CHAPTER 12

12.1 (a) The first iteration can be implemented as

$$x_1 = \frac{41 + 0.4x_2}{0.8} = \frac{41 + 0.4(0)}{0.8} = 51.25$$

$$x_2 = \frac{25 + 0.4x_1 + 0.4x_3}{0.8} = \frac{25 + 0.4(51.25) + 0.4(0)}{0.8} = 56.875$$

$$x_3 = \frac{105 + 0.4x_2}{0.8} = \frac{105 + 0.4(56.875)}{0.8} = 159.6875$$

Second iteration:

$$x_1 = \frac{41 + 0.4(56.875)}{0.8} = 79.6875$$

$$x_2 = \frac{25 + 0.4(79.6875) + 0.4(159.6875)}{0.8} = 150.9375$$

$$x_3 = \frac{105 + 0.4(150.9375)}{0.8} = 206.7188$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{79.6875 - 51.25}{79.6875} \right| \times 100\% = 35.69\%$$

$$\varepsilon_{a,2} = \left| \frac{150.9375 - 56.875}{150.9375} \right| \times 100\% = 62.32\%$$

$$\varepsilon_{a,3} = \left| \frac{206.7188 - 159.6875}{206.7188} \right| \times 100\% = 22.75\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	51.25	100.00%	
	x_2	56.875	100.00%	
	x_3	159.6875	100.00%	100.00%
2	x_1	79.6875	35.69%	
	x_2	150.9375	62.32%	
	x_3	206.7188	22.75%	62.32%

3	x_1	126.7188	37.11%	
	x_2	197.9688	23.76%	
	x_3	230.2344	10.21%	37.11%
4	x_1	150.2344	15.65%	
	x_2	221.4844	10.62%	
	x_3	241.9922	4.86%	15.65%
5	x_1	161.9922	7.26%	
	x_2	233.2422	5.04%	
	x_3	247.8711	2.37%	7.26%
6	x_1	167.8711	3.50%	
	x_2	239.1211	2.46%	
	x_3	250.8105	1.17%	3.50%

Thus, after 6 iterations, the maximum error is 3.5% and we arrive at the result: x_1 = 167.8711, x_2 = 239.1211 and x_3 = 250.8105.

(b) The same computation can be developed with relaxation where λ = 1.2.

First iteration:

$$x_1 = \frac{41 + 0.4x_2}{0.8} = \frac{41 + 0.4(0)}{0.8} = 51.25$$

Relaxation yields: $x_1 = 1.2(51.25) - 0.2(0) = 61.5$

$$x_2 = \frac{25 + 0.4x_1 + 0.4x_3}{0.8} = \frac{25 + 0.4(61.5) + 0.4(0)}{0.8} = 62$$

Relaxation yields: $x_2 = 1.2(62) - 0.2(0) = 74.4$

$$x_3 = \frac{105 + 0.4x_2}{0.8} = \frac{105 + 0.4(62)}{0.8} = 168.45$$

Relaxation yields: $x_3 = 1.2(168.45) - 0.2(0) = 202.14$

Second iteration:

$$x_1 = \frac{41 + 0.4(62)}{0.8} = 88.45$$

Relaxation yields: $x_1 = 1.2(88.45) - 0.2(61.5) = 93.84$

$$x_2 = \frac{25 + 0.4(93.84) + 0.4(202.14)}{0.8} = 179.24$$

Relaxation yields: $x_2 = 1.2(179.24) - 0.2(74.4) = 200.208$

$$x_3 = \frac{105 + 0.4(200.208)}{0.8} = 231.354$$

Relaxation yields: $x_3 = 1.2(231.354) - 0.2(202.14) = 237.1968$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{93.84 - 61.5}{93.84}\right| \times 100\% = 34.46\%$$

$$\varepsilon_{a,2} = \left|\frac{200.208 - 74.4}{200.208}\right| \times 100\% = 62.84\%$$

