Discrete-Time Dynamical Systems as State-Space Models

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My Talk at a Glance: What to Expect from It?

1. What are discrete-time dynamical systems?
2. Linkage between state-space models and discrete-time dynamical systems models
3. Examples of state-space models
   - Coding examples in dynr
4. Examples of regime-switching state-space models
   - Coding examples in dynr

Today's Workshop at a Glance: What to Expect from It?

- What this workshop isn't:
  - A comprehensive workshop on each of the four packages covered
  - A full course on dynamic or dynamical systems models, or state-space models

Today's Workshop at a Glance: What to Expect from It?

- What this workshop isn't:
  - A comprehensive workshop on each of the four packages covered
  - A full course on dynamic or dynamical systems models, or state-space models
- BUT we will cover the *essential survival basics to get you started*.
- We will focus on providing *highlights* of each package and *linkages* among them.

Discrete-Time Dynamical Systems Models

We express the regularity of the dynamics in terms of one-step-ahead difference equations of the form:

$$\text{Processes}_{i,t} = \text{Processes}_{i,t-1} + \text{Change functions } (.) + \text{Random noise} \qquad (1)$$

i indexes person; t is a discrete-valued time index; "(.)" denotes all elements that affect the change processes of interest (e.g., the processes of interest at some previous time points, covariates, parameters)

- In discrete-time dynamical systems models, time is represented as integers.
- Data are usually equally spaced.
- $\text{Processes}_{i,t}$ can be a vector of observed or latent processes
- The change functions may be linear or nonlinear
- The change functions may be person-specific

State-Space Models

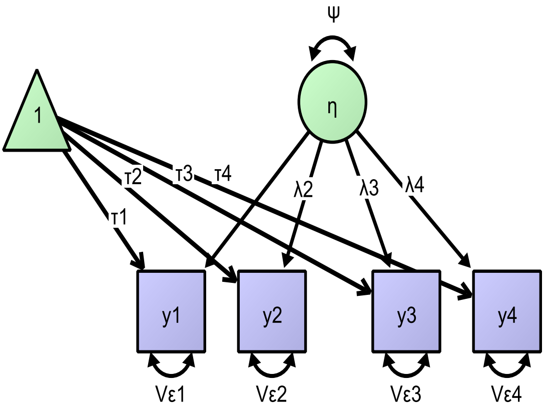
- State-space models can be regarded as a modeling framework that is composed of:
  1. a measurement model linking some observed (manifest) variables to some unobserved (latent) variables, and
  2. a dynamic model describing the evolution of the latent variables over time.
- The dynamic model in a state-space model takes on the form of the one-step-ahead equation similar to the form shown in (1).
- Virtually all linear, discrete-time dynamic models can be expressed in state-space form.

ASIDE I: Tutorials on Essential Matrix Operations to Understand a Model Presented in Matrix Form

- Overall Review and Practice
  https://www.khanacademy.org/math/precalculus/precalc-matrices
- Basic Concept (4mins)
  https://www.opened.com/video/introduction-to-the-matrix-khan-academy/181810
- Basic Operations (5mins)
  https://www.opened.com/video/matrices-basic-matrix-operations-add-subtract-multiply-by/116163
- Multiplication (10mins)
  https://www.opened.com/video/linear-algebra-matrix-multiplication/336137
- Matrix equation (10mins)
  https://www.opened.com/video/ex-1-solve-a-system-of-two-equations-using-a-matrix-equation/891337

ASIDE II: Introduction to Path Diagram Notations in the Structural Equation Modeling Framework

What do these squares, circles, triangle and paths mean?



ASIDE III: Tutorials on R

Check out GettingStartedonR.mp4 and StepbyStepRStudioExample.mp4. Document type

- http://www.cyclismo.org/tutorial/R/
- https://www.datacamp.com/courses/free-introduction-to-r

Video clips with exercise

- Only need the first section and it is free https://www.datacamp.com/courses/introduction-to-data
- Collection of short video clips by subject: http://www.twotorials.com/ http://dist.stat.tamu.edu/pub/rvideos/

ASIDE IV: How to Install dynr

- Check out WhatsfordynrInstallationandOverview.mp4.
- dynr can be installed by typing *install.packages('dynr')* at the console to install it from CRAN
- HOWEVER, to get dynr to work, there are some installation steps you need to first go through the steps in: Link to installation instructions

Linear State-Space Models: Measurement Model

$$\boldsymbol{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}. \tag{2}$$

- $\boldsymbol{y}_{it} = p_k \times 1$ vector of manifest variables
- $\boldsymbol{\eta}_{it} = w_k \times 1$ vector of latent variables
- $\boldsymbol{\Lambda}_k = p_k \times w_k$ factor loading matrix,
- $\boldsymbol{\tau}_k = p_k \times 1$ vector of intercepts, and
- $\boldsymbol{\epsilon}_{it} = p_k \times 1$ vector of measurement errors
- The subscript k (as opposed to s, as in the SEM model) is used to indicate elements in a state-space model.
- Vectors of fixed exogenous variables may be added to the measurement model (not shown here), but available in the R package *dynr*.

Linear State-Space Models: Dynamic Model

$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}, \tag{3}$$

- $\boldsymbol{\nu}_k = w_k \times 1$ vector of intercepts
- $\boldsymbol{B}_k = w_k \times w_k$ regression matrix relating latent variables to each other
- $\boldsymbol{\zeta}_{it} = w_k \times 1$ vector of residuals or dynamic noise
- Vectors of fixed exogenous variables may be added to the dynamic model (not shown here), but available in the R package *dynr*.

Linear State-Space Models: Noise and Initial Condition Structures

$$
\begin{aligned}
\boldsymbol{\eta}_{i1|0} &\sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) && \text{Initial condition} \\
\boldsymbol{\epsilon}_{it} &\sim N(\mathbf{0}, \boldsymbol{\Theta}_k) && \text{Measurement errors} \\
\boldsymbol{\zeta}_{it} &\sim N(\mathbf{0}, \boldsymbol{\Psi}_k) && \text{Dynamic noise/uncertainties}
\end{aligned}
$$

Modeling Equations and Log Likelihood Functions Used in the SEM and the State–Space Frameworks

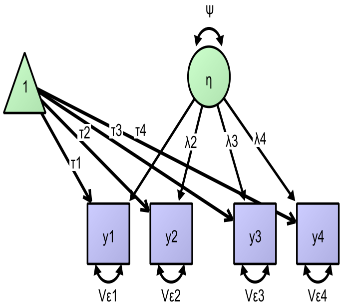
	Equations	
SEM		
Measurement	$y_i = \tau_s + \Lambda_s \eta_i + \epsilon_i$	
Structural	$\eta_i = \nu_s + B_s \eta_i + \zeta_i$	
Log likelihood	$LL_{RML}(\theta_s) =$	
	$\frac{1}{2}\sum_{i=1}^{N}[-p_{si}\log(2\pi) - \log\lvert\Sigma\rvert - (y_i - \mu)'\Sigma^{-1}(y_i - \mu)]$	
	$\mu = \tau_s + \Lambda_s(I - B_s)^{-1}\nu_s$	
	$\Sigma = \Lambda_s(I - B_s)^{-1}\Psi_s(I - B_s)^{-1'}\Lambda_s' + \Theta_s$	
State-space		
Measurement	$y_{it} = \tau_k + \Lambda_k \eta_{it} + \epsilon_{it}$	
Dynamic	$\eta_{it} = \nu_k + B_k \eta_{i,t-1} + \zeta_{it}$	
Log likelihood	$LL_{KF}(\theta_k) =$	
	$\frac{1}{2}\sum_{i=1}^{N}\sum_{t=1}^{T}[-p_k\log(2\pi) - log\lvert G_t\rvert - (e_{i,t\mid t-1}')G_t^{-1}(e_{i,t\mid t-1})]$	
	$e_{i,t\mid t-1} = y_{it} - (\Lambda_k \eta_{i,t\mid t-1} + \tau_k)$	Innovation vector
	$G_t = \Lambda_k P_{t\mid t-1}\Lambda_k' + \Theta_k$	Innovation covariance matrix

Adapted from Chow, Ho, Hamaker, and Dolan (2010). SEM, 17, 303-332.

