11.1 Cartesian Coordinates in Space
- Know how to sketch xy-plane, xz-plane, yz-plane
- 3 axes divide 3D space into 8 octants, the first octant is where x,y,z > 0
- Know how to plot a point in 3D space (First locate x,y on xy-plane, then go up/down)
- Distance formula: $P = (x_0, y_0, z_0)$, $Q = (x_1, y_1, z_1)$
$$\text{dist} = |\mathbf{PQ}| = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}$$
- Circle/Disk vs. Sphere/Closed. Distinguish equations for them and sketch
- Sphere: Supposed $P = (x_0, y_0, z_0)$ is the center of a sphere and R is the radius
$$(\mathbf{x} - \mathbf{x_0})^2 + (\mathbf{y} - \mathbf{y_0})^2 + (\mathbf{z} - \mathbf{z_0})^2 = \mathbf{R}^2$$

11.2 Vectors in Space
- Notation: $\bar{u} = 8\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ where $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, $\hat{k} = (0,0,1)$
- Note: Write $\bar{v} = 4\hat{\imath} + 0\hat{\jmath} - 1\hat{k}$ instead of $\bar{v} = 4\hat{\imath} - 1\hat{k}$
- Must know to compute 2D vectors with trigonometry (Memorize values of sin/cos from 0–π)
- Associated Definitions:
    - (A) Zero-vector is all 0's, denoted by $\bar{0}$
    - (B) Supposed P = (x_0, y_0, z_0) and Q = (x_1, y_1, z_1)
      $\overrightarrow{PQ} = $ "Q - P" $= (x_1 - x_0)\hat{\imath} + (y_1 - y_0)\hat{\jmath} + (z_1 - z_0)\hat{k}$
    - (C) Given $\bar{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, we define the length (aka magnitude/norm):
      $\left\|\bar{v}\right\| = \sqrt{a^2 + b^2 + c^2}$
    - (D) A unit vector is a vector with length 1
    - (E) Supposed we have a vector $\bar{v}$ and we want the unit vector which points in the same direction as $\bar{v}$:
      $$\frac{\bar{v}}{\left\|\bar{v}\right\|}$$
    - (F) Two nonzero vectors are parallel if one is a scalar multiple of the other

11.3 The Dot Product (Scalar Product)
- Definition: Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$
  $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$
  $\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- Product Properties:
    - (A) $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$
    - (B) $\bar{a} \cdot (\bar{b} \pm \bar{c}) = \bar{a} \cdot \bar{b} \pm \bar{a} \cdot \bar{c}$
    - (C) $\bar{a} \cdot \bar{a} = \left\|a\right\|^2 = a_1^2 + a_2^2 + a_3^2$
- Additional Properties:
    - (A) $\bar{a} \cdot \bar{b} = \left\|\bar{a}\right\| \cdot \left\|\bar{b}\right\| \cdot \cos\theta$, where θ = angle between $\bar{a}, \bar{b}$
    - (B) $\bar{a} \cdot \bar{b} = 0$ iff $\bar{a} \perp \bar{b}$
    - (C) $\cos\theta = \frac{\bar{a} \cdot \bar{b}}{\left\|\bar{a}\right\| \cdot \left\|\bar{b}\right\|}$
- Vector Projection:
  $$\mathbf{Proj}_{\bar{b}}\bar{a} = \frac{\bar{a} \cdot \bar{b}}{\bar{b} \cdot \bar{b}} \cdot \bar{b}$$

11.4 The Cross Product
- Pre-definition: Define $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- Definition: Given $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$
  $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$
  $\bar{a} \times \bar{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\imath} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$ (Trick $\begin{vmatrix} a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{vmatrix}$)
- Product Properties:
    - (A) $\bar{a} \times (\bar{b} \pm \bar{c}) = \bar{a} \times \bar{b} \pm \bar{a} \times \bar{c}$
    - (B) $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$ /anticommutativity/
- Additional Properties:
    - (A) $\left\|a \times b\right\| = \left\|\bar{a}\right\| \left\|\bar{b}\right\| \sin\theta$
    - (B) $\bar{a} \times \bar{b} = \bar{0}$ iff $\bar{a}$ and $\bar{b}$ are parallel
    - (C) $\bar{a} \times \bar{b}$ is $\perp$ to both $\bar{a}$ and $\bar{b}$ via right-hand rule!

