CHAPTER 17

17.1 A table of integrals can be consulted to determine

$$\int \tanh dx = \frac{1}{a} \ln \cosh ax$$

Therefore,

$$\int_0^t \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}}t\right) dt = \sqrt{\frac{gm}{c_d}} \sqrt{\frac{m}{gc_d}} \left[\ln \cosh\left(\sqrt{\frac{gc_d}{m}}t\right)\right]_0^t$$

$$\sqrt{\frac{gm^2}{gc_d^2}} \left[\ln \cosh\left(\sqrt{\frac{gc_d}{m}}t\right) - \ln \cosh(0)\right]$$

Since cosh(0) = 1 and ln(1) = 0, this reduces to

$$\frac{m}{c_d} \ln \cosh\left(\sqrt{\frac{gc_d}{m}}t\right)$$

17.2 **(a)** The analytical solution can be evaluated as

$$\int_0^4 (1 - e^{-2x})\, dx = \left[x + 0.5e^{-2x}\right]_0^4 = 4 + 0.5e^{-2(4)} - 0 - 0.5e^{-2(0)} = 3.500167731$$

(b) single application of the trapezoidal rule

$$(4-0)\frac{0 + 0.999665}{2} = 1.99329 \qquad (\varepsilon_t = 42.88\%)$$

(c) composite trapezoidal rule

<u>n = 2:</u>

$$(4-0)\frac{0 + 2(0.981684) + 0.999665}{4} = 2.96303 \qquad (\varepsilon_t = 15.35\%)$$

<u>n = 4:</u>

$$(4-0)\frac{0 + 2(0.86466 + 0.981684 + 0.99752) + 0.999665}{8} = 3.3437 \qquad (\varepsilon_t = 4.47\%)$$

(d) single application of Simpson's 1/3 rule

$$(4-0)\frac{0+4(0.981684)+0.999665}{6}=3.28427 \quad (\varepsilon_t = 6.17\%)$$

(e) composite Simpson's 1/3 rule (n = 4)

$$(4-0)\frac{0+4(0.86466+0.99752)+2(0.981684)+0.999665}{12}=3.47059 \quad (\varepsilon_t = 0.84\%)$$

(f) Simpson's 3/8 rule.

$$(4-0)\frac{0+3(0.930517+0.995172)+0.999665}{8}=3.388365 \quad (\varepsilon_t = 3.19\%)$$

17.3 (a) The analytical solution can be evaluated as

$$\int_0^{\pi/2}(6+3\cos x)\,dx=\left[6x+3\sin x\right]_0^{\pi/2}=6(\pi/2)+3\sin(\pi/2)-6(0)-3\sin(0)=12.424778$$

(b) single application of the trapezoidal rule

$$\left(\frac{\pi}{2}-0\right)\frac{9+6}{2}=11.78097 \quad (\varepsilon_t = 5.18\%)$$

(c) composite trapezoidal rule

<u>n = 2:</u>

$$\left(\frac{\pi}{2}-0\right)\frac{9+2(8.12132)+6}{4}=12.26896 \quad (\varepsilon_t = 1.25\%)$$

<u>n = 4:</u>

$$\left(\frac{\pi}{2}-0\right)\frac{9+2(8.77164+8.12132+7.14805)+6}{8}=12.386125 \quad (\varepsilon_t = 0.3111\%)$$

(d) single application of Simpson's 1/3 rule

$$\left(\frac{\pi}{2}-0\right)\frac{9+4(8.12132)+6}{6}=12.4316 \quad (\varepsilon_t = 0.0550\%)$$

(e) composite Simpson's 1/3 rule (n = 4)

$$\left(\frac{\pi}{2}-0\right)\frac{9+4(8.7716+7.14805)+2(8.12132)+6}{12}=12.42518 \quad (\varepsilon_t = 0.0032\%)$$

(f) Simpson's 3/8 rule.

$$\left(\frac{\pi}{2}-0\right)\frac{9+3(8.59808+7.5)+6}{8}=12.42779 \quad (\varepsilon_t = 0.0243\%)$$