SEM vs. State-Space Framework

- Traditionally, longitudinal models in the SEM framework are implemented by specifying repeated measurements of each variable as different variables.
- When time-based dynamics are ignored (i.e., when B_K is a matrix of zeros) or when cross-sectional data ($T = 1$) are involved, the two modeling frameworks are identical.

Confirmatory Factor Analysis Model



Measurement model

$$\boldsymbol{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}$$

$$\begin{bmatrix} y_{1it} \\ y_{2it} \\ y_{3it} \\ y_{4it} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix} + \begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \begin{bmatrix} \eta_{it} \end{bmatrix} + \begin{bmatrix} \epsilon_{1it} \\ \epsilon_{2it} \\ \epsilon_{3it} \\ \epsilon_{4it} \end{bmatrix}$$

Dynamic model

$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}$$

$$\begin{bmatrix} \eta_{it} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} \eta_{i,t-1} \end{bmatrix} + \begin{bmatrix} \zeta_{it} \end{bmatrix}$$

SEM vs. State-Space Framework

Table: SEM estimates were from LISREL; state-space estimates were from mkfm6 by Conor Dolan. Download mkfm6

Matrices	True values	SEM estimates (SE)	State-space estimates (SE)
$\Lambda_s = \Lambda_k$	$\begin{bmatrix} 1 \\ 1.2 \\ .7 \\ .8 \end{bmatrix}$	$\begin{bmatrix} =1 \\ 1.2823(.0864) \\ .8227(.0700) \\ .8417(.0877) \end{bmatrix}$	$\begin{bmatrix} =1 \\ 1.2823(.0854) \\ .8227(.0697) \\ .8417(.0876) \end{bmatrix}$
$E(\epsilon_i \epsilon_i') = \Theta_s = \Theta_k$	$\begin{bmatrix} 5 \\ 4 \\ 6 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 5.3627(.7151) \\ 2.1878(.7762) \\ 6.0199(.6934) \\ 11.4892(1.2303) \end{bmatrix}$	$\begin{bmatrix} 5.3627(.7037) \\ 2.1879(.7698) \\ 6.0199(.6978) \\ 11.4891(1.2264) \end{bmatrix}$
$\tau_s = \tau_k$	$\begin{bmatrix} 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$	$\begin{bmatrix} 2.7074(.2868) \\ 3.6790(.3193) \\ 4.9759(.2602) \\ 6.6474(.3113) \end{bmatrix}$	$\begin{bmatrix} 2.7074(.2861) \\ 3.6790(.3185) \\ 4.9759(.2595) \\ 6.6474(.3105) \end{bmatrix}$
$\Psi_s = \Psi_k$	$\begin{bmatrix} 10 \end{bmatrix}$	$\begin{bmatrix} 11.0045(1.6204) \end{bmatrix}$	$\begin{bmatrix} 11.0045(1.6121) \end{bmatrix}$

Adapted from Chow et al. (2010). SEM, 17, 303-332.

How can I get started?

- The R package *dynr* on CRAN (Ou, Hunter, & Chow, under review) can be used to perform state-space modeling
- Other *dynr* demos and tutorials are available on the QuantDev website → Resources → dynr-package-linear-and-nonlinear-dynamic-modeling-r

For a quick snapshot of the unique strengths of *dynr* see WhatsfordynrInstallationandOverview.mp4

What's for dynr?

Why make a new package for this?

- DYNamic modeling in R (other team members: Mike Hunter, Lu Ou, Linying Ji, Meng Chen, Hui-Ju Hung) - now available on CRAN; demo examples available on the QuantDev website at https://quantdev.ssri.psu.edu/ → Resources
- Linear and nonlinear models
- Continuous- and discrete-time models
- Handles regime-switching properties
- Single- and multiple-subject data
- Intuitive interface
- Fast
- Great reporting

dynr preparation

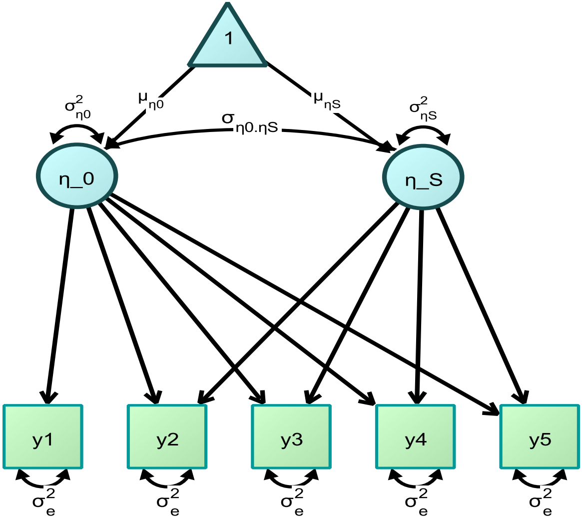
- Gather data with `dynr.data()`
- Prepare *recipes* with
  - `prep.measurement()`
  - `prep.*Dynamics()`
  - `prep.initial()`
  - `prep.noise()`
  - `prep.regimes()` (optional)
- *Mix* recipes and data into a model with `dynr.model()`
- *Cook* model with `dynr.cook()`
- *Serve* results with
  - `summary()`
  - `plot()`
  - `dynr.ggplot()`
  - `plotFormula()`
  - `printex()`

Examples

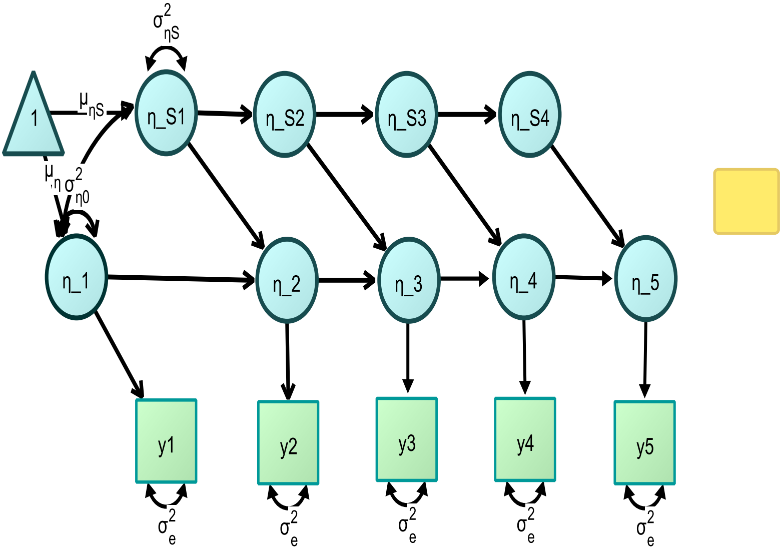
Linear growth curve model

Check out GrowthCurveModel.R

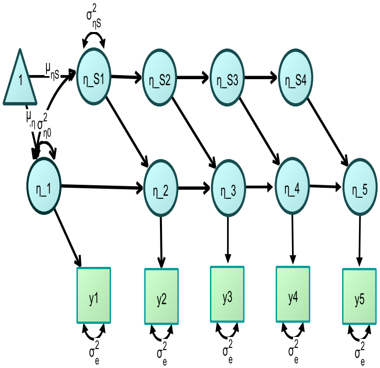
Growth Curve Model as a SEM



Growth Curve Model in One Possible State-Space Form



(1) Growth Curve Model: State-Space Measurement Model



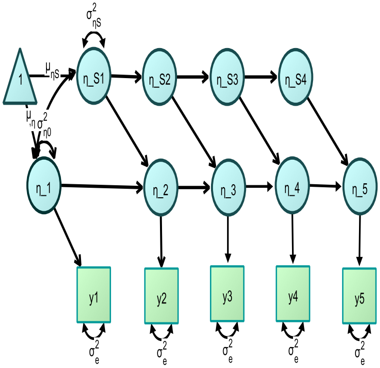
Measurement model (deterministic portion)