$$\varepsilon_{a,3} = \left|\frac{237.1968 - 202.14}{237.1968}\right| \times 100\% = 14.78\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	relaxation	ε_a	maximum ε_a
1	x_1	51.25	61.5	100.00%	
	x_2	62	74.4	100.00%	
	x_3	168.45	202.14	100.00%	100.000%
2	x_1	88.45	93.84	34.46%	
	x_2	179.24	200.208	62.84%	
	x_3	231.354	237.1968	14.78%	62.839%
3	x_1	151.354	162.8568	42.38%	
	x_2	231.2768	237.49056	15.70%	
	x_3	249.99528	252.55498	6.08%	42.379%
4	x_1	169.99528	171.42298	5.00%	
	x_2	243.23898	244.38866	2.82%	
	x_3	253.44433	253.6222	0.42%	4.997%

Thus, relaxation speeds up convergence. After 6 iterations, the maximum error is 4.997% and we arrive at the result: x_1 = 171.423, x_2 = 244.389 and x_3 = 253.622.

12.2 The first iteration can be implemented as

$$x_1 = \frac{27 - 2x_2 + x_3}{10} = \frac{27 - 2(0) + 0}{10} = 2.7$$

$$x_2 = \frac{-61.5 + 3x_1 - 2x_3}{-6} = \frac{-61.5 + 3(2.7) - 2(0)}{-6} = 8.9$$

$$x_3 = \frac{-21.5 - x_1 - x_2}{5} = \frac{-21.5 - (2.7) - 8.9}{5} = -6.62$$

Second iteration:

$$x_1 = \frac{27 - 2(8.9) - 6.62}{10} = 0.258$$

$$x_2 = \frac{-61.5 + 3(0.258) - 2(-6.62)}{-6} = 7.914333$$

$$x_3 = \frac{-21.5 - (0.258) - 7.914333}{5} = -5.934467$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{0.258 - 2.7}{0.258} \right| \times 100\% = 947\%$$

$$\varepsilon_{a,2} = \left| \frac{7.914333 - 8.9}{7.914333} \right| \times 100\% = 12.45\%$$

$$\varepsilon_{a,3} = \left| \frac{-5.934467 - (-6.62)}{-5.934467} \right| \times 100\% = 11.55\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	2.7	100.00%	
	x_2	8.9	100.00%	
	x_3	-6.62	100.00%	100%
2	x_1	0.258	946.51%	
	x_2	7.914333	12.45%	
	x_3	-5.93447	11.55%	946%
3	x_1	0.523687	50.73%	
	x_2	8.010001	1.19%	
	x_3	-6.00674	1.20%	50.73%
4	x_1	0.497326	5.30%	
	x_2	7.999091	0.14%	
	x_3	-5.99928	0.12%	5.30%
5	x_1	0.500253	0.59%	
	x_2	8.000112	0.01%	
	x_3	-6.00007	0.01%	0.59%

Thus, after 5 iterations, the maximum error is 0.59% and we arrive at the result: x_1 = 0.500253, x_2 = 8.000112 and x_3 = –6.00007.

12.3 The first iteration can be implemented as

$$x_1 = \frac{27 - 2x_2 + x_3}{10} = \frac{27 - 2(0) + 0}{10} = 2.7$$

$$x_2 = \frac{-61.5 + 3x_1 - 2x_3}{-6} = \frac{-61.5 + 3(0) - 2(0)}{-6} = 10.25$$

$$x_3 = \frac{-21.5 - x_1 - x_2}{5} = \frac{-21.5 - 0 - 0}{5} = -4.3$$

Second iteration:

$$x_1 = \frac{27 - 2(10.25) - 4.3}{10} = 0.22$$

$$x_2 = \frac{-61.5 + 3(2.7) - 2(-4.3)}{-6} = 7.466667$$

$$x_3 = \frac{-21.5 - (2.7) - 10.25}{5} = -6.89$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{0.22 - 2.7}{0.258}\right| \times 100\% = 1127\%$$

$$\varepsilon_{a,2} = \left|\frac{7.466667 - 10.25}{7.466667}\right| \times 100\% = 37.28\%$$

$$\varepsilon_{a,3} = \left|\frac{-6.89 - (-4.3)}{-6.89}\right| \times 100\% = 37.59\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	2.7	100.00%	
	x_2	10.25	100.00%	
	x_3	-4.3	100.00%	100.00%
2	x_1	0.22	1127.27%	
	x_2	7.466667	37.28%	
	x_3	-6.89	37.59%	1127.27%

