S$  and an operation  $\star$  which satisfies the associativity property  $(a \star b) \star c = a \star (b \star c), \forall a, b, c \in S$ . Eg, the set of natural numbers provided with addition is a semigroup.

A semigroup is called Feller semigroup if element  $T_t$  satisfies :

- Operation in semigroup is composition operation and  $T_{t+s} = T_t \circ T_s$
- $\|T_t f\| \leq \|f\|$
- $\lim_{t \downarrow 0} \|T_t f - f\| = 0$

- Usually, we define operator  $T_f$  by  $T_f(x) = \mathbb{E}^x[f(X_t)] = \mathbb{E}[f(X_t)|X_0 = x]$ . Then :

$$\begin{aligned}\mathcal{A}f(x) &= \lim_{t \downarrow 0} \frac{\mathbb{E}[f(X_t)|X_0 = x] - f(x)}{t} \\ &= \lim_{t \downarrow 0} \frac{\mathbb{E}[f(X_t) - f(X_0)|X_0 = x]}{t} \\ &= \frac{\mathbb{E}[df]}{dt}(X_0) \\ &= \frac{\mathbb{E}[df]}{dt}(x)\end{aligned}$$

We show in short that  $T = (T_t)_{t \geq 0}$  is a semigroup :

$$\begin{aligned}T_t \circ T_s f(x) &= \mathbb{E}[T_s f(X_t) | X_0 = x] \\ &= \mathbb{E}[\mathbb{E}[f(X_{t+s}) | X_t] | X_0 = x] \\ &= \mathbb{E}[f(X_{t+s}) | X_0 = x] \text{ (by 2.2.13.3)} \\ &= T_{t+s} f(x)\end{aligned}$$

where we applied the homogeneity property :

$$\mathbb{E}[f(X_{t+s}) | X_t] = \mathbb{E}[f(X_s) | X_0]$$

### 18.3.1 Explanation for infinitesimal generator

- *Idea behind infinitesimal generator* : Consider two processes  $X_t$  and  $Y_t = f(X_t)$  ( $f$  must be in  $D(\mathcal{A})$ ). Given instant  $t_0$ , the process  $X_{t_0} = x$  and the process  $Y_{t_0} = f(X_{t_0}) = f(x)$ . The goal of infinitesimal generator

is to have the **slope or derivative of expected value** for  $Y_t$  at instant  $t_0$ . Therefore, we can observe the general trend of  $Y_t$ , it means if we are at instant  $t_0$  with  $f(Y_{t_0} = f(x))$ , we can have a vision in general for  $Y_{t_0+\Delta}$  where  $0 < \Delta \ll 1$ , if  $Y_{t_0+\Delta}$  increases or decreases in general after a small period.

- *Why  $Y_t = f(X_t)$  ?:* For any process  $Y_t$ , Thus calculating infinitesimal generator  $\mathcal{A}Y_t$  may be a challenge and if we know  $X_t$  is a basic process (such Ito) and a function  $f$  that satisfies some constraints ( $f \in D(\mathcal{A})$ ), then we can calculate  $\mathcal{A}Y_t$ .
- *Why  $\lim_{t \downarrow 0}$  ? :* Since we only want to know expected value of slope in forward.
- In case of finance, if we know the stochastic process of price, then we can expect the trend in next days.

## 18.4 Examples

### 18.4.1 Infinitesimal generator of Brownian Motion

Let  $X_t = B_t$  be standard Brownian motion, then by using Ito's formula reduced (4.16.3.6):

$$df(X_t) = \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} dt + \left( \frac{\partial f}{\partial X_t} \right) dB_t$$

Calculate the infinitesimal generator  $\mathcal{A}f(X_{t_0})$  where  $X_{t_0} = x$ .

We'll take expectations on both sides :

$$\begin{aligned} \mathbb{E}[df(X_t)] &= \mathbb{E} \left[ \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \right] \mathbb{E}[dt] + \mathbb{E} \left[ \frac{\partial f}{\partial X_t} \right] \mathbb{E}[dB_t] \\ &= \mathbb{E} \left[ \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \right] dt \quad (\text{since } dt \text{ constant and } \mathbb{E}[dB_t] = 0) \end{aligned}$$

Hence, evaluate at  $t_0$  when  $X_{t_0} = x$  :

$$\begin{aligned} \frac{\mathbb{E}[df]}{dt}(x) &= \mathbb{E} \left[ \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2}(x) \right] \\ &= \frac{1}{2} f''(x) \end{aligned}$$

### 18.4.2 Infinitesimal generator of Geometric Brownian Motion

Let's return to Geometric Brownian Motion (in 4.16.3.9), which means  $X_t$  satisfies :

$$dX_t = aX_t dt + bX_t dB_t$$

Calculate the infinitesimal generator  $\mathcal{A}f(X_{t_0})$  where  $X_{t_0} = x$ .

By using Ito's formula reduced (4.16.3.6):

$$df(X_t) = \left( \frac{\partial f}{\partial X_t} aX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} b^2 X_t^2 \right) dt + \left( \frac{\partial f}{\partial X_t} bX_t \right) dB_t$$

We'll take expectations on both sides :

$$\begin{aligned} \mathbb{E}[df(X_t)] &= \mathbb{E} \left[ \frac{\partial f}{\partial X_t} aX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} b^2 X_t^2 \right] \mathbb{E}[dt] + \mathbb{E} \left[ \frac{\partial f}{\partial X_t} bX_t \right] \mathbb{E}[dB_t] \\ &= \mathbb{E} \left[ \frac{\partial f}{\partial X_t} aX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} b^2 X_t^2 \right] dt \quad (\text{since } dt \text{ constant and } \mathbb{E}[dB_t] = 0) \end{aligned}$$

Hence, evaluate at  $t_0$  when  $X_{t_0} = x$  :

$$\begin{aligned} \frac{\mathbb{E}[df]}{dt}(x) &= \mathbb{E} \left[ \frac{\partial f}{\partial X_t}(x)ax + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2}(x)b^2x^2 \right] \\ &= f'(x)ax + \frac{1}{2} f''(x)b^2x^2 \end{aligned}$$

#### 18.4.3 Infinitesimal generator of Ito process

This case is the general case of two above process. Let  $X_t$  be an Ito process (4.16.3.1):

$$X_t = \mu_t dt + \sigma_t dB_t$$

Calculate the infinitesimal generator  $\mathcal{A}f(t_0, X_{t_0})$  where  $X_{t_0} = x$ .

By using Ito's formula (4.16.3.4):

$$df(t, X_t) = \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 \right) dt + \left( \frac{\partial f}{\partial X_t} \sigma_t \right) dB_t$$

We'll take expectations on both sides :

$$\begin{aligned} \mathbb{E}[df(t, X_t)] &= \mathbb{E} \left[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 \right] \mathbb{E}[dt] + \mathbb{E} \left[ \frac{\partial f}{\partial X_t} \sigma_t \right] \mathbb{E}[dB_t] \\ &= \mathbb{E} \left[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \sigma_t^2 \right] dt \quad (\text{since } dt \text{ constant and } \mathbb{E}[dB_t] = 0) \end{aligned}$$

Hence, evaluate at  $t_0$  when  $X_{t_0} = x$  with  $\mu_{t_0}$  and  $\sigma_{t_0}$  :

$$\begin{aligned} \frac{\mathbb{E}[df]}{dt}(t_0, x) &= \mathbb{E} \left[ \frac{\partial f}{\partial t}(t_0, x) + \frac{\partial f}{\partial X_t}(t_0, x)\mu_{t_0} + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2}(t_0, x)\sigma_{t_0}^2 \right] \\ &= f_t(t_0, x) + f_x(t_0, x)\mu_{t_0} + \frac{1}{2} f_{xx}(t_0, x)\sigma_{t_0}^2 \end{aligned}$$

#### 18.4.4 Infinitesimal generator of continuous Markov process

Find the infinitesimal generator of continuous Markov process (4.7), with transition probability  $p_{ij}$  and mean sojourn time in state  $i$  is  $\frac{1}{\lambda_i}$ .

Let's decompose :

$$\mathbb{E}^i[f(X_t)] = \mathbb{E}[f(X_t), N(t) = 0 | X_0 = i] + \mathbb{E}[f(X_t), N(t) = 1 | X_0 = i] + \mathbb{E}[f(X_t), N(t) \geq 2 | X_0 = i]$$

where  $N(t)$  is the number of transition from 0 to  $t$  with time arrival  $T_1, T_2, \dots$  and increment interval  $Y_1 = T_1, Y_2 = T_2 - T_1, Y_3 = T_3 - T_2, \dots$

We treat for each case :

- For  $N(t) = 0$  :

$$\begin{aligned} & \mathbb{E}[f(X_t), N(t) = 0 | X_0 = i] - f(i) \\ &= \sum_j f(j)P(X_t = j, N(t) = 0 | X_0 = i) - f(i) \\ &= \sum_j f(j)P(X_t = j | N(t) = 0 | X_0 = i)P(N(t) = 0 | X_0 = i) - f(i) \quad (\text{Bayes}) \\ &= f(i)P(X_t = i | N(t) = 0 | X_0 = i)P(N(t) = 0 | X_0 = i) - f(i) \\ &= f(i)P(N(t) = 0 | X_0 = i) - f(i) \\ &= f(i)(P(T_1 > t) | X_0 = i) - 1 \\ &= f(i)(e^{-\lambda_i t} - 1) \\ &= -f(i)\lambda_i t + o(t) \quad (\text{if } t \downarrow 0) \end{aligned}$$

We note an important remark if  $t \downarrow 0$ :

$$P(N(t) = 0 | X_0 = i) = e^{-\lambda_i t} = 1 - \lambda_i t + o(t)$$

- For  $N(t) = 1$  :

$$\begin{aligned} & \mathbb{E}[f(X_t), N(t) = 1 | X_0 = i] \\ &= \sum_j f(j)P(X_t = j, N(t) = 1 | X_0 = i) \\ &= \sum_j f(j)P(X_{T_1} = j, N(t) = 1 | X_0 = i) \\ &= \sum_j f(j)P(N(t) = 1, X_{T_1} = j | X_0 = i) \\ &= \sum_j f(j)P(N(t) = 1 | X_{T_1} = j | X_0 = i)P(X_{T_1} = j | X_0 = i) \quad (\text{Bayes}) \\ &= \sum_j f(j)p_{ij}P(N(t) = 1 | X_{T_1} = j | X_0 = i) \quad (j \neq i) \\ &= \sum_j f(j)p_{ij}P(T_1 < t, T_2 > t | X_{T_1} = j | X_0 = i) \end{aligned}$$

$$\begin{aligned}
&= \sum_j f(j) p_{ij} \int_0^t P(T_1 = x \mid X_0 = i) P(Y_2 > t - x \mid X_{T_1} = j) dx \quad (\text{independent increment}) \\
&= \sum_j f(j) p_{ij} \int_0^t \lambda_i e^{-\lambda_i x} e^{-\lambda_j(t-x)} dx \\
&= \sum_j f(j) p_{ij} \lambda_i e^{-\lambda_j t} \frac{e^{(\lambda_j - \lambda_i)t} - 1}{\lambda_j - \lambda_i} \\
&= \sum_j f(j) p_{ij} \lambda_i t + o(t) \quad (\text{if } t \downarrow 0)
\end{aligned}$$

We note an important remark if  $t \downarrow 0$ :

$$P(N(t) = 1 \mid X_{T_1} = j \mid X_0 = i) = \lambda_i t + o(t) = P(N(t) = 1 \mid X_0 = i)$$

since  $\lambda_i t + o(t)$  does not depend on  $j$ .

- For  $N(t) \geq 2$ , from the two above remarks, we have if  $t \downarrow 0$ :

$$\begin{aligned}
P(N(t) \geq 2 \mid X_0 = i) &= 1 - P(N(t) = 1 \mid X_0 = i) - P(N(t) = 0 \mid X_0 = i) \\
&= 1 - (\lambda_i t + o(t)) - (1 - \lambda_i t + o(t)) \\
&= o(t)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}[f(X_t), N(t) \geq 2 \mid X_0 = i] \\
&= \sum_j f(j) P(X_t = j, N(t) \geq 2 \mid X_0 = i) \\
&= \sum_j f(j) P(X_t = j \mid N(t) \geq 2 \mid X_0 = i) P(N(t) \geq 2 \mid X_0 = i) \\
&= \sum_j f(j) P(X_t = j \mid N(t) \geq 2 \mid X_0 = i) o(t) \quad (\text{if } t \downarrow 0) \\
&= o(t)
\end{aligned}$$

since

$$\begin{aligned}
&P(X_t = j \mid N(t) \geq 2 \mid X_0 = i) \\
&= P(X_{T_n} = j, X_{T_{n-1}} = k, \dots, X_{T_1} = l \mid X_0 = i) \\
&= P(X_{T_n} = j \mid X_{T_{n-1}} = k) \dots P(X_{T_1} = l \mid X_0 = i) \quad (\text{Bayes and Markov})
\end{aligned}$$

does not depend on  $t$ .

Check in case  $n = 2$  :

$$\mathbb{E}[f(X_t), N(t) = 2 \mid X_0 = i]$$

$$\begin{aligned}
&= \sum_j f(j) P(X_t = j, N(t) \geq 2 \mid X_0 = i) \quad (j \neq i) \\
&= \sum_j f(j) P(X_{T_2} = j, N(t) \geq 2 \mid X_0 = i) \\
&= \sum_{j,k} f(j) P(X_{T_2} = j, X_{T_1} = k, N(t) \geq 2 \mid X_0 = i) \quad (k \neq i, j) \\
&= \sum_{j,k} f(j) P(N(t) \geq 2, X_{T_2} = j, X_{T_1} = k \mid X_0 = i) \\
&= \sum_{j,k} f(j) P(N(t) \geq 2 \mid X_{T_2} = j, X_{T_1} = k \mid X_0 = i) P(X_{T_2} = j, X_{T_1} = k \mid X_0 = i) \text{(Bayes)} \\
&= \sum_{j,k} f(j) P(t > T_2 > T_1 \mid X_{T_2} = j, X_{T_1} = k, X_0 = i) P(X_{T_2} = j \mid X_{T_1} = k \mid X_0 = i) P(X_{T_1} = k \mid X_0 = i) \\
&= \sum_{j,k} f(j) P(t > T_2 > T_1 \mid X_{T_2} = j, X_{T_1} = k, X_0 = i) P(X_{T_2} = j \mid X_{T_1} = k) P(X_{T_1} = k \mid X_0 = i) \\
&= \sum_{j,k} f(j) P(t > T_2 > T_1 \mid X_{T_2} = j, X_{T_1} = k, X_0 = i) p_{kj} p_{ik} \\
&= \sum_{j,k} f(j) p_{kj} p_{ik} \int_0^t P(T_1 = x \mid X_0 = i) \int_x^t P(Y_2 = y \mid X_{T_1} = k) P(Y_3 > t - x - y \mid X_{T_2} = j) dx dy \\
&= \sum_{j,k} f(j) p_{kj} p_{ik} \int_0^t \int_x^t \lambda_i e^{-\lambda_i x} \lambda_k e^{-\lambda_k y} (1 - e^{-\lambda_j (t-x-y)}) dx dy \\
&\quad \sum_{j,k} f(j) p_{kj} p_{ik} o(t) \quad (\text{if } t \downarrow 0)
\end{aligned}$$

Finally :

$$\mathcal{A}f(i) = -\lambda_i f(i) + \sum_{j \neq i} p_{ij} \lambda_j f(j)$$

## 18.5 Kolmogorov backward and forward equations

We remind that we define operator  $T_f$  in 4.18.3 by :

$$T_t f(x) = \mathbb{E}^x[f(X_t)] = \mathbb{E}[f(X_t) \mid X_0 = x]$$

Therefore, the **important note** here is that the operator  $T_t$  signifies the **expected value** of a stochastic process  $f(X_t)$  at the time  $t$ , while we know  $X_0 = x$ .

Now, for Kolmogorov backward and forward equations (KBE and KFE), let's consider two operators  $T_t$ ,  $T_{t+\Delta}$  which signify respectively the expected value at the time  $t$  and  $t + \Delta$ . The goal of KBE and KFE is to have the **slope or derivative expected value** for a stochastic process  $f(X_t)$  at instant  $t$  (or  $\lim_{\Delta \downarrow 0} \frac{T_{t+\Delta} f - T_t f}{\Delta}$ ), while we know  $X_0 = x$  :

$$(T_t f)' = \lim_{\Delta \downarrow 0} \frac{T_{t+\Delta} f - T_t f}{\Delta} = \lim_{\Delta \downarrow 0} \frac{\frac{T_\Delta T_t f - T_0 T_t f}{\Delta}}{\Delta} = \lim_{\Delta \downarrow 0} \frac{T_\Delta - T_0}{\Delta} T_t f = \mathcal{A} T_t f \quad (\text{KFE})$$

$$= \lim_{\Delta \downarrow 0} \frac{T_t T_\Delta f - T_t T_0 f}{\Delta} = \lim_{\Delta \downarrow 0} T_t \frac{T_\Delta - T_0}{\Delta} f = T_t \mathcal{A} f \quad (\text{KBE})$$

with  $T_0$  being the identity operator and  $\lim_{\Delta \downarrow 0} \frac{T_\Delta - T_0}{\Delta} f = \lim_{\Delta \downarrow 0} \frac{T_\Delta f - f}{\Delta}$  is thus infinitesimal generator  $\mathcal{A}f$ .

As we can see infinitesimal generator as a kind of derivative (for stochastic process), therefore we also have Taylor expansion :

$$\begin{aligned} T_t f &= T_0 f + \sum_{n=1}^{\infty} \frac{t^n}{n!} (T_t f)_{t=0}^{(n)} \\ &= f + \sum_{n=1}^{\infty} \frac{t^n}{n!} (\mathcal{A})^{(n)} T_0 f \\ &= e^{t\mathcal{A}} f \end{aligned}$$

### 18.5.1 Example

Let's return to example on infinitesimal generator of continuous Markov process 4.18.4.4, if using a family function  $f_k$  such that  $f_k(x) = \mathbb{1}_k(x)$ , then on one hand:

$$T_t f_k(i) = \mathbb{E}[f_k(X_t) | X_0 = i] = \mathbb{E}[\mathbb{1}_k(X_t) | X_0 = i] = P(X_t = k | X_0 = i) = p_{ik}(t)$$

where  $p_{ik}(t)$  is transition probability with time  $t$ , do not confuse with  $p_{ik}$  which is a transition matrix without time.

On other hand :

$$\begin{aligned} T_t \mathcal{A} f_k(i) &= -\lambda_i \mathbb{E}[f_k(X_t) | X_0 = i] + \sum_{j \neq i} p_{ij} \lambda_j \mathbb{E}[f_k(X_t) | X_0 = j] \\ &= -\lambda_i p_{ik}(t) + \sum_{j \neq i} p_{ij} \lambda_j p_{jk}(t) \end{aligned}$$

By Kolmogorov backward equation :

$$(T_t f_k)' = T_t \mathcal{A} f_k$$

This implies :

$$p'_{ik}(t) = -\lambda_i p_{ik}(t) + \sum_{j \neq i} p_{ij} \lambda_j p_{jk}(t)$$

or

$$P'(t) = QP(t)$$

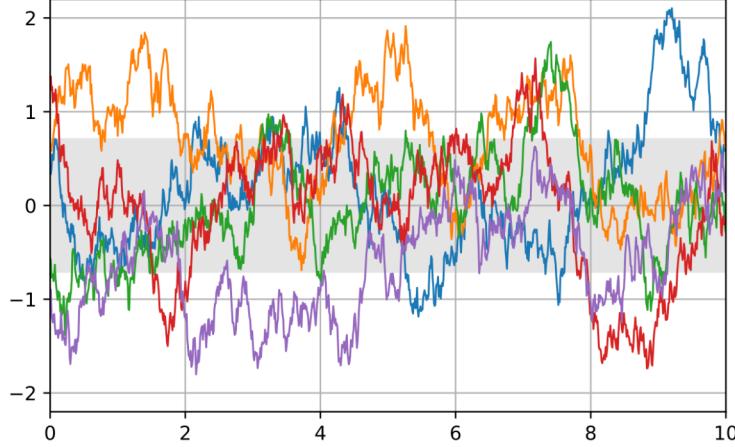


Figure 4.5: Ornstein Uhlenbeck simulations with  $\theta = 1$ ,  $\sigma = 1$  and  $\mu = 0$ .

where

$$P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1r}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2r}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{r1}(t) & p_{r2}(t) & \dots & p_{rr}(t) \end{bmatrix}$$

and

$$Q = \begin{bmatrix} -\lambda_1 & p_{12}\lambda_1 & \dots & p_{1r}\lambda_1 \\ p_{21}\lambda_2 & -\lambda_2 & \dots & p_{2r}\lambda_2 \\ \vdots & \vdots & \ddots & \vdots \\ p_{r1}\lambda_r & p_{r2}\lambda_r & \dots & -\lambda_r \end{bmatrix}$$

We find the same result as in section 4.7.4.

## 19 Ornstein-Uhlenbeck process

The Ornstein–Uhlenbeck process (O-U process)  $x_t$  is defined by the following SDE :

$$dx_t = \theta(\mu - x_t)dt + \sigma dW_t$$

where  $\theta > 0, \sigma > 0, \mu$  are parameters and  $W_t$  denotes the Wiener process.

The keypoint of O-U process is that the process will oscillate around  $\mu$ . This is because of drift  $\theta(\mu - x_t)$  : if  $x_t$  is less than  $\mu$ , then the drift make the process go up and vice-versa.

## 19.1 Explicit solution

This O-U process can be also solved by Ito's formula 4.16.3 and we have :

$$x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma}{\sqrt{2\theta}} W_{1-e^{-2\theta t}}$$

## 19.2 Mathematical properties

The first moment (i.e. the mean) is (conditioned on an initial value  $x_0$ ):

$$\mathbb{E}[x_t | x_0] = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$$

The second moment is :

$$Cov(x_s, x_t) = \frac{\sigma^2}{2\theta} \left( e^{-\theta|t-s|} - e^{-\theta(t+s)} \right)$$

## 19.3 Application

To simulate prices, GBM is used (see 4.16.3.9 and 6.4.2). However, for many financial metrics are not prices, e.g interest rates or volatility, O-U process is preferred.

# 20 Equivalent measures

Let's start with an example.

## 20.1 Illustrative example

### 20.1.1 Statement

Let  $W_t^n(\omega), t \in [0, T]$  a process be defined by

$$W_t^n(\omega) = \sqrt{\delta} \sum_{i=1}^k Y_i(\omega)$$

where :

- Time discretization :  $\delta = \frac{T}{n}$ .
- Time instant  $t = k\delta$ .
- $\omega = (\omega_1, \dots, \omega_n)$  represents the combined outcome of  $n$  consecutive binary draws.
- Each draw  $\omega_i$  may be  $u$  (up) or  $d$  (down) with random variable  $Y_i$  such as :

$$Y_i = \begin{cases} 1 & \text{if } w_i = u \\ -1 & \text{if } w_i = d \end{cases}$$

- $\omega_i$  are independent.

Let  $\Omega$  be the set of all possible  $\omega$ , let  $\mathcal{F}$  be the appropriate  $\sigma$ -field ( $\sigma$ -algebra 2.1.1), and let  $\mathbb{P}$  and  $\mathbb{Q}$  be two probability measures (2.1.7.3.2) on  $(\Omega, \mathcal{F})$ , such that :

$$\begin{aligned}\mathbb{P}(\omega_i = u) &= p = \frac{1}{2} \\ \mathbb{Q}(\omega_i = u) &= q = \frac{1}{2}(1 - \gamma\sqrt{\delta})\end{aligned}$$

Next, we study the  $\lim_{n \rightarrow \infty} W_t^n$  w.r.t to measures  $P$  and  $Q$ .

- Under measure  $P$  :

$$\begin{aligned}\mathbb{E}[Y_i] &= 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0 \\ \text{Var}(Y_i) &= 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} - \mathbb{E}[Y_i]^2 = 1\end{aligned}$$

Then  $\lim_{n \rightarrow \infty} W_t^n \sim \mathcal{N}(0, \delta k) = \mathcal{N}(0, t)$ , which is thus the standard Brownian motion (4.8).

- Under measure  $Q$  :

$$\begin{aligned}\mathbb{E}[Y_i] &= 1 \times \frac{1}{2}(1 - \gamma\sqrt{\delta}) + (-1) \times \frac{1}{2}(1 + \gamma\sqrt{\delta}) = -\gamma\sqrt{\delta} \\ \text{Var}(Y_i) &= 1^2 \times \frac{1}{2}(1 - \gamma\sqrt{\delta}) + (-1)^2 \times \frac{1}{2}(1 + \gamma\sqrt{\delta}) - \mathbb{E}[Y_i]^2 = 1 - \gamma^2\delta\end{aligned}$$

Then  $\lim_{n \rightarrow \infty} W_t^n \sim \mathcal{N}(-\gamma\sqrt{\delta}\sqrt{\delta}k, (1 - \gamma^2\delta)k\delta) = \mathcal{N}(-\gamma t, t - t\gamma^2\delta) = \mathcal{N}(-\gamma t, t)$ , since  $\delta \rightarrow 0$  for latter term. Thus, this is Brownian motion with drift  $-\gamma$  (since  $-\gamma$  relates to  $dt$ ).

### 20.1.2 Switching transformation

Then, we try to find a way to transform  $\lim_{n \rightarrow \infty} W_t^n$  from  $\mathbb{Q}$  measure to  $\mathbb{P}$  measure. This can be done by playing with the drift, always under  $\mathbb{Q}$  measure, let's:

$$\begin{aligned}\tilde{W}_t^n(\omega) &= W_n^n(\omega) + \gamma t \\ &= \sqrt{\delta} \sum_{i=1}^k Y_i + \gamma k \delta \\ &= \sqrt{\delta} \sum_{i=1}^k (Y_i + \gamma\sqrt{\delta}) \\ &= \sqrt{\delta} \sum_{i=1}^k (\tilde{Y}_i)\end{aligned}$$

This means that under  $\mathbb{Q}$  measure :

- $\tilde{W}_t^n \sim \mathcal{N}(0, t)$ .

- In order transform to measure  $\mathbb{P}$ , we just modify the random variable :

$$\tilde{Y}_i = Y_i + \gamma\sqrt{\delta} = \begin{cases} 1 + \gamma\sqrt{\delta} & \text{if } \omega_i = u \\ -1 + \gamma\sqrt{\delta} & \text{if } \omega_i = d \end{cases}$$

Inversely, in the similar manner, we can transform  $\lim_{n \rightarrow \infty} W_t^n$  from  $\mathbb{P}$  measure to  $\mathbb{Q}$  by subtracting  $\gamma t$ .

Therefore, the meaning of equivalent measures is that, by a transformation, we can have the same process.

## 20.2 Definition

We say that two measures  $\mathbb{P}$  and  $\mathbb{Q}$  (2.1.7) are equivalent if they are qualitatively similar. Specifically, they agree on which events have measure zero.

### Absolute continuity reminder

Let's :

$$\mathbb{P}_0 := \{A \in \mathcal{F} \mid \mathbb{P}(A) = 0\}$$

$$\mathbb{Q}_0 := \{A \in \mathcal{F} \mid \mathbb{Q}(A) = 0\}$$

Then the measure  $\mathbb{Q}$  is said to be absolutely continuous to the measure  $\mathbb{P}$  if and only if  $\mathbb{Q}_0 \supseteq \mathbb{P}_0$ . This is denoted as  $\mathbb{Q} \ll \mathbb{P}$  (see 2.2.11.2 for more details).

We say that  $\mathbb{P}$  and  $\mathbb{Q}$  are equivalent, noted  $\mathbb{P} \sim \mathbb{Q}$  if and only if  $\mathbb{P}_0 = \mathbb{Q}_0$  or

$$\begin{cases} \mathbb{P} \ll \mathbb{Q} \\ \mathbb{Q} \ll \mathbb{P} \end{cases}.$$

## 20.3 Girsanov's theorem

The Girsanov theorem tells how stochastic processes become if we change its associated measure. Girsanov theorem is especially important in the theory of financial mathematics. This is because it tells how to convert a *process from* the *physical measure*  $\mathbb{P}$  (1.28.2), which describes the probability that an underlying asset price will take a particular value or values, *to its corresponding process* under the *Risk neutral* measure  $\mathbb{Q}$ . This transformation is a very useful tool for evaluating the value of derivatives (options).

### 20.3.1 Significance

Girsanov's theorem is important in the general theory of stochastic processes since it enables the key result that

If the measure  $\mathbb{Q}$  is absolutely continuous to the measure  $\mathbb{P}$ , which means  $\mathbb{Q} \ll \mathbb{P}$ , then every  $\mathbb{P}$ -martingale is a  $\mathbb{Q}$ -martingale. Thus, this is still true with semimartingale (4.15), but we usually use with martingale.

### 20.3.2 Statement of theorem

1. Let  $X_t$  be continuous local martingale with probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ .
2. We define the stochastic exponential (also called Doléans-Dade exponential)  $Z_t$  of  $X_t$  by :

$$Z_t = \mathcal{E}(X_t) = \exp \left( X_t - \frac{1}{2} \langle X \rangle_t \right)$$

where  $\langle X \rangle_t$  is quadratic variation (4.8.2). Note that  $Z_t$  is a strictly positive local martingale and is also the solution for the following SDE :

$$dZ_t = Z_t dX_t$$

3. We define the measure  $\mathbb{Q}$  in probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$  such that for given  $\mathcal{F}_t$  :

$$\mathbb{Q}(A) = \int_A Z_t(\omega) d\mathbb{P}(\omega)$$

4. If  $Y_t$  is a local martingale under measure  $\mathbb{P}$  then the process

$$\tilde{Y}_t = Y_t - \langle Y, X \rangle_t$$

is its equivalent local martingale under the measure  $\mathbb{Q}$ .

### 20.3.3 Alternative statement

1. Let's define the process  $X_t$  by:

$$X_t = \int_0^t U_s dW_s$$

where  $W_t$  is a Brownian motion and  $U_s$  is a process, both under measure  $\mathbb{P}$ .

2. A necessary and sufficient condition for  $Z_t = \mathcal{E}(X_t)$  to be a martingale is Novikov's condition which requires that

$$\mathbb{E}_{\mathbb{P}} \left[ \exp \left( 1/2 \int_0^T Y_s^2 ds \right) \right] < \infty$$

Then we can create the measure  $\mathbb{Q}$  as above.

3. Then

$$\tilde{W}_t = W_t - \int_0^t U_s ds$$

is a Brownian motion under the measure  $\mathbb{Q}$ .

#### 20.3.4 Corollary 1

Given two measures  $\mathbb{P}$  and  $\mathbb{Q}$  such that  $\mathbb{Q} \ll \mathbb{P}$  and a process  $Y_t$  under measure  $\mathbb{P}$ . Then there exists  $\tilde{Y}_t$ , the corresponding of  $Y_t$  under measure  $\mathbb{Q}$ .

**Proof**

By Radon–Nikodym theorem (2.2.11.4) there exists  $f$  such that :

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = f$$

$f$  is also called Radon–Nikodym derivative and it corresponds to  $Z_t$  in 4.20.3.2. We can think that from  $Z_t$ , we can infer  $X_t$  and  $\tilde{Y}_t = Y_t - \langle Y, X \rangle_t$ .

#### 20.3.5 Corollary 2

If  $X$  is a continuous local martingale and  $W$  is Brownian motion under measure  $\mathbb{P}$  then

$$\tilde{W}_t = W_t - \langle W, X \rangle_t$$

is standard Brownian motion under measure  $\mathbb{Q}$ .

**Proof**

From Girsanov's theorem (4.20.3.2),  $\tilde{W}_t$  is local martingale under measure  $\mathbb{Q}$ . Furthermore, we have  $\langle \tilde{W} \rangle_t = \langle W \rangle_t = t$  (measure  $\mathbb{Q}$ ), then it follows by Levy's characterization,  $\tilde{W}_t$  is Brownian motion under measure  $\mathbb{Q}$ .

#### 20.3.6 Application to finance

An application of Girsanov's theorem is to convert price simulation by Geometric Brownian Motion (6.4.2) from physical measure  $\mathbb{P}$  to the one under risk-neutral probability  $\mathbb{Q}$ .

- The evolution of the stock price :

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where  $\mu$  is return rate,  $\sigma$  is volatility of underlying asset,  $r$  is risk-free rate and  $W_t$  is a standard Brownian motion under then physical measure  $\mathbb{P}$ .

- We define :

$$X_t = \int_0^t \frac{r - \mu}{\sigma} dW_s$$

then

$$\begin{aligned} \langle X \rangle_t &= \int_0^t dX_s \times dX_s \\ &= \int_0^t \frac{r - \mu}{\sigma} dW_s \times \frac{r - \mu}{\sigma} dW_s \\ &= \int_0^t \left( \frac{r - \mu}{\sigma} \right)^2 dW_s^2 \end{aligned}$$

$$= \int_0^t \left( \frac{r - \mu}{\sigma} \right)^2 ds \text{ (see 4.8.1)}$$

and

$$\begin{aligned} Z_t &= \mathcal{E}(X_t) \\ &= \exp \left( X_t - \frac{1}{2} \langle X \rangle_t \right) \\ &= \exp \left( \int_0^t \frac{r - \mu}{\sigma} dW_s - \int_0^t \left( \frac{r - \mu}{\sigma} \right)^2 ds \right) \end{aligned}$$

- We define the measure  $\mathbb{Q}$  for given  $\mathcal{F}_t$  :

$$\mathbb{Q}(A) = \int_A Z_t(\omega) d\mathbb{P}(\omega)$$

- Since  $W_t$  is a standard Brownian motion (martingale) under then physical measure  $\mathbb{P}$ , then

$$\begin{aligned} \tilde{W}_t &= W_t - \langle W, X \rangle_t \\ &= W_t - \int_0^t dW_s \times \frac{r - \mu}{\sigma} dW_s \\ &= W_t - \int_0^t \frac{r - \mu}{\sigma} ds \\ &= W_t - \frac{r - \mu}{\sigma} t \end{aligned}$$

is standard Brownian motion under measure  $\mathbb{Q}$  (see proof 4.20.3.5) and  $dW_t = d\tilde{W}_t + \frac{r - \mu}{\sigma} t$ .

- Then the evolution of the stock price under measure  $\mathbb{Q}$  is :

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ &= \mu S_t dt + \sigma S_t \left( d\tilde{W}_t + \frac{r - \mu}{\sigma} t \right) \\ &= \mu S_t dt + \sigma S_t \left( d\tilde{W}_t + \frac{r - \mu}{\sigma} t \right) \\ &= r S_t dt + \sigma S_t d\tilde{W}_t \end{aligned}$$

Therefore,

$$\mathbb{E}_{\mathbb{Q}} \left[ \frac{dS_t}{S_t} \right] = \mathbb{E}_{\mathbb{Q}} \left[ r dt + \sigma d\tilde{W}_t \right] = r dt$$

This shows that the measure  $\mathbb{Q}$  is risk neutral measure, because the underlying price  $S_t$  have the return rate which equals to risk-free rate  $r$  (see definition of risk neutral measure 1.28.3).

# Chapter 5

## Optimal control

### 1 Deterministic optimal control

#### 1.1 Brachistochrone curve

This is a classical example for the optimal control.

##### 1.1.1 Statement

The word *brachistochrone* designates a curve in a vertical plane on which a heavy material point placed in a uniform gravitational field, sliding **without friction and without initial speed**, presents a **minimum travel time** among all the curves joining two fixed points: we speak of problem of the brachistochrone curve (see figure 5.1).

Our challenge is to find out the form or equation that describe the curve corresponding to optimal time. The input is two points : initial point  $(0, 0)$  and terminal point  $(a, b)$  where  $b < 0$ .

##### 1.1.2 Solution

Since the initial velocity is 0, let  $h > 0$  be the altitude loss (ordinate) from point  $(0, 0)$  then the actual velocity is:

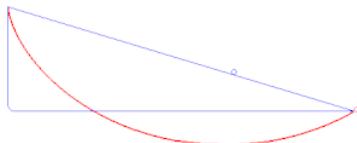


Figure 5.1: Brachistochrone illustration. The blue one is the minimum distance and the red one is minimum time.

$$v = \sqrt{2gh}$$

The law of refraction, according to Fermat's principle, indicates that, throughout its trajectory, a light ray obeys the rule :

$$\frac{\sin \theta}{v} = \text{Constant}$$

where  $\theta$  represents the angle between the vertical axis and the direction of velocity.

**An important property that if this condition is satisfied, then we have the optimal time.** In figure 5.1, the blue trajectory have the constant  $\theta$  and  $v$  is accelerated (hence not constant), then  $\frac{\sin \theta}{v}$  is not constant, so this trajectory do not correspond to the optimal time.

Let  $y < 0$  is the ordinate, in term of time, the optimal condition infers that :

$$\frac{\sin \theta}{\sqrt{-2gy}} = C$$

Let  $D$  be the altitude loss that the particle passes through the horizontal, then  $\theta = \frac{\pi}{2}$  and  $y = -D$

$$\frac{1}{\sqrt{2gD}} = C$$

From two above equations:

$$\frac{\sin \theta}{\sqrt{-2gy}} = \frac{1}{\sqrt{2gD}}$$

Since  $\sin \theta = \frac{dx}{\sqrt{dx^2+dy^2}}$  and  $y' = \frac{dy}{dx}$ , we obtain the following differential equation :

$$(1 + y'^2)y = -D$$

### 1.1.3 Solving the differential equation

Suppose the  $D$  is known, then the solution for this well-known problem, by posing  $y' = \cot(\frac{\alpha}{2})$ :

$$\begin{cases} x(\alpha) = \frac{D}{2}(\alpha - \sin(\alpha)) + c \\ y(\alpha) = -\frac{D}{2}(1 - \cos(\alpha)) \end{cases}$$

At the initial point  $y = 0$ , then  $\alpha = 0$ , then  $x(0) = 0$  infers that  $c = 0$ . Then we need to find  $D$  by using the terminal point  $(a, b)$  :

$$\begin{cases} a = \frac{D}{2}(\alpha - \sin(\alpha)) \\ b = -\frac{D}{2}(1 - \cos(\alpha)) \end{cases}$$

We can find first  $\alpha$  and then  $D$ .

More solutions can be found in document “Curves & Cash: Finding the Optimal Path Using Control Theory”.

## 1.2 Euler–Lagrange equation

Given the functional  $J$  (which takes as input a function and outputs a function) :

$$J(x) = \int_{t_0}^{t_1} \mathcal{L}(t, x(t), \dot{x}(t)) dt$$

We wish to find a function  $x(t)$  that **extremize**  $J$ , where :

- $x(t_0) = x_0$  and  $x(t_1) = x_1$  fixed.
- $x : [t_0, t_1] \rightarrow E$  is of class  $\mathcal{C}^1$ , where  $E$  is a *normed vector space* ([https://en.wikipedia.org/wiki/Normed\\_vector\\_space](https://en.wikipedia.org/wiki/Normed_vector_space)).
- $\mathcal{L} : [t_0, t_1] \times E^2 \rightarrow \mathbb{R}$  is of class  $\mathcal{C}^1$

$\mathcal{L}$  is also call Lagrange function.

Then a necessary condition (stationary point or Euler equation) for  $x(t)$  is :

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = 0$$

### 1.2.1 Example 1

A standard example is finding the real-valued function  $y(x)$  on the interval  $[a, b]$ , such that  $y(a) = c$  and  $y(b) = d$ , for which the *path length along the curve* traced by  $y$  is as short as possible :

$$\begin{aligned} &= \min_y \int_a^b \sqrt{dx^2 + dy^2} \\ &= \min_y \int_a^b \sqrt{1 + y'^2} dx \\ &= \min_y \int_a^b \mathcal{L}(x, y, y') \end{aligned}$$

Apply the Euler–Lagrange equation :

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y} &= \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial y'} \right) \\ \Leftrightarrow 0 &= \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right) \\ \Leftrightarrow C &= \frac{y'}{\sqrt{1 + y'^2}} \\ \Rightarrow y' &= \frac{C}{\sqrt{1 - C^2}} := A \\ \Leftrightarrow y(x) &= Ax + B \end{aligned}$$

It means is a straight line from  $(a, y(a))$  to  $(b, y(b))$ .

### 1.2.2 Example 2

Euler–Lagrange equation is used in Horn–Schunck method for calculating the optical flow.

### 1.2.3 Example 3

We discover an other solution for Brachistochrone problem. Let  $x(t)$  and  $y(t)$  are abscissa and ordinate of the heavy material point. From the kinetic energy and the potential energy, we must have :

$$\begin{aligned} \frac{\dot{x}(t)^2 + \dot{y}(t)^2}{2} &= gy(t) \\ \Leftrightarrow \frac{(dx)^2 + (dy)^2}{2(dt)^2} &= gy \\ \Leftrightarrow dt &= \sqrt{\frac{(dx)^2 + (dy)^2}{2gy}} = \frac{\sqrt{1 + \frac{(dy)^2}{(dx)^2}}}{\sqrt{2gy}} dx \end{aligned}$$

We need to optimize the total time from point A to point B :

$$\int_A^B dt = \int_A^B \frac{\sqrt{1 + y'(x)}}{\sqrt{2gy}} dx = \int_A^B \mathcal{L}(x, y, y') dx$$

Euler–Lagrange equation gives that :

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial y'} \right)$$

After, we find the same equation as in section 5.1.1.

## 1.3 Optimal control by Pontryagin's maximum principle

### 1.3.1 General optimal control problem

Consider a *dynamical system* or *dynamical constraint* :

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

where  $\mathbf{x}$  denotes **state vector** (vertical) and  $\mathbf{u}$  denotes **control vector**. The goal is to optimize a *performance index*  $I(\mathbf{x}(t), \mathbf{u}(t), t)$  at each point time between 0 and  $T$ , which means:

$$J = \Psi(T, \mathbf{x}(T)) + \int_0^T I(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

where  $\Psi(T, \mathbf{x}(T))$  is the ending cost at state  $\mathbf{x}(T)$ .

The goal is to **find an optimal control policy function**  $\mathbf{u}^*(t)$  to optimize the objective function  $J$ .

### 1.3.2 Hamiltonian

We **define** the Hamiltonian by :

$$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) \equiv I(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda^T(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

where  $\lambda(t)$ , refers to **costate variables**.

### 1.3.3 Pontryagin's maximum principle

Pontryagin's maximum principle say that a *necessary condition* to solve the optimal control problem or to optimize  $J$  :

1. The Hamiltonian must satisfy :

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda, t) \geq H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t), \quad \forall \mathbf{u}(t)$$

This implies, by the first-order for  $\mathbf{u}(t)$ :

$$\frac{\partial H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t)}{\partial \mathbf{u}} = 0$$

2. The costate  $\lambda(t)$  must be the solution of costate equation:

$$\dot{\lambda}(t) = -\frac{\partial H(\mathbf{x}, \mathbf{u}, \lambda(t), t)}{\partial \mathbf{x}}$$

3. The terminal condition on  $T$  :

$$\frac{\partial \Psi(T, \mathbf{x}(T))}{\partial T} + H(T) = 0$$

4. If  $\mathbf{x}(T)$  is free, it means  $\mathbf{x}(T)$  is not given previously in the announcement of problem and we can chose as we want, there is an additional equation :

$$\lambda(T) = \frac{\partial \Psi(T, \mathbf{x}(T))}{\partial \mathbf{x}}$$

All above equations and the dynamical constraint form a system of equations, which is only **necessary conditions** for solving  $\mathbf{u}(t)$ . It depends on the system ( $I$  and  $f$ ) that we can have solutions or not. A **sufficient condition** for the maximum of  $J$  is the concavity of the Hamiltonian evaluated at the solution, i.e. :

$$H_{\mathbf{uu}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t) \leq 0$$

where  $H_{\mathbf{uu}} = \frac{\partial^2 H}{\partial \mathbf{u}^2}$ .

### 1.3.4 Demonstration for Pontryagin's maximum principle

Let's break the integral into a partition :

$$\int_0^T I(\mathbf{x}(t), \mathbf{u}(t), t) dt = \sum_{n \rightarrow \infty} I(\mathbf{x}(t_{n,n+1}), \mathbf{u}(t_{n,n+1}), t_{n,n+1})(t_{n+1} - t_n)$$

where  $t_n \leq t_{n,n+1} \leq t_{n+1}$ .

Then we can maximize  $I(\mathbf{x}(t_{n,n+1}), \mathbf{u}(t_{n,n+1}), t_{n,n+1})$  for **each time**  $t_{n,n+1}$ . Let  $t_{n+1} - t_n$  be fixed and we treat independently for each interval  $[t_{n+1}, t_n]$ . Using  $\lambda(t_{n,n+1})$  as Lagrangian multiplier, we need to maximize :

$$I(\mathbf{x}(t_{n,n+1}), \mathbf{u}(t_{n,n+1}), t_{n,n+1}) + \lambda(t_{n,n+1}) (\mathbf{f}(\mathbf{x}(t_{n,n+1}), \mathbf{u}(t_{n,n+1}), t_{n,n+1}) - \dot{\mathbf{x}}(t_{n,n+1})) , \quad \forall t_{n,n+1}$$

Back to integral notation, it means we need to maximize for  $[0, T]$ :

$$\begin{aligned} & \sum_0^{n \rightarrow \infty} [I(\mathbf{x}(t_{n,n+1}), \mathbf{u}(t_{n,n+1}), t_{n,n+1}) + \lambda(t_{n,n+1}) (\mathbf{f}(\mathbf{x}(t_{n,n+1}), \mathbf{u}(t_{n,n+1}), t_{n,n+1}) - \dot{\mathbf{x}}(t_{n,n+1}))] (t_{n+1} - t_n) \\ &= \int_0^T (I(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda(t) (\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t))) dt \\ &= \int_0^T (I(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)) dt - \int_0^T \lambda(t) \dot{\mathbf{x}}(t) dt \\ &= \int_0^T (I(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)) dt + \int_0^T \dot{\lambda}(t) \mathbf{x}(t) dt - \lambda(T) \mathbf{x}(t) |_0^T \\ &= \int_0^T (I(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) + \dot{\lambda}(t) \mathbf{x}(t)) dt - \lambda(T) \mathbf{x}(T) + \lambda(0) \mathbf{x}(0) \\ &= \sum_0^{n \rightarrow \infty} [I(\mathbf{x}(t_{n,n+1}), \mathbf{u}(t_{n,n+1}), t_{n,n+1}) + \lambda(t_{n,n+1}) \mathbf{f}(\mathbf{x}(t_{n,n+1}), \mathbf{u}(t_{n,n+1}), t_{n,n+1}) + \\ & \quad \dot{\lambda}(t_{n,n+1}) \mathbf{x}(t_{n,n+1})] (t_{n+1} - t_n) - \lambda(T) \mathbf{x}(T) + \lambda(0) \mathbf{x}(0) \end{aligned}$$

where we did use the integrating by parts (to remove  $\dot{\mathbf{x}}(t)$ ) and rebreak the integral into a partition. We define the **extended Hamiltonian** (which is inside of the above integral) :

$$\begin{aligned} \tilde{H}(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) &\equiv I(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) + \dot{\lambda}(t) \mathbf{x}(t) \\ &= H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) + \dot{\lambda}(t) \mathbf{x}(t) \end{aligned}$$

Then the objective function

$$J = \Psi(T, \mathbf{x}(T)) + \int_0^T I(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

$$= \sum_0^{n \rightarrow \infty} [\tilde{H}(\mathbf{x}(t_{n,n+1}), \mathbf{u}(t_{n,n+1}), \lambda(t_{n,n+1}), t_{n,n+1})](t_{n+1} - t_n) + \Psi(T, \mathbf{x}(T)) - \lambda(T)\mathbf{x}(T) + \lambda(0)\mathbf{x}(0)$$

To maximize each  $\tilde{H}(\mathbf{x}(t_{n,n+1}), \mathbf{u}(t_{n,n+1}), \lambda(t_{n,n+1}), t_{n,n+1})$  at each instant  $t_{n,n+1}$ , or in general  $t$ , we must have the following necessary conditions:

- Taking derivative w.r.t  $\mathbf{u}$  :

$$\begin{aligned} & \frac{\partial \tilde{H}(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t)}{\partial \mathbf{u}} = 0 \\ \Leftrightarrow & \frac{\partial H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t)}{\partial \mathbf{u}} = 0 \end{aligned}$$

- Taking derivative w.r.t  $\mathbf{x}$  :

$$\begin{aligned} & \frac{\partial \tilde{H}(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t)}{\partial \mathbf{x}} = 0 \\ \Leftrightarrow & \frac{\partial H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t)}{\partial \mathbf{x}} = -\dot{\lambda}(t) \end{aligned}$$

Then we find the two first conditions as above section. We do not write the  $\frac{\partial \tilde{H}}{\partial \lambda}$  since it is complicated as the present of  $\dot{\lambda}$ , but it will lead to the dynamical constraint.

The resting is to maximize  $J$  at time  $t = T$ :

- If  $\mathbf{x}(T)$  is not fixed, then we can take the differential for variable  $\mathbf{x}(T)$  ( $\frac{\partial \Psi(T, \mathbf{x}(T))}{\partial \mathbf{x}}$ ) to maximize :

$$\Psi(T, \mathbf{x}(T)) - \lambda(T)\mathbf{x}(T),$$

which implies the fourth condition in Pontryagin's maximum principle :

$$\frac{\partial \Psi(T, \mathbf{x}(T))}{\partial \mathbf{x}} = \lambda(T)$$

- Note that  $J$  can be also rewritten by :

$$J = \int_0^T \tilde{H}(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) dt + \Psi(T, \mathbf{x}(T)) - \lambda(T)\mathbf{x}(T) + \lambda(0)\mathbf{x}(0)$$

Then we take the derivative for  $T$ , this implies:

$$\begin{aligned} & \tilde{H}(\mathbf{x}(T), \mathbf{u}(T), \lambda(T), T) + \frac{\partial \Psi(T, \mathbf{x}(T)) \partial \mathbf{x}(T)}{\partial \mathbf{x} \partial T} + \frac{\partial \Psi(T, \mathbf{x}(T))}{\partial T} - \frac{\partial \lambda(T)}{\partial T} \mathbf{x}(T) - \frac{\partial \mathbf{x}(T)}{\partial T} \lambda(T) = 0 \\ \Leftrightarrow & \tilde{H}(\mathbf{x}(T), \mathbf{u}(T), \lambda(T), T) + \frac{\partial \Psi(T, \mathbf{x}(T))}{\partial T} - \frac{\partial \lambda(T)}{\partial T} \mathbf{x}(T) + \frac{\partial \mathbf{x}(T)}{\partial T} \left( \frac{\partial \Psi(T, \mathbf{x}(T))}{\partial \mathbf{x}} - \lambda(T) \right) = 0 \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow H(\mathbf{x}(T), \mathbf{u}(T), \lambda(T), T) + \dot{\lambda}(T)\mathbf{x}(T) + \frac{\partial\Psi(T, \mathbf{x}(T))}{\partial T} - \frac{\partial\lambda(T)}{\partial T}\mathbf{x}(T) = 0 \\ &\Leftrightarrow H(\mathbf{x}(T), \mathbf{u}(T), \lambda(T), T) + \frac{\partial\Psi(T, \mathbf{x}(T))}{\partial T} = 0 \end{aligned}$$

Note that  $\frac{\partial\mathbf{x}(T)}{\partial T} \left( \frac{\partial\Psi(T, \mathbf{x}(T))}{\partial\mathbf{x}} - \lambda(T) \right) = 0$  since if  $X(T)$  is fixed, we have  $\frac{\partial\mathbf{x}(T)}{\partial T} = 0$  and if  $X(T)$  is not fixed, we have  $\frac{\partial\Psi(T, \mathbf{x}(T))}{\partial\mathbf{x}} - \lambda(T) = 0$ . Here is the third equation in Pontryagin's maximum principle.

#### 1.3.4.1 Transversality conditions

Transversality conditions means necessary terminal (boundary) conditions to maximize  $J$ , for **only costate variables**. In general, usually  $\Psi(T, \mathbf{x}(T)) = 0$ . If  $X(T)$  is not fixed, then the transversality condition is just  $\lambda(T) = 0$ . If  $X(T)$  is fixed, we have no transversality condition.

#### 1.3.5 Relation with Euler-Lagrange equation

In case that if :

$$\dot{\mathbf{x}}(t) = \mathbf{u}(t),$$

the optimal control is reduced to the Euler-Lagrange problem. Using the two first necessary conditions:

- $\frac{\partial H}{\partial \mathbf{u}} = 0 \Leftrightarrow \frac{\partial I}{\partial \mathbf{u}} + \lambda \frac{\partial f}{\partial \mathbf{u}} = 0 \Leftrightarrow \frac{\partial I}{\partial \mathbf{u}} = -\lambda$  or  $\frac{\partial I}{\partial \mathbf{x}} = -\lambda$
- $\frac{\partial H}{\partial \mathbf{x}} = -\dot{\lambda} \Leftrightarrow \frac{\partial I}{\partial \mathbf{x}} + \lambda \frac{\partial f}{\partial \mathbf{x}} = -\dot{\lambda} \Leftrightarrow \frac{\partial I}{\partial \mathbf{x}} = -\dot{\lambda}$

From the above two equation, we have Euler-Lagrange equation :

$$\frac{\partial I}{\partial \mathbf{x}} = \frac{d}{dt} \frac{\partial I}{\partial \dot{\mathbf{x}}}$$

### 1.4 Example of Hamiltonian: Ramsey–Cass–Koopmans model

This is a neoclassical model of economic growth.

#### 1.4.1 Statement

By assumption, the only productive factors are capital :

- The capital  $K(t)$
- The labour force  $L(t)$ , can be seen as the population.  $L$  is assumed to grow at a constant rate  $n$ , i.e.  $\dot{L} = \frac{dL}{dt} = nL$ , implying that  $L(t) = L_0 e^{nt}$ .

Next, we suppose that :

- $C(t)$  denote aggregate (total) consumption at instant  $t$ . Then the average consumption (per capita or per labour) is  $c(t) = \frac{C(t)}{L(t)}$ .

- $Y(t)$  denote aggregate (total) production at instant  $t$ . Then the average production is  $y(t) = \frac{Y(t)}{L(t)}$ .
- $k(t) = \frac{K(t)}{L(t)}$  is the capital per person (also called capital intensity).
- Depreciation rate  $\delta$  (constant) for the decay of capital.

Then the capital at time instant  $t + \Delta_t$  (in small  $\Delta_t$ , we suppose  $Y(t), C(t)$  constant) is calculated from time instant  $t$  by :

$$K(t + \Delta_t) = K(t) + (Y(t) - \delta K(t) - C(t)) \Delta_t$$

Or

$$\lim_{\Delta_t \rightarrow 0} \frac{K(t + \Delta_t) - K(t)}{\Delta_t} = \dot{K}(t) = Y(t) - \delta K(t) - C(t)$$

Or in *per capita* :

$$\frac{\dot{K}(t)}{L(t)} = \frac{Y(t)}{L(t)} - \delta \frac{K(t)}{L(t)} - \frac{C(t)}{L(t)}$$

Or

$$\frac{\dot{K}(t)L(t) - \dot{L}(t)K(t)}{L^2(t)} + \frac{\dot{L}(t)K(t)}{L^2(t)} = \frac{Y(t)}{L(t)} - \delta k(t) - c(t)$$

Or

$$\dot{k}(t) = y(t) - (\delta + n)k(t) - c(t)$$

Thus, the average production  $y(t)$  can be expressed as a function of capital intensity  $k(t)$  such as  $y(t) = f(k(t))$ , e.g.,  $y(t) = k(t)^{0.5}$ . In particular, a common choice for  $f$  is Cobb–Douglas production function. However, we can use any production function  $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  satisfying the *Inada conditions* :

- $f(0_n) = 0$
- The function  $f$  is concave, i.e. the Hessian matrix of  $f$  is negative-semi definite.
- $\lim_{x_i \rightarrow 0} \frac{\partial f(\mathbf{x})}{\partial x_i} = +\infty$
- $\lim_{x_i \rightarrow +\infty} \frac{\partial f(\mathbf{x})}{\partial x_i} = 0$

Finally, we have :

$$\dot{k}(t) = f(k(t)) - (\delta + n)k(t) - c(t)$$

Next, we discover the objective function, which tries to maximize the utility  $U$  of the consumption  $C(t)$

$$\int_0^\infty e^{-\rho t} U(C(t)) dt$$

which is also called social welfare function. Note that  $U(C(t)) = U(L(t)c(t)) = L(t)u(c(t))$ , where  $u$  is a utility function that takes  $c(t)$  as input. Then maximizing the above objective function is equivalent to :

$$\int_0^\infty e^{-(\rho-n)t} u(c(t)) dt$$

### 1.4.2 Solution

In resume, we want to

$$\max_c \int_0^\infty e^{-(\rho-n)t} u(c) dt$$

where  $\rho > n$  to ensure the limite of integral. Subject to:

$$c(t) = f(k(t)) - (\delta + n)k(t) - \dot{k}(t)$$

or for short :

$$c = f(k) - (\delta + n)k - \dot{k}$$

Then the objective of Ramsey–Cass–Koopmans model is **to find the optimal consumption  $c(t)$** .

**Hamiltonian application.** Let's define the Hamiltonian function :

$$H(k(t), c(t), \lambda(t)) = e^{-(\rho-n)t} u(c) + \lambda (f(k) - (\delta + n)k - c)$$

Then we need :

- $\frac{\partial H}{\partial c} = 0 :$

$$e^{-(\rho-n)t} u'(c) - \lambda = 0$$

- $\frac{\partial H}{\partial \lambda} = \dot{k} :$

$$f(k) - (\delta + n)k - c = \dot{k}$$

- $\frac{\partial H}{\partial k} = -\dot{\lambda}(t):$

$$\lambda (f'(k) - \delta - n) = -\lambda'$$

Note that  $f'(k)$  is not constant. If we take the derivative (in term of  $t$ ) of the first condition :

$$-(\rho - n)e^{-(\rho-n)t} u'(c) + e^{-(\rho-n)t} u''(c)c'(t) - \lambda'(t) = 0$$

Then using the third equation:

$$\begin{aligned} e^{-(\rho-n)t} u'(c) (f'(k) - \delta - n) &= (\rho - n)e^{-(\rho-n)t} u'(c) - e^{-(\rho-n)t} u''(c)c'(t) \\ \Leftrightarrow u'(c) (f'(k) - \delta - n) &= (\rho - n)u'(c) - u''(c)c'(t) \\ \Leftrightarrow u''(c)c'(t) &= (-f'(k) + \delta + \rho) u'(c) \end{aligned}$$

$$\Leftrightarrow c'(t) = (f'(k) - \delta - \rho) c \left( -\frac{u'(c)}{cu''(c)} \right)$$

$$\Leftrightarrow c'(t) = \sigma(c) (f'(k) - \delta - \rho) c$$

where  $\sigma(c) = -\frac{u'(c)}{cu''(c)} = -\frac{d \ln c}{d \ln(u'(c))}$  is called *elasticity of intertemporal substitution* (which is the inverse of relative risk aversion in 6.9.4.2). In some literature,  $f'(k)$  can be noted as  $f_k(k)$ .

The above equation and the second equation form a system of differential equation:

$$\begin{cases} \dot{k} = f(k) - (n + \delta)k - c = g_1(k, c) \\ \dot{c} = \sigma(c) (f'(k) - \delta - \rho) c = g_2(k, c) \end{cases}$$

is called *Ramsey-Cass-Koopmans dynamical system*. If using matrix notation :

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} -(n + \delta) & -1 \\ 0 & \sigma(c) (f'(k) - \delta - \rho) \end{bmatrix} \begin{bmatrix} k \\ c \end{bmatrix} + \begin{bmatrix} f(k) \\ 0 \end{bmatrix}$$

Unfortunately, this system is **not always linear** since there are  $\sigma(c)$ ,  $f'(k)$  and  $f(k)$ . If the utility function  $u(c) = \ln(c)$ , then  $\sigma(c) = 1$ .

**Study of steady state** : where  $\dot{k} = 0$  and  $\dot{c} = 0$ , we have three solutions  $(k_0, c_0)$ :

1.  $\begin{cases} f'(k_0) = \delta + \rho \\ c_0 = f(k_0) - (n + \delta)k_0 \end{cases}$
2.  $\begin{cases} c_0 = 0 \\ k_0 = 0 \end{cases}$
3.  $\begin{cases} c_0 = 0 \\ f(k_0) = (n + \delta)k_0, k_0 \neq 0 \end{cases}$

The second point is trivial because  $f(k) = 0$  (by Inada condition) but the third point is more complicated. We need to show that there exists  $k_0 \neq 0$  such that  $f(k_0) = (n + \delta)k_0$ .

From  $\lim_{k \rightarrow 0^+} f'(k) = +\infty$  and  $\lim_{k \rightarrow +\infty} f'(k) = 0$  (Inada conditions), then by the *intermediate value theorem*, there exists  $k_1$  such that  $f'(k_1) = n + \delta$  or  $k_1$  is a critical point of function  $h(k) = f(k) - (n + \delta)k$ . We have :

- $f(k)$  is concave and  $(n + \delta)k$  is linear, then  $h(k)$  is concave (by using the definition of concave), then  $h''(k) < 0$ . This infers that  $h$  has only critical point  $k_1$ .
- $h'(k) > 0, \forall k \in [0, k_1]$  and  $h'(k) < 0, \forall k \in (k_1, +\infty)$  since  $\lim_{k \rightarrow 0^+} f'(k) = +\infty$  and  $\lim_{k \rightarrow +\infty} f'(k) = 0$ .

- $\lim_{k \rightarrow +\infty} h(k) = -\infty$ . Proof : Since  $\lim_{k \rightarrow +\infty} h'(k) = -(n + \delta) < 0$ , then there exists  $m$  and  $r < 0$  such that  $\forall k > m$ ,  $h'(k) < r$ . Using the *mean value theorem*, there exists  $c \in (m, k)$  such that

$$\frac{h(k) - h(m)}{k - m} = h'(c) < r$$

or  $h(k) < h(m) + r(k - m)$ . As  $m$  is fixed, then if  $k \rightarrow +\infty$  then  $h(k) \rightarrow -\infty$

From all above statements, we have the variation table :

$k$	0	$k_1$	$+\infty$
$h''(k)$		-	
$h'(k)$	+	0	-
$h(k)$	0	$h(k_1)$	$-\infty$

This shows that there exists uniquely  $k_0 > k_1 > 0$  such that  $h(k_0) = 0$  or  $f(k_0) = (n + \delta)k_0$ .

**Due to the non-linearity**, we study the property of steady state by approximating (or linearizing)  $g_1(k, c)$  and  $g_2(k, c)$ . Using Taylor series (up to 1 order) and by using Jacobian matrix (for first derivative) :

$$\begin{aligned} \begin{bmatrix} g_1(k, c) \\ g_2(k, c) \end{bmatrix} &= \mathbf{J}(k_0, x_0) \begin{bmatrix} k - k_0 \\ c - c_0 \end{bmatrix} \\ \begin{bmatrix} g_1(k, c) \\ g_2(k, c) \end{bmatrix} &= \begin{bmatrix} \frac{\partial g_1(k_0, c_0)}{\partial k} & \frac{\partial g_1(k_0, c_0)}{\partial c} \\ \frac{\partial g_2(k_0, c_0)}{\partial k} & \frac{\partial g_2(k_0, c_0)}{\partial c} \end{bmatrix} \begin{bmatrix} k - k_0 \\ c - c_0 \end{bmatrix} \\ \begin{bmatrix} g_1(k, c) \\ g_2(k, c) \end{bmatrix} &= \begin{bmatrix} f'(k_0) - (n + \delta) & -1 \\ \sigma(c_0)f''(k_0)c_0 & (f'(k_0) - \delta - \rho)(\sigma'(c_0)c_0 + \sigma(c_0)) \end{bmatrix} \begin{bmatrix} k - k_0 \\ c - c_0 \end{bmatrix} \end{aligned}$$

Also, we check the concavity of the Hamiltonian with respect to control vector  $c$  (the sufficient condition) :

$$H_{cc}(k(t), c(t), \lambda(t)) = e^{-(\rho-n)t} u''(c(t)),$$

which is always negative since  $u''(c) < 0, \forall c$ . This shows that each path  $c(t)$  is optimal path for the maximization problem.

#### Phase portrait study

We first draw isolines. The isoline here is the line that  $k' = 0$  or  $c' = 0$ , in figure 5.2.

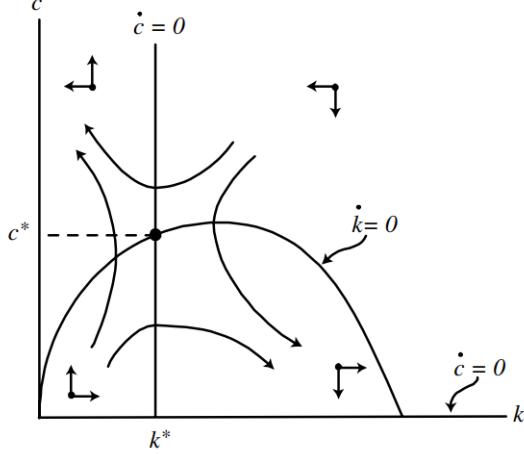


Figure 5.2: Isolines and some representative trajectories. We draw orthogonal arrows when passing isoline. The orthogonal arrows show the evolution of  $k$  and  $c$  when time increases.

**For the first case,**  $\begin{cases} f'(k_0) = \delta + \rho \\ c_0 = f(k_0) - (n + \delta)k_0 \end{cases}$

$$\mathbf{J} = \begin{bmatrix} \rho - n & -1 \\ \sigma(c_0)f''(k_0)c_0 & 0 \end{bmatrix}$$

$\det(\mathbf{J}) = -\sigma(c_0)f''(k_0)c_0 < 0$  since :

- $c_0 > 0$
- $\sigma(c_0) = -\frac{u'(c_0)}{cu''(c_0)} > 0$  since  $u'(c_0) > 0$  and  $u''(c_0) < 0$ .
- $f''(k_0) < 0$  since  $f$  is concave (such as  $-x^2$ ) by Inada condition.

Since  $\det(\mathbf{J}) < 0$ , the first steady state is a **saddle point and unstable** (Liapounov, since there is at least one eigenvalue is positive). See more the phase portrait of a linear differential system in stability theory [https://en.wikipedia.org/wiki/Stability\\_theory](https://en.wikipedia.org/wiki/Stability_theory). This is the only interior steady state, and for this point, we have two arms (figure 5.3), one is stable (eigenvalue is negative), one is unstable (eigenvalue is positive).

**For the second case,**  $\begin{cases} c_0 = 0 \\ k_0 = 0 \end{cases}$

$$\mathbf{J} = \begin{bmatrix} f'(k_0) - (n + \delta) & -1 \\ 0 & (f'(k_0) - \delta - \rho)\sigma(c_0) \end{bmatrix}$$

Since  $f'(k_0) \rightarrow +\infty$ , then  $\det(\mathbf{J}) > 0, \text{tr}(\mathbf{J}) > 0$ . We have also  $\Delta = \text{tr}(\mathbf{J})^2 - 4\det(\mathbf{J}) > 0$  because  $(a + b)^2 \geq 4ab$ , the equality happens when  $a = b$ , which

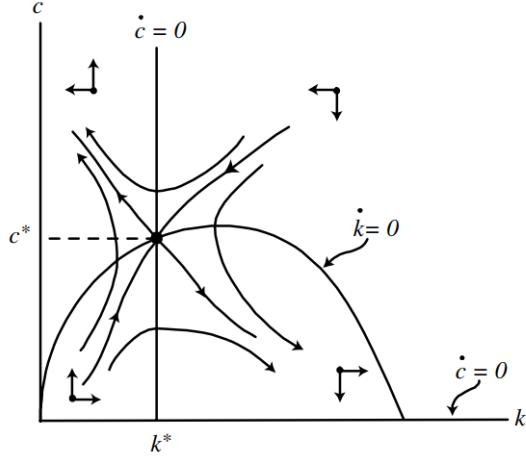


Figure 5.3: Arms (trajectories connected to an interior steady state), normally, there are two arms for an interior steady state.

means  $f'(k_0) - (n + \delta) = (f'(k_0) - \delta - \rho) \sigma(c_0)$  or  $f'(k_0)(1 - \sigma(c_0)) = n - \rho$ , but this can not be possible since  $f'(k_0) \rightarrow +\infty$  and  $\rho > n$ . Then this steady state is a **source** that pushes.

**For the third case,** 
$$\begin{cases} c_0 = 0 \\ f(k_0) = (n + \delta)k_0, k_0 > 0 \end{cases}$$

$$\mathbf{J} = \begin{bmatrix} f'(k_0) - (n + \delta) & -1 \\ 0 & (f'(k_0) - \delta - \rho) \sigma(c_0) \end{bmatrix}$$

Using the above variation table, we see that  $h'(k_0) = f'(k_0) - (n + \delta) < 0$ . This infers  $f'(k_0) - \delta - \rho < 0$  since  $\rho > n$ . Therefore,  $\det(\mathbf{J}) > 0$  and  $\text{tr}(\mathbf{J}) < 0$ . Then we can have a **sink** or a **degenerate sink** if  $f'(k_0)(1 - \sigma(c_0)) = n - \rho$ .

We see these three steady state in the phase portrait (figure 5.4). The first point is in the top, the second point is the origin and the third point is on the abscissa. In case of we can not find the explicite solution, the phase portrait can show the evolution of a solution.

Consider some following representative trajectories :

- *Profligacy* :  $c(t)$  increases and  $k(t)$  decreases, on the left of isoline  $c' = 0$ , except the upward stable arm.
- *Wasteful over-accumulation or all saving* : converging to the third steady state  $(k_0, 0)$  where  $f(k_0) = (n + \delta)k_0$ .
- Stable arms to interior steady state.

**For the profligate trajectories**, from the first differential equation  $\dot{k} = f(k) - (n + \delta)k - c$ . For each  $k$ , we define  $c_k$  such that  $f(k) - (n + \delta)k - c_k = 0$ , then  $\dot{k} = f(k) - (n + \delta)k - c_k - (c - c_k)$ . As  $c(t)$  increases, then there exists a moment

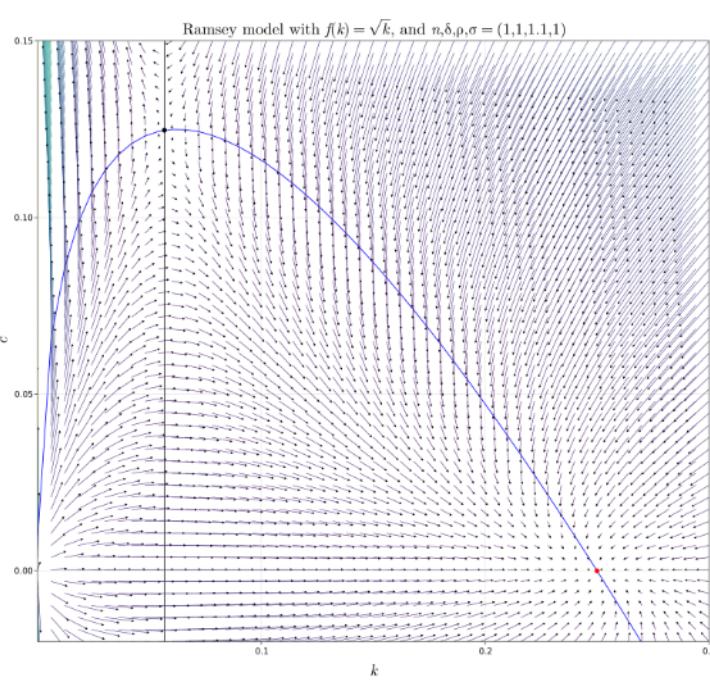


Figure 5.4: Phase portrait of Ramsey Cass Koopmans model. There are three steady states (points). Abscissa is  $k$  and ordinate is  $c$ . The arrows show the optimal trajectorys of  $k(t), c(t)$  when  $t \rightarrow +\infty$ . Each optimal trajectory is defini by initial value  $k(t_0), c(t_0)$ . Note that if  $k(t_0) = k$  is given, then we need to choose the control variable  $c(t_0)$  to have the optimal trajectory.

from that  $k(t)' < 0$  (upper the isoline  $k' = 0$ ). Then  $k(t)$  go to negative, which is not feasible.

**For the all saving trajectories**, we remind the transversality condition 5.1.3.4.1 :

$$\lim_{t \rightarrow +\infty} k(t)\lambda(t) = 0$$

and combining with the first-order condition of Hamiltonian, we must have :

$$\lim_{t \rightarrow +\infty} k(t)e^{-(\rho-n)t}u_c(c(t)) = 0$$

Remind an equation that we have from above :

$$\begin{aligned} u''(c)c'(t) &= (-f'(k) + \delta + \rho)u'(c) \\ \Leftrightarrow \frac{u''(c)c'(t)}{u'(c)} &= (-f'(k) + \delta + \rho) \\ \Leftrightarrow \frac{\partial \ln(u'(c))}{\partial t} &= (-f'(k) + \delta + \rho) \end{aligned}$$

or

$$u'(c) = \exp \left( \int_0^t (-f'(k) + \delta + \rho)ds + u'(c(0)) \right)$$

Since  $k(t) \rightarrow k_0$  **of third steady state**, then we have :

$$\begin{aligned} &\lim_{t \rightarrow +\infty} k(t)e^{-(\rho-n)t}u_c(c(t)) \\ &= \lim_{t \rightarrow +\infty} k_0 e^{-(\rho-n)t} e^{((-f'(k_0)+\delta+\rho)t+u'(c(0)))} \\ &= \lim_{t \rightarrow +\infty} b k_0 e^{(-f'(k_0)+\delta+n)t} \end{aligned}$$

As  $k_0$  in this case is on the right of  $k_1$  ( $k_0 > k_1$ ) from the above demonstration and  $f'$  decreases beyond of  $k_1$ , then we must have  $-f'(k_0) + \delta + n > 0$ . This shows that  $\lim_{t \rightarrow +\infty} k(t)\lambda(t) \rightarrow +\infty$ , equivalently, all saving trajectories violate the transversality condition and they are not optimal trajectories.

Finally, there is only trajectories **stable arms to interior steady state** are optimal.

Note that, steady states do not mean equilibrium. For more detailed, read *Lecture Notes on The Optimal Growth Problem* of Todd Keister at [https://mpra.ub.uni-muenchen.de/1461/1/MPRA\\_paper\\_1461.pdf](https://mpra.ub.uni-muenchen.de/1461/1/MPRA_paper_1461.pdf).

## 1.5 Linear-quadratic regulator

The case where :

- System dynamics are described by a set of **linear** differential equations.
- The cost is described by a **quadratic** function.

For horizon, we have finite (bounded) and infinite case (unbounded). For time, we have continuous and discrete time. So we have 4 combinations, they are lightly different about the cost function, the boundary condition, the parameter matrix  $Q, R, \dots$ . We discover here the continuous time, combined with finite and infinite horizon. At first, let's discover a indispensable tool for Linear-quadratic regulator (LQR), the Riccati differential equation.

### 1.5.1 Riccati equation

In the narrowest sense, it means that any *first-order* ( $y'$ ) *ordinary differential equation* ( $y'$  in term of only  $x$ ) is *quadratic* in the unknown function :

$$y'(x) = q_0(x) + q_1(x)y(x) + q_2(x)y^2(x)$$

To solve this problem, we can first convert it to a *second-order ordinary differential equation* :

$$u'' - Ru' + Su = 0.$$

where  $y = -u'/(q_2 u)$ ,  $R = q_1 + \frac{q'_2}{q_2}$  and  $S = q_2 q_0$ . Then we solve it as a standard second-order differential equation.

### 1.5.2 Finite-horizon, continuous-time

#### 1.5.2.1 Announcement

Cost function to minimize or we need to find  $u(t)$  that minimize :

$$J = x^T(t_1)F(t_1)x(t_1) + \int_{t_0}^{t_1} (x^T Q x + u^T R u + 2x^T N u) dt$$

Dynamic constraint :

$$\dot{x} = Ax + Bu$$

#### 1.5.2.2 Solution

Let's consider that  $u$  to minimize the cost is under linear form of state :

$$u = -Kx,$$

By replacing  $u = -Kx$  in the cost function, we can find  $K$ , which is given by:

$$K = R^{-1}(B^T P(t) + N^T),$$

where  $P$  is found by solving the continuous time Riccati differential equation:

$$A^T P(t) + P(t)A - (P(t)B + N)R^{-1}(B^T P(t) + N^T) + Q = -\dot{P}(t),$$

with the boundary condition:

$$P(t_1) = F(t_1).$$

### 1.5.3 Finite-horizon, continuous-time

#### 1.5.3.1 Announcement

Cost function to minimize is lightly different :

$$J = \int_0^\infty (x^T Q x + u^T R u + 2x^T N u) dt$$

But the dynamic constraint is the same :

$$\dot{x} = Ax + Bu$$

#### 1.5.3.2 Solution

The solution is the same above, except that

$$A^T P + PA - (PB + N)R^{-1}(B^T P + N^T) + Q = 0,$$

in this case, we have **no longer** the Riccati differential equation but the algebraic Riccati equation.

#### 1.5.4 Note

In linear-quadratic regulator, we can have **explicite form** of  $u(t)$  and  $x(t)$  since the cost function and the dynamic constraint is more simple, for example, compared to Ramsey-Cass-Koopmans model (or optimal growth model). Remind that in the latter, we need the phase portrait (evaluated at each point) to know approximately the evolution of  $u(t)$  and  $x(t)$ .

Futhermore, LQR is a particular case of optimal control problem. If we apply the sec:Pontryagin's maximum principle (5.1.3.3) or Hamilton-Jacobi-Bellman equation (5.1.6), we refind the same way for the solution.

## 1.6 Hamilton-Jacobi-Bellman

The Hamilton-Jacobi-Bellman (HJB) equation is a *differential equation* that provides **necessary and sufficient** conditions for deterministic optimality of a control with respect to a loss function. Furthermore, it can be generalized to stochastic systems.

#### 1.6.1 Annoucement

Consider the following problem in deterministic optimal control over the time period  $[0, T]$ :

$$V(x(0), 0) = \min_u \int_0^T I(x(t), u(t), t) dt + \phi(x(T))$$

subject to (dynamic system):

$$\frac{dx(t)}{dt} = f(x(t), u(t), t)$$

### 1.6.2 Value function

To apply HJB equation, it is necessary to define the **value function**.

For the problem of optimal control, the value function represents the optimal cost over the interval  $[t, T]$  when started at the time  $t$ . For example, the value function of the above problem (in announcement) is :

$$\begin{aligned} V(x(t), t) &= \min_u \int_t^T I(x(s), u(s), s) ds + \phi(x(T), T) \\ &= \int_t^T I(x(s), u^*(s), s) ds + \phi(x(T), T) \end{aligned}$$

The value function can be interpreted as the cost to finish the process, and is thus also referred as “cost-to-go function”. Note that here  $x(t)$  is just an initial value at time  $t$ .

### 1.6.3 Hamilton–Jacobi–Bellman equation

For each instant  $t$ , we have to solve the following partial differential equation (PDE), which called **Hamilton–Jacobi–Bellman equation** :

$$\min_{u(t)} \left\{ I(x(t), u(t), t) + \left[ \frac{\partial V(x(t), t)}{\partial x} \right]^T f(x(t), u(t), t) \right\} + \frac{\partial V(x(t), t)}{\partial t} = 0$$

subject to the terminal condition :

$$V(x(T), T) = \phi(x(T), T)$$

Remind that the Hamiltonian for optimal control problem is expressed by:

$$H(x(t), u(t), \lambda(t), t) = I(x(t), u(t), t) + \lambda^T(t) f(x(t), u(t), t)$$

Then HJB equation can be rewritten as :

$$\min_{u(t)} H[x(t), u(t), V_x(x(t), t), t] + V_t(x(t), t) = 0$$

Suppose that the optimal  $u^*(t) = g\left(x(t), \frac{\partial V(x(t), t)}{\partial x}, t\right)$ , we replace  $u^*(t)$  in  $H$  and we need to solve a problem which is known as Hamilton–Jacobi equation :

$$S[x(t), V_x(x(t), t), t] + V_t(x(t), t) = 0$$

where  $S[x(t), V_x(x(t), t), t] = H[x(t), g\left(x(t), \frac{\partial V(x(t), t)}{\partial x}, t\right), V_x(x(t), t), t]$ . This explain why the name Hamilton–Jacobi–Bellman.

#### 1.6.4 Proof for HJB equation

The **Bellman's principle of optimality** is just split a problem into elementary problems. We may apply a lot of times this mechanism in dynamic programming but do not know the name. Given a small  $dt$  we must have :

$$\begin{aligned} V(x(t), t) &= \min_{u(s), s \in [t, t+dt]} \int_t^{t+dt} I(x(s), u(s), s) ds + V(x(t+dt), t+dt) \\ &= \min_{u(t)} I(x(t), u(t), t) dt + V(x(t+dt), t+dt) \end{aligned}$$

By Taylor expansion :

$$\begin{aligned} V(x(t+dt), t+dt) &= V(x(t), t) + \left[ \frac{\partial V(x(t), t)}{\partial x} \right]^T \dot{x}(t) dt + \frac{\partial V(x(t), t)}{\partial t} dt + o(dt) \\ &= V(x(t), t) + \left[ \frac{\partial V(x(t), t)}{\partial x} \right]^T f(x(t), u^*(t), t) dt + \frac{\partial V(x(t), t)}{\partial t} dt + o(dt) \end{aligned}$$

By replacing into the above equation, we refind HJB differential equation :

$$\min_{u(t)} \left\{ I(x(t), u(t), t) + \left[ \frac{\partial V(x(t), t)}{\partial x} \right]^T f(x(t), u(t), t) \right\} + \frac{\partial V(x(t), t)}{\partial t} = 0$$

#### 1.6.5 Relation to Pontryagin's maximum principle

On one hand, by costate equation (one of necessary condition) in Pontryagin's maximum principle (5.1.3.3) :

$$I_x(x, u^*(t), t) + f_x(x, u^*(t), t) \lambda(t) + \dot{\lambda}(t) = 0$$

On the other hand, from HJB equation, suppose that  $u(t)$  is optimal, then we differentiate both sides with respect to  $x$  and note that  $\dot{x}(t) = f(x(t), u(t), t)$  :

$$\begin{aligned} I_x(x(t), u^*(t), t) + \frac{\partial^2 V(x(t), t)}{\partial x^2} f(x(t), u^*(t), t) + f_x(x(t), u^*(t), t) \frac{\partial V(x(t), t)}{\partial x} + \frac{\partial^2 V(x(t), t)}{\partial t \partial x} &= 0 \\ \Leftrightarrow I_x(x(t), u^*(t), t) + \frac{\partial^2 V(x(t), t)}{\partial x^2} \dot{x}(t) + f_x(x(t), u^*(t), t) \frac{\partial V(x(t), t)}{\partial x} + \frac{\partial^2 V(x(t), t)}{\partial t \partial x} &= 0 \\ \Leftrightarrow I_x(x(t), u^*(t), t) + f_x(x(t), u^*(t), t) \frac{\partial V(x(t), t)}{\partial x} + \frac{\partial^2 V(x(t), t)}{\partial x^2} \dot{x}(t) + \frac{\partial^2 V(x(t), t)}{\partial t \partial x} &= 0 \\ \Leftrightarrow I_x(x(t), u^*(t), t) + f_x(x(t), u^*(t), t) \frac{\partial V(x(t), t)}{\partial x} + \frac{\partial}{\partial t} \frac{\partial V(x(t), t)}{\partial x} &= 0 \end{aligned}$$

By identifying from above two equations, we have :

$$\lambda(t) = \frac{\partial V(x(t), t)}{\partial x}$$

### 1.6.6 Procedure for deterministic HJB approach

1. Form the Hamiltonian :

$$H = I(x(t), u(t), t) + \left[ \frac{\partial V(x(t), t)}{\partial x} \right]^T f(x(t), u(t), t)$$

2. Optimize  $H$  with respect to  $u$ , i.e, solve for  $u^*(t)$  such that :

$$\frac{\partial H}{\partial u}(u^*(t)) = 0$$

and now  $u^*(t)$  is expressed by  $g\left(x(t), \frac{\partial V(x(t), t)}{\partial x}, t\right)$ .

3. Replace  $u^*(t) = g\left(x(t), \frac{\partial V(x(t), t)}{\partial x}, t\right)$  in  $H$ .

4. Solve the Hamilton-Jacobi equation, which is a partial differential equation  
for  $V$  with respect to  $x(t)$  and  $t$ :

$$I(x(t), g\left(x(t), \frac{\partial V(x(t), t)}{\partial x}, t\right), t) + \left[ \frac{\partial V(x(t), t)}{\partial x} \right]^T f(x(t), g\left(x(t), \frac{\partial V(x(t), t)}{\partial x}, t\right), t) + \frac{\partial V(x(t), t)}{\partial t} = 0$$

subject to the terminal condition :

$$V(x(T), T) = \phi(x(T), T)$$

Thus, to solve this PDE is a challenge. Usually we apply the finite difference method 5.3.1 and the finite element method 5.3.2 in section 5.3.

5. Use the solution  $V$  from above, calculate  $\frac{\partial V(x(t), t)}{\partial x}$  and obtain the final expression  $u^*(t) = g\left(x(t), \frac{\partial V(x(t), t)}{\partial x}, t\right) = h(x(t), t)$ .

6. Get optimal state by solving differential equation :

$$x'(t) = f(x(t), h(x(t), t), t)$$

subject to eventual initial or terminal conditions  $x(0) = x_0, x(T) = x_T$ .  
Then we infer  $u^*(t)$  and  $J^*(t)$ .

### 1.6.7 Example

Minimize :

$$J = \int_0^2 \frac{1}{2} u(t)^2 dt$$

subject to (dynamic system):

$$\begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}$$

where initial and final states are :

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

### Solution

1. Form the Hamiltonian :

$$H = \frac{1}{2}u^2 + \frac{\partial V}{\partial x_1}x_2 + \frac{\partial V}{\partial x_2}u$$

2. Optimize  $H$  with respect to  $u$  :

$$\begin{aligned} \frac{\partial H}{\partial u}(u^*) &= 0 \\ \Leftrightarrow u^* &= -\frac{\partial V}{\partial x_2} \end{aligned}$$

3. Now with  $u^* = -\frac{\partial V}{\partial x_2}$  :

$$H = \frac{\partial V}{\partial x_1}x_2 - \frac{1}{2} \left[ \frac{\partial V}{\partial x_2} \right]^2$$

4. We need to solve the HJB equation:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}x_2 - \frac{1}{2} \left[ \frac{\partial V}{\partial x_2} \right]^2 = 0$$

s.t

$$V(x_1(T), x_2(T), T) = 0$$

By supposing that  $V$  has form :

$$V(x_1, x_2, t) = ax_1 + bx_2 + cx_1x_2 + dx_1t + ex_2t + g(t)$$

By replacing in HJB equation, we obtain :

$$V(x_1, x_2, t) = ax_1 + bx_2 - ax_2t + \frac{1}{6}(a^2t^3 - 3abt^2 + 3b^2t) + C$$

where  $a, b, C$  are constant, and by  $V(x_1(T), x_2(T), T) = 0$ , we have

$$ax_1(2) + bx_2(2) - 2ax_2(2) + \frac{1}{6}(8a^2 - 12ab + 6b^2) + C = 0$$

or

$$a + \frac{4}{3}a^2 - 2ab + b^2 + C = 0$$

5. Now we can express  $u$  in term of  $t$ :

$$u^*(t) = -\frac{\partial V}{\partial x_2} = -(b - at)$$

6. We solve the following differential equation to obtain  $x$  :

$$\begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ at - b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ at - b \end{bmatrix}$$

We infers that :

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{6}at^3 - \frac{1}{2}bt^2 + c_1t + c_2 \\ \frac{1}{2}at^2 - bt + c_1 \end{bmatrix}$$

By boundary conditions, we get  $a = 3, b = 4, c_1 = 2, c_2 = 1$ .

Finally,

- $u^*(t) = 3t - 4$
- $a + \frac{4}{3}a^2 - 2ab + b^2 + C = 0$ , then  $C = -7$ .
- The optimal  $J^* = V(x_1(0), x_2(0), 0) = ax_1(0) + bx_2(0) + C = 3 \times 1 + 4 \times 2 - 7 = 4$

We recheck :

$$J^* = \int_0^2 \frac{1}{2}(3t - 4)^2 dt = \frac{1}{18}(3t - 4)^3|_0^2 = \frac{8 + 64}{18} = 4$$

## 1.7 Which method to use ?

We have seen a general optimal control problem and have seen two methods to solve it : Pontryagin's maximum principle and HJB equation. A raised question is which method to use ? Here is a little comparison between these two methods :

- Pontryagin's maximum principle provides only **necessary** conditions while HJB equation provides a **necessary and sufficient** condition.
- Pontryagin's maximum principle **can not be extended** for **stochastic** optimal control problems while HJB equation **does**.
- Pontryagin's Maximum Principle is potentially **more computationally efficient** in that the conditions which it specifies only need to hold over a **particular trajectory** while HJB equation needs to hold over the **entire state space** to be valid.

## 2 Stochastic optimal control

This is also optimal control but there are **additional stochastic factors** such as noise, either in observations in evolution of the system.

### 2.1 Kalman filter

In statistics and control theory, Kalman filtering, also known as *linear quadratic estimation* (LQE), is an algorithm that uses a series of measurements observed over time, including statistical noise and other inaccuracies. This algorithm produces estimates of unknown variables that tend to be **more accurate than those based on a single measurement alone**, by estimating a joint probability distribution over the variables for each timeframe.

The algorithm works by a **two-phase process** :

1. The Kalman filter produces estimates of the current state variables, along with their uncertainties.
2. Once the outcome of the next measurement (necessarily corrupted with some error, including random noise) is observed, these estimates are updated using a weighted average. The bigger weights are given to estimates with greater certainty.

The algorithm is recursive. It can operate in real time, using three main factors : the measurements or observations  $\mathbf{z}$ , the state  $\mathbf{x}$  and the certainty matrix  $\mathbf{P}$  (covariance matrix of errors for state  $\mathbf{x}$ ).

#### 2.1.0.1 Announcement

We consider the following announcement, with two phases characterized by *evolution equation* and *observation equation*:

**Evolution equation** :

$$\mathbf{x}_k = \mathbf{F}_k \mathbf{x}_{k-1} + \mathbf{B}_k \mathbf{u}_k + \mathbf{w}_k$$

where:

- $k$  is time instant
- $\mathbf{x}_k$  is the state vector
- $\mathbf{F}_k$  is the state-transition matrix
- $\mathbf{B}_k$  is the controller (model) matrix
- $\mathbf{u}_k$  is the input vector for the controller
- $\mathbf{w}_k$  is supposed to be a centered normal noise

### Observation equation

$$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k$$

where:

- $\mathbf{z}_k$  is the observation vector (for what are observed)
- $\mathbf{H}_k$  is the observation matrix
- $\mathbf{v}_k$  is supposed to be a centered normal noise

$\mathbf{x}_0, \mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{v}_1, \dots, \mathbf{v}_k$  are all assumed to be mutually independent.

#### 2.1.0.2 Objective

The objective is to have an estimation  $\hat{\mathbf{x}}_k$  of  $\mathbf{x}_k$  with the best (minimum) variance of error, for each  $k$ , given  $\mathbf{F}_k, \mathbf{B}_k, \mathbf{H}_k \dots$  except  $\mathbf{x}_k, \mathbf{w}_k, \mathbf{v}_k$ . For the two latter, we only know their distribution (centered normal).

#### 2.1.0.3 Algorithm

The notation  ${}_{n|m}$  represents : at instant  $n$  given observations from 0 up to instant  $m$  (included), where  $m \leq n$ . Now, we understand that  $\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k|k}$ .

**Phase 1, Prediction.** Given the estimate  $\hat{\mathbf{x}}_{k-1|k-1}$ , the objective of this phase is just to estimate the state vector and the certainty matrix (or covariance matrix of errors). **Before the observation  $k$** , the evolution equation *without noise* is :

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_k \mathbf{u}_k$$

and then the covariance matrix of errors :

$$\begin{aligned} \mathbf{P}_{k|k-1} &= Cov(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}) \\ &= Cov(\mathbf{F}_k \mathbf{x}_{k-1} + \mathbf{B}_k \mathbf{u}_k + \mathbf{w}_k - \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} - \mathbf{B}_k \mathbf{u}_k) \\ &= Cov(\mathbf{F}_k (\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1}) + \mathbf{w}_k) \\ &= Cov(\mathbf{F}_k (\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1})) + Cov(\mathbf{w}_k) \\ &= \mathbf{F}_k Cov(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1}) \mathbf{F}_k^T + \mathbf{Q}_k \\ &= \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k \end{aligned}$$

where  $\mathbf{Q}_k = Cov(\mathbf{w}_k)$

#### Phase 2, Update or adjustment with new observation

The objective of this phase is to update the state vector and the covariance matrix of errors, **after the observation  $k$** . Let's define a new notation  $\tilde{\mathbf{y}}_k$  called innovation, which is thus the difference between the observation and the prediction of observation *without noise* :

$$\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$$

We expect to use an unknown matrix  $\mathbf{K}_k$  to weight the innovation  $\tilde{\mathbf{y}}_k$ , in order to update  $\hat{\mathbf{x}}_{k|k-1}$  into  $\hat{\mathbf{x}}_{k|k}$ , by :

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{y}}_k$$

Now we need a criterion for determining  $\mathbf{K}_k$ . Let's take the one that minimizes :

$$\mathbb{E} [\|\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\|^2]$$

For the utility, we first calculate the covariance matrix become:

$$\begin{aligned} \mathbf{P}_{k|k} &= Cov(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}) \\ &= Cov(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1} - \mathbf{K}_k \tilde{\mathbf{y}}_k) \\ &= Cov(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1} - \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1})) \\ &= Cov(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1} - \mathbf{K}_k(\mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k) + \mathbf{K}_k \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}) \\ &= Cov((\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}) - \mathbf{K}_k \mathbf{v}_k) \\ &= Cov((\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})) + Cov(\mathbf{K}_k \mathbf{v}_k) \\ &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) Cov(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k Cov(\mathbf{v}_k) \mathbf{K}_k^T \\ &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1} (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T \end{aligned}$$

where  $\mathbf{R}_k = Cov(\mathbf{v}_k)$ .

Note that  $\mathbb{E} [\|\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\|^2] = \text{tr}(\mathbf{P}_{k|k})$ . Then we perform the first-order optimal condition, let's remind first some properties of trace and its derivative :

- $\text{tr}(ABCD) = \text{tr}(BCDA) = \text{tr}(CDAB) = \text{tr}(DABC)$
- $\nabla_A \text{tr}(AB) = \nabla_A \text{tr}(B^T A^T) = B^\top$
- $\nabla_A \text{tr}(ABA^\top C) = C^\top AB^\top + CAB$

With the note that covariance matrix  $\mathbf{P}$  and  $\mathbf{R}$  are sysmetric, we have :

$$\begin{aligned} \frac{\partial \text{tr}(\mathbf{P}_{k|k})}{\partial \mathbf{K}_k} &= \frac{\partial \text{tr}(\mathbf{P}_{k|k-1} - \mathbf{K}_k \mathbf{H}_k \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}(\mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{H}_k \mathbf{P}_{k|k-1}(\mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T)}{\partial \mathbf{K}_k} \\ &= -2\mathbf{P}_{k|k-1} \mathbf{H}_k^T + 2\mathbf{K}_k \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + 2\mathbf{K}_k \mathbf{R}_k \end{aligned}$$

This given :

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k)^{-1} = \mathbf{P}_{k|k-1} \mathbf{H}_k^T (\mathbf{S}_k)^{-1}$$

where

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k$$

Now with the optimal  $\mathbf{K}_k$ , which implies  $\mathbf{K}_k \mathbf{S}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T$ , we replace it into the above formula of  $\mathbf{P}_{k|k}$ , this leads to :

$$\begin{aligned}\mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{K}_k \mathbf{H}_k \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1} (\mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{S}_k \mathbf{K}_k^T \\ &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1} \quad (\text{the last two terms cancel out})\end{aligned}$$

#### 2.1.0.4 Resumed algorithm

**Phase 1:**

- $\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_k \mathbf{u}_k$
- $\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k$

**Phase 2 :**

- $\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$  (innovation)
- $\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k$  (also be the covariance of innovation  $\tilde{\mathbf{y}}_k$ , trivially proved)
- $\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1}$  (Kalman gain)
- $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{y}}_k$  (update state)
- $\mathbf{P}_{k|k} = (I - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$  (update covariance)

**Initializing :** There are two terms need to be initialized :  $\hat{\mathbf{x}}_{0|0}$  and  $\mathbf{P}_{0|0}$ .

- If the initial state  $\mathbf{x}_0$  is known, we set :

$$\hat{\mathbf{x}}_{0|0} = \mathbf{x}_0$$

and

$$\mathbf{P}_{0|0} = 0_{p \times p},$$

where  $p$  is the dimension of state  $\mathbf{x}$ .

- If the initial state is unknown, we may initialize these two term to any value :
  - For example, we may provide an initial guess, for example  $\hat{\mathbf{x}}_{0|0} = 0_p$  and initialize the covariance matrix  $\mathbf{P}_{0|0} = bI$  with quite large values  $b$ . If our initializations are good,  $\hat{\mathbf{x}}$  and  $\mathbf{P}$  will be stabilized when  $k \rightarrow \infty$ .
  - If we know in advance  $\mathbf{z}_1, \dots, \mathbf{z}_N$ , we can use them to estimate  $\hat{\mathbf{x}}_{0|0}$ , for example :

$$\hat{\mathbf{x}}_{0|0} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{z}_k - \mathbf{H}_1 \mathbf{F}_1 \mathbf{x}\|_2^2$$

In a more sophisticated manner, we can take :

$$\hat{\mathbf{x}}_{0|0} = \mathbb{E} \left[ \underset{\mathbf{x}}{\operatorname{argmin}} \| \mathbf{z}_1 - \mathbf{H}_1 \mathbf{F}_1 \mathbf{x} \|_2^2 \right]$$

which means we take mean for different values  $k$ . For  $\mathbf{P}_{0|0}$ , it can be initialized with suitable variances on its diagonal, for example :

$$\mathbf{P}_{0|0} = \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma_p^2 \end{bmatrix}$$

where  $\sigma_i^2$  must be theoretically  $\text{Var}(\mathbf{x}_0[i] - \hat{\mathbf{x}}_{0|0}[i])$ . In practice, we never know  $\mathbf{x}_0$ . We make an hypothesis that  $\text{Var}(\mathbf{x}_0[i] - \hat{\mathbf{x}}_{0|0}[i])$  is less than  $\text{Var}(\hat{\mathbf{x}}_{0|0}[i])$  and the latter can be estimated by :

$$\text{Var} \left( \underset{\mathbf{x}}{\operatorname{argmin}} \| \mathbf{z}_1 - \mathbf{H}_1 \mathbf{F}_1 \mathbf{x} \|_2^2 \right)$$

#### 2.1.0.5 Discretization for non-linear case

In the above problem, we have  $\mathbf{x}_k = \mathbf{F}_k \mathbf{x}_{k-1}$ .  $\mathbf{F}_k$  is constant from instant  $k$  to  $k+1$  and it means a *linear dynamical system*. For the non-linear case, we can take a discretization such that the duration between instant  $k$  to  $k+1$  is short and then we can consider  $\mathbf{F}_k$  is constant. If  $F$  is something differentiable then we can take:

$$\mathbf{F}_k = \frac{d\mathbf{x}(t)}{dt}$$

where  $t$  is anything between instant  $k$  to  $k+1$ .

#### 2.1.0.6 Continuous case

The continuous time version of Kalman filtering is also called Kalman–Bucy filtering.

##### Dynamic system

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{w}(t) \\ \mathbf{z}(t) &= \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}(t) \end{aligned}$$

##### Solution

1. We find the covariance matrix  $\mathbf{P}(t)$  by solving the following differential equation (Riccati equation):

$$\mathbf{P}'(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^\top(t) + \mathbf{Q}(t) - \mathbf{P}(t)\mathbf{H}^\top(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)\mathbf{P}^\top(t)$$

2. Then we get the Kalman gain :

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^T(t)\mathbf{R}^{-1}(t)$$

3. Finally, find the estimate of state by solving :

$$\hat{\mathbf{x}}'(t) = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{K}(t)(\mathbf{z}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t))$$

## 2.2 Linear-quadratic-Gaussian control

The linear-quadratic-Gaussian (LQG) control problem is one of the most fundamental (stochastic) optimal control problems.

### 2.2.0.1 Announcement

Consider the continuous-time linear dynamic system :

$$\begin{cases} \dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + \mathbf{v}(t) \\ \mathbf{y}(t) = C(t)\mathbf{x}(t) + \mathbf{w}(t) \end{cases},$$

as in Kalman filter (sec 5.2.1),  $\mathbf{x}$  is state vector,  $\mathbf{u}$  is control vector,  $\mathbf{y}$  is observation vector,  $\mathbf{v}$  and  $\mathbf{w}$  are white Gaussian noise.  $A(t), B(t), C(t)$  are parameters.

Given this dynamic system, then the objective is to find the control  $\mathbf{u}(t)$  such that the following cost function is minimized:

$$J = \mathbb{E} \left[ \mathbf{x}^T(T)F\mathbf{x}(T) + \int_0^T \mathbf{x}^T(t)Q(t)\mathbf{x}(t) + \mathbf{u}^T(t)R(t)\mathbf{u}(t) dt \right],$$

$$F \geq 0, \quad Q(t) \geq 0, \quad R(t) > 0,$$

The final time  $T$  may be either finite or infinite. If  $T$  tends to infinity the first term  $\mathbf{x}^T(T)F\mathbf{x}(T)$  of the cost function becomes negligible and irrelevant to the problem. Indeed, LQG problem is a stochastic version of LQR problem (sec 5.1.5) with additional noise in state and observation vectors.

### 2.2.0.2 Certainty equivalence

This is a property show that, if :

- Dynamic system is linear.
- Noise is additive ( $x + \varepsilon$ ), in observation or in evolution.
- Objective function is the expected value of a quadratic form.
- Control system is centralized. It means a system that has a single central authority that makes decisions and controls the entire system. In contrast, a decentralized control system distributes decision-making authority across multiple entities or nodes. For example, see section 5.2.3.

Then **the optimal control solution** in this case is the same as would be obtained in the **absence of the additive noise**, which means LQR problem (sec 5.1.5). More over, the quadratic cost allows for the optimal control vector  $\mathbf{u}(t)$  to be linear functions of the observations  $\mathbf{y}(t)$  :

$$\mathbf{u}(t) = M(t)\mathbf{y}(t) + N(t)$$

#### 2.2.0.3 Solution

The solution for LQG controller, by the certainty equivalence, can be seen a combination of two steps:

1. Using Kalman filter to find the estimate  $\hat{\mathbf{x}}$  of state vector  $\mathbf{x}$ .
2. Then applying LQR for  $\hat{\mathbf{x}}$  instead of  $\mathbf{x}$ .

### 2.3 Witsenhausen's counterexample

This example is shown in figure 5.5.

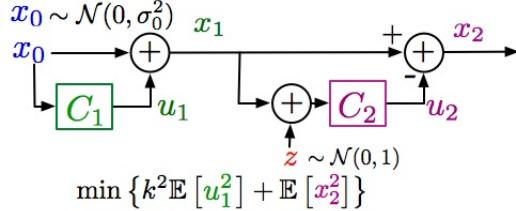


Figure 5.5: Diagram for Witsenhausen's counterexample and its cost function.  $C_1$  and  $C_2$  are two controllers.

Input for controller :

- The first controller  $u_1$  take the observation  $x_0$  as input :  $u_1 = f(x_0)$
- The second controller  $u_2$  take the observation  $y_1 = x_1 + z$  as input :  $u_2 = g(y_1)$

Since the controller  $(C_1)$  and  $(C_2)$  can not communicate with each other (information of  $C_1$  can not be seen for  $C_2$ ), then the control system is **decentralized**.

#### 2.3.1 Annoucement

Dynamic system :

$$\begin{cases} x_1 = x_0 + u_1, & x_0 \sim \mathcal{N}(0, \sigma_0^2) \\ x_2 = x_1 - u_2 \end{cases}$$

Observation :

$$\begin{cases} x_0 \\ y_1 = x_1 + z, \quad z \sim \mathcal{N}(0, 1) \end{cases}$$

$z$  and  $x_0$  are independent. Cost function to minimize:

$$J = k^2 E[u_1^2] + E[x_2^2]$$

### 2.3.2 Notes

We are curious to check if the certainty equivalence is true for this case, where the system is **decentralized**.

- In case of certainty equivalence, we find the solution for control vector where there is no noise  $z$  :

$$\begin{aligned} x_2 &= x_1 - g(x_1 + z) \\ &= x_0 + f(x_0) - g(x_0 + f(x_0) + z) \\ &= x_0 + f(x_0) - g(x_0 + f(x_0)) \end{aligned}$$

If we take the following configuration for control vector:

$$\begin{cases} f(a) = 0 \text{ (zero mapping)} \\ g(a) = a \text{ (identical function)} \end{cases}$$

This implies  $x_2 = 0$  and  $J = k^2 E[u_1^2] + E[x_2^2] = k^2 0 + 0 = 0$ . As  $J \geq 0$ , then this configuration is a solution. Finally, apply this solution with uncertain factor (noise). This implies that  $x_2 = -z$  and  $J = E[x_2^2] = E[(-z)^2] = 1$ .

- However, let's try an other simple solution,  $f$  is always zero mapping and now  $g(a) = 0.5a$ , then  $x_2 = 0.5x_0 - 0.5z$  and  $J = E[(0.5x_0 - 0.5z)^2] = E[(0.5x_0)^2] + E[(0.5z)^2] = \frac{1}{4}(\sigma_0^2 + 1)$ . If  $\sigma_0 = 1$ , then  $J = \frac{1}{2}$  and it is a better solution then the one of certainty equivalence.

## 2.4 Stochastic Hamilton–Jacobi–Bellman

### 2.4.1 Annoucement

The optimal control in this cas is the extension of the deterministic cas 5.1.6, let's consider the following problem :

$$\min_u \mathbb{E} \left[ \int_0^T I(X_t, u(t), t) dt + \phi(X_T, T) \right]$$

where :

- $(X_t)_{t \in [0, T]}$  is the stochastic process.
- $u(t)$  the steering or control function.

subject to the dynamic system, where  $X_t$  is usually Ito process (4.16.3.1):

$$dX_t = f(X_t, u(t), t)dt + \sigma(X_t, u(t), t)dW_t$$

#### 2.4.2 Value function

Let's define firstly the value function :

$$V(x(t), t) = \min_u \mathbb{E} \left[ \int_t^T I(X_s, u(s), s)ds + \phi(X_T) \right]$$

with  $X_t = x(t)$ , where  $x(t)$  is deterministic.

#### 2.4.3 Stochastic Hamilton–Jacobi–Bellman equation

$$\min_u \left\{ I(x, u, t) + V_x f + \frac{1}{2} V_{xx} \sigma^2 \right\} + V_t = 0$$

subject to the terminal condition :

$$V(x(T), T) = \mathbb{E}[\phi(X_T, T)] = \phi(x(T), T)$$

Note that, if  $\sigma = 0$ , which means now the stochastic problem becomes deterministic one, then we refind the equation in deterministic case (5.1.6.3). The stochastic HJB equation can be rewritten by :

- With Hamiltonian, here we have an additional term  $\frac{1}{2}V_{xx}\sigma^2$  since  $\sigma$  contains  $u$  :

$$\min_u H[x, u, V_x, V_{xx}, t] + V_t = 0$$

- With  $\mathcal{A}$  representing the stochastic differentiation operator or infinitesimal generator (4.18) :

$$\min_u \{I(x, u, t) + \mathcal{A}V\} = 0$$

This is because the infinitesimal generator for  $V(x(t), t)$  in section 4.18.4.3 is :

$$\mathcal{A}V(x(t), t) = V_t + V_x f + \frac{1}{2} V_{xx} \sigma^2$$

#### 2.4.4 Deriving the equation

First, by using Bellman's principle of optimality :

$$\begin{aligned} V(x(t), t) &= \min_u \mathbb{E} \left[ \int_t^{t+dt} I(X_s, u(s), s)ds \right] + \mathbb{E}[V(x(t) + dX_t, t + dt)] \\ &= \min_u \mathbb{E}[I(X_t, u(t), t)dt] + \mathbb{E}[V(x(t) + dX_t, t + dt)] \end{aligned}$$

$$= \min_u I(x(t), u(t), t) dt + \mathbb{E}[V(x(t) + dX_t, t + dt)]$$

Second, by Taylor expansion:

$$\begin{aligned} V(x(t) + dX_t, t + dt) &= V(x(t), t) + V_t dt + V_x dX_t + \frac{1}{2} V_{xx}(dX_t)^2 + o(dt) \quad (\text{Ito 4.16.3}) \\ &= V(x(t), t) + V_t dt + V_x f dt + V_x \sigma dW_t + \frac{1}{2} V_{xx} \sigma^2 dt \end{aligned}$$

By replacing into the above equation, note that  $\mathbb{E}[dW_t] = 0$ , we refind stochastic HJB equation :

$$\min_u \left\{ I(x(t), u(t), t) + V_x f + \frac{1}{2} V_{xx} \sigma^2 \right\} + V_t = 0$$

#### 2.4.5 Multi-dimensional case

In this case, the stochastic process  $\mathbf{X}_t$  and  $\mathbf{u}$  are multi-dimensional :

$$\min_u \mathbb{E} \left[ \int_0^T I(\mathbf{X}_t, \mathbf{u}(t), t) dt + \phi(\mathbf{X}_T, T) \right]$$

subject to the dynamic system where  $\mathbf{X}_t$  is a multidimensional Ito process :

$$d\mathbf{X}_t = f(\mathbf{X}_t, \mathbf{u}(t), t) dt + \Gamma(\mathbf{X}_t, \mathbf{u}(t), t) dB_t$$

Then the stochastic HJB equation is :

$$\min_u \left\{ I(\mathbf{X}_t, \mathbf{u}(t), t) + V_{\mathbf{x}}^T f + \frac{1}{2} \text{tr}(V_{\mathbf{xx}} \Gamma \Gamma^T) \right\} + V_t$$

where  $V_{\mathbf{xx}}$  is Hessian matrix.

#### 2.4.6 Procedure for stochastic HJB approach

The procedure for stochastic HJB approach is the same as in the deterministic case (5.1.6.6), except that :

- For Hamiltonian  $H$ , now we have an additional term related to  $V_{\mathbf{xx}}$  :

$$H(\mathbf{x}, \mathbf{u}, V_{\mathbf{x}}, V_{\mathbf{xx}}, t) = I(\mathbf{x}, \mathbf{u}, t) + V_x f + \frac{1}{2} \text{tr}(V_{\mathbf{xx}} \Gamma \Gamma^T) \sigma^2$$

- For HJB equation, the same as above, we have an additional term related to  $V_{\mathbf{xx}}$ . In some cases, this HJB equation which is a PDE problem can be solved by Feynman–Kac 5.3.3.

- For final step, now we need to solve for a *stochastic* differential equation (instead of ordinary one):

$$dX_t = f(X_t, h(X_t, t), t) dt + \sigma(X_t, h(X_t, t), t) dW_t$$

where  $u^*(t) = h(X_t, t)$ .

In some papers,  $u(t)$  is also called control process, but it **is not** stochastic process, thus it describes just how we control given  $X_t$  at time  $t$  :  $u^* = h(X_t, t)$ . In most cases we will need to impose some constraints on  $u(t)$  to keep the (optimal control) problem tractable, such that :

- The SDE problem in final step must have unique solution.
- $u^*$  follows the feedback control law, which means we need only  $X_t$  and  $t$  to have  $u^*$  or  $u^*(t) = h(X_t, t)$ .

#### 2.4.7 Example : Production planning model

This example is extracted from : <https://personal.utdallas.edu/~sethi/Prosper/Control-Tex-Chapte13.pdf>

Let's

- $X_t$  the inventory level at time  $t$  (state variable).
- $u(t)$  the production quantity at time  $t$  (control vector).
- $a$  the optimal production quantity.
- $C$  the salvage value (residual value) per unit of inventory at time  $T$ .
- $S$  the demand quantity by client.
- $\sigma$  constant diffusion coefficient.

We want to optimize for the production quantity and the inventory level, along with salvage value :

$$\min_u \mathbb{E} \left[ \int_0^T ((u(t) - a)^2 + X_t^2) dt - CX_T \right]$$

with dynamic system :

$$dX_t = (u(t) - S)dt + \sigma dW_t$$

where  $dW_t$  is interpreted the random by “sales returns”, “inventory spoilage”, etc.

#### Solution

1. Form the Hamiltonian :

$$H(x, u, V_x, V_{xx}, t) = (u - a)^2 + x^2 + V_x(u - S) + \frac{1}{2}V_{xx}\sigma^2$$

2. Optimize  $H$  with respect to  $u$  :

$$\begin{aligned} \frac{\partial H}{\partial u}(u^*) &= 0 \\ \Leftrightarrow u^* &= a - \frac{V_x}{2} \end{aligned}$$

3. Now with  $u^* = a - \frac{V_x}{2}$  :

$$H = -\frac{1}{4}V_x^2 + x^2 + (a - S)V_x + \frac{1}{2}V_{xx}\sigma^2$$

4. We need to solve the HJB equation:

$$-\frac{1}{4}V_x^2 + x^2 + (a - S)V_x + \frac{1}{2}V_{xx}\sigma^2 + V_t = 0$$

s.t

$$V(x(T), T) = Cx(T)$$

... By Feynman-Kac 5.3.3.

## 2.5 Stochastic programming

Stochastic programming is a mathematical programming framework that is used to solve optimization problems involving uncertainty. A stochastic program is an optimization problem in which some or all problem parameters are uncertain, but follow known probability distributions. This framework contrasts with deterministic optimization, in which all problem parameters are assumed to be known exactly. Because many real-world decisions involve uncertainty, stochastic programming has found applications in a broad range of areas ranging from finance to transportation to energy optimization.

Example :

<http://cgm.cs.mcgill.ca/~avis/courses/567/roussos/intro.pdf>

## 3 Partial differential equation

We remind ODE and PDE forms:

- **Ordinary differential equation (ODE)** is a differential equation whose unknown(s) consists of one (or more) function(s) of **one variable** (here  $x$ ) and involves the derivatives of those functions :

$$a_0(x)u + a_1(x)u' + a_2(x)u'' + \dots + a_n(x)u^{(n)} + b(x) = 0$$

- **Partial differential equation (PDE)** Have **more than one independent variable** (here  $x, y, z$ ) and usually in contrast with ODE. For example :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

### 3.1 Finite difference method

Finite-difference methods (FDM) are a class of numerical techniques for solving differential equations by approximating derivatives with finite differences. Both spatial and temporal domains (if applicable) are discretized, or broken into a finite number of steps. The value of the solution at these discrete points is approximated by solving algebraic equations containing finite differences and values from nearby points.

#### 3.1.1 Finite differences

3 types of finite differences, with  $h > 0$  :

- Forward difference, denoted  $\Delta_h[f]$ :

$$\Delta_h[f](x) = f(x + h) - f(x)$$

- Backward difference, denoted  $\nabla_h[f]$ :

$$\nabla_h[f](x) = f(x) - f(x - h)$$

- Central difference, denoted  $\delta_h[f]$ :

$$\delta_h[f](x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$$

#### 3.1.2 An example of finite difference method

Considering the following problem :

$$u'(x) - \tau u(x) = 0, \quad \forall x \in [0, 1], \quad u(0) = u_0$$

By using forward finite difference with step  $h$  :

$$\frac{u_{n+1} - u_n}{h} - \tau u_n = 0, \quad \forall 0 \leq n \leq M,$$

where  $M = \frac{1}{h} - 1$  and  $u_n = u(nh)$ .

Then:

$$u_{n+1} = (1 + h\tau)u_n = (1 + h\tau)^{n+1}u_0$$

Or:

$$u_n = (1 + h\tau)^n u_0 \underset{h \rightarrow 0}{=} e^{h\tau n} u_0$$

On other hand, we know that the solution exact of DE  $u' = \tau u$  is :

$$u(x) = u_0 e^{\tau x}$$

and if we discretize this solution, we get the same as the above solution :

$$u_n = u(nh) = u_0 e^{\tau nh}$$

## 3.2 Finite element method

We start by the weak formulation in linear system of equations, which is important in the finite element method (FEM). Then we discover (FEM) by two boundary value problems (BVP).

### 3.2.1 Weak formulation

In a weak formulation, equations or conditions are no longer required to hold absolutely (and this is not even well defined) and has instead weak solutions only with respect to certain "test vectors".

Let  $V$  be a (Banach) space, let  $V'$  be the dual space of  $V$  (which preserves the linearity from  $V$ ).  $A: V \rightarrow V'$  be an **linear operator** and  $f \in V'$ .

The objective is that we try to find  $u \in V$  which satisfies :

$$Au = f$$

under **weak formulation**, which means we try to find  $u$  which satisfies :

$$\langle Au, v \rangle = \langle f, v \rangle, \quad \forall v \in V$$

where  $\langle \cdot, \cdot \rangle$  is an inner product.  $v$  here is called test vector.

#### Intuitive understanding

The main point is that if  $\dim(V) = n$  (of finite dimension) and if we have

$$\langle a, v_i \rangle = \langle b, v_i \rangle, \quad \forall i = 1, \dots, n$$

where  $(v_i)_{i=1,\dots,n}$  are  $n$  independent vectors, then we get  $a = b$ , or in the case above  $Au = f$ .

#### Example

For example,  $V$  is space of smooth functions,  $V'$  is space of functions that have their primitive and  $A(u) = u'$ , which is the derivative of  $u \in V$  and is a linear operator. Given  $f \in V'$ , we want to find  $u$  such that :

$$Au = u' = f$$

In weak formulation, if we need only to find  $u$  such that :

$$\langle Au, v \rangle = \langle f, v \rangle, \quad \forall v \in V$$

where  $\langle \cdot, \cdot \rangle$  is an inner product.

### Approach to find $\mathbf{u}$

Let  $v_1, \dots, v_n \in \mathbb{R}^n$  be  $n$  linear independent vectors. We decompose  $u$  into this new basis vector  $v_1, \dots, v_n$ :

$$u = \sum_j^n u_j^v v_j$$

where  $u_i^v$  are scalar. By definition of weak formulation, we need to find  $u$  that satisfies:

$$\begin{aligned} \langle Au, v_i \rangle &= \langle f, v_i \rangle, \quad \forall i = 1, \dots, n \\ \Leftrightarrow \langle A \sum_j^n u_j^v v_j, v_i \rangle &= \langle f, v_i \rangle \\ \Leftrightarrow \sum_j^n \langle Av_j, v_i \rangle u_j^v &= \langle f, v_i \rangle \end{aligned}$$

Here, we need to solve  $\mathbf{u}^v \in \mathbb{R}^n$  with :

$$\mathbf{A}^v \mathbf{u}^v = \mathbf{f}^v$$

where  $\mathbf{A}_{ij}^v = \langle Av_j, v_i \rangle$  and  $\mathbf{f}_i^v = \langle f, v_i \rangle$ . Note that once we find out  $\mathbf{u}^v$ , we need to change back to  $u$  by :

$$u = \sum_j^n u_j^v v_j$$

#### 3.2.2 Illustrative examples

$$\mathbf{P1} : \begin{cases} u''(x) = f(x), & x \in (0, 1) \\ u(0) = u(1) = 0 \end{cases}$$

where  $f$  is given,  $u$  is an unknown function of  $x$ .

$$\mathbf{P2} : \begin{cases} u_{xx}(x, y) + u_{yy}(x, y) = f(x, y), & (x, y) \in \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

where  $\Omega$  is a connected open region in the  $(x, y)$  and  $\partial\Omega$  is its boundary. Note that here are two particular cases of Poisson's equation.

### 3.2.3 General method

Two steps :

- In the first step, one rephrases the original BVP in its weak form. Little to no computation is usually required for this step. The transformation is done by hand on paper.
- The second step is the discretization, where the weak form is discretized in a finite-dimensional space.

After this second step, we have concrete formula for a large but finite-dimensional linear problem whose solution will approximately solve the original BVP. This finite-dimensional problem is then implemented on a computer.

#### Step 1 : Weak formulation

**For P1,**  $u$  is solution if for every smooth functions  $v$ , we have :

$$\begin{aligned}\langle f, v \rangle &= \langle u'', v \rangle \\ \int_0^1 f v dx &= \int_0^1 u'' v dx\end{aligned}$$

As we need to find  $u(x)$  where  $0 < x < 1$ , then we can take  $v(0) = v(1) = 0$ . Applying the integral by parts :

$$\begin{aligned}\int_0^1 f v dx &= \int_0^1 u'' v dx \\ &= u' v|_0^1 - \int_0^1 u' v' dx \\ &= -\langle u', v' \rangle \\ &:= -\phi(u, v)\end{aligned}$$

**For P2,**  $u$  is solution if for every smooth functions  $v$ , we have :

$$\begin{aligned}\langle f, v \rangle &= \langle u'', v \rangle \\ \int_{\Omega} f v ds &= \int_{\Omega} u'' v ds\end{aligned}$$

$s$  here can be considered as small area (since we are in 2 dimensional). As we need to find  $u(x, y)$  where  $(x, y)$  are not on the boundary  $\partial\Omega$ , then we can take  $v(x, y) = 0$  if  $(x, y) \in \partial\Omega$ . We have :

$$\int_{\Omega} f v ds = \int_{\Omega} \Delta u v ds$$

$$\begin{aligned}
&= \oint_{\partial\Omega} \nabla u \cdot \nabla v dl - \int_{\Omega} \nabla u \cdot \nabla v ds \quad (\text{first identity of Green}) \\
&= 0 - \langle \nabla u, \nabla v \rangle \\
&:= -\phi(u, v)
\end{aligned}$$

### Step 2 : Discretization

This is the main idea of FEM.

**For P1**, one discretizes the segment  $[0, 1]$  into  $n$  segments  $[x_i, x_{i+1}]$ , where  $x_0 < x_1 < \dots < x_{n+1}$  ( $x_{i+1} - x_i$  are not necessarily equidistant). Given function  $f$ , if  $n$  is big enough, one can consider that function  $f$ , for each  $x_i$  to  $x_{i+1}$ , is approximately a linear function (figure 5.6) :

$$f_{i,i+1}^a(x) = \frac{f(x_{i+1})(x - x_i) - f(x_i)(x - x_{i+1})}{x_{i+1} - x_i}$$

This approximation is something like theory of integration.

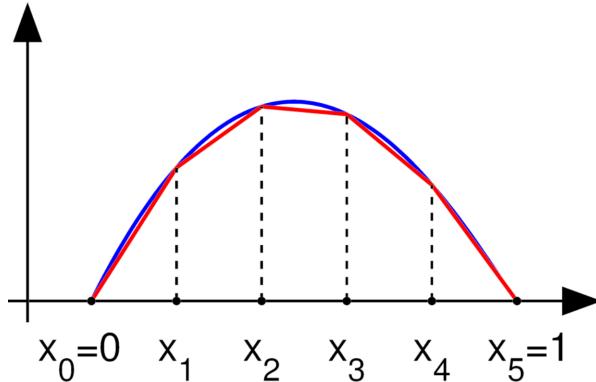


Figure 5.6: Linear approximation for curved function.

The advantage of piecewise linear approximation is that we can construct (find) a basis that can generate any piecewise linear function. A common basis is composed of tent functions.

#### Tent function basis

$$v_k(x) = \begin{cases} \frac{x - x_{k-1}}{x_k - x_{k-1}} & \text{if } x \in [x_{k-1}, x_k] \\ \frac{x_{k+1} - x}{x_{k+1} - x_k} & \text{if } x \in [x_k, x_{k+1}] \\ 0 & \text{otherwise} \end{cases}$$

where  $k = 0, \dots, n + 1$ . Note that if  $k = 0$  or  $k = n + 1$ , we use only the half right or half left of tent function. The number of discretized points is equal to the number of tent functions.

Here is an example show that, for any piecewise linear function, we can reconstruct it by tent functions (figure 5.7). The bold segment is the linear

approximation (for any curved function). Each two dashed line that meet on top or at a discretized point represents a tent basis function. Indeed, for any bold segment, we can reconstruct it by the sum of its two under “diagonal” dashed segments.

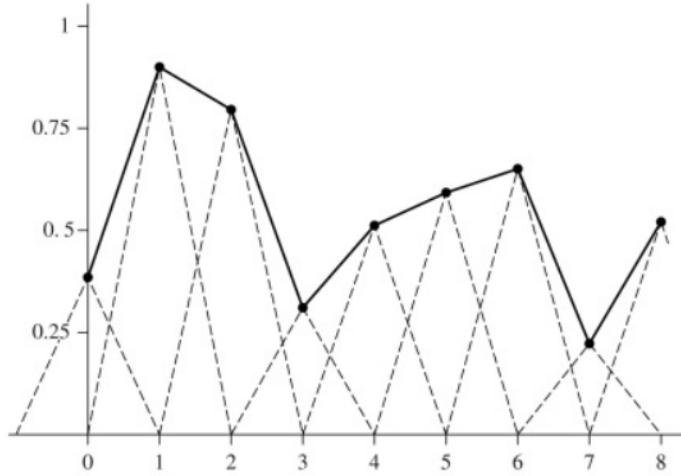


Figure 5.7: An example that show all piecewise linear function can be decomposed into tent basis functions, in case  $y = f(x)$ .

**For P2 :** The discretization is shown as in 5.8. The points  $(x, y)$  are set on the Oxy-plan and in the definition domain of function (5.8 lower figure). A question raised here is how we can connect discretized points  $(x, y, f(x, y))$  to define a piecewise linear function ?

An intuitive is to use each 3 discretized points  $(x, y, f(x, y))$  to form a piecewise linear function (5.8 upper figure). But how can we choose the sets of 3 discretized points without interseccion on projection in Oxy-plan ? Fortunately, this is always possible since for any polygon, we can devide it into many triangles without interseccion.

#### Tent function basis

As in **P1**, we adapt for tent function for **P2**. The tent function basis is a pyramid, formed by a discretized point and all of its neighbor discretized points projected on Oxy-plan. Then the number of discretized point is equal to the number of tents. For a tent, its “inclined facade” is each surface limited by 3 points : 2 connected points are on Oxy-plan and the resting point is the discretized point (which is top of tent). Remind that piecewise linear approximation is a surface with 3 discretized points. For reconstructing a surface, we sum its three under “inclined facade”.

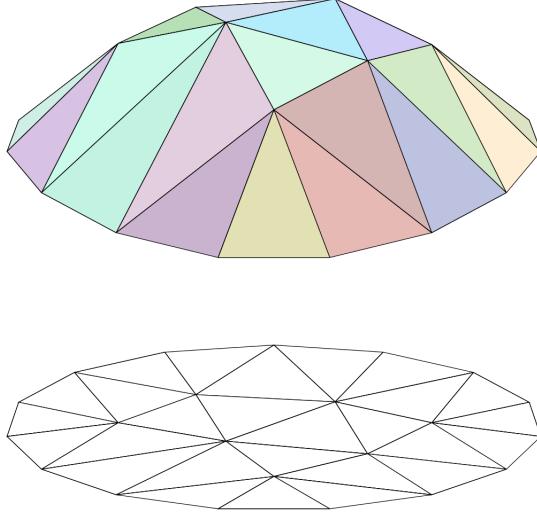


Figure 5.8: An example that show all piecewise linear function can be decomposed into tent basis functions, in case  $z = f(x, y)$ .

### 3.2.4 Property of tent function basis

**P1:**

$$\langle v_j, v_k \rangle = \int_0^1 v_j v_k dx$$

and

$$\phi(v_j, v_k) = \int_0^1 v'_j v'_k dx$$

will be zero for almost all  $j, k$  if  $|j - k| > 1$ , which means if two tent function basis  $v_j$  and  $v_k$  are not neighbor, then  $v_j(x)v_k(x) = 0, \forall x$  and  $v'_j(x)v'_k(x) = 0, \forall x$ .

**P2:**

$$\langle v_j, v_k \rangle = \int_{\Omega} v_j v_k dx$$

and

$$\phi(v_j, v_k) = \int_{\Omega} \nabla v'_j \nabla v'_k dx$$

will be zero if  $x_j$  and  $x_k$  if there is no intersection between their two bases, then  $v_j(x, y)v_k(x, y) = 0, \forall x, y$  and  $v'_j(x, y)v'_k(x, y) = 0, \forall x, y$ .

### 3.2.5 Matrix form of the problem

Let's represent  $u(x)$  and  $f(x)$  in term of tent function basis :

$$u(x) = \sum_{k=1}^n u_k v_k, \quad u_k \text{ is scalar}$$

$$f(x) = \sum_{k=1}^n f_k v_k, \quad f_k \text{ is scalar}$$

Then for each  $v_j$ , we must have :

$$\begin{aligned} -\phi(u, v_j) &= \langle f, v_j \rangle \\ \Leftrightarrow -\int_0^1 u' v'_j dx &= \int_0^1 f v_j dx \\ \Leftrightarrow -\sum_{k=1}^n u_k \int_0^1 v'_k v'_j dx &= \sum_{k=1}^n f_k \int_0^1 v_k v_j dx \end{aligned}$$

Then for all  $v_j$ , we have :

$$\begin{aligned} -\mathbf{L}[u_1, \dots, u_n]^T &= \mathbf{M}[f_1, \dots, f_n]^T \\ \Leftrightarrow -\mathbf{Lu} &= \mathbf{Mf} \end{aligned}$$

where  $\mathbf{L}[i, j] = \phi(v_i, v_j)$  and  $\mathbf{M}[i, j] = \langle v_i, v_j \rangle$ . Finally, we can solve  $-\mathbf{Lu} = \mathbf{Mf}$  for variable  $\mathbf{u}$ .

In practice way, if  $n$  is sufficiently big, we do not need to decompose  $f$  into basis function, we can solve alternatively :

$$-\mathbf{Lu} = \mathbf{b}$$

where  $\mathbf{b}[i] = \int_0^1 f v_i dx$ .

### 3.2.6 FEM and FDF comparison

Finite element method (FEM) and finite difference method (FDM) are both numerical methods used for solving partial differential equations (PDEs) and other mathematical problems.

On one hand, FDM involves discretizing the domain into a grid of points and approximating the derivatives in the PDEs using finite differences between the grid points. The PDEs are then replaced by a set of algebraic equations, which can be solved numerically using matrix algebra.

On the other hand, in FEM, the domain of the problem is divided into smaller connected elements (zones), which form a mesh. The solution to the PDE is then approximated by representing it as a linear combination of basis functions over each element of the mesh. The coefficients of the basis functions are determined by solving a set of linear equations.

One advantage of FEM is that it can handle irregular geometries more easily than FDM. FEM also tends to produce more accurate solutions, especially for problems with complex geometry. However, FEM can be more computationally expensive than FDM, and the formulation of the basis functions can be more complex.

In summary, the choice between FEM and FDM depends on the specific problem being solved and the available computational resources.

### 3.3 Feynman–Kac formula

It's about solving for a kind of parabolic partial differential equation, where there is the term of second order partial differential. Let's consider :

$$\frac{\partial V}{\partial t}(x, t) + \mu(x, t) \frac{\partial V}{\partial x}(x, t) + \frac{1}{2} \sigma^2(x, t) \frac{\partial^2 V}{\partial x^2}(x, t) - \beta(x, t) V(x, t) + I(x, t) = 0,$$

subject to the terminal condition  $V(x, T) = \psi(x)$ .

#### 3.3.1 Solution

With  $\mu, \sigma, \psi, \beta, I$  are known functions, the solution is :

$$V(x, t) = \mathbb{E} \left[ e^{- \int_t^T \beta(X_\tau, \tau) d\tau} \psi(X_T) + \int_t^T e^{- \int_t^\tau \beta(X_s, s) ds} I(X_\tau, \tau) d\tau \middle| X_t = x \right]$$

where  $X_t$  is an Itô process satisfying :

$$dX_t = \mu(X, t) dt + \sigma(X, t) dW_t$$

Note that, in a particular case that  $\beta(x, t) = 0$ , which means the stochastic HJB equation 5.2.4.3, then :

$$V(x, t) = \mathbb{E} \left[ \psi(X_T) + \int_t^T I(X_\tau, \tau) d\tau \middle| X_t = x \right]$$

### 3.4 Fokker–Planck equation

This is a partial differential equation that describes the probability density function  $P(x, t)$  of process  $X_t$  at instant  $t$  :

$$\frac{\partial}{\partial t} P(x, t) = - \frac{\partial}{\partial x} [D_1(x, t) P(x, t)] + \frac{\partial^2}{\partial x^2} [D_2(x, t) P(x, t)]$$

where  $D_1(x, t)$  is drift and  $D_2(x, t)$  is diffusion coefficient.

#### 3.4.1 For Ito process

If  $X_t$  is an Ito process, we have :

$$dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dW_t$$

Then the Fokker–Planck equation in this case is :

$$\frac{\partial}{\partial t} P(x, t) = - \frac{\partial}{\partial x} [\mu(x, t) P(x, t)] + \frac{\partial^2}{\partial x^2} \left[ \frac{\sigma(x, t)^2}{2} P(x, t) \right]$$

where we have replaced  $D_1(x, t)$  by  $\mu(x, t)$  and  $D_2(x, t)$  by  $\frac{\sigma(x, t)^2}{2}$ .

### 3.4.2 Example

We discover the density function  $P(x, t)$  for each particular process.

#### 3.4.2.1 Wiener process

A standard scalar Wiener process is generated by the stochastic differential equation

$$dX_t = dW_t$$

Here the drift term  $\mu = 0$  and the diffusion coefficient is  $\frac{1}{2}$ . Thus the corresponding Fokker–Planck equation is

$$\frac{\partial P(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 P(x, t)}{\partial x^2}$$

If the initial condition is  $P(x, 0) = \delta(x)$ , the solution is :

$$P(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

This is thus the gaussian function with mean 0 and variance  $t$ .

#### 3.4.2.2 Ornstein–Uhlenbeck proces

The Ornstein–Uhlenbeck process (4.19) is a process defined as

$$dX_t = -\theta X_t dt + \sigma dW_t$$

The corresponding Fokker–Planck equation is :

$$\frac{\partial P(x, t)}{\partial t} = \theta \frac{\partial}{\partial x} (x P(x, t)) + \frac{\sigma^2}{2} \frac{\partial^2 P(x, t)}{\partial x^2}$$

With initial condition  $P(x, t_0) = \delta(x - x_0)$  at instant  $t_0$ , the solution is :

$$P(x, t | x_0, t_0) = \sqrt{\frac{\theta}{\pi\sigma^2(1 - e^{-2\theta(t-t_0)})}} \exp \left[ -\frac{\theta}{\sigma^2} \frac{(x - x_0 e^{-\theta(t-t_0)})^2}{1 - e^{-2\theta(t-t_0)}} \right]$$

## 4 Merton's portfolio problem

This section is reported from “On Merton’s Portfolio Problem” at <https://umu.diva-portal.org/smash/get/diva2:1667781/FULLTEXT01.pdf>

After discovering stochastic optimal control, it is now the time for Merton’s portfolio problem. There are two main approaches to this problem : the traditional approach and the modern one. The traditional method is most likely to be favoured. We present here only the traditional approach which solves the problem by using the Hamilton-Jacobi-Bellman equation. .

## 4.1 Announcement

We start with notations :

- $T$ : ending time.
- $X_t$  : wealth of portfolio.
- $c(X_t = x, t)$  : amount of wealth to consume at instant  $t$  and given  $X_t = x$ .  
 $c$  must be non-negative, i.e., we can not inject capital.
- $\beta$  : the subjective discount rate.
- $r$  : risk free rate.
- $\mu, \sigma$  : the expected return and volatility of the stock.
- $w(X_t = x, t)$  : weight on risky asset at instant  $t$  and given  $X_t = x$ .  $w$  has no constraint, which means we can short sell the risky asset to place a larger weight in the risk free asset ( $w < 0$ ) as well as borrow money to place a larger weight in the risky assets ( $w > 1$ ).
- $u(x)$  : utility function (6.8). There are several standard ones. We usually in this case the constant relative risk aversion (CRRA)  $u(x) = \frac{x^p}{p}$ , where  $p \in (0, 1]$ .
- $\epsilon > 0$  the coefficient attached on bequest term to adjust the important between consumption and bequest.

The **objective function** is defined as follow :

$$\begin{aligned} & \max_{c,w} \mathbb{E} \left[ \int_0^T e^{-\beta t} u(c(X_t, t)) dt + \epsilon e^{-\beta T} u(X_T) \right] \\ &= \max_{c,w} \mathbb{E} \left[ \int_0^T e^{-\beta t} c(X_t, t)^p dt + \epsilon e^{-\beta T} X_T^p \right] \end{aligned}$$

with the initial condition  $X_0 = x_0$  and the constraint  $c(X_t, t) > 0$ .

The **dynamic system** is given by :

$$X_t = w(X_t, t) (\mu X_t dt + \sigma X_t dW_t) + (1 - w(X_t, t)) r X_t dt - c(X_t, t) dt$$

where the first term is risky part modeled by Geometric Brownian Motion, the second is risk free part and the third is by consumption. We can rewrite dynamic system by :

$$X_t = (w(X_t, t)(\mu - r)X_t + rX_t - c(X_t, t)) dt + w(X_t, t)\sigma X_t dW_t$$

## 4.2 Solution

1. Form the Hamiltonian :

$$H(x, c, w, t) = e^{-\beta t} c^p + V_x(w(\mu - r)x + rx - c) + \frac{1}{2} V_{xx} w^2 \sigma^2 x^2$$

2. Optimize  $H$  with respect to  $c$  and  $w$  :

$$\begin{aligned} \frac{\partial H}{\partial c}(c^*) &= 0 & \frac{\partial H}{\partial w}(w^*) &= 0 \\ \Leftrightarrow e^{-\beta t} p(c^*)^{p-1} - V_x &= 0 & \Leftrightarrow V_x(\mu - r)x + V_{xx} w^* \sigma^2 x^2 &= 0 \\ \Leftrightarrow c^* &= \left( e^{\beta t} \frac{V_x}{p} \right)^{\frac{1}{p-1}} & \Leftrightarrow w^* &= -\frac{V_x(\mu - r)}{V_{xx} \sigma^2 x} \end{aligned}$$

3. Now with  $c^*$  and  $w^*$  in  $H$ :

$$\begin{aligned} H &= e^{-\beta t} \left( e^{\beta t} \frac{V_x}{p} \right)^{\frac{p}{p-1}} + V_x \left( -\frac{V_x(\mu - r)^2}{V_{xx} \sigma^2} + rx - \left( e^{\beta t} \frac{V_x}{p} \right)^{\frac{1}{p-1}} \right) + \frac{1}{2} \frac{V_x^2(\mu - r)^2}{V_{xx} \sigma^2} \\ &= e^{-\beta t} \left( e^{\beta t} \frac{V_x}{p} \right)^{\frac{p}{p-1}} + V_x \left( rx - \left( e^{\beta t} \frac{V_x}{p} \right)^{\frac{1}{p-1}} \right) - \frac{1}{2} \frac{V_x^2(\mu - r)^2}{V_{xx} \sigma^2} \end{aligned}$$

4. We need to solve the HJB equation:

$$e^{-\beta t} \left( e^{\beta t} \frac{V_x}{p} \right)^{\frac{p}{p-1}} + V_x \left( rx - \left( e^{\beta t} \frac{V_x}{p} \right)^{\frac{1}{p-1}} \right) - \frac{1}{2} \frac{V_x^2(\mu - r)^2}{V_{xx} \sigma^2} + V_t = 0$$

s.t the terminal boundary condition :

$$V(x(T), T) = \epsilon e^{-\beta T} x(T)^p$$

By supposing that our solution for  $V(x, t)$  can be express by the following form :

$$V(x, t) = e^{-\beta t} f(t) x^p$$

After substituting in HJB equation and simplifying, we have the following :

$$e^{-\beta t} x^p \left( \frac{df}{dt} + f(t) \left( \frac{p(\mu - r)^2}{2\sigma^2(1-p)} + rp - \beta \right) + f(t)^{-\frac{p}{1-p}} (1-p) \right) = 0$$

Let  $D = \frac{p(\mu - r)^2}{2\sigma^2(1-p)} + rp - \beta$ , then we have :

$$\frac{df}{dt} + Df(t) + f(t)^{-\frac{p}{1-p}} (1-p) = 0$$

This is known as a differential Bernoulli equation (DBE), which has form  $y'(x) + P(x)y(x) = Q(x)y^n(x)$ , where  $n$  is a **real** number. The solution

for DBE is to do a change of variable  $z = y^{1-n}$  in that the initial DBE become a linear differential equation.

Come back to our case, let  $h(t) = f(t)^{1+\frac{p}{1-p}} = f(t)^{\frac{1}{1-p}}$ , then we have  $f(t) = h(t)^{1-p}$  and  $\frac{df}{dt} = (1-p)h^{-p}\frac{dh}{dt}$ :

$$(1-p)h^{-p}\frac{dh}{dt} + Dh^{1-p} + (1-p)h^{-p} = 0$$

$$\Leftrightarrow (1-p)\frac{dh}{dt} + Dh + (1-p) = 0$$

This linear differential equation have solution :

$$h(t) = Ce^{-D\frac{t}{1-p}} - \frac{1-p}{D}$$

This leads to the solution for  $V(x, t)$  :

$$V(x, t) = e^{-\beta t}f(t)x^p = e^{-\beta t}h(t)^{1-p}x^p = e^{-\beta t}\left(Ce^{-D\frac{t}{1-p}} - \frac{1-p}{D}\right)^{1-p}x^p$$

By the terminal boundary condition  $V(x(T), T) = \epsilon e^{-\beta T}x(T)^p$  :

$$e^{-\beta T}\left(Ce^{-D\frac{T}{1-p}} - \frac{1-p}{D}\right)^{1-p}x(T)^p = \epsilon e^{-\beta T}x(T)^p$$

$$\Leftrightarrow C = \left(\epsilon^{\frac{1}{1-p}} + \frac{1-p}{D}\right)e^{\frac{DT}{1-p}}$$

hence,

$$h(t) = \left(\epsilon^{\frac{1}{1-p}} + \frac{1-p}{D}\right)e^{\frac{D(T-t)}{1-p}} - \frac{1-p}{D}$$

Then

$$V(x, t) = e^{-\beta t}\left(\left(\epsilon^{\frac{1}{1-p}} + \frac{1-p}{D}\right)e^{\frac{D(T-t)}{1-p}} - \frac{1-p}{D}\right)^{1-p}x^p$$

5. We first check the non-negative condition, which means  $c^* = \left(e^{\beta t}\frac{V_x}{p}\right)^{\frac{1}{p-1}} \geq 0$ . We need  $V_x \leq 0 \Leftrightarrow e^{-\beta t}f(t)x^p \leq 0 \Leftrightarrow f(t) \geq 0 \Leftrightarrow h(t) \geq 0$ . We show that this is always the case. Let  $z = \frac{1-p}{D}$  and  $\alpha = T - t \leq 0$ , then we need to show that

$$q(z) = (\epsilon + z)e^{\frac{\alpha}{z}} - z \geq 0$$

- $z > 0$  : then  $e^{\frac{\alpha}{z}} > 1$ , then  $(\epsilon + z)e^{\frac{\alpha}{z}} > z$ .
- $0 > z > -\epsilon$  : then  $(\epsilon + z)e^{\frac{\alpha}{z}} > 0$  and  $-z > 0$ .

- $z \leq -\epsilon$ : then  $q'(z) = e^{\frac{\alpha}{z}} \left(1 - \frac{\epsilon\alpha}{z^2} - \frac{\alpha}{z}\right) - 1$  and  $q''(z) = e^{\frac{\alpha}{z}} \left(\frac{\alpha^2}{x^4}\right) (\epsilon + z + 2\epsilon z)$ .  
 Since  $\epsilon + z + 2\epsilon z < 0$ , then  $q''(z) < 0$ . This implies that  $0 = q'(-\infty) \geq q'(-\infty < z \leq -\epsilon)$ . This also implies  $q(-\infty < z \leq -\epsilon) \geq q(-\epsilon) = \epsilon > 0$ .

Now we can write the explicit form for  $c^*$  and  $w^*$  :

$$c^* = \left( e^{\beta t} \frac{V_x}{p} \right)^{\frac{1}{p-1}} = \left( \left( \epsilon^{\frac{1}{1-p}} + \frac{1-p}{D} \right) e^{\frac{D(T-t)}{1-p}} - \frac{1-p}{D} \right)^{-1} x$$

where  $D = \frac{p(\mu-r)^2}{2\sigma^2(1-p)} + rp - \beta$ .

$$w^* = -\frac{V_x(\mu-r)}{V_{xx}\sigma^2 x} = \frac{\mu-r}{\sigma^2(1-p)}$$

6. Get optimal state by solving differential equation :

$$X_t = (w^*(X_t, t)(\mu - r)X_t + rX_t - c^*(X_t, t)) dt + w^*(X_t, t)\sigma X_t dW_t$$

with  $X_0 = x_0$ . For the reason of complexity, we admit that this SDE have unique solution and the initial condition  $X_0 = x_0$  have us to identify parameters in the explicit form of  $X_t$ .

## 5 Optimal stopping

Do not confuse with optional stopping theorem (4.10.13). The optimal stopping or early stopping is about choosing a time to take a particular action, in order to maximise an expected reward or minimise an expected cost. Optimal stopping can be found in many fields such as statistics, economics and finance (American options, see 1.10.8.2).

### 5.1 Secretary problem

This key example of an optimal stopping problem is the secretary problem.

#### 5.1.1 Context

A boss wants to recruit a secretary and he know the total number of candidates. For everyone, after the interview, he must decide whether to hire or not. If yes, he completes the recruitment process without seeing the other candidates. Otherwise, he does not have the possibility of calling the candidate back later. In the context of this problem, the recruiter does not have access to an intrinsic value of the candidates (such as “this candidate is worth 7/10”). After an interview, he can only compare a candidate with previous ones (for example “this candidate is better than the first, but less good than the second”).

The goal is to define a strategy that maximizes the probability of hiring the best candidate.

### 5.1.2 Optimal policy

The best strategy consists of rejecting 37% of first candidates (or, more precisely, a  $\frac{1}{e}$  proportion), then hiring the first candidate who is better than the best among previous ones. This policy also called “the 37% rule”.

In other words, suppose that we have 100 candidates. We perfome interviews with 37 random candidates all reject them all. We note  $A$  the best among them as a reference. From the 38<sup>th</sup> candidate, if one is better than  $A$ , we hire this one and finish the recruitment process. If no one is better than  $A$ , we are obligated to hire the last one (100<sup>th</sup> candidate).

### 5.1.3 Deriving the optimal policy

Let  $r - 1$  be the number of first candidates that will be rejected. Then we need to maximize  $L(r)$  which is the following sum

$$\begin{aligned} L(r) &= \sum_{i=r}^n P(i \text{ is selected} \& i \text{ is the best}) \\ &= \sum_{i=r}^n P(i \text{ is selected} \mid i \text{ is the best})P(i \text{ is the best}) \\ &= \sum_{i=r}^n P(\text{the best among first } i-1 \text{ is in first } r-1) \times \frac{1}{n} \\ &= \sum_{i=r}^n \frac{r-1}{i-1} \frac{1}{n} = \sum_{i=r}^n \frac{r-1}{n} \frac{1}{\frac{i-1}{n}} \frac{1}{n} \end{aligned}$$

Suppose that  $n$  is big enough, then we can consider this discret problem as a continuous problem, with  $x = \frac{r-1}{n}$ ,  $t = \frac{i-1}{n}$  and  $dt = \frac{1}{n}$ . Maximizing  $L(r)$  is equivalent to maximize :

$$H(x) = x \int_x^1 \frac{1}{t} dt = -x \ln(x)$$

Solving  $H'(x) = 0$  or  $\ln(x) + 1 = 0$ , we get  $x = \frac{1}{e}$ . Equivalently,  $r - 1 = \frac{n}{e}$ .

## 5.2 American options pricing

In the trading of options, the holder of an American option is allowed to exercise the right to buy (or sell) the underlying asset at a predetermined price at any time before or at the expiry date. Therefore, *the valuation of American options* is essentially an optimal stopping problem.

We may need to discover first the variational inequality, which is thus a necessary condition for optimization problem : [https://en.wikipedia.org/wiki/Variational\\_inequality](https://en.wikipedia.org/wiki/Variational_inequality)

Then the details for pricing American options are described in : <https://personal.ntu.edu.sg/nprivault/MA5182/american-options.pdf>

## Chapter 6

# Maths quantitative

### 1 Time series

Given a time serie signal (or a process)  $X_t$ , usually the price of an asset, it can contain the main following components :

- $T_t$  the trend component at time t, which reflects the long-term progression of the series (secular variation). A trend exists when there is a persistent increasing or decreasing direction in the data. The trend component does not have to be linear.
- $C_t$  the cyclical component at time t, which reflects repeated but non-periodic fluctuations. The duration of these fluctuations depend on the nature of the time series.
- $S_t$  the seasonal component at time t, reflecting seasonality (seasonal variation). A seasonal pattern exists when a time series is influenced by seasonal factors. Seasonality occurs over a fixed and known period (e.g., the quarter of the year, the month, or day of the week).
- $I_t$ , the irregular component (or “noise”) at time t, which describes random, irregular influences. It represents the residuals or remainder of the time series after the other components have been removed.

**Usually, we combine the trend and cycle** into a single trend-cycle component (sometimes called the trend for simplicity).

If  $X_t$  is additive model:

$$X_t = T_t + S_t + I_t$$

If  $X_t$  is multiplicative model:

$$X_t = T_t \times S_t \times I_t$$

An additive model would be used when the variations around the trend do not vary with the level of the time  $t$  (do not depend on  $t$ ), whereas a multiplicative model would be appropriate if the trend is proportional to the level of the time  $t$ .

We can switch between additive model and multiplicative model by using logarithm and exponential :

$$X_t = T_t \times S_t \times I_t \Leftrightarrow \ln(X_t) = \ln(T_t) + \ln(S_t) + \ln(I_t)$$

Good document can be found here <https://otexts.com/fpp2/>

## 1.1 Model selection

In the case that ACF and PACF plots can not chose which model to use, we need other criteria. Here we show some popular criteria to evaluate statistical models. Note that, to use these criteria, models must be fitted before, which means we must have value of parameters of model.

### 1.1.1 Akaike Information Criterion (AIC)

The Akaike Information Criterion (AIC) is a score to asses statistical models, which fit to a set of data. With  $k$  number of parameters and  $\hat{\mathcal{L}}$  is maximum likelihood of fitted model :

$$AIC = -2 \ln(\hat{\mathcal{L}}) + 2k$$

The preferred model has the minimum AIC among models. We can see that the AIC grows when the number of parameters,  $k$ , increases, but it reduces if the maximum likelihood increases. In short, AIC privileges model that has better maximum likelihood and lower number of parameters.

### 1.1.2 Akaike Information Criterion with correction (AICc)

$$AICc = AIC + \frac{2k(k+1)}{n-k-1}$$

where  $n$  is the number of observations or sample size.

### 1.1.3 Bayesian Information Criterion (BIC)

$$BIC = -2 \ln(\hat{\mathcal{L}}) + k \ln(n)$$

where  $n$  is the number of observations. The difference between BIC and AIC is that BIC penalizes the number of parameters by the size of observation.

## 1.2 Classical decomposition

The classical decomposition method originated in the 1920s. It is a relatively simple procedure, and forms the starting point for most other methods of time series decomposition. We first begin with the moving averages filter, which helps to extract the trend-cycle component.

### 1.2.1 Moving averages filter

A moving averages (MA) of order  $n$ , denoted by  $n$ -MA can be written as:

$$\hat{T}_t^n = \frac{1}{n} \sum_{i=-\lfloor \frac{n}{2} \rfloor}^{\lfloor \frac{n-1}{2} \rfloor} X_{t+i}$$

where we use  $\hat{T}_t$  for denoting the result after the moving average filter since this is an estimate of trend-cycle  $T_t$  and  $\lfloor x \rfloor$  for the floor function. The number of concerned elements of  $X$  is  $\lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1 = n$ . We say a filter MA is symmetric if the number of concerned elements after instant  $t$  is equal to the one before instant  $t$ . Then  $n$ -MA is symmetric if  $n$  is odd.

If we use a  $n$ -MA filter followed by  $m$ -MA filter, denoted  $m \times n$ -MA then we get :

$$\begin{aligned} \hat{T}_t^{m \times n} &= \frac{1}{m} \sum_{i=-\lfloor \frac{m}{2} \rfloor}^{\lfloor \frac{m-1}{2} \rfloor} \hat{T}_{t+i}^n \\ &= \frac{1}{m} \left( \sum_{i=-\lfloor \frac{m}{2} \rfloor}^{\lfloor \frac{m-1}{2} \rfloor} \frac{1}{n} \sum_{j=-\lfloor \frac{n}{2} \rfloor}^{\lfloor \frac{n-1}{2} \rfloor} X_{t+i+j} \right) \\ &= \frac{1}{mn} \left( \sum_{i=-\lfloor \frac{m}{2} \rfloor}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{j=-\lfloor \frac{n}{2} \rfloor}^{\lfloor \frac{n-1}{2} \rfloor} X_{t+i+j} \right) \end{aligned}$$

For  $m \times n$ -MA being symmetric :

- The number of concerned elements is  $\lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1 = m + n - 1$ . Then a necessary condition for symmetric is that  $m + n - 1$  must be odd, equivalently  $m, n$  are both even or both odd.
- The number of concerned elements before instant  $t$  is  $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$  and the one after instant  $t$  is  $\lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor$ . In the case that  $m, n$  are both odd, then  $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor$  or  $m \times n$ -MA is sysmetric.
- In the case that  $m, n$  are both even, then  $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor + 2$ , or we need to push forward  $i$  or  $j$  one unit to get  $m \times n$ -MA sysmetric

(recentering). For example,

$$\hat{T}_t^{2 \times 2} = \frac{1}{4} \left( \sum_{i=-1}^0 \sum_{j=-1}^0 X_{t+i+j} \right) = \frac{1}{4} X_{t-2} + \frac{1}{2} X_{t-1} + \frac{1}{4} X_t$$

Then we need to recenter by pushing forward  $i$  one unit, to get

$$\hat{T}_t^{2 \times 2} = \frac{1}{4} \left( \sum_{i=0}^1 \sum_{j=-1}^0 X_{t+i+j} \right) = \frac{1}{4} X_{t-1} + \frac{1}{2} X_t + \frac{1}{4} X_{t+1}$$

Since a MA of even order is not symmetric, then to make it symmetric, usually we apply a 2-MA to after this MA of even order and recentering. For example a 2-MA follows a 4-MA.

#### 1.2.1.1 Trend-cycle estimation with seasonality data

The most common use of moving averages filter is for estimating the trend-cycle from seasonal data. Consider the centered  $2 \times 4$ -MA :

$$\hat{T}_t^{2 \times 4} = \frac{1}{8} X_{t-2} + \frac{1}{4} X_{t-1} + \frac{1}{4} X_t + \frac{1}{4} X_{t+1} + \frac{1}{8} X_{t+2}$$

When applied to quarterly data, each quarter of the year is weighted by  $\frac{1}{4}$  since the  $t-2$  and  $t+2$  quarters signifie the same quarter but in two consecutive years. Consequently, the seasonal variation will be averaged out and the resulting values of  $\hat{T}_t^{2 \times 4}$  will have little or no seasonal variation remaining. A similar effect would be obtained using a  $2 \times 8$ -MA or a  $2 \times 12$ -MA to quarterly data.

Therefore, if the seasonal period is even and of order  $m$ , we use a centered  $2 \times m$ -MA to estimate the trend-cycle. If the seasonal period is odd and of order  $m$ , we just use a  $m$ -MA to estimate the trend-cycle.

#### 1.2.1.2 With real time data

Note that, in case of real time data, MA can serve as an indicator in analysis of time series, where we only of the observation until the present  $t$  :

$$\hat{T}_t^n = \frac{1}{n} \sum_{i=t-n+1}^t X_i$$

#### 1.2.2 Description of classical decomposition

Suppose that our data  $X_t$  is seasonal, the procedure of classical decomposition is the following :

1. **Trend-cycle component estimation.** As described in 6.1.2.1.1, we extract  $\hat{T}_t$

2. **Getting detrended series.**  $X_T - \hat{T}_t$ .
3. **Seasonal component estimation.** We simply average the detrended values for each given season. For example, with monthly data, the seasonal component for March is the average of all the detrended March values in the data. These seasonal component values are then adjusted to ensure that they add to zero. The seasonal component is obtained by stringing together these monthly values, and then replicating the sequence for each year of data. This gives  $\hat{S}_t$ .
4. **The irregular component** is calculated by subtracting the estimated seasonal and trend-cycle components :  $\hat{I}_t = X_t - \hat{T}_t - \hat{S}_t$ .

### 1.2.3 Methods based on classical decomposition

- X11 method originated in the US Census Bureau and Statistics Canada. It has many extra steps in order to overcome the drawbacks of the classical decomposition. In particular, trend-cycle estimates are available for all observations including the end points and the seasonal component is allowed to vary slowly over time. X11 also has some sophisticated features for handling trading day variation, holiday effects... X11 is highly robust to outliers and level shifts in the time series.
- SEATs stands for “Seasonal Extraction in ARIMA Time Series” (ARIMA 6.1.9). This procedure was developed at the Bank of Spain, and is now widely used by government agencies around the world. The procedure works only with quarterly and monthly data. So seasonality of other kinds, such as daily data, or hourly data, or weekly data, require an alternative approach.
- STL is a versatile and robust method for decomposing time series. STL is an acronym for “Seasonal and Trend decomposition using Loess”, while Loess is a method for estimating nonlinear relationships. The STL method was developed in 90s.

X11 and SEATs are detailed in the book “*Seasonal adjustment methods and real time trend-cycle estimation*”

### 1.2.4 Measuring strength of trend and seasonality

This is about to measure the strength of trend and seasonality, compared to irregular component, in a time series. Recall that the decomposition is written as :

$$X_t = \hat{T}_t + \hat{S}_t + \hat{I}_t$$

To measure the strength of trend compared to irregular component, we use the following metric :

$$F_T = \max \left( 0, 1 - \frac{\text{Var}(\hat{I}_t)}{\text{Var}(\hat{T}_t + \hat{I}_t)} \right)$$

The idea is that strength of a component is proportional to its variance. Then the strength of a trend is expressed by ( $Var(\hat{T}_t)$ ) and we compare it with the variance of noise or irregular component ( $Var(\hat{I}_t)$ ).

For the same reasoning, we have the strength of seasonality compared to irregular component :

$$F_S = \max \left( 0, 1 - \frac{Var(\hat{I}_t)}{Var(\hat{S}_t + \hat{I}_t)} \right)$$

If  $F_T > F_S$ , then the time series is more trending than seasonal.

### 1.2.5 Forecasting with decomposition

Suppose that after decomposing, we have  $\hat{T}_t, \hat{S}_t, \hat{I}_t$ . A simple idea to use for forecasting is :

- Forecasting for the trend  $\hat{T}_t$ .
- Replicating for the seasonality  $\hat{S}_t$ .
- Estimating a distribution for irregular component  $\hat{I}_t$  and sampling future value
- Summing all above component.

In some case, if we can estimate  $\hat{S}_t$  first, then we remove  $\hat{S}_t$  from  $X_t$  to get  $\hat{T}_t + \hat{I}_t$  which is also called **seasonally adjusted component**. The latter can be used for models such as ARIMA (6.1.9).

## 1.3 Exponential smoothing

Exponential smoothing means models that use weights decaying exponentially as the observations get older. In other words, the more recent the observation the higher the associated weight. More or less, they can keep a little bit information from the first observations.

### 1.3.1 Simple exponential smoothing (SES)

This is the most basic form in exponential smoothing :

$$\begin{cases} l_0 = c \\ l_t = \alpha X_t + (1 - \alpha)l_{t-1} \quad (t \geq 1) \end{cases}$$

where  $l_t$  the **level** (or the smoothed value) of  $X_t$ . Suppose that we have observations until time  $T$ , then the forecast value at  $T + 1$  is :

$$\begin{aligned} X_{T+1|T} &= l_T \\ &= \alpha X_T + \alpha(1 - \alpha)X_{T-1} + \alpha(1 - \alpha)^2X_{T-2} + \dots + \alpha(1 - \alpha)^{T-1}X_1 + \alpha(1 - \alpha)^T c \end{aligned}$$

This show the exponential form (or exponential weights). In general, SES takes the flat forecast :

$$X_{T+h|T} = l_T, \quad h = 1, 2, \dots$$

### 1.3.1.1 Fitting

We need to estimate the value for  $c$  and  $\alpha$ . This is usually done by optimizing the **one-step training error**, which means to minimize the sum of the squared errors (SSE). Hence, we find the values of the unknown parameters and the initial values that minimise :

$$\sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T (X_t - \hat{X}_{t|t-1})^2$$

This involves a non-linear minimisation problem, and we need to use an optimisation tool to solve it.

### 1.3.1.2 Example

Given the time series data of total oil production in millions of tonnes for Saudi Arabia. We estimate  $\alpha$  and  $c$ , then forecasting for severel future years, in table 6.1.

Year	Time	Observation	Level
	$t$	$X_t$	$l_t = \alpha X_t + (1 - \alpha)l_{t-1}$
1995	0		446.59
1996	1	445.36	445.57
1997	2	453.20	451.93
1998	3	454.41	454.00
1999	4	422.38	427.63
2000	5	456.04	451.32
2001	6	440.39	442.20
		Forecasting	
	$h$	$\hat{X}_{T+h T} = l_T, T = 6$	
2002	1	442.20	
2003	2	442.20	
2004	3	442.20	

Table 6.1: Forecasting the total oil production in millions of tonnes for Saudi Arabia using simple exponential smoothing. Suppose that we find  $\alpha = 0.83$  and  $c = 446.59$

### 1.3.1.3 Notes

For SES, if we use  $c = X_0$  with a given  $\alpha$ , then we have an exponential moving average (EMA) filter, which is something compatible to moving average (MA)

filter 6.1.2.1.2 for real time data and can serve as an indicator.

Note that, EMA and MA are filter and SES **is rather a model**, which requires fitting for parameters  $c$  and  $\alpha$ .

Finally, SES **is not compatible** with time serie data where there is a trend or seasonlity. This is because SES use the flat forecast, which does not evolve while trend is something that evolves.

### 1.3.2 Holt's linear trend method

This is an extended version of SES to allow the forecasting of data with a trend. Holt's linear trend method is also called *double exponential smoothing* and involves two smoothing equations (one for the level and one for the trend):

$$\begin{cases} l_0 = c \\ b_0 = d \\ l_t = \alpha X_t + (1 - \alpha)(l_{t-1} + b_{t-1}) & \text{(level equation)} \\ b_t = \beta(l_t - l_{t-1}) + (1 - \beta)b_{t-1} & \text{(trend equation)} \end{cases}$$

and for forecasting :

$$\hat{X}_{T+h|T} = l_T + hb_T$$

This forecast is linear (w.r.t.  $h$ ) and displays a constant trend (increasing or decreasing) indefinitely into the future.

The assumption here is that the time series data  $X_t$  can be decomposed into level  $l_t$  and the trend  $b_t$  :

- The level  $l_t$  is estimated by the weighted average of observation  $X_t$  and the forecast  $\hat{X}_{t|t-1}$ , as with the simple exponential smoothing. Here,  $l_t = \alpha X_t + (1 - \alpha)\hat{X}_{t|t-1} = \alpha X_t + (1 - \alpha)(l_{t-1} + b_{t-1})$ .
- The trend  $b_t$  is estimated by updating the variation in level  $l_t - l_{t-1}$  into the previous estimate of the trend  $b_{t-1}$ .

For fitting in Holt's linear trend method, this is the same as SES in 6.1.3.1.1. If the value of  $\alpha$  is very close to one, showing that the level reacts strongly to each new observation. If  $\beta$  is very closed to 0, then  $b_t$  is close to  $b_{t-1}$ , this means the trend varies slowly or may do not vary.

#### 1.3.2.1 Using as filter

The Holt's linear trend method can serve as a filter with  $l_0 = X_0, b_0 = X_1 - X_0$  and  $\alpha, \beta$  are given :

$$\begin{cases} l_0 = X_0 \\ b_0 = X_1 - X_0 \\ l_t = \alpha X_t + (1 - \alpha)(l_{t-1} + b_{t-1}) & \text{(level equation)} \\ b_t = \beta(l_t - l_{t-1}) + (1 - \beta)b_{t-1} & \text{(trend equation)} \end{cases}$$

### 1.3.3 Damped trend method

Since forecasts generated by Holt's linear method display a constant trend and this is inappropriate for a long forecast. To overcome this drawback, in Damped trend method, we introduce a parameter that "dampens" the trend to a flat line some time in the future. Damped trend method is represented by :

$$\begin{cases} l_0 = c \\ b_0 = d \\ l_t = \alpha X_t + (1 - \alpha)(l_{t-1} + \phi b_{t-1}) & (\text{level equation}) \\ b_t = \beta(l_t - l_{t-1}) + (1 - \beta)\phi b_{t-1} & (\text{trend equation}) \end{cases}$$

and for forecasting :

$$\hat{X}_{T+h|T} = l_T + (\phi + \phi^2 + \dots + \phi^h)b_T$$

If  $\phi = 1$ , the method is identical to Holt's linear method. For  $0 < \phi < 1$ , it dampens the trend so that it approaches a constant some time in the future, which means  $\lim_{h \rightarrow \infty} \hat{X}_{T+h|T} = l_T + \frac{\phi}{1-\phi}b_T$ . This also means that short-run forecasts are trended while long-run forecasts are constant.

In the forecast  $\hat{X}_{T+1|T} = l_T + \phi b_T$ , we use only a part of  $b_T$ , which means  $\phi b_T$  instead of the totality of  $b_T$  in Holt's linear trend method. Respectively, in level and trend equations, we keep only a part of  $b_T$ , which means  $\phi b_T$ .

For fitting damped trend method, this is the same as SES in 6.1.3.1.1. In practice,  $\phi$  is usually greater than 0.8 as the damping has a very strong effect for smaller values. Values of  $\phi$  close to 1 will mean that a damped model is not able to be distinguished from a non-damped model. For these reasons, we usually optimize SSE with a condition that  $\phi$  is between 0.98 and 0.8 or in a simple manner, we just fix or estimate  $\phi$  first, for example,  $\phi = 0.9$ , then optimize just a problem of Holt's linear trend.

### 1.3.4 Holt-Winters' seasonal method

As both Holt's linear trend method and damped trend method is still not compatible with time serie data where there is seasonality, this leads to the appearance of Holt-Winters' seasonal method to capture the seasonality.

The Holt-Winters seasonal method is also called triple exponential smoothing since it involves three smoothing equations, one for the level  $l_t$ , one for the trend  $b_t$  and one for the seasonal component  $s_t$ , with corresponding smoothing parameters  $\alpha, \beta, \gamma$ . We use  $m$  to denote the frequency of the seasonality, i.e., the number of seasons in a year. For example, for quarterly data  $m = 4$ , and for monthly data  $m = 12$ .

There are two variations to this method that differ in the nature of the seasonality. The additive method is preferred when the seasonal variations are roughly constant through the series, while the multiplicative method is preferred when the seasonal variations are changing proportional to the instant  $t$  of the series.

- Additive version :

$$\begin{cases} l_0 \\ b_0 \\ s_0, \dots, s_{m-1} \\ l_t = \alpha(X_t - s_{t-m}) + (1 - \alpha)(l_{t-1} + b_{t-1}) & \text{(level equation)} \\ b_t = \beta(l_t - l_{t-1}) + (1 - \beta)b_{t-1} & \text{(trend equation)} \\ s_t = \gamma(X_t - l_{t-1} - b_{t-1}) + (1 - \gamma)s_{t-m} & \text{(seasonal equation)} \end{cases}$$

subtract to :

$$\sum_{i=0}^{m-1} s_{i+km} = 0, \quad \forall k \geq 0$$

which means within each year, the seasonal component will add up to approximately zero. For forecasting :

$$\hat{X}_{T+h|T} = l_T + h b_T + s_{T+h-m(k+1)}$$

where  $k = \lfloor \frac{h-1}{m} \rfloor$ , which ensures that the estimates of the seasonal indices used for forecasting come from the final year of the sample.

The equation for the seasonal component can be often expressed as:

$$s_t = \gamma(X_t - l_t) + (1 - \gamma)s_{t-m}$$

This is because :

$$\begin{aligned} s_t &= \gamma(X_t - l_t) + (1 - \gamma)s_{t-m} \\ &= \gamma(X_t - \alpha(X_t - s_{t-m}) - (1 - \alpha)(l_{t-1} + b_{t-1})) + (1 - \gamma)s_{t-m} \\ &= \gamma(1 - \alpha)(X_t - l_{t-1} - b_{t-1}) + (1 - \gamma(1 - \alpha))s_{t-m} \end{aligned}$$

where  $\gamma(1 - \alpha)$  can be consider as  $\gamma$ .

- By analogy, we have the multiplicative version :

$$\begin{cases} l_0 \\ b_0 \\ s_0, \dots, s_{m-1} \\ l_t = \alpha \frac{X_t}{s_{t-m}} + (1 - \alpha)(l_{t-1} + b_{t-1}) & \text{(level equation)} \\ b_t = \beta(l_t - l_{t-1}) + (1 - \beta)b_{t-1} & \text{(trend equation)} \\ s_t = \gamma \frac{X_t}{l_{t-1} + b_{t-1}} + (1 - \gamma)s_{t-m} & \text{(seasonal equation)} \end{cases}$$

subtract to :

$$\sum_{i=0}^{m-1} s_{i+km} = m, \quad \forall k \geq 0$$

which means within each year, the seasonal component will sum up to approximately  $m$ . For forecasting :

$$\hat{X}_{T+h|T} = (l_T + hb_T)s_{T+h-m(k+1)}$$

where  $k = \lfloor \frac{h-1}{m} \rfloor$ .

In both versions, for level equation, we remove the expected seasonal component by  $s_{t-m}$  from  $X_t$  before averaging and by analogy, for seasonal equation, we remove expected sum of level and trend  $l_{t-1} + b_{t-1}$  from  $X_t$  before averaging.

#### 1.3.4.1 Holt-Winters' damped method

Damped trend method (6.1.3.3) can integrate with both additive and multiplicative Holt-Winters' methods. A method that often provides accurate and robust forecasts for seasonal data is the Holt-Winters method with a damped trend and multiplicative seasonality.

#### 1.3.5 Taxonomy of exponential smoothing methods

In table 6.2, we see all combinations to have a exponential smoothing method. There are several methods we have already seen by decoding the short hand into name of method in table 6.3.

Trend \ Seasonality	$N$ (None)	$A$ (Additive)	$M$ (Multiplicative)
$N$ (None)	$(N, N)$	$(N, A)$	$(N, M)$
$A$ (Additive)	$(A, N)$	$(A, A)$	$(A, M)$
$A$ (Additive damped)	$(A_d, N)$	$(A_d, A)$	$(A_d, M)$

Table 6.2: A classification of exponential smoothing methods.

Short hand	Method
$(N, N)$	Simple exponential smoothing
$(A, N)$	Holt's linear method
$(A_d, N)$	Additive damped trend method
$(A, A)$	Additive Holt-Winters' method
$(A, M)$	Multiplicative Holt-Winters' method
$(A_d, M)$	Holt-Winters' damped method

Table 6.3: Code to name of method.

Note that for seasonal component we have additive and multiplicative version, while in trend component we use only additive version, because the multiplicative trend methods often tend to produce poor forecasts.

### 1.3.5.1 Error-Trend-Seasonality (ETS)

Up to now, we only consider variations for trend and seasonality component, while error component is additive by default. Therefore, to make an Error-Trend-Seasonality (ETS) model, we take into account two types for error :

- $A$  (Additive) :

$$\varepsilon_t = X_t - \hat{X}_{t|t-1}$$

which is thus the difference between ground true and the prediction.

- $M$  (Multiplicative) :

$$\varepsilon_t = \frac{X_t - \hat{X}_{t|t-1}}{\hat{X}_{t|t-1}}$$

which is thus the difference devided by the prediction.

### 1.3.6 Viewed under innovations state space models

Let's start with ETS( $A, N, N$ ), a simple exponential smoothing with additive errors :

$$\begin{cases} l_t = \alpha X_t + (1 - \alpha)l_{t-1} \\ \hat{X}_{t+1|t} = l_t \end{cases}$$

We can rewrite :

$$\begin{aligned} l_t &= \alpha X_t + (1 - \alpha)l_{t-1} \\ &= l_{t-1} + \alpha(X_t - l_{t-1}) \\ &= l_{t-1} + \alpha(X_t - \hat{X}_{t|t-1}) \\ &= l_{t-1} + \alpha\varepsilon_t \end{aligned}$$

Then ETS( $A, N, N$ ) becomes:

$$\begin{cases} l_t = l_{t-1} + \alpha\varepsilon_t \\ X_t = l_{t-1} + \varepsilon_t \end{cases}$$

In the same manner, we can express ETS( $A, A, N$ ) by :

$$\begin{cases} l_t = l_{t-1} + b_{t-1} + \alpha\varepsilon_t \\ b_t = b_{t-1} + \beta\varepsilon_t \\ X_t = l_{t-1} + b_{t-1} + \varepsilon_t \end{cases}$$

Under this representation and the assumption that residuals  $\varepsilon_t$  are  $NID(0, \Sigma^2)$ , where NID stands for normally and independently distributed, we have an innovations state space model. Note that

- All ETS models can be written in innovations state space form.

- Additive and multiplicative (error) versions give the same point forecasts but different prediction intervals.

We remind that innovations state space models are similar to state models with Kalman filter (5.2.1). Here  $l_t, b_t, \dots$  are state and  $X_t$  is measurement. We consider only the online inference, which means we predict for state at instant  $T + 1$ , then we have the measurement at instant  $T + 1$  before predicting state at instant  $T + 2$  and so on.

Using Kalman filter with ETS model helps to get better estimated states  $\hat{l}_t, \hat{b}_t, \dots$ , then better  $\hat{X}_t$ . Note that we need to estimate the variance  $\sigma^2$  of noise  $\varepsilon_t$  before applying Kalman filter. A more details about the combination between ETS and Kalman can be found in <https://robjhyndman.com/uwafiles/9-StateSpaceModels.pdf>

### 1.3.6.1 Local level model

This is a state model corresponds to ETS( $A, N, N$ ), where  $\alpha\varepsilon_t$  is replaced by  $\xi_t$ :

$$\begin{cases} l_t = l_{t-1} + \xi_t \\ X_t = l_{t-1} + \varepsilon_t \end{cases}$$

Here instead of estimating  $\alpha$ , we estimate  $\varepsilon_t$  and  $\xi_t$  which are white noise.

### 1.3.6.2 Local linear trend model

This is a state model corresponds to ETS( $A, A, N$ ), where  $\alpha\varepsilon_t$  and  $\beta\varepsilon_t$  are replaced by  $\xi_t$  and  $\zeta_t$ :

$$\begin{cases} l_t = l_{t-1} + b_{t-1} + \xi_t \\ b_t = b_{t-1} + \zeta_t \\ X_t = l_{t-1} + b_{t-1} + \varepsilon_t \end{cases}$$

Here instead of estimating  $\alpha$  and  $\beta$ , we estimate  $\varepsilon_t$ ,  $\xi_t$  and  $\zeta_t$  which are white noise.

## 1.3.7 Procedure for ETS models

### 1.3.7.1 Smoothing parameters estimation

First of all, a Box-Cox transformation 3.4.3 may be applied to time series data.

Assume that errors (additive or multiplicative) are normally distributed. For an additive error model, MLE gives the same results as minimising the sum of squared errors (see 3.2.12.1.3). However, different results will be obtained for multiplicative error models. By default in R, we estimate the smoothing parameters of ETS models by MLE.

### 1.3.7.2 Model selection

For a comparison between models after fitting, we can use metrics or scores AIC, AICc, BIC as described in section 6.1.1.

Note that, models with multiplicative errors are only used when the data are strictly positive. If not, it can make numerically unstable estimation.

### 1.3.7.3 Prediction intervals

- **Forecast variance** can be obtained by using estimated variance in training (fitting). In figure 6.1, we give the formulas for the additive error ETS models, which are the simplest.

Model	Forecast variance: $\sigma_h^2$
(A,N,N)	$\sigma_h^2 = \sigma^2 [1 + \alpha^2(h - 1)]$
(A,A,N)	$\sigma_h^2 = \sigma^2 \left[ 1 + (h - 1) \{ \alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h - 1) \} \right]$
(A,A <sub>d</sub> ,N)	$\sigma_h^2 = \sigma^2 \left[ 1 + \alpha^2(h - 1) + \frac{\beta\phi h}{(1-\phi)^2} \{ 2\alpha(1 - \phi) + \beta\phi \} \right. \\ \left. - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)} \{ 2\alpha(1 - \phi^2) + \beta\phi(1 + 2\phi - \phi^h) \} \right]$
(A,N,A)	$\sigma_h^2 = \sigma^2 \left[ 1 + \alpha^2(h - 1) + \gamma k(2\alpha + \gamma) \right]$
(A,A,A)	$\sigma_h^2 = \sigma^2 \left[ 1 + (h - 1) \{ \alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h - 1) \} \right. \\ \left. + \gamma k \{ 2\alpha + \gamma + \beta m(k + 1) \} \right]$
(A,A <sub>d</sub> ,A)	$\sigma_h^2 = \sigma^2 \left[ 1 + \alpha^2(h - 1) + \gamma k(2\alpha + \gamma) \right. \\ \left. + \frac{\beta\phi h}{(1-\phi)^2} \{ 2\alpha(1 - \phi) + \beta\phi \} \right. \\ \left. - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)} \{ 2\alpha(1 - \phi^2) + \beta\phi(1 + 2\phi - \phi^h) \} \right. \\ \left. + \frac{2\beta\gamma\phi}{(1-\phi)(1-\phi^m)} \{ k(1 - \phi^m) - \phi^m(1 - \phi^{mk}) \} \right]$

Figure 6.1: **Forecast variance** expressions for each additive state space model, where  $\sigma^2$  is the residual variance,  $m$  is the seasonal period, and  $k = \lfloor \frac{h-1}{m} \rfloor$  (i.e., the number of complete years in the forecast period prior to time  $T+h$ ). Source : <https://otexts.com/fpp2/ets-forecasting.html>

- For both for the additive and multiplicative error ETS models, we can estimate **forecast variance** through validation data or simulated future paths as in 6.1.11.

## 1.4 Hodrick–Prescott filter

$X_t$  is decomposed into  $X_t = \tau_t + c_t$ , where  $\tau_t$  trend component or smoothed version of  $X_t$  and  $c_t$  is cyclical component. In standard Hodrick–Prescott (HP)

filter, the trend component is determined by :

$$\min_{\tau} \left( \sum_{t=1}^T (X_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \right)$$

The first term, try to make  $X_t$  closed to  $\tau_t$  and the second term prevents  $\tau$  to varies too fast. Hodrick and Prescott suggest 1600 as a value for  $\lambda$  for quarterly data. In practice,  $\lambda = 100$  for yearly data and  $\lambda = 14,400$  for monthly data are commonly used.

Let's consider only  $\tau_t$ , we need to optimize :

$$(X_t - \tau_t)^2 + (\tau_{t+1} + \tau_{t-1} - 2\tau_t)^2 + (\tau_{t+2} + \tau_t - 2\tau_{t+1})^2 + (\tau_t + \tau_{t-2} - 2\tau_{t+1})^2$$

By first-order condition :

$$\lambda\tau_{t-2} - 4\lambda\tau_{t-1} + (1 + 6\lambda)\tau_t - 4\lambda\tau_{t+1} + \lambda\tau_{t+2} = X_t$$

Then the Hodrick–Prescott filter is explicitly given by solving:

$$[\lambda L^2 - 4\lambda L + (1 + 6\lambda) - 4\lambda L^{-1} + \lambda L^{-2}] \tau_t = X_t$$

where  $L$  denotes the lag operator

#### 1.4.1 Properties

The Hodrick–Prescott filter will only be optimal when:

- Data exists in a  $I(2)$  trend ( $I(2)$  is order of integration sec 3.4.7.2). If one-time permanent shocks or growth rate splits occur, the filter will generate shifts in the trend that do not actually exist.
- Noise in data is approximately normally distributed.
- The standard HP filter is a two-side version, which means we have the forward side  $\tau_{t+1}$  and also the backward side  $\tau_{t-1}$  in filter. Hence, it should not be used when estimating models such as Kalman filter or likelihood-based methods, which are based on only **recursive state-space representations** (backward). A solution is to use one-sided HP filter version (only backward) :

$$\min_{\tau} \left( \sum_{t=1}^T (X_t - \tau_t)^2 + \lambda \sum_{t=3}^T [(\tau_t - \tau_{t-1}) - (\tau_{t-1} - \tau_{t-2})]^2 \right)$$

- A quite well-known working paper titled “Why You Should Never Use the Hodrick-Prescott Filter” presents evidences against using the HP filter :
  - The HP filter produces series with spurious dynamic relations that have no basis in the underlying data-generating process.

- A one-sided version of the filter reduces but does not eliminate spurious predictability and moreover produces series that do not have the properties sought by most potential users of the HP filter.
- Statistically, the good value for  $\lambda$  is vastly at odds with common practice, e.g., a value for  $\lambda$  far below 1600 for quarterly data.

## 1.5 Autoregressive model

$\text{AR}(p)$  :

$$X_t = \sum_{i=1}^p \phi_i X_{t-i} + \varepsilon_t$$

where  $\varepsilon_t$  are white noise and have the same finite variance.

We have several fitting methods to find  $\phi_i$  :

- Ordinary Least Squares (OLS)
- Maximum Likelihood Estimation (MLE)
- Yule-Walker with autocovariance

MLE is very general and can apply to various model fitting, in this case, we have a hypothesis that  $\varepsilon_t$  follows normal distribution, hence :

$$f(X_t; \phi_1, \dots, \phi_p) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(X_t - \sum_{i=1}^p \phi_i X_{t-i})^2}{2\sigma^2}\right)$$

Note that, before using the AR model, we need to check if there is an unit root (sec 3.4.7.1) in the signal by Dickey–Fuller test (sec 3.4.7.3) or augmented Dickey–Fuller test (sec 3.4.8). If exists an unit root, a differencing is required to remove the unit root.

### 1.5.1 Yule-Walker equation

With model  $\text{AR}(p)$ :

$$X_t = \sum_{i=1}^p \phi_i X_{t-i} + \varepsilon_t$$

We define the autocovariance  $\gamma_m = \text{Cov}(X_t, X_{t-m})$ , then :

$$\begin{aligned} \gamma_m &= \text{Cov}(X_t, X_{t-m}) \\ &= \text{Cov}\left(\sum_{i=1}^p \phi_i X_{t-i} + \varepsilon_t, X_{t-m}\right) \\ &= \sum_{i=1}^p \phi_i \gamma_{m-i} + \text{Cov}(\varepsilon_t, X_{t-m}) \end{aligned}$$

Here we make a hypothesis that the white noises  $\varepsilon_t$  are independent of each other and  $X_{t-j}$  (where  $j \neq 0$ ) are independent of  $\varepsilon_t$ . Then we have :

$$Cov(\varepsilon_t, X_{t-m}) = \begin{cases} \sigma_\varepsilon^2 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

Taking at least  $p$  equations  $m = 1, \dots, p$  to solve for  $p$  value  $\phi_1, \dots, \phi_p$  and taking  $m = 0$  for calculating  $\sigma_\varepsilon$ . Yule-Walker equation is thus a solution by Method of Moments (3.2.13.1), as we can consider covariance as a type of moment of order 2.

### 1.5.2 Constraints for stationary forecast

We normally restrict AR models to forecast stationary data, in which case some constraints on the values of the parameters are required :

- For an AR(1) model:  $|\phi_1| < 1$ .
- For an AR(2) model:  $|\phi_2| < 1, \phi_1 + \phi_2 < 1, \phi_1 - \phi_2 < 1$ .
- For an AR( $p$ ) when  $p \geq 3$ , the restrictions are much more complicated. In some programming such as R or Matlab, there are tools which takes care of these restrictions.

## 1.6 Moving Average model

This is about Moving Average (MA) model and not to be confused with Moving Average filter in 6.1.2.1.

MA( $q$ ):

$$X_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$

where  $\varepsilon_t, \dots, \varepsilon_{t-q} \stackrel{iid}{\sim} \mathcal{N}(0, \sigma_w^2)$

### 1.6.1 MA(1)

In case  $q = 1$  or MA(1):

$$X_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

- $E(X_t) = \mu$
- $Var(X_t) = (1 + \theta_1^2) \sigma_\varepsilon^2$
- $ACF(X_t, X_{t-1}) = \frac{\theta_1}{1 + \theta_1^2}$
- $ACF(X_t, X_{t-h}) = 0$ , if  $h > 1$ .

We remind the way to calculate ACF (Autocorrelation function) :

$$ACF(X_t, X_{t-h}) = \frac{Cov(X_t, X_{t-h})}{\sigma_X^2}$$

There is no unique connection between values of  $\theta_1$  and  $ACF(h = 1)$  in MA(1). It means that, for example, given value of  $ACF(h = 1) = 0.4$ , it may happen that  $\theta_1 = 0.5$  or  $\theta_1 = 2$ .

### 1.6.2 Invertibility

Thus, AR model and MA can be transformed, one to other if several constraints are satisfied.

**From AR to MA :**

$$\begin{aligned} x_t &= \phi_1 x_{t-1} + \varepsilon_t \\ &= \phi_1(\phi_1 x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \phi_1^2 x_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\ &= \dots \\ &= \phi_1^h x_{t-h} + \phi_1^{h-1} \varepsilon_{t-(h-1)} + \dots + \phi_1^3 \varepsilon_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \end{aligned}$$

If  $|\phi_1| < 1$  and  $h$  is sufficiently big, we can remove the term  $\phi_1^h x_{t-h}$  and it becomes MA model.

**From MA to AR :**

$$\begin{aligned} x_t &= \varepsilon_t + \theta_1 \varepsilon_{t-1} \\ &= \varepsilon_t + \theta_1 x_{t-1} - \theta_1^2 \varepsilon_{t-2} \\ &= \dots \\ &= \varepsilon_t + \theta_1 x_{t-1} - \theta_1^2 x_{t-2} + \theta_1^3 x_{t-3} - \theta_1^4 x_{t-4} + \dots + \theta_1^{2h-1} x_{t-(2h-1)} - \theta_1^{2h} x_{t-2h} \end{aligned}$$

If  $|\theta_1| < 1$  then from a sufficient big  $h$ , we can cut off small signals.

### 1.6.3 Constraints for stationary forecast and invertibility

Again, as AR model, we need to impose constraints to forecast stationary data and these constraints are also necessary and sufficient conditions for invertibility :

- For an MA(1) model:  $|\theta_1| < 1$ .
- For an MA(2) model:  $|\theta_2| < 1, \theta_1 + \theta_2 > -1, \theta_1 - \theta_2 < 1$ .
- For an MA( $p$ ) when  $p \geq 3$ , the restrictions are much more complicated.  
Again, we can use Matlab or R.

#### 1.6.4 Fitting

For a MA(2) :  $X_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$ . Given  $y$  as observations, we need to find  $\theta$  and  $\varepsilon$ .

$$\begin{bmatrix} X_2 \\ X_3 \\ \vdots \\ X_{t+1} \end{bmatrix} \approx \begin{bmatrix} \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_{t+1} \end{bmatrix} + \theta_1 \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_t \end{bmatrix} + \theta_2 \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \vdots \\ \varepsilon_{t-1} \end{bmatrix} = \begin{bmatrix} \varepsilon_2 & \varepsilon_1 & \varepsilon_0 \\ \varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \vdots & \vdots & \vdots \\ \varepsilon_{t+1} & \varepsilon_t & \varepsilon_{t-1} \end{bmatrix} \times \begin{bmatrix} 1 \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

By using the following iterative procedure, we fit alternatively  $\varepsilon$  and  $\theta$  :

1. Initializing  $\theta_1, \theta_2, \varepsilon_0, \varepsilon_1$
2. Calculating for  $\varepsilon_2, \dots, \varepsilon_{t+1}$  with given data  $X_t$
3. Consider that  $[\varepsilon_2, \dots, \varepsilon_{t+1}]^T$  is noise, then fitting for only  $\theta$  by minimizing  
with LS as : 
$$\begin{bmatrix} X_2 \\ X_3 \\ \vdots \\ X_{t+1} \end{bmatrix} \approx \begin{bmatrix} \varepsilon_1 & \varepsilon_0 \\ \varepsilon_2 & \varepsilon_1 \\ \vdots & \vdots \\ \varepsilon_t & \varepsilon_{t-1} \end{bmatrix} \times \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$
4. Repeat from step 2.

A python code for this is found in sec 7.2.

### 1.7 AR or MA ?

Given a data to fit, to know which model to use (AR or MA or none of them), at a first try, we can see Autocorrelation Function (ACF) plot and Partial Autocorrelation Function (PACF) plot. First of all, we discover the concept of Partial correlation

#### 1.7.1 Partial correlation

Given a vector space  $\mathbb{R}^n$ . Let  $\mathcal{Z}$  be a subspace of  $\mathbb{R}^n$  and  $x, y \in \mathbb{R}^n$  two vectors. Then the partial correlation between  $x$  and  $y$  with respect to  $\mathcal{Z}$ , denoted  $\rho_{xy|\mathcal{Z}}$ , is the correlation between the residual  $r_x$  and  $r_y$ , which are obtained by removing component of  $x$  and  $y$  in  $\mathcal{Z}$ .

Suppose that  $\mathcal{Z}$  is of dimension  $k \leq n$ , with a base matrix  $Z \in \mathbb{R}^{n \times k}$  :

$$Z = [z_1, z_2, \dots, z_k], z_i \in \mathbb{R}^n$$

Then the residuals  $r_x, r_y$  are defined by :

- $r_x = x - Zw_x$ , where  $w_x = \underset{w \in \mathbb{R}^k}{\operatorname{argmin}} \|x - Zw\|_2^2$
- $r_y = y - Zw_y$ , where  $w_y = \underset{w \in \mathbb{R}^k}{\operatorname{argmin}} \|y - Zw\|_2^2$

Finally, the partial correlation between  $x$  and  $y$  is :

$$\rho_{xy|\mathcal{Z}} = \text{corr}(r_x, r_y)$$

An geometrical interpretation for partial correlation is showed by figure 6.2.

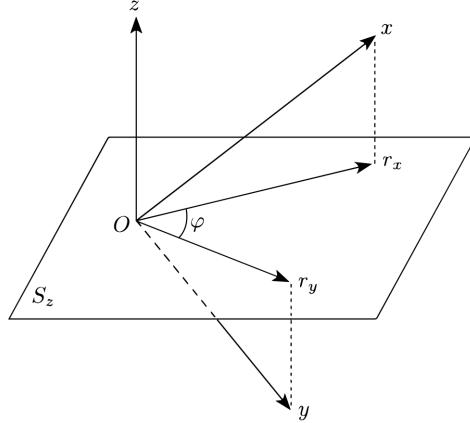


Figure 6.2: In space  $\mathbb{R}^3$ .  $\mathcal{Z} = \text{span}(\{(0, 0, 1)\})$  or the space is generated by only the axe  $Oz$ . Two signals  $x$  et  $y$  are first projected into  $Oz$ , then we calculate the correlation between their residuals (after removing component projected in  $Oz$ ).

Here, may we remind correlation tests in 3.4.2.3.1

### 1.7.2 ACF and PACF

We plot two corrgrams (or correlograms), one for ACF and one for PACF. Corrgrams are a graphical representation of one or more correlations between data series.

- $ACF(X_t, X_{t-h})$ .
- $PACF(X_t, X_{t-h})$ , with  $\mathcal{Z} = [X_{t-1}, \dots, X_{t-h+1}]$ . These measure the relationship between  $X_t$  and  $X_{t-h}$  after removing the effects of lags  $1, 2, 3, \dots, h-1$

#### 1.7.2.1 AR's signature in ACF and PACF

<https://spureconomics.com/interpreting-acf-and-pacf-plots/>

The signature of AR model through ACF and PACF plots are shown in figure 6.3, in which :

- The ACF is exponentially decaying or sinusoidal.

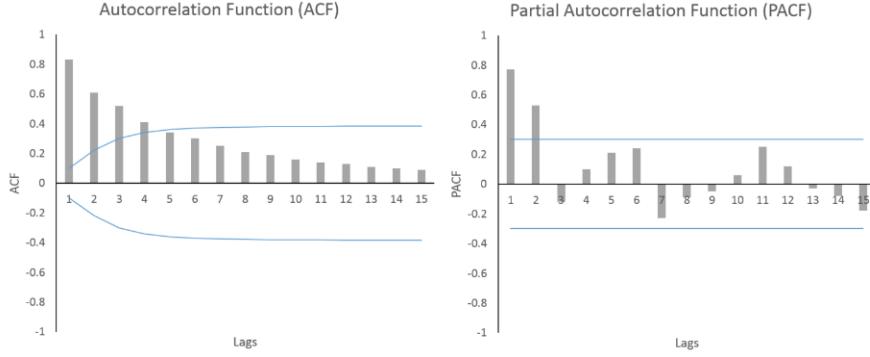


Figure 6.3: AR's signature in ACF and PACF.

- There is a significant spike at lag  $p$  in the PACF, but none beyond lag  $p$ . Here  $p$  is the order in  $\text{AR}(p)$  model.

To know if a point in PACF is a significant spike or not, we can use thresholds  $\frac{\pm 1.96}{\sqrt{n}}$ .

#### 1.7.2.2 MA signature in ACF and PACF

The signature of MA model through ACF and PACF plots are shown in figure 6.4, in which :

- The PACF is exponentially decaying or sinusoidal.
- There is a significant spike at lag  $q$  in the ACF, but none beyond lag  $q$ . Here  $q$  is the order in  $\text{MA}(q)$  model.

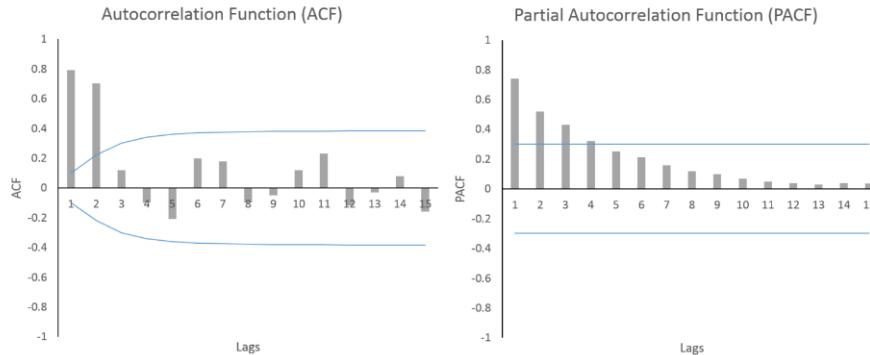


Figure 6.4: MA's signature in ACF and PACF.

### 1.7.2.3 Neither AR nor MA

If the shape of ACF and PACF plots are not the same in figure 6.3 and 6.4, neither AR nor MA can fit the data.

## 1.8 ARMA

This is a model that combines AR and MA models:

$$X_t = \varepsilon_t + \sum_{i=1}^p \phi_i X_{t-i} + \sum_{i=1}^q \theta_i \varepsilon_{t-i}$$

### 1.8.1 Signature of ARMA with ACF and PACF plots

Given time serie data, a way to know if ARMA is a good choice is to observe the ACF and PACF plots. If there are significant spikes at first lags in both ACF and PACF (as in figure 6.5), this can be a sign for using ARMA model.

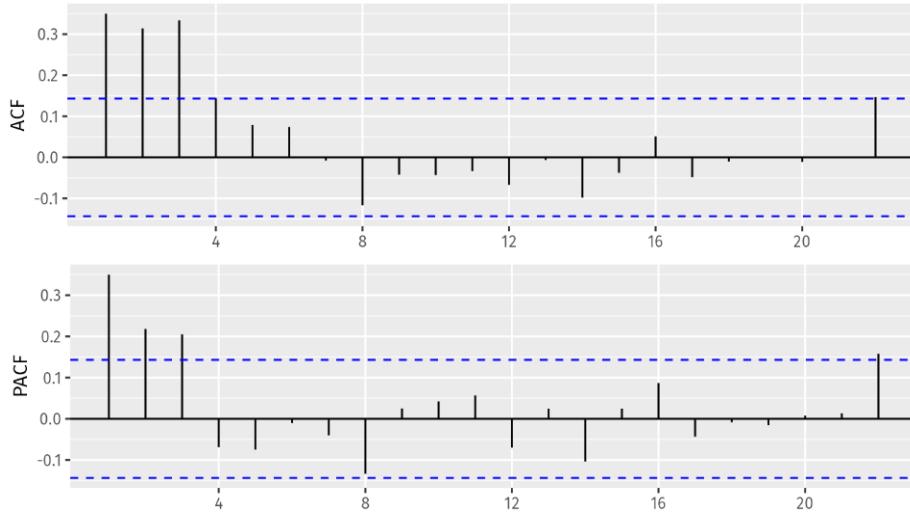


Figure 6.5: ACF and PACF for of a time serie data that can fit to an ARMA model

Finding appropriate values of  $p$  and  $q$  in the  $\text{ARMA}(p, q)$  model can be facilitated by ACF (for  $q$ ) and PACF (for  $p$ ) as in AR and MA model (6.1.7.2). However, in ARMA we need to deal with a “range” of models, for example in figure 6.5, we find that  $p = q = 3$ , then we need to consider all models ARMA with  $(p, q)$ , where  $p, q \in \{1, 2, 3, 4, 5\}$ .

### 1.8.2 Fitting

ARMA models in general can be, after choosing  $p$  and  $q$ , fitted by least squares regression to find the values of the parameters which minimize the error term.

Let's take an example for ARMA(1, 2), with only  $X$  are observation :

$$\begin{aligned} \begin{bmatrix} X_2 \\ X_3 \\ \vdots \\ X_{t+1} \end{bmatrix} &\approx \phi_1 \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_t \end{bmatrix} + \begin{bmatrix} \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_{t+1} \end{bmatrix} + \theta_1 \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_t \end{bmatrix} + \theta_2 \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \vdots \\ \varepsilon_{t-1} \end{bmatrix} \\ &= \begin{bmatrix} X_1 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 \\ X_2 & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \vdots & \vdots & \vdots & \vdots \\ X_t & \varepsilon_{t+1} & \varepsilon_t & \varepsilon_{t-1} \end{bmatrix} \times \begin{bmatrix} \phi_1 \\ 1 \\ \theta_1 \\ \theta_2 \end{bmatrix} \end{aligned}$$

- We can use iterative procedure as in MA (6.1.6.4) to estimate  $\varepsilon, \phi, \theta$ .
- Generalized Method of Moments ( 3.2.13) can also be used. Thus it is similar to Yule-Walker equation (6.1.5.1).

### 1.8.3 Ljung-Box Test

Ljung-Box Test is a type of statistical test to know whether there is a cluster or group of non-zero autocorrelation score in a **time serie**. In other words, Ljung-Box Test aims to know if there is at least a dependence between two instants in a time serie or they are all independent. Instead of testing at each distinct lag, Ljung-Box Test performs the overall scenario, with entire number of lags.

- $H_0$ : The data are independently distributed.
- $H_a$ : The data are not independently distributed and they exhibit a serial correlation.

We define :

$$Q = n(n+2) \sum_{k=1}^h \frac{\hat{p}_k^2}{n-h}$$

where  $n$  is the sample size,  $\hat{p}_k$  is the sample autocorrelation at lag  $k$ , and  $h$  is the number of lags being tested. Under  $H_0$  the statistic  $Q$  asymptotically follows a  $\chi^2_{(h)}$ .

The Ljung-Box test is commonly used in ARMA model 6.1.8.

### 1.8.4 Evaluation

After fitting, we may need to evaluate :

- Which couple of order  $(p, q)$  to use in ARMA for the best performance ?

- Is ARMA( $p, q$ ) model is a good choice for the given data time serie ?

For the first question we can use criteria such as AIC, AICc, BIC and for the second question, we use Ljung–Box Test.

#### 1.8.4.1 Estimating orders for ARMA

In order to determine which order  $(p, q)$  of the ARMA model is appropriate for a series, we need to use the AIC, AICc or BIC (6.1.1) across a subset of values for  $(p, q)$  :

- AIC :

$$AIC = -2 \ln(\hat{\mathcal{L}}) + 2(p + q + r + 1)$$

where  $r = 1$  if there is a constant  $\mu$  in ARMA, otherwise  $r = 0$ . Note that we need to add one for  $\sigma^2$ , the variance for residuals, as we have supposed that these residuals follow the same normal distribution with zero mean and finite variance.

- AICc :

$$AICc = -2 \ln(\hat{\mathcal{L}}) + \frac{2(p + q + r + 1)(p + q + r + 2)}{n - p - q - r - 2}$$

where  $n$  is the number of observations, which means the given data is  $X_1, \dots, X_n$ .

- BIC :

$$AIC = -2 \ln(\hat{\mathcal{L}}) + (p + q + r + 1) \ln(n)$$

For example, if an ARMA(3,0) after fitting give AICc=340.7 and an ARMA(1,3) after fitting gives AICc=342.1, then we should use ARMA(3,0).

#### 1.8.4.2 Use Ljung–Box test

Suppose that we have found good values for  $(p, q)$  by the above process, then we can apply the Ljung–Box Test to determine if a good fit has been achieved.

Note that Ljung–Box test is applied to the **residuals** of a fitted ARMA model, not the original time serie data. Here we want to test the hypothesis that the residuals from the ARMA model are indepedent.

For ARMA( $p, q$ ), the degrees of freedom need to be adjusted to reflect the parameter estimation (see 3.4.1.2). Since we number of unknown parameters is  $p + q + r + 1$ , then the degrees of freedom should be set to  $h - p - q - r - 1$ , where  $h$  is the number of lags being tested.

It is generally considered good practice to find the smallest values of  $p$  and  $q$  which can provide an acceptable fit to the data.

### 1.8.5 Notes

We remind that ARMA models are only applied when time serie data shows evidence of weakly stationary (4.2.1.1).

First, transformations such as logarithms can help to stabilise the variance of a time series and then we can apply differencing to stabilise mean.

The procedure that we use ACF and PACF to chose order  $p, d, q$ , then fit and evaluate ARMA model is called Box–Jenkins approach.

## 1.9 ARIMA

Autoregressive integrated moving average (ARIMA) is extended version of autoregressive moving average (ARMA) model. The I (for “integrated”) indicates that the data values have been replaced with the difference between their values and the previous values (and this differencing process may have been performed more than once). This is for objective to remove unit root.

### 1.9.1 Inference or forecasting

We will illustrate the procedure of inference through model ARIMA(2,1,1), which means:

$$\begin{aligned} (1 - \phi_1 L - \phi_2 L^2)(1 - L)X_t &= (1 + \theta_1 L)\hat{\varepsilon}_t \\ \Leftrightarrow (1 - (1 + \phi_1)L + (\phi_1 - \phi_2)L^2 + \phi_2 L^3) X_t &= (1 + \theta_1 L)\hat{\varepsilon}_t \\ \Leftrightarrow X_t &= (1 + \phi_1)X_{t-1} - (\phi_1 - \phi_2)X_{t-2} + \phi_2 X_{t-3} + \hat{\varepsilon}_t + \theta_1 \hat{\varepsilon}_{t-1} \end{aligned}$$

where  $L$  is backshift or lag operator. Suppose that we have time serie data until instant  $T$  and we want to forecast  $X_{T+1}, X_{T+2}, \dots$

For  $X_{T+1}$ , we set  $\hat{\varepsilon}_{T+1} = 0$ , this is because we have considered the mean of  $\hat{\varepsilon}$  is 0 or this mean is removed by using  $\mu$  in MA model part, then :

$$X_{T+1} = (1 + \phi_1)X_T - (\phi_1 - \phi_2)X_{T-1} + \phi_2 X_{T-2} + \theta_1 \hat{\varepsilon}_T$$

For  $X_{T+2}$ , we first set  $\hat{\varepsilon}_{T+2} = 0$ . If we do not have the true value of  $X_{T+1}$ , then we set  $\hat{\varepsilon}_{T+1} = 0$ , otherwise, we have true value of  $X_{T+1}$ , we can infer  $\hat{\varepsilon}_{T+1}$ , then :

$$X_{T+2} = (1 + \phi_1)X_{T+1} - (\phi_1 - \phi_2)X_T + \phi_2 X_{T-1} + \theta_1 \hat{\varepsilon}_{T+1}$$

and so on.

#### 1.9.1.1 Prediction intervals

The main key to get prediction intervals is to estimate **forecast variance**  $\sigma^2$  of forecast residuals. Here we precise two ways have this forecast variance :

- **By using estimated variance in training (fitting).** Let  $\hat{\sigma}$  be the estimated variance in training. Suppose that the used model is ARIMA(0, 0, q) and we have known data until time  $T$ , then:

$$X_{T+h} = \hat{\varepsilon}_{T+h} + \sum_{i=1}^q \theta_i \hat{\varepsilon}_{T+h-i}$$

For hypothesis that  $\varepsilon_t$  are independent, then the forecast variance  $\hat{\sigma}_f^2$  is :

$$\hat{\sigma}_f^2 = \text{Var}(X_{T+h}) = \hat{\sigma}^2 \left(1 + \sum_{i=1}^q \theta_i^2\right)$$

Then a 95% prediction interval is given by  $X_{T+h} \pm 1.96\hat{\sigma}_f$ . For a general case of ARIMA( $p, d, q$ ) or seasonal ARIMA, it is much more complicated in this way. **Note that, in this manner, we suppose that**  $\hat{\sigma}$  the estimated variance in training is the same as the one of forecast residuals  $\hat{\varepsilon}_{T+h}$ , where  $h \geq 1$ .

- **By estimating this value through validation data or simulated future paths,** see 6.1.11.

### 1.9.2 Seasonal ARIMA

When the seasonality shows in a time series data, we apply the seasonal ARIMA, in which the seasonal-differencing is applied to eliminate the seasonal component.

Seasonal ARIMA is denoted by ARIMA( $p, d, q$ )( $P, D, Q$ ) $_m$ , where  $m$  refers to the number of periods in each season, and the uppercase  $P, D, Q$  refer to the autoregressive, differencing, and moving average terms for the seasonal part of the ARIMA model.

$$(1 - \sum_{i=1}^p \phi_i L^i)(1 - L)^d (1 - \sum_{i=1}^P \Phi_i L^{mi})(1 - L^m)^D X_t = (1 + \sum_{i=1}^q \theta_i L^i)(1 + \sum_{i=1}^Q \Theta_i L^{mi}) \varepsilon_t$$

where  $(1 - L^m)^D$  means the seasonal-differencing. For example, ARIMA(1, 1, 1)(1, 1, 1) $_4$  model (without a constant) is for quarterly data ( $m = 4$ ):

$$(1 - \phi_1 L)(1 - L)(1 - \Phi_1 L^4)(1 - L^4)X_t = (1 + \theta_1 L)(1 + \Theta_1 L^4)\varepsilon_t$$

#### 1.9.2.1 Signature of seasonal ARIMA

The seasonal part of an AR or MA model will be seen in the seasonal lags (after seasonal differencing) of the PACF and ACF.

For example, an ARIMA(0, 0, 0)(0, 0, 1) $_{12}$  model will show :

- A spike at lag 1 in the ACF but no other significant spikes.

- Exponential decay in the seasonal lags of the PACF.

Similarly, an ARIMA(0,0,0)(1,0,0)<sub>12</sub> model will show:

- Exponential decay in the seasonal lags of the ACF.
- A significant spike at lag 1 in the PACF.

### 1.9.2.2 Procedure

The solving procedure for seasonal ARIMA is almost the same as for non-seasonal ARIMA, except that we need to deal with seasonal AR and MA terms.

The process is best illustrated via example <https://otexts.com/fpp2/seasonal-arima.html>.

## 1.10 ARIMA vs ETS

It is a commonly held myth that ARIMA models are more general than exponential smoothing.

- Linear ETS models are particular cases of ARIMA models, while non-linear ETS models have no equivalent ARIMA counterparts.
- ETS models do not need stationary time series data, while ARIMA models do.
- The AICc (6.1.1) is useful for selecting models in both ARIMA and ETS. However, it cannot be used to compare between ETS and ARIMA models because they are in different model classes, and the likelihood is computed in different ways.

## 1.11 Prediction intervals

Beside the forecasting the expected value of future time series data, we also need to estimate prediction intervals. We discover two cases, one is with estimated variance in training (fitting) and one is with validation data, to have **forecast variance**.

### 1.11.1 With estimated variance in training (fitting)

#### 1.11.1.1 Explicit relation

In this case, we can express explicitly the forecast variance in term of estimated variance in training and parameters of model as we in 6.1.3.7.3 and 6.1.9.1.1.

### 1.11.1.2 Implicit relation

If there is no explicit relation between forecast variance and estimated variance in training or it is hard to establish this relation, we can use the simulated future paths. The latter is about the Monte Carlo process. We sample noise with estimated variance in training  $\sigma^2$  and evaluate the process to have a simulated future path. For example, with ETS( $A, N, N$ ) model

$$l_t = l_{t-1} + \alpha \varepsilon_t X_t = l_{t-1} + \varepsilon_t$$

we can have future path  $X_t$  while sampling  $\varepsilon_t$  from  $\mathcal{N}(0, \sigma^2)$ . We repeat this process many times to have many simulated future paths, from which we infer the forecast variance. To have the sampling distribution of forecast variance, we can use bootstrap (3.2.1.2).

### 1.11.2 With validation data

Now consider that we separate it into training part and validation part. The training part is used to fit the model and the validation part is used to estimate forecast variance. We consider two following cases. the first one that residuals or irregular component are normally and independently distributed and the second one is the general case.

#### 1.11.2.1 Residuals are normally and independently distributed

- Since general residuals follow a normal distribution  $\mathcal{N}(0, \sigma^2)$ , then we must have that validation residuals follow a normal distribution  $\mathcal{N}(0, \sigma_v^2)$ . Here  $\sigma^2$  is the variance of general residuals and  $\sigma_v^2$  is the variance of validation residuals and these two values can be slightly different. Our objective is then to check the normality of validation residuals and get  $\sigma_v^2$ .
- To get validation residuals, we use our trained or fitted model to forecast value  $\hat{X}_{T+h|T}$  for instants  $T + h$  in validation data. The validation residuals are thus the difference between ground true  $X_{T+h}$  in validation data and their corresponding forecasted value  $\hat{X}_{T+h|T}$ , which means  $X_{T+h} - \hat{X}_{T+h|T}$ .
- We check if these validation residuals follow a normal distribution by using Anderson-Darling, Kolmogorov-Smirnov, and D'Agostino K-squared tests or observe the P-P plot (probability-probability plot).
- If validation residuals pass these tests for normality, we just calculate the forecast variance  $\sigma_v^2$  from validation residuals and use it, for example, to give a 95% prediction intervals by  $\hat{X}_{T+h|T} \pm 1.96\sigma_v$ .

#### 1.11.2.2 General case

In the general case, we do not know the distribution of validation residuals but we suppose that, in the most simple case, they follow a normal distribution

$\mathcal{N}(0, \sigma_v^2)$ . This can be done as in above (6.1.11.2.1), where we remove the normality test step.

## 2 Useful concepts

### 2.1 Cholesky decomposition for matrix inversion

The covariance matrix  $\Sigma$  is always Hermitian positive-definite then we can apply the Cholesky decomposition by writing :

$$\Sigma = LL^T$$

where  $L$  is a lower triangular matrix with strict positive diagonal entries. We have some important properties :

- $L^{-1}$  is also a lower triangular matrix with strict positive diagonal entries.
- $(L^T)^{-1} = (L^{-1})^T$  (transposition and inverse are interchangeable).
- $\Sigma^{-1} = (L^T)^{-1}L^{-1} = (L^{-1})^T L^{-1}$

### 2.2 Return rate

#### 2.2.1 Arithmetic and log return rate

The arithmetic return rate can be calculated by :

- Closed price at day  $(k)$  and at day  $(k - 1)$  :

$$R^{(k)} = \frac{\text{Closed price day } k - \text{Closed price day } (k - 1)}{\text{Closed price day } (k - 1)}$$

- Future closed price of asset  $p_k$  at  $k$  days after today (here instead of past data in the case above, we can use future data) :

$$R^{(k)} = \frac{p_k - p_{k-1}}{p_{k-1}}$$

- For a period  $(k)$ , we may take into account of dividend:

$$R^{(k)} = \frac{\text{Ending Price} - \text{Beginning Price} + \text{Dividend}}{\text{Beginning Price}}$$

The logarithmic rate of return is calculated by:

$$R = \log \left( \frac{\text{Ending value}}{\text{Beginning value}} \right)$$

## Overall return

Also called cumulative return. In case of arithmetic return :

$$1 + R = (1 + R_1)(1 + R_2) \cdots (1 + R_n)$$

In case of logarithmic return :

$$R = R_1 + R_2 + \cdots + R_n$$

### 2.2.2 Annualization

Usually and conventionally, return is desired to be expressed in a period of a year. Let  $R$  be return rate over a pediode of  $t$ , where  $t$  is expressed in year unity. Then the annualized return  $r$  is :

$$r = \frac{R}{t}$$

For example, if we have a return rate  $R = 6\%$  over one year and half, then the annualized return  $r = 4\%$ . In trading, we usually infer the annualized return rate  $r$  from daily return rate  $R$ . Note that we have about 252 trading days per year. If  $R = 0.08\%$ , then  $r = \frac{0.08\%}{252} = 20.16\%$

Note that the above formula is true of we have without any reinvestment. In the opposite case, with reinvestment, we need to take into account the effect of compounding. The new relationship in this case becomes :

$$1 + R = (1 + r)^t$$

which can be used to convert the return  $R$  to a **compound** rate of return  $r$  :

$$r = (1 + R)^{\frac{1}{t}} - 1 = \sqrt[t]{1 + R} - 1$$

For example, a  $R = 33.1\%$  return over 3 years is equivalent to a compound rate of return  $r$  of  $\sqrt[3]{1.331} - 1 = 10\%$  per year with reinvestment.

**Important note :**

- The returns for periods of less than one year must not be annualized. This is because an annualized rate of return over a period of less than one year is statistically unlikely to be indicative of the annualized rate of return over the long run, where there is risk involved. In addition, annualizing a return over a period of less than one year might be interpreted as *suggesting* that the rest of the year is most likely to have the same rate of return, but the latter is not completely true.
- Do not to confuse annual returns with annualized returns.

### 2.2.3 Average rate of return

- **Without reinvestment**, we apply the *arithmetic average* rate of return over  $n$  time periods of **equal length** :

$$\bar{r} = \frac{1}{n} \sum_{i=1}^n r_i = \frac{1}{n}(r_1 + \cdots + r_n)$$

- **With reinvestment or compounding is performed**, we apply the *geometric average* rate of return over  $n$  time periods of **equal length** :

$$\bar{r}_{\text{geometric}} = \left( \prod_{i=1}^n (1 + r_i) \right)^{\frac{1}{n}} - 1$$

#### 2.2.3.1 Time-weighted return

Suppose an investor have return rate 2% for first year and 5% for a periode of next two years.

In case of the arithmetic average, we suppose that the return is 2.5% for second year and 2.5% for the third year or 1% for the second year and 4% for the third year, whatever, then the arithmetic average is  $(2 + 5)/3 = 2.33\%$ . This looks intuitively correct **but there is a problem** : the return of 5% for a periode of last two years is **with reinvestment**. It means the investor withdraws his profit at the end of second year but this is not true.

Consequently, we have the concept of time-weighted return **only for geometric average** :

$$\begin{aligned}\bar{r} &= ((1 + r_1)(1 + r_2)(1 + r_3))^{\frac{1}{3}} - 1 \\ &= ((1 + r_1)(1 + r_{23}))^{\frac{1}{3}} - 1 \\ &= 2.3\%\end{aligned}$$

with  $(1 + r_2)(1 + r_3) = 1 + r_{23}$  since it means the overall return in second and third year.

Here may we raise a question that why we call time-weighted return ? since we see here no coefficient weighted by time-periode. Let's use log return, with  $r^{\log} = \ln(1 + r^{\text{arithmetic}})$ , then :

$$\begin{aligned}\bar{r}^{\log} &= \frac{r_1^{\log} + r_{23}^{\log}}{3} \\ &= \frac{r_1^{\log} + 2\bar{r}_{23}^{\log}}{3}\end{aligned}$$

where  $\bar{r}_{23}^{\log}$  is the annualized return of second and third year.

Here we see clearly the coefficient weighted by time-periode.

#### 2.2.4 Compound interest

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

where :

- $A$  is the final amount.
- $P$  is the original principal amount.
- $r$  is the nominal annual interest rate (return rate).
- $n$  is the compounding frequency (month or quarter by default).
- $t$  is the overall length of time the interest is applied (expressed using the same time units as  $r$ , usually years).

Example : with  $P = \$200000$ ,  $r = 0.04$ (4%),  $n = 12$  (months),  $t = 20$  (years). Then  $A = \$444516$ , it means that after 20 years, the initial capital \$200K is equivalent to \$444.5K

#### 2.2.5 Mensuality

This help to calculate the mensuality  $m$  that the borrower need to repay the lender.

$$m = \frac{\frac{Pr}{n} \left(1 + \frac{r}{n}\right)^{nt}}{\left(1 + \frac{r}{n}\right)^{nt} - 1} = \frac{\frac{Pr}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

We reuse the same configuration as example of compound interest (6.2.2.4), then with a loan of \$200K, the mensuality  $m = \$1212$ . The total (borrowed capital and cost) is  $m \times n \times t = \$290870$  or the initial capital \$200K is equivalent to only \$290.8K

Here may raise a question, why there is a big different for profit for the lender (\$90.8K) and the profit in example in 6.2.2.4 (\$244.5K) ? This is because the lender **receive a mensuality** each month.

If the borrower need to buy assurance then, they must pay :

$$m = \frac{\frac{Pr}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}} + \frac{Pa}{n}$$

where  $a$  is the assurance rate.

#### 2.2.6 Equivalence of borrowed capital

Let's consider two credits with  $P_1, r_1$  and  $P_2, r_2$ . The objective is to find the equivalent borrowed capitals  $P_1$  and  $P_2$  given two distinct rates  $r_1$  and  $r_2$  and the same mensuality  $m_1 = m_2$  :

$$\frac{P_1 r_1}{1 - \left(1 + \frac{r_1}{n}\right)^{-nt}} = \frac{P_2 r_2}{1 - \left(1 + \frac{r_2}{n}\right)^{-nt}}$$

$$P_1 = P_2 \frac{r_2}{r_1} \frac{1 - (1 + \frac{r_1}{n})^{-nt}}{1 - (1 + \frac{r_2}{n})^{-nt}}$$

Suppose that an apartment cost \$200K at 2010 and we can borrow the bank all \$200K at rate 2% to buy this apartment. At 2012 the rate has raised to 4% and we have always the same salary (our capacity of repayment is unchanged), what is the equivalence that we can borrowed ?

Apply  $P_2 = \$200K$ ,  $r_2 = 0.02$ ,  $r_1 = 0.04$  Then  $P_1 = \$166.9K$

### 2.2.7 External flow

Suppose an investor transfers \$500 into a portfolio at the beginning of Year 1, and another \$1,000 at the beginning of Year 2, and the portfolio has a total value of \$1,500 at the end of the Year 3. The net gain over the three-year period is zero, so intuitively, we might expect that the overall return over the whole 3-year period to be 0%.

There is one important thing that need to be considered, the **external flow** by injecting \$1,000 at the beginning of Year 2.

Let's  $M$  is the value of portfolio just before the injection. Then the overall return :

$$1 + R = \frac{M}{500} \times \frac{1500}{M + 1000}$$

If  $M = 800$ , then  $R = 33\%$  and not 0% by intuition. Therefore, the external flow (time in and out of money and the amount of money) is important for calculating overall return rate.

### 2.2.8 Money-weighted rate of return

Let  $C_t$  be cash flow (1.7), where  $t$  is real non-negative number. PV means present value. The money-weighted rate of return (also called the **internal rate of return**)  $r$  must satify that PV Outflows equals to PV Inflows, which means :

$$\begin{aligned} \text{PV Outflows} &= \sum_{t=0}^n \frac{C_t}{(1+r)^t} \\ &= \text{PV Inflows} \end{aligned}$$

#### Example

An investment manager purchases a stock today for \$100. He intends to hold the stock for three years and collects \$5 in dividends each year. At the end of the third year, he expects to be able to sell the stock for \$150. What is the money-weighted return on this investment portfolio ?

Solution :

- Identify inflows and outflows. Inflows: Dividends (\$5 in years 1, 2, and 3) and the sale of stock (\$150 in year 3). Outflows: Purchase of stock \$100.

- Set PV outflows = PV inflows :

$$\frac{5}{1+r} + \frac{5}{(1+r)^2} + \frac{5+150}{(1+r)^3} = 100$$

- Solving for  $r$ , we get  $r = 18.88\%$ .

The dividend discount model (6.3.13) is a particular case and has more or less mechanism of money-weighted rate of return.

### Notes

- There **not always** exists an real solution  $r$  to the equation PV outflows = PV inflows.
- There may also be more than one real solution to the equation, requiring some interpretation to determine the most appropriate one.
- The money-weighted rate of return over multiple sub-periods **can not obtained** by combining the one of each sub-period as time-weighted return 6.2.2.3.1, which means we always need to solve the equation PV outflows = PV inflows to obtain it.

#### 2.2.9 Synthese example

	Year 1	Year 2	Year 3	Year 4	Year 5
Shares owned before distribution	70.373	71.676	74.125	76.859	84.752
Dividend per share	\$0.26	\$0.29	\$0.30	\$0.50	\$0.53
Capital gain distribution per share	\$0.06	\$0.39	\$0.47	\$1.86	\$1.12
Total distribution per share	\$0.32	\$0.68	\$0.77	\$2.36	\$1.65
Total distribution	\$22.52	\$48.73	\$57.10	\$181.73	\$141.60
Share price at distribution	\$17.28	\$19.90	\$20.88	\$22.98	\$21.31
Shares purchased (total distribution / price)	1.303	2.449	2.734	7.893	6.562
Shares owned after distribution	71.676	74.125	76.859	84.752	<b>\$91.314</b>
Share price at end of year	\$17.50	\$19.49	\$20.06	\$20.62	<b>\$19.90</b>

Table 6.4: A balanced mutual fund during boom times.

Suppose that an investor make an initial investment \$1,000 at end of year 0, share price \$14.21 at end of Year 0. The information of this investment is given in table 6.4.

Let's see the first column. We the the number of shares is  $\frac{1000}{14.21} = 70.373$  shares. The two next lines is the dividend per share and capital gain distribution per share.

Capital gain distribution occurs when investment funds sell securities within their portfolios at a profit. This profit are then distributed to investors. Capital gain distribution is **specific to investment funds**.

The total distribution per share is the sum of dividend and capital gain distribution. The total distribution is calculated by the total distribution per share times shares owned.

If the investor performs the reinvestment, he uses the total distribution to buy more shares, in this case the number of additional shares is  $\frac{22.52}{17.28} = 1.303$  shares. Then his number of shares increases to  $70.373 + 1.303 = 71.676$ .

- If the investor reinvested all distributions, he would own 91.314 shares valued at \$19.90 per share. The return over the five-year period is  $\frac{19.90 \times 91.314 - 1000}{1000} = 1 = 81.71\%$ . The geometric average annual total return is  $\sqrt[5]{1.8171} - 1 = 12.69\%$ .
- If the investor did not reinvest, he would have received total distributions (cash payments) of \$5.78 per share. The return over the five-year period for such an investor would be  $\frac{19.90 + 5.78}{14.21} - 1 = 80.72\%$ . Note that, we do not take into account of time value of money. In reality, the investor can use his cash payments to invest in risk-free asset. The arithmetic average rate of return would be  $\frac{80.72}{5} = 16.14\%$  per year.

#### 2.2.10 When use arithmetic return and logarithmic return

- The arithmetic return is **without reinvestment** and is **not compounded**. We apply arithmetic return when there is only **one single period**. For example, we can compare returns of different assets at a given year.
- The logarithmic return is **with reinvestment** and is all about **continuous compounding**. Then we apply logarithmic return in case of **multip-time periods** with reinvestment. In general, log return is better in approximately normal distribution and when working with distribution. For example, see 6.2.2.3.1.

### 2.3 Cointegration

First of all, cointegration **has nothing to do** with the integral concept  $\int f(x)dx$ .

Given a collection of  $k$  time series variables  $(X_1, X_2, \dots, X_k)$ . The collection is said to be **co-integrated** if :

- $(X_1, X_2, \dots, X_k)$  are integrated of the same order  $d$  (see 3.4.7.2).
- There is a linear combination of this collection that has order of integration less than  $d$ . This means there exist coefficients  $a_1, a_2, \dots, a_k$  such that  $a_1 X_1 + \dots + a_k X_k$  is integrated of order less than  $d$ .

In this case, we note  $(X_1, X_2, \dots, X_k) \sim CI(d, \dots, d)$ . Cointegration has become an important property in contemporary time series analysis. Time series often have trends—either deterministic or stochastic.

For application, cointegration helps detect the (long-term) relationship between two or more time series. For example, a stock market index and the price

of its associated futures contract move through time, each roughly following a random walk. Testing the hypothesis that if there is a relationship between the futures price and the spot price could now be done by testing if there is a cointegration between two series.

### 2.3.1 Intuitive interpretation

Suppose that we have two price time series  $x(t)$  and  $y(t)$  of  $I(1)$  and they are cointegrated. Now we decompose  $x(t)$  and  $y(t)$  into a nonstationary component  $\nu$  and a stationary component  $\varepsilon$  as follows :

$$\begin{aligned}x(t) &= \nu_x(t) + \varepsilon_x(t) \\y(t) &= \nu_y(t) + \varepsilon_y(t)\end{aligned}$$

Now we construct the cointegrated series  $z(t)$ , or the spread between  $x(t)$  and  $y(t)$ . Spread here is understood as a combination (see 1.11) :

$$\begin{aligned}z(t) &= \alpha x(t) + \beta y(t) \\&= (\alpha \nu_x(t) + \beta \nu_y(t)) + (\alpha \varepsilon_x(t) + \beta \varepsilon_y(t)) \\&= \alpha \varepsilon_x(t) + \beta \varepsilon_y(t)\end{aligned}$$

where cointegration condition implies  $(\alpha \nu_x(t) + \beta \nu_y(t)) = 0$  and the linear combination of two stationary series is also stationary.

The price time series  $x(t)$  and  $y(t)$  share dependent nonstationary components  $\nu_x(t) = -\frac{\beta}{\alpha} \nu_y(t)$ , which may include trend, seasonal ...

For application, if two assets are cointegrated, the non-stationary components are dependent. For example, good candidates for cointegrated pairs could be:

- Stocks that belong to the same sector.
- WTI crude oil and Brent crude oil.
- AUD/USD and NZD/USD.

### 2.3.2 Comparison with correlation

#### 2.3.2.1 Distance

Let's consider the following illustration :

- A drunk man leaves the pub with his dog.
- He forgets to put a leash on his dog, and so he and the dog begin to stumble home. Eventually, they reach a crosswalk, and realizing his dog is untethered, the drunk man leashes his dog before crossing the road.
- The dog and the man cross the road, and walk awhile toward his brownstone. Eventually the dog realizes they are almost home and the dog begins tugging strongly at the leash, effectively pulling the man the final few yards to home.

When the man and the dog first leave the pub, their paths are **correlated**. They generally **move in the same direction**, but the **distance between the dog and the man has no actual limit**.

When the man leashes his dog, they become **cointegrated**. Now, while their **direction is still the same**, the **distance from one to another is less than a threshold**. The dog cannot move beyond the length of the leash from the man.

Finally, when the dog begins pulling the man home, the two become **causal**. We can fully explain the path of the man if you know the path of the dog.

<https://www.quora.com/What-is-the-difference-between-correlation-and-cointegration-Is-cointegration-a-good-thing>

### 2.3.2.2 Short and long term

The difference between correlation and cointegration can be seen in figure 6.6 :

- On the left : The two signals are moving in same direction. Hence, they are correlated. The difference between two signals increased from 0 to 100. Thus, they are not cointegrated.
- On the right : The difference between two signals moves from positive to zero, from zero to negative and so on. Thus, the stocks are cointegrated. However, the blue moves up and down but the green hardly changes. Hence, they are not correlated.

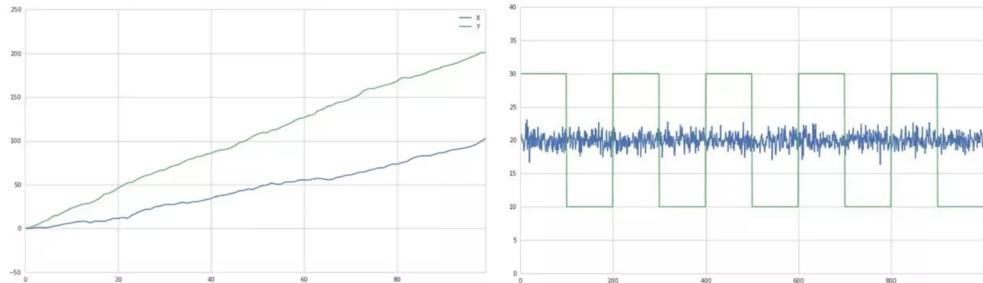


Figure 6.6: Difference between correlation and cointegration. Left : Two signals are correlated but not cointegrated. Right : Two signals are cointegrated but not correlated.

Correlation is a **short-term** measure that reflects the immediate relationship between two signals. This means if one signal changes its direction (up or down), the other one, in short-term, need to change in the same direction.

Cointegration, on the other hand, is a **long-term** measure that reflects the stable relationship between two assets over time. This means if one signal changes its direction (up or down), the other one may not change in the same direction in short term, but in long term, it needs to change to keep the distance between their nonstationary components unchanged.

Both correlation and cointegration are important measures for evaluating pairs trading opportunities (1.12.1). However, they serve different purposes (short-term and long-term) and should be used together to get a complete picture of the relationship between two assets. A high degree of correlation and cointegration between two assets indicates that is likely to persist over time and can be exploited for profit.

### 2.3.2.3 Implication

Correlation has no well-defined relationship with cointegration. Cointegrated series might have low correlation, and highly correlated series might not be cointegrated at all. Often, variables with high correlation will also be cointegrated, and vice versa, but this isn't always the case.

### 2.3.3 Engle–Granger two-step method test

Suppose that we have two time series  $x_t$  and  $y_t$ , we need to check if they are cointegrated. In this case, we apply for  $d = 1$ .

1. The first step of this method is to pretest if  $x_t$  and  $y_t$  are non-stationary, which means  $x_t$  and  $y_t$  are  $I(1)$ . This can be done by DF test (3.4.7.3) or ADF test (3.4.8). If ok, we pass to the next step.
2. If these  $x_t$  and  $y_t$  are cointegrated then there will exist coefficients  $\mu$  and  $\beta$  such that:

$$y_t = \mu + \beta x_t$$

where  $u_t$  is stationary with zero mean. We apply OLS for this regression problem to find  $\mu$  and  $\beta$ . Calculating  $\hat{u}_t = y_t - \hat{\mu} + \hat{\beta}x_t$ , which are thus residuals.

3. Then, we can run an ADF test on  $\hat{u}_t$ . However, since  $\hat{\beta}$  and  $\hat{\mu}$  are estimated, the distribution for  $\hat{u}$  is not the same as the usual DF-distributions and will depend on the number of regressors (here,  $x_t$ ). Additional parameters in regression will shift the DF-distributions to the left.

## 3 Modern portfolio theory

This theory is also called *mean-variance analysis*, which is used to refer to mathematical frameworks to construct an optimal portfolio by studying the trade-off between expected return (mean) and risk (variance). We first discover Markowitz's assumptions, which are foundational to Modern portfolio theory.

### 3.1 Markowitz's assumptions

- Risk of a portfolio is based on the variability of returns. Usually, we use the std.

- An investor is **risk averse** : This means that given two portfolios that offer the same expected return, investors will prefer the less risky one.
- An investor prefers to increase consumption : This means instead of allocating their resources towards long-term investments or savings, he may choose to spend a larger portion of their income or wealth on immediate consumption goods and experiences.
- The investor's utility function is concave and increasing 6.8, due to their risk aversion and consumption preference.
- Analysis is based on single period model of investment.
- An investor either *maximizes their portfolio expected return for a given level of risk* or *minimizes their risk for a given expected return*.
- An investor is *rational* in nature.

### 3.2 Mathematical modeling

- $N$  is the number of assets.
- $r_i$  is the return on asset  $i$ .
- $w_i$  is the weighting of component asset  $i$  in a portfolio. Thus  $w_i = \frac{X_i}{X}$  where  $X_i, X$  are respectively the amount (money) of asset  $i$  and the amount of the portfolio.
- $\sigma_i$  is the standard deviation of the periodic returns on an asset  $i$ .
- $\sigma_{ij}$  is the covariance of the periodic returns on the two assets  $i$  and  $j$ . Alternatively,  $\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$ , where  $\rho_{ij}$  is the correlation.

Then the expected return and variance of the portfolio :

- $\mu_p = \sum_{i=1}^N w_i \mathbb{E}[r_i] = \mathbb{E}[r]^T w$
- $\sigma_p^2 = \sum_{i=1}^N w_i^2 \sigma_i^2 + \sum_i \sum_j w_i w_j \sigma_{ij} = w^T \Sigma w$ , where  $\Sigma$  is the covariance matrix whose elements are  $\sigma_{ij}$ .

### 3.3 Efficient frontier

For a given “risk tolerance”  $q \in [0, \infty)$  as parameter, the **efficient frontier** is found by minimizing the following expression:

$$\min_w \frac{1}{2} w^T \Sigma w - q \mathbb{E}[r]^T w$$

subject to

$$\sum_i w_i = 1$$

An **alternative problem** is by parametrizing on the expected portfolio return  $\mu$  :

$$\min_w \frac{1}{2} w^T \Sigma w$$

subject to :

$$\mathbb{E}[r]^T w = \mu$$

Thus, we can understand that there is a correspondance between  $q$  and  $\mu$ .

### 3.3.1 Solution

We use Lagrangian for the alternative problem :

$$\min_w \frac{1}{2} w^T \Sigma w + \lambda(\mathbb{E}[r]^T w - \mu) + \eta(\mathbf{1}^T w - 1)$$

Then we need to solve the first-order conditions, for obtaining the optimal  $w$  :

$$\begin{cases} \Sigma w + \lambda \mathbb{E}[r] + \eta \mathbf{1} \\ \mathbb{E}[r]^T w = \mu \\ \mathbf{1}^T w = 1 \end{cases} \Leftrightarrow \begin{bmatrix} \Sigma & \mathbb{E}[r] & \mathbf{1} \\ \mathbb{E}[r]^T & 0 & 0 \\ \mathbf{1}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ \mu \\ 1 \end{bmatrix} \Leftrightarrow M \begin{bmatrix} w \\ \lambda \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ \mu \\ 1 \end{bmatrix}$$

This is a system equations of  $N + 2$  equations and  $N + 2$  variables (with  $\lambda$  and  $\eta$ ). We can have the solution by using computer. However, to get more insight and solve manually this problem.

### 3.3.2 Graphic representation

From above system of equation, we get  $w = M_{[:,2]}^{-1}\mu + M_{[:,3]}^{-1}$ . This means that for a given  $\mu$ , the optimal variance of portfolio  $w^T \Sigma w$  is :

$$w^T \Sigma w = \sum_{i=1}^N \sum_{j=1}^N (M_{[i,2]}^{-1}\mu + M_{[i,3]}^{-1})(M_{[j,2]}^{-1}\mu + M_{[j,3]}^{-1})\sigma_{ij} = a\mu^2 + b\mu + c$$

or

$$\sigma_p = \sqrt{w^T \Sigma w} = \sqrt{a\mu^2 + b\mu + c}$$

Then  $(\sigma_p, \mu)$  form a **hyperbola**. We present this hyperbola in figure 6.7. Conversely, by this graph, with a given  $\sigma_p$ , we can have 0,1 or 2 optimal expected returns, depends on value of given  $\sigma_p$ .

The *upper part* of the hyperbolic boundary is the **efficient frontier**. This is logic because for a given  $\sigma_p$ , a rational investor must take the higher expected return. In this figure we find also the captial allocation line, which is presented in next subsection.

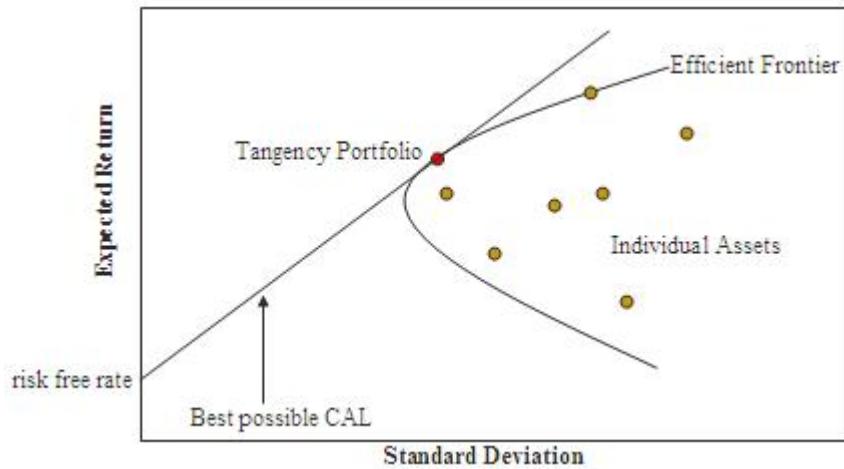


Figure 6.7: Efficient frontier and the capital allocation line (CAL).

### 3.4 Sharpe ratio

The Sharpe ratio (also known as the Sharpe index, the Sharpe measure, and the reward-to-variability ratio) is the **quantity to measure the performance** of an investment such as a security or portfolio compared to a risk-free asset as reference. The Sharpe ratio is defined by :

$$S = \frac{\mathbb{E}[r - r_f]}{\sqrt{Var(r - r_f)}}$$

where  $r$  is return rate of security or portfolio and  $r_f$  is return rate of risk-free asset. Note that,  $r_f$  can change after a period. If  $r_f$  is fixed, then :

$$S = \frac{\mathbb{E}[r] - r_f}{\sqrt{Var(r)}} = \frac{\mathbb{E}[r] - r_f}{\sigma}$$

where  $\sigma = \sqrt{Var(r)}$  signify the volatility.

#### 3.4.1 Example

Suppose we have two investments  $a$  and  $b$ .  $a$  is with an expected return of 12% and a standard deviation of 10% and  $b$  is with an expected return of 9% and a standard deviation of 7%. Given the risk-free return rate is 3%, according to the sharp ratio, which investment that we choose ?

We calculate the sharp ratio for each investment :

$$S_a = \frac{0.12 - 0.03}{0.1} = 0.9$$

$$S_b = \frac{0.09 - 0.03}{0.07} = 0.857$$

Then we choose  $a$ .

### 3.4.2 Optimal sharp ratio

Given a portfolio with a list of securities, with their statistique data. we want to optimize the portfolio for the best sharp ratio.

Suppose that a portfolio is composed of  $n$  securities with return rate  $i$ , with their coressponding weights  $\gamma_i$ . Then we try to optimize :

$$\max_{(\gamma_i)_{(i=1,\dots,n)}} \frac{\mathbb{E}[\sum_{i=1}^n \gamma_i r_i - r_f]}{\sqrt{Var(\sum_{i=1}^n \gamma_i r_i)}}$$

where the variance can be devepped as

$$Var(\sum_{i=1}^n \gamma_i r_i) = \sum_{i=1}^n \gamma_i^2 Var(r_i) + \sum_{i=1}^n \sum_{j=1, j \neq i}^n 2\gamma_i \gamma_j Cov(r_i, r_j)$$

subject to condition :

$$\sum \gamma_i = 1$$

If we impose  $\gamma_i \geq 0$ , we do not allow for short sell. An optimization solution can be seen in 6.3.10.1

## 3.5 Capital allocation line

If we have **only risky assets** in our portfolio, then the **efficient frontier** help us to observe what is the optimal expected return for a given risk and conversly, what is the optimal risk for a given expected return.

Now if we have **risky assets and a risk-free asset**, then we need to observe the **capital allocation line (CAL)** to observe the same thing.

Given a portfolio  $b$  that contains only risky assets and portfolio  $c$  contains portfolio  $b$  and a risk-free asset with return rate  $r_f$ . For risky assets, we can establish an optimal point on the efficient frontier with (expected) return rate  $r_b$  and  $\sigma_b$  and weighted with the risk-free asset :

$$\begin{cases} r_c = wr_b + (1-w)r_f \\ \sigma_c = w\sigma_b \end{cases}$$

which implies equation for a capital allocation line :

$$r_c - r_f = \frac{r_b - r_f}{\sigma_b} \sigma_c,$$

which describes the relation between risk premium  $r_c - r_f$  and the volatility (risk)  $\sigma_b$  is a linear relation, with slope  $\frac{r_b - r_f}{\sigma_b}$ , which is known as the “reward-to-variability ratio” or “sharp ratio” (6.3.4).

However, the CAL  $(\sigma_c, \frac{r_b - r_f}{\sigma_b} \sigma_c + r_f)$  is not optimal. This is because we have chosen any point  $(\sigma_b, r_b)$  on efficent frontier. If the chosen point  $(\sigma_b^*, r_b^*)$  is the

point of tangency between the line start from  $(0, r_f)$  and the efficient frontier, see figure 6.7. In this case, we have the best possible CAL :

$$r_c - r_f = \frac{r_b^* - r_f}{\sigma_b^*} \sigma_c,$$

Summing up, for an optimal  $r_c$  given  $\sigma_c$  or vice versa, we need to combine the best possible CAL (until the point of tangency) and the efficient frontier (from the point of tangency).

Note that,  $w$  is not obligatory between 0 and 1. In the prolongation line:

- If  $w > 1$  means that we invest more than what we have to asset  $b$  by borrowing money with a risk-free rate  $r_f$ . This can be seen as a good deal.
- If  $w < 0$  means that we invest more in risk-free asset by borrowing money with a rate that equal to return rate of risky assets  $r_b$ . This is not possible for a rational investor.

### 3.6 Mutual fund separation theorem

Also called two mutual fund theorem. This theorem states that any portfolio on the efficient frontier can be generated by holding a combination of any two given portfolios on the frontier. These two given portfolios are thus the “mutual funds”. Note that, here mutual funds do not include risk-free assets in their portfolios.

If the location of the desired portfolio on the frontier is between the locations of the two mutual funds, both mutual funds will be held in positive quantities. If the desired portfolio is outside the range spanned by the two mutual funds, then one of the mutual funds must be sold short (held in negative quantity) while the size of the investment in the other mutual fund must be greater than the amount available for investment (the excess being funded by the borrowing from the other fund).

#### 3.6.1 Application

In practice, given an optimal point (risk & expected return), if we want construct a portfolio at this point can have a lot of cost transaction since the number of securities can be reach to a hundreds or a thousands. An alternative way is to use the mutual fund separation theorem. Indeed, mutual funds are something optimal, which means they are all located on the efficient frontier. This is because, there are professional teams, analysis, ... behind them.

Therefore, we can take two mutual funds and weight them to have a portfolio at our desired optimal point. In this case, the number of mutual funds (two) is much smaller than the number of individual assets in the portfolio.

There are some known benchmark mutual funds such as :

- Vanguard Total Stock Market Index Fund

- Fidelity Contrafund
- American Funds Growth Fund of America
- ...

### 3.7 Market portfolio

The market portfolio refers to a theoretical portfolio that includes all assets available in the market, weighted by their market capitalization or value. Here, market portfolio does not hold any risk free asset. Thus market portfolio is selected by ETF index (1.2) such as S&P 500, MSCI World, FTSE, ...

Market portfolio is different from portfolio in mutual fund since assets in market portfolio are weighted by their market capitalization and not as weighted for a desired optimal point risk-return.

In graph representation, the market portfolio is the point of tangency (red point) in figure 6.7. Hence market portfolio is also located in efficient frontier. The latter may be understood that the market portfolio always reflects to given risk-free rate (to be the point of tangency) and the market portfolio is optimized by itself (rational investor) to be located on efficient frontier.

### 3.8 Capital market line

The capital market line (CML) is thus the best possible CAL (6.3.5).

Given  $p$  is a combination of the market portfolio and the risk-free asset, as the CAL, in CML, we have the expected return rate  $r_p$  is a function of  $\sigma_p$  :

$$\text{CML} : \sigma_p \mapsto r_f + \sigma_p \frac{r_M - r_f}{\sigma_M}$$

where  $(\sigma_M, r_M)$  are risk-return point of market portfolio. Note that this line continues after point  $M$  (figure 6.8).

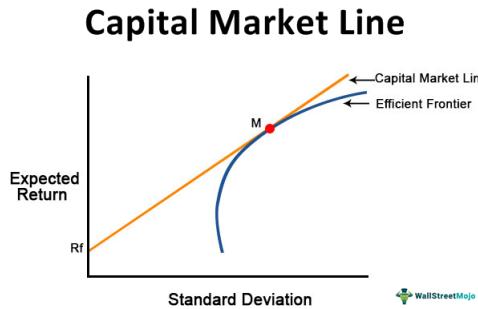


Figure 6.8: Capital Market Line

On one hand, a stock picking rule of thumb is to buy assets whose Sharpe ratio (6.3.4) will be above the slope of CML ( $\frac{r_M - r_f}{\sigma_M}$ ) and sell those whose Sharpe

ratio will be below. On the other hand, if the efficient market hypothesis (1.33) is true, it follows that it's impossible to beat the market. Therefore, all portfolios should have a Sharpe ratio less than or equal to the market's.

If the efficient market hypothesis is false, there are portfolios whose Sharpe ratio will be bigger than the market's then it means an abnormal extra return over the market's return at a given level of risk. This is called the alpha 6.3.12.

### 3.9 Diversification

Suppose that we have a portfolio (can be a single security) with a return rate  $r$  a high volatility  $\sigma$  and we want to reduce the volatility. A solution is to diversify this portfolio by adding an additional security with return rate  $r_a$  and volatility  $\sigma_a$ . Then the sharp ratio for new portfolio is :

$$S = \frac{\mathbb{E}[wr + (1-w)r_a]}{\sqrt{Var(wr + (1-w)r_a)}} = \frac{\mathbb{E}[wr + (1-w)r_a]}{\sqrt{w^2\sigma^2 + 2w(1-w)Cov(r, r_a) + (1-w)^2\sigma_a^2}}$$

where  $w$  is the weight attached to the actual portfolio.

Hence, we see that to reduce the volatility of new portfolio, for a given weight  $w$ , it depends on  $Cov(r, r_a)$  and  $\sigma_a^2$  of the additional security  $a$ . If the volatility  $\sigma_a$  is fixed, then we should choose  $a$  such that  $Cov(r, r_a) < 0$ , which means a decorrelation with the actual portfolio.

Note that, diversification does not means to improve the sharp ratio since we also to consider the average return rate of addtional security  $\mathbb{E}[r_a]$ .

#### 3.9.1 Specific risk

Also called unsystematic risk or diversifiable risk. Specific risk is the risk associated with **individual** asset. We can apply a diversification by adding an additional asset to form a portfolio with lower then the specific risk.

#### 3.9.2 Systematic risk

Also called undiversifiable risk or aggregate risk. This risk refers to the risk common to all securities in a market. Systematic risk cannot be diversified away since the portfolio now is the market.

Within the market portfolio, *asset specific risk* will be diversified away to the extent possible. Systematic risk is therefore **equated with** the risk of the market portfolio.

### 3.10 Capital asset pricing model

In finance, the capital asset pricing model (CAPM) is a model used to determine a theoretically required **appropriate rate of return** of an asset in a market, given

- The risk-free rate  $r_f$ .

- The expected return the market  $\mathbb{E}[r_M]$ .
- The sensitivity between risk premium (of asset  $i$ ,  $\mathbb{E}[r_i] - r_f$ ) and market premium ( $\mathbb{E}[r_M] - r_f$ ), noted  $\beta_i$ .

The CAPM is usually expressed by :

$$\mathbb{E}[r_i] - r_f = \beta_i (\mathbb{E}[r_M] - r_f)$$

### 3.10.1 Deriving the formula

We set up the notation as in section 6.3.2. Let's consider the sharp ratio (6.3.4) of a portfolio with return  $r_p$  and std  $\sigma_p$  :

$$S_p = \frac{\mathbb{E}[r_p] - r_f}{\sigma_p} = \frac{\sum_i w_i (\mathbb{E}[r_i] - r_f)}{\sigma_p}$$

with

$$\begin{aligned}\mathbb{E}[r_p] &= \sum_i w_i \mathbb{E}[r_i] \\ \sigma_p &= \sqrt{\sum_i \sum_j w_i w_j \text{Cov}(r_i, r_j)}\end{aligned}$$

To maximize  $S_p$  with respect to  $w_i$ , we take the first order conditions :

$$\frac{\partial S_p}{\partial w_i} = \frac{(\mathbb{E}[r_i] - r_f)\sigma_p - (\mathbb{E}[r_p] - r_f)\frac{1}{\sigma_p} \left( \sum_j w_j \text{Cov}(r_i, r_j) \right)}{\sigma_p^2} = 0$$

Let  $w_i^*$  be optimal weights (here for a system of above equations), then this portfolio  $p$  is optimized and it becomes the portfolio or the market  $M$ , because we know investors rational and always try to optimize the market, then :

$$\begin{aligned}\mathbb{E}[r_i] - r_f &= (\mathbb{E}[r_M] - r_f) \frac{1}{\sigma_M^2} \left( \sum_j w_j^* \text{Cov}(r_i, r_j) \right) \\ &= (\mathbb{E}[r_M] - R_f) \frac{\text{Cov}(r_i, r_M)}{\text{Var}(r_M)} \\ &= (\mathbb{E}[r_M] - R_f) \beta_i\end{aligned}$$

where  $\beta_i = \frac{\text{Cov}(r_i, r_M)}{\text{Var}(r_M)}$ .

#### 3.10.1.1 Remark

We note that  $\beta_i = \frac{\text{Cov}(r_i, r_M)}{\text{Var}(r_M)}$  is also the solution for following problem :

- A stochastic linear regression (see 3.2.12.1.1):

$$r_i - r_f = \beta_i(r_M - r_f) \quad \text{or} \quad r_i = \beta_i r_M$$

- A least square linear regression (see 3.2.12.1.2):

$$\min_{\beta_i} \sum_k \left( r_i^{(k)} - \beta_i r_M^{(k)} \right)^2$$

- The expected market rate of return ( $\mathbb{E}[r_M]$ ) is usually estimated by measuring the **arithmetic average** of the historical returns of market portfolio (e.g. S&P 500).
- The risk free rate of return ( $r_f$ ) is usually the arithmetic average of historical risk free rates of return and not the current risk free rate of return.
- $\beta$  are normally calculated with 5 past years, and used to infer for future value.
- To get  $\beta_i$  for asset  $i$ , we could just look up a stock's beta online (Yahoo Finance and FinViz). An example of calculating  $\beta$  : <https://www.mlq.ai/capital-asset-pricing-model-python/>

### 3.10.2 Example

The objective is to infer required rate of return or equity discount rate. Let's take an example :

The current risk-free rate is 2% on a short-term U.S. Treasury. The long-term average rate of return for the market is 10%. Let's say Company A has a beta of 1.50, meaning that it is riskier than the overall market (which has a beta of 1). To invest in Company A :

$$\mathbb{E}[r_A] = 2\% + 1.50X(10\% - 2\%) = 14\%$$

Company B has a beta of 0.50, which implies that it is less risky than the overall market. To invest in Company B:

$$\mathbb{E}[r_B] = 2\% + 0.50X(10\% - 2\%) = 6\%$$

### 3.10.3 Security market line

The security market line (SML) is thus the relation between  $\mathbb{E}[r]$  and  $\beta$  by CAPM :

$$\mathbb{E}[r] = (\mathbb{E}[r_M] - r_f)\beta + r_f$$

We thus see this relation in example 6.3.10.2 and this can be described by the figure 6.9.

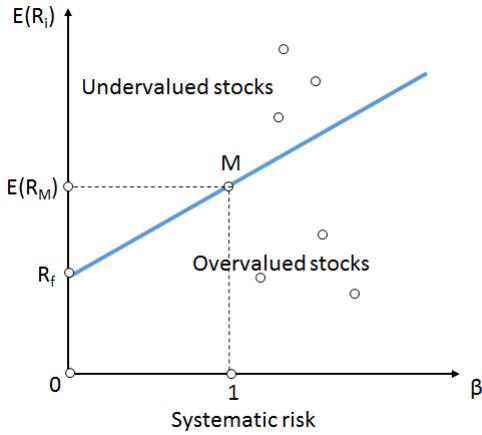


Figure 6.9: The security market line. The x-axis represents  $\beta$ , and the y-axis represents the expected return. Given  $\beta_i$  of stock  $i$ , along with the market premium (slope of SML)  $\mathbb{E}[r_M] - r_f$ , we can check the actual stock  $i$  is overvalued or undervalued.

### 3.10.4 Notes

CAPM assumes a particular form of functions, in which only first and second moments matter (the relations between variable can be only expected value or variance).

Despite its failing numerous empirical tests, and the existence of more modern approaches to asset pricing and portfolio selection (such as arbitrage pricing theory and Merton's portfolio problem), the CAPM still remains popular due to its simplicity and utility in a variety of situations.

#### 3.10.4.1 Beta analysis

If you feel that your portfolio is too conservative or stagnant - that you're missing out on gains because your investments don't move very much - then it's likely that your portfolio has a  $\beta < 1$ . Adding stocks with  $\beta > 1$  would increase return but also volatility (risk).

Conversely, if you feel that your portfolio is too risky - that you can't stomach its big upward and downward swings - then there's a good chance that your portfolio has a  $\beta > 1$ . Adding stocks with  $\beta < 1$  would help decrease volatility but also return.

If  $|\beta| > 1$  we say that this asset is aggressive and if  $|\beta| < 1$  we say that this asset is defensive.

Some investments, such as put options, have negative  $\beta$ , meaning that they would be expected to move in the opposite direction of the market index.

### 3.10.4.2 Limitations of beta analysis

Beta analysis can be a useful way to manage the level of risk in your portfolio, but like any financial technique, it's not perfect.

One major drawback of beta is that it's a backward-looking metric. It's calculated based on past returns, which may not be consistent with future returns. Beta's backward-looking nature also means it's not very useful for evaluating young asset with a short record of data.

It's also worth noting that  $\beta$  only measures the sensibility related to systematic risk, which means the asset's respond to upturns or downturns of the market.  $\beta$  doesn't include other fundamental information, e.g., if the asset is a stock of company, we have earnings, cash flow, debt, dividends,... These information is not directly reflected by  $\beta$ .

### 3.10.5 Asset pricing

Given  $\mathbb{E}[P_i^T]$  the future price of security or portfolio  $i$  (can be obtained by simulation or by the future market ...). Let's consider 2 cases:

- If we know  $\mathbb{E}[r_i]$ , which is the expected return rate associated with this security (usually annual), from time instant 0 to  $T$ , then the present value  $P_i^0$  can be obtained by using  $r_i$  as discounted rate:

$$P_i^0 = \frac{\mathbb{E}[P_i^T]}{1 + \mathbb{E}[r_i]}$$

- If we do not know  $\mathbb{E}[r_i]$ , we can apply the CAPM to get  $\mathbb{E}[r_i]$  or we can also apply the following formula (if we know  $Cov(P_i^T, r_M)$ ) to get the price  $P_i^0$  :

$$P_i^0 = \frac{1}{1 + r_f} \left( \mathbb{E}[P_i^T] - \frac{Cov(P_i^T, r_M)}{Var(r_M)} (\mathbb{E}[r_M] - r_f) \right)$$

#### Demonstration

$$\begin{aligned} P_i^0 &= \frac{1}{1 + r_f} \left( \mathbb{E}[P_i^T] - \frac{Cov(P_i^T, r_M)}{Var(r_M)} (\mathbb{E}[r_M] - r_f) \right) \\ \Leftrightarrow 1 &= \frac{1}{1 + r_f} \left( \frac{\mathbb{E}[P_i^T]}{P_i^0} - \frac{Cov(\frac{P_i^T}{P_i^0}, r_M)}{Var(r_M)} (\mathbb{E}[r_M] - r_f) \right) \\ \Leftrightarrow 1 &= \frac{1}{1 + r_f} \left( \mathbb{E}[1 + r_i] - \frac{Cov(\frac{P_i^T}{P_i^0} - 1, r_M)}{Var(r_M)} (\mathbb{E}[r_M] - r_f) \right) \\ \Leftrightarrow 1 &= \frac{1}{1 + r_f} \left( \mathbb{E}[1 + r_i] - \frac{Cov(r_i, r_M)}{Var(r_M)} (\mathbb{E}[r_M] - r_f) \right) \end{aligned}$$

$$\begin{aligned}\Leftrightarrow 1 &= \frac{1}{1+r_f} (\mathbb{E}[1+r_i] - (\mathbb{E}[r_i] - r_f)) \\ \Leftrightarrow 1 &= \frac{1}{1+r_f} (1+r_f)\end{aligned}$$

### 3.11 Arbitrage pricing theory model

Arbitrage pricing theory (APT) is a multi-factor model for asset pricing which relates various macro-economic (systematic) risk variables to the pricing of financial assets. Proposed by economist Stephen Ross in 1976, it is widely believed to be an improved alternative to its predecessor, the Capital Asset Pricing Model (CAPM) in section 6.3.10.

APT is founded upon the law of one price. The law of one price (LOOP) states that in the absence of trade frictions (such as transport costs and tariffs), and under conditions of free competition and price flexibility (where no individual sellers or buyers have power to manipulate prices and prices can freely adjust), identical goods sold in different locations must sell for the same price when prices are expressed in a common currency.

APT argues that when **opportunities for arbitrage are exhausted** in a given period, then the expected return of an asset is a linear function of various factors where sensitivities of each factor is represented by beta coefficient.

#### 3.11.1 Factor analysis

Factor analysis (factor structure , factor model, risk factors) is a statistical method used to describe variability of observed variables in terms of a potentially lower number of unobserved variables called factors. For example, it is possible that variations in six observed variables mainly reflect the variations in two unobserved (underlying) variables. Factor analysis searches for such joint variations in response to unobserved latent variables. The observed variables are modelled as linear combinations of the potential factors plus "error" terms, hence factor analysis can be thought of as a special case of *errors-in-variables models*.

In finance, we use factor structure to express return in term of systematic risks. We will see it clearly in multifactor pricing equation.

#### 3.11.2 Multifactor pricing equation

In APT model, risky asset returns are said to follow a **factor structure** if they can be expressed as:

$$r_j = a_j + \beta_{j1}f_1 + \beta_{j2}f_2 + \dots + \beta_{jn}f_n + \epsilon_j$$

where:

- $r_j$  is return for asset  $j$ .

- $a_j$  is a constant for asset  $j$ .
- $f_1, \dots, f_n$  are (systematic) factors (risks). A systematic factor, also known as a systematic risk factor or a market factor, refers to a common source of risk that affects a broad set of assets or securities within a particular market or industry.
- $\beta_{j_n}$  is the sensitivity of the asset to factor  $n$ , also called factor loading.
- $\epsilon_j$  is the idiosyncratic random shock of asset (noise), with mean zero and  $Var(\epsilon_j) = \sigma_j$

In this multifactor pricing equation, parameters need to estimate are :

- For each asset  $j : a_j, \beta_{j_1}, \dots, \beta_{j_n}$  and  $\sigma_j$ .
- For factors :  $\mathbb{E}[f_1], \dots, \mathbb{E}[f_n]$  and  $Cov(f_k, f_l)$  where  $k, l \in \{1, \dots, n\}$ .

### 3.11.2.1 Quick analysis

Always in this multifactor pricing equation, constant  $a_j$  can be understood as sum of two components : expected return for accepting unsystematic risk (6.3.9.1) and expected return for accepting systematic risk (6.3.9.2), equivalently  $a_j = \mathbb{E}_{unsystematic}[r_j] + \mathbb{E}_{systematic}[r_j]$ .

- $\mathbb{E}_{unsystematic}[r_j]$  is to recompose for unsystematic risk related to  $\sigma_j$
- $\mathbb{E}_{systematic}[r_j]$  is to recompose for unsystematic risks related to  $f_1, \dots, f_n$ .

### 3.11.3 The APT equation(s)

We remind that in a **well-diversified** portfolio, there is no systematic risk. Unsystematic risk is often tied to a specific company or industry and can be avoided by building a well-diversified portfolio. Then for a well-diversified portfolio  $j$  with several sources of systematic risk, the expected return (for accepting systematic risk)  $\mathbb{E}[r_j] = \mathbb{E}_{systematic}[r_j]$  of above equation is given by:

$$\mathbb{E}[r_j] = r_f + \beta_{j_1}\lambda_1 + \beta_{j_2}\lambda_2 + \dots + \beta_{j_n}\lambda_n$$

where:

- $\mathbb{E}[r_j]$  is the expected return on a well-diversified portfolio  $j$ .
- $\lambda_n$  is the risk premium of the factor.
- $r_f$  is the risk-free rate.

In APT equation, if portfolio  $j$  is independent with all systematic risks, which means  $\beta_{j_1} = \beta_{j_n} = 0$  then  $\mathbb{E}[r_j] = r_f$ . Therefore  $\mathbb{E}[r_j]$  is equal to the risk-free rate plus premiums by accepting systematic risks.

### 3.11.3.1 Mimicking portfolio

From APT equation, if we take a portfolio composed of only factor  $k$  as asset, we must have :

$$\mathbb{E}[f_k] = r_f + \lambda_k$$

where  $\beta_k = 1$  and  $(n - 1)$  other  $\beta$  are 0.

Then we can rewrite APT equation by :

$$\mathbb{E}[r_j] - r_f = \beta_{j1}(\mathbb{E}[f_1] - r_f) + \beta_{j2}(\mathbb{E}[f_2] - r_f) + \dots + \beta_{jn}(\mathbb{E}[f_n] - r_f)$$

Here if there is only one factor, it is exactly CAPM model 6.3.10.

### 3.11.4 Model by matrix

For a set of  $m$  assets with returns  $r \in \mathbb{R}^m$ , constants  $a \in \mathbb{R}^m$ , factor loadings (parameters)  $B \in \mathbb{R}^{m \times n}$ , factors  $F \in \mathbb{R}^n$ , error (noise)  $\varepsilon \in \mathbb{R}^m$ , a general (matrix) factor model that is used in APT is:

$$r = a + BF + \varepsilon$$

where  $\varepsilon \sim N(0, \Psi)$  and  $F \sim N(\mu, \Omega)$ ,  $\mu$  is the expected risk premium vector and  $\Omega$  is the factor covariance matrix. Assuming that the noise terms and factors are uncorrelated, the mean and covariance for the returns are respectively:

$$E(r) = a + B\mu$$

$$\begin{aligned} Cov(r) &= Var(BF) + 2Cov(BF, \varepsilon) + Var(\varepsilon) \\ &= Var(BF) + Var(\varepsilon) \\ &= B\Omega B^T + \Psi \end{aligned}$$

**Solving for matrix  $B$**  : It is generally assumed that we know the factors  $F$  in a model, which allows least squares to be utilized. If factors  $F$  are not known, we assume that the factors are latent variables and use :

- Principal component analysis (PCA)
- Correspondence analysis (CA)
- Multiple correspondence analysis (MCA)
- Factor analysis of mixed data (FAMD)

### 3.11.5 Example

Assume that we want to apply the APT formula for a well-diversified portfolio  $a$ . The riskless rate of return is 2%. Two similar well-diversified portfolios (indices) are the S&P 500 and the Dow Jones Industrial Average (DJIA). Two factors considered here are inflation and gross domestic product (GDP). The betas of inflation and GDP on the S&P 500 are 0.5 and 3.3, respectively. The betas of inflation and GDP on the DJIA are 1 and 4.5, respectively. The S&P 500 has expected return of 10%, and the DJIA has expected return of 8%.

$$\begin{cases} \mathbb{E}[r_{S\&P}] = 0.02 + \beta_{S\&P}^{inflation} \lambda_{inflation} + \beta_{S\&P}^{GDP} \lambda_{GDP} \\ \mathbb{E}[r_{DJIA}] = 0.02 + \beta_{DJIA}^{inflation} \lambda_{inflation} + \beta_{DJIA}^{GDP} \lambda_{GDP} \end{cases}$$

or

$$\begin{cases} 0.1 = 0.02 + 0.5\lambda_{inflation} + 3.3\lambda_{GDP} \\ 0.08 = 0.02 + \lambda_{inflation} + 4.5\lambda_{GDP} \end{cases}$$

Solving this system, we have  $\lambda_{inflation} = -0.154285$  and  $\lambda_{GDP} = 0.047619$ , then expected return for portfolio  $a$  is :

$$\mathbb{E}[r_a] = 0.02 + \beta_a^{inflation}(-0.154285) + \beta_a^{GDP}0.047619$$

## 3.12 Alpha

We saw in CAPM (6.3.10) and in APT (6.3.11) the beta, which the sensitivity to non-diversifiable or systematic factor. In this section, we try to understand alpha. Before going into definition of alpha, we discover some helpful terminologies :

### 3.12.1 Helpful terminologies

*Passive Management:* Also known as *passive investing* or *index investing*, passive management aims to replicate the performance of a specific market index or benchmark. The goal is to achieve a return that closely matches the performance of the chosen index, rather than outperforming it. Passive managers typically employ a buy-and-hold strategy and minimize trading activity. Examples of passive management strategies include:

- **Index Funds:** These funds replicate the performance of a specific market index, such as the S&P 500. They hold a diversified portfolio of securities that closely mimics the index composition.
- **Exchange-Traded Funds (ETFs):** ETFs are similar to index funds but trade on stock exchanges (e.g. S&P 500) like individual stocks. They provide exposure to various asset classes or market sectors and track specific indices.

- Target-Date Funds: These funds are commonly used for retirement savings and automatically adjust the asset allocation over time based on a predetermined target retirement date.

*Active Management:* Active management involves making investment decisions with the goal of **outperforming** the market or a specific benchmark. Active managers employ research, analysis, and trading strategies to select securities they believe will generate superior returns. They aim to beat the market by identifying mispriced assets or taking advantage of market trends. Active management typically involves higher costs, including management fees and transaction costs, as compared to passive management. Additionally, the performance of active managers can vary, with some outperforming the market and others underperforming. Examples of active management strategies include:

- Mutual Funds: Mutual funds are professionally managed investment vehicles that actively select and manage a diversified portfolio of securities. Fund managers make ongoing investment decisions to generate returns higher than the benchmark.
- Hedge Funds: Hedge funds are investment partnerships that employ various strategies to generate absolute returns. Hedge fund managers actively seek opportunities in multiple markets, often using complex investment techniques.
- Quantitative Strategies: These strategies rely on computer-based algorithms and quantitative models to make investment decisions. They analyze large amounts of data and historical patterns to identify trading opportunities.

*Active return*, also known as active performance or excess return, refers to the difference in investment performance between an actively managed portfolio and a benchmark or passive investment strategy. It measures the extent to which an investment portfolio has outperformed or underperformed its benchmark. Active return is calculated by subtracting the return of the benchmark from the return of the portfolio. The formula for active return is as follows:

$$\text{ActiveReturn} = \text{PortfolioReturn} - \text{BenchmarkReturn}$$

A positive active return indicates that the portfolio has outperformed the benchmark, while a negative active return signifies underperformance. Active return is a key metric used to evaluate the success of active management strategies. It helps investors assess whether the additional risks and costs associated with active management have led to superior investment performance compared to a passive approach.

It's important to note that active return is just one aspect of evaluating an actively managed portfolio. Other factors, such as risk-adjusted return (e.g. Sharp ratio (6.3.4), Treynor Ratio, ...), consistency of performance, and investment style, ... should also be considered when assessing the effectiveness of an active management strategy.

### 3.12.2 Definition of alpha

Alpha is a **measure** of the *active return* on an investment. An alpha of 1% means the investment is 1% better than the benchmark market during the same period; a negative alpha means the investment underperforms the market. Alpha, along with beta, is one of two key coefficients in the capital asset pricing model (6.3.10) used in modern portfolio theory. Alpha is closely related to other quantities that contain risk such as *standard deviation*, *R-squared* (3.2.12.3) and *Sharpe ratio*.

As funds (using Active Management) include various fees, these funds has to maintain an alpha greater than its fees in order to provide positive gains, compared with a standard index fund. Historically, the vast majority of traditional funds have had negative alphas, which has led to a *capital flight* ( money rapidly flows out of investment) to index funds and to non-traditional hedge funds.

It is also possible to analyze a portfolio of investments and calculate a theoretical performance alpha, most commonly using the capital asset pricing model (CAPM).

#### 3.12.2.1 Definition of alpha in single index model

The alpha coefficient ( $\alpha_i$ ) is a parameter in the **single index model** (SIM) :

$$R_{i,t} - R_f = \alpha_i + \beta_i(R_{M,t} - R_f) + \varepsilon_{i,t}$$

where the following inputs are:

- $t$  : is time instant
- $R_i$  : the portfolio return
- $R_M$ : the market return
- $R_f$ : the risk-free rate of return
- $\beta_i$ : the beta of the portfolio

In this definition, alpha and beta are both parameter to be estimated by fitting the SIM with data of portfolio return and market return.

Alpha is also the *intercept* of the **security characteristic line** (SCL), as in figure 6.10 (do not confuse with SML in figure 6.9)

It can be shown that in an efficient market, the expected value of the alpha coefficient is zero (the assumption in CAPM).

#### 3.12.2.2 Definition by Jensen or Jensen's alpha

In this case, alpha is calculated based by the same formula as before :

$$\alpha_i = R_{i,t} - R_f - \beta_i(R_{M,t} - R_f)$$

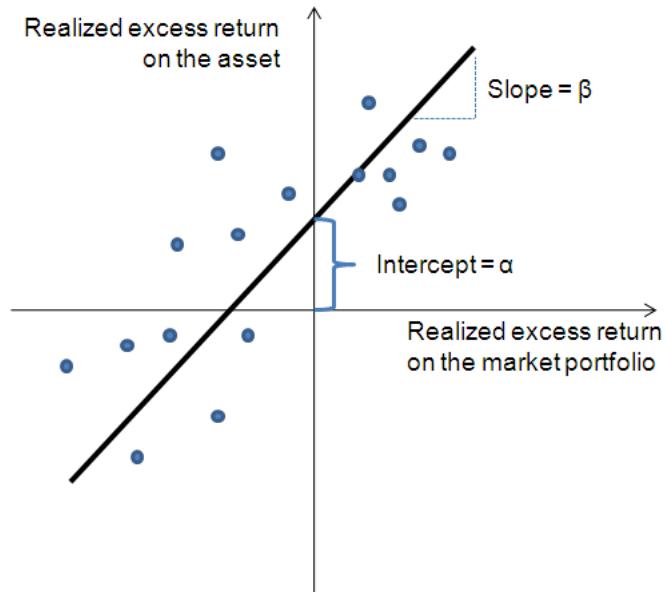


Figure 6.10: Security characteristic line

However, in this definition, only beta had been estimated from CAPM, and alpha is just inferred with the formula.

#### Example

Let's us assume Faye's portfolio return is 8%, the risk-free rate is 11%, and a beta of 1.8% against a benchmark of 9%. Then alpha is :

$$0.08 - 0.11 - 1.8 \times (0.09 - 0.11) = 0.006$$

We observe that even the return is less than risk-free rate, we can have positive alpha.

### 3.13 Dividend discount model

In finance and investing, the dividend discount model (DDM) is a method of valuing the price of a company's stock based on **the fact that its stock is worth the sum of all of its future dividend payments, discounted back to their present value**. It's means we buy a stock in order to exchange a right to get dividend forever. This DDM is sometimes referred to as the Gordon Growth Model (GGM).

Let's note:

- $P$  is the current stock price.
- $g$  is the constant growth rate for the dividends.

- $r$  is the constant cost of equity capital (return rate).
- $D_t$  is the value of dividends at the end of the  $t$  first periods.

Assumption: the price of stock, which is the money that investor must pay (never reselling after), is equal to the sum infinity of discounted dividends (back to present value). Hence:

$$P = \sum_{t=1}^{+\infty} \frac{D_t}{(1+r)^t} = \sum_{t=1}^{+\infty} \frac{D_1(1+g)^t}{(1+r)^t}$$

As usually (hypothesis, but not always true) that  $g < r$  means that growth rate of dividend is less than one of stock's price. Using geometric suite, we have :

$$P = \frac{D_1}{r-g}$$

Conversely, if we know stock's price  $P$  but not the cost of equity (return rate)  $r$ , then  $r$  can be inferred by :

$$r = \frac{D_1}{P} + g$$

### 3.13.1 Example

A company is expected to pay an annual dividend of \$3 next year, and its stock is currently trading at \$100 a share. The company has been steadily raising its dividend each year at a 4% growth rate. Then the return rate can be inferred by :

$$r = (3/100) + 0.04 = 7\%$$

## 3.14 Fama-French three-factor model

This model is an extension of the traditional CAPM described in section 6.3.10 and it is compatible to APT model 6.3.11. In CAPM, we use only one variable  $\beta$  to express the return of a portfolio in term of the return of the market. In contrast, the Fama-French model uses three variables. Fama and French started with the observation that two classes of stocks have tended to do better than the market :

- Stock with small caps (market capitalization)
- Stock with high book-to-market ratio (B/P or B/M) (see 1.19)

They then added these two factors to CAPM to reflect a portfolio's exposure to these two classes :

$$\mathbb{E}[r_i] - r_f = \beta_i(\mathbb{E}[r_M] - r_f) + s_i \cdot SMB + h_i \cdot HML + \alpha_i$$

Here  $r_i$  is rate of return of portfolio  $i$ ,  $r_f$  is the risk-free return rate, and  $r_m$  is the return of the market portfolio. The parameter  $\beta$  is analogous to the classical one but not equal to it, since there are now two additional factors to do some of the work. SMB stands for Small (market capitalization) Minus Big and HML for High (book-to-market ratio) Minus Low.

As mentioned before, this model is inspired by an observation, then there is no theoretical demonstration. However, this model have been showed to be correct in empirical analysis.

The SMB factor is the (monthly) average return of the small-cap stocks minus the average return of the large-cap stocks. The HML factor is calculated as the (monthly) average return of high B/M stocks minus the average return of low B/M stocks.

### 3.14.1 Estimating parameters

Given  $E(r_i), r_f, E(r_m)$ , SMB, HML, the parameters need to be estimated is  $\theta = \{\beta, s, h, \alpha\}$ . One of methods is using OLS 3.2.12.2 or OLS casted as Method of moments (3.2.13.5.2) or generalized method of moments 3.2.13 if the number of moment is bigger than 4.

## 3.15 Forward price

Forward price is the price at which two parties agree to buy or sell an asset at a future date. This price is agreed upon at the time when contract made, but the actual exchange of the asset and payment occurs on the future date specified in the contract. The forward price is typically based on the current market price of the asset, adjusted for any costs or interest rates that may be applicable.

$$F = S_0 e^{(r+s-q)T} - \sum_{i=1}^N D_i e^{(r+s-q)(T-t_i)}$$

where :

- $F$  is the forward price to be paid at time  $T$ .
- $r$  is the risk-free interest rate.
- $s$  is the storage cost.
- $q$  is the convenience yield. Its represents the benefits for holding asset.
- $S_0$  is the spot price of the asset (i.e. what it would sell for at time 0 or simply current price).
- $D_i$  is a dividend that is guaranteed to be paid at time  $t_i$  where  $0 < t_i < T$ .

For example, the asset is a house. Holding a house can make a rental benefit of 2% per annum, then 2% is the convenience yield  $q$ . However, we need to pay land tax of 1% (imagine the house need a ground to base on, and this ground is of state). Then the land tax of 1% is storage cost. In general, we need to consider all win and loss while holding an asset.

### 3.16 Weighted average cost of capital

The weighted average cost of capital (WACC) is a financial metric that represents the average rate of return a company needs to generate in order to cover the costs of its various sources of financing, including **equity** and **debt**. It provides a measure of the minimum acceptable return a company should achieve on its investments. Both equity and debt are used to raise capital.

#### 3.16.1 General definition

$$WACC = \frac{\sum_{i=1}^N r_i \cdot MVO_i}{\sum_{i=1}^N MVO_i} = w_i r_i$$

where  $N$  is the number of sources of capital, which include equity (popular or preferred security,...) and (bond, debt,...),  $r_i$  is the **required** rate of return (or cost of equity) security  $i$ , MVO means the market value of all outstanding securities.

The market value of all outstanding securities refers to the total value of all publicly traded securities (contrary to no-publicly traded securities) issued by a company or available in the market.

**Example** For example, let's consider a company with the following outstanding securities:

Common Stock:

- Market price per share: \$50
- Number of shares outstanding: 10 million

Preferred Stock:

- Market price per share: \$100
- Number of shares outstanding: 2 million

Bonds:

- Market price per bond: \$1,000
- Number of bonds outstanding: 5,000

Then

- MVO of common stock =  $\$50 \times 10,000,000 = \$500,000,000$

- MVO of preferred stock =  $\$100 \times 2,000,000 = \$200,000,000$

- MVO of bonds =  $\$1,000 \times 5,000 = \$5,000,000$

Finally

$$WACC = \frac{500}{705}r_{CS} + \frac{200}{705}r_{PS} + \frac{5}{705}r_B$$

where  $r_{CS}$ ,  $r_{PS}$ ,  $r_B$  are respectively the rate of return of common stock, preferred stock and the interest rate of bond.

### 3.16.2 Short definition

$$WACC = \frac{E}{D+E}R_e + \frac{D}{D+E}R_d$$

where  $E$  is the total shareholder's equity,  $D$  is the total debt,  $R_e$  is the cost of equity,  $R_d$  is the cost of debt.

Sometime, there is an intervention of the *corporate tax rate*  $t$ . The corporate tax rate refers to the percentage of profits that corporations are required to pay in taxes on their taxable income. The corporate tax rate can vary between countries and jurisdictions and may be subject to changes over time.

The main point is that the profit is taxable and the debt is not. Hence, if we want a return after tax of  $a\%$ , then the return before tax is  $\frac{a}{1-t}$ . Then

$$\begin{aligned} WACC \frac{1}{(1-t)} &= \frac{E}{D+E}R_e \frac{1}{(1-t)} + \frac{D}{D+E}R_d \\ \Leftrightarrow WACC &= \frac{E}{D+E}R_e + \frac{D}{D+E}R_d(1-t) \end{aligned}$$

**Example** A company that have the following state

- Equity: \$4,000,000
- Debt: \$6,000,000
- Cost of equity ( $R_e$ ): 10%
- Cost of debt ( $R_d$ ): 5%
- Corporate tax rate: 30%

Next, calculate the WACC using the formula:

$$WACC = (0.40 \cdot 0.10) + (0.60 \cdot 0.05)(1 - 0.30)$$

$$WACC = 0.061 = 6.1\%$$

This means that the company needs to generate a return of at least 6.1% on its investments to cover its financing costs and provide value to its shareholders.

## 4 Stock simulation

*Why do we usually use returns and not prices to model financial data in time series analysis ?* Because prices usually have a unit root, while returns can be assumed to be stationary and it is eventually considered to be normal distribution.

Next, we use Monte Carlo to simulate evolution of prices, give an initial price  $t_0$ .

### 4.1 By arithmetic return

Let's note return rates of given portfolio, with arithmetic return by  $r_n = \frac{S_{n+1} - S_n}{S_n}$ , where  $S_n$  is the price of stock at instant  $n$ . Its  $\mu$  and  $\sigma$  are estimated by past data :

$$\begin{cases} \mu = \mathbb{E}[r_n] \\ \sigma = \sqrt{\text{Var}[r_n]} \end{cases}$$

Then, for a future price simulation, we can sample :

$$\frac{S_{n+1}}{S_n} - 1 \leftarrow \mathcal{N}(\mu, \sigma^2)$$

With a given  $S_0$ , we can have a deterministic evolution of price from  $S_0$  to  $S_N$ , notably with  $0 \leq k \leq N$  :

$$S_k = S_0 ((1 + \mu) + \varepsilon \sigma)^k$$

where  $\varepsilon$  is sampled from  $\mathcal{N}(0, 1)$ .

### 4.2 By log return with GBM

Discretization : suppose that the time from instant  $n$  to  $(n + 1)$  is  $\Delta$ . Then we can express  $r$  as a function of time  $t$ :

$$r(t) = \mu \frac{t}{\Delta} + \sigma B_{\frac{t}{\Delta}}$$

where  $B$  means Brownian motion. Note that if  $t = \Delta$ , we get  $r \leftarrow \mathcal{N}(\mu, \sigma^2)$ . For the simplicity, we set  $\Delta$  as a unit as 1 :

$$r(t) = \mu t + \sigma B_t$$

where  $\mu$  and  $\sigma$  are estimated by

$$\begin{cases} \mu = \mathbb{E}[r_n] \\ \sigma = \sqrt{\text{Var}[r_n]} \end{cases}$$

with  $r_n = \log(\frac{S_{n+1}}{S_n})$ .

Taking the derivative in term  $t$  :

$$dr(t) = \mu dt + \sigma dB_t$$

Remind that  $r(t)$  is the return rate after a **time length**  $t$  from initial instant  $t_0$ , then we have:

$$r(t) = \log\left(\frac{S_{t_0+t}}{S_{t_0}}\right)$$

For simplicity, set  $t_0 = 0$  and take the derivative in term  $t$  for  $r(t)$ :

$$dr(t) = \frac{dS_t}{S_t}$$

Hence, by matching  $\frac{dS_t}{S_t}$  and  $\mu dt + \sigma dB_t$  via  $dr(t)$ :

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

**This leads to the origine for the Geometric Brownian Motion** (see 4.16.3.9).

By applying the result of GBM :

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right)$$

or if  $t = \Delta \rightarrow 0$ :

$$\frac{S_\Delta - S_0}{S_0} = \log \frac{S_\Delta}{S_0} = (\mu - \frac{1}{2}\sigma^2)\Delta + \sigma B_\Delta$$

where  $(\mu - \frac{1}{2}\sigma^2)$  is also called **drift**.

**Simulation.** For a future price simulation, we can sample :

$$\frac{S_{n+1}}{S_n} = \exp\left((\mu - \frac{1}{2}\sigma^2) + \sigma x\right)$$

where  $x \leftarrow \mathcal{N}(0, 1)$  or taking directly :

$$S_{n+1} = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)n + \sigma \sqrt{n}x\right)$$

where  $x \leftarrow \mathcal{N}(0, 1)$ .

### 4.3 Which simulation to use ?

We may come back to section 6.2.2.10 to see when use use arithmetic return and logarithmic return. For simulation :

- If in case without reinvestment and the periode is long such as year, we can apply arithmetic return.
- If in case with reinvestment and the periode is short such as day, we apply logarithmic return. For stock simulation by GBM, it is obligatory to use logarithmic return.

## 5 Option pricing

### 5.1 Black–Scholes model

This model is to calculate value of options (1.10) and only for **European** options. Hypothesis for Black–Scholes model (but can be relaxed):

- Efficient market
- Absence of transition cost
- No dividend payment
- Know Volatility and risk-free

E.g, an investor want to realize an call option :

- Share price > Strike price : Exercise
- Strike price > Share price : Don't Exercise

#### 5.1.1 In and out the money

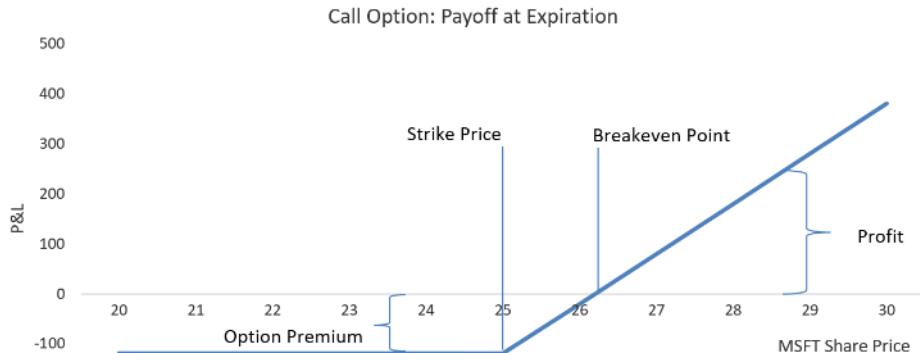


Figure 6.11: Call option payoff. 100 shares, cost for call option is \$120. Strike price is \$25. “In-the-money” is the zone after strike price and “out-of-the-money” is the zone before strike price point. “At the price” is on strike price point.

“In-the-money” is a term used in options trading to describe a situation where an option has intrinsic value. An option is considered “in-the-money” if the current market price of the underlying asset, such as a stock, is favorable in comparison to the option’s strike price.

For example, if an investor holds a *call option* with a strike price of \$50 and the stock price is currently trading at \$60, the call option is “in-the-money” because the holder has the right to buy the stock at \$50, which is lower than its current market price. The same, if an investor holds a *put option* with a strike

price of \$50 and the stock price is currently trading at \$40, the put option is “in-the-money” because the holder has the right to sell the stock at \$50, which is higher than its current market price.

Being “in-the-money” means the option has intrinsic value and, if exercised, would result in a profit for the option holder. Conversely, if an option is “out-of-the-money”, it means it has no intrinsic value and would result in a loss if exercised.

### 5.1.2 Black–Scholes reasoning

Let  $C$  be the cost for buying an option (option premium). The Black-Scholes formula calculates  $C$  from the following five data:

- $S_0$  is the current value of the stock.
- $T$  is the time remaining for the option before expiration.
- $K$  is the strike price set by the option.
- $r$  is the risk-free rate.
- $\sigma$  or  $s$  is the annualized volatility of the stock price. It is the volatility of daily log return, multiplied by the number of trading days of year. The volatility  $\sigma$  is difficult to assess. Two analysts may have significantly different values of  $\sigma$ .

For an option under **risk neutral measure** (1.28.3), the cost multiplied by interest rate after the time  $T$  must be equal to the expected value of payoff :

$$C \exp(rT) = \mathbb{E}[(S_T - K)^+]$$

equivalently,

$$\begin{aligned} C \exp(rT) &= \mathbb{E}[(S_T - K) \mathbb{1}_{S_T > K}] \\ &= \mathbb{E}(S_T \mathbb{1}_{S_T > K}) - \mathbb{E}(K \mathbb{1}_{S_T > K}) \\ &= \mathbb{E}(S_T \mathbb{1}_{S_T > K}) - P(S_T > K)K \end{aligned}$$

Applying stock simulation (in sec 6.4.2) :

$$S_T = S_0 \exp \left( \left( r - \frac{1}{2}s^2 \right) T + sW_T \right)$$

where there is no  $\mu$  (estimated return rate) in this GBM since we did transform to **risk neutral measure** see Girsanov’s theorem 4.20.3.6. Furthermore, in Black-Scholes equation (6.6.3), we do not also have this term  $\mu$ .

- For the second term  $P(S_T > K)K$ :

$$\begin{aligned}
P(S_T > K) &= P \left( S_0 \exp \left( (r - \frac{1}{2}s^2)T + sW_T \right) > K \right) \\
&= P \left( (r - \frac{1}{2}s^2)T + sW_T > \ln(\frac{K}{S_0}) \right) \\
&= P \left( \frac{1}{\sqrt{T}}W_T > \frac{\ln(\frac{K}{S_0}) - (r - \frac{1}{2}s^2)T}{s\sqrt{T}} \right) \\
&= P \left( \frac{1}{\sqrt{T}}W_T < \frac{(r - \frac{1}{2}s^2)T - \ln(\frac{K}{S_0})}{s\sqrt{T}} \right)
\end{aligned}$$

Let's  $d_2 = \frac{(r - \frac{1}{2}s^2)T - \ln(\frac{K}{S_0})}{s\sqrt{T}}$  then  $P(S_T > K) = N(d_2)$  where  $N$  is cumulative distribution function of normal distribution.

- For the first term  $\mathbb{E}(S_T \mathbb{1}_{S_T > K})$ , let  $f_{S_T}$  be the density function of  $S_T$  which is **log-normal distribution**:

$$f_{S_T}(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp \left( -\frac{(\ln x - \mu)^2}{2\sigma^2} \right)$$

where  $\mu = \ln(S_0) + (r - \frac{1}{2}s^2)T$  and  $\sigma = s\sqrt{T}$ . Then

$$\begin{aligned}
\mathbb{E}(S_T \mathbb{1}_{S_T > K}) &= \int_K^{+\infty} x f_{S_T}(x) dx \\
&= \int_K^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp \left( -\frac{(\ln x - \mu)^2}{2\sigma^2} \right) dx \\
&= \int_K^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp \left( -\frac{(\ln x - \mu)^2}{2\sigma^2} \right) dx \\
&= \int_{\frac{\ln K - \mu}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{y^2}{2} \right) \exp(\sigma y + \mu) dy \\
&= \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{y^2}{2} + \sigma y + \mu \right) dy \\
&= \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2}(y - \sigma)^2 + \mu - \frac{\sigma^2}{2} \right) dy \\
&= \int_{-d_2 - \sigma}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2}z^2 \right) \exp \left( \mu + \frac{\sigma^2}{2} \right) dz \\
&= N(d_1) \exp \left( \mu + \frac{\sigma^2}{2} \right) \\
&= N(d_1) \exp(\ln S_0 + rT)
\end{aligned}$$

$$= N(d_1)S_0 \exp(rT)$$

$$\text{where } d_1 = d_2 + \sigma = \frac{(r + \frac{1}{2}s^2)T - \ln(\frac{K}{S_0})}{s\sqrt{T}}.$$

Finally, we get the Black–Scholes formula :

$$C = N(d_1)S_0 - N(d_2)K \exp(-rT)$$

## 5.2 Binomial options pricing model

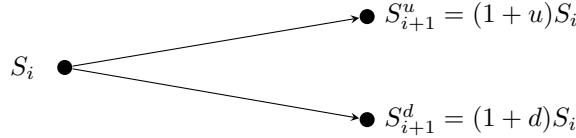
Invented first by Cox, Ross et Rubinstein (1979). There are also other versions.

- In finance, the binomial options pricing model (BOPM) provides a generalizable numerical method for the valuation of options.
- The BOPM approach has been widely used since it is able to handle a variety of conditions for which other models cannot easily be applied. This is largely because the BOPM is based on the description of an underlying instrument over a period of time (with discretization) rather than a single point.
- As Black-Scholes model can only applicable to EU options, then for US options or Bermudan options, which holder exercise the option at any time (before the expiration date), we must have a tool to value options. Hence BOPM.
- Although computationally slower than the Black–Scholes formula, it is more accurate, particularly for longer-dated options on securities with dividend payments.
- For options with several sources of uncertainty (e.g., real options) and for options with complicated features (e.g., Asian options), BOPM are less practical due to several difficulties, and Monte Carlo option models are commonly used instead.

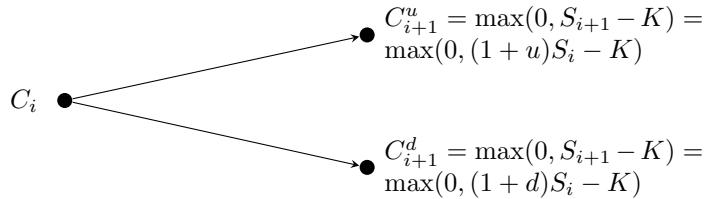
We discover BOPM through single-period, two-period and multi-period cases.

### 5.2.1 Single-period

Given an underlying instrument  $A$ , let  $S_i$  be its price and  $C_i$  be the option price of this underlying instrument  $A$ , both at instant  $i$ . As named “binomial”, we consider only two directions, the price goes up and goes down after an instant unit, with respectively rate  $u$  and  $d$ . Hence :



and for a call option :



**Keypoint.** If we pay \$\\$C\_i\$ to buy option price of underlying instrument \$A\$ to get a payoff, then **it is equivalent** to create a portfolio of \$\\$C\_i\$ composed of underlying instrument \$A\$ and a risk free asset to get a same payoff.

Let \$l\$ be the quantity of underlying instrument \$A\$ and \$F\$ be the amount of risk free asset with risk free rate \$r\$ (after one period), we then have the following equations :

$$\begin{aligned} C_i &= lS_i + F \\ C_{i+1}^u &= lS_i(1+u) + F(1+r) \\ C_{i+1}^d &= lS_i(1+d) + F(1+r) \end{aligned}$$

Here we have three unknown variables \$l, F, C\_i\$ and the rest is known. Solving this gives :

$$\begin{aligned} l &= \frac{C_{i+1}^u - C_{i+1}^d}{(u-d)S_i} = \frac{C_{i+1}^u - C_{i+1}^d}{S_{i+1}^u - S_{i+1}^d} \\ F &= \frac{(1+u)C_{i+1}^d - (1+d)C_{i+1}^u}{(u-d)(1+r)} \\ C_i &= lS_i + F \end{aligned}$$

For a call option :

- \$C\_{i+1}^u \geq C\_{i+1}^d\$ and obviously \$S\_{i+1}^u > S\_{i+1}^d\$, then \$l \geq 0\$.
- In addition,

$$l = \frac{\max(0, S_i(1+u) - K) - \max(0, S_i(1+d) - K)}{(u-d)S_i},$$

then it is easy to prove that \$l \leq 1\$.

- About amount of risk free  $F$  :

$$\begin{aligned} F &= \frac{(1+u)C_{i+1}^d - (1+d)C_{i+1}^u}{(u-d)(1+r)} \\ &= \frac{(1+u)\max(0, S_i(1+d) - K) - (1+d)\max(0, S_i(1+u) - K)}{(u-d)(1+r)} \\ &\leq 0 \end{aligned}$$

This means that, in the equivalent portfolio, we need to **borrow** money.

Until here, we have the option price is :

$$C_i = lS_i + F$$

If we express  $C_i$  by  $C_{i+1}^u$  and  $C_{i+1}^d$  :

$$\begin{aligned} C_i &= lS_i + F \\ &= \frac{C_{i+1}^u - C_{i+1}^d}{(u-d)S_i} S_i + \frac{(1+u)C_{i+1}^d - (1+d)C_{i+1}^u}{(u-d)(1+r)} \\ &= \frac{C_{i+1}^u - C_{i+1}^d}{(u-d)} + \frac{(1+u)C_{i+1}^d - (1+d)C_{i+1}^u}{(u-d)(1+r)} \\ &= \frac{\frac{r-d}{u-d}C_{i+1}^u + \frac{u-r}{u-d}C_{i+1}^d}{(1+r)} \\ &= \frac{pC_{i+1}^u + (1-p)C_{i+1}^d}{(1+r)} \end{aligned}$$

where  $p = \frac{r-d}{u-d}$  which can be understood as probability that price of underlying instrument goes up. This equation also means that the option price is the expected payoff, discounted to the instant  $i$ .

### 5.2.1.1 Interpreting $l$

Thus  $l$  is  $\Delta$  in Greeks (see 6.6.1.1) :

$$\Delta = \frac{\partial C}{\partial S} = l$$

$l$  is also called the riskless hedge ratio. Suppose that we have a portfolio of value  $f$  that contains  $n_A$  underlying instrument  $A$  and other ones:

$$f = n_A S_A + n_B S_B + n_c S_c + \dots$$

Then the sensibility of portfolio w.r.t. price of  $A$  is  $\frac{\partial f}{\partial S_A} = n_A$ , to reduce this sensibility to 0 (**we do not want significant variation when  $S_A$  changes**), one way to do that is to write (sell)  $\frac{n_A}{l}$  call options of underlying instrument  $A$ , with price  $C_A$  :

$$f = n_A S_A - \frac{n_A}{l} C_A + n_B S_B + n_c S_c + \dots$$

now the sensibility of portfolio w.r.t. price of  $A$  becomes  $\frac{\partial f}{\partial S_A} = n_A - \frac{n_A}{l} \frac{\partial C_A}{\partial S_A} = 0$ .

Note that this riskless hedge ratio  $l$  changes over time, then to keep the sensibility closed to 0, we need the dynamic hedging, which means to change the (configuration of) portfolio when stock price moves.

### 5.2.1.2 Binomial simulation

Suppose that we have a stock  $X$  that has expected return  $\mu$  and volatility  $\sigma$ . If we want to simulate this stock with a binomial model :

- with probability  $p$  of increasing rate  $u$ .
- with probability  $1 - p$  of decreasing rate  $d$ .

Then we must have :

$$\begin{cases} pu + (1 - p)d = \mu \\ pu^2 + (1 - p)d^2 - \mu^2 = \sigma^2 \end{cases}$$

or

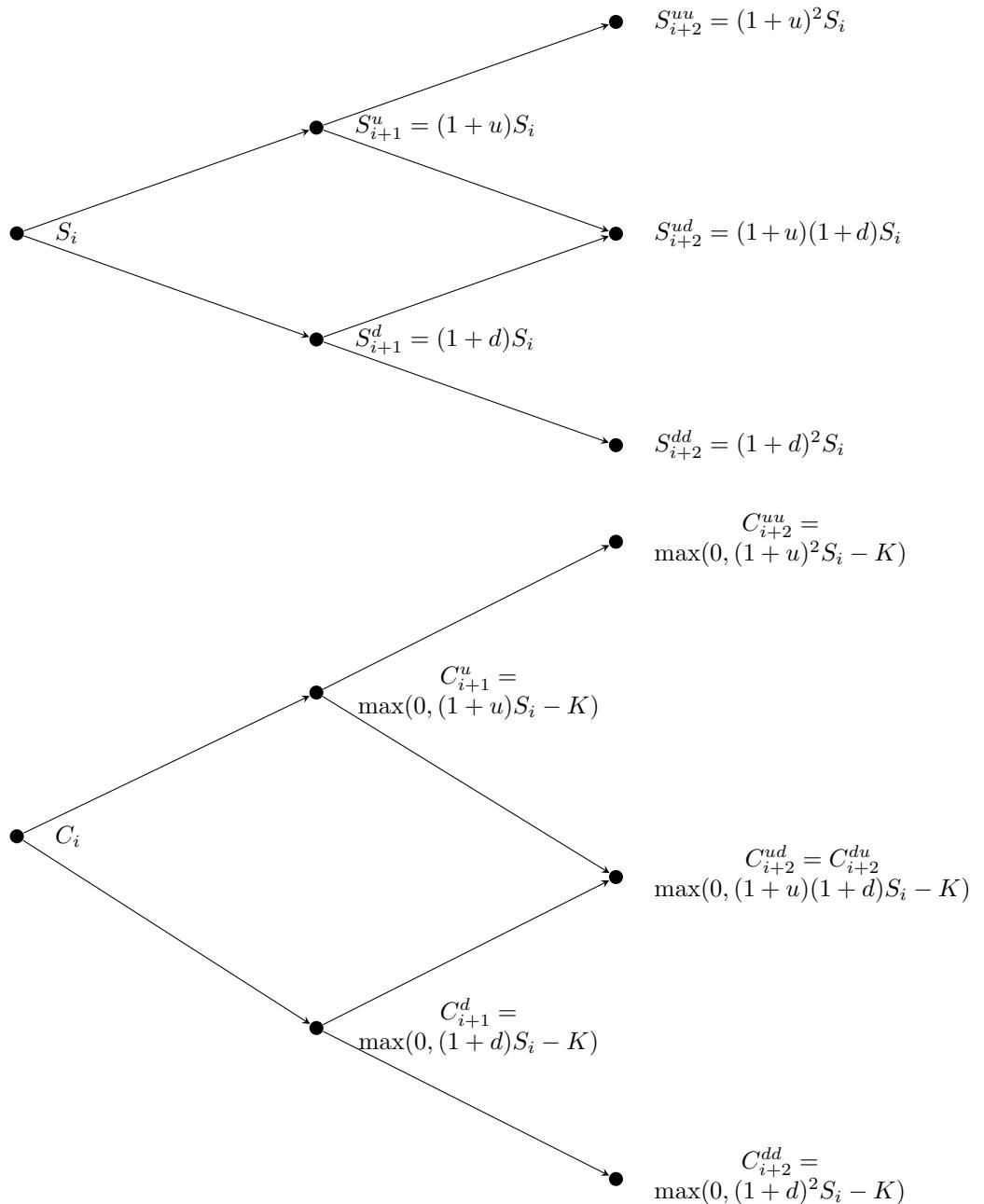
$$\begin{cases} pu + (1 - p)d = \mu \\ p(1 - p)(u - d)^2 = \sigma^2 \end{cases}$$

Solving this system, we get :

$$\begin{cases} u = \sigma \frac{\sqrt{1-p}}{\sqrt{p}} + \mu \\ d = -\sigma \frac{\sqrt{p}}{\sqrt{1-p}} + \mu \end{cases}$$

### 5.2.2 Two-period

In this configuration, we go forward until instant  $i + 2$ , then go backward to instant  $i$ . Here are schemas for price and option.

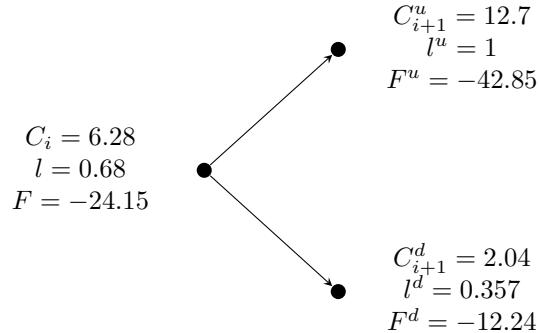


Using the same mechanism as in single period, we must have the following equations :

$$C_i = lS_i + F$$

$$\begin{aligned}
C_{i+1}^u &= lS_i(1+u) + F(1+r) \\
C_{i+1}^d &= lS_i(1+d) + F(1+r) \\
C_{i+1}^u &= l^u S_{i+1}^u + F^u \\
C_{i+2}^{uu} &= l^u S_{i+1}^u(1+u) + F^u(1+r) \\
C_{i+2}^{ud} &= l^u S_{i+1}^u(1+d) + F^u(1+r) \\
C_{i+1}^d &= l^d S_{i+1}^d + F^d \\
C_{i+2}^{du} &= l^d S_{i+1}^d(1+u) + F^d(1+r) \\
C_{i+2}^{dd} &= l^d S_{i+1}^d(1+d) + F^d(1+r)
\end{aligned}$$

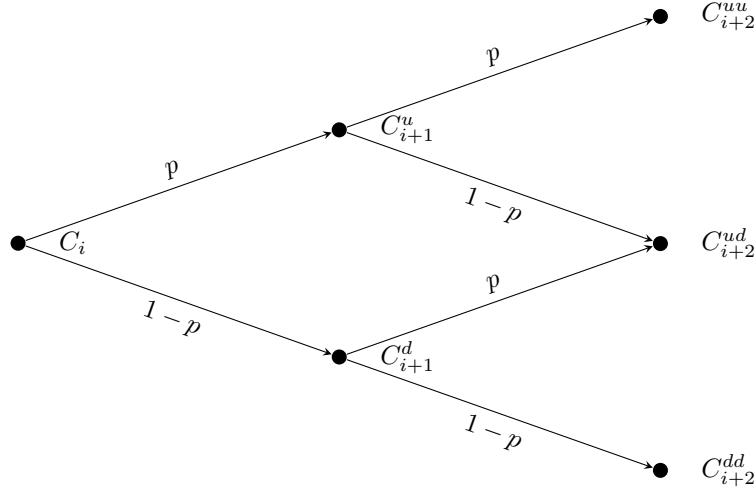
Thus for nodes which are not final nodes, we have three equations, with three variables. Here the variables are  $(C_i, l, F)$ ,  $(C_{i+1}^u, l^u, F^u)$ ,  $(C_{i+1}^d, l^d, F^d)$ . For example, in the following schema that we have solved for all variables :



We see that  $l$  and  $F$  change when prices  $S$  changes. For a call option :

- If price increases,  $l$  increases and  $F$  decreases, which means we have, in equivalence, more shares and more borrowing capability.
- If price decreases,  $l$  decreases and  $F$  increases, which means we have, in equivalence, less shares and less borrowing capability.

Note that the probability  $p = \frac{r-d}{u-d}$  that price of the underlying instrument increases is unchanging since  $r, d, u$  are the same. Then we have :



$$\begin{aligned}
 C_i &= \frac{pC_{i+1}^u + (1-p)C_{i+1}^d}{(1+r)} \\
 C_{i+1}^u &= \frac{pC_{i+2}^{uu} + (1-p)C_{i+2}^{ud}}{(1+r)} \\
 C_{i+1}^d &= \frac{pC_{i+2}^{du} + (1-p)C_{i+2}^{dd}}{(1+r)}
 \end{aligned}$$

### 5.2.3 Multi-period

In this configuration, we go forward until instant  $n$ , then go backward to instant 0.

Three main steps:

- Step 1: Create the binomial price tree. From  $S_0$  price of underlying instrument at the valuation date, we create  $S_{i,k}$  where  $i = 1, \dots, n$  with  $n$  is the total discretized intervals and  $k$  means for vertical index as the following. From case  $S_{i,k}$  we go to case  $S_{i+1,k}$  if price increases and go to case  $S_{i+1,k+1}$  if price decreases. Let  $p$  be the probability that price raises,  $u \geq 1$ ,  $d \leq 1$  are respectively increasing and decreasing rate, we have the figure 6.12.

- Step 2: Find **binomial value** at each final node  $P_{n,k}$  :

$$P_{n,k} = \begin{cases} \max(S_{n,k} - K, 0) & \text{if call option} \\ \max(K - S_{n,k}, 0) & \text{if put option} \end{cases}$$

where  $K$  is the strike price.

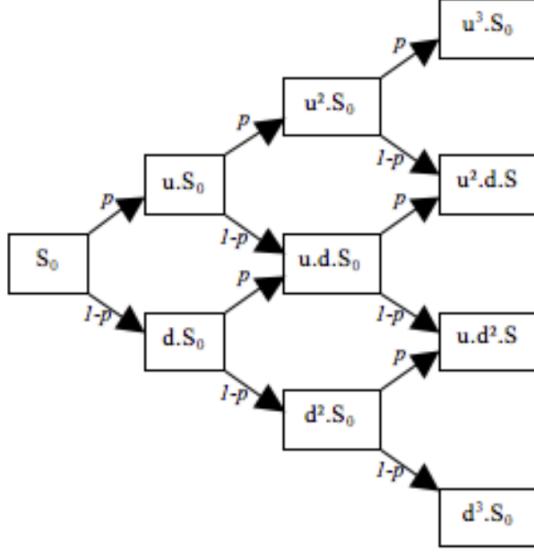


Figure 6.12: Binary tree with  $p$  is probability that price raises.

- Step 3: Find binomial value at earlier nodes. The binomial value is then found for each node, starting at the penultimate time step, and working back to the first node of the tree, which means :

$$P_{i,k}e^{(r\Delta t)} = p \times P_{i+1,k} + (1 - p) \times P_{i+1,k+1}$$

while  $i = n - 1, \dots, 0$

Depending on the nature of early exercise in each the style of the option, we have :

- For a European option, there is no early exercise, then the binomial value applies at all nodes. The binomial value  $P_{i,k}$  represents the fair price of the derivative at a particular time  $i$  and  $k$  times decresing from starting (i.e at each node).
- For an American option, since there is early exercise, the new binomial value is updated by the greater value between binomial value and exercise value. The exercise value is  $\max(S_{i,k} - K, 0)$  for call option and  $\max(K - S_{i,k}, 0)$  for put option. For example, with call option :

if  $P(i, k) < \max(S_{i,k} - K, 0)$  then  $P(i, k) = \max(S_{i,k} - K, 0)$ .

We need to update the binomial value  $P(i, k)$  because the investors are rational, they know what is a greater payoff between binomial value (loss, we must pay for buying option) and exercise value (gain) at node  $(i, k)$ .

The update is to guarantee that we always have fair price, which means the loss must equals to gain. This update is also an advantage of time discritizating, which does not exist in Black-Scholes model.

- For a Bermudan option (1.10.8.3) the binomial value is updated only at instants where early exercise is allowed.

After backward calculation with eventual updates (for American and Bermudan options),  $P_{0,0}$  is the option price given by BOPM method.

### 5.2.3.1 Parameters setting

In BOPM,  $p, u, d$  are chosen such that the price evolution of BOPM **has the same attitude** with the price evolution given by geometric Brownian motion (6.4.2) with rate  $r$  and volatility  $\sigma$ . Let  $S_A(n)$  be the price simulated by BOPM and  $S_B(t)$  be the price simulated by the geometric Brownian motion, hence :

- $S_A(n) = S_0 \prod_{i=1}^n (uX_i + (1 - X_i)d)$  where  $X_i \sim \text{Bernoulli}(p)$  and  $X_i$  are independent.
- $S_B(t) = S_0 \exp \left( (r - \frac{\sigma^2}{2})t + \sigma B_t \right)$

The same attitude is translated into following conditions :

$$\begin{cases} \mathbb{E}[S_A(n)] = \mathbb{E}[S_B(n\Delta_t)] \\ \text{Var} \left[ \log \left( \frac{S_A(n)}{S_0} \right) \right] = \text{Var} \left[ \log \left( \frac{S_B(n\Delta_t)}{S_0} \right) \right] \end{cases}$$

where  $t = n\Delta_t$  means that we discretize by  $\Delta_t$ .

#### Expected value

- The expected value of price simulated by the GBM :

– Forward price simulation :

$$S_B(t) = S_0 \exp \left( (r - \frac{\sigma^2}{2})t + \sigma B_t \right)$$

– As the above  $S_B(t)$  follows the log normal distribution, then :

$$\mathbb{E}[S_B(t)] = S_0 \exp \left( (r - \frac{\sigma^2}{2})t + \frac{\sigma^2 t}{2} \right) = S_0 \exp(rt) = S_0 \exp(rn\Delta_t)$$

- The expected value of price simulated by the binomial distribution :

$$S_A(n) = S_0 \prod_{i=1}^n (uX_i + (1 - X_i)d)$$

where  $X_i \sim \text{Bernoulli}(p)$  and  $X_i$  are independent. The expected value :

$$\mathbb{E}[S_A(n)] = S_0 \prod_{i=1}^n \mathbb{E}[uX_i + (1 - X_i)d] = S_0(pu + (1 - p)d)^n$$

Hence by identifying :

$$\begin{aligned} S_0 \exp(rn\Delta_t) &= S_0(pu + (1-p)d)^n \\ \Leftrightarrow p &= \frac{e^{r\Delta_t} - d}{u - d} \end{aligned}$$

Note that  $p$  must be in  $(0, 1)$  then  $\frac{e^{r\Delta_t} - d}{u - d}$  has to be between 0 and 1.

### Variance

- By the geometric Brownian motion:

$$Var \left[ \log \left( \frac{S_B(n\Delta_t)}{S_0} \right) \right] = Var[\sigma B_t] = \sigma^2 t = \sigma^2 n\Delta_t$$

- By the binomial distribution :

$$\begin{aligned} Var \left[ \log \left( \frac{S_A(n)}{S_0} \right) \right] &= Var \left[ \sum_{i=1}^n \log(uX_i + (1-X_i)d) \right] \\ &= np(1-p)(\log(u) - \log(p))^2 \end{aligned}$$

Hence :

$$p(1-p)(\log(u) - \log(p))^2 = \sigma^2 \Delta_t$$

Usually, beside two above conditions, we need one more equation since we have 3 variables, let's impose  $u = \frac{1}{d}$ .

Solving for  $u$  and  $d$  may be complicated, then we just test the solution. Thus, Cox-Ross-Rubinstein set  $u$  and  $d$  by

$$\begin{cases} u = e^{\sigma\sqrt{\Delta_t}} \\ d = e^{-\sigma\sqrt{\Delta_t}} \end{cases}$$

We show that  $u$  and  $d$  satisfies the above equation, by noting that  $\Delta_t$  is very small, then :

$$\begin{aligned} p &= \frac{e^{r\Delta_t} - d}{u - d} \\ &= \frac{1 + r\Delta_t - (1 - \sigma\sqrt{\Delta_t})}{(1 + \sigma\sqrt{\Delta_t}) - (1 - \sigma\sqrt{\Delta_t})} \\ &= \frac{\sigma\sqrt{\Delta_t}}{2\sigma\sqrt{\Delta_t}} \\ &= \frac{1}{2} \end{aligned}$$

and

$$p(1-p)(\log(u) - \log(p))^2 = \sigma^2 \Delta_t$$

### 5.3 Monte Carlo methods for option pricing

The BOPM is more or less a Monte Carlo simulation. With a more general view, Monte Carlo can be used to simulate multiple possible scenarios of the underlying asset's price movement over time. For example, in BOPM we have only two directions *up* and *down* with Bernoulli distribution, but we can think to more direction with more sophisticated distribution. The Monte Carlo method for option pricing involves the following steps:

1. Model the underlying asset's price movement using a stochastic process, such as the geometric Brownian motion model.
2. Generate a large number of simulated asset price paths using the stochastic process and random number generators.
3. Calculate the payoff of the option at the expiration date for each simulated price path.
4. Discount the payoffs back to the present using a risk-free interest rate.
5. Average the discounted payoffs to obtain the estimated option value.

The Monte Carlo method is particularly useful for pricing options with complex payoffs or when closed-form solutions are not available, such as American options. It also allows for the incorporation of various market factors and assumptions, such as volatility and interest rates, into the pricing model.

However, Monte Carlo simulations can be computationally intensive and may require significant computing power and time to produce accurate results. They also rely on the accuracy of the underlying stochastic model and the quality of the random number generators used in the simulation.

## 6 Greeks (finance)

In finance, “Greeks” refer to a set of mathematical measures used to quantify the sensitivity of derivative securities (options) to various factors. These factors include changes in the price, in time decay, in volatility, in interest rates (of the underlying asset).

Sensitivity here means the variation of  $Y = f(X)$  while  $X$  changes. In short, we want to know if  $X$  is changed into  $X + dX$ , what is the corresponding change  $dY = f(X + dX) - f(X)$  or equivalently the slope  $\frac{\partial Y}{\partial X}$ ?

The above concept is known as first-order Greeks. There are also the second-order Greeks  $\frac{\partial^2 Y}{\partial X \partial Y}$  or third-order Greeks.

Almost Greeks can be used in hedging, for example, delta-hedging, vega-hedging, ...

## 6.1 First-order Greeks

### 6.1.1 Delta

$\Delta$  measures the rate of change of the theoretical option value with respect to changes in the underlying asset's price. Delta is the first derivative of option price  $V$  with respect to the underlying instrument's price  $S$ :

$$\Delta = \frac{\partial V}{\partial S}$$

or we can write :

$$dV = \Delta dS$$

We have seen  $\Delta$  in 6.5.2.1.1. Since the option cost of a share is always much smaller than the underlying instrument's price ( $V \ll S$ ), then we have  $|\Delta| \leq 1$ . Conventionally, we express  $\Delta$  as the variation of option  $dV$ , given that  $dS = \$1$ . For a vanilla option,  $\Delta$  can be positive or negative, to facilitate understanding the profit or loss :

- $0 \leq \Delta \leq 1$  describes a long call (or a short put). Since in this context, we expect that the gain of profit is in the **same direction** with the increase of underlying instrument's price.
- $-1 \leq \Delta \leq 0$  describes a long put (or a short call). Since in this context, we expect that the gain of profit is in the **opposite direction** with the increase of underlying instrument's price.

The total  $\Delta$  of a complex portfolio  $f$  of positions  $A, B, \dots$  w.r.t. **an underlying instrument**  $X$  can be calculated by simply taking the sum of  $\Delta$  for each individual position w.r.t. this underlying instrument  $X$ :

$$\Delta = \frac{\partial f}{\partial S_X} = n_A \frac{\partial S_A}{\partial S_X} + n_B \frac{\partial S_B}{\partial S_X} + \dots$$

#### 6.1.1.1 Delta of sum of long call and short put

If a portfolio  $f$  is composed of a long call and a short put (of the same underlying instrument) :  $f = C - P$ , then  $\Delta$  of  $f$  is equal to 1. This is by the put-call parity in section 1.10.7, we have

$$C - P = S - D \cdot K$$

where  $C, P$  is the price of call and put option,  $S$  is spot price of underlying instrument,  $D$  is discounted rate,  $K$  is strike price. Then

$$\Delta = \frac{\partial f}{\partial S} = \frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1$$

### 6.1.1.2 Share equivalent understanding

The acquired options are commonly presented as the number of shares of underlying instrument, by using  $\Delta$ , see 6.5.2.1.1. If using Black-Scholes model 6.5.1,  $\Delta = N(d_1)$ .

For example, suppose a portfolio of 100 American call options on underlying instrument  $X$ , each call option have  $\Delta = 0.25$ . Then this portfolio will gain or lose value just like 25 shares of  $X$  as the price changes for *small price movements*.

The sign of  $\Delta$  and percentage notation are often dropped, .e.g, in the above example, we say 25 delta call instead a call option on underlying instrument  $X$ .

### 6.1.1.3 Delta hedging or Delta neutral

It means to adjust the portfolio, that aims to keep the overall  $\Delta$  of our portfolio close to zero, for related underlying instruments

For example, if the  $\Delta$  of a portfolio of options in  $X$  (expressed as shares of the underlying) is +2.75, the trader would be able to delta hedge the portfolio by selling short 2.75 shares of the underlying. This portfolio will then retain its value regardless of which direction the price of  $X$  moves.

To make sense, delta hedging must give a positive profit in a long term and not only to keep  $\Delta$  be 0. For example, buy one share  $A$  then short one share  $A$  does not make sense.

Note that :

- $\Delta$  can vary in term of time. Therefore, **the frequency of adjustments** depends on various factors, including the volatility of the underlying asset, time remaining until expiration, and the desired level of risk management.
- In practice, maintaining  $\Delta = 0$  is very complex because there are risks associated with re-hedging on large movements in the underlying instrument's price, and research indicates portfolios tend to have lower cash flows if re-hedged too frequently.
- Delta hedging is not a perfect strategy, and it cannot completely eliminate all risks. However, it can help traders and investors manage and reduce their exposure to price fluctuations in the underlying asset while holding options positions.

### 6.1.1.4 As a proxy for probability

The (absolute value of)  $\Delta$ , which is between 0 and 1, can reflects more or less *the probability that the option will expire in-the-money*. For example, if a call option has a delta of 0.15, the trader might estimate that the option has approximately a 15% chance of expiring in-the-money. Similarly, if a put contract has a delta of -0.25, the trader might expect the option to have a 25% probability of expiring in-the-money. At-the-money calls and puts have a delta of approximately 0.5 and -0.5.

Indeed, the actual (true) probability of an option finishing in the money is its dual delta, which is the first derivative of option price with respect to strike :

$$\frac{\partial V}{\partial K}$$

### 6.1.2 Vega

Vega measures sensitivity to volatility  $\sigma$  and it is the derivative of the option value with respect to the volatility of the underlying asset.

$$\mathcal{V} = \frac{\partial V}{\partial \sigma}$$

Vega is not the name of any Greek letter. The glyph used is a non-standard majuscule version of the Greek letter nu  $\nu$ , written as  $\mathcal{V}$ . Vega is typically expressed as the amount that option price will gain or lose as volatility of underlying instrument rises or falls by 1 percentage point.

- All options (both calls and puts) will gain value with rising volatility.
- Vega can be an important Greek to monitor for an option trader, especially in volatile markets, since the value of some option strategies can be particularly sensitive to changes in volatility.
- In option straddle (1.14.1), the value of at-the-money is extremely dependent on changes to volatility.
- The symbol kappa,  $\kappa$ , is sometimes used (by academics) instead of vega.

### 6.1.3 Theta

Theta  $\Theta$ , measures the sensitivity of option price  $V$  w.r.t time to expiry  $\tau$  :

$$\Theta = -\frac{\partial V}{\partial \tau}$$

As option price is the sum of intrinsic value and extrinsic value and only the latter depends on  $\tau$  (see 1.10.5.2), then  $\Theta$  measures the sensitivity of extrinsic value w.r.t time to expiry  $\tau$ .

Extrinsic value is *monotonically increasing* with time to expiry, which means that if time to expiry increases (decreases) then extrinsic value increases (decreases). Therefore,  $\Theta$  is always negative for both call and put options.

- Big negative  $\Theta$  means that extrinsic value decreases very fast while time passes (time to expiry decreases). In the contrast, small negative  $\Theta$  means that extrinsic value decreases slowly.
- In a portfolio,  $\Theta$  is negative for long options and positive for short options.

- The total theta for a portfolio of options can be determined by summing the thetas for each individual position.
- Sometime  $\Theta$  is expressed with  $\tau$  which is defined in years. By convention, it is usual to divide the result by the number of days in a year, to arrive at the amount an option's price will move, after one day.

#### 6.1.3.1 An exception is a deep in-the-money

In some situations, specially deep in-the-money, it can happen that extrinsic value increases from instant  $t$  to  $t + 1$  while time to expiry decreases, which is controversial to the monotonically increasing. This is because the increasing of extrinsic value is from other factor, not from time to expiry. For example, let's denote the function of extrinsic value at instant  $t$  by  $E_t$  then :

$$E_t = E_T f(T - t, \dots) \exp(-(T - t)r)$$

where  $r$  is discounting rate,  $T$  is the maturity,  $\tau = T - t$  is time to expiry,  $f$  contains all factors that extrinsic value  $E_t$  depends on. Here we can see that, from  $t$  to  $t + 1$ , although  $f(T - t, \dots)$  decreases,  $\exp(-(T - t)r)$  increases and if the latter is stronger than the former, extrinsic value increases.

#### 6.1.4 Rho

Rho,  $\rho$ , measures sensitivity of option price to the risk-free interest rate.

$$\rho = \frac{\partial V}{\partial r}$$

Except under extreme circumstances, the value of an option is less sensitive to changes in the risk-free interest rate than to changes in other parameters. For this reason, rho is the least used of the first-order Greeks.

Rho is typically expressed as the amount of money that option will gain or lose as the risk-free interest rate rises or falls by 1.0% per annum.

#### 6.1.5 Lambda

Lambda,  $\lambda$  or omega  $\Omega$ , or **elasticity** is the percentage change in option value per percentage change in the underlying price :

$$\lambda = \Omega = \frac{\frac{\partial V}{\partial S}}{\frac{S}{V}} = \frac{\partial V}{\partial S} \times \frac{S}{V} = \Delta \times \frac{S}{V}$$

It is similar to the concept of delta (6.6.1.1) but expressed in percentage terms rather than absolute terms.

### 6.1.6 Epsilon

Epsilon,  $\epsilon$  (also known as psi,  $\psi$ ), is the sensibility of option price w.r.t the dividend yield rate  $q$  of underlying instrument :

$$\epsilon = \psi = \frac{\partial V}{\partial q}$$

Obviously, this sensitivity can only be applied to derivative instruments of equity products (which have dividend yield).

## 6.2 Second-order Greeks

### 6.2.1 Gamma

Gamma ( $\Gamma$ ) measures the rate of change in the  $\Delta$  with respect to changes in the underlying price. Gamma is also the second order derivative of the value function with respect to the underlying price :

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$$



Figure 6.13: Gamma's evolution with underlying price. Long put is in red and long call is in green.)

In figure 6.13, we see the properties of Gamma :

- Long options (call or put) have positive Gamma and short options have negative Gamma. This can be proved by using two-period BOPM 6.5.2.2.

- Gamma is biggest when the price is approximately at-the-money (ATM) and diminishes the further out when the price go either in-the-money (ITM) or out-of-the-money (OTM). This may be a sign for detect the strike price.
- Gamma is convexity in term of underlying price.
- When a trader seeks to establish an effective delta-hedge for a portfolio, the trader may also seek to neutralize the portfolio's Gamma (necessary condition). We see that, in figure 6.13, if delta is 0, this implies that Gamma is 0, however if Gamma is 0, we are not sure that delta is 0. Hedging with Gamma is known effective **over a wider range of underlying price movements**.

### 6.2.2 Vanna

Vanna, also referred to as DdeltaDvol or DvegaDspot is a second-order derivative of option price, to the underlying spot price and to volatility :

$$\text{Vanna} = \frac{\partial \Delta}{\partial \sigma} = \frac{\partial \mathcal{V}}{\partial S} = \frac{\partial^2 V}{\partial S \partial \sigma}$$

Vanna can be useful to monitor a delta-hedged portfolio or a vega-hedged portfolio because vanna help to observe the effectiveness of delta-hedge against volatility changes or the effectiveness of a vega-hedge against change in the underlying spot price.

## 6.3 Black-Scholes equation

Black-Scholes equation is a partial differential equation (PDE) that describes the price evolution of options, or more generally, derivatives. Let  $V(t, S)$  be the price of the option as a function of stock price  $S$  and time  $t$  :

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS \frac{\partial V}{\partial S}$$

where  $r$  is the risk-free interest rate, and  $\sigma$  is the volatility of the stock. At the maturity time  $t = T$ , if the stock price  $S = s$ , then  $V(T, s) = g_K(s)$ , where  $g_K$  is parameterized by strike price  $K$ .

### 6.3.1 Interpretation

Black-Scholes equation can be rewritten by :

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS \frac{\partial V}{\partial S}$$

where on the left hand side :

- $\frac{\partial V}{\partial t}$  is thus something similar to  $\Theta$  in sec 6.6.1.3. The difference here is that we use time instant  $t$  instead of time to expiry  $\tau$ .

- $\frac{\partial^2 V}{\partial S^2}$  is thus  $\Gamma$  in sec 6.6.2.1, which measures the convexity of the derivative value with respect to the underlying value.

On the right hand side,  $\frac{\partial V}{\partial S}$  is thus Delta (6.6.1.1) and it is equivalent to the number of shares of the underlying asset (see 6.5.2.1). Always from the latter,  $V$  is equivalent to a portfolio that contains  $\frac{\partial V}{\partial S}$  shares of underlying asset and an amount  $F$  of riskless asset. Hence,  $V - S \frac{\partial V}{\partial S} = F$  is riskless.

On the left hand side, it is about the sum of theta and a term incorporating gamma. For an option, this theta is typically negative since as  $t$  increases to maturity, the extrinsic value of option (1.10.5.2) decreases. Gamma is typically positive and so the gamma term reflects the gains in holding the option.

The equation states that *over any infinitesimal time interval* the loss from theta and the gain from the gamma term must offset each other so that the result is equivalent to a riskless portfolio.

### 6.3.2 Derivation

- For stock price simulated by GBM (6.4.2), we have :

$$\frac{dS}{S} = \mu dt + \sigma dW$$

By discretizing for small time interval  $\Delta_t$  :

$$\Delta S = \mu S \Delta t + \sigma S \Delta W$$

- Note that option price  $V$  depends on two variables  $t$  and  $S$ . We may think  $V$  depends also on strike price  $K$  but it is rather a condition at time  $T$  :  $V(T, s) = g_K(s)$ . By Ito's formula for  $V(t, S)$  (see 4.16.3.4) :

$$dV = \left( \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial V}{\partial S} \sigma S dW$$

By discretizing for small time interval  $\Delta_t$  :

$$\Delta V = \left( \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) \Delta_t + \frac{\partial V}{\partial S} \sigma S \Delta W$$

- As previously mentioned and in 6.5.2.1.1, we consider the delta-hedge portfolio, consisting of being long one option and short  $\frac{\partial V}{\partial S}$  shares at time  $t$  :

$$F = V - \frac{\partial V}{\partial S} S$$

For a small time interval  $\Delta_t$  from  $t$  to  $t + \Delta_t$ , we can consider that  $\frac{\partial V}{\partial S}$  is a constant, then :

$$\Delta F = \Delta V - \frac{\partial V}{\partial S} \Delta S$$

On one hand, by replacing  $\Delta V$  and  $\Delta S$  :

$$\begin{aligned}\Delta F &= \left( \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) \Delta_t + \frac{\partial V}{\partial S} \sigma S \Delta W - \frac{\partial V}{\partial S} (\mu S \Delta_t + \sigma S \Delta W) \\ &= \left( \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) \Delta_t\end{aligned}$$

On the other hand,  $F$  is riskless asset, then :

$$\begin{aligned}\Delta F &= r \Delta_t F \\ &= r \Delta_t \left( V - \frac{\partial V}{\partial S} S \right)\end{aligned}$$

From both hands, we have Black-Scholes equation :

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 = rV - r \frac{\partial V}{\partial S} S$$

In order that this equation makes sense, we need assumptions (of the Black-Scholes model) :

1. The option price function  $V$  is differentiable with respect to  $t$ .
2. The option price function  $V$  is twice differentiable with respect to underlying price  $S$ .

Note that it depends on the choice of payoff function  $g_K(s)$  at expiry time  $T$  as a condition to solve for the explicit form of  $V$ .

The Black-Scholes equation is a particular case of Feynman-Kac formula (5.3.3) and it can be solved numerically by finite difference method (5.3.1).

### 6.3.3 American options

In this case, our Black-Scholes equation becomes an inequality :

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \leq rV - r \frac{\partial V}{\partial S} S$$

together with condition at the maturity time  $t = T$  and  $S = s$  :  $V(T, s) \geq g_K(s)$ .

We have the inequality because  $\Delta F \leq r \Delta_t F$  in derivation 6.6.3.2.

## 7 Kelly criterion

### 7.1 Gambling formula

$$f^* = p - \frac{q}{b} = p - \frac{1-p}{b}$$

where:

- $f^*$  is the fraction of the current bankroll (amount of money) to wager.
- $p$  is the probability of a win.
- $q$  is the probability of a loss ( $q = 1 - p$ )
- $b$  is the proportion of the bet gained with a win. E.g., if betting \$10 on a 2-to-1 odds bet (upon win you are returned \$30, means you win \$20), then  $b = \$20/\$10 = 2$ . Note that, if losing, we lose all what we bet, so  $b$  is also called  $b$  against 1.

We consider the three following cases :

- If a bet offers a 60% chance of winning the bet ( $p = 0.60, q = 0.40, b = 1$ ), then the player should wager 20% of his budget at each opportunity ( $f^* = 0.20$ ), in order to maximize long-term rates of capital growth.
- If the player has no advantage (his average statistical gain equals his average statistical loss), i.e. if  $b = \frac{q}{p}$ , then the rule is to bet nothing.
- If the player is at a disadvantage (his average statistical gain is less than his average statistical loss), i.e. if  $b < \frac{q}{p}$ , then the formula yields a negative result, indicating that the player should take the other side of the bet.

## 7.2 With losing rate c

Always with the case of only two results, for the investissement, there is a small difference from the above formula, where we have  $b$  against 1. Now, we introduce the parameter  $c$ , which is the ratio part of amount if we lose. E.g, if we invest 2\$, in case of losing, we lose 0.4\$, then  $c = 0.2$ . The new formula is:

$$f^* = \frac{p}{c} - \frac{q}{b}$$

### 7.2.1 Proof

- $f, p, q, b, c$  are defined as above.
- We start with wealth of  $A_0$ , then at the game  $n$ , we have the wealth  $A_n$ .
- Suppose that we bet  $n \rightarrow \infty$  times, in that we have  $pn$  times we win and  $qn$  times we lose.

Then

$$A_n = A_0(1 + fb)^{pn}(1 - fc)^{qn}$$

And the average growth rate  $r$  is :

$$r = \sqrt[n]{\frac{A_n}{A_0}} = (1 + fb)^p(1 - fc)^q$$

$r$  is also called expected geometric growth rate. We want to find the maximum  $r$  of this curve (as a function of  $f$ ), which involves finding the derivative of the equation. This is more easily accomplished by taking the logarithm of each side first. The resulting equation is:

$$E = \log(r) = p \log(1 + fb) + q \log(1 - fc)$$

Solving for  $\frac{dE}{df} = 0$ , we get :

$$f^* = \frac{p}{c} - \frac{q}{b}$$

### 7.2.2 If $p$ is unknown

In this case  $p$  is unknown, then we can not directly calculate for  $f^*$ . However, we have a trick to progressively update to have  $f^*$ . Suppose that we play  $N$  games and win  $K$  games, then we know  $b$  and  $c$ .  $f$  is the current fraction of bankroll.

$$A_n = A_0(1 + fb)^K(1 - fc)^{N-K}$$

We need to find optimal  $\Delta$  by taking first order of  $\ln(\frac{A_n}{A_0})$  :

$$\begin{aligned} \frac{Kb}{1 + fb} - \frac{(N - K)c}{1 - fc} &= 0 \\ \Leftrightarrow Kb - fKbc - (N - K)c - (N - K)cfb &= 0 \\ \Leftrightarrow Kb - (N - K)c - Ncfb &= \\ \Leftrightarrow f &= \frac{Kb - (N - K)c}{Ncb} \end{aligned}$$

Then :

$$f = \frac{Kb - (N - K)c}{Ncb}$$

This means that after each game, we update  $N$  and  $K$  and then use the value  $\frac{Kb - (N - K)c}{Ncb}$ .

## 7.3 Betting on multiple possible outcomes

Example: pari-mutuel betting on horse racing. Each horse  $i$  has a probability  $p_i$  to win, and proportion  $b_i$  in case of winning. Let  $f_i$  is the bet (fraction) for each horse  $i$ . Then if the horse  $i$  win, we lose a fraction  $(1 - \sum_{j=1, j \neq i}^N f_j)$  and win a fraction  $f_i b_i$ . The expected geometric growth rate  $r$  :

$$r = (1 - \sum_{j=2}^N f_j + f_1 b_1)^{p_1} \dots (1 - \sum_{j=1}^{N-1} f_j + f_N b_N)^{p_N}$$

$$= \prod_{i=1}^N (1 - \sum_{j=1}^N f_j + f_i + f_i b_i)^{p_i}$$

The optimal bets is the solution of the following system of equations :

$$\begin{cases} f_i^* = \max(p_i - R/b_i, 0) \\ R = 1 - \sum_{j=1}^N f_j^* \end{cases}$$

## 7.4 Application to the stock market

In mathematical finance, if a portfolio has **security weights** that maximize the **expected geometric growth rate**, then the portfolio is growth optimal.

### 7.4.1 One risky asset and one risk free asset

Apply the stock simulation for a given risky asset (in 6.4.2), with estimated mean return rate  $\mu$  and estimated volatility  $\sigma$  (from one periode) :

$$S_t = S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right)$$

Now suppose that we use a fraction (weight)  $f$  on this risky asset and the rest  $1-f$  for the remaining asset, with the return rate  $r$ . Let  $R_c$  be the expected return of the combinaison of this risky asset and the free risk asset, then :

$$\begin{cases} \mathbb{E}[R_c] = f\mu + (1-f)r \\ \sigma_{R_c} = f\sigma \end{cases}$$

Then the stock simulation for combination :

$$S_t^C = S_0^C \exp \left( \left( f\mu + (1-f)r - \frac{(f\sigma)^2}{2} \right) t + f\sigma W_t \right)$$

Then the expected log return  $R_1^C$  for a period time of length 1 :

$$G(f) = f\mu + (1-f)r - \frac{(f\sigma)^2}{2}$$

Then the optimal  $f^*$  is obtained by first- order derivative of  $G$  :

$$f^* = \frac{\mu - r}{\sigma^2}, \quad 0 \leq f \leq 1$$

and

$$G(f^*) = r + \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2$$

Do not confuse the optimal  $f^* = \frac{\mu - r}{\sigma^2}$  with the Sharp ratio  $\frac{\mu - r}{\sigma}$  (6.3.4).

The keypoint for the difference between optimal weight found by this method and the efficient frontier (sec 6.3.3) is :

- In this method, all investors do not care about risk, while the efficient frontier does.
- In this method, the expected return is calculated based on Geometric Brownian motion, which use parameters estimated from logarithmic returns.

#### 7.4.2 Multi risky asset and one risk free asset

Let  $R_c$  is the expected return of the combinaison of multi risky assets and the free risk asset, then :

$$\left\{ \begin{array}{l} \mathbb{E}[R_c] = f_1\mu_1 + \dots + f_N\mu_N + (1 - \sum_{i=1}^N f_i)r \\ \sigma_{R_c} = \sqrt{f_1^2\sigma_1^2 + \dots + f_N^2\sigma_N^2 + 2\sum_i^N \sum_{j>i}^N f_i f_j \sigma_{ij}} \end{array} \right.$$

Then the expected log return  $R_1^C$  for a period time of length 1 :

$$f_1\mu_1 + \dots + f_N\mu_N + (1 - \sum_{i=1}^N f_i)r - \frac{f_1^2\sigma_1^2 + \dots + f_N^2\sigma_N^2 + 2\sum_i^N \sum_{j>i}^N f_i f_j \sigma_{ij}}{2}$$

Then the optimal  $f^*$  is :

$$\left\{ \begin{array}{l} u_1 - r - f_1\sigma_1^2 - \sum_{j=1, j \neq 1}^N f_j \sigma_{1j} = 0 \\ u_2 - r - f_2\sigma_2^2 - \sum_{j=1, j \neq 2}^N f_j \sigma_{2j} = 0 \\ \vdots \\ u_N - r - f_N\sigma_N^2 - \sum_{j=1, j \neq N}^N f_j \sigma_{Nj} = 0 \\ 0 \leq \sum_i^N f_i \leq 1 \end{array} \right.$$

or let  $\Sigma$  be the covariance matrix among risky assets then :

$$\Sigma f^* = \mu - r, \quad 0 \leq \sum_i^N f_i \leq 1$$

#### 7.4.3 Garbage In, Garbage Out

Computations of growth optimal portfolios can suffer tremendous garbage in, garbage out problems. The “Garbage In, Garbage Out” (GIGO) problem refers to the concept that the quality of the output is determined by the quality of the input data it receives. In simpler terms, if you provide a system with inaccurate, incomplete, or faulty data, the output or predictions generated by the system will also be inaccurate, unreliable, or nonsensical. In other words, *ex-post* performance of a growth-optimal portfolio may differ fantastically from the *ex-ante* prediction. We observe Garbage In, Garbage Out in the following example.

#### 7.4.3.1 A realistic example

Let's consider a mythical stock  $X$  that has a mean annual return of  $\mu = 10.7\%$  and an annual standard deviation of  $\sigma = 12.4\%$ . In addition suppose we are able to borrow at a risk-free interest rate of 3%.

We can calculate :

- The Sharp ratio :

$$\frac{0.107 - 0.03}{0.124} = 0.62$$

This Sharp ratio is less than 1, which means not a good investment.

- The optimal grow rate :

$$0.03 + \frac{1}{2} \cdot 0.62^2 = 0.22$$

- The optimal Kelly  $f^*$  :

$$f^* = \frac{0.107 - 0.03}{0.124^2} = 5.01$$

Thus this optimal Kelly is equivalent to a leverage that says for a 100,000\$ portfolio we should borrow an additional 401,000\$ USD to have a total portfolio value of 501,000\$. In practice it is **unlikely** that our brokerage would let us trade with such substantial margin and so the Kelly Criterion would need to be adjusted. Here :

- Garbage In : the estimation of means and standard deviations are always subject to uncertainty.
- Garbage Out : optimal Kelly  $f^*$ .

In practice many traders tend to use a more conservative leverage such as the Kelly Criterion divided by two, affectionately known as “half-Kelly”.

## 8 Utility

In economics, utility refers to the pleasure, satisfaction or happiness that individuals derive from consuming goods and services. Utility is a subjective measure, meaning it varies from person to person and cannot be directly observed or measured. Instead, economists use utility as a theoretical concept to understand and analyze individuals' preferences and decision-making.

### 8.1 Utility function

A utility function is math function to measure the level of satisfaction from consuming different combinations (also called *alternative or bundle*) of goods and services. The utility function assigns a numerical value to each possible alternative, reflecting the individual's subjective preference for that alternative.

### 8.1.1 Example

Suppose James has utility function  $U = \sqrt{xy}$  such that  $x$  is the number of apples and  $y$  is the number of chocolates. Alternative  $A$  has 9 apples and 16 chocolates; alternative  $B$  has 13 apples and 13 chocolates. Putting the values  $x, y$  into the utility function yields  $\sqrt{9 \times 16} = 12$  for alternative  $A$  and  $\sqrt{13 \times 13} = 13$  for  $B$ , so James prefers alternative  $B$ .

In general economic terms, a utility function ranks preferences concerning a set of goods and services.

### 8.1.2 Conditions required for utility function

For a finite set of alternatives, these require only that the preference ordering is complete (so the individual is able to determine which of any two alternatives is preferred or that they are indifferent).

If the set of alternatives is not finite, we may have a **continuous** utility function that maps from alternative space to the real space.

By default, utility function is **increasing**.

## 8.2 Cardinal or ordinal utility

Cardinal utility and ordinal utility are two different approaches to measuring and comparing the level of satisfaction or preferences of individuals.

### 8.2.1 Cardinal utility

Cardinal utility states that the utilities obtained from consumption **can be measured and are representable by numbers**. When cardinal utility is assumed, the magnitude of utility differences is an important quantity. For example, suppose a cup of orange juice has utility of 120 *utils*, a cup of tea has a utility of 80 *utils*, and a cup of water has a utility of 40 *utils*. With cardinal utility, it can be concluded that the cup of orange juice is better than the cup of tea by exactly the same amount by which the cup of tea is better than the cup of water. However, we cannot conclude, that the cup of tea is two thirds of the goodness of the cup of juice, because this conclusion would depend not only on magnitudes of utility differences, but also on the “zero” of utility. For example, if the zero of utility was located at -40 *utils*, then a cup of orange juice would be 160 *utils* more than zero, a cup of tea 120 *utils* more than zero. Hence, we have a ratio of three fourths instead of two thirds. A cardinal utility function can be transformed to another utility function by a positive linear transformation (multiplying by a positive number, and adding some other number).

### 8.2.2 Ordinal utility

Instead of giving numbers over different bundles, ordinal utilities are only **the rankings** of utilities received from different bundles of goods or services. For example, ordinal utility could tell that having two ice creams provide a greater

utility to individuals in comparison to one ice cream but could not tell exactly how much extra utility received by the individual. Ordinal utility does not require individuals to specify how much extra utility he or she received from the preferred bundle of goods or services in comparison to other bundles. They are only required to tell which bundles they prefer.

If a function  $u(x) > 0$  is ordinal, it is equivalent to the function  $u(x)^2$ . In contrast, if  $u(x)$  is cardinal, it is not equivalent to  $u(x)^2$ , since  $|u(x_1) - u(x_2)| \neq |u(x_1)^2 - u(x_2)^2|$ .

### 8.3 Indifference curves

Indifference curves are graphical representations that depict **different combinations** of goods or services that yield the **same level of utility** for an individual. See figure 6.14.

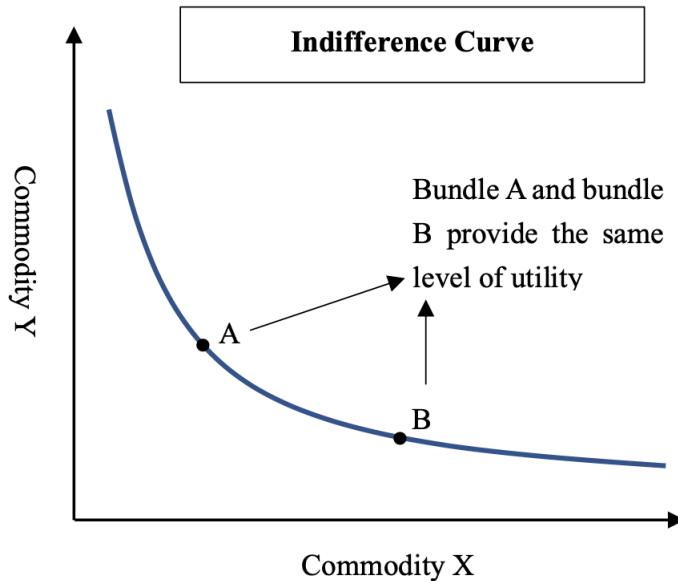


Figure 6.14: The blue line represents an indifference curve, on which bundles (composed by a quantity of commodity X and a quantity of commodity Y) have the same utility.

### 8.4 Marginal Utility

Marginal utility refers to the additional utility gained from consuming an additional unit of a good or service. When individuals consume more of a

particular good, the marginal utility tends to decrease. For instance, the first slice of pizza may provide high utility, but as more slices are consumed, the satisfaction from each additional slice diminishes.

**This is also a reason for that utility function must be concave.** By the marginal utility, we have if  $w_1 < w_2$  :

$$\begin{cases} U(w_1 + \Delta) - U(w_1) > U(w_2 + \Delta) - U(w_2) & \text{if } \Delta > 0 \\ U(w_1) - U(w_1 + \Delta) > U(w_2) - U(w_2 + \Delta) & \text{if } \Delta < 0 \end{cases}$$

In the both case :

$$\frac{U(w_1 + \Delta) - U(w_1)}{\Delta} > \frac{U(w_2 + \Delta) - U(w_2)}{\Delta}$$

or

$$U'(w_1) > U'(w_2)$$

This means the derivative function  $U'$  is strictly monotonically decreasing or  $U'' < 0$ , then  $U$  must be concave (this is a property of concave function). Since  $U$  is a concave function, then the Jensen's inequality (2.2.8.3) can be applied :

$$U(\mathbb{E}[X]) \geq \mathbb{E}[U(X)]$$

where  $X$  is a random variable.

## 8.5 Utility maximization

In economic theory, individuals are assumed to be *rational decision-makers* who aim to maximize their utility based on their preferences and budget constraints. Utility maximization involves allocating resources (income or wealth) to different goods and services in a way that maximizes overall satisfaction. For example, we want to maximize the total utility for an individual w.r.t quantity of apples ( $x_1$ ) and oranges ( $x_2$ ), with a constraint on budget :

$$\max_{x_1, x_2} [u_1(x_1) + u_2(x_2)]$$

s.t.

$$x_1 p_1 + x_2 p_2 = S$$

where  $p_1, p_2$  are price for an unity of apple and an unity of orange,  $S$  is our budget.

## 8.6 Expected utility

Given the probability distribution of possible outcomes  $P(\omega)$  and the associated utility function of outcome  $U(\omega)$ . The expected utility is calculated by the law of the unconscious statistician (2.2.2) :

$$\mathbb{E}[U] = \sum_{\omega \in \Omega} P(\omega) \times U(\omega)$$

## 8.7 St. Petersburg paradox

The St. Petersburg paradox is a paradox related to probability and decision theory in economics. It consists of a lottery game modeled by a random variable whose mathematical expectation is infinite, but for which participants would only agree to pay a small amount of money to play. Different approaches have been proposed to resolve this paradox.

### 8.7.1 Announcement

**The game :** The player pays an initial value  $c$  to the bank (dealer). A coin is tossed heads or tails as long as it comes up *tails*. The game **ends** when first *head* appear and then the bank gives the payoff to the player. This gain is initially \$2, doubled for each appearance of tail. Thus, the payout is \$2 if *head* appear on the first toss, \$4 if *head* appears on the second, \$8 on the third, \$16 on the fourth, etc. So if *head* first appears on the  $n^{th}$  toss, the bank pays  $\$2^n$ . A sample calculation for expected value for gain :

$$\mathbb{E} = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \frac{1}{16} \cdot 16 + \dots = 1 + 1 + 1 + 1 + \dots = \infty$$

The paradox comes here, the player just put a finite amount of money but he can get infinity payoff (gain).

**The question :** What is the player's initial bet so that the game is fair, i.e. so that the player's initial bet is equal to his expectation of winning, and that neither the bank nor the player are advantaged by this game ?

### 8.7.2 Solution by using utility function

The classical resolution of the paradox involves an utility function. Let's suppose that we use the utility function  $U(w) = \ln(w)$  (also known as *log utility*), where  $w$  is player's (gambler's) total wealth. Using log utility function means that if  $w_1, w_2$  are big, hence the distance between  $\ln(w_1) - \ln(w_2)$  is much less than  $w_1 - w_2$ . To find the bet that the player must put  $c$ , we need to resolve (for  $c$ ) :

$$\sum_{k=1}^{+\infty} \frac{1}{2^k} [\ln(w + 2^k - c)] = \ln(w)$$

which means the expected utility of gain if gambler plays is equal to utility of his total wealth. Here, the initial bet  $c$  depends on the total wealth of each gambler  $w$ .

### 8.7.3 Solution by finite amount of bank

If we assume that the bank only has a finite amount, the calculation of expected gain is the same as in the announcement, except that the series is no longer infinite. For example, if we assume that the bank has "only"  $2^N$  euros,

then it cannot pay more if heads appear after  $N + 1$  rolls. The expected gain is :

$$\mathbb{E} = \sum_{k=1}^N \frac{1}{2^k} \cdot 2^k = N$$

So if the bank has \$1050000, then the bet that the player must put is  $c = N = \log_2^{1050000} \approx \$20$ .

## 8.8 Popular utility functions

### 8.8.1 Exponential utility

$$u(w) = \begin{cases} 1 - e^{-\alpha w} & \text{if } \alpha \neq 0 \\ c & \text{if } \alpha = 0 \end{cases}$$

$\alpha$  is a constant that represents the degree of risk preference :

- $\alpha > 0$  for risk aversion.
- $\alpha = 0$  for Risk neutrality.
- $\alpha < 0$  risk-loving.

In situations where only risk aversion is allowed, the formula is often simplified to :

$$u(w) = 1 - e^{-\alpha w}$$

where  $\alpha > 0$ .

### 8.8.2 Isoelastic utility function

Also called power utility function :

$$u(w) = \begin{cases} \frac{w^{1-\rho}-1}{1-\rho} & \text{if } \rho > 0 \text{ and } \rho \neq 1 \\ \ln(w) & \text{if } \rho = 1 \end{cases}$$

### 8.8.3 Hyperbolic absolute risk aversion (HARA)

This is an utility function that is usually used thanks to its flexibility.

$$u(w) = \frac{1-\gamma}{\gamma} \left( \frac{aw}{1-\gamma} + b \right)^\gamma$$

- Utility is linear (the risk neutral case) if  $\gamma \rightarrow 1$ .
- Utility is quadratic, with increasing absolute risk aversion (IARA) if  $\gamma = 2$ .

- The exponential utility function (6.8.8.1), which has constant absolute risk aversion, occurs if  $b = 1$  and  $\gamma \rightarrow -\infty$ .
- The power utility function (6.8.8.2) occurs if  $\gamma < 1$  and  $a = 1 - \gamma$ .

## 9 Risk aversion

Risk aversion refers to the tendency of individuals or investors to prefer lower-risk choices, even if the higher-risk choices offer potentially higher returns. An investor with high risk aversion is someone who prioritizes capital preservation and is willing to accept potentially lower returns in exchange for a higher level of certainty and security.

### 9.1 Example

A person is given the choice between two scenarios: one with a guaranteed payoff, and one with a risky payoff with same average value. In the former scenario, the person receives \$50. In the uncertain scenario, a fair coin is tossed to decide whether the person receives \$100 or nothing. The expected payoff for both scenarios is \$50, meaning that an individual who was insensitive to risk would not care whether they took the guaranteed payment or the gamble. However, individuals may have different risk attitudes :

- Risk averse (or risk avoiding) : they would accept a guaranteed payment less than \$50 (for example, \$45) for a stable situation, instead of taking a toss.
- Risk neutral : if they are indifferent between the bet and a guaranteed payment of \$50.
- Risk loving (or risk seeking) : they would accept the tossing coin even when the guaranteed payment is more than \$50 (for example, \$55). In the other manner, they would accept the tossing coin even in favorable tossing, they receive only \$90 instead of \$100.

### 9.2 Certainty equivalent

The certainty equivalent (CE) is amount that would make indifferent between taking the guaranteed amount and the uncertain one. The CE is different for each person. Using the above example, then the certainty equivalent in three cases can be :

- Risk averse (or risk avoiding) : \$45
- Risk neutral : \$50
- Risk loving (or risk seeking) : \$55.

The risk premium (as described in 1.28.2.1) in this case, is the difference between the expected value and the certainty equivalent :

$$RP = \text{Expected Value} - \text{Certainty Equivalent}$$

With the above example:

- For risk-averse individuals, risk premium is positive.
- For Risk neutral persons it is zero.
- For risk-loving individuals their risk premium is negative.

Given a utility of money function  $U$  of an individual, then the CE is defined by :

$$U(CE) = \mathbb{E}[U(X)]$$

Then we can see three risk premium (positive, negative, zero) for these three type of people in the figure 6.15.

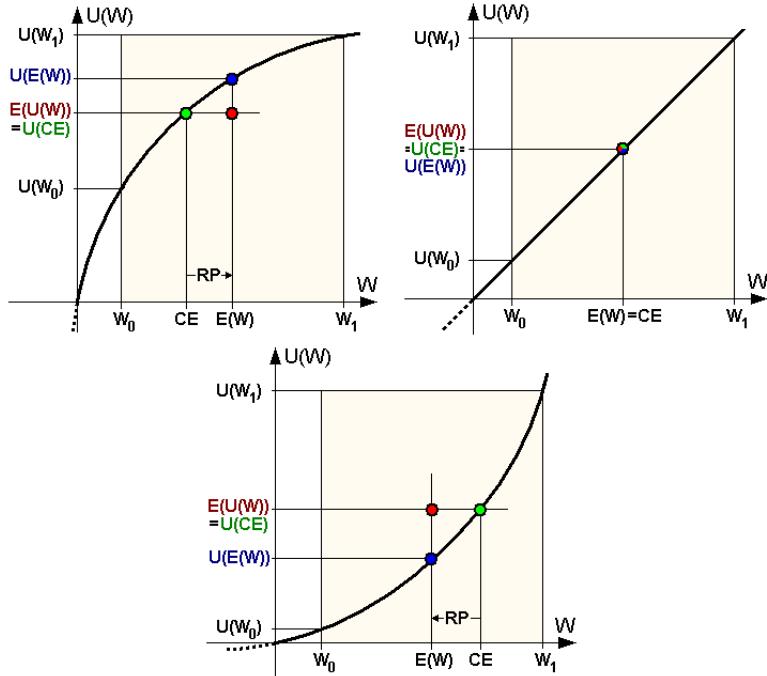


Figure 6.15: Risk premium for three types : risk averse (top-left), risk neutral (top-right), risk-loving (bot). Suppose that the wealth  $W$  is equally distributed from  $W_0$  to  $W_1$ . The mean of wealth  $\mathbb{E}[W]$  is on abscissa axe. The mean of utility of wealth  $\mathbb{E}[U(W)]$  is the green point. By going down from this green point, we have CE.

## 9.3 Utility of money

Respectively to three cases : risk-averse, Risk neutral, risk-loving, we can have three models about utility function for money. These utility functions have two key properties: **slope** and **concavity**.

### 9.3.1 Concavity

- For risk-averse individuals : the utility function is a concave function. This concave function has the diminishing marginal utility property, which mean if  $x_3 - x_2 = x_2 - x_1 > 0$ , then  $U(x_3) - U(x_2) < U(x_2) - U(x_1)$ .
- For Risk neutral persons : the utility function is a linear function. If  $x_3 - x_2 = x_2 - x_1 > 0$ , then  $U(x_3) - U(x_2) = U(x_2) - U(x_1)$ .
- For risk-loving individuals : the utility function is a convex function. If  $x_3 - x_2 = x_2 - x_1 > 0$ , then  $U(x_3) - U(x_2) > U(x_2) - U(x_1)$ .

Note that, in section 6.8.4, we find that utility function need to be concave. This is because we make an assumption that utility function has marginal utility property. This is not the case for risk-loving individuals, since they feel more is better.

We know that for risk averse individuals, a sure amount would always be preferred over a risky bet having the same expected value. Moreover, for risky bets that have the same expected value, they would prefer the one with less risk. This means if we consider a bet as a distribution, then among two distributions of the same mean, they would prefer the less spread out one (less variance). In figure 6.5, they would prefer the bet 1.

State	<i>head</i>	<i>tail</i>
Bet 1	-5	5
Bet 2	-10	10

Table 6.5: Two bets have same expected value, but bet 2 is more spread out than bet 1.

### 9.3.2 Upward slope

The upward slope controls the intensity of risk-aversion or risk loving, in figure 6.15:

- Risk-averse individual : high upward slope at the beginning and then low upward slope.
- Risk loving individual : low upward slope at the beginning and then low upward slope.

The high upward slope implies that the person feels that more is better : a larger amount received yields greater utility. If there are two bets, the first bet is first-order stochastically dominant over the second bet (2.7.2), which means the probability mass of the second bet is pushed to the right to form the first bet, risk loving or risk neutral individuals would prefer the first bet, since it gives a better expected value, even with more risk (more variance), see an example in table 6.6.

State	1	2	3	4	5	6
Bet 1	\$-1	\$-1	\$-1	\$1	\$1	\$1
Bet 2	\$2	\$2	\$2	\$-1	\$-1	\$-1

Table 6.6: Two bets in which the first bet is first-order stochastically dominant over the second bet. The state is given by a fair dice.

## 9.4 Measures of risk aversion under expected utility theory

Given an utility function, there are various measures of the risk aversion.

### 9.4.1 Absolute risk aversion

In figure 6.15 (top-left), the higher the curvature of  $u(c)$ , the higher the risk aversion. A simple measure for this curvature can be the second derivative  $u''(c)$ . However, if  $u(c)$  suffers an affine transformation, then  $u''(c)$  can vary. In order to keep the measure unchanged with respect to these transformations, we introduce *absolute risk aversion* (ARA), also known as the *coefficient of absolute risk aversion*, defined as :

$$A(c) = -\frac{u''(c)}{u'(c)}$$

For example, if  $u(c) = \alpha + \beta \ln(c)$ , so  $A(c) = 1/c$ . Note  $A(c)$  does not depend on  $\alpha$  and  $\beta$  so affine transformations of  $u(c)$  do affect  $A(c)$ .

#### 9.4.1.1 Proof for risk aversion related

Suppose that we are in case of risk averse individuals. Let  $\pi$  be the risk premium then  $\pi = \mathbb{E}[X] - CE = \bar{X} - CE$ . We have the definition of  $CE$  as in 6.9.2 :

$$u(CE) = \mathbb{E}[u(X)] \quad \text{or} \quad u(\bar{X} - \pi) = \mathbb{E}[u(X)]$$

One one hand, using the first order approximation :

$$u(\bar{X} - \pi) \approx u(\bar{X}) - \pi u'(\bar{X})$$

On other hand, using the Taylor expansion for  $u(X)$  at  $\bar{X}$ :

$$u(X) \approx u(\bar{X}) + (X - \bar{X})u'(\bar{X}) + \frac{1}{2}(X - \bar{X})^2u''(\bar{X})$$

then we take the expected value of two sides:

$$\mathbb{E}[u(X)] \approx u(\bar{X}) + \frac{1}{2}Var(X)u''(\bar{X})$$

From both hands :

$$u(\bar{X}) - \pi u'(\bar{X}) = u(\bar{X}) + \frac{1}{2}Var(X)u''(\bar{X})$$

or

$$\pi = -\frac{1}{2} \frac{u''(\bar{X})}{u'(\bar{X})} Var(X)$$

Here we observe that the relation between the risk premium  $\pi$  and the risk, expressed by  $Var(X)$ . Suppose that 2 investors invest in the asset  $X$ , each investor have his own utility function  $u_1$  and  $u_2$ . If the first investor is more risk-averse than the second, we must have  $CE_1 < CE_2$  or  $\pi_1 > \pi_2$ , this implies that  $-\frac{u''_1(\bar{X})}{u'_1(\bar{X})} > -\frac{u''_2(\bar{X})}{u'_2(\bar{X})}$ . Then  $-\frac{u''(c)}{u'(c)}$  reflects the risk-aversion.

#### 9.4.2 Relative risk aversion

The relative risk aversion (RRA) or coefficient of relative risk aversion is defined as :

$$R(c) = cA(c) = -c \frac{u''(c)}{u'(c)}$$

Unlike ARA whose units are in  $\$^{-1}$ , RRA is a dimensionless quantity, which allows it to be applied universally.

#### 9.4.3 Risk tolerance

The risk tolerance, noted  $T(c)$  is the **reciprocal** of absolute risk aversion  $A(c)$  :

$$T(c) = \frac{1}{A(c)}$$

## 10 Intertemporal portfolio choice

Intertemporal portfolio choice refers to the process of determining how to allocate investment resources over time in order to optimize an investor's wealth or utility. It involves making decisions about the composition and allocation of assets across different periods, taking into account factors such as risk, return, and individual preferences for consumption or wealth accumulation.

Typically the criterion is the expected value of some concave function of the value of the portfolio after a certain number of time periods—that is, the expected utility of final wealth. Alternatively, it may be a function of the various levels of goods and services consumption that are attained by withdrawing some funds from the portfolio after each time period.

We discover in the section for the time discret case, for the continuous time case, see the Merton's portfolio problem in chapter 5.4 for example. In time discret we have Time-independent and Time-dependent cases.

## 10.1 Time-independent decisions

We suppose that a period of time can be divided into  $T$  time intervals, where the stochastic properties (return rates) in each interval is independent with others. Let  $W_0$  be the initial wealth and let the *stochastic portfolio rate of growth* in interval  $t$  be  $R_t$ . Then the wealth after  $T$  intervals is:

$$W_T = W_0 R_1 R_2 \dots R_T$$

where  $R_t$  can be decomposed into the weighted sum of underlying asset  $i$ :

$$R_t = w_{1t} r_{1t} + w_{2t} r_{2t} + \dots + w_{it} r_{it} + \dots + w_{nt} r_{nt}$$

where  $r_{it}$  refers to the **stochastic rate of growth** of underlying asset  $i$  at interval  $t$ . The weights must satisfy  $\sum_{i=1}^n w_{it} = 1$ . Now associating the wealth with an utility function, which means  $u(W)$ . Then the objective is to find  $w_{it}$  for each interval  $t$  to maximize the expected utility  $\mathbb{E}[u(W_T)]$  or equivalently :

$$\max_{w_{it}} \mathbb{E}[u(W_0 R_1 R_2 \dots R_T)]$$

### 10.1.1 In case of log utility

If we use the log utility, which means  $U(W) = \ln(W)$ :

$$\begin{aligned} & \max_{w_{it}} \mathbb{E}[U(W_0 R_1 R_2 \dots R_T)] \\ &= \max_{w_{it}} \mathbb{E}[\ln(W_0 R_1 R_2 \dots R_T)] \\ &= \max_{w_{it}} \mathbb{E}[\ln(W_0) + \ln(R_1) + \dots + \ln(R_T)] \\ &= \max_{w_{it}} \mathbb{E}[\ln(W_0)] + \mathbb{E}[\ln(R_1)] + \dots + \mathbb{E}[\ln(R_T)] \quad \text{Time (interval) independent} \\ &= \max_{w_{it}} \ln(W_0) + \sum_{t=1}^T \mathbb{E}[\ln(w_{1t} r_{1t} + w_{2t} r_{2t} + \dots + w_{it} r_{it} + \dots + w_{nt} r_{nt})] \end{aligned}$$

### 10.1.2 In case of power utility

If we use the log utility, which means  $U(W) = aW^a$ , where  $0 < a < 1$ :

$$\begin{aligned} & \max_{w_{it}} \mathbb{E}[U(W_0 R_1 R_2 \dots R_T)] \\ &= \max_{w_{it}} aW_0^a \prod_{t=1}^T \mathbb{E}[(w_{1t} r_{1t} + w_{2t} r_{2t} + \dots + w_{it} r_{it} + \dots + w_{nt} r_{nt})^a] \end{aligned}$$

## 10.2 Time-dependent decisions

As per the above, the expected utility of final wealth with a power utility function is

$$\max_{w_{it}} \mathbb{E}[U(W_0 R_1 R_2 \dots R_T)]$$

If there is not serial independence of rate of growth through time, then the expectations operator **cannot be applied separately** to the various multiplicative terms. Thus, the probability distribution of rate of growth for the various assets depends on their previous-period realizations, and so cannot be determined in advance. Not only depending on previous-period realizations, the optimal portfolio in a particular interval depends also the knowledge of how decisions will be made in future intervals.

The solution can be obtained by :

- **Dynamic programming.** We can view the above problem in more simply case :  $\max_{w_{it}} \mathbb{E}[U(W_0 R_1 R_2)]$ . Given a configuration of  $w_{i1}$  in  $R_1$ , then the optimal configuration  $w_{i2}^*$  that maximizes only  $R_2$  is depend on  $w_{i1}$ . Here the dynamic programming is used to find the optimal configuration for  $(\sum_{i=1}^n w_{i1} r_{i1})(\sum_{j=1}^n w_{j2} r_{j2})$ . This method becomes complex very quickly if there are more than a few time periods  $T$  or more than a few assets  $n$ .
- **Dollar cost averaging.** Take  $w_{i1} = w_{i2} = \dots = w_{iT}, \forall i$ . Indeed, Dollar cost averaging method means that the investor regularly invests a fixed amount of money into a particular asset or a fixed configuration of portfolio at regular intervals. Example, each month, the investor by a fixed \$500 amount of bitcoin. The basic idea behind dollar cost averaging is to reduce the impact of market volatility.

## 11 Analysis methods

Analysis methods consists of fundamental analysis and technical analysis. These two approaches used to analyze financial assets such as stocks, bonds, and currencies.

**Fundamental analysis** (FA) focuses on evaluating the intrinsic value of an asset by analyzing its underlying economic, financial, and qualitative factors such as the company's financial statements, industry trends, management team, and macroeconomic factors like interest rates, inflation, and GDP growth. The goal is to determine the asset's true worth, and whether it is undervalued or overvalued in the market. Fundamental analysts use this information to make **long-term** investment decisions based on the asset's fundamentals, looking to buy assets that are undervalued and sell them when they become overvalued.

On the other hand, **technical analysis** (TA) focuses on studying past market data, primarily charts, to identify patterns and trends that could indicate future price movements. Technical analysts use various tools and techniques such as moving averages, trend lines, and technical indicators to identify support and

resistance levels, and potential entry and exit points. The goal is to identify **short-term** price movements and profit from them. Technical analysts assume that market movements are not random, and patterns in price and volume data can reveal valuable insights.

Both approaches have their own advantages and limitations and are often used in combination to make investment decisions. We see in table 6.7 a comparison between these two analysis.

	<b>Fundamental analysis</b>	<b>Technical analysis</b>
Focus on	The intrinsic value of a security by examining the underlying economic, financial, and industry factors that affect its value	The past market data, primarily price and volume, to identify patterns and trends that can be used to predict future price movements.
Data used	Financial statements, economic indicators, and industry trends to determine the value of a security.	Charts, graphs, and other visual representations of market data to identify patterns and trends.
Risk	Considered a lower-risk approach to investing because it involves a deeper understanding of the underlying factors that drive market trends.	Considered a higher-risk approach because it relies on predicting future price movements based on past data.
Timeframe	Long-term approach that considers the overall health and prospects of a company or industry.	Short-term approach that focuses on identifying short-term trading opportunities.
Application	More commonly used by long-term investors who are interested in buying and holding securities for an extended period of time.	More commonly used by short-term traders who are looking to profit from short-term price movements.

Table 6.7: Comparison between fundamental analysis and technical analysis.

## 11.1 Technical analysis

Properties:

- Supply & demand for an asset determine its price.
- The future price of a asset is more or less consistent with its past behavior.
- There are rational and irrational market participants. Irrational means that it is led by emotion (Herd mentality, Animal spirit, Social proof, Fear of missing out FOMO).
- Psychology or sentiment of investor is also a key factor.

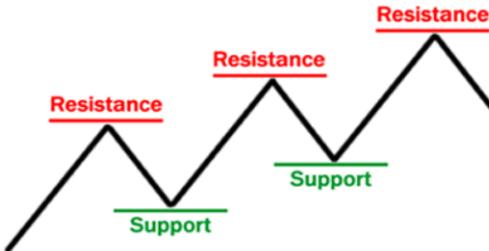


Figure 6.16: Supports and resistances

- Technical analysis is widely used in the analysis of commodities, currencies than FA. This is because these assets don't have income flow (e.g dividend).
- A problem may arise if volume trading of considered assets are low (illiquid).
- TA can be used on a stand-alone way or in combination with FA.

The following is definition of basic concepts :

- Trend : Upward, downward, stepway.
- Retracement : ending of a trend, which can be trend reversal or sideway movement.
- Support is a price level which protects an asset's price from going under.
- Resistance is the opposite with support. It means the price level which prevents an asset's price from going upper. See figure 6.16.
- Change in polarity. This is the idea that once a support level is breached, its becomes resistance level and vice versa.

## 11.2 Chart patterns

The art of trade is in reading charts correctly by spotting partterns and that aren't obvious for all traders. We present here several popular partterns :

- Head and shoulder and its inverse version.
- Double / Triple tops and their inverse version.
- Ascending or descending triangle.
- Bullish rectangle or bearish rectangle.



Figure 6.17: Head and shoulder pattern.



Figure 6.18: Double tops.



Figure 6.19: Ascending triangle.

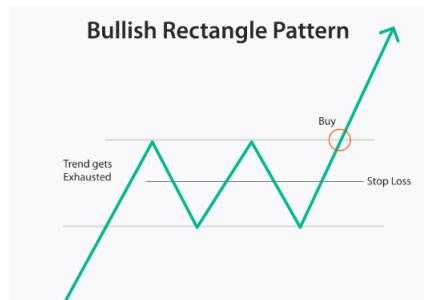


Figure 6.20: Bullish rectangle.

## 11.3 Indicators

Sometime called oscillator, which is calculated based on :

- Price.
- Volume.
- Momentum.
- Market sentiment.
- ...

### 11.3.1 Momentum

$$\text{Close Price}(p) - \text{Close Price}(p - n)$$

Usually,  $p$  is index for today and  $p - n$  is index for  $n$  days ago.

### 11.3.2 Rate of change (ROC)

$$\text{ROC} = \left( \frac{\text{Close Price}(p)}{\text{Close Price}(p - n)} - 1 \right) \times 100$$

Relation between ROC and price : When ROC switches from negative to positive, price moves upward and vice versa.

### 11.3.3 Relative Strength Index (RSI)

Let  $C$  be the price and  $i$  be time instant. We decompose the momentum of  $C_i$  into upward trend  $U_i$  and downward trend  $D_i$  :

- If  $C_i > C_{i-1}$  :  $U_i = C_i - C_{i-1}$  and  $D_i = 0$ .
- If  $C_i < C_{i-1}$  :  $U_i = 0$  and  $D_i = C_{i-1} - C_i$ .

Then RSI is defined by :

$$\text{RSI}(n) = \frac{F_n(U)}{F_n(U) + F_n(D)} \times 100$$

where  $F_n$  is low-pass filter, usually smoothed or modified moving average (SMMA or MMA), or exponential moving average (EMA),  $n$  is the parameter which indicates window size of filter.

One reminds the properties of RSI :

- RSI identify the power of a movement (upward or downward) :
  - If the market is steadily rising or sharply rising, the RSI tends towards 100.

- If the market is steadily falling or sharply falling, the RSI tends towards 0.
- RSI indicate whether you are in a situation of overbought or oversold.  
Using the following classical strategy to trade :
  - (oversold) IF CURRENT RSI  $\leq 30$  : BUY SIGNAL
  - (overbought) IF CURRENT RSI  $\geq 70$  : SELL SIGNAL

#### 11.3.4 Larry Connors

This means a trading strategie by combining SMA and RSI.

- Sell if the current price is above SMA(200) and RSI(2) is greater than 90.
- Buy if the current price is below SMA(200) and RSI(2) is less than 10.

#### 11.3.5 Bollinger Bands

:

Bollinger bands display a graphical band (the envelope maximum and minimum of moving averages). If the current price  $C_i$  is out of these bands, it is overvalued or undervalued. The width of Bollinger band at instant  $i$  is defined by :

$$F_n(C_i) \pm k\sigma_n(C_i)$$

where  $F_n$  is a low-pass filter (SMA by default, EMA is a common second choice),  $\sigma_n$  is the standard deviation of last  $n$  prices. See figure 6.21 for an illustration.



Figure 6.21: Bollinger bands with classical paramters  $n = 20$  and  $k = 2$ . Blue line means SMA.

### 11.3.6 Stochastic oscillator

It is an oscillator composed of two lines (components)  $\%K$  and  $\%D$  :

$$\%K(i) = \frac{\text{Close Price}_i - L_n}{H_n - L_n} \times 100$$

where  $L_n$ ,  $H_n$  are lowest and highest price of last  $n$  days. Typically,  $n$  is 5, 9, or 14. Here instead of analyzing the price, we analyse the ratio between the price and its range.

$$\%D_N(i) = \text{SMA}_N(K) = \frac{\%K(i+N) + \dots + \%K(i)}{N}$$

Typically,  $N$  is 3. The indicator  $\%K$  can go from 0 to 100 very quickly.  $\%D$  is the result of applying low pass filter on  $\%K$ . Trading strategie by stochastic oscillator can be :

- When  $\%K$  crosses upward  $\%D$  : Buy signal
- When  $\%K$  crosses downward  $\%D$  : Sell signal
- When  $\%K$  crosses upward 20 : Buy signal
- When  $\%K$  crosses downward 80 : Sell signal

### 11.3.7 Moving Average Convergence Divergence (MACD)

The MACD indicator is a collection of three time series calculated from historical price data, most often the closing price. These three series are: the *MACD series*, the *signal series*, and the *histogram series* (or divergence series) is the difference between the two formers:

- The MACD series is the difference between a fast (short period) EMA and a slow (longer period) EMA of the price series :

$$\text{MACD} = \text{EMA}_{12}(C) - \text{EMA}_{26}(C)$$

- The signal series is an **EMA of the MACD series** :

$$\text{Signal} = \text{EMA}_9(\text{MACD})$$

- Histogram = MACD - Signal.

In figure 6.22, we observe clearly all these three series. We observe also the intersections between  $\text{EMA}_{12}(C)$  and  $\text{EMA}_{26}(C)$  correspond with intersections between MACD series and 0.

Trading strategie by MACD :

- MACD series goes downward and crosses signal series, equivalently histogram series change sign from positive to negative : Sell signal.

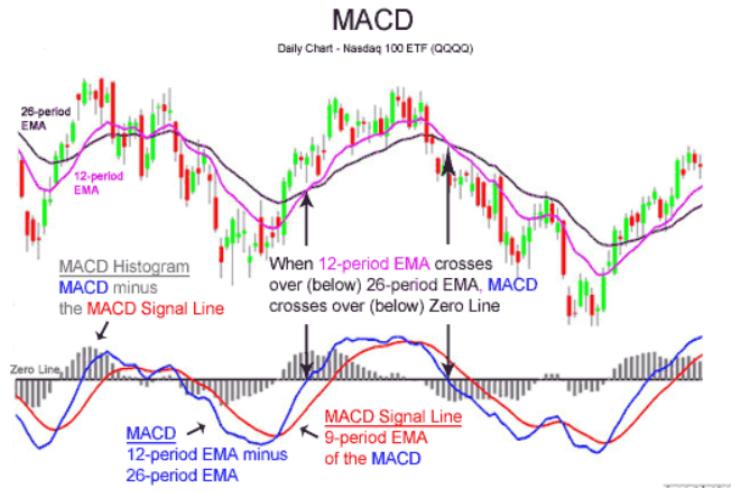


Figure 6.22:  $EMA_{12}(C)$  is represented by pink line.  $EMA_{26}(C)$  is represented by black line. MACD series is represented by blue line at the bottom. Signal is represented by red line. Histogram series is represented by column chart.

- MACD series goes upward and crosses signal series, equivalently histogram series change sign from negative to positive : Buy signal.

The keypoint about MACD is that this is a second order analysis : The first order (velocity) is MACD, then second order (acceleration) is obtained by the difference between MACD and itself applied by a low-pass filter.

## 11.4 Non-price-based indicators

In all above indicators, there is always the price in the formula of indicator. Here we discover indicators which are not based on price.

### 11.4.1 Put Call ratio

The put-call ratio  $P$  is calculated by dividing the number of traded put options (sell) by the number of traded call options (buy).

- $P = 1$  then bull = bear .
- $P > 0.7$  then bearish environment.
- $P < 0.5$  then bullish environment.

The lowest level of put-call ratio was 0.39, set in March 2000 at the peak of the dot-com bubble.

### 11.4.2 VIX

VIX is the ticker symbol and the popular name for the Chicago Board Options Exchange's CBOE Volatility Index. VIX is a popular measure of 30-day expected volatility of the S&P500 index.

$$VIX = \sigma = \sqrt{\frac{2e^{r\tau}}{\tau} \left( \int_0^F \frac{P(K)}{K^2} dK + \int_F^\infty \frac{C(K)}{K^2} dK \right)}$$

where:

- $r$  is the risk-free rate.
- $\tau$  is the number of average days in a month (30 days).
- $F$  is the 30-day forward price on the S&P 500.
- $P(K)$  and  $C(K)$  are prices for puts and calls with strike  $K$  and 30 days to maturity.

An other formula for calculate VIX is given in figure 6.23 :

$$\sigma^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T} \left[ \frac{F}{K_0} - 1 \right]^2$$

where

$\sigma$	$VIX \text{ index} = \sigma \times 100$	$\Delta K_i$	Interval between strike prices – half the difference between the strike on either side of $K_i$ :
$T$	Time to expiration (in years)	$\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$	
$F$	Option-implied forward price	$R$	Risk-free interest rate to expiration
$K_0$	First strike equal to or otherwise immediately below the forward index level, $F$	$Q(K_i)$	The midpoint of the bid-ask spread for each option with strike $K_i$ .
$K_i$	Strike price of the $i^{\text{th}}$ out-of-the-money option; a call if $K_i > K_0$ and a put if $K_i < K_0$ ; both put and call if $K_i = K_0$ .		

Figure 6.23: VIX formula.

Interpretation of VIX :

- High VIX is a signal for downward market
- Low VIX is a signal for upward market

#### **11.4.3 Margin debt level**

The margin debt level (1.6) can be used as an indicator for analyzing the market :

- High margin debt level → more lending : Bullish
- Low margin debt level → less lending : Bearish

#### **11.4.4 Mutual fund cash level**

Mutual fund cash level is the percentage of a mutual fund's total assets that are held in cash or cash equivalents. Most mutual funds keep approximately 5% of the portfolio in cash and equivalents in order to handle transactions and day-to-day redemptions of shares. Observing the cash level of funds is a way to measure investor for analyzing the market :

- When cash level is low : Bullish
- When cash level is high : Bearish

#### **11.4.5 TRIN or arms index**

$$\text{TRIN} = \frac{\text{Advancing Stocks}/\text{Declining Stocks}}{\text{Advancing Volume}/\text{Declining Volume}}$$

where :

- Advancing Stocks : Number of stocks that are higher on the day.
- Declining Stocks : Number of stocks that are lower on the day.
- Advancing Volume : Total volume of all advancing stocks.
- Declining Volume : Total volume of all declining stocks.

Relation with market :

- $\text{TRIN} = 1$  : Same advancing and declining
- $\text{TRIN} > 1$  : More advancing → bullish
- $\text{TRIN} < 1$  : More declining → bearish

### **11.5 Business cycle**

Business cycles (also called cycle periode) are intervals of *expansion* followed by *recession or contraction* in economic performance.

- Some analysts consider the four year US presidential election cycle to be relevant.
- Others consider the 10, 18 or 54 year periode (called contra wave) to be of more relevant.

### 11.5.1 Elliott wave

The Elliott wave is a form of technical analysis that financial traders use to analyze financial market cycles and forecast market trends by identifying extremes in investor psychology and price levels, such as highs and lows, by looking for patterns in prices. We see an example of Elliott wave in figure 6.24.

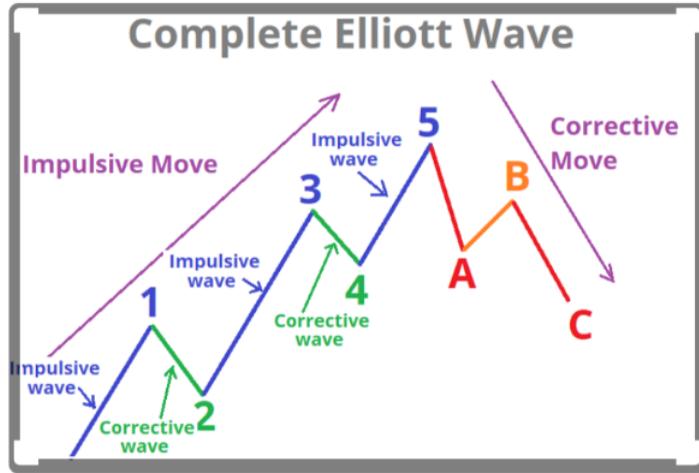


Figure 6.24: Elliott waves : In a bull market (impulsive move) we can observe 5 waves pattern characterized by 3 distinct strong upward (impulsive waves) with 2 retracement waves (correction waves) between 3 impulse waves. After the bull market, we see that it is the bear market (corrective move) with three waves : and two downward waves and one corrective wave.

Other scenarios of Elliott wave can be seen in figure 6.25. The length of wave can be long (year or even century) or small (day, month).

#### 11.5.1.1 Relation with Fibonacci sequence

There is a relationship between the Elliott Wave Principle and the Fibonacci sequence (0, 1, 1, 2, 3, 5, 8, ...). First we define Fibonacci ratios by dividing one number in the Fibonacci sequence by another number in the sequence (no necessary to be consecutive) and this converges, for example :

- In case between  $n$  and  $n - 1$  :

$$\frac{3}{2} = 1.5, \frac{5}{3} = 1.666, \frac{8}{5} = 1.6, \frac{13}{8} = 1.625, \frac{21}{13} = 1.615$$

$$\text{or } \lim_{n \rightarrow \infty} \frac{F_n}{F_{n-1}} \rightarrow 1.618$$

- In case between  $n - 1$  and  $n$  :

$$\frac{2}{3} = 0.666, \frac{3}{5} = 0.6, \frac{5}{8} = 0.625, \frac{8}{13} = 0.615, \frac{13}{21} = 0.619$$

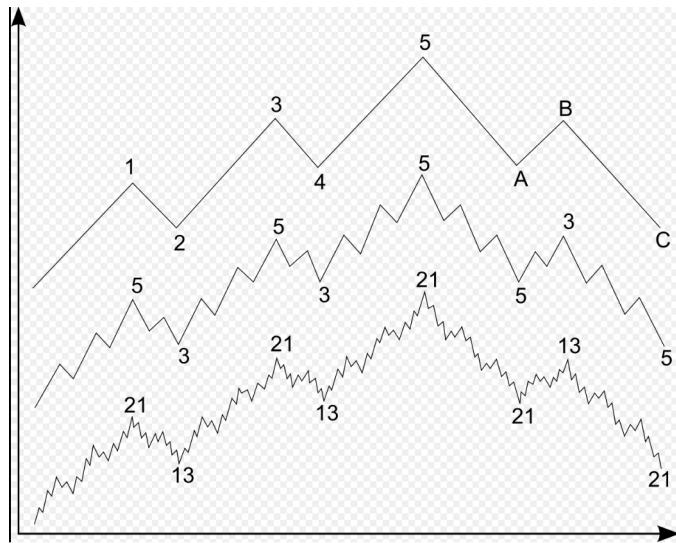


Figure 6.25: Other scenarios of Elliott wave.

$$\text{or } \lim_{n \rightarrow \infty} \frac{F_{n-1}}{F_n} \rightarrow 0.618.$$

- In case between  $n$  and  $n - 2$ :

$$\frac{5}{2} = 2.5, \frac{8}{3} = 2.66, \frac{13}{5} = 2.6, \frac{21}{8} = 2.625, \frac{34}{13} = 2.615$$

$$\text{or } \lim_{n \rightarrow \infty} \frac{F_n}{F_{n-2}} \rightarrow 2.618.$$

- In case between  $n - 2$  and  $n$ :

$$\frac{2}{5} = 0.4, \frac{3}{8} = 0.375, \frac{5}{13} = 0.384, \frac{8}{21} = 0.380, \frac{13}{34} = 0.382$$

$$\text{or } \lim_{n \rightarrow \infty} \frac{F_{n-2}}{F_n} \rightarrow 0.382.$$

Note that in case between  $n - 1$  and  $n$  or between  $n$  and  $n + 1$ , we have the Golden ratio, which is the solution of equation  $1 + \frac{1}{x} = x$  or  $x = \frac{a+b}{a} = \frac{a}{b}$ . The most commonly used Fibonacci ratios in Elliott Wave analysis are 0.618, 1.618, 2.618, 0.382 and 0.236. Then the relation with Elliott wave is that the **lengths of the waves** tend to equal to the length of preceding wave multiplied by a Fibonacci ratio. Fibonacci ratios are believed to represent key levels of support and resistance in financial markets.

# Chapter 7

## Annex

### 1 Diffusion problem

The diffusion problem first an example of Brownian Motion.

Let  $\rho(x, t)$  be the number density, which means number of particles per unit volume around  $x$  at the time  $t$ . For simplest case,  $x$  is one-dimensional space.

At the same position  $x$ , after a time interval  $\tau$  :  $\rho(x, t + \tau)$ . We construct the diffusion equation by expressing  $\rho(x, t + \tau)$  in two ways.

In the first way, by Taylor series w.r.t time  $t$ :

$$\rho(x, t + \tau) = \rho(x, t) + \tau \frac{\partial \rho(x, t)}{\partial t} + \dots$$

In the second way, by Taylor series w.r.t position  $x$  :

$$\begin{aligned} \rho(x, t + \tau) &= \mathbb{E}_\Delta(\rho(x - \Delta, t)) \\ &= \int_{-\infty}^{\infty} \rho(x - \Delta, t) \phi(\Delta) d\Delta \\ &= \int_{-\infty}^{\infty} \left[ \rho(x, t) - \Delta \frac{\partial \rho(x, t)}{\partial x} + \frac{\Delta^2}{2} \frac{\partial^2 \rho(x, t)}{\partial x^2} - \dots \right] \phi(\Delta) d\Delta \\ &= \rho(x, t) \int_{-\infty}^{\infty} \phi(\Delta) d\Delta - \frac{\partial \rho(x, t)}{\partial x} \int_{-\infty}^{\infty} \Delta \phi(\Delta) d\Delta + \frac{\partial^2 \rho(x, t)}{\partial x^2} \int_{-\infty}^{\infty} \frac{\Delta^2}{2} \phi(\Delta) d\Delta - \dots \\ &= \rho(x, t) + \frac{\partial^2 \rho(x, t)}{\partial x^2} \int_{-\infty}^{\infty} \frac{\Delta^2}{2} \phi(\Delta) d\Delta + HigherOrderEvenMoments \end{aligned}$$

where  $\Delta$  is magnitude of jump (from  $x - \Delta$  to  $x$ ),  $\phi(\Delta)$  is probability density function, so the probability for a jump  $\Delta$  is  $\phi(\Delta)d\Delta$ . In short, the density  $\rho(x, t + \tau)$  is obtained by the expected value of all possible previous jumps.

Note that  $\phi(\Delta)$  is symmetric, then in the second way, all odd elements (related to  $\Delta^{(2k+1)}$ ) are 0. Also, we have  $\int_{-\infty}^{\infty} \phi(\Delta) d\Delta = 1$ . Hence, by combining the first and the second and taking until second order for  $\frac{\partial \rho}{\partial x}$  :

$$\tau \frac{\partial \rho(x, t)}{\partial t} = \frac{\partial^2 \rho(x, t)}{\partial x^2} \int_{-\infty}^{\infty} \frac{\Delta^2}{2} \phi(\Delta) d\Delta$$

Let's coefficient of diffusion  $D = \int_{-\infty}^{\infty} \frac{\Delta^2}{2\tau} \phi(\Delta) d\Delta$ , the above equation become :

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}$$

Note that in small time  $\tau, D$  is considered as constant. Then the solution for diffusion equation :

$$\rho(x, t) = \frac{N}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Consider during a time  $\tau$ ,  $D(\tau) = \frac{\sigma^2}{2\tau}$ , where  $\sigma^2$  is the second moment of probability density function (pdf)  $\phi$ . Hence, we observe that  $\sigma^2(t)$  in the solution  $\rho(x, t)$  is  $\sigma^2(t) = 2Dt = 2\frac{\sigma^2}{2\tau}\tau = \frac{t}{\tau}\sigma^2$ .

In analogy with discret case, if 2 variable independent  $X$  and  $Y$   $\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2$ , we have  $\sigma^2(2\tau) = 2\sigma^2$

Let  $x(t)$  is the distance of the particle from its starting position  $x(0)$  at time  $t$ . The displacement of  $x(t)$  is characterized by the mean quadratic  $\mathbb{E}(x^2(t)) = 2Dt$ .

## 2 Experiment code for fitting MA

```
import numpy as np

""" For MA(1) """
N = 10000
theta = 0.5
eps = np.random.randn(N+1)
y = eps[1:] + theta * eps[:N]

# Solution
E_esp = np.zeros(N+1)
E_theta = 0
for j in range(100):
    for i in range(N):
        E_esp[i+1] = y[i] - E_theta * E_esp[i]
        E_theta = np.linalg.lstsq(E_esp[:N][:, None], y)[0][0]

print(E_theta)
# Check by variance
var_y = np.std(y) ** 2
theta_pos = np.sqrt(var_y - 1)
# Note that theta can be -np.sqrt(var_y - 1)
```

```

""" For MA(2) """
N = 10000
theta_1 = 0.25
theta_2 = 0.75
eps = np.random.randn(N+2)
y = eps[2:] + theta_1*eps[1:-1] + theta_2 * eps[:-2]

# Solution
E_esp = np.zeros(N+2)
E_theta_1 = 0
E_theta_2 = 0
for j in range(100):
    for i in range(N):
        E_esp[i+2] = y[i] - E_theta_1 * E_esp[i+1] - E_theta_2 * E_esp[i]
E_theta_2, E_theta_1 = np.linalg.lstsq(np.concatenate((E_esp[:-2][:,None], E_esp[2:]), axis=1), y, rcond=None)[0]

print(E_theta_2, E_theta_1)

var_y = np.std(y)**2
print(1 + theta_1**2 + theta_2**2)
print(var_y)

```