$E(\boldsymbol{y}_{it}|\boldsymbol{\eta}_{it}) = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it}$

$$y_{it} = [0] + [1 \quad 0] \begin{bmatrix} \eta_{it} \\ \eta_{Sit} \end{bmatrix}$$

Specified via *prep.measurement* in *dynr*

(2) Growth Curve Model: State-Space Dynamic Model



Dynamic model (deterministic)

$$\boldsymbol{E}(\boldsymbol{\eta}_{it}|\boldsymbol{\eta}_{i,t-1}) = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1}$$

In matrix form:
prep.matrixDynamics()

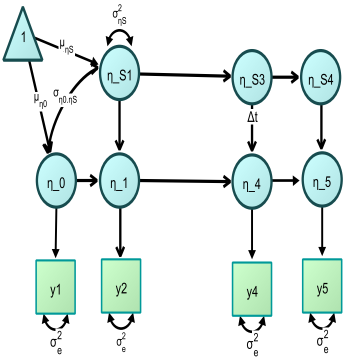
$$\begin{bmatrix} \eta_{it} \\ \eta_{Sit} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_{i,t-1} \\ \eta_{Si,t-1} \end{bmatrix}$$

As formulas:
prep.formulaDynamics()

$$\eta_{it} = \eta_{i,t-1} + \eta_{Si,t-1}$$

$$\eta_{Sit} = \eta_{Si,t-1}$$

Growth Curve Model to Irregularly Spaced Data



Dynamic model (deterministic)

$E(\boldsymbol{\eta}_{it}|\boldsymbol{\eta}_{i,t-1}) = \boldsymbol{\nu}_k + \boldsymbol{B}_k\boldsymbol{\eta}_{i,t-1}$

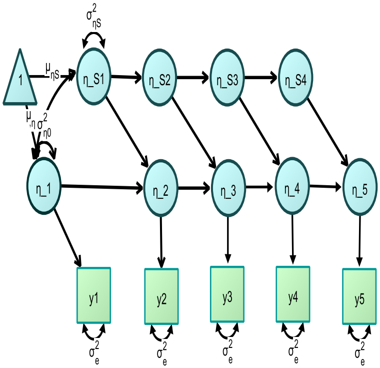
In matrix form: prep.matrixDynamics()

$$\begin{bmatrix} \eta_{it} \\ \eta_{Sit} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & \Delta_{it} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_{i,t-1} \\ \eta_{Si,t-1} \end{bmatrix}$$

As formulas: prep.formulaDynamics()

$$\eta_{it} = \eta_{i,t-1} + \Delta_{it}\eta_{Si,t-1}$$
$$\eta_{Sit} = \eta_{Si,t-1}$$

(3) Growth Curve Model: State-Space Initial Condition Model



Initial condition (IC)

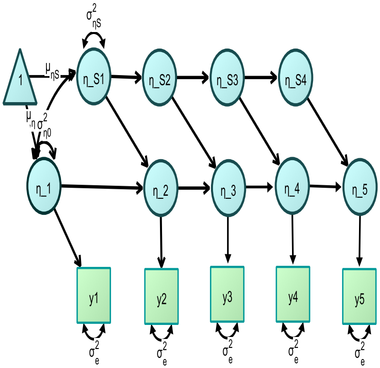
$\boldsymbol{\eta}_{i1|0} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$

Specified via *prep.initial* in *dynr*):

$$\begin{bmatrix} \eta_{i1} \\ \eta_{Si1} \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_{\eta 0} \\ \mu_{\eta S} \end{bmatrix}, \begin{bmatrix} \sigma^2_{\eta 0} & \\ \sigma_{\eta 0.\eta S} & \sigma^2_{\eta S} \end{bmatrix}\right)$$

Fixed effects of intercept and slope IC means; var-cov parameters for between-person random effects in IC covariance matrix, Σ_0

(4) Growth Curve Model: State-Space Measurement/Process Noise Structure



Noise structure

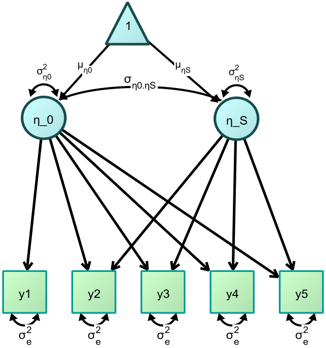
$$\epsilon_{it} \sim N(\mathbf{0}, \mathbf{\Theta}_k)$$
$$\zeta_{it} \sim N(\mathbf{0}, \mathbf{\Psi}_k)$$

Specified via *prep.noise* in *dynr*):

- $\mathbf{\Psi}_k$ is a matrix of zeros
- $\mathbf{\Theta}_k$ consists of σ^2_e

No stochastic process noises in standard growth curve models

Growth Curve Model in Another State-Space Form



Measurement model

$$\boldsymbol{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}$$

$$y_{it} = \begin{bmatrix}0\end{bmatrix} + \begin{bmatrix}1 & Time_{it}\end{bmatrix} \begin{bmatrix}\eta_{it} \\ \eta_{Sit}\end{bmatrix} + \epsilon_{it}$$

Linear trend as time-varying factor loadings

Dynamic model

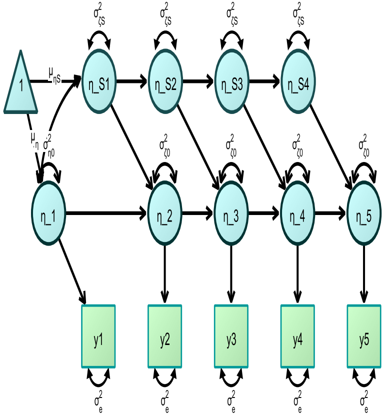
$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}$$

$$\begin{bmatrix}\eta_{it} \\ \eta_{Sit}\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix} + \begin{bmatrix}1 & \underline{0} \\ 0 & 1\end{bmatrix} \begin{bmatrix}\eta_{i,t-1} \\ \eta_{Si,t-1}\end{bmatrix} + \begin{bmatrix}0 \\ 0\end{bmatrix}$$

Examples

Related Variations/Extensions

Local Linear Trend model



Measurement model

$$\boldsymbol{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}$$

$$y_{it} = [0] + [1 \quad 0] \begin{bmatrix} \eta_{it} \\ \eta_{Sit} \end{bmatrix} + [0]$$

η_{it} is now a time-varying intercept (local level); η_{Sit} is a local slope (trend).

Dynamic model

$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}$$

$$\begin{aligned} \eta_{it} &= \eta_{i,t-1} + \eta_{Si,t-1} + \zeta_{0it} \\ \eta_{Sit} &= \eta_{Si,t-1} + \zeta_{Sit} \end{aligned}$$