3	x_1	0.517667	57.50%	
	x_2	7.843333	4.80%	
	x_3	-5.83733	18.03%	57.50%
4	x_1	0.5476	5.47%	
	x_2	8.045389	2.51%	
	x_3	-5.9722	2.26%	5.47%
5	x_1	0.493702	10.92%	
	x_2	7.985467	0.75%	
	x_3	-6.0186	0.77%	10.92%
6	x_1	0.501047	1.47%	
	x_2	7.99695	0.14%	
	x_3	-5.99583	0.38%	1.47%

Thus, after 6 iterations, the maximum error is 1.47% and we arrive at the result: x_1 = 0.501047, x_2 = 7.99695 and x_3 = –5.99583.

12.4 The first iteration can be implemented as

$$c_1 = \frac{3800 + 3c_2 + c_3}{15} = \frac{3800 + 3(0) + 0}{15} = 253.3333$$

$$c_2 = \frac{1200 + 3c_1 + 6c_3}{18} = \frac{1200 + 3(253.3333) + 6(0)}{18} = 108.8889$$

$$c_3 = \frac{2350 + 4c_1 + c_2}{12} = \frac{2350 + 4(253.3333) + 108.8889}{12} = 289.3519$$

Second iteration:

$$c_1 = \frac{3800 + 3(108.889) + 289.3519}{15} = 294.4012$$

$$c_2 = \frac{1200 + 3(294.4012) + 6(289.3519)}{18} = 212.1842$$

$$c_3 = \frac{2350 + 4(294.4012) + 212.1842}{12} = 311.6491$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{294.4012 - 253.3333}{294.4012} \right| \times 100\% = 13.95\%$$

$$\varepsilon_{a,2} = \left| \frac{212.1842 - 108.8889}{212.1842} \right| \times 100\% = 48.68\%$$

$$\varepsilon_{a,3} = \left| \frac{311.6491 - 289.3519}{311.6491} \right| \times 100\% = 7.15\%$$

The remainder of the calculation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	253.3333	100.00%	
	x_2	108.8889	100.00%	
	x_3	289.3519	100.00%	100.00%
2	x_1	294.4012	13.95%	
	x_2	212.1842	48.68%	
	x_3	311.6491	7.15%	48.68%
3	x_1	316.5468	7.00%	
	x_2	223.3075	4.98%	
	x_3	319.9579	2.60%	7.00%
4	x_1	319.3254	0.87%	
	x_2	226.5402	1.43%	
	x_3	321.1535	0.37%	1.43%
5	x_1	320.0516	0.23%	
	x_2	227.0598	0.23%	
	x_3	321.4388	0.09%	0.23%

Note that after several more iterations, we arrive at the result: x_1 = 320.2073, x_2 = 227.2021 and x_3 = 321.5026.

12.5 The equations must first be rearranged so that they are diagonally dominant

$$-8x_1 + x_2 - 2x_3 = -20$$

$$2x_1 - 6x_2 - x_3 = -38$$

$$-3x_1 - x_2 + 7x_3 = -34$$

(a) The first iteration can be implemented as

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(2.5) + 0}{-6} = 7.166667$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(2.5) + 7.166667}{7} = -2.761905$$

Second iteration:

$$x_1 = \frac{-20 - 7.166667 + 2(-2.761905)}{-8} = 4.08631$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.08631) + (-2.761905)}{-6} = 8.155754$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.08631) + 8.155754}{7} = -1.94076$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{4.08631 - 2.5}{4.08631} \right| \times 100\% = 38.82\%$$

$$\varepsilon_{a,2} = \left| \frac{8.155754 - 7.166667}{8.155754} \right| \times 100\% = 12.13\%$$

$$\varepsilon_{a,3} = \left| \frac{-1.94076 - (-2.761905)}{-1.94076} \right| \times 100\% = 42.31\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
0	x_1	0		
	x_2	0		
	x_3	0		
1	x_1	2.5	100.00%	
	x_2	7.166667	100.00%	
	x_3	-2.7619	100.00%	100.00%
2	x_1	4.08631	38.82%	
	x_2	8.155754	12.13%	
	x_3	-1.94076	42.31%	42.31%
3	x_1	4.004659	2.04%	
	x_2	7.99168	2.05%	
	x_3	-1.99919	2.92%	2.92%

