11.1 Cartesian Coordinates in Space

- Know how to sketch xy-plane, xz-plane, yz-plane
- 3 axes divide 3D space into 8 octants, the first octant is where x,y,z > 0
- Know how to plot a point in 3D space (First locate x,y on xy-plane, then go up/down)
- Distance formula: $P = (x_0, y_0, z_0)$, $Q = (x_1, y_1, z_1)$

  dist = |PQ| $= \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}$
- Circle/Disk vs. Sphere/Closed. Distinguish equations for them and sketch
- Sphere: Supposed $P = (x_0, y_0, z_0)$ is the center of a sphere and R is the radius

  $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$

11.2 Vectors in Space

- Notation: $\bar{u} = 8\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ where $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, $\hat{k} = (0,0,1)$
- Note: Write $\bar{v} = 4\hat{\imath} + 0\hat{\jmath} - 1\hat{k}$ instead of $\bar{v} = 4\hat{\imath} - 1\hat{k}$
- Must know to compute 2D vectors with trigonometry (Memorize values of sin/cos from 0-π)
- Associated Definitions:
  - (A) Zero-vector is all 0's, denoted by $\bar{0}$
  - (B) Supposed P = (x_0, y_0, z_0) and Q = (x_1, y_1, z_1)

    $\overrightarrow{PQ}$ = "Q - P" = $(x_1 - x_0)\hat{\imath} + (y_1 - y_0)\hat{\jmath} + (z_1 - z_0)\hat{k}$
  - (C) Given $\bar{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, we define the length (aka magnitude/norm):

    $\|\bar{v}\| = \sqrt{a^2 + b^2 + c^2}$
  - (D) A unit vector is a vector with length 1
  - (E) Supposed we have a vector $\bar{v}$ and we want the unit vector which points in the same direction as $\bar{v}$:

    $$\frac{\bar{v}}{\|\bar{v}\|}$$
  - (F) Two nonzero vectors are parallel if one is a scalar multiple of the other

11.3 The Dot Product (Scalar Product)

- Definition: Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

  $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

  $\bar{a} \cdot \bar{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Product Properties:
  - (A) $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$
  - (B) $\bar{a} \cdot (\bar{b} \pm \bar{c}) = \bar{a} \cdot \bar{b} \pm \bar{a} \cdot \bar{c}$
  - (C) $\bar{a} \cdot \bar{a} = \|a\|^2 = a_1^2 + a_2^2 + a_3^2$
- Additional Properties:
  - (A) $\bar{a} \cdot \bar{b} = \|\bar{a}\| . \|\bar{b}\| . \cos\theta$, where θ = angle between $\bar{a}, \bar{b}$
  - (B) $\bar{a} \cdot \bar{b}$ **= 0 iff** $\bar{a} \perp \bar{b}$
  - (C) $\cos\theta = \frac{\bar{a} \cdot \bar{b}}{\|\bar{a}\| . \|\bar{b}\|}$
- Vector Projection:

  $Proj_{\bar{b}}\bar{a} = \frac{\bar{a} \cdot \bar{b}}{\bar{b} \cdot \bar{b}} . \bar{b}$

11.4 The Cross Product

- Pre-definition: Define $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- Definition: Given $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

  $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

  $\bar{a} \times \bar{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$ (Trick $\begin{vmatrix} a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{vmatrix}$)
- Product Properties:
  - (A) $\bar{a} \times (\bar{b} \pm \bar{c}) = \bar{a} \times \bar{b} \pm \bar{a} \times \bar{c}$
  - (B) $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$ /anticommutativity/
- Additional Properties:
  - (A) $\|\bar{a} \times \bar{b}\| = \|\bar{a}\| \|\bar{b}\| \sin\theta$
  - (B) $\bar{a} \times \bar{b} = \bar{0}$ **iff** $\bar{a}$ **and** $\bar{b}$ **are parallel**
  - (C) $\bar{a} \times \bar{b}$ **is** $\perp$ **to both** $\bar{a}$ **and** $\bar{b}$ via right-hand rule!

11.5 Lines in Space

- Idea: Start with a single point $P = (x_0, y_0, z_0)$ and a direction vector $\bar{L} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

If we attached $\bar{L}$ to P, we see a line that goes forever!