17.4 (a) The analytical solution can be evaluated as

$$\int_{-2}^{4} (1-x-4x^3+2x^5)\,dx = \left[x-\frac{x^2}{2}-x^4+\frac{x^6}{3}\right]_{-2}^{4}$$

$$= 4-\frac{4^2}{2}-4^4+\frac{4^6}{3}-(-2)+\frac{(-2)^2}{2}+(-2)^4-\frac{(-2)^6}{3}=1104$$

(b) single application of the trapezoidal rule

$$(4-(-2))\frac{-29+1789}{2}=5280 \quad (\varepsilon_t = 378.3\%)$$

(c) composite trapezoidal rule

<u>n = 2:</u>

$$(4-(-2))\frac{-29+2(-2)+1789}{4}=2634 \quad (\varepsilon_t = 138.6\%)$$

<u>n = 4:</u>

$$(4-(-2))\frac{-29+2(1.9375+(-2)+131.3125)+1789}{8}=1516.875 \quad (\varepsilon_t = 37.4\%)$$

(d) single application of Simpson’s 1/3 rule

$$(4-(-2))\frac{-29+4(-2)+1789}{6}=1752 \quad (\varepsilon_t = 58.7\%)$$

(e) composite Simpson’s 1/3 rule (n = 4)

$$(4-(-2))\frac{-29+4(1.9375+131.3125)+2(-2)+1789}{12}=1144.5 \quad (\varepsilon_t = 3.6685\%)$$

(f) Simpson’s 3/8 rule.

$$(4-(-2))\frac{-29+3(1+31)+1789}{8}=1392 \quad (\varepsilon_t = 26.09\%)$$

(g) Boole’s rule.

$$(4-(-2))\frac{7(-29)+32(1.9375)+12(-2)+32(131.3125)+7(1789)}{90}=1104 \quad (\varepsilon_t=0\%)$$

17.5 (a) Analytical solution:

$$\int_0^{0.6} 2e^{-1.5x}dx=\left[-1.33333e^{-1.5x}\right]_0^{0.6}=-0.54209-(-1.33333)=0.79124$$

(b) Trapezoidal rule

$$I=(0.05-0)\frac{2+1.8555}{2}+(0.15-0.05)\frac{1.8555+1.597}{2}+(0.25-0.15)\frac{1.597+1.3746}{2}$$

$$+(0.35-0.25)\frac{1.3746+1.1831}{2}+(0.475-0.35)\frac{1.1831+0.9808}{2}$$

$$+(0.6-0.475)\frac{0.9808+0.8131}{2}=0.79284$$

$$\varepsilon_t=\left|\frac{0.79124-0.79284}{0.79124}\right|\times 100\%=0.2022\%$$

(c) Trapezoidal/Simpson's rules

$$I=(0.05-0)\frac{2+1.8555}{2}+(0.35-0.05)\frac{1.8555+3(1.597+1.3746)+1.1831}{8}$$

$$+(0.6-0.35)\frac{1.1831+4(0.9808)+0.8131}{6}=0.791282$$

$$\varepsilon_t=\left|\frac{0.79124-0.791282}{0.79124}\right|\times 100\%=0.0052\%$$

17.6 (a) Analytical solution:

$$\int_{-1}^{1}\int_0^2 (x^2-2y^2+xy^3)\ dx\,dy=\int_{-1}^{1}\left[\frac{x^3}{3}-2y^2x+y^3\frac{x^2}{2}\right]_0^2 dy$$

$$=\int_{-1}^{1}\frac{8}{3}-4y^2+2y^3\ dy=\left[\frac{8}{3}y-\frac{4}{3}y^3+\frac{1}{2}y^4\right]_{-1}^{1}=2.666667$$

(b) Sweeping across the x dimension:

<u>$y = -1$:</u>

$$I=(2-0)\frac{-2+2(-2)+0}{4}=-3$$

<u>$y = 0$:</u>

$$I = (2-0)\frac{0+2(1)+4}{4} = 3$$

<u>$y = 1$:</u>

$$I = (2-0)\frac{-2+2(0)+4}{4} = 1$$

These values can then be integrated along the *y* dimension:

$$I = (1-(-1))\frac{-3+2(3)+1}{4} = 2$$

$$\varepsilon_t = \left|\frac{2.666667-2}{2.666667}\right| \times 100\% = 25\%$$

(c) Sweeping across the *x* dimension:

<u>$y = -1$:</u>

$$I = (2-0)\frac{-2+4(-2)+0}{6} = -3.33333$$

<u>$y = 0$:</u>

$$I = (2-0)\frac{0+4(1)+4}{6} = 2.666667$$

<u>$y = 1$:</u>

$$I = (2-0)\frac{-2+4(0)+4}{6} = 0.666667$$

These values can then be integrated along the *y* dimension:

$$I = (1-(-1))\frac{-3.33333+4(2.666667)+0.666667}{6} = 2.666667 \qquad \varepsilon_t = 0\%$$

Which is perfect.

17.7 (a) Analytical solution:

$$\int_{-2}^{2}\int_{0}^{2}\int_{-3}^{1}(x^3-3yz)\ dx\,dy\,dz = \int_{-2}^{2}\int_{0}^{2}\left[\frac{x^4}{4}-3yzx\right]_{-3}^{1}\ dy\,dz$$

$$= \int_{-2}^{2}\int_{0}^{2} -20-12yz\ dy\,dz = \int_{-2}^{2}\left[-20y-6y^2z\right]_0^2\ dz = \int_{-2}^{2} -40-24z\ dz$$

$$= \left[-40z-12z^2\right]_{-2}^{2} = -160$$

(b) For z = –2, sweeping across the *x* dimension:

<u>$z = -2$; $y = 0$:</u>

$$I = (1-(-3))\frac{-27+4(-1)+1}{6} = -20$$

<u>$z = -2$; $y = 1$:</u>

$$I = (1-(-3))\frac{-21+4(5)+7}{6} = 4$$

<u>$z = -2$; $y = 2$:</u>

$$I = (1-(-3))\frac{-15+4(11)+13}{6} = 28$$

These values can then be integrated along the y dimension:

$$I = (2-0)\frac{-20+4(4)+28}{6} = 8$$

For $z = 0$, similar calculations yield

<u>$z = 0$; $y = 0$:</u> $I = -20$
<u>$z = 0$; $y = 1$:</u> $I = -20$
<u>$z = 0$; $y = 2$:</u> $I = -20$

These values can then be integrated along the y dimension:

$$I = (2-0)\frac{-20+4(-20)-20}{6} = -40$$

For $z = 2$, similar calculations yield

<u>$z = 2$; $y = 0$:</u> $I = -20$
<u>$z = 2$; $y = 1$:</u> $I = -44$
<u>$z = 2$; $y = 2$:</u> $I = -68$

These values can then be integrated along the y dimension:

$$I = (2-0)\frac{-20+4(-44)-68}{6} = -88$$

Finally, these results can be integrated along the z dimension,

$$I = (2-(-2))\frac{8+4(-40)-88}{6} = -160 \qquad \varepsilon_t = 0\%$$

17.8 (a) The trapezoidal rule can be implemented as,

$$I = (2-1)\frac{5+6}{2} + (3.25-2)\frac{6+5.5}{2} + \bullet\bullet\bullet = 60.425\frac{\text{m}\cdot\text{min}}{\text{s}} \times \frac{60\,\text{s}}{\text{min}} = 3{,}625.5\,\text{m}$$

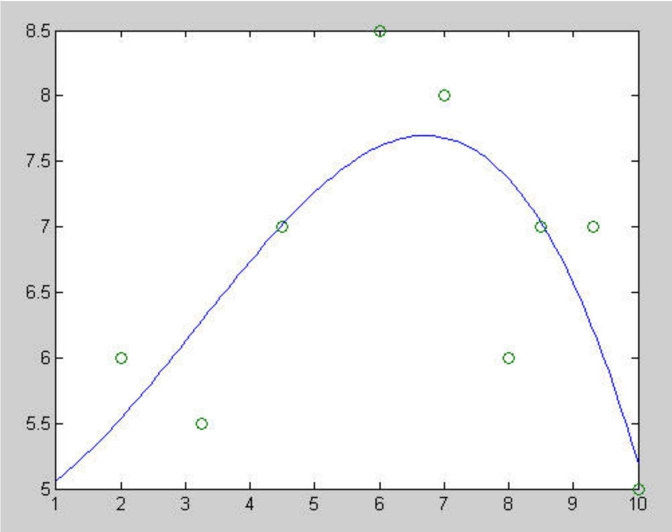
(b) The polynomial can be fit as,

```
>> format long
>> t = [1 2 3.25 4.5 6 7 8 8.5 9.3 10];
>> v = [5 6 5.5 7 8.5 8 6 7 7 5];
>> p = polyfit(t,v,3)

p =
  -0.01671980042198   0.16014730974888   0.10763952867661
4.81477984921060
```