Local Linear Trend Model

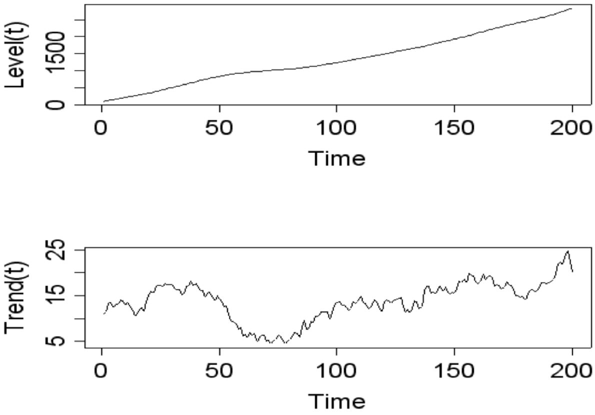
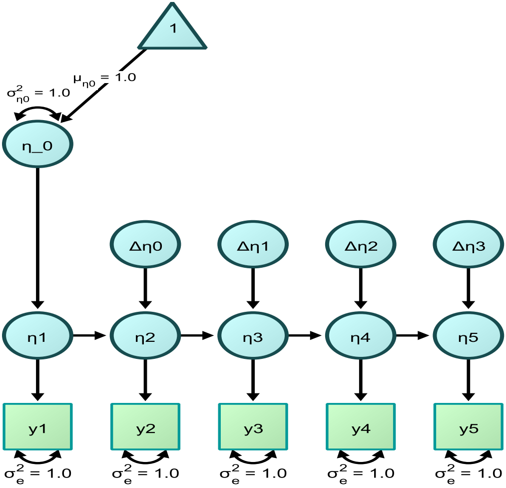
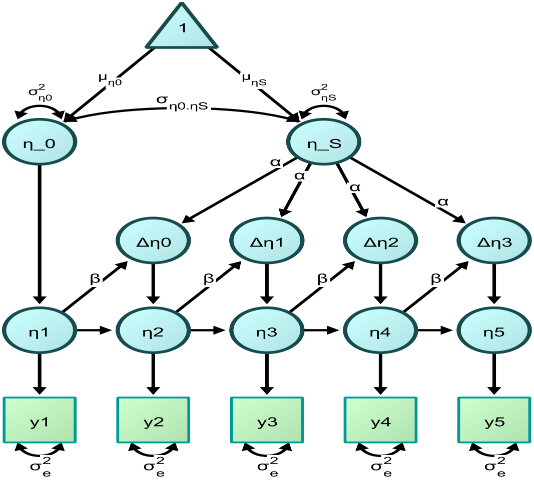


Figure: The stochastic local trend captures the uncertainties in within-person changes (slopes) over time.

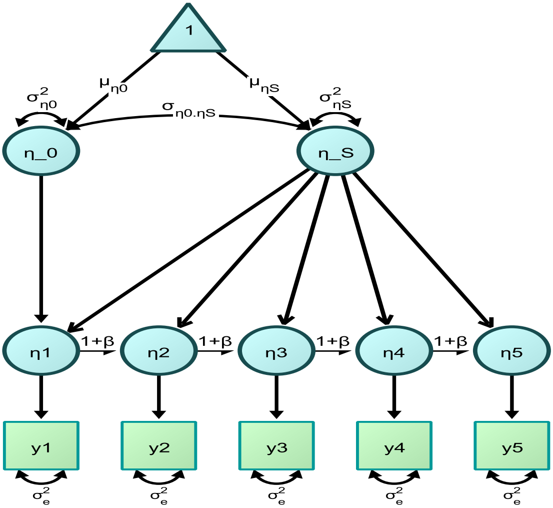
Univariate Latent Change Score Model with No Change



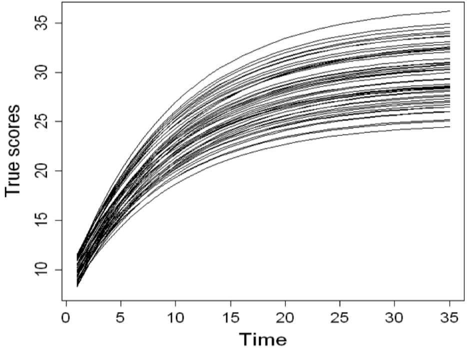
Univariate Dual Change Score Model (McArdle & Hamagami, 2001)



Equivalent Dual Change Score Model without the Latent Change Components (Chow et al., 2010)

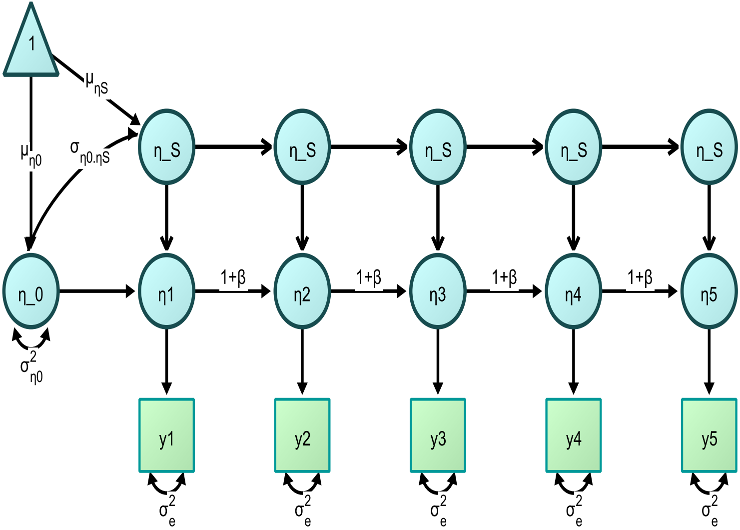


Simulated Trajectories from the Dual Change Score Model

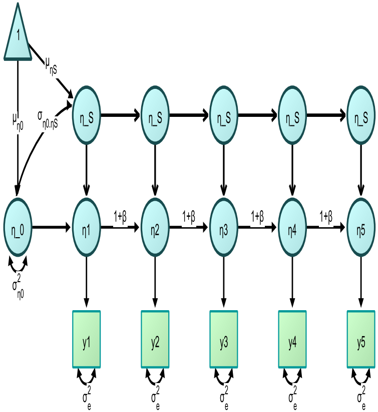


The dual change score model and its bivariate extension have been used to examine lead-lag relations between constructs such as lifespan developmental changes in intelligence and related performance measures (Ferrer & McArdle, 2004; Ghisletta & Lindenberger, 2003).

Dual Change Score Model in State-Space Form



Dual Change Score Model in State-Space Form



Measurement model

$$\boldsymbol{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}$$

$$y_{it} = \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \eta_{it} \\ \eta_{Sit} \end{bmatrix} + \epsilon_{it}$$

Dynamic model

$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}$$

$$\begin{aligned} \eta_{it} &= (1+\beta)\eta_{i,t-1} + \eta_{Si,t-1} \\ \eta_{Sit} &= \eta_{Si,t-1} \end{aligned}$$

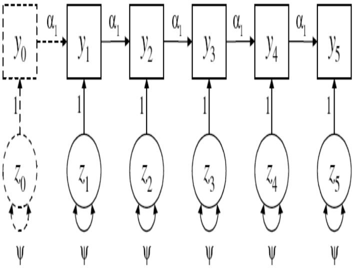
Examples

Time Series Models (Single- or multi-subject)

- These models are meant to describe within-person changes.
- When applied to longitudinal panel data or multi-subject time series data, we are assuming that all individuals in the sample can be described by the exact same model.
- In other words, we are trying to extract change mechanisms that are invariant over subjects (may or may not be tenable).
- Random effects extensions possible.

Autoregressive Model of Order 1 (AR(1) Model)

Figure: From Browne and Nesselroade (2005)



Assumes that all systematic trends have been removed. The process of interest fluctuates around a zero intercept (set point).

Measurement model

$$\boldsymbol{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}$$

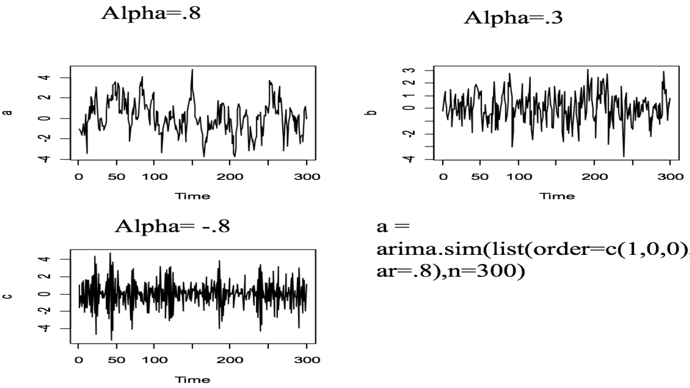
$$y_{it} = [0] + [1]\,[\eta_{it}]$$

Dynamic model

$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}$$

$$\eta_{it} = \alpha_1 \eta_{i,t-1} + z_{it}$$

AR(1) Coefficient as a Reflection of Inertia



a =
arima.sim(list(order=c(1,0,0),
ar=.8),n=300)

In the context of affective dynamics, higher positive AR(1) weight has been likened to higher inertia, sluggishness, or "getting stuck" in extreme affective states (Hamaker, Asparouhov, Brose, Schmiedek, & Muthén, 2017, under review; Koval, Kuppens, Allen, & Sheeber, 2012; Kuppens, Allen, & Sheeber, 2010; Suls, Green, & Hillis, 1998).