Thus, after 3 iterations, the maximum error is 2.92% and we arrive at the result: x_1 = 4.004659, x_2 = 7.99168 and x_3 = –1.99919.

(b) The same computation can be developed with relaxation where λ = 1.2.

First iteration:

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$

Relaxation yields: $x_1 = 1.2(2.5) - 0.2(0) = 3$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(3) + 0}{-6} = 7.333333$$

Relaxation yields: $x_2 = 1.2(7.333333) - 0.2(0) = 8.8$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(3) + 8.8}{7} = -2.3142857$$

Relaxation yields: $x_3 = 1.2(-2.3142857) - 0.2(0) = -2.7771429$

<u>Second iteration:</u>

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 8.8 + 2(-2.7771429)}{-8} = 4.2942857$$

Relaxation yields: $x_1 = 1.2(4.2942857) - 0.2(3) = 4.5531429$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.5531429) - 2.7771429}{-6} = 8.3139048$$

Relaxation yields: $x_2 = 1.2(8.3139048) - 0.2(8.8) = 8.2166857$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.5531429) + 8.2166857}{7} = -1.7319837$$

Relaxation yields: $x_3 = 1.2(-1.7319837) - 0.2(-2.7771429) = -1.5229518$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{4.5531429 - 3}{4.5531429}\right| \times 100\% = 34.11\%$$

$$\varepsilon_{a,2} = \left|\frac{8.2166857 - 8.8}{8.2166857}\right| \times 100\% = 7.1\%$$

$$\varepsilon_{a,3} = \left|\frac{-1.5229518 - (-2.7771429)}{-1.5229518}\right| \times 100\% = 82.35\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	relaxation	ε_a	maximum ε_a

1	x_1	2.5	3	100.00%	
	x_2	7.3333333	8.8	100.00%	
	x_3	-2.314286	-2.777143	100.00%	100.000%
2	x_1	4.2942857	4.5531429	34.11%	
	x_2	8.3139048	8.2166857	7.10%	
	x_3	-1.731984	-1.522952	82.35%	82.353%
3	x_1	3.9078237	3.7787598	20.49%	
	x_2	7.8467453	7.7727572	5.71%	
	x_3	-2.12728	-2.248146	32.26%	32.257%
4	x_1	4.0336312	4.0846055	7.49%	
	x_2	8.0695595	8.12892	4.38%	
	x_3	-1.945323	-1.884759	19.28%	19.280%
5	x_1	3.9873047	3.9678445	2.94%	
	x_2	7.9700747	7.9383056	2.40%	
	x_3	-2.022594	-2.050162	8.07%	8.068%
6	x_1	4.0048286	4.0122254	1.11%	
	x_2	8.0124354	8.0272613	1.11%	
	x_3	-1.990866	-1.979007	3.60%	3.595%

Thus, relaxation actually seems to retard convergence. After 6 iterations, the maximum error is 3.595% and we arrive at the result: $x_1 = 4.0122254$, $x_2 = 8.0272613$ and $x_3 = -1.979007$.

12.6 As ordered, none of the sets will converge. However, if Set 1 and 2 are reordered so that they are diagonally dominant, they will converge on the solution of (1, 1, 1).

Set 1:
$$\begin{aligned} 9x + 3y + z &= 13 \\ 2x + 5y - z &= 6 \\ -6x \qquad + 8z &= 2 \end{aligned}$$

Set 2:
$$\begin{aligned} 4x + 2y - 2z &= 4 \\ x + 5y - z &= 5 \\ x + y + 6z &= 8 \end{aligned}$$

At face value, because it is not strictly diagonally dominant, Set 2 would seem to be divergent. However, since it is very close to being diagonally dominant, a solution can be obtained.