- Parametric form: Suppose we have $\boldsymbol{P = (x_0, y_0, z_0)}$ and $\boldsymbol{\bar{L} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}}$, the parametric equations of the corresponding line are:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \qquad \text{where } t = \textit{any number} \\ z &= z_0 + ct \end{aligned}$$

- Vector equation of a line: All we do is put x,y,z from above into a vector:

$$\bar{r}(t) = (x_0 + at)\hat{\imath} + (y_0 + bt)\hat{\jmath} + (z_0 + ct)\hat{k}$$

- Symmetric Equation:
    + Normal case $(a, b, c \neq 0)$

      $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

    + Special case A: (either one of $a, b, c = 0$)

      Ex: $P = (1,2,3)$ and $\bar{L} = 0\hat{\imath} + 8\hat{\jmath} + 7\hat{k}$

      Here's the parametric form: $x = 1 + 0t$, $y = 2 + 8t$, $z = 3 + 7t$

      => Symmetric form: $\frac{y-2}{8} = \frac{z-3}{7}, x = 1$

    + Special case B: (2 of $a, b, c = 0$)

      Ex: $P = (1,2,3)$ and $\bar{L} = 42\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

      Here's the parametric form: $x = 1 + 42t$, $y = 2$, $z = 3$

      => Symmetric form: $y = 2, z = 3$ (No need to mention x)

- Distance between a point and line: Suppose we have a line with point P and direction $\bar{L}$ and suppose Q is some other point, then the perpendicular distance from Q to the line:

  $$dist = \frac{\left\|\overrightarrow{PQ} \times \bar{L}\right\|}{\|\bar{L}\|}$$ (Note: We can extract P and $\bar{L}$ from a line given its form)

11.6 Planes in Space

- Definition: A plane is a flat surface extending forever in two directions
- What soft of info could give us a plane?
    + a point + a perpendicular line
    + 3 points
    + 2 agreeable lines
- Equation: Start with a point $P = (x_0, y_0, z_0)$ and a normal vector $\bar{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

  we get a plane containing P and $\perp$ to $\bar{n}$

  $$a.(x - x_0) + b.(y - y_0) + c.(z - z_0) = 0$$

- Notes:

  (A) This can be rewritten: $2x + 5y - 3z = 18$ (Lost point but still get normal vector)

  (B) In this form, we've lost the "original point"

  (C) We still see $\bar{n} = 2\hat{\imath} + 5\hat{\jmath} - 3\hat{k}$ from the coefficients

  (D) We can still find points on the plane - any point satisfying the equation

  Example: (9,0,0) or (0,0,-6) or ...

  (E) This equation is equivalent to $4x + 10y - 6z = 36$

  This changes $\bar{n}$ but that's fine - the plane is unchanged!

  (F) If you're not given a point and a vector, you must obtain them

  Example: Suppose you're given 3 points P, Q, R => **Normal vector** $\boldsymbol{\bar{n} = \overrightarrow{PQ} \times \overrightarrow{PR}}$

- Pictures: Suppose our plane is $ax + by - cz = d$

  (A) Two of (a, b, c) = 0

  Ex: $2z = 10$ => $z = 5$ (xy-plane but up at $z = 5$)

  (B) One of (a, b, c) = 0

  Ex: 2x + 4y = 8 (first draw the line as if $z = 0$, then extend up/down)

  (C) None of (a, b, c) = 0

  Ex: x + 2y + 4z = 0 (int: x = 8, y = 4, z = 2, then connect them)

- Distance: Suppose a plane has point P and normal vector $\bar{n}$ and Q is another point

  $$\text{dist} = \frac{|\overrightarrow{PQ} \cdot \bar{n}|}{\|\bar{n}\|}$$

12.1 Vector-Valued Functions

- Definition: A vector-valued function (VVF) is a function of the form
  $\bar{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ Note: $x(t), y(t), z(t)$ can be any function (like cos, sin)
- Used to describe: (A) Position of an object (B) Velocity (C) Acceleration
- Some common graph in 2D (must know how to draw them)
  + $\bar{r}(t) = (2+3t)\hat{\imath} + (0-t)\hat{\jmath}$ for $0 \le t \le 2$ => line segment from (2,0) to (8,-2)
  + $\bar{r}(t) = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath}$ for $0 \le t \le 2\pi$ => full circle from (1,0) to (0,1) to (-1,0)...
  + $\bar{r}(t) = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath}$ for $0 \le t \le \pi$ => half circle from (1,0) to (0,1) to (-1,0)
  + $\bar{r}(t) = (\cos(2t))\hat{\imath} + \left(\sin(2t)\right)\hat{\jmath}$ for $0 \le t \le \pi$ => full circle
  + $\bar{r}(t) = (4 + 2\cos t)\hat{\imath} + (3 + \sin t)\hat{\jmath}$ for $0 \le t \le \pi$ => semi-ellipse with center at (4,3) stretch twice in x
- Some common graph in 3D (must know how to draw them)
  + $\bar{r}(t) = 3\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath} + 2\hat{k}$ for $0 \le t \le 2\pi$ => circle of radius 3 at z = 0
  + $\bar{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\hat{k}$ for $t \ge 0$ => start at (1,0,0) spirals up (radius = 1)

