In the following, we run our algorithm Algorithm 3 GuessingBivar($\boldsymbol{u}$) on the same example described in example_iterative.md. The Maple code is available at https://github.com/ktran11/CrecbiseqGuessing/blob/main/maple/bivariate_recursive.mpl.

Context

Let $\mathbb{K} = \mathbb{F}_{97}$. We consider the ideal of relation I_u generated by the following reduced Gröbner basis G_u

$$\begin{aligned}
&x^5 + 60x^4 + 45x^3 + 77x^2 + 10x + 28,\\
&yx^3 + yx^2 + 84yx + 69y + 30x^4 + 59x^3 + 27x^2 + 44x + 61,\\
&y^2x^2 + 34y^2x + 42y^2 + 12yx^2 + 20yx + 19y + 28x^4 + 4x^3 + 18x^2 + 43x + 70,\\
&y^3 + 88y^2x + 86y^2 + 84yx^2 + 33yx + 31y + 88x^4 + 65x^3 + 96x^2 + 67x + 45
\end{aligned}$$

We have the parameter $d_x = 5, d_y = 3$ and set $D_x = 10, D_y = 6$.

We build a sequence $\mathbf{u} = (u_{i,j})_{(i,j)\in\mathbb{N}^2}$ with random initial terms that we extend using the relations given by G_u.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	77	17	3	8	13	86	20	70	49	70	50
$\vdots$	11	9	61	23	24	93	19	7	89	10	16
$\vdots$	3	20	67	78	50	75	79	26	6	34	73
$\vdots$	38	80	91	65	0	38	41	4	56	4	17
$\vdots$	22	40	1	35	47	41	39	63	43	41	80
$j=1$	79	95	10	45	66	74	42	75	57	90	64
$j=0$	29	5	31	39	23	18	58	53	38	36	83
$u_{i,j}$	$i=0$	$i=1$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

First we compute the C-relation on the first slice,

$j=0$	29	5	31	39	23	18	58	53	38	36	83
$u_{i,j}$	$i=0$	$i=1$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

which can be read off from the polynomial $lc(r_{-1})$.

$f_0 = GuessingUnivar(lc(r_{-1})) = x^5 + 60x^4 + 45x^3 + 77x^2 + 10x + 28$

Input of the Half-Gcd-Seq algorithm

$$r_{-1} := (29x^{10} + 5x^9 + 31x^8 + 39x^7 + 23x^6 + 18x^5 + 58x^4 + 53x^3 + 38x^2 + 36x + 83)y^7,$$

$$
\begin{aligned}
r_0 :=&(29x^{10}+5x^9+31x^8+39x^7+23x^6+18x^5+58x^4+53x^3+38x^2+36x+83)y^6\\
&+(79x^{10}+95x^9+10x^8+45x^7+66x^6+74x^5+42x^4+75x^3+57x^2+90x+64)y^5\\
&+(22x^{10}+40x^9+x^8+35x^7+47x^6+41x^5+39x^4+63x^3+43x^2+41x+80)y^4\\
&+(38x^{10}+80x^9+91x^8+65x^7+38x^5+41x^4+4x^3+56x^2+4x+17)y^3\\
&+(3x^{10}+20x^9+67x^8+78x^7+50x^6+75x^5+79x^4+26x^3+6x^2+34x+73)y^2\\
&+(11x^{10}+9x^9+61x^8+23x^7+24x^6+93x^5+19x^4+7x^3+89x^2+10x+16)y\\
&+(77x^{10}+17x^9+3x^8+8x^7+13x^6+86x^5+20x^4+70x^3+49x^2+70x+50)1,
\end{aligned}
$$

$f_0 = x^5+60x^4+45x^3+77x^2+10x+28$ and $k = \lfloor D_y/2 \rfloor = 3$.

Root Call Input: $(r_{-1}, r_0, f_0, 3)$

We have $d := \lceil k/2 \rceil = 2$ so the first recursive call has for input

$$
r_{-1}\lceil_{2(d-1)} := (29x^{10}+5x^9+31x^8+39x^7+23x^6+18x^5+58x^4+53x^3+38x^2+36x+83)y^2
$$

and

$$
\begin{aligned}
r_0\lceil_{2(d-1)-1} :=&(29x^{10}+5x^9+31x^8+39x^7+23x^6+18x^5+58x^4+53x^3+38x^2+36x+83)y\\
&+(79x^{10}+95x^9+10x^8+45x^7+66x^6+74x^5+42x^4+75x^3+57x^2+90x+64)
\end{aligned}
$$

In the recursive call with Input: $(r_{-1}\lceil_2, r_0\lceil_1, f_0, 1)$

We have $d := \lceil 1/2 \rceil = 1$ and

$$
(r_{-1}\lceil_2)\lceil_0 = (29x^{10}+5x^9+31x^8+39x^7+23x^6+18x^5+58x^4+53x^3+38x^2+36x+83)
$$

and $r_0 = 0$.