The third set is not diagonally dominant and will diverge for most orderings. However, the following arrangement will converge albeit at a very slow rate:

Set 3:
$$\begin{aligned} -3x + 4y + 5z &= 6 \\ 2y - z &= 1 \\ -2x + 2y - 3z &= -3 \end{aligned}$$

12.7 The equations to be solved are

$$f_1(x, y) = -x^2 + x + 0.5 - y$$

$$f_2(x, y) = x^2 - y - 5xy$$

The partial derivatives can be computed and evaluated at the initial guesses

$$\frac{\partial f_{1,0}}{\partial x} = -2x + 1 = -2(1.2) + 1 = -1.4 \qquad \frac{\partial f_{1,0}}{\partial y} = -1$$

$$\frac{\partial f_{2,0}}{\partial x} = 2x - 5y = 2(1.2) - 5(1.2) = -3.6 \qquad \frac{\partial f_{2,0}}{\partial y} = -1 - 5x = -1 - 5(1.2) = -7$$

They can then be used to compute the determinant of the Jacobian for the first iteration is

$$-1.4(-7) - (-1)(-3.6) = 6.2$$

The values of the functions can be evaluated at the initial guesses as

$$f_{1,0} = -1.2^2 + 1.2 + 0.5 - 1.2 = -0.94$$

$$f_{2,0} = 1.2^2 - 5(1.2)(1.2) - 1.2 = -6.96$$

These values can be substituted into Eq. (12.12) to give

$$x_1 = 1.2 - \frac{-0.94(-3.6) - (-6.96)(-1)}{6.2} = 1.26129$$

$$x_2 = 1.2 - \frac{-6.96(-1.4) - (-0.94)(-3.6)}{6.2} = 0.174194$$

The computation can be repeated until an acceptable accuracy is obtained. The results are summarized as

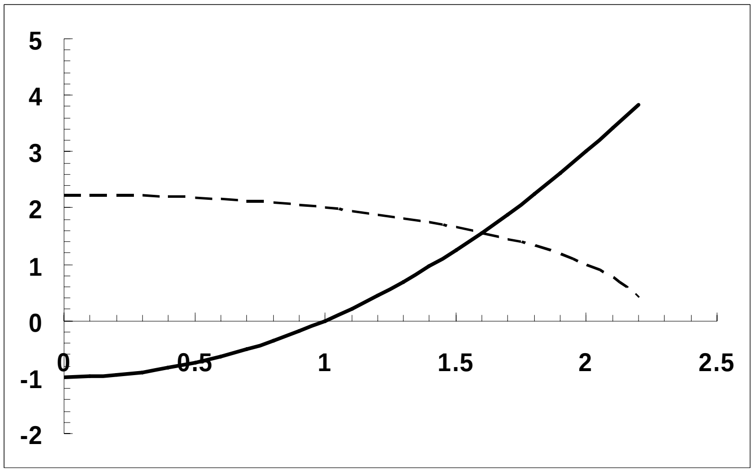
iteration	x	y	ε_{a1}	ε_{a2}
0	1.2	1.2		
1	1.26129	0.174194	4.859%	588.889%
2	1.234243	0.211619	2.191%	17.685%
3	1.233319	0.212245	0.075%	0.295%
4	1.233318	0.212245	0.000%	0.000%

12.8 (a) The equations can be set up in a form amenable to plotting as

$$y = x^2 - 1$$

$$y = \sqrt{5 - x^2}$$

These can be plotted as



Thus, a solution seems to lie at about $x = y = 1.6$.

(b) The equations can be solved in a number of different ways. For example, the first equation can be solved for x and the second solved for y. For this case, successive substitution does not work

First iteration:

$$x = \sqrt{5 - y^2} = \sqrt{5 - (1.5)^2} = 1.658312$$
$$y = (1.658312)^2 - 1 = 1.75$$

Second iteration:

$$x = \sqrt{5 - (1.75)^2} = 1.391941$$
$$y = (1.391941)^2 - 1 = 0.9375$$

Third iteration:

$$x = \sqrt{5 - (0.9375)^2} = 2.030048$$
$$y = (2.030048)^2 - 1 = 3.12094$$

Thus, the solution is moving away from the solution that lies at approximately $x = y = 1.6$.

