CHAPTER 10

10.1 The flop counts for *LU* decomposition can be determined in a similar fashion as was done for Gauss elimination. The major difference is that the elimination is only implemented for the left-hand side coefficients. Thus, for every iteration of the inner loop, there are n multiplications/divisions and $n - 1$ addition/subtractions. The computations can be summarized as

Outer Loop k	Inner Loop i	Addition/Subtraction flops	Multiplication/Division flops
1	2, n	$(n-1)(n-1)$	$(n-1)n$
2	3, n	$(n-2)(n-2)$	$(n-2)(n-1)$
.	.		
.	.		
.	.		
k	$k+1$, n	$(n-k)(n-k)$	$(n-k)(n+1-k)$
.	.		
.	.		
.	.		
$n-1$	n, n	(1)(1)	(1)(2)

Therefore, the total addition/subtraction flops for elimination can be computed as

$$\sum_{k=1}^{n-1}(n-k)(n-k)=\sum_{k=1}^{n-1}\left[n^2-2nk+k^2\right]$$

Applying some of the relationships from Eq. (8.14) yields

$$\sum_{k=1}^{n-1}\left[n^2-2nk+k^2\right]=\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$$

A similar analysis for the multiplication/division flops yields

$$\sum_{k=1}^{n-1}(n-k)(n+1-k)=\frac{n^3}{3}-\frac{n}{3}$$

$$\left[n^3+O(n^2)\right]-\left[n^3+O(n)\right]+\left[\frac{1}{3}n^3+O(n^2)\right]=\frac{n^3}{3}+O(n^2)$$

Summing these results gives

$$\frac{2n^3}{3}-\frac{n^2}{2}-\frac{n}{6}$$

For forward substitution, the numbers of multiplications and subtractions are the same and equal to

$$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2} = \frac{n^2}{2} - \frac{n}{2}$$

Back substitution is the same as for Gauss elimination: $n^2/2 - n/2$ subtractions and $n^2/2 + n/2$ multiplications/divisions. The entire number of flops can be summarized as

	Mult/Div	Add/Subtr	Total
Forward elimination	$\frac{n^3}{3} - \frac{n}{3}$	$\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$	$\frac{2n^3}{3} - \frac{n^2}{2} - \frac{n}{6}$
Forward substitution	$\frac{n^2}{2} - \frac{n}{2}$	$\frac{n^2}{2} - \frac{n}{2}$	$n^2 - n$
Back substitution	$\frac{n^2}{2} + \frac{n}{2}$	$\frac{n^2}{2} - \frac{n}{2}$	n^2
Total	$\frac{n^3}{3} + n^2 - \frac{n}{3}$	$\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$	$\frac{2n^3}{3} + \frac{3n^2}{2} - \frac{7n}{6}$

The total number of flops is identical to that obtained with standard Gauss elimination.

10.2 Equation (10.6) is

$$[L]\{[U]\{x\} - \{d\}\} = [A]\{x\} - \{b\} \tag{10.6}$$

Matrix multiplication is distributive, so the left-hand side can be rewritten as

$$[L][U]\{x\} - [L]\{d\} = [A]\{x\} - \{b\}$$

Equating the terms that are multiplied by $\{x\}$ yields,

$$[L][U]\{x\} = [A]\{x\}$$

and, therefore, Eq. (10.7) follows

$$[L][U] = [A] \tag{10.7}$$

Equating the constant terms yields Eq. (10.8)

$$[L]\{d\} = \{b\} \tag{10.8}$$

10.3 The matrix to be evaluated is

$$\begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}$$

Multiply the first row by $f_{21} = -3/10 = -0.3$ and subtract the result from the second row to eliminate the a_{21} term. Then, multiply the first row by $f_{31} = 1/10 = 0.1$ and subtract the result from the third row to eliminate the a_{31} term. The result is

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0.8 & 5.1 \end{bmatrix}$$

Multiply the second row by $f_{32} = 0.8/(-5.4) = -0.148148$ and subtract the result from the third row to eliminate the a_{32} term.