Autoregressive Model of Order 1 (AR(1) Model) with Individual Differences in Set Point

Measurement model

$$\boldsymbol{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}$$

Within-person centering

$$y_{it} = \begin{bmatrix}0\end{bmatrix} + \begin{bmatrix}1 & 1\end{bmatrix} \begin{bmatrix}\eta_{it} \\ \text{SetP}_{0it}\end{bmatrix}$$

- SetPt_{0i} is the person-specific latent mean or set-point (a latent variable)

Dynamic model

$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}$$

$$\begin{aligned} \eta_{it} &= \alpha_1 \eta_{i,t-1} + z_{it} \\ \text{SetP}_{0it} &= \text{SetP}_{0i,t-1} \end{aligned}$$

Initial condition (IC)

$$\eta_{i1|0} \sim N(\mu_0, \Sigma_0)$$

Between-person centering

- Person-specific trait variables may be included here
- These variables may be (grand-mean) centered to ease interpretation

$$\begin{bmatrix}\eta_{i1} \\ \text{SetP}_{0i1}\end{bmatrix} \sim N\left(\begin{bmatrix}0 \\ B_0 + B_1 \text{trait}_i\end{bmatrix}, \begin{bmatrix}\sigma^2_{\eta 0} & \\ \sigma_{\eta 0,\mu_0} & \sigma^2_{\mu 0}\end{bmatrix}\right)$$

Autoregressive Model of Order 2 (AR(2) Model)

Measurement model

$$\boldsymbol{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}$$

$$y_{it} = \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \eta_{it} \\ \eta_{i,t-1} \end{bmatrix}$$

Dynamic model

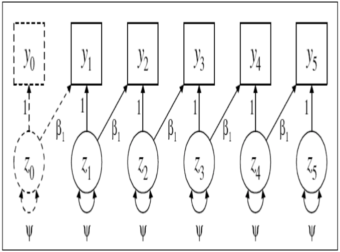
$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}$$

$$\begin{aligned} \eta_{it} &= \alpha_1 \eta_{i,t-1} + \alpha_2 \eta_{i,t-2} + z_{it} \\ \eta_{i,t-1} &= \eta_{i,t-1} \end{aligned}$$

$$\begin{bmatrix} \eta_{it} \\ \eta_{i,t-1} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \eta_{i,t-1} \\ \eta_{i,t-2} \end{bmatrix} + \begin{bmatrix} z_{it} \\ 0 \end{bmatrix}$$

Moving Average Model of Order 1 (MA(1) Model)

Figure: From Browne and Nesselroade (2005)



The process of interest fluctuates around a zero intercept (set point) as driven by random shocks, z_{it} and $z_{i,t-1}$.

Measurement model

$$\mathbf{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}$$

$$y_{it} = [0] + [1 \quad \beta_1] \begin{bmatrix} z_{it} \\ z_{i,t-1} \end{bmatrix}$$

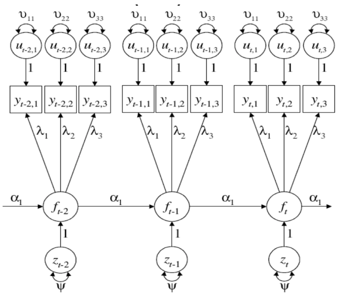
Dynamic model

$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}$$

$$\begin{bmatrix} z_{it} \\ z_{i,t-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_{i,t-1} \\ z_{i,t-2} \end{bmatrix} + \begin{bmatrix} z_{it} \\ 0 \end{bmatrix}$$

Process Factor Analysis Model with AR(1) Relations at the Factor Level (PFA(1,0))

Figure: From Browne and Nesselroade (2005) for a single factor



The process of interest fluctuates around a zero intercept (set point) at the latent level following a vector AR(1) process.

Measurement model

$$\boldsymbol{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}$$

$$\begin{bmatrix} y_{1it} \\ y_{2it} \\ y_{3it} \\ y_{4it} \\ y_{5it} \\ y_{6it} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ 0 & 1 \\ 0 & \lambda_{42} \\ 0 & \lambda_{52} \end{bmatrix} \begin{bmatrix} f_{1it} \\ f_{2it} \end{bmatrix} + \epsilon_{it}$$

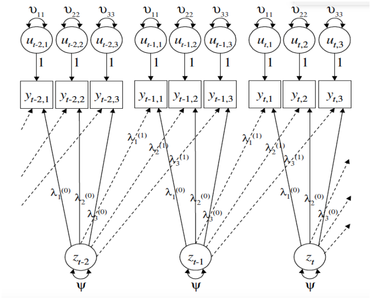
Dynamic model

$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}$$

$$\begin{bmatrix} f_{1it} \\ f_{2it} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} f_{1i,t-1} \\ f_{2i,t-1} \end{bmatrix} + \begin{bmatrix} z_{1it} \\ z_{2it} \end{bmatrix}$$

Shock Factor Analysis Model of Order 1

Figure: From Browne and Nesselroade (2005) for a single shock factor and up to lag-1 loadings



The latent factors are shock factors that have lagged factor loadings on the observed variables. A model first proposed by Molenaar (1985).

Measurement model

$$\boldsymbol{y}_{it} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}$$

$$y_{it} = \begin{bmatrix} \lambda_{10} & \lambda_{11} & 0 & 0 \\ \lambda_{20} & \lambda_{21} & 0 & 0 \\ \lambda_{30} & \lambda_{31} & 0 & 0 \\ 0 & 0 & \lambda_{40} & \lambda_{41} \\ 0 & 0 & \lambda_{50} & \lambda_{51} \\ 0 & 0 & \lambda_{60} & \lambda_{61} \end{bmatrix} \begin{bmatrix} z_{1it} \\ z_{1i,t-1} \\ z_{2it} \\ z_{2i,t-1} \end{bmatrix} + \epsilon_{it}$$

Dynamic model

$$\boldsymbol{\eta}_{it} = \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it}$$

$$\begin{bmatrix} z_{1it} \\ z_{1i,t-1} \\ z_{2it} \\ z_{2i,t-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_{1i,t-1} \\ z_{1i,t-2} \\ z_{2i,t-1} \\ z_{2i,t-2} \end{bmatrix} + \begin{bmatrix} \zeta_{1it} \\ 0 \\ \zeta_{2it} \\ 0 \end{bmatrix}$$

Identification: Shock factor variances fixed at 1.0.