The first recursive call is on the input $(r_{-1}\lceil_2)\lceil_0, 0, f_0, 0)$ and it outputs $I_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, [], []$

On line 4, we update the input by computing

$$
\begin{bmatrix} r_{-1}\lceil_2 \\ r_0\lceil_1 \end{bmatrix} = I_2 \begin{bmatrix} r_{-1}\lceil_2 \\ r_0\lceil_1 \end{bmatrix}
$$

The remainders do not satisfy the condition $\deg_y(r_{-1}) - 1 > \deg_y(r_0)$ so the computation continues.

and we compute, as in the iterative version, the quotient matrix Q_0 and the relation f_0 by calling the algorithm

$$
\begin{aligned}
Q_0, f_0 &:= QuoTwoVar(r_{-1}\lceil_2, r_0\lceil_1, f_0)\\
&= \begin{bmatrix} 0 & 1 \\ 96 & y+4x^4+53x^3+33x^2+92x+59 \end{bmatrix}, x^5+60x^4+45x^3+77x^2+10x+28
\end{aligned}
$$

We apply the transformation on the input and obtain

$$
\begin{bmatrix} r_0\lceil_1 \\ r_1\lceil_{-1} \end{bmatrix} := Q_0 \cdot_{f_0} \begin{bmatrix} r_{-1}\lceil_2 \\ r_0\lceil_1 \end{bmatrix}
$$

The second recursive call starts with the input $((r_0\lceil_1)\lceil_0, 0, f_0, 0)$ and outputs $I_2, [], []$.

Finally, we return in the root call with the output $Q_0, [Q_0], [f_0]$.

Second part of the Root call Input $(r_{-1}, r_0, f_0, 3)$

We update the input with the matrix Q_0 and obtain

$$\begin{aligned} r_0 :=& (29x^{10} + 5x^9 + 31x^8 + 39x^7 + 23x^6 + 18x^5 + 58x^4 + 53x^3 + 38x^2 + 36x + 83)y^6 \\ &+ (79x^{10} + 95x^9 + 10x^8 + 45x^7 + 66x^6 + 74x^5 + 42x^4 + 75x^3 + 57x^2 + 90x + 64)y^5 \\ &+ (22x^{10} + 40x^9 + x^8 + 35x^7 + 47x^6 + 41x^5 + 39x^4 + 63x^3 + 43x^2 + 41x + 80)y^4 \\ &+ (38x^{10} + 80x^9 + 91x^8 + 65x^7 + 38x^5 + 41x^4 + 4x^3 + 56x^2 + 4x + 17)y^3 \\ &+ (3x^{10} + 20x^9 + 67x^8 + 78x^7 + 50x^6 + 75x^5 + 79x^4 + 26x^3 + 6x^2 + 34x + 73)y^2 \\ &+ (11x^{10} + 9x^9 + 61x^8 + 23x^7 + 24x^6 + 93x^5 + 19x^4 + 7x^3 + 89x^2 + 10x + 16)y \\ &+ (77x^{10} + 17x^9 + 3x^8 + 8x^7 + 13x^6 + 86x^5 + 20x^4 + 70x^3 + 49x^2 + 70x + 50)1, \end{aligned}$$

and

$$\begin{aligned} r_1 =& (9x^{10} + 10x^9 + 38x^8 + 53x^7 + 42x^6 + 62x^5 + 28x^4 + 14x^3 + 49x^2 + 44x + 15)y^5 + \\ & (11x^{10} + 37x^9 + 72x^8 + 38x^7 + 91x^6 + 91x^5 + 22x^4 + 23x^3 + 95x^2 + 44x + 89)y^4 + \\ & (48x^{10} + 83x^9 + 14x^8 + 81x^7 + 87x^5 + 47x^4 + 17x^3 + 23x^2 + 59x + 37)y^3 + \\ & (37x^{10} + 73x^9 + 59x^8 + 83x^7 + 12x^6 + 3x^5 + 52x^4 + 32x^3 + 49x^2 + 77x + 1)y^2 + \\ & (32x^{10} + 27x^9 + 69x^8 + 14x^7 + 87x^6 + 87x^5 + 78x^4 + 94x^3 + 58x^2 + 50x + 38)y + \\ & 86x^{10} + 54x^9 + 63x^8 + 26x^7 + 53x^6 + 73x^5 + 52x^4 + 3x^3 + 10x^2 + 31x + 36 \end{aligned}$$

