11.1 Cartesian Coordinates in Space

- Know how to sketch xy-plane, xz-plane, yz-plane
- 3 axes divide 3D space into 8 octants, the first octant is where x,y,z > 0
- Know how to plot a point in 3D space (First locate x,y on xy-plane, then go up/down)
- Distance formula: $P=(x_0,y_0,z_0)$, $Q=(x_1,y_1,z_1)$

  dist = |PQ| = $\sqrt{(x_0-x_1)^2+(y_0-y_1)^2+(z_0-z_1)^2}$
- Circle/Disk vs. Sphere/Closed. Distinguish equations for them and sketch
- Sphere: Supposed $P=(x_0,y_0,z_0)$ is the center of a sphere and R is the radius

  $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$

11.2 Vectors in Space

- Notation: $\bar{u}=8\hat{\imath}+2\hat{\jmath}-3\hat{k}$ where $\hat{\imath}=(1,0,0)$, $\hat{\jmath}=(0,1,0)$, $\hat{k}=(0,0,1)$
- Note: Write $\bar{v}=4\hat{\imath}+0\hat{\jmath}-1\hat{k}$ instead of $\bar{v}=4\hat{\imath}-1\hat{k}$
- Must know to compute 2D vectors with trigonometry (Memorize values of sin/cos from 0-π)
- Associated Definitions:
  - (A) Zero-vector is all 0's, denoted by $\bar{0}$
  - (B) Supposed P = (x_0,y_0,z_0) and Q = (x_1,y_1,z_1)

    $\overrightarrow{PQ}$ = "Q - P" = $(x_1-x_0)\hat{\imath}+(y_1-y_0)\hat{\jmath}+(z_1-z_0)\hat{k}$
  - (C) Given $\bar{v}=a\hat{\imath}+b\hat{\jmath}+c\hat{k}$, we define the length (aka magnitude/norm):

    $\|\bar{v}\|=\sqrt{a^2+b^2+c^2}$
  - (D) A unit vector is a vector with length 1
  - (E) Supposed we have a vector $\bar{v}$ and we want the unit vector which points in the same direction as $\bar{v}$:

    $$\frac{\bar{v}}{\|\bar{v}\|}$$
  - (F) Two nonzero vectors are parallel if one is a scalar multiple of the other

11.3 The Dot Product (Scalar Product)

- Definition: Let $\bar{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$

  $\bar{b}=b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}$

  $\bar{a}\cdot\bar{b}=a_1b_1+a_2b_2+a_3b_3$
- Product Properties:
  - (A) $\bar{a}\cdot\bar{b}=\bar{b}\cdot\bar{a}$
  - (B) $\bar{a}\cdot(\bar{b}\pm\bar{c})=\bar{a}\cdot\bar{b}\pm\bar{a}\cdot\bar{c}$
  - (C) $\bar{a}\cdot\bar{a}=\|a\|^2=a_1^2+a_2^2+a_3^2$
- Additional Properties:
  - (A) $\bar{a}\cdot\bar{b}=\|\bar{a}\|.\|\bar{b}\|.\cos\theta$, where θ = angle between $\bar{a},\bar{b}$
  - (B) $\bar{a}\cdot\bar{b}$ **= 0 iff** $\bar{a}\perp\bar{b}$
  - (C) $\cos\theta=\frac{\bar{a}\cdot\bar{b}}{\|\bar{a}\|.\|\bar{b}\|}$
- Vector Projection:

  $Proj_{\bar{b}}\bar{a}=\frac{\bar{a}\cdot\bar{b}}{\bar{b}\cdot\bar{b}}.\bar{b}$

11.4 The Cross Product

- Pre-definition: Define $\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$
- Definition: Given $\bar{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$

  $\bar{b}=b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}$

  $\bar{a}\times\bar{b}=\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\imath}+\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\hat{\jmath}+\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$ (Trick $\begin{vmatrix} a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{vmatrix}$)
- Product Properties:
  - (A) $\bar{a}\times(\bar{b}\pm\bar{c})=\bar{a}\times\bar{b}\pm\bar{a}\times\bar{c}$
  - (B) $\bar{a}\times\bar{b}=-\bar{b}\times\bar{a}$ /anticommutativity/
- Additional Properties:
  - (A) $\|\bar{a}\times\bar{b}\|=\|\bar{a}\|\,\|\bar{b}\|\sin\theta$
  - (B) $\bar{a}\times\bar{b}=\bar{0}$ **iff** $\bar{a}$ **and** $\bar{b}$ **are parallel**
  - (C) $\bar{a}\times\bar{b}$ **is** $\perp$ **to both** $\bar{a}$ **and** $\bar{b}$ via right-hand rule!