The cubic can be plotted along with the data,

```
>> tt = linspace(1,10);
>> vv = polyval(p,tt);
>> plot(tt,vv,t,v,'o')
```



This equation can be integrated to yield

$$x = \int_1^{10} -0.01672t^3 + 0.16015t^2 + 0.10764t + 4.81478\,dt$$

$$= \left[-0.00418t^4 + 0.053382t^3 + 0.05382t^2 + 4.81478t\right]_1^{10}$$

$$= 60.19533 \frac{\text{m}\cdot\text{min}}{\text{s}} \times \frac{60\text{ s}}{\text{min}} = 3{,}611.72\text{ m}$$

17.9

z	$w(z)$	$\rho gw(z)(60 - z)$	$\rho gzw(z)(60 - z)$
60	200	0	0
50	190	1.8639E+07	9.3195E+08

40	175	3.4335E+07	1.3734E+09
30	160	4.7088E+07	1.4126E+09
20	135	5.2974E+07	1.0595E+09
10	130	6.3765E+07	6.3765E+08
0	122	7.1809E+07	0

$$f_t = 60\frac{7.1809 + 4(6.3765 + 4.7088 + 1.8639) + 2(5.2974 + 3.4335) + 0}{3(6)} \times 10^7 = 2.5480 \times 10^9$$

$$\int_0^D \rho g z w(z)(D - z)\,dz = 60\frac{0 + 4(0.63765 + 1.4126 + 0.93195) + 2(1.0595 + 1.3734) + 0}{3(6)} \times 10^9 = 5.5982 \times 10^{10}$$

$$d = \frac{5.5982 \times 10^{10}}{2.5480 \times 10^9} = 21.971$$

17.10 (a) Trapezoidal rule:

$$f = 30\frac{0 + 2(54.937 + 51.129 + 36.069 + 27.982 + 19.455) + 13.311}{2(6)} = 996.1363$$

$$f = \frac{30\dfrac{0 + 2(274.684 + 511.292 + 586.033 + 559.631 + 486.385) + 399.332}{2(6)}}{996.1363} = \frac{13088.45}{996.1363} = 13.139\text{ m}$$

(b) Simpson’s 1/3 rule:

$$f = 30\frac{0 + 4(54.937 + 36.069 + 19.455) + 2(51.129 + 27.982) + 13.311}{3(6)} = 1042.294$$

$$f = \frac{30\dfrac{0 + 4(274.684 + 586.033 + 486.385) + 2(511.292 + 559.631) + 399.332}{3(6)}}{1042.294} = \frac{13215.97}{1042.294} = 12.6797\text{ m}$$

17.11 The values needed to perform the evaluation can be tabulated:

Height l, m	**Force, $F(l)$, N/m**	**$l \times F(l)$**
0	0	0
30	340	10200
60	1200	72000
90	1600	144000
120	2700	324000
150	3100	465000
180	3200	576000
210	3500	735000
240	3800	912000

Because there are an even number of equally-spaced segments, we can evaluate the integrals with the multi-segment Simpson's 1/3 rule.

$$F = (240 - 0)\frac{0 + 4(340 + 1600 + 3100 + 3500) + 2(1200 + 2700 + 3200) + 3800}{24} = 521{,}600$$

$$I = (240 - 0)\frac{0 + 4(10200 + 144000 + 465000 + 735000) + 2(72000 + 324000 + 576000) + 912000}{24}$$

$$= 82{,}728{,}000$$

The line of action can therefore be computed as

$$d = \frac{82{,}728{,}000}{521{,}600} = 158.6043$$

17.12 (a) Analytical solution:

$$M = \int_0^{11} 5 + 0.25x^2 \, dx = \left[5x + 0.083333x^3\right]_0^{11} = 165.9167$$