The remainders do not satify the condition of line 5 so we can compute Q_1, f_1 with $QuoTwoVar(r_0!! \upharpoonleft_2, r_1!! \upharpoonleft_1, f_0)$ and we obtain

$$Q_1 = \begin{bmatrix} 0 & 1 \\ 16x^4 + 31x^3 + 64x^2 + 52x + 82 & y + 25x^2 + 90x + 78 \end{bmatrix}, \quad f_1 = x^3 + x^2 + 84x + 69$$

We apply the transformation and obtain

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} := Q_1 \cdot_{f_0} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}$$

with

$$\begin{aligned} r_2 =& (84x^{10} + 42x^9 + 88x^8 + 94x^7 + 92x^6 + 5x^5 + 40x^4 + 79x^3 + 96x^2 + 14x + 51)y^4 \\ &+ (41x^{10} + 90x^9 + 68x^8 + 19x^7 + 87x^6 + 27x^5 + 84x^4 + 84x^3 + 18x^2 + 31x + 33)y^3 \\ &+ (64x^{10} + 73x^9 + 68x^8 + 54x^7 + 61x^6 + 23x^5 + 51x^4 + 16x^3 + 30x^2 + 54x + 8)y^2 \\ &+ (13x^{10} + 38x^9 + 13x^8 + 55x^7 + 90x^6 + 80x^5 + 77x^4 + 38x^3 + 11x^2 + 21x + 69)y \\ &+ 37x^{10} + 17x^9 + 61x^8 + 30x^7 + 48x^6 + 52x^5 + 54x^4 + 65x^3 + 83x^2 + 41x + 93 \end{aligned}$$

Then, we have the second recursive call on the input $(r_1\upharpoonleft_2, r_2\upharpoonleft_1, f_0, 1)$

In the recursive call with Input: $(r_1\upharpoonleft_2, r_2\upharpoonleft_1, f_0, 1)$

As before, we have $d := \lceil 1/2 \rceil = 1$ and

$$(r_1\upharpoonleft_2)\upharpoonleft_0 = (9x^{10} + 10x^9 + 38x^8 + 53x^7 + 42x^6 + 62x^5 + 28x^4 + 14x^3 + 49x^2 + 44x + 15)$$

and $r_2 = 0$.

The first recursive call is on the input $(r_1\upharpoonleft_2)\upharpoonleft_0, 0, f_0, 0)$ and it outputs $I_2, [], []$

On line 4, we update the input by computing

$$\begin{bmatrix} r_1\upharpoonright_2 \\ r_2\upharpoonright_1 \end{bmatrix} = I_2 \begin{bmatrix} r_1\upharpoonright_2 \\ r_2\upharpoonright_1 \end{bmatrix}$$

The remainders do not satisfy the condition $\deg_y(r_1) - 1 > \deg_y(r_2)$ so the computation continues.

and we compute the quotient matrix Q_2 and the relation f_2 by calling the algorithm

$$\begin{aligned} Q_2, f_2 &:= QuoTwoVar(r_1\upharpoonright_2, r_2\upharpoonright_1, f_0) \\ &= \begin{bmatrix} 0 & 1 \\ 72x^2 + 46x + 2 & y + 72 + 67x \end{bmatrix}, x^2 + 34x + 42 \end{aligned}$$

We apply the transformation on the input and obtain

$$\begin{bmatrix} r_2\upharpoonright_1 \\ r_3\upharpoonright_{-1} \end{bmatrix} := Q_2 \cdot_{f_0} \begin{bmatrix} r_1\upharpoonright_2 \\ r_2\upharpoonright_1 \end{bmatrix}$$

The second recursive call starts with the input $((r_0\upharpoonright_1)\upharpoonright_0, 0, f_0, 0)$ and outputs $I_2, [], []$.

Finally, we return in the root call with the output $Q_2, [Q_2], [f_2]$.

Back to the root call

In the first recursive call, we have $R, T, \mathcal{F}_0 := Q_0, [Q_0], [f_0]$ and we have compute the quotient matrix and the relation Q_1, f_1 during the root call.