An alternative solution involves solving the second equation for x and the first for y. For this case, successive substitution does work

First iteration:

$$x = \sqrt{y + 1} = \sqrt{1.5 + 1} = 1.581139$$
$$y = \sqrt{5 - x^2} = \sqrt{5 - (1.581139)^2} = 1.581139$$

Second iteration:

$$x = \sqrt{1.581139} = 1.606592$$
$$y = \sqrt{5 - (1.606592)^2} = 1.555269$$

Third iteration:

$$x = \sqrt{5 - (1.555269)^2} = 1.598521$$
$$y = (1.598521)^2 - 1 = 1.563564$$

After several more iterations, the calculation converges on the solution of x = 1.600485 and y = 1.561553.

(c) The equations to be solved are

$$f_1(x, y) = x^2 - y - 1$$

$$f_2(x, y) = 5 - y^2 - x^2$$

The partial derivatives can be computed and evaluated at the initial guesses

$$\frac{\partial f_{1,0}}{\partial x} = 2x \qquad \frac{\partial f_{1,0}}{\partial y} = -1$$

$$\frac{\partial f_{2,0}}{\partial x} = -2x \qquad \frac{\partial f_{2,0}}{\partial y} = -2y$$

They can then be used to compute the determinant of the Jacobian for the first iteration is

$$-1.4(-7) - (-1)(-3.6) = 6.2$$

The values of the functions can be evaluated at the initial guesses as

$$f_{1,0} = -1.2^2 + 1.2 + 0.5 - 1.2 = -0.94$$

$$f_{2,0} = 1.2^2 - 5(1.2)(1.2) - 1.2 = -6.96$$

These values can be substituted into Eq. (12.12) to give

$$x_1 = 1.2 - \frac{-0.94(-3.6) - (-6.96)(-1)}{6.2} = 1.26129$$

$$x_2 = 1.2 - \frac{-6.96(-1.4) - (-0.94)(-3.6)}{6.2} = 0.174194$$

The computation can be repeated until an acceptable accuracy is obtained. The results are summarized as

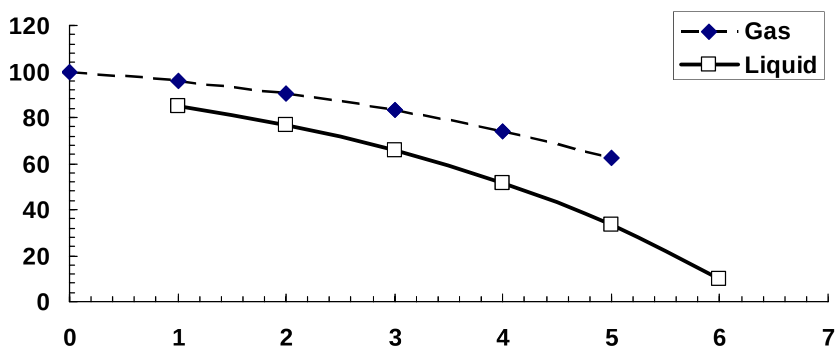
iteration	ξ	ψ	ε_{a1}	ε_{a2}
0	1.5	1.5		
1	1.604167	1.5625	6.494%	4.000%
2	1.600489	1.561553	0.230%	0.061%
3	1.600485	1.561553	0.000%	0.000%