(b) Trapezoidal rule:

$$I = (1 - 0)\frac{5 + 5.25}{2} + (2 - 1)\frac{5.25 + 6}{2} + \bullet\bullet\bullet = 166.375$$

(c) Simpson's rule:

$$I = (2 - 0)\frac{5 + 4(5.25) + 6}{6} + (4 - 2)\frac{6 + 4(7.25) + 9}{6} + \bullet\bullet\bullet = 165.9167$$

17.13 We can set up a table that contains the values comprising the integrand

***x*, cm**	**ρ, g/cm^3**	**A_c, cm^2**	**$\rho \times A_c$, g/cm**
0	4	100	400
200	3.95	103	406.85
300	3.89	106	412.34
400	3.8	110	418
600	3.6	120	432
800	3.41	133	453.53
1000	3.3	150	495

We can integrate this data using a combination of the trapezoidal and Simpson's rules,

$$I = (200 - 0)\frac{400 + 406.85}{2} + (400 - 200)\frac{406.85 + 4(412.34) + 418}{6}$$

$$+ (1000 - 400)\frac{418 + 3(432 + 453.53) + 495}{8} = 430{,}877.9 \text{ g} = 430.8779 \text{ kg}$$

17.14 We can set up a table that contains the values comprising the integrand

***t*, hr**	***t*, d**	**rate (cars/4 min)**	**rate (cars/d)**
7:30	0.312500	18	6480
7:45	0.322917	24	8640
8:00	0.333333	14	5040
8:15	0.343750	24	8640
8:45	0.364583	21	7560
9:15	0.385417	9	3240

We can integrate this data using a combination of Simpson's 3/8 and 1/3 rules. This yields the number of cars that go through the intersection between 7:30 and 9:15 (1.75 hrs),

$$I = (0.34375 - 0.3125)\frac{6480 + 3(8640 + 5040) + 8640}{8}$$

$$+ (0.385417 - 0.34375)\frac{8640 + 4(7560) + 3240}{6}$$

$$= 219.375 + 292.5 = 511.875 \text{ cars}$$

The number of cars going through the intersection per minute can be computed as

$$\frac{511.875 \text{ cars}}{1.75 \text{ hr}}\frac{\text{hr}}{60 \text{ min}} = 4.875\frac{\text{cars}}{\text{min}}$$

17.15 We can use Simpson's 1/3 rule to integrate across the *y* dimension,

<u>*x* = 0:</u>

$$I = (4 - 0)\frac{-2 + 4(-4) - 8}{6} = -17.3333$$

<u>*x* = 4:</u>

$$I = (4 - 0)\frac{-1 + 4(-3) - 8}{6} = -14$$

<u>*x* = 8:</u>

$$I = (4 - 0)\frac{4 + 4(1) - 6}{6} = 1.3333$$

<u>*x* = 12:</u>

$$I = (4 - 0)\frac{10 + 4(7) + 4}{6} = 28$$

These values can then be integrated along the *x* dimension with Simpson's 3/8 rule:

$$I = (12-0)\frac{-17.3333 + 3(-14 + 1.3333) + 28}{8} = -41$$

17.16
```
>> t=[0 10 20 30 35 40 45 50];
>> Q=[4 4.8 5.2 5.0 4.6 4.3 4.3 5.0];
>> c=[10 35 55 52 40 37 32 34];
>> Qc=Q.*c;
>> M=trapz(t,Qc)

M =
  9.5185e+003
```

The problem can also be solved with a combination of Simpson's 3/8 and 1/3 rules:

```
>> M=(30-0)*(Qc(1)+3*(Qc(2)+Qc(3))+Qc(4))/8;
>> M=M+(50-30)*(Qc(4)+4*(Qc(5)+Qc(7))+2*Qc(6)+Qc(8))/12

M =
  9.6235e+003
```

Thus, the answers are 9.5185 g (trapezoidal rule) and 9.6235 g (Simpson's rules).