In the second recursive call, we have $S, U, \mathcal{F}_1 := Q_2, [Q_2], [f_2]$.

The output of the root call is then $Q_2 Q_1 Q_0 \text{rem} f_0, [Q_0, Q_1, Q_2], [f_0, f_1, f_2]$

At the end of the call Half-Gcd-Seq, we have enough information to compute a Gröbner basis of $I(u)$.

`GuessingBivar`

The following description follows the iterative version of the algorithm, we have described in example_iterative.md.

`GuessingBivar` initialization sets f_0 as the first element g_0 of the Grobner basis.

The next computations correspond to the `for` loop on line 6 of Algorithm 3 `GuessingBivar(u)`.

First loop

The matrix product $Q_1 Q_0$ is

$$\begin{bmatrix} 96 & y + 4x^4 + 53x^3 + 33x^2 + 92x + 59 \\ 96y + 72x^2 + 7x + 19 & y^2 + (4x^4 + 53x^3 + 58x^2 + 85x + 40)y + 72x^4 + 67x^3 + 74x^2 + 23x + 83 \end{bmatrix}$$

whose entry on the first line, second column is $t_1 = y + 4x^4 + 53x^3 + 33x^2 + 92x + 59$. Its modular product $g_1 := t_1 f_1 \bmod f_0$ is the second element of the Grobner basis.

$$g_1 = (x^3 + x^2 + 84x + 69)y + 30x^4 + 59x^3 + 27x^2 + 44x + 61$$

Second loop

The matrix product

$$Q_2 Q_1 Q_0 = \begin{bmatrix} s_2 & t_2 \\ s_3 & t_3 \end{bmatrix}$$

where

$$\begin{aligned}
s_2 =& 96y + 72x^2 + 7x + 19 \\
t_2 =& y^2 + (4x^4 + 53x^3 + 58x^2 + 85x + 40)y + 72x^4 + 67x^3 + 74x^2 + 23x + 83 \\
s_3 =& 96y^2 + (72x^2 + 37x + 44)y + 71x^3 + 52x^2 + 82x + 8 \\
t_3 =& y^3 + (4x^4 + 53x^3 + 58x^2 + 55x + 15)y^2 + (53x^4 + 74x^3 + 51x^2 + 78x + 20)y \\
& + 33x^4 + 14x^3 + 70x^2 + 87x + 71
\end{aligned}$$

Its modular product $g_2 := t_2 f_2 \bmod f_0$ is the third element of the Grobner basis.

$$g_2 = (x^2 + 34x + 42)y^2 + (58x^4 + 62x^3 + 38x^2 + 90x + 4)y + 50x^4 + 39x^3 + 36x^2 + 34x + 94$$

After the loop

The loop stops after its second iteration because the degree sequence is no longer normal. The last step 11 of Algorithm 3 `GuessingBivar(u)` appends $g_3 := t_{d_y} = t_3$ to the Grobner basis.

Output

The minimal Grobner basis on output is $G := \{f_0, g_1, g_2, t_{d_y}\}$, with

$$\begin{aligned}
f_0 =& x^5 + 60x^4 + 45x^3 + 77x^2 + 10x + 28 \\
g_1 =& (x^3 + x^2 + 84x + 69)y + 30x^4 + 59x^3 + 27x^2 + 44x + 61, \\
g_2 =& (x^2 + 34x + 42)y^2 + (58x^4 + 62x^3 + 38x^2 + 90x + 4)y + 50x^4 + 39x^3 + 36x^2 + 34x + 94, \\
t_{d_y} =& y^3 + (4x^4 + 53x^3 + 58x^2 + 55x + 15)y^2 + (53x^4 + 74x^3 + 51x^2 + 78x + 20)y \\
& + 33x^4 + 14x^3 + 70x^2 + 87x + 71
\end{aligned}$$

This Grobner basis does not match G_u because G is only minimal whereas G_u is reduced, *i.e* minimal and the elements in the basis are reduced w.r.t. the other ones.

Note that G has exactly the same leading monomials as all the ones in G_u. Furthermore, the `Maple` command `Groebner:-InterReduce(G,plex(y,x),characteristic=97)` yields the same polynomials as in G_u, hence the output G is a minimal Gröbner of $I(u)$.

Note that the elements in our output Grobner basis have support in the (d_x, d_y)-box, *i.e.* their partial degrees are bounded by d_x and d_y.