CHAPTER 18

18.1 The integral can be evaluated analytically as,

$$I = \int_1^2 \left(2x + \frac{3}{x}\right)^2 dx = \int_1^2 4x^2 + 12 + 9x^{-2} \, dx$$

$$I = \left[\frac{4x^3}{3} + 12x - \frac{9}{x}\right]_1^2 = \frac{4(2)^3}{3} + 12(2) - \frac{9}{2} - \frac{4(1)^3}{3} - 12(1) + \frac{9}{1} = 25.8333$$

The tableau depicting the implementation of Romberg integration to ε_s = 0.5% is

iteration →	1	2	3
ε_t →	6.9355%	0.1613%	0.0048%
ε_a →		1.6908%	0.0098%
1	27.62500000	25.87500000	25.83456463
2	26.31250000	25.83709184	
4	25.95594388		

Thus, the result is 25.83456.

18.2 **(a)** The integral can be evaluated analytically as,

$$I = \left[-0.01094x^5 + 0.21615x^4 - 1.3854x^3 + 3.14585x^2 + 2x\right]_0^8 = 34.87808$$

(b) The tableau depicting the implementation of Romberg integration to ε_s = 0.5% is

iteration →	1	2	3	4
ε_t →	20.1699%	42.8256%	0.0000%	0.0000%
ε_a →		9.9064%	2.6766%	0.000000%
1	27.84320000	19.94133333	34.87808000	34.87808000
2	21.91680000	33.94453333	34.87808000	
4	30.93760000	34.81973333		
8	33.84920000			

Thus, the result is exact.

(c) The transformations can be computed as

$$x = \frac{(8+0) + (8-0)x_d}{2} = 4 + 4x_d \qquad dx = \frac{8-0}{2} dx_d = 4dx_d$$

These can be substituted to yield

$$I = \int_{-1}^{1} \left[-0.0547(4+4x_d)^4 + 0.8646(4+4x_d)^3 - 4.1562(4+4x_d)^2 + 6.2917(4+4x_d) + 2\right] 4dx_d$$

The transformed function can be evaluated using the values from Table 18.1

$$I = 0.5555556\, f(-0.774596669) + 0.8888889\, f(0) + 0.5555556\, f(0.774596669) = 34.87808$$

which is exact.

(d)
```
>> format long
>> y = inline('-0.0547*x.^4+0.8646*x.^3-4.1562*x.^2+6.2917*x+2');
>> I = quad(y,0,8)

I =
  34.87808000000000
```

18.3 Although it's not required, the analytical solution can be evaluated simply as

$$I = \int_0^3 xe^x\, dx = \left[e^x(x-1)\right]_0^3 = 41.17107385$$

(a) The tableau depicting the implementation of Romberg integration to ε_s = 0.5% is

iteration →	1	2	3
ε_t →	119.5350%	5.8349%	0.1020%
ε_a →		26.8579%	0.3579%
1	90.38491615	43.57337260	41.21305531
2	55.27625849	41.36057514	
4	44.83949598		

which represents a percent relative error of 0.102 %.

(b) The transformations can be computed as

$$x = \frac{(3+0)+(3-0)x_d}{2} = 1.5 + 1.5x_d \qquad dx = \frac{3-0}{2}dx_d = 1.5dx_d$$

These can be substituted to yield

$$I = \int_{-1}^{1} \left[(1.5+1.5x_d)e^{1.5+1.5x_d}\right] 1.5dx_d$$

The transformed function can be evaluated using the values from Table 18.1

$$I = f(-0.577350269) + f(0.577350269) = 39.6075058$$

which represents a percent relative error of 3.8 %.

(c) Using MATLAB

```
>> format long
```

```
>> I = quad(inline('x.*exp(x)'),0,3)

I =
  41.17107385090233
```

which represents a percent relative error of 1.1×10^{-8} %.

```
>> I = quadl(inline('x.*exp(x)'),0,3)

I =
  41.17107466800178
```

which represents a percent relative error of 2×10^{-6} %.