12.3 Derivatives and Integrals of Vector-Valued Functions

- Derivative: $\bar{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ => $\boldsymbol{\bar{r}'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}}$
- Application of Derivative: Use $\bar{r}(t)$ to describe location of an object at time t
  + $\boldsymbol{\bar{v}(t) = \bar{r}'(t)}$ = velocity => vector
  + $\boldsymbol{s(t) = \left|\left|\bar{v}(t)\right|\right|}$ = speed => not a vector
  + $\boldsymbol{\bar{a}(t) = \bar{v}'(t) = \bar{r}''(t)}$ = acceleration => vector

Note: $\bar{v}(t)$ encapsulates both speed and direction of motion: $\bar{v}(t)$ is tangent to $\bar{r}(t)$, pointing to direction of motion
$\bar{a}(t)$ encapsulates both change in speed also how fast/in what way velocity is changing:
  + slowing down if angle($\bar{v}(t), \bar{a}(t)$) > 90° and speeding up if angle($\bar{v}(t), \bar{a}(t)$) < 90°
  + $\bar{v}(t)$ is also "rotating" based on the direction of $\bar{a}(t)$

- Integration: $\boldsymbol{\int\left(t\hat{\imath} + \cos t\,\hat{\jmath} + 2\hat{k}\right)dt = \frac{1}{2}t^2\hat{\imath} + \sin t\,\hat{\jmath} + 2t\hat{k} + \bar{C}}$ Note: $\bar{C}$ is a vector
- Derivative on Dot and Cross Product:
  $\boldsymbol{(\overline{r_1}(t) \cdot \overline{r_2}(t))' = \overline{r_1}'(t) \cdot \overline{r_2}(t) + \overline{r_1}(t) \cdot \overline{r_2}'(t)}$
  $\boldsymbol{\left(\overline{r_1}(t) \times \overline{r_2}(t)\right)' = \overline{r_1}'(t) \times \overline{r_2}(t) + \overline{r_1}(t) \times \overline{r_2}'(t)}$

12.4 Curve vs. Parameterization

- Definition: (A) A parameterization of a curve is a VVF $\bar{r}(t)$ = ...
  (B) A curve is the graph of the parameterization
- Closed: (A) A param $\bar{r}(t)$ defined for $a \le t \le b$ is cloed if:
  + $\bar{r}(a) = \bar{r}(b)$ (start=end) and provided
  + it does not contact itself ∞ many times (Note: start/end should touch only one time)
- Smooth: (A) A param $\bar{r}(t)$ is smooth if:
  + $\bar{r}$ is differentiable
  + $\bar{r}$' exists wherever $\bar{r}$ exists
  + $\bar{r}$' must be continuous
  + $\boldsymbol{\bar{r}'(t)}$ **cannot** = $\boldsymbol{\bar{0}}$ except it is permitted to be $\bar{0}$ at the endpoints (if there are)
- Piecewise Smooth:
  (A) A param is piecewise smooth if the t-values can be broken into finitely many sub-intervals and the param is smooth on each
- A curve is closed/smooth/piecewise smooth if and only if its param is
- Use of piecewise smooth: If C is piecewise smooth on [a,b] then:
  length of C = L = $\boldsymbol{\int_a^b \left|\left|\bar{r}'(t)\right|\right| dt}$

12.5 Tangential and Normal Components of Acceleration

- Definition: For a VVF $\bar{r}(t)$, the tangent vector is $\boldsymbol{\bar{T}(t) = \frac{\bar{v}(t)}{||\bar{v}(t)||}}$ (Recall: $\bar{v}(t) = \bar{r}'(t)$)
  and the normal vector is $\boldsymbol{\bar{N}(t) = \frac{\bar{T}'(t)}{||\bar{T}'(t)||}}$
- Tangential and Normal Components of Acceleration:
  $\boldsymbol{\bar{a}(t) = a_T\bar{T} + a_N\bar{N}}$ where a_T = tan comp of acc = $\boldsymbol{\frac{\bar{v}\cdot\bar{a}}{||\bar{v}||}}$
  a_N = normal comp of acc = $\boldsymbol{\frac{||\bar{v}\times\bar{a}||}{||\bar{v}||}}$
  a_T = measure of how much acceleration is in direction of motion
  a_N = measure of how much acceleration is perpendicular to direction of motion

Note: $a_T = 0$ means no speed change, only direction change
$a_T > 0$ means the object is speeding up, $a_T < 0$ means the object is slowing down
$a_N = 0$ means no direction change, only speed change
a_N cannot be negative