Lecture 1: Real Numbers

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Introduction

Today, we're going to solidify our understanding of the number line. We'll start from what we know (rational numbers) and introduce the properties that complete the number line, giving us the real numbers. The goal is to understand *why* we need real numbers beyond rationals.

1 Building Blocks: From Natural Numbers to Rational Numbers

- **Natural Numbers ($\mathbb{N}$):** Counting numbers $\{1, 2, 3, \dots\}$. Basic arithmetic (addition, multiplication).
- **Integers ($\mathbb{Z}$):** Include negatives and zero $\{\dots, -2, -1, 0, 1, 2, \dots\}$. Allows for subtraction.
- **Rational Numbers ($\mathbb{Q}$):** Fractions p/q, where $p \in \mathbb{Z}, q \in \mathbb{N}$. Allows for division (except by zero).

Key Idea: $\mathbb{Q}$ forms a **field** (can add, subtract, multiply, divide) and an **ordered set** (can compare numbers, $a < b$, $a = b$, or $a > b$).

2 The Problem with Rational Numbers: "Gaps" in the Number Line

Motivation: While rationals seem to fill the number line, they have "holes."

- Consider $\sqrt{2}$. We know $x^2 = 2$ has no rational solution.
- Geometrically, $\sqrt{2}$ is the length of the diagonal of a unit square. It exists! But it's not a rational number.
- This means there are points on our intuitive number line that aren't represented by rational numbers. We need to "fill these gaps."

3 Ordered Fields: A Foundation for Real Numbers

Definition 3.1. *An **ordered field** is a set of numbers that behaves like $\mathbb{Q}$ in terms of arithmetic and order.*

- *It's a **field**: You can add, subtract, multiply, and divide (except by zero), and these operations follow standard rules (associativity, commutativity, etc.).*
- *It's **ordered**: You can compare any two numbers (one is greater than, less than, or equal to the other), and this order is compatible with arithmetic.*

Motivation: We want to build the real numbers on a solid algebraic and order-based foundation. $\mathbb{Q}$ is our prime example of an ordered field.

4 Completeness: Filling the Gaps

To fill the gaps, we introduce the idea of "completeness." This relies on the concepts of bounds, supremum, and infimum.

- **Upper Bound:** A number M is an upper bound for a set S if all elements in S are less than or equal to M.
- **Supremum (Least Upper Bound, LUB):** The smallest of all upper bounds. If a set has a supremum, it's like finding the "edge" of the set from above.
  - **Motivation:** If a set of numbers is growing but doesn't go on forever (it's "bounded above"), it should "converge" to a specific value, its "edge." In $\mathbb{Q}$, this doesn't always happen (e.g., $\{x \in \mathbb{Q} \mid x^2 < 2\}$ should have $\sqrt{2}$ as its edge, but $\sqrt{2}$ isn't rational).
- **Lower Bound:** A number m is a lower bound for a set S if all elements in S are greater than or equal to m.
- **Infimum (Greatest Lower Bound, GLB):** The largest of all lower bounds.

Definition 4.1 (The Supremum Property (Completeness Axiom)). *Every non-empty set of real numbers that is bounded above has a supremum that is also a real number.*

Motivation: This is the crucial axiom that fills the "holes." It guarantees that if a set of numbers is "approaching" a value from below (and is bounded), that "limiting" value *must* exist within our number system. This is what allows for concepts like limits, continuity, and convergence in calculus.

5 Defining the Real Numbers ($\mathbb{R}$)

Definition 5.1. *The set of **real numbers**, $\mathbb{R}$, is defined as the unique ordered field that contains the rational numbers $\mathbb{Q}$ and satisfies the **supremum property**.*

Motivation: We are saying: take the rational numbers, add the "completeness" property to them (which means no more holes!), and what you get is the real numbers. The "uniqueness" means that this set of properties perfectly defines $\mathbb{R}$ – there's only one mathematical structure that fits.

6 $\mathbb{R}$ is Not Algebraically Closed

Motivation: Just because $\mathbb{R}$ is "complete" doesn't mean it contains roots for *all* polynomial equations.

Definition 6.1. *A field is **algebraically closed** if every non-constant polynomial with coefficients in that field has at least one root in that field.*

Example: Consider $x^2 + 1 = 0$. This means $x^2 = -1$.

- In $\mathbb{R}$, we know that for any x, $x^2 \geq 0$. So $x^2 = -1$ has no real solution.
- **Conclusion:** $\mathbb{R}$ is *not* algebraically closed. We need complex numbers ($\mathbb{C}$) for that. This shows that "completeness" (no gaps on the number line) is different from "algebraic closure" (all polynomial roots exist).

7 Fundamental Properties of $\mathbb{R}$: Archimedean Property and Density of Rationals

These properties are direct consequences of the completeness of $\mathbb{R}$.

7.1 Archimedean Property

Theorem 7.1. *For any positive real number x, there exists a natural number $n \in \mathbb{N}$ such that $n > x$.*

Motivation: This property formalizes the idea that the natural numbers "stretch" indefinitely. No matter how large a real number x is, you can always count past it. It also implies that for any tiny positive number ϵ, you can find a fraction $1/n$ that's even smaller. This is crucial for convergence proofs (e.g., showing a sequence approaches a limit).

7.2 Density of Rationals

Theorem 7.2. *If $x, y \in \mathbb{R}$ and $x < y$, then there exists a rational number $q \in \mathbb{Q}$ such that $x < q < y$.*

Motivation: This tells us that even though $\mathbb{Q}$ has "holes," it's incredibly "dense" within $\mathbb{R}$. No matter how close two real numbers are, you can always "squeeze" a rational number between them. This means rationals can approximate any real number as closely as we desire, which is fundamental for many areas of analysis.

8 Summary

We started with rational numbers and saw their limitation (gaps). We then introduced the concept of an ordered field and, crucially, the **supremum property (completeness)**, which defines the real numbers by filling these gaps. We also noted that while $\mathbb{R}$ is complete, it's not algebraically closed. Finally, we saw two important consequences of completeness: the **Archimedean property** (natural numbers can exceed any real number) and the **density of rationals** (rationals are "everywhere" in $\mathbb{R}$). These ideas form the bedrock for calculus and advanced mathematical analysis.

9 Problems for Practice

1. Let $S \subset \mathbb{R}$ and $a > 0$. Prove that
   (a) $\sup(a + S)a + \sup S$
2. Prove Archimedean Property
3. Prove Density property of rationals.