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

Therefore, the *LU* decomposition is

$$[L]\{U\} = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix}\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

Multiplying [*L*] and [*U*] yields the original matrix as verified by the following MATLAB session,

```
>> L = [1 0 0;-0.3 1 0;0.1 -0.148148 1];
>> U = [10 2 -1;0 -5.4 1.7;0 0 5.351852];
>> A = L*U

A =
   10.0000    2.0000   -1.0000
   -3.0000   -6.0000    2.0000
    1.0000    1.0000    5.0000
```

10.4 The *LU* decomposition can be computed as

$$[L]\{U\} = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix}\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

Forward substitution:

$$\{d\} = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix}\begin{Bmatrix} 27 \\ -61.5 \\ -21.5 \end{Bmatrix}$$

$$d_1 = 27$$

$$d_2 = -61.5 + 0.3(27) = -53.4$$

$$d_3 = -21.5 - 0.1(27) - (-0.148148)(-53.4) = -32.11111$$

Back substitution:

$$\{x\} = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 27 \\ -53.5 \\ -32.11111 \end{Bmatrix}$$

$$x_3 = \frac{-32.11111}{5.351852} = -6$$

$$x_2 = \frac{-53.4 - 1.7(-6)}{-5.4} = 8$$

$$x_1 = \frac{27 - 2(8) - (-1)(-6)}{10} = 0.5$$

For the alternative right-hand-side vector, forward substitution is implemented as

$$\{d\} = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix} \begin{Bmatrix} 12 \\ 18 \\ -6 \end{Bmatrix}$$

$$d_1 = 12$$

$$d_2 = 18 + 0.3(12) = 21.6$$

$$d_3 = -6 - 0.1(12) - (-0.148148)(18) = -4$$

Back substitution:

$$\{x\} = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} 12 \\ 21.6 \\ -4 \end{Bmatrix}$$

$$x_3 = \frac{-4}{5.351852} = -0.747405$$

$$x_2 = \frac{21.6 - 1.7(-0.747405)}{-5.4} = -4.235294$$

$$x_1 = \frac{12 - 2(-4.235294) - (-1)(-0.747405)}{10} = 1.972318$$

10.5 The system can be written in matrix form as

$$[A]=\begin{bmatrix}2 & -6 & -1\\ -3 & -1 & 7\\ -8 & 1 & -2\end{bmatrix} \qquad \{b\}=\begin{Bmatrix}-38\\ -34\\ -20\end{Bmatrix}$$

Partial pivot:

$$[A]=\begin{bmatrix}-8 & 1 & -2\\ -3 & -1 & 7\\ 2 & -6 & -1\end{bmatrix} \qquad \{b\}=\begin{Bmatrix}-20\\ -34\\ -38\end{Bmatrix}$$

Forward eliminate

$f_{21} = -3/(-8) = 0.375$ $\qquad$ $f_{31} = 2/(-8) = -0.25$

$$[A]=\begin{bmatrix}-8 & 1 & -2\\ 0 & -1.375 & 7.75\\ 0 & -5.75 & -1.5\end{bmatrix}$$

Pivot again

$$[A]=\begin{bmatrix}-8 & 1 & -2\\ 0 & -5.75 & -1.5\\ 0 & -1.375 & 7.75\end{bmatrix} \qquad \{b\}=\begin{Bmatrix}-20\\ -38\\ -34\end{Bmatrix}$$

$f_{21} = -0.25$ $\qquad$ $f_{31} = 0.375$

Forward eliminate

$f_{32} = -1.375/(-5.75) = 0.23913$

$$[A]=\begin{bmatrix}-8 & 1 & -2\\ 0 & -5.75 & -1.5\\ 0 & 0 & 8.108696\end{bmatrix}$$

Therefore, the *LU* decomposition is

$$[L]\{U\}=\begin{bmatrix}1 & 0 & 0\\ -0.25 & 1 & 0\\ 0.375 & 0.23913 & 1\end{bmatrix}\begin{bmatrix}-8 & 1 & -2\\ 0 & -5.75 & -1.5\\ 0 & 0 & 8.108696\end{bmatrix}$$

Forward elimination

$$\{d\}=\begin{bmatrix}1 & 0 & 0\\ -0.25 & 1 & 0\\ 0.375 & 0.23913 & 1\end{bmatrix}\begin{Bmatrix}-20\\ -38\\ -34\end{Bmatrix}$$

$$d_1 = -20$$

$$d_2 = -38 - (-0.25)(-20) = -43$$

$$d_3 = -34 - 0.375(-20) - 0.23913(-43) = -16.21739$$

Back substitution:

$$\begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 8.108696 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -20 \\ -43 \\ -16.21739 \end{Bmatrix}$$

$$x_3 = \frac{-16.21739}{8.108696} = -2$$

$$x_2 = \frac{-43 - (-1.5)(-2)}{-5.75} = 8$$

$$x_1 = \frac{-20 - 1(8) - (-2)(-2)}{-8} = 4$$

10.6 Here is an M-file to generate the *LU* decomposition without pivoting

```
function [L, U] = LUNaive(A)
% LUNaive(A):
%   LU decomposition without pivoting.
% input:
%   A = coefficient matrix
% output:
%   L = lower triangular matrix
%   U = upper triangular matrix

[m,n] = size(A);
if m~=n, error('Matrix A must be square'); end
L = eye(n);
U = A;
% forward elimination
for k = 1:n-1
  for i = k+1:n
    L(i,k) = U(i,k)/U(k,k);
    U(i,k) = 0;
    U(i,k+1:n) = U(i,k+1:n)-L(i,k)*U(k,k+1:n);
  end
end
```