18.4 The exact solution can be evaluated simply as

```
>> format long
>> erf(1.5)

ans =
   0.96610514647531
```

(a) The transformations can be computed as

$$x=\frac{(1.5+0)+(1.5-0)x_d}{2}=0.75+0.75x_d \qquad dx=\frac{1.5-0}{2}dx_d=0.75dx_d$$

These can be substituted to yield

$$I=\frac{2}{\sqrt{\pi}}\int_{-1}^{1}\left[e^{-(0.75+0.75x_d)^2}\right]0.75dx_d$$

The transformed function can be evaluated using the values from Table 18.1

$$I=f(-0.577350269)+f(0.577350269)=0.974173129$$

which represents a percent relative error of 0.835 %.

(b) The transformed function can be evaluated using the values from Table 18.1

$$I=0.5555556\,f(-0.774596669)+0.8888889\,f(0)+0.5555556\,f(0.774596669)=0.965502083$$

which represents a percent relative error of 0.062 %.

18.5 (a) The tableau depicting the implementation of Romberg integration to ε_s = 0.5% is

iteration →	1	2	3	4
ε_a →		19.1131%	1.0922%	0.035826%
1	199.66621287	847.93212300	1027.49455856	1051.60670352

2	685.86564547	1016.27190634	1051.22995126
4	933.67034112	1049.04507345	
8	1020.20139037		

Note that if 8 iterations are implemented, the method converges on a value of 1053.38523686. This result is also obtained if you use the composite Simpson's 1/3 rule with 1024 segments.

(b) The transformations can be computed as

$$x = \frac{(30+0)+(30-0)x_d}{2} = 15 + 15x_d \qquad dx = \frac{30-0}{2} dx_d = 15 dx_d$$

These can be substituted to yield

$$I = 200 \int_{-1}^{1} \left[\frac{15+15x_d}{22+15x_d} e^{-2.5(15+15x_d)/30} \right] 15 dx_d$$

The transformed function can be evaluated using the values from Table 18.1

$$I = f(-0.577350269) + f(0.577350269) = 1162.93396$$

(c) Interestingly, the `quad` function encounters a problem and exceeds the maximum number of iterations

```
>> format long
>> I = quad(inline('200*x/(7+x)*exp(-2.5*x/30)'),0,30)
Warning: Maximum function count exceeded; singularity likely.
(Type "warning off MATLAB:quad:MaxFcnCount" to suppress this warning.)
> In quad at 88

I =
    1.085280043451920e+003
```

The `quadl` function converges rapidly, but does not yield a very accurate result:

```
>> I = quadl(inline('200*x/(7+x)*exp(-2.5*x/30)'),0,30)

I =
    1.055900924411335e+003
```

18.6 The integral to be evaluated is

$$I = \int_0^{1/2} \left(10e^{-t} \sin 2\pi t\right)^2 dt$$

(a) The tableau depicting the implementation of Romberg integration to ε_s = 0.1% is

iteration→	1	2	3	4
ε_a→		25.0000%	2.0824%	0.025340%

1	0.00000000	20.21768866	15.16502516	15.41501768
2	15.16326649	15.48081663	15.41111155	
4	15.40142910	15.41546811		
8	15.41195836			

(b) The transformations can be computed as

$$x = \frac{(0.5+0)+(0.5-0)x_d}{2} = 0.25 + 0.25x_d \qquad dx = \frac{0.5-0}{2}dx_d = 0.25dx_d$$

These can be substituted to yield

$$I = \int_{-1}^{1} \left[10e^{-(0.25+0.25x_d)} \sin 2\pi(0.25+0.25x_d)\right]^2 0.25dx_d$$

For the two-point application, the transformed function can be evaluated using the values from Table 18.1

$$I = f(-0.577350269) + f(0.577350269) = 7.684096 + 4.313728 = 11.99782$$

For the three-point application, the transformed function can be evaluated using the values from Table 18.1

$$I = 0.5555556\, f(-0.774596669) + 0.8888889\, f(0) + 0.5555556\, f(0.774596669)$$

$$= 0.5555556(1.237449) + 0.8888889(15.16327) + 0.5555556(2.684915) = 15.65755$$

(c)
```
>> format long
>> I = quad(inline('(10*exp(-x).*sin(2*pi*x)).^2'),0,0.5)

I =
  15.41260804934509
```