11.5 Lines in Space

- Idea: Start with a single point $P = (x_0, y_0, z_0)$ and a direction vector $\bar{L} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

If we attached $\bar{L}$ to P, we see a line that goes forever!

- Parametric form: Suppose we have $\boldsymbol{P = (x_0, y_0, z_0)}$ and $\boldsymbol{\bar{L} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}}$, the parametric equations of the corresponding line are:

$$\boldsymbol{x = x_0 + at}$$
$$\boldsymbol{y = y_0 + bt} \qquad \text{where } t = \textit{any number}$$
$$\boldsymbol{z = z_0 + ct}$$

- Vector equation of a line: All we do is put x,y,z from above into a vector:

$$\boldsymbol{\bar{r}(t) = (x_0 + at)\hat{\imath} + (y_0 + bt)\hat{\jmath} + (z_0 + ct)\hat{k}}$$

- Symmetric Equation:
  + Normal case $(a, b, c \neq 0)$

    $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

  + Special case A: (either one of $a, b, c = 0$)

    Ex: $P = (1,2,3)$ and $\bar{L} = 0\hat{\imath} + 8\hat{\jmath} + 7\hat{k}$

    Here's the parametric form: $x = 1 + 0t$, $y = 2 + 8t$, $z = 3 + 7t$

    => Symmetric form: $\frac{y-2}{8} = \frac{z-3}{7}, x = 1$

  + Special case B: (2 of $a, b, c = 0$)

    Ex: $P = (1,2,3)$ and $\bar{L} = 42\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

    Here's the parametric form: $x = 1 + 42t$, $y = 2$, $z = 3$

    => Symmetric form: $\boldsymbol{y = 2, z = 3}$ (No need to mention x)

- Distance between a point and line: Suppose we have a line with point P and direction $\bar{L}$ and suppose Q is some other point, then the perpendicular distance from Q to the line:

$$\boldsymbol{dist = \frac{\left\|\overrightarrow{PQ} \times \bar{L}\right\|}{\|\bar{L}\|}}$$ (Note: We can extract P and $\bar{L}$ from a line given its form)

11.6 Planes in Space

- Definition: A plane is a flat surface extending forever in two directions
- What soft of info could give us a plane?
  + a point + a perpendicular line
  + 3 points
  + 2 agreeable lines
- Equation: Start with a point $P = (x_0, y_0, z_0)$ and a normal vector $\bar{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

  we get a plane containing P and $\perp$ to $\bar{n}$

$$\boldsymbol{a.(x - x_0) + b.(y - y_0) + c.(z - z_0) = 0}$$

- Notes:

  (A) This can be rewritten: $2x + 5y - 3z = 18$ (Lost point but still get normal vector)

  (B) In this form, we've lost the "original point"

  (C) We still see $\bar{n} = 2\hat{\imath} + 5\hat{\jmath} - 3\hat{k}$ from the coefficients

  (D) We can still find points on the plane - any point satisfying the equation

  Example: (9,0,0) or (0,0,-6) or ...

  (E) This equation is equivalent to $4x + 10y - 6z = 36$

  This changes $\bar{n}$ but that's fine - the plane is unchanged!

  (F) If you're not given a point and a vector, you must obtain them

  Example: Suppose you're given 3 points P, Q, R => **Normal vector** $\boldsymbol{\bar{n} = \overrightarrow{PQ} \times \overrightarrow{PR}}$

- Pictures: Suppose our plane is $ax + by - cz = d$

  (A) Two of (a, b, c) = 0

  Ex: $2z = 10$ => $z = 5$ (xy-plane but up at $z = 5$)

  (B) One of (a, b, c) = 0

  Ex: 2x + 4y = 8 (first draw the line as if $z = 0$, then extend up/down)

  (C) None of (a, b, c) = 0

  Ex: x + 2y + 4z = 0 (int: x = 8, y = 4, z = 2, then connect them)