Test with Prob. 10.3

```
>> A = [10 2 -1;-3 -6 2;1 1 5];
>> [L,U] = LUnaive(A)

L =
```

```
    1.0000         0         0
   -0.3000    1.0000         0
    0.1000   -0.1481    1.0000

U =
   10.0000    2.0000   -1.0000
         0   -5.4000    1.7000
         0         0    5.3519
```

Verification that [*L*][*U*] = [*A*].

```
>> L*U

ans =
   10.0000    2.0000   -1.0000
   -3.0000   -6.0000    2.0000
    1.0000    1.0000    5.0000
```

Check using the `lu` function,

```
>> [L,U]=lu(A)

L =
    1.0000         0         0
   -0.3000    1.0000         0
    0.1000   -0.1481    1.0000

U =
   10.0000    2.0000   -1.0000
         0   -5.4000    1.7000
         0         0    5.3519
```

10.7 The result of Example 10.4 can be substituted into Eq. (10.14) to give

$$[A]=[U]^T[U]=\begin{bmatrix} 2.44949 & & \\ 6.123724 & 4.1833 & \\ 22.45366 & 20.9165 & 6.110101 \end{bmatrix}\begin{bmatrix} 2.44949 & 6.123724 & 22.45366 \\ & 4.1833 & 20.9165 \\ & & 6.110101 \end{bmatrix}$$

The multiplication can be implemented as in

$$a_{11}=2.44949^2=6.000001$$

$$a_{12}=6.123724\times 2.44949=15$$

$$a_{13}=22.45366\times 2.44949=55.00002$$

$$a_{21}=2.44949\times 6.123724=15$$

$$a_{22}=6.123724^2+4.1833^2=54.99999$$

$$a_{22} = 22.45366 \times 6.123724^2 + 20.9165 \times 4.1833 = 225$$

$$a_{31} = 2.44949 \times 22.45366 = 55.00002$$

$$a_{32} = 6.123724 \times 22.45366 + 4.1833 \times 20.9165 = 225$$

$$a_{33} = 22.45366^2 + 20.9165^2 + 6.110101^2 = 979.0002$$

10.8 (a) For the first row (i = 1), Eq. (10.15) is employed to compute

$$u_{11} = \sqrt{a_{11}} = \sqrt{8} = 2.828427$$

Then, Eq. (10.16) can be used to determine

$$u_{12} = \frac{a_{12}}{u_{11}} = \frac{20}{2.828427} = 7.071068$$

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{15}{2.828427} = 5.303301$$

For the second row (i = 2),

$$u_{22} = \sqrt{a_{22} - u_{12}^2} = \sqrt{80 - (7.071068)^2} = 5.477226$$

$$u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}} = \frac{50 - 7.071068(5.303301)}{5.477226} = 2.282177$$

For the third row (i = 3),

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2} = \sqrt{60 - (5.303301)^2 - (2.282177)^2} = 5.163978$$

Thus, the Cholesky decomposition yields

$$[U] = \begin{bmatrix} 2.828427 & 7.071068 & 5.303301 \\ & 5.477226 & 2.282177 \\ & & 5.163978 \end{bmatrix}$$

The validity of this decomposition can be verified by substituting it and its transpose into Eq. (10.14) to see if their product yields the original matrix [A]. This is left for an exercise.

(b)
```
>> A = [8 20 15;20 80 50;15 50 60];
>> U = chol(A)

U =
```

```
    2.8284    7.0711    5.3033
         0    5.4772    2.2822
         0         0    5.1640
```

(c) The solution can be obtained by hand or by MATLAB. Using MATLAB:

```
>> b = [50;250;100];
>> d=U'\b

d =

   17.6777
   22.8218
   -8.8756

>> x=U\d

x =
   -2.7344
    4.8828
   -1.7187
```

10.9 Here is an M-file to generate the Cholesky decomposition without pivoting

```
function U = cholesky(A)
% cholesky(A):
%   cholesky decomposition without pivoting.
% input:
%   A = coefficient matrix
% output:
%   U = upper triangular matrix
[m,n] = size(A);
if m~=n, error('Matrix A must be square'); end
for i = 1:n
  s = 0;
  for k = 1:i-1
    s = s + U(k, i) ^ 2;
  end
  U(i, i) = sqrt(A(i, i) - s);
  for j = i + 1:n
    s = 0;
    for k = 1:i-1
      s = s + U(k, i) * U(k, j);
    end
    U(i, j) = (A(i, j) - s) / U(i, i);
  end
end
```