18.7 The integral to be evaluated is

$$I = \int_0^{0.75} 10\left(1 - \frac{r}{0.75}\right)^{1/7} 2\pi r\, dr$$

(a) The tableau depicting the implementation of Romberg integration to ε_s = 0.1% is

iteration →	1	2	3	4
ε_a →		25.0000%	1.0725%	0.098313%
1	0.00000000	10.67030554	12.88063803	13.74550712
2	8.00272915	12.74249225	13.73199355	
4	11.55755148	13.67014971		
8	13.14200015			

(b) The transformations can be computed as

$$x = \frac{(0.75+0)+(0.75-0)x_d}{2} = 0.375 + 0.375x_d \qquad dx = \frac{0.75-0}{2}dx_d = 0.375dx_d$$

These can be substituted to yield

$$I = \int_{-1}^{1}\left[10\left(1-\frac{0.375+0.375x_d}{0.75}\right)^{1/7} 2\pi(0.375+0.375x_d)\right]^2 0.375dx_d$$

For the two-point application, the transformed function can be evaluated using the values from Table 18.1

$$I = f(-0.577350269) + f(0.577350269) = 14.77171$$

(c)

```
>> format long
>> I = quad(inline('10*(1-x/0.75).^(1/7)*2*pi.*x'),0,0.75)

I =
  14.43168560836254
```

18.8 The integral to be evaluated is

$$I = \int_2^8 (9 + 4\cos^2 0.4t)(5e^{-0.5t} + 2e^{0.15t})\, dt$$

(a) The tableau depicting the implementation of Romberg integration to ε_s = 0.1% is

iteration →	1	2	3	4
ε_a →		7.4179%	0.1054%	0.001212%
1	411.26095167	317.15529472	322.59571622	322.34570788
2	340.68170896	322.25568988	322.34961426	
4	326.86219465	322.34374398		
8	323.47335665			

(b)

```
>> format long
>> y = inline('(9+4*cos(0.4*x).^2).*(5*exp(-0.5*x)+2*exp(0.15*x))')
>> I=quadl(y,2,8)

I =
    3.223483672542467e+002
```

18.9 **(a)** The integral can be evaluated analytically as,

$$\int_{-2}^{2}\left[\frac{x^3}{3} - 3y^2x + y^3\frac{x^2}{2}\right]_0^4 dy$$

$$\int_{-2}^{2} \frac{(4)^3}{3} - 3y^2(4) + y^3 \frac{(4)^2}{2}\, dy$$

$$\int_{-2}^{2} 21.33333 - 12y^2 + 8y^3\, dy$$

$$\left[21.33333y - 4y^3 + 2y^4\right]_{-2}^{2}$$

$$21.33333(2) - 4(2)^3 + 2(2)^4 - 21.33333(-2) + 4(-2)^3 - 2(-2)^4 = 21.33333$$

(b) The operation of the `dblquad` function can be understood by invoking `help`,

```
>> help dblquad
```

A session to use the function to perform the double integral can be implemented as,

```
>> dblquad(inline('x.^2-3*y.^2+x*y.^3'),0,4,-2,2)

ans =
   21.3333
```

18.10

```
>> F=@(x) (1.6*x-0.045*x.^2).*cos(-0.00055*x.^3+0.0123*x.^2+0.13*x);
>> W=quad(F,0,30)

W =
 -157.0871
```

18.11 The integral to be determined is

$$I = \int_{0}^{1/2} (5e^{-1.25t}\sin 2\pi t)^2\, dt$$

Change of variable:

$$x = \frac{0.5+0}{2} + \frac{0.5-0}{2}x_d = 0.25 + 0.25x_d$$

$$dx = \frac{0.5-0}{2}dx_d = 0.25dx_d$$

$$I = \int_{-1}^{1} (5e^{-1.25(0.25+0.25x_d)}\sin 2\pi(0.25+0.25x_d))^2\ 0.25\, dx_d$$

Therefore, the transformed function is

$$f(x_d) = 0.25\,(5e^{-1.25(0.25+0.25x_d)}\sin 2\pi(0.25+0.25x_d))^2$$

Five-point formula:

$$I = 0.236927 f(-0.90618) + 0.478629 f(-0.53847) + 0.568889 f(0) + 0.478629 f(0.53847) + 0.236927 f(0.90618)$$

$$= 0.236927(0.127087) + 0.478629(2.059589) + 0.568889(3.345384) + 0.478629(1.050662) + 0.236927(0.040941) = 3.431617$$