- Distance: Suppose a plane has point P and normal vector $\bar{n}$ and Q is another point

$$\boldsymbol{dist = \frac{|\overrightarrow{PQ} \cdot \bar{n}|}{\|\bar{n}\|}}$$

12.1 Vector-Valued Functions

- Definition: A vector-valued function (VVF) is a function of the form
  $\bar{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ Note: $x(t), y(t), z(t)$ can be any function (like cos, sin)
- Used to describe: (A) Position of an object (B) Velocity (C) Acceleration
- Some common graph in 2D (must know how to draw them)
  + $\bar{r}(t) = (2+3t)\hat{\imath} + (0-t)\hat{\jmath}$ for $0 \le t \le 2$ => line segment from (2,0) to (8,-2)
  + $\bar{r}(t) = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath}$ for $0 \le t \le 2\pi$ => full circle from (1,0) to (0,1) to (-1,0)...
  + $\bar{r}(t) = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath}$ for $0 \le t \le \pi$ => half circle from (1,0) to (0,1) to (-1,0)
  + $\bar{r}(t) = (\cos(2t))\hat{\imath} + (\sin(2t))\hat{\jmath}$ for $0 \le t \le \pi$ => full circle
  + $\bar{r}(t) = (4+2\cos t)\hat{\imath} + (3+\sin t)\hat{\jmath}$ for $0 \le t \le \pi$ => semi-ellipse with center at (4,3) stretch twice in x
- Some common graph in 3D (must know how to draw them)
  + $\bar{r}(t) = 3\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath} + 2\hat{k}$ for $0 \le t \le 2\pi$ => circle of radius 3 at $z = 2$
  + $\bar{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\hat{k}$ for $t \ge 0$ => start at (1,0,0) spirals up (center=z-axes, r=1)

12.3 Derivatives and Integrals of Vector-Valued Functions

- Derivative: $\bar{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ => $\boldsymbol{\bar{r}'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}}$
- Application of Derivative: Use $\bar{r}(t)$ to describe location of an object at time t
  + $\boldsymbol{\bar{v}(t) = \bar{r}'(t)}$ = velocity => vector
  + $\boldsymbol{s(t) = ||\bar{v}(t)||}$ = speed => not a vector
  + $\boldsymbol{\bar{a}(t) = \bar{v}'(t) = \bar{r}''(t)}$ = acceleration => vector

Note: $\bar{v}(t)$ encapsulates both speed and direction of motion: $\bar{v}(t)$ is tangent to $\bar{r}(t)$, pointing to direction of motion
$\bar{a}(t)$ encapsulates both change in speed also how fast/in what way velocity is changing:
  + slowing down if angle($\bar{v}(t),\bar{a}(t)$) > 90° and speeding up if angle($\bar{v}(t),\bar{a}(t)$) < 90°
  + $\bar{v}(t)$ is also "rotating" based on the direction of $\bar{a}(t)$

- Integration: $\boldsymbol{\int (t\hat{\imath} + \cos t\,\hat{\jmath} + 2\hat{k})dt = \frac{1}{2}t^2\hat{\imath} + \sin t\,\hat{\jmath} + 2t\hat{k} + \bar{C}}$ Note: $\bar{C}$ is a vector
- Derivative on Dot and Cross Product:
  $$(\overline{r_1}(t) \cdot \overline{r_2}(t))' = \overline{r_1}'(t) \cdot \overline{r_2}(t) + \overline{r_1}(t) \cdot \overline{r_2}'(t)$$
  $$(\overline{r_1}(t) \times \overline{r_2}(t))' = \overline{r_1}'(t) \times \overline{r_2}(t) + \overline{r_1}(t) \times \overline{r_2}'(t)$$

12.4 Curve vs. Parameterization

- Definition: (A) A parameterization of a curve is a VVF $\bar{r}(t)$ = ...
  (B) A curve is the graph of the parameterization
- Closed: (A) A param $\bar{r}(t)$ defined for $a \le t \le b$ is cloed if:
  + $\bar{r}(a) = \bar{r}(b)$ (start=end) and provided
  + it does not contact itself ∞ many times (Note: start/end should touch only one time)
- Smooth: (A) A param $\bar{r}(t)$ is smooth if:
  + $\bar{r}$ is differentiable
  + $\bar{r}$' exists wherever $\bar{r}$ exists
  + $\bar{r}$' must be continuous
  + $\boldsymbol{\bar{r}'(t)}$ **cannot** = $\bar{0}$ except it is permitted to be $\bar{0}$ at the endpoints (if there are)
- Piecewise Smooth:
  (A) A param is piecewise smooth if the t-values can be broken into finitely many sub-intervals and the param is smooth on each
- A curve is closed/smooth/piecewise smooth if and only if its param is
- Use of piecewise smooth: If C is piecewise smooth on [a,b] then:
  length of C = L = $\boldsymbol{\int_a^b ||\bar{r}'(t)||dt}$

