1 Real Numbers and Ordered Fields

1. Prove the following statements are true in every ordered field.

   (a) If $x > 0$ then $-x < 0$, and vice versa.

   (b) If $x > 0$ and $y < z$ then $xy < xz$.

   (c) If $x < 0$ and $y < z$ then $xy > xz$.

   (d) If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.

   (e) If $0 < x < y$ then $0 < 1/y < 1/x$.

2. Let $S \subset \mathbb{R}$ and $a \in \mathbb{R}$. Prove that

   (a) $\sup(a + S) = a + \sup S$

   (b) If $a > 0$ then $\sup(aS) = a \sup S$

   (c) If $a > 0$ then $\sup(aS) = a \inf S$

3. Prove that supremum property implies infimum property.

4. Use Archimedean Property to prove the following

   (a) If $S := \{1/n : n \in \mathbb{N}\}$, then $\inf S = 0$.

   (b) If $t > 0$, there exists $n_t \in \mathbb{N}$ such that $0 < 1/n_t < t$.

   (c) If $y > 0$, there exists $n_y \in \mathbb{N}$ such that $n_y - 1 \leq y \leq n_y$.

   (d) If $S := \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\right\}$, find $\inf S$ and $\sup S$.

5. Prove that between any two real numbers there is an irrational number.

6. Let $I_n := [0, 1/n]$ for $n \in \mathbb{N}$. Prove that

$$\bigcap_{n=1}^{\infty} I_n = \{0\}.$$

7. Let $J_n := (0, 1/n)$ for $n \in \mathbb{N}$. Prove that

$$\bigcap_{n=1}^{\infty} J_n = \emptyset.$$

8. Let $K_n := (n, \infty)$ for $n \in \mathbb{N}$. Prove that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

2 Sequence and Series

1. Find $K(\epsilon)$ for the following sequences

   (a) $\frac{1}{5n+1}$; $\epsilon = \frac{1}{6}$

   (b) $\frac{2}{n^3} + 5$; $\epsilon = \frac{1}{8}$

   (c) $\frac{3n}{2n+1}$; $\epsilon = \frac{1}{10}$

   (d) $\frac{n^2+n}{2n^2-1}$; $\epsilon = \frac{1}{5}$

   (e) $(-1)^n \frac{1}{n}$; $\epsilon = \frac{1}{61}$

2. Porve that limit of a sequence if it exist is unique.

3. Evaluate the following limits (also prove it using the definition of limit):

   (a) $\lim_{n\to\infty} \frac{3n}{2n+1}$

   (b) $\lim_{n\to\infty} \frac{n}{n^2+1}$

   (c) $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n})$

4. Let x_n, y_n be two sequences such that $x_n \to x$ and $y_n \to y$. Prove that:

   (a) $x_n + y_n \to x + y$

   (b) $x_n - y_n \to x - y$

   (c) $x_n y_n \to xy$

   (d) $x_n/y_n \to x/y$ if $y \neq 0$

5. Prove that if $x_n \to x$ then $|x_n| \to |x|$ and $\sqrt{|x_n|} \to \sqrt{|x|}$.

6. Prove that for any real number x, there exists a sequence of rational numbers (x_n) such that $x_n \to x$.

7. Let $Y = (y_n)$ be defined inductively by $y_1 := 1$, $y_{n+1} := \frac{1}{4}(2y_n + 3)$ for $n \geq 1$. Show that $\lim Y = 3/2$.

8. Prove that if $\sum x_n$ converges, then $x_n \to 0$.