Regime-Switching Models

- A regime–switching longitudinal model consists of several latent (unobserved) classes–or "regimes."
- Within each class, a submodel is used to described the distinct change patterns associated with the regime.
- Each "regime" can be thought of as one of the stages or phases of a dynamic process.
- Individuals can switch between classes or regimes over time.
- The changes that unfold within a regime are continuous in nature

Application to Facial Electromyography (EMG) Data (Yang & Chow, 2010)

See example in RSLinearDiscrete.R

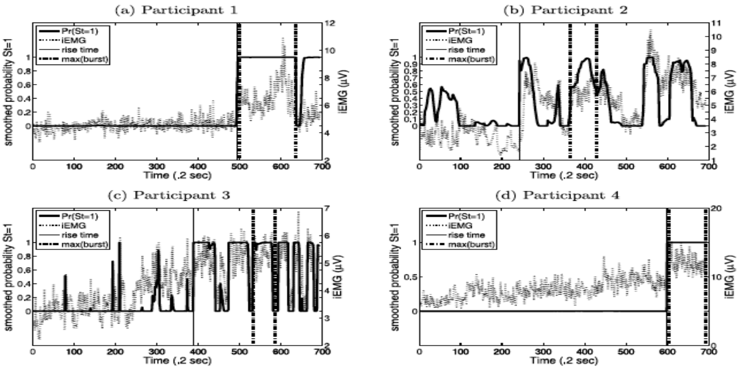


FIGURE 3.
The smoothed probabilities of being in the activation phase $\Pr(S_t = 1|Y)$ for the four participants based on the final selected model are shown in panels (a)–(d), respectively. According to the final selected model, the *longest durations of burst* were estimated to be 28.6, 12.6, 10.6 and 18.6 seconds for Participants 1 to 4, respectively; They were marked as the period between the two dashed vertical lines in panels (a)–(d). The *rise times* were estimated to be 98.8, 77.8, 42.0 and 119.8 seconds for Participants 1 to 4, respectively. They were marked as the period between the first time point and the solid vertical line in panels (a)–(d).

Regime Switching in the Development of Human Intelligence (Fluid and Crystallized Intelligence Factors)

Ferrer and colleagues (Ferrer & McArdle, 2004; Ferrer et al., 2007) found that the coupling from fluid intelligence to reading tended to attenuate from childhood to adolescence.

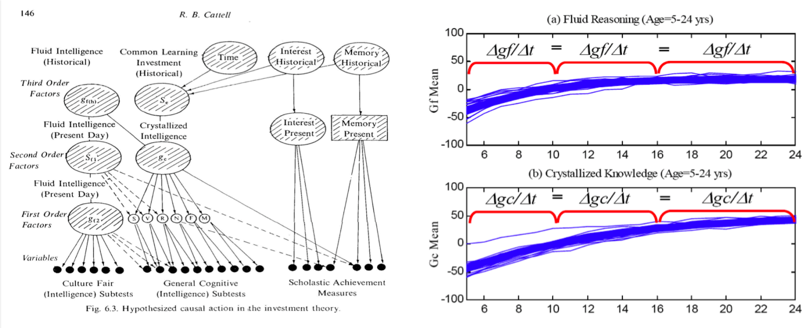
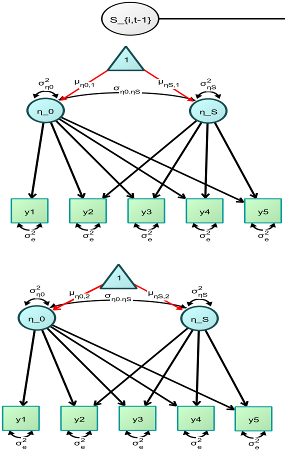


Figure: Figures from a talk by Ferrer (2005). For details see Ferrer et al. (2007).

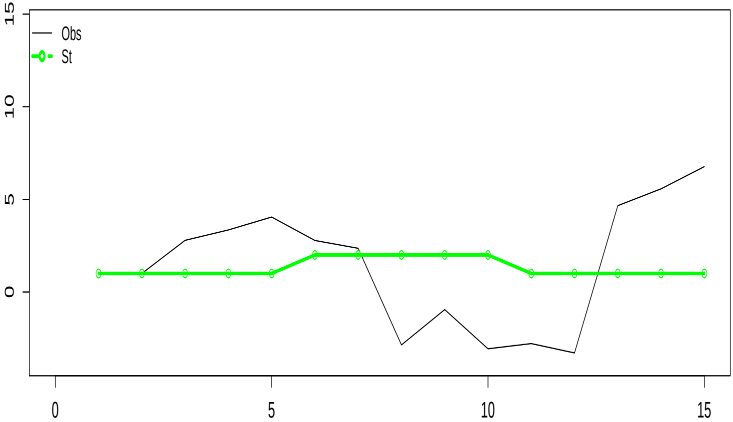
Regime-Switching Growth Curve Model

- Model proposed by Dolan and colleagues (Dolan, Schmittmann, Lubke, & Neale, 2005; Schmittmann, Dolan, van der Maas, & Neale, 2005)



Regime-Switching Growth Curve Model

- A growth mixture model is equivalent to a RS growth curve model in which there is no switching between regimes (i.e., there are no changes in class membership).



Regime-Switching Model

Define S_{it} to be person i's latent regime indicator at time t, k here denotes the regime value, $\boldsymbol{x}_{it}$ denotes a vector of covariates.

$$\mathbf{y}_{it|S_{it}=k} = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k \boldsymbol{\eta}_{it} + \boldsymbol{\epsilon}_{it}, \quad \boldsymbol{\epsilon}_{it} \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Omega}_k\right), \quad \text{Measurement model}$$

$$\boldsymbol{\eta}_{it|S_i=k} = \boldsymbol{f}_k(\boldsymbol{\eta}_{it}, \boldsymbol{x}_{it}) + \zeta_i, \quad \zeta_i \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Psi}_k\right), \quad \text{Structural model}$$

The initial class (or regime) probabilities can be represented using a multinomial logistic regression model as

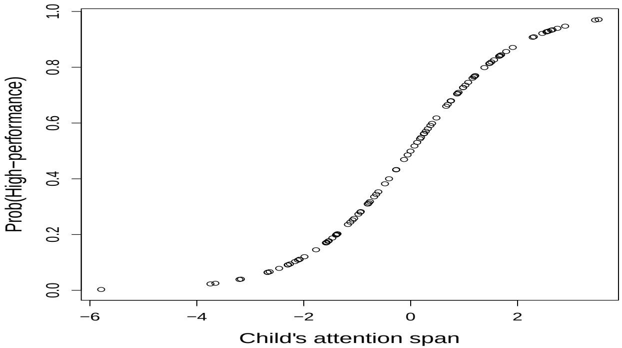
$$\Pr(S_{i1} = k|\boldsymbol{x}_{i1}) \triangleq \pi_{k,i1} = \frac{\exp(a_{k1}+\boldsymbol{b}'_{k1}\boldsymbol{x}_{i1})}{\sum_{s=1}^{K_1} \exp(a_{s1}+\boldsymbol{b}'_{s1}\boldsymbol{x}_{i1})}, \tag{4}$$

A multinomial logistic regression model is then used to describe each individual i's transition in class membership from time t-1 to time t as

$$\Pr(S_{it} = k|S_{i,t-1} = j, \boldsymbol{x}_{it}) \triangleq \pi_{jk,it} = \frac{\exp(a_{kt}+a_{\Delta jkt}+\boldsymbol{b}'_{jkt}\boldsymbol{x}_{it})}{\sum_{s=1}^{K_t} \exp(a_{st}+a_{\Delta jst}+\boldsymbol{b}'_{jst}\boldsymbol{x}_{it})} \tag{5}$$

What Is Multinomial Logistic Function Again?

Consider the form of the relationship between a predictor (e.g., child's attention span) and the probability of starting out in a particular regime (high-performing regime) relative to starting out in other possible regimes. This relationship is commonly assumed to follow the form of Equation (4) or (5) give by:



Initial Regime or Class Probabilities

The initial class (or regime) probabilities can be represented using a multinomial logistic regression model as

$$\Pr(S_{i1} = k|\boldsymbol{x}_{i1}) \triangleq \pi_{k,i1} = \frac{\exp(a_{k1} + \boldsymbol{b}'_{k1}\boldsymbol{x}_{i1})}{\sum_{s=1}^{K_1}\exp(a_{s1} + \boldsymbol{b}'_{s1}\boldsymbol{x}_{i1})},$$

$$\begin{aligned}
S_{it} &= \text{individual } i\text{'s class membership at time } t \\
\boldsymbol{S}_i &= [S_{i1}, \ldots, S_{iT}]' = \text{person } i\text{'s class history from time 1 to } T \\
K_1 &= \text{the number of classes at time 1} \\
a_{k1} &= \text{the logit intercept at time 1} \\
\boldsymbol{x}_{i1} &= \text{a vector of fixed covariates for person } i \text{ at time 1} \\
\boldsymbol{b}_{k1} &= \text{a vector of logit slopes}
\end{aligned}$$

Typically, a_{K1} is set to 0 so the last class is used as the reference class.