17.17 A table can be set up to hold the values that are to be integrated:

y, m	*H*, m	*U*, m/s	*UH*, m^2/s
0	0.5	0.03	0.015
2	1.3	0.06	0.078
4	1.25	0.05	0.0625
5	1.7	0.12	0.204
6	1	0.11	0.11
9	0.25	0.02	0.005

The cross-sectional area can be evaluated using a combination of Simpson's 1/3 rule and the trapezoidal rule:

$$A_c = (4-0)\frac{0.5 + 4(1.3) + 1.25}{6} + (6-4)\frac{1.25 + 4(1.7) + 1}{6} + (9-6)\frac{1 + 0.25}{2}$$

$$= 4.633333 + 3.016667 + 1.875 = 9.525\ \text{m}^2$$

The flow can be evaluated in a similar fashion:

$$Q = (4-0)\frac{0.015 + 4(0.078) + 0.0625}{6} + (6-4)\frac{0.0625 + 4(0.204) + 0.11}{6} + (9-6)\frac{0.11 + 0.005}{2}$$

$$= 0.259667 + 0.3295 + 0.1725 = 0.761667\ \frac{\text{m}^3}{\text{s}}$$

17.18 First, we can estimate the areas by numerically differentiating the volume data. Because the values are equally spaced, we can use the second-order difference formulas from Fig. 23.1 to compute the derivatives at each depth. For example, at the first depth, we can use the forward difference to compute

$$A_s(0) = -\frac{dV}{dz}(0) = -\frac{-1{,}963{,}500 + 4(5{,}105{,}100) - 3(9{,}817{,}500)}{8} = 1{,}374{,}450 \text{ m}^2$$

For the interior points, second-order centered differences can be used. For example, at the second point at (z = 4 m),

$$A_s(4) = -\frac{dV}{dz}(4) = -\frac{1{,}963{,}500 - 9{,}817{,}500}{8} = 981{,}750 \text{ m}^2$$

The other interior points can be determined in a similar fashion

$$A_s(8) = -\frac{dV}{dz}(8) = -\frac{392{,}700 - 5{,}105{,}100}{8} = 589{,}050 \text{ m}^2$$

$$A_s(12) = -\frac{dV}{dz}(12) = -\frac{0 - 1{,}963{,}500}{8} = 245{,}437.5 \text{ m}^2$$

For the last point, the second-order backward formula yields

$$A_s(16) = -\frac{dV}{dz}(16) = -\frac{3(0) - 4(392{,}700) + 1{,}963{,}500}{8} = -49{,}087.5 \text{ m}^2$$

Since this is clearly a physically unrealistic result, we will assume that the bottom area is 0. The results are summarized in the following table along with the other quantities needed to determine the average concentration.

***z*, m**	***V*, m^3**	***c*, g/m^3**	***A_s*, m^2**	***$c \times A_s$***
0	9817500	10.2	1374450.0	14019390
4	5105100	8.5	981750.0	8344875
8	1963500	7.4	589050.0	4358970
12	392700	5.2	245437.5	1276275
16	0	4.1	0	0

The necessary integrals can then be evaluated with the multi-segment Simpson's 1/3 rule,

$$\int_0^z A_s(z)\,dz = (16-0)\frac{1{,}374{,}450 + 4(981{,}750 + 245{,}437.5) + 2(589{,}050) + 0}{12} = 9{,}948{,}400 \text{ m}^3$$

$$\int_0^z c(z)A_s(z)\,dz = (16-0)\frac{14{,}019{,}390 + 4(8{,}344{,}875 + 1{,}276{,}275) + 2(4{,}358{,}970) + 0}{12} = 81{,}629{,}240 \text{ g}$$

The average concentration can then be computed as

$$\bar{c} = \frac{\int_0^Z c(z) A_s(z)\,dz}{\int_0^Z A_s(z)\,dz} = \frac{81{,}629{,}240}{9{,}948{,}400} = 8.205263 \frac{\text{g}}{\text{m}^3}$$

17.19

```
>> x=[0 1 2.7 3.8 3.7 3. 1.4];
>> th=[0 30 60 90 120 150 180];
>> F=cos(th*pi/180);
>> W=cumtrapz(x,F)

W =
         0    0.9330    2.0941    2.3691    2.3941    2.8722    4.3651

>> plot(th,W)
```