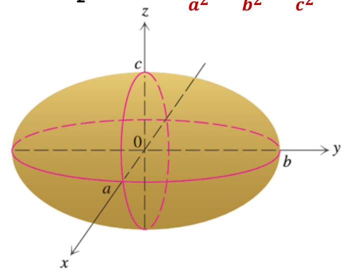
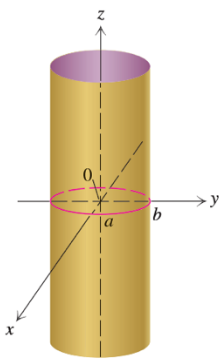
13.1 Functions of Several Variables

- Graphing examples:
  - (A) plane: $\boldsymbol{ax + by + cz = d}$
  - (B) sphere: $\boldsymbol{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2}$ => center at (x_0, y_0, z_0) & radius = R
  - (C) ellipsoid: $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$ or $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$ => center at (x_0, y_0, z_0)



Note: x-radius = a, y-radius = b, z-radius = c

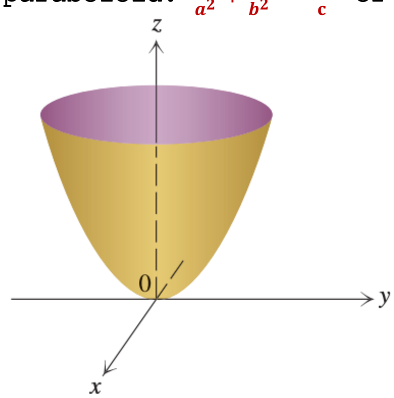
  - (D) cylinder: $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ or $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ or $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ or $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$ or etc.



Assume $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$

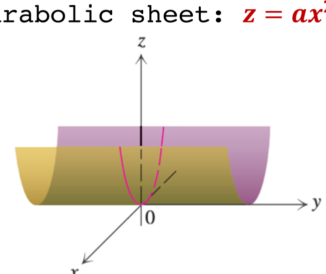
Note: - center at (x_0, y_0)
- x-radius = a, y-radius = b
- if $a = b$, circular cylinder
  if $a \neq b$, elliptic cylinder

  - (E) paraboloid: $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z+d}{c}}$ or etc.

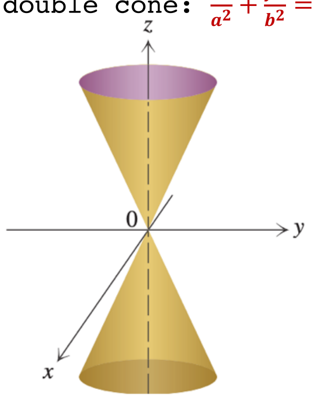


Note: - if $a = b$, circular paraboloid
  if $a \neq b$, elliptic paraboloid
- if $c > 0$, open up (in picture)
  if $c < 0$, open down
- if $d > 0$, shift down
  if $d < 0$, shift up

  - (F) parabolic sheet: $\boldsymbol{z = ax^2 + b}$ or $y = ax^2 + b$ or etc.



  - (E) double cone: $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2}$ single cone: $z = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$



- Level curves: Supposed we have a function f(x,y). If we choose a z-value, say z=c, and we plot the 2D result **f(x,y) = c**. We get a level curve.
- Level surfaces: Same idea for f(x,y,z)

13.3 Partial Derivatives

- Definition: Suppose f is a function of several variables. Define the partial derivative of f with respect to x to be the derivative treating x as the variable and all other variables as constant. Denote: $\frac{\partial f}{\partial x}$ or $f_x(x, y, \ldots)$
- Example: $f(x,y) = xe^{xy}$ => $f_x(x,y) = e^{xy} + xye^{xy}$ $f_y(x,y) = x^2e^{xy}$
- Interpretation: At a point (x_0, y_0)
  + $f_x(x_0, y_0)$ is the slope of tangent line in the x-direction (change in x-direction)
  + $f_y(x_0, y_0)$ is the slope of tangent line in the y-direction (change in y-direction)
- Second partial derivatives:
  + $f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2}$ or $f_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x}$ or $f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2}$ or $f_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y}$
  + Note: $f_{xy} = f_{yx}$ or $f_{xyz} = f_{zxy}$ or etc. in this class

13.5 Directional Derivatives

- Definition: Suppose f is a function and $\bar{u}$ is **unit vector** indicating direction. We define the directional derivative of f in the direction of ū:
  + (3D) if $\bar{u} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$: $D_{\bar{u}}f(x,y,z) = af_x + bf_y + cf_z$ (scalar)
  + (2D) if $\bar{u} = a\hat{\imath} + b\hat{\jmath}$: $\boldsymbol{D_{\bar{u}}f(x,y) = af_x + bf_y}$

13.6 The Gradient

- Definition: Suppose we have $f(x,y,z)$, define the gradient of f to be the vector: $grad\, f = \boldsymbol{\nabla f = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}}$ (the directional derivative will be largest in this direction)
- Maximum directional derivative will equal $||\nabla f||$
- $\nabla f(x_0, y_0) \perp$ level curve for f at (x_0, y_0), $\nabla f(x_0, y_0, z_0) \perp$ level surface for f at (x_0, y_0, z_0)

13.4 The Chain Rule

- First, draw a tree based on the relation of functions. Then, use chain rule.
- Example: $w = x^2y + \frac{x}{y}$ & $y = x^3 + x$ => $\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}.\frac{dy}{dx}$

13.7 Tangent Planes and Approximations

- Equation of tangent plane at (x_0, y_0): $\boldsymbol{f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)}$
- We can approximate a value at (x_1, y_1) which is close to (x_0, y_0) by the tangent plane:
  $f(x_1, y_1) \approx f(x_0, y_0) + f_x(x_0, y_0)(x_1 - x_0) + f_y(x_0, y_0)(y_1 - y_0)$

13.8 Extreme Values

- Relative Min/Max:
  (A) Find critical points solving $f_x = 0$ & $f_y = 0$
  (B) Define the discriminant function: $\boldsymbol{D(x,y) = f_{xx}(x,y).f_{yy}(x,y) - [f_{xy}(x,y)]^2}$
  For each critical point (x,y):
    - If $\boldsymbol{D(x,y) = +}$: Either relative min or relative max
      + If $\boldsymbol{f_{xx}(x,y)/f_{yy}(x,y) = +}$: **min**
      + If $\boldsymbol{f_{xx}(x,y)/f_{yy}(x,y) = -}$: **max**
    - If $\boldsymbol{D(x,y) = -}$: **Saddle point**
    - If $D(x,y) = 0$: No conclusion (check calculation again)
- Absolute Min/Max:
  (A) Find critical points (x,y) which lie in given region => find $f(x,y)$
  (B) **Find min/max of f on the edges of region**
  (C) Choose min/max from among the results of (A) & (B)

13.9 Lagrange Multipliers

- Suppose we have a function $f(x,y)$ and a constraint $g(x,y) = 0$. To find max and min, we do:
  (A) Solve the system for all possible (x,y):
    $\boldsymbol{f_x = \lambda g_x}$
    $\boldsymbol{f_y = \lambda g_y}$
    $\boldsymbol{g(x,y) = 0}$ (constraint)
  (B) Compute $f(x,y)$ for all those possible points and pick out min/max