Log Odds Calculation: 2-Class Example, No Covariate

$K_1 = 2$, thus: $\Pr(S_{i1} = 1|\boldsymbol{x}_{i1}) \stackrel{\Delta}{=} \pi_{1,i1}$
$= \frac{\exp(a_{k1}+\boldsymbol{b}'_{k1}\boldsymbol{x}_{i1})}{\sum_{s=1}^{K_1}\exp(a_{s1}+\boldsymbol{b}'_{s1}\boldsymbol{x}_{i1})} = \frac{\exp(a_{11})}{\exp(a_{11})+\exp(a_{21})}$, with $a_{21} = 0$ for identification

S_{i1}	Log odds	Exp $\rightarrow$ odds	Pr $=$ exp/sum
#1	a_{11}	$\exp(a_{11})$	$\frac{\exp(a_{11})}{sum}$
#2	$a_{21} = 0$	$\exp(a_{21}) = \exp(0) = 1$	$\frac{\exp(0)}{sum}$
$\sum$	Not of interest	sum$=\exp(a_{11}) + 1$	1.00

Other helpful concepts:

- Odds to be in class 1 $= \frac{\Pr(\text{class 1})}{\Pr(\text{not in class 1})} = \exp(a_{11})$. For instance, if the $\exp(a_{11}) = 2$, then the odds of starting out in class 1 are 2:1.
- Equal odds to be in both classes, 1:1, are obtained when $a_{11} = 0$.
- For estimation purposes, it is easier to work with the log odds.

Transition Probabilities

A multinomial logistic regression model is used to describe each individual i's transition in class membership from time t-1 to time t as

$$\Pr(S_{it} = k | S_{i,t-1} = j, \boldsymbol{x}_{it}) \triangleq \pi_{jk,it} = \frac{\exp(a_{kt} + a_{\Delta jkt} + \boldsymbol{b}'_{jkt}\boldsymbol{x}_{it})}{\sum_{s=1}^{K_t} \exp(a_{st} + a_{\Delta jst} + \boldsymbol{b}'_{jst}\boldsymbol{x}_{it})}$$

$\pi_{jk,it}$ = individual i's probability of transitioning from class j to class k at time t

K_t = the number of classes at time t

a_{kt} = the logit intercept for the kth class at time t

$\boldsymbol{x}_{it}$ = a vector of fixed covariates

$\boldsymbol{b}_{jkt}$ = a vector of logit slopes

$a_{\Delta,jkt}$ = deviation in logodds due to transitioning into class k from class j than from the reference class

Transition Log Odds Calculation: 2-Class Example

$K_t = 2$, no covariate, thus $\boldsymbol{b}'_{jkt} = \boldsymbol{b}'_{jst} = \boldsymbol{0}$:

$$\Pr(S_{it} = k | S_{i,t-1} = j) \quad \stackrel{\Delta}{=} \pi_{jk,it} = \frac{\exp(a_{kt} + a_{\Delta jkt} + \boldsymbol{b}'_{jkt}\boldsymbol{x}_{it})}{\sum_{s=1}^{K_t} \exp(a_{st} + a_{\Delta jst} + \boldsymbol{b}'_{jst}\boldsymbol{x}_{it})}$$

	$S_{it}\#1$	$S_{it}\#2$
$S_{i,t-1}\#1$	$a_{1t} + a_{\Delta 11t}$	0
$S_{i,t-1}\#2$	a_{1t}	0

Each row should sum to 1.00:

$$\Pr(S_{it} = 1 | S_{i,t-1} = 1) \quad = \quad \pi_{11} = \frac{\exp(a_{1t} + a_{\Delta 11t})}{[\exp(a_{1t} + a_{\Delta 11t}) + \exp(0)]}$$

$$\Pr(S_{it} = 2 | S_{i,t-1} = 1) \quad = \quad \pi_{12} = \frac{\exp(0)}{[\exp(a_{1t} + a_{\Delta 11t}) + \exp(0)]} = 1 - \pi_{11}$$

$$\Pr(S_{it} = 1 | S_{i,t-1} = 2) \quad = \quad \pi_{21} = \frac{\exp(a_{1t})}{[\exp(a_{1t}) + \exp(0)]}$$

$$\Pr(S_{it} = 2 | S_{i,t-1} = 2) \quad = \quad \pi_{22} = \frac{\exp(0)}{[\exp(a_{1t}) + \exp(0)]} = 1 - \pi_{21}$$

RS Facial EMG Model

Example in RSLinearDiscrete.R

We allowed the intercept, μ_{yS_t}; the regression slope associated with the effect of self-report on facial EMG, β_{S_t}; and the AR(1) parameter, ϕ_{St}, to be regime-dependent as:

$$
\begin{aligned}
\text{EMG}_t &= \mu_{yS_t} + \beta_{S_t}\text{self}_t \\
&+ \text{lEMG}_t, \\
\text{lEMG}_t &= \phi_{S_t}\text{lEMG}_{t-1} + \zeta_t.
\end{aligned}
$$

Measurement model

$E(\boldsymbol{y}_{it}|\boldsymbol{\eta}_{it}) = \boldsymbol{\tau}_k + \boldsymbol{\Lambda}_k\boldsymbol{\eta}_{it}$

```
recMeas <- prep.measurement(
  #Factor loading matrix consists only of
  #fixed scalar value = 1
  values.load=rep(list(
                  matrix(1, 1, 1)), 2),
  values.int=list(matrix(0, 1, 1),
                  matrix(1, 1, 1)),
  params.int=list(matrix('mu_0', 1, 1),
                  matrix('mu_1', 1, 1)),
  values.exo=list(matrix(0, 1, 1),
                  matrix(1, 1, 1)),
  params.exo=list(matrix('beta_0', 1, 1),
                  matrix('beta_1', 1, 1)),
  obs.names = c('EMG'),
  state.names=c('lEMG'),
  exo.names=c("self"))
```

RS Facial EMG Model

We allowed the intercept, μ_{yS_t}; the regression slope associated with the effect of self-report on facial EMG, β_{S_t}; and the AR(1) parameter, ϕ_{S_t}, to be regime-dependent as:

$$
\begin{aligned}
y_t &= \mu_{yS_t} + \beta_{S_t}\text{SelfRprt}_t + e_t, \\
e_t &= \phi_{S_t} e_{t-1} + \zeta_t \\
\zeta_t &\sim N(0, dynNoise)
\end{aligned} \tag{6}
$$

Dynamic model, noise cov structure

$$
\begin{aligned}
\boldsymbol{\eta}_{it} &= \boldsymbol{\nu}_k + \boldsymbol{B}_k \boldsymbol{\eta}_{i,t-1} + \boldsymbol{\zeta}_{it} \\
\boldsymbol{\epsilon}_{it} &\sim N(\mathbf{0}, \boldsymbol{\Theta}_k) \\
\boldsymbol{\zeta}_{it} &\sim N(\mathbf{0}, \boldsymbol{\Psi}_k)
\end{aligned}
$$

```
recDyn <- prep.matrixDynamics(
  values.dyn=list(matrix(.1, 1, 1),
                  matrix(.8, 1, 1)),
  params.dyn=list(matrix('phi_0', 1, 1),
                  matrix('phi_1', 1, 1)),
  isContinuousTime=FALSE)

recNoise <- prep.noise(
  values.latent=matrix(1, 1, 1),
  params.latent=matrix('dynNoise', 1, 1),
  values.observed=matrix(0, 1, 1),
  params.observed=matrix('fixed', 1, 1))
```

