

1 Problems set 1(Real Numbers)

1. Prove the following statements are true in every ordered field.

- (a) If $x > 0$ then $-x < 0$, and vice versa.
- (b) If $x > 0$ and $y < z$ then $xy < xz$.
- (c) If $x < 0$ and $y < z$ then $xy > xz$.
- (d) If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.
- (e) If $0 < x < y$ then $0 < 1/y < 1/x$.

2. Let $S \subset \mathbb{R}$ and $a \in \mathbb{R}$. Prove that

- (a) $\sup(a + S) = a + \sup S$
- (b) If $a > 0$ then $\sup(aS) = a \sup S$
- (c) If $a < 0$ then $\sup(aS) = a \inf S$

3. Prove that supremum property implies infimum property.

4. Use Archimedean Property to prove the following

- (a) If $S := \{1/n : n \in \mathbb{N}\}$, then $\inf S = 0$.
- (b) If $t > 0$, there exists $n_t \in \mathbb{N}$ such that $0 < 1/n_t < t$.
- (c) If $y > 0$, there exists $n_y \in \mathbb{N}$ such that $n_y - 1 \leq y \leq n_y$.
- (d) If $S := \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.

5. Prove that between any two real numbers there is an irrational number.

6. Let $I_n := [0, 1/n]$ for $n \in \mathbb{N}$. Prove that

$$\bigcap_{n=1}^{\infty} I_n = \{0\}.$$

7. Let $J_n := (0, 1/n)$ for $n \in \mathbb{N}$. Prove that

$$\bigcap_{n=1}^{\infty} J_n = \emptyset.$$

8. Let $K_n := (n, \infty)$ for $n \in \mathbb{N}$. Prove that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

2 Problem set 2(Sequences)

1. Find $K(\epsilon)$ for the following sequences

(a) $\frac{1}{5n+1}$; $\epsilon = \frac{1}{6}$

(b) $\frac{2}{n^3} + 5$; $\epsilon = \frac{1}{8}$

(c) $\frac{3n}{2n+1}$; $\epsilon = \frac{1}{10}$

(d) $\frac{n^2+n}{2n^2-1}$; $\epsilon = \frac{1}{5}$

(e) $(-1)^n \frac{1}{n}$; $\epsilon = \frac{1}{61}$

2. Prove that limit of a sequence if it exist is unