In the following, we run a small example of an iterative version of our algorithm Algorithm 3 GuessingBivar($\boldsymbol{u}$).

Context

Let $\mathbb{K} = \mathbb{F}_{97}$

We consider the ideal of relation I_u generated by the following reduced Gröbner basis G_u

$$\begin{aligned}
&x^5 + 60x^4 + 45x^3 + 77x^2 + 10x + 28, \\
&yx^3 + yx^2 + 84yx + 69y + 30x^4 + 59x^3 + 27x^2 + 44x + 61, \\
&y^2x^2 + 34y^2x + 42y^2 + 12yx^2 + 20yx + 19y + 28x^4 + 4x^3 + 18x^2 + 43x + 70, \\
&y^3 + 88y^2x + 86y^2 + 84yx^2 + 33yx + 31y + 88x^4 + 65x^3 + 96x^2 + 67x + 45
\end{aligned}$$

We have the parameter $d_x = 5, d_y = 3$ and set $D_x = 10, D_y = 6$.

Input of the pseudo-Euclidean algorithm

$$r_{-1} := (29x^{10} + 5x^9 + 31x^8 + 39x^7 + 23x^6 + 18x^5 + 58x^4 + 53x^3 + 38x^2 + 36x + 83)y^7$$

and

$$\begin{aligned}
r_0 := &(29x^{10} + 5x^9 + 31x^8 + 39x^7 + 23x^6 + 18x^5 + 58x^4 + 53x^3 + 38x^2 + 36x + 83)y^6 \\
&+ (79x^{10} + 95x^9 + 10x^8 + 45x^7 + 66x^6 + 74x^5 + 42x^4 + 75x^3 + 57x^2 + 90x + 64)y^5 \\
&+ (22x^{10} + 40x^9 + x^8 + 35x^7 + 47x^6 + 41x^5 + 39x^4 + 63x^3 + 43x^2 + 41x + 80)y^4 \\
&+ (38x^{10} + 80x^9 + 91x^8 + 65x^7 + 38x^5 + 41x^4 + 4x^3 + 56x^2 + 4x + 17)y^3 \\
&+ (3x^{10} + 20x^9 + 67x^8 + 78x^7 + 50x^6 + 75x^5 + 79x^4 + 26x^3 + 6x^2 + 34x + 73)y^2 \\
&+ (11x^{10} + 9x^9 + 61x^8 + 23x^7 + 24x^6 + 93x^5 + 19x^4 + 7x^3 + 89x^2 + 10x + 16)y \\
&+ (77x^{10} + 17x^9 + 3x^8 + 8x^7 + 13x^6 + 86x^5 + 20x^4 + 70x^3 + 49x^2 + 70x + 50)1
\end{aligned}$$

Computation of the quotient matrix Q_0 and the relation of $v_{*,0}$

First we compute the C-relation on the first slice $lc(r_0)$

$f_0 = GuessingUnivar(lc(r_0)) = x^5 + 60x^4 + 45x^3 + 77x^2 + 10x + 28$

Since $lc(r_0) = lc(r_{-1})$ we deduce that $a_0 = -1$ and

$$\begin{aligned}
r_1^{tmp} := &yr_0 + a_0 r_{-1} \\
= &(79x^{10} + 95x^9 + 10x^8 + 45x^7 + 66x^6 + 74x^5 + 42x^4 + 75x^3 + 57x^2 + 90x + 64)y^6 \\
&+ (22x^{10} + 40x^9 + x^8 + 35x^7 + 47x^6 + 41x^5 + 39x^4 + 63x^3 + 43x^2 + 41x + 80)y^5 \\
&+ (38x^{10} + 80x^9 + 91x^8 + 65x^7 + 38x^5 + 41x^4 + 4x^3 + 56x^2 + 4x + 17)y^4 \\
&+ (3x^{10} + 20x^9 + 67x^8 + 78x^7 + 50x^6 + 75x^5 + 79x^4 + 26x^3 + 6x^2 + 34x + 73)y^3 \\
&+ (11x^{10} + 9x^9 + 61x^8 + 23x^7 + 24x^6 + 93x^5 + 19x^4 + 7x^3 + 89x^2 + 10x + 16)y^2 \\
&+ (77x^{10} + 17x^9 + 3x^8 + 8x^7 + 13x^6 + 86x^5 + 20x^4 + 70x^3 + 49x^2 + 70x + 50)y
\end{aligned}$$

$b_0 := HankelSolver(lc(r_0), lc(r_1^{tmp}), f_0)$:

We can then compute the quotient matrix

$$Q_0 = \begin{bmatrix} 0 & 1 \\ a_0 & y - b_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 96 & y + 4x^4 + 53x^3 + 33x^2 + 92x + 59 \end{bmatrix}$$

also obtain the next remainder

$$\begin{aligned} r_1 :=& a_0 r_{-1} + (y - b_0) r_0 \\ =& (9x^{10} + 10x^9 + 38x^8 + 53x^7 + 42x^6 + 62x^5 + 28x^4 + 14x^3 + 49x^2 + 44x + 15)y^5 + \\ & (11x^{10} + 37x^9 + 72x^8 + 38x^7 + 91x^6 + 91x^5 + 22x^4 + 23x^3 + 95x^2 + 44x + 89)y^4 + \\ & (48x^{10} + 83x^9 + 14x^8 + 81x^7 + 87x^5 + 47x^4 + 17x^3 + 23x^2 + 59x + 37)y^3 + \\ & (37x^{10} + 73x^9 + 59x^8 + 83x^7 + 12x^6 + 3x^5 + 52x^4 + 32x^3 + 49x^2 + 77x + 1)y^2 + \\ & (32x^{10} + 27x^9 + 69x^8 + 14x^7 + 87x^6 + 87x^5 + 78x^4 + 94x^3 + 58x^2 + 50x + 38)y + \\ & 86x^{10} + 54x^9 + 63x^8 + 26x^7 + 53x^6 + 73x^5 + 52x^4 + 3x^3 + 10x^2 + 31x + 36 \end{aligned}$$

Computation of Q_1, f_1 and r_2

With the same method, we can compute

$$Q_1 = \begin{bmatrix} 0 & 1 \\ 16x^4 + 31x^3 + 64x^2 + 52x + 82 & y + 25x^2 + 90x + 78 \end{bmatrix}, \quad f_1 = x^3 + x^2 + 84x + 69$$

and the remainder r_2, whose coefficient in y^5 has been eliminated by construction,

$$\begin{aligned} r_2 :=& a_1 r_0 + (y - b_1) r_1 \\ =& (84x^{10} + 42x^9 + 88x^8 + 94x^7 + 92x^6 + 5x^5 + 40x^4 + 79x^3 + 96x^2 + 14x + 51)y^4 \\ & + (41x^{10} + 90x^9 + 68x^8 + 19x^7 + 87x^6 + 27x^5 + 84x^4 + 84x^3 + 18x^2 + 31x + 33)y^3 \\ & + (64x^{10} + 73x^9 + 68x^8 + 54x^7 + 61x^6 + 23x^5 + 51x^4 + 16x^3 + 30x^2 + 54x + 8)y^2 \\ & + (13x^{10} + 38x^9 + 13x^8 + 55x^7 + 90x^6 + 80x^5 + 77x^4 + 38x^3 + 11x^2 + 21x + 69)y \\ & + 37x^{10} + 17x^9 + 61x^8 + 30x^7 + 48x^6 + 52x^5 + 54x^4 + 65x^3 + 83x^2 + 41x + 93 \end{aligned}$$

Computation of Q_2, f_2 and r_3

We continue with the computation of the remainder r_3, which requires the quotient matrix Q_2 and the C-relation f_2

$$Q_2 = \begin{bmatrix} 0 & 1 \\ 72x^2 + 46x + 2 & y + 72 + 67x \end{bmatrix}, \quad f_2 = x^2 + 34x + 42$$

and the remainder r_3, whose coefficient in y^4 has been eliminated by construction,

$$\begin{aligned} r_3 :=& a_2 r_1 + (y - b_2) r_2 \\ =& (23x^{10} + 62x^9 + 38x^8 + 40x^7 + 35x^6 + 58x^5 + 31x^4 + 4x^3 + 92x^2 + 53x + 41)y^2 \\ & + (93x^{10} + 3x^9 + 88x^8 + 41x^7 + 49x^6 + 64x^5 + 19x^4 + 3x^3 + 62x^2 + 42x + 43)y \\ & + 34x^{10} + 4x^9 + 94x^8 + 27x^7 + 83x^6 + 28x^5 + 87x^4 + 38x^3 + 54x^2 + 6x + 2 \end{aligned}$$

The degree sequence is no longer normal

At this step, we have $2 = \deg(r_3) \neq \deg(r_2) - 1 = 3$.

We can stop the computation of the remainders, we have enough informations to reconstruct a Gröbner basis of the ideal of relation.

`GuessingBivar`

`GuessingBivar` initialization sets f_0 as the first element g_0 of the Grobner basis.

The next computations correspond to the `for` loop on line 6 of Algorithm 3 `GuessingBivar(u)` .

First loop

The matrix product Q_1Q_0 is

$$\begin{bmatrix} 96 & y+4x^4+53x^3+33x^2+92x+59 \\ 96y+72x^2+7x+19 & y^2+(4x^4+53x^3+58x^2+85x+40)y+72x^4+67x^3+74x^2+23x+83 \end{bmatrix}$$

whose entry on the first line, second column is $t_1 = y+4x^4+53x^3+33x^2+92x+59$. Its modular product $g_1 := t_1 f_1 \bmod f_0$ is the second element of the Grobner basis.

$$g_1 = (x^3+x^2+84x+69)y+30x^4+59x^3+27x^2+44x+61$$

Second loop

The matrix product $Q_2Q_1Q_0 = \begin{bmatrix} s_2 & t_2 \\ s_3 & t_3 \end{bmatrix}$ where

$$\begin{aligned} s_2 =& 96y+72x^2+7x+19 \\ t_2 =& y^2+(4x^4+53x^3+58x^2+85x+40)y+72x^4+67x^3+74x^2+23x+83 \\ s_3 =& 96y^2+(72x^2+37x+44)y+71x^3+52x^2+82x+8 \\ t_3 =& y^3+(4x^4+53x^3+58x^2+55x+15)y^2+(53x^4+74x^3+51x^2+78x+20)y \\ &+33x^4+14x^3+70x^2+87x+71 \end{aligned}$$

Its modular product $g_2 := t_2 f_2 \bmod f_0$ is the third element of the Grobner basis.

$$g_2 = (x^2+34x+42)y^2+(58x^4+62x^3+38x^2+90x+4)y+50x^4+39x^3+36x^2+34x+94$$

After the loop

The loop stops after its second iteration because the degree sequence is no longer normal. The last step 11 of Algorithm 3 `GuessingBivar(u)` appends $g_3 := t_{d_y} = t_3$ to the Grobner basis.

Output

The minimal Grobner basis on output is $G := \{f_0, g_1, g_2, t_{d_y}\}$, with

$$\begin{aligned} f_0 =& x^5+60x^4+45x^3+77x^2+10x+28 \\ g_1 =& (x^3+x^2+84x+69)y+30x^4+59x^3+27x^2+44x+61, \\ g_2 =& (x^2+34x+42)y^2+(58x^4+62x^3+38x^2+90x+4)y+50x^4+39x^3+36x^2+34x+94, \\ t_{d_y} =& y^3+(4x^4+53x^3+58x^2+55x+15)y^2+(53x^4+74x^3+51x^2+78x+20)y \\ &+33x^4+14x^3+70x^2+87x+71 \end{aligned}$$

This Grobner basis do not match G_u because G is only minimal whereas G_u is reduced, *i.e* minimal and the elements in the basis are reduced w.r.t. the other ones.

Note that G has exactly the same leading monomials as all the ones in G_u. Furthermore, the `Maple` command `Groebner:-InterReduce(G,plex(y,x),characteristic=97)` yields the same polynomials as in G_u, hence the output G is a minimal Gröbner of $I(u)$.