12.9 The mass balances can be expressed in matrix form as

$$\begin{bmatrix} 2.8 & 0 & 0 & 0 & 0 & -0.8 & 0 & 0 & 0 & 0 \\ -2 & 2.8 & 0 & 0 & 0 & 0 & -0.8 & 0 & 0 & 0 \\ 0 & -2 & 2.8 & 0 & 0 & 0 & 0 & -0.8 & 0 & 0 \\ 0 & 0 & -2 & 2.8 & 0 & 0 & 0 & 0 & -0.8 & 0 \\ 0 & 0 & 0 & -2 & 2.8 & 0 & 0 & 0 & 0 & -0.8 \\ -0.8 & 0 & 0 & 0 & 0 & 1.8 & -1 & 0 & 0 & 0 \\ 0 & -0.8 & 0 & 0 & 0 & 0 & 1.8 & -1 & 0 & 0 \\ 0 & 0 & -0.8 & 0 & 0 & 0 & 0 & 1.8 & -1 & 0 \\ 0 & 0 & 0 & -0.8 & 0 & 0 & 0 & 0 & 1.8 & -1 \\ 0 & 0 & 0 & 0 & -0.8 & 0 & 0 & 0 & 0 & 1.8 \end{bmatrix} \begin{Bmatrix} c_{G1} \\ c_{G2} \\ c_{G3} \\ c_{G4} \\ c_{G5} \\ c_{L1} \\ c_{L2} \\ c_{L3} \\ c_{L4} \\ c_{L5} \end{Bmatrix} = \begin{Bmatrix} 200 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{Bmatrix}$$

These equations can then be solved. The results are tabulated and plotted below:

Reactor	Gas	Liquid
0	100	
1	95.73328	85.06649
2	90.2475	76.53306
3	83.19436	65.5615
4	74.12603	51.45521
5	62.46675	33.31856
6		10



12.10 Substituting centered difference finite differences, the Laplace equation can be written for the node (1, 1) as

$$0 = \frac{T_{21} - 2T_{11} + T_{01}}{\Delta x^2} + \frac{T_{12} - 2T_{11} + T_{10}}{\Delta y^2}$$

Because the grid is square ($\Delta x = \Delta y$), this equation can be expressed as

$$0 = T_{21} - 4T_{11} + T_{01} + T_{12} + T_{10}$$

The boundary node values (T_{01} = 100 and T_{10} = 75) can be substituted to give

$$4T_{11} - T_{12} - T_{21} = 175$$

The same approach can be written for the other interior nodes. When this is done, the following system of equations results

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{12} \\ T_{21} \\ T_{22} \end{Bmatrix} = \begin{Bmatrix} 175 \\ 125 \\ 75 \\ 25 \end{Bmatrix}$$

These equations can be solved using the Gauss-Seidel method. For example, the first iteration would be

$$T_{11} = \frac{175 + T_{12} + T_{21}}{4} = \frac{175 + 0 + 0}{4} = 43.75$$

$$T_{12} = \frac{125 + T_{11} + T_{22}}{4} = \frac{125 + 43.75 + 0}{4} = 42.1875$$

$$T_{21} = \frac{75 + T_{11} + T_{22}}{4} = \frac{75 + 43.75 + 0}{4} = 29.6875$$

$$T_{22} = \frac{25 + T_{12} + T_{21}}{4} = \frac{25 + 42.1875 + 29.6875}{4} = 24.21875$$

The computation can be continued as follows:

iteration	**unknown**	**value**	ε_a	**maximum** ε_a
1	x_1	43.75	100.00%	
	x_2	42.1875	100.00%	
	x_3	29.6875	100.00%	
	x_4	24.21875	100.00%	100.00%
2	x_1	61.71875	29.11%	
	x_2	52.73438	20.00%	
	x_3	40.23438	26.21%	
	x_4	29.49219	17.88%	29.11%
3	x_1	66.99219	7.87%	
	x_2	55.37109	4.76%	
	x_3	42.87109	6.15%	
	x_4	30.81055	4.28%	7.87%
4	x_1	68.31055	1.93%	
	x_2	56.03027	1.18%	
	x_3	43.53027	1.51%	
	x_4	31.14014	1.06%	1.93%
5	x_1	68.64014	0.48%	

x_2	56.19507	0.29%	
x_3	43.69507	0.38%	
x_4	31.22253	0.26%	0.48%

Thus, after 5 iterations, the maximum error is 0.48% and we are converging on the final result: T_{11} = 68.64, T_{12} = 56.195, T_{21} = 43.695, and T_{22} = 31.22.