Therefore, the RMS current can be computed as

$$I_{\text{RMS}} = \sqrt{3.431617} = 1.852463$$

18.12 (a)

```
>> F1=@(t) (sin(2*pi*t)).^2;
>> Irms=sqrt(quad(F1,0,1))

Irms =
    0.7071

>> R=5;
>> P=Irms^2*R

P =
    2.5000
```

(b)

```
>> ib=@(t) sin(2*pi*t);
>> Vb=@(t) 5*ib(t)-1.25*ib(t).^2;
>> Pb=@(t) ib(t).*V2(t);
>> P=quad(Pb,0,1)

P =
    2.5000
```

Interestingly, the power is identical. The reason for this can be seen by inspecting each of the power functions. For **(a)**, the power function is

$$P = I^2 R = 5(\sin 2\pi t)^2$$

For **(b)**, it is

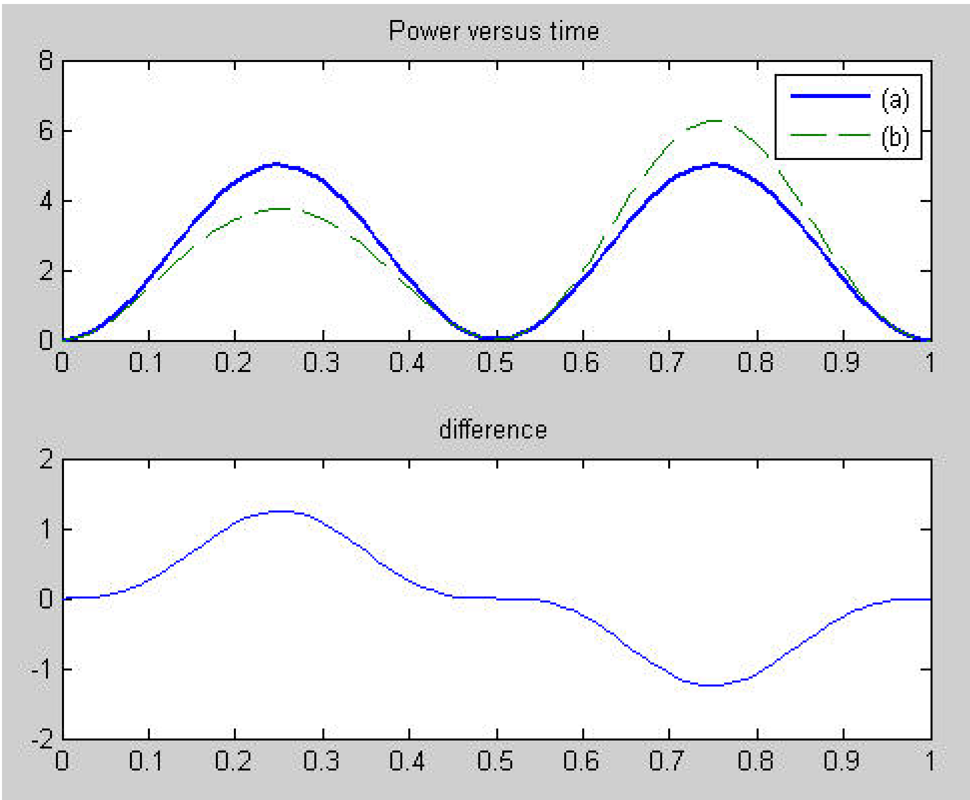
$$P = IV = I(5I - 1.25I^2) = 5I^2 - 6.25I^3$$

$$P = 5(\sin 2\pi t)^2 - 6.25(\sin 2\pi t)^3$$

A plot can be developed of both functions along with their difference.

```
t=linspace(0,1);
```

```
Pa=@(t) 5*(sin(2*pi*t)).^2;
P1=Pa(t);
P2=Pb(t);
subplot(2,1,1),plot(t,P1,t,P2,'--')
legend('(a)','(b)'),title('Power versus time')
subplot(2,1,2),plot(t,delta)
title('difference')
```



As can be seen, the difference is symmetrical across the period. Therefore, the positive and negative discrepancies cancel.