RS Facial EMG Model

In prep.initial, we specify the initial conditions as:

$$\begin{aligned} IEMG_{1|0} &\sim N(0, 1) \\ \log\left(\frac{\Pr(R1)}{(1 - \Pr(R1))}\right) &= 10 \quad (7) \end{aligned}$$

Initial condition (IC)

$\eta_{i1|0} \sim N(\boldsymbol{\mu}_{0k}, \boldsymbol{\Sigma}_{0k})$,
Logit parameters for S_{i1}

```
recIni <- prep.initial(
  values.inistate=matrix(0, 1, 1),
  params.inistate=matrix('fixed', 1, 1),
  values.inicov=matrix(1, 1, 1),
  params.inicov=matrix('fixed', 1, 1),
  values.regimep=c(10, 0),
  params.regimep=c('fixed', 'fixed'))
```

RS Models

Regime transition LOs

$\Pr(S_{it}|S_{i,t-1},\mathbf{x}_{it})$

Spec1

	$S_{it}\#1$	$S_{it}\#2$
$S_{i,t-1}\#1$	$a_{1t}+a_{\Delta 11t}$	0
$S_{i,t-1}\#2$	a_{1t}	0

Spec2

	$S_{it}\#1$	$S_{it}\#2$
$S_{i,t-1}\#1$	a_{11t}	0
$S_{i,t-1}\#2$	a_{1t}	0

Spec3

	$S_{it}\#1$	$S_{it}\#2$
$S_{i,t-1}\#1$	a_{1t}	0
$S_{i,t-1}\#2$	$a_{1t}+a_{\Delta 21t}$	0

```
Spec1 <- prep.regimes(
  values=matrix(0, 2, 2),
  params=matrix(c('a_Delta11t','fixed',
                  'a1t', 'fixed'),
                ncol=2, byrow=TRUE),
  deviation=TRUE)

Spec2 <- prep.regimes(
  values=matrix(0, 2, 2),
  params=matrix(c('a11t', 'fixed',
                  'a1t',  'fixed'),
                ncol=2, byrow=TRUE))

Spec3 <- prep.regimes(
  values=matrix(0, 2, 2),
  params=matrix(c('a1t','fixed',
                  'a_Delta21t', 'fixed'),
                ncol=2, byrow=TRUE),
  deviation=TRUE, refRow=1)
```

Nonlinear Dynamic Factor Analysis Model with Regime Switching

Example in RSNonlinearDiscrete.R

PE = positive emotion; NE = negative emotion

$$
\begin{aligned}
PE_{it} &= a_P PE_{i,t-1} + b_{PN,c_{it}} NE_{i,t-1} + \zeta_{PE,it} \\
NE_{it} &= a_N NE_{i,t-1} + b_{NP,c_{it}} PE_{i,t-1} + \zeta_{NE,it} \qquad (8) \\
b_{PN,c_{it}} &= \begin{cases} 0 & \text{if } c_{it} = 0 \\ b_{PN0}\left(\frac{\exp(\mathrm{abs}(NE_{i,t-1}))}{1+\exp(\mathrm{abs}(NE_{i,t-1}))}\right) & \text{if } c_{it} = 1, \end{cases} \\
b_{NP,S_{it}} &= \begin{cases} 0 & \text{if } c_{it} = 0 \\ b_{NP0}\left(\frac{\exp(\mathrm{abs}(PE_{i,t-1}))}{1+\exp(\mathrm{abs}(PE_{i,t-1}))}\right) & \text{if } c_{it} = 1, \end{cases} \qquad (9)
\end{aligned}
$$

Chow & Zhang (2013, *Psychometrika*). Model adapted from an earlier model presented by Chow, Tang, Yuan, Song, and Zhu (2011), *BJMSP*.

RS Nonlinear Dynamic Factor Analysis Model

dynr provides built-in, automatic differentiation procedures to "linearize" nonlinear dynamic models prior to model fitting using functions available in the *Deriv* library. The example here shows one instance where manual jacobian function is provided.

```
formula=list(
  list(PE~a1*PE, NE~a2*NE),
  list(PE~a1*PE+c12*(exp(abs(NE)))/(1+exp(abs(NE)))*NE,
       NE~a2*NE+c21*(exp(abs(PE)))/(1+exp(abs(PE)))*PE))

jacob=list(
  list(PE~PE~a1,NE~NE~a2),
  list(PE~PE~a1,
       PE~NE~c12*(exp(abs(NE))/(exp(abs(NE))+1)+
                  NE*sign(NE)*exp(abs(NE))/(1+exp(abs(NE))^2)),
       NE~NE~a2,
       NE~PE~c21*(exp(abs(PE))/(exp(abs(PE))+1)+
                  PE*sign(PE)*exp(abs(PE))/(1+exp(abs(PE))^2))))

dynm<-prep.formulaDynamics(formula=formula,
                           startval=c(a1=.3,a2=.4,c12=-.5,c21=-.5),
                           isContinuousTime=FALSE,jacobian=jacob)
```

Links among the Different Modeling Approaches

	Discrete-time dynamical systems	Continuous-time dynamical systems
Deterministic	* Ordinary difference eqn models * Latent difference score models (McArdle & Hamagami, 2001) * Equally spaced growth curve models * Chaotic models of population density (e.g., logistic growth model)	* Ordinary differential eqn model * Differential structural eqn model (Boker, Neale, & Rausch, 2008) * Damped/coupled oscillators model (Boker & Graham, 1998) * Continuous-time growth curve model * Chaotic models (e.g., Lorenz eqs)
Stochastic	* State-space models * Time series models * Dynamic factor analysis models (Molenaar, 1985; Nesselroade et al., 2001) Nesselroade et al., 2001) * Longitudinal mediation models (Cole & Maxwell, 2003) * Nonlinear state-space models	* Exact discrete time models * Ornstein Uhlenbeck model Oravecz et al. 2009 * Stochastic damped oscillator model (Oud, 2007) * Stochastic catastrophe models (Cobb & Zacks, 1985) * Nonlinear stochastic differential eqs

Note. The examples listed are non-exhaustive.

What's for dynr?

Why make a new package for this?

		Discrete-Time	Continuous-Time
Single-Regime	linear	State-Space Model	Stochastic diffy eq model (SDE)
		KF	Hybrid KF
		dynr, **OpenMx**, **GIMME**, *SsfPack*, *MKFM6*, **pomp**,**KFAS**, **dlm**, **dse**	**dynr**, **OpenMx**, **ctsem**, **pomp**
	nonlinear	Nonlinear state-space model	Nonlinear SDE
		EKF	Hybrid EKF
		dynr, **pomp**, *SsfPack*	**dynr**, **pomp**
Multiple-Regime	linear	RS state-space model	RS SDE
		Kim filter	Hybrid Kim filter
		dynr, Various *GAUSS* and *MATLAB* codes	**dynr** only
	nonlinear	Nonlinear RS state-space model	Nonlinear RS SDE
		Extended Kim filter	Hybrid extended Kim filter
		dynr only	**dynr** only

Note. KF=Kalman filter, EKF=Extended Kalman filter, RS =regime-switching, Kim filter = KF+Hamilton filter+Collapse procedure Kim and Nelson (1999), Extended Kim filter was proposed by Chow and Zhang (2013); the hybrid extended Kim filter is proposed in **?**

My Talk at a Glance: What Did We Talk about?

1. What are discrete-time dynamical systems?
2. Linkage between state-space models and discrete-time dynamical systems models
3. Examples of state-space models
   - Detailed dynr example using growth curve model (GrowthCurveModel.R)
   - Other coding examples provided but not covered: CFA.r, UnivariateDualChangeScoreModel.R, PFAindynr.Rmd
4. Examples of regime-switching state-space models
   - Somewhat detailed dynr example using a facial EMG-inspired RS AR model (RSLinearDiscrete.R, RSLinearDiscreteExample.pdf)
   - Other coding examples provided but not covered: RSNonlinearDiscrete.R, RSNonlinearDiscreteExample.pdf (RS nonlinear dynamic model of emotions)

References I

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