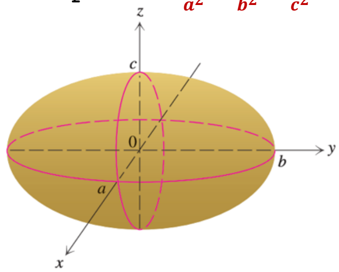
12.5 Tangential and Normal Components of Acceleration

- Definition: For a VVF $\bar{r}(t)$, the tangent vector is $\boldsymbol{\bar{T}(t) = \frac{\bar{v}(t)}{||\bar{v}(t)||}}$ (Recall: $\bar{v}(t) = \bar{r}'(t)$)
  and the normal vector is $\boldsymbol{\bar{N}(t) = \frac{\bar{T}'(t)}{||\bar{T}'(t)||}}$
- Tangential and Normal Components of Acceleration:
  $\boldsymbol{\bar{a}(t) = a_T\bar{T} + a_N\bar{N}}$ where a_T = tan comp of acc = $\frac{\bar{v}\cdot\bar{a}}{||\bar{v}||}$
  a_N = normal comp of acc = $\frac{||\bar{v}\times\bar{a}||}{||\bar{v}||}$
  a_T = measure of how much acceleration is in direction of motion
  a_N = measure of how much acceleration is perpendicular to direction of motion

Note: $a_T = 0$ means no speed change, only direction change
$a_T > 0$ means the object is speeding up, $a_T < 0$ means the object is slowing down
$a_N = 0$ means no direction change, only speed change
a_N cannot be negative

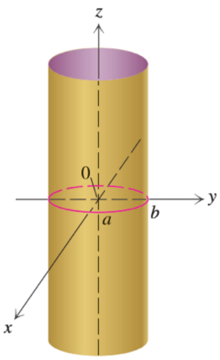
13.1 Functions of Several Variables

- Graphing examples:
  - (A) plane: $\boldsymbol{ax+by+cz=d}$
  - (B) sphere: $\boldsymbol{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2}$ => center at (x_0, y_0, z_0) & radius = R
  - (C) ellipsoid: $\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1}$ or $\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$ => center at (x_0, y_0, z_0)



Note: x-radius = a, y-radius = b, z-radius = c

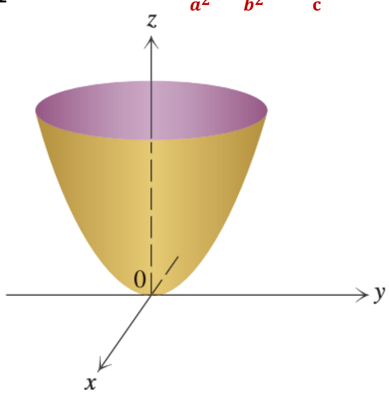
  - (D) cylinder: $\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ or $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$ or $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ or $\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$ or etc.



Assume $\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$

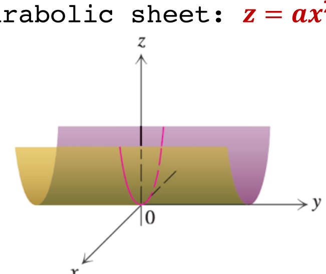
Note:
- center at (x_0, y_0)
- x-radius = a, y-radius = b
- if $a=b$, circular cylinder
  if $a \neq b$, elliptic cylinder

  - (E) paraboloid: $\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z+d}{c}}$ or etc.

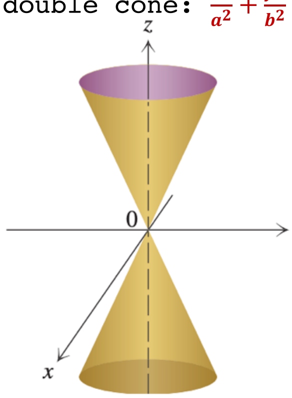


Note:
- if $a=b$, circular paraboloid
  if $a \neq b$, elliptic paraboloid
- if $c>0$, open up (in picture)
  if $c<0$, open down
- if $d>0$, shift down
  if $d<0$, shift up

  - (F) parabolic sheet: $\boldsymbol{z=ax^2+b}$ or $y=ax^2+b$ or etc.