18.13 The average voltage can be computed as

$$\bar{V} = \frac{\int_0^{60} i(t)R(i)\,dt}{60}$$

We can use the formulas to generate values of $i(t)$ and $R(i)$ and their product for various equally-spaced times over the integration interval as summarized in the table below. The last column shows the integral of the product as calculated with Simpson's 1/3 rule.

t	*i*(*t*)	*R*(*i*)	*i*(*t*) × *R*(*i*)	Simpson's 1/3
0	3600.000	36469.784	131291223	
6	2950.461	29916.029	88266063	1075071847
12	2288.787	23235.215	53180447	

18	1726.549	17553.332	30306694	381186392
24	1260.625	12839.641	16185971	
30	878.355	8966.984	7876196	102019847
36	569.294	5830.320	3319166	
42	327.533	3370.360	1103904	15944048
48	151.215	1568.912	237242	
54	41.250	436.372	18000	618486
60	0.000	0.000	0	
			Sum →	**1574840619**

The average voltage can therefore be computed as

$$\bar{V} = \frac{1{,}574{,}840{,}619}{60} = 2.6247344 \times 10^7$$

18.14 We can use the trapezoidal rule to integrate the data. For example at $t = 0.2$, the voltage is computed as

$$V(0.2) = \frac{1}{10^{-5}}(0.2 - 0)\frac{0.0002 + 0.0003683}{2} = 5.683$$

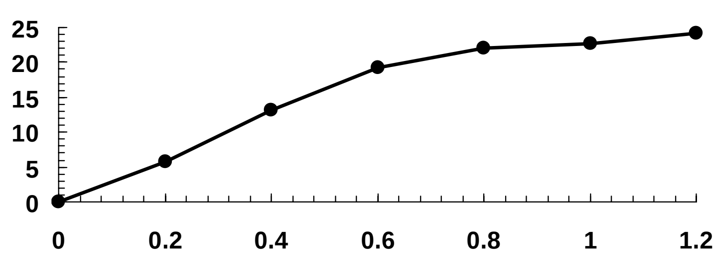
At $t = 0.4$, we add this value to the integral from $t = 0.2$ to 0.4,

$$V(0.4) = 5.683 + \frac{1}{10^{-5}}(0.4 - 0.2)\frac{0.0003683 + 0.0003819}{2} = 13.185$$

The remainder of the values can be computed in a similar fashion,

***t*, s**	***i*, A**	**Trap**	***V*(*t*)**
0	0.0002000	0	0
0.2	0.0003683	5.683	5.683
0.4	0.0003819	7.502	13.185
0.6	0.0002282	6.101	19.286
0.8	0.0000486	2.768	22.054
1	0.0000082	0.568	22.622
1.2	0.0001441	1.523	24.145

The plot of voltage versus time can be developed as



18.15 The work is computed as the product of the force times the distance, where the latter can be determined by integrating the velocity data (recall Eq. PT6.5),

$$W = F\int_0^t v(t)\,dt$$

A table can be set up holding the velocities at evenly spaced times over the integration interval. The Simpson's 1/3 rule can then be used to integrate this data as shown in the last column of the table

t	*v*	**Simp 1/3 rule**
0	0	
1	4	8
2	8	
3	12	24
4	16	
5	17	34.6667
6	20	
7	25	50.6667
8	32	
9	41	82.6667
10	52	
11	65	130.6667
12	80	
13	97	194.6667
14	116	
	Sum →	**525.3333**

Thus, the work can be computed as

$$W = 200\text{ N}(525.3333\text{ m}) = 105{,}066.7\text{ N}\cdot\text{m}$$

18.16 As in the plot, the initial point is assumed to be $e = 0$, $s = 40$. We can then use a combination of the trapezoidal and Simpsons rules to integrate the data as

