MATH241 NOTES

11.1 Some Basics

- Know how to sketch xy-plane, xz-plane, yz-plane
- 3 axes divide 3D space into 8 octants, the first octant is where x,y,z > 0
- Know how to plot a point in 3D space (First locate x,y on xy-plane, then go up/down)
- Distance formula: $P = (x_0, y_0, z_0)$, $Q = (x_1, y_1, z_1)$
  dist = $|PQ| = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}$
- Circle/Disk vs. Sphere/Closed. Distinguish equations for them and sketch
- Sphere: Supposed $P = (x_0, y_0, z_0)$ is the center of a sphere and R is the radius
  $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$

11.2 Vectors

- Notation: $\bar{u} = 8\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ where $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, $\hat{k} = (0,0,1)$
- Note: Write $\bar{v} = 4\hat{\imath} + 0\hat{\jmath} - 1\hat{k}$ instead of $\bar{v} = 4\hat{\imath} - 1\hat{k}$
- Must know to compute 2D vectors with trigonometry (Memorize values of sin/cos from 0-π)
- Associated Definitions:
  - (A) Zero-vector is all 0's, denoted by $\bar{0}$
  - (B) Supposed P = (x_0, y_0, z_0) and Q = (x_1, y_1, z_1)
    $\overrightarrow{PQ}$ = "Q - P" = $(x_1 - x_0)\hat{\imath} + (y_1 - y_0)\hat{\jmath} + (z_1 - z_0)\hat{k}$
  - (C) Given $\bar{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, we define the length (aka magnitude/norm):
    $||\bar{v}|| = \sqrt{a^2 + b^2 + c^2}$
  - (D) A unit vector is a vector with length 1
  - (E) Supposed we have a vector $\bar{v}$ and we want the unit vector which points in the same direction as $\bar{v}$:
    $$\frac{\bar{v}}{||\bar{v}||}$$
  - (F) Two nonzero vectors are parallel if one is a scalar multiple of the other

11.3 Dot Product (Scalar Product)

- Definition: Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$
  $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$
  $\bar{a} \cdot \bar{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Product Properties:
  - (A) $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$
  - (B) $\bar{a} \cdot (\bar{b} \pm \bar{c}) = \bar{a} \cdot \bar{b} \pm \bar{a} \cdot \bar{c}$
  - (C) $\bar{a} \cdot \bar{a} = ||a||^2 = a_1^2 + a_2^2 + a_3^2$
- Additional Properties:
  - (A) $\bar{a} \cdot \bar{b} = ||\bar{a}||.||\bar{b}||.\cos\theta$, where θ = angle between $\bar{a}, \bar{b}$
  - (B) $\bar{a} \cdot \bar{b}$ iff $\bar{a} \perp \bar{b}$
  - (C) $\cos\theta = \frac{\bar{a} \cdot \bar{b}}{||\bar{a}||.||\bar{b}||}$
- Vector Projection:
  $Proj_{\bar{b}}\bar{a} = \frac{\bar{a} \cdot \bar{b}}{\bar{b} \cdot \bar{b}}.\bar{b}$

11.4 Cross Product

- Pre-definition: Define $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- Definition: Given $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$
  $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$
  $\bar{a} \times \bar{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\imath} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$ (Trick $\begin{vmatrix} a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{vmatrix}$)
- Product Properties:
  - (A) $\bar{a} \times (\bar{b} \pm \bar{c}) = \bar{a} \times \bar{b} \pm \bar{a} \times \bar{c}$
  - (B) $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$ /anticommutativity/
- Additional Properties:
  - (A) $||a \times b|| = ||\bar{a}||\,||\bar{b}||\,sin\,\theta$
  - (B) $\bar{a} \times \bar{b} = \bar{0}$ iff $\bar{a}$ and $\bar{b}$ are parallel
  - (C) $\bar{a} \times \bar{b}$ is $\perp$ to both $\bar{a}$ and $\bar{b}$ via right-hand rule!

11.5 Lines in Space

- Idea: Start with a single point $P = (x_0, y_0, z_0)$ and a direction vector $\bar{L} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

If we attached $\bar{L}$ to P, we see a line that goes forever!

- Parametric form: Suppose we have $P = (x_0, y_0, z_0)$ and $\bar{L} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, the parametric equations of the corresponding line are:

$$x = x_0 + at$$
$$y = y_0 + bt \qquad \text{where } t = \textit{any number}$$
$$z = z_0 + ct$$

- Vector equation of a line: All we do is put x,y,z from above into a vector:

$$\bar{r}(t) = (x_0 + at)\hat{\imath} + (y_0 + bt)\hat{\jmath} + (z_0 + ct)\hat{k}$$

- Symmetric Equation:
  + Normal case ($a, b, c \neq 0$)

    $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

  + Special case A: (either one of $a, b, c = 0$)

    Ex: $P = (1,2,3)$ and $\bar{L} = 0\hat{\imath} + 8\hat{\jmath} + 7\hat{k}$

    Here's the parametric form: $x = 1 + 0t$, $y = 2 + 8t$, $z = 3 + 7t$

    => Symmetric form: $\frac{y-2}{8} = \frac{z-3}{7}, x = 1$

  + Special case B: (2 of $a, b, c = 0$)

    Ex: $P = (1,2,3)$ and $\bar{L} = 42\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

    Here's the parametric form: $x = 1 + 42t$, $y = 2$, $z = 3$

    => Symmetric form: $y = 2, z = 3$ (No need to mention x)

- Distance between a point and line: Suppose we have a line with point P and direction $\bar{L}$ and suppose Q is some other point, then the perpendicular distance from Q to the line:

$$dist = \frac{\left\| \overrightarrow{PQ} \times \bar{L} \right\|}{\|\bar{L}\|}$$

(Note: We can extract P and $\bar{L}$ from a line given its form)