11.5 Lines in Space

- Idea: Start with a single point $P = (x_0, y_0, z_0)$ and a direction vector $\bar{L} = a\hat{i} + b\hat{j} + c\hat{k}$
  If we attached $\bar{L}$ to P, we see a line that goes forever!
- Parametric form: Suppose we have $P = (x_0, y_0, z_0)$ and $\bar{L} = a\hat{i} + b\hat{j} + c\hat{k}$, the parametric equations
  of the corresponding line are:

$$x = x_0 + at$$
$$y = y_0 + bt \qquad \text{where } t = any\ number$$
$$z = z_0 + ct$$

- Vector equation of a line: All we do is put x,y,z from above into a vector:

$$\bar{r}(t) = (x_0 + at)\hat{i} + (y_0 + bt)\hat{j} + (z_0 + ct)\hat{k}$$

- Symmetric Equation:
    + Normal case $(a, b, c \neq 0)$
      $$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$
    + Special case A: (either one of $a, b, c = 0$)
        Ex: $P = (1,2,3)$ and $\bar{L} = 0\hat{i} + 8\hat{j} + 7\hat{k}$
        Here's the parametric form: $\quad x = 1 + 0t$
        $$y = 2 + 8t$$
        $$z = 3 + 7t$$
        => Symmetric form: $\frac{y-2}{8} = \frac{z-3}{7}, x = 1$
    + Special case B: (2 of $a, b, c = 0$)
        Ex: $P = (1,2,3)$ and $\bar{L} = 42\hat{i} + 0\hat{j} + 0\hat{k}$
        Here's the parametric form: $\quad x = 1 + 42t$
        $$y = 2$$
        $$z = 3$$
        => Symmetric form: $y = 2, z = 3$ (No need to mention x)
- Distance between a point and line: Suppose we have a line with point P and direction $\bar{L}$
  and suppose Q is some other point, then the perpendicular distance from Q to the line:

$$dist = \frac{\|\vec{PQ} \times \bar{L}\|}{\|\bar{L}\|} \quad \text{(Note: We can extract } P \text{ and } \bar{L} \text{ from a line given its form)}$$

11.6 Planes in Space

- Definition: A plane is a flat surface extending forever in two directions
- What soft of info could give us a plane?
    + a point + a perpendicular line
    + 3 points
    + 2 agreeable lines
- Equation: Start with a point $P = (x_0, y_0, z_0)$ and a normal vector $\bar{n} = a\hat{i} + b\hat{j} + c\hat{k}$
        we get a plane containing P and $\perp$ to $\bar{n}$
        $$a.(x - x_0) + b.(y - y_0) + c.(z - z_0) = 0$$
- Notes:
    (A) This can be rewritten: $2x + 5y - 3z = 18$ (Lost point but still get normal vector)
    (B) In this form, we've lost the "original point"
    (C) We still see $\bar{n} = 2\hat{i} + 5\hat{j} - 3\hat{k}$ from the coefficients
    (D) We can still find points on the plane – any point satisfying the equation
        Example: (9,0,0) or (0,0,-6) or ...
    (E) This equation is equivalent to $4x + 10y - 6z = 36$
        This changes $\bar{n}$ but that's fine – the plane is unchanged!
    (F) If you're not given a point and a vector, you must obtain them
        Example: Suppose you're given 3 points P, Q, R => **Normal vector $\bar{n} = \vec{PQ} \times \vec{PR}$**
- Pictures: Suppose our plane is $ax + by - cz = d$
    (A) Two of (a, b, c) = 0
        Ex: $2z = 10 \Rightarrow z = 5$ $\quad$ (xy-plane but up at $z = 5$)
    (B) One of (a, b, c) = 0
        Ex: 2x + 4y = 8 $\quad$ (first draw the line as if $z = 0$, then extend up/down)
    (C) None of (a, b, c) = 0
        Ex: x + 2y + 4z = 0 $\quad$ (int: x = 8, y = 4, z = 2, then connect them)
- Distance: Suppose a plane has point P and normal vector $\bar{n}$ and Q is another point

$$dist = \frac{|\vec{PQ} \cdot \bar{n}|}{\|\bar{n}\|}$$