Test with Prob. 10.8

```
>> A = [8 20 15;20 80 50;15 50 60];
>> cholesky(A)

ans =
```

```
    2.8284    7.0711    5.3033
         0    5.4772    2.2822
         0         0    5.1640
```

Check with the `chol` function

```
>> U = chol(A)

U =
    2.8284    7.0711    5.3033
         0    5.4772    2.2822
         0         0    5.1640
```

10.10 MATLAB can be used to generate the *LU* decomposition

```
>> A=[3 -2 1;2 6 -4;-1 -2 5];
>> b=[-10 44 -26]';
>> [L,U]=lu(A)

L =
    1.0000         0         0
    0.6667    1.0000         0
   -0.3333   -0.3636    1.0000
U =
    3.0000   -2.0000    1.0000
         0    7.3333   -4.6667
         0         0    3.6364
```

Therefore,

$$\begin{bmatrix} 1 & & \\ 0.6667 & 1 & \\ -0.3333 & -0.3636 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 \\ & 7.3333 & -4.6667 \\ & & 3.6364 \end{bmatrix}$$

The forward substitution can be implemented as

```
>> d=L\b

d =
  -10.0000
   50.6667
  -10.9091
```

Back substitution yields the final solution

```
>> x=U\d

x =
    1.0000
    5.0000
   -3.0000
```

We can verify this result using left division

```
>> x=A\b

x =
    1.0000
    5.0000
   -3.0000
```

Thus, $x_1 = 1$, $x_2 = 5$, and $x_3 = -3$

10.11 (a) Multiply first row by $f_{21} = 3/8 = 0.375$ and subtract the result from the second row to give

$$\begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 2 & 3 & 9 \end{bmatrix}$$

Multiply first row by $f_{31} = 2/8 = 0.25$ and subtract the result from the third row to give

$$\begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 2.5 & 8.75 \end{bmatrix}$$

Multiply second row by $f_{32} = 2.5/6.25 = 0.4$ and subtract the result from the third row to give

$$[U] = \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 0 & 8.1 \end{bmatrix}$$

As indicated, this is the U matrix. The L matrix is simply constructed from the f's as

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.375 & 1 & 0 \\ 0.25 & 0.4 & 1 \end{bmatrix}$$

Merely multiply $[L][U]$ to yield the original matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.375 & 1 & 0 \\ 0.25 & 0.4 & 1 \end{bmatrix}\begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 0 & 8.1 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 1 \\ 3 & 7 & 2 \\ 2 & 3 & 9 \end{bmatrix}$$

(b) The determinant is equal to the product of the diagonal elements of $[U]$:

$D = 8 \times 6.25 \times 8.1 = 405$

(c) Solution with MATLAB:

```
>> A=[8 2 1;3 7 2;2 3 9];
>> [L,U]=lu(A)

L =
    1.0000         0         0
    0.3750    1.0000         0
    0.2500    0.4000    1.0000

U =
    8.0000    2.0000    1.0000
         0    6.2500    1.6250
         0         0    8.1000

>> L*U

ans =
     8     2     1
     3     7     2
     2     3     9

>> det(A)

ans =
   405
```

10.12 (a) The determinant is equal to the product of the diagonal elements of [U]:

$D = 3 \times 7.3333 \times 3.6364 = 80$

(b) Forward substitution:

```
>> L=[1 0 0;0.6667 1 0;-0.3333 -0.3636 1];
>> U=[3 -2 1;0 -7.3333 -4.6667;0 0 3.6364];
>> b=[-10 44 -26]';
>> d=L\b

d =
  -10.0000
   50.6670
  -10.9105
```

Back substitution:

```
>> x=U\d

x =
   -5.6664
   -4.9998
   -3.0004
```

10.13 Using MATLAB:

```
>> A=[2 -1 0;-1 2 -1;0 -1 2];
>> U=chol(A)

U =
    1.4142   -0.7071         0
         0    1.2247   -0.8165
         0         0    1.1547
```

The result can be validated by

```
>> U'*U

ans =
    2.0000   -1.0000         0
   -1.0000    2.0000   -1.0000
         0   -1.0000    2.0000
```

10.14 Using MATLAB:

```
>> A=[9 0 0;0 25 0;0 0 4];
>> U=chol(A)

U =
     3     0     0
     0     5     0
     0     0     2
```

Thus, the factorization of this diagonal matrix consists of another diagonal matrix where the elements are the square root of the original. This is consistent with Eqs. 10.15 and 10.16, which for a diagonal matrix reduce to

$$u_{ii} = \sqrt{a_{ii}}$$

$$u_{ij} = 0 \quad \text{for } i \neq j$$