  - (E) double cone: $\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2}$ single cone: $z=\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}$



- Level curves: Supposed we have a function f(x,y). If we choose a z-value, say z=c, and we plot the 2D result **f(x,y) = c**. We get a level curve.
- Level surfaces: Same idea for f(x,y,z)

13.3 Partial Derivatives

- Definition: Suppose f is a function of several variables. Define the partial derivative of f with respect to x to be the derivative treating x as the variable and all other variables as constant. Denote: $\frac{\partial f}{\partial x}$ or $f_x(x, y, \dots)$
- Example: $f(x,y) = xe^{xy}$ => $f_x(x,y) = e^{xy} + xye^{xy}$ $f_y(x,y) = x^2e^{xy}$
- Interpretation: At a point (x_0, y_0)
  + $f_x(x_0, y_0)$ is the slope of tangent line in the x-direction (change in x-direction)
  + $f_y(x_0, y_0)$ is the slope of tangent line in the y-direction (change in y-direction)
- Second partial derivatives:
  + $f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2}$ or $f_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x}$ or $f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2}$ or $f_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y}$
  + Note: $f_{xy} = f_{yx}$ or $f_{xyz} = f_{zxy}$ or etc. in this class

13.5 Directional Derivatives

- Definition: Suppose f is a function and $\bar{u}$ is **unit vector** indicating direction. We define the directional derivative of f in the direction of ū:
  + (3D) if $\bar{u} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$: $D_{\bar{u}}f(x,y,z) = af_x + bf_y + cf_z$ (scalar)
  + (2D) if $\bar{u} = a\hat{\imath} + b\hat{\jmath}$: $\boldsymbol{D_{\bar{u}}f(x,y) = af_x + bf_y}$

13.6 The Gradient

- Definition: Suppose we have $f(x,y,z)$, define the gradient of f to be the vector:

$grad\, f = \boldsymbol{\nabla f = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}}$ (the directional derivative will be largest in this direction)
- Maximum directional derivative will equal $||\nabla f||$
- $\nabla f(x_0, y_0) \perp$ level curve for f at (x_0, y_0), $\nabla f(x_0, y_0, z_0) \perp$ level surface for f at (x_0, y_0, z_0)

13.4 The Chain Rule

- First, draw a tree based on the relation of functions. Then, use chain rule.
- Example: $w = x^2y + \frac{x}{y}$ & $y = x^3 + x$ => $\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} . \frac{dy}{dx}$

13.7 Tangent Planes and Approximations

- Equation of tangent plane at (x_0, y_0): $\boldsymbol{f(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$
- We can approximate a value at (x_1, y_1) which is close to (x_0, y_0) by the tangent plane:

  $f(x_1, y_1) \approx f(x_0, y_0) + f_x(x_0, y_0)(x_1 - x_0) + f_y(x_0, y_0)(y_1 - y_0)$

13.8 Extreme Values

- Relative Min/Max:
  (A) Find critical points solving $f_x = 0$ & $f_y = 0$
  (B) Define the discriminant function: $\boldsymbol{D(x,y) = f_{xx}(x,y).f_{yy}(x,y) - [f_{xy}(x,y)]^2}$
  For each critical point (x,y):
    - If $\boldsymbol{D(x,y) = +}$: Either relative min or relative max
      + If $\boldsymbol{f_{xx}(x,y)/f_{yy}(x,y) = +}$: **min**
      + If $\boldsymbol{f_{xx}(x,y)/f_{yy}(x,y) = -}$: **max**
    - If $\boldsymbol{D(x,y) = -}$: **Saddle point**
    - If $D(x,y) = 0$: No conclusion (check calculation again)
- Absolute Min/Max:
  (A) Find critical points (x,y) which lie in given region => find $f(x,y)$
  (B) **Find min/max of *f* on the edges of region**
  (C) Choose min/max from among the results of (A) & (B)

13.9 Lagrange Multipliers

- Suppose we have a function $f(x,y)$ and a constraint $g(x,y) = 0$. To find max and min, we do:
  (A) Solve the system for all possible (x,y):
    $\mathbf{f_x = \lambda g_x}$
    $\boldsymbol{f_y = \lambda g_y}$
    $\boldsymbol{g(x,y) = 0}$ (constraint)
  (B) Compute $f(x,y)$ for all those possible points and pick out min/max