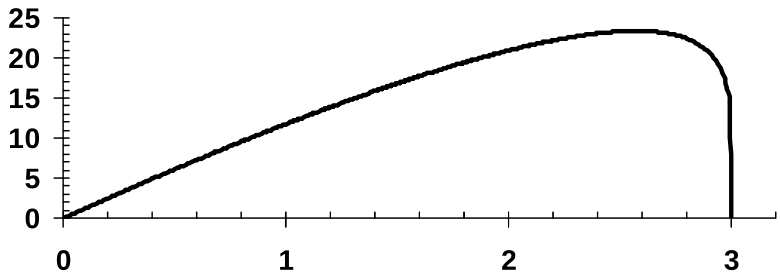
$$I = (0.02-0)\frac{40+40}{2} + (0.05-0.02)\frac{40+37.5}{2} + (0.25-0.05)\frac{37.5+4(43+60)+2(52)+60}{12} =$$

$$0.8 + 1.1625 + 4.358333 + 5.783333 = 12.10417$$

18.17 The function to be integrated is

$$Q = \int_0^3 2\left(1-\frac{r}{r_0}\right)^{1/6}(2\pi r)\,dr$$

A plot of the integrand can be developed as



As can be seen, the shape of the function indicates that we must use fine segmentation to attain good accuracy. Here are the results of using a variety of segments.

n	*Q*
2	25.1896
4	36.1635
8	40.9621
16	43.0705
32	44.0009
64	44.4127
128	44.5955
256	44.6767
512	44.7128
1024	44.7289
2048	44.7361
4096	44.7392
8192	44.7407
16384	44.7413
32768	44.7416
65536	44.7417
131072	44.7418
262144	44.7418

Therefore, the result to 4 significant figures appears to be 44.7418. The same evaluation can be performed simply with MATLAB

```
>> vA=@(r) 2*(1-r/3).^(1/6)*2*pi.*r;
>> Q=quad(vA,0,3)

Q =
   44.7418
```

18.18 The work is computed as

$$W = \int_0^x F\,dx$$

A table can be set up holding the forces at the various displacements. A combination of Simpson's 1/3 and 3/8 rules can be used to determine the integral as shown in the last two columns,

x, m	*F*, N	Integral	Method
0	0		
0.05	10		
0.1	28	1.133333	Simp 1/3
0.15	46		
0.2	63	4.583333	Simp 1/3
0.25	82		
0.3	110		
0.35	130	14.41875	Simp 3/8
	Sum→	**20.13542**	

18.19 The distance traveled is equal to the integral of velocity

$$y = \int_{t_1}^{t_2} v(t)\, dt$$

A table can be set up holding the velocities at evenly spaced times (h = 1) over the integration interval. The Simpson's 1/3 rule can then be used to integrate this data as shown in the last column of the table

t	*v*	Simp 1/3 rule
0	0	
1	6	19.33333
2	34	
3	84	175.3333
4	156	
5	250	507.3333
6	366	
7	504	1015.333
8	664	
9	846	1699.333
10	1050	
11	1045	2090
12	1040	
13	1035	2070
14	1030	
15	1025	2050
16	1020	
17	1015	2030
18	1010	
19	1005	2010
20	1000	
21	1052	2105.333
22	1108	
23	1168	2337.333
24	1232	
25	1300	2601.333

26	1372	
27	1448	2897.333
28	1528	
29	1612	3225.333
30	1700	
	Sum →	**26833.33**

Since the underlying functions are second order or less, this result should be exact. We can verify this by evaluating the integrals analytically,

$$y = \int_0^{10} 11t^2 - 5t \; dt = \left[3.66667t^3 - 2.5t^2\right]_0^{10} = 3416.667$$

$$y = \int_{10}^{20} 1100 - 5t \; dt = \left[1100t - 2.5t^2\right]_{10}^{20} = 10{,}250$$

$$y = \int_{20}^{30} 50t + 2(t-20)^2 \; dt = \left[\frac{2}{3}t^3 - 15t^2 + 800t\right]_{20}^{30} = 13{,}166.67$$

The total distance traveled is therefore 3416.667 + 10,250 + 13,166.67 = 26,833.33.

18.20 6-segment trapezoidal rule:

$$y = (30-0)\frac{0 + 2(97.422 + 207.818 + 333.713 + 478.448 + 646.579) + 844.541}{2(6)} = 10{,}931.25$$

6-segment Simpson's 1/3 rule:

$$y = (30-0)\frac{0 + 4(97.422 + 333.713 + 646.579) + 2(207.818 + 478.448) + 844.541}{3(6)} = 10{,}879.88$$

6-point Gauss quadrature:

$$y = 10{,}879.61914$$

Romberg integration:

	1	2	3	4
n	ε_a →	4.0634%	0.0098%	0.0000291%
1	12668.10909	10896.96330	10879.82315	**10879.62077**
2	11339.74974	10880.89441	10879.62393	
4	10995.60824	10879.70333		
8	10908.67956			

MATLAB:

```
>> format long
>> v=@(t) 1800*log(160000./(160000-2500*t))-9.8*t;
```

```
>> y=quad(v,0,30)

y =
    1.087961940490141e+004
```

18.21 (a) Create the following M function:

```
>> y=@(x) 1/sqrt(2*pi)*exp(-(x.^2)/2);
>> Q=quad(y,-1,1)

Q =
    0.6827

>> Q=quad(y,-2,2)

Q =
    0.9545
```

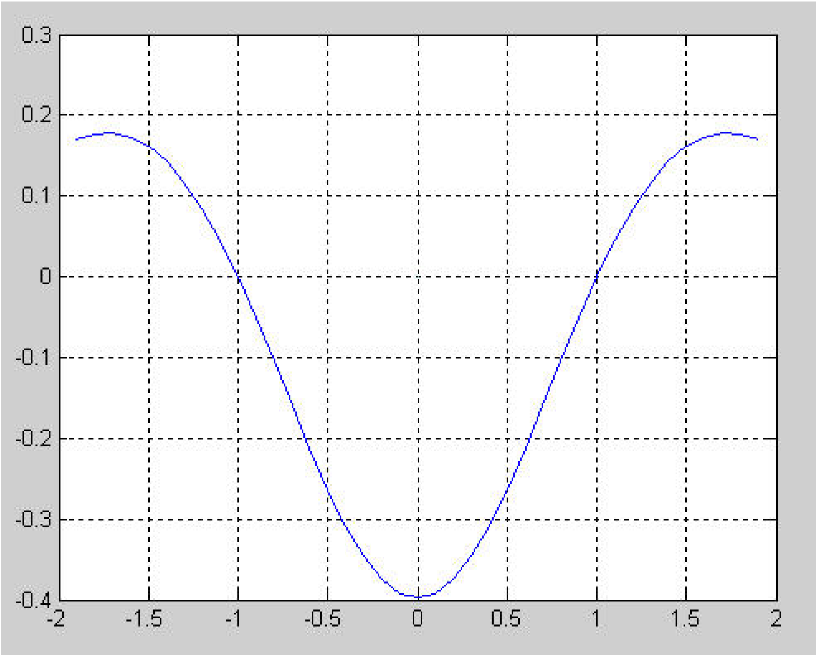
Thus, about 68.3% of the area under the curve falls between –1 and 1 and about 95.45% falls between –2 and 2.

(b) The inflection point is indicated by a zero second derivative. Recall from Chap. 4 (p. 103), that the second derivative can be approximated by

$$f''(x_i) \cong \frac{f(x_{i+1}) - 2f(x_{i+1}) + f(x_{i-1})}{h^2}$$

The following script uses this formula to compute the second derivative and generate a plot of the results,

```
x=-2:.1:2;
y=1/sqrt(2*pi)*exp(-(x.^2)/2);
xx=zeros(length(x)-2);
d2ydx2=xx;
for i=2:length(x)-1
  xx(i-1)=x(i);
  d2ydx2(i-1)=(y(i-1)-2*y(i)+y(i+1))/(x(i)-x(i-1))^2;
end
plot(xx,d2ydx2);grid
```



Thus, inflection points ($d^2y/dx^2 = 0$) occur at –1 and 1.

Note that in the next chapter we will introduce the `diff` function which provides an alternative way to make the same assessment. Here is a script that illustrates how this might be done:

```
x=-2:.1:2;
f=y(x);
d=diff(f)./diff(x);
xx=-1.95:.1:1.95;
d2=diff(d)./diff(xx);
xxx=-1.9:.1:1.9;
plot(xxx,d2,'o')
```

18.22

	1	**2**	**3**
n	ε_a →	**7.9715%**	**0.0997%**
1	1.34376994	1.97282684	1.94183605
2	1.81556261	1.94377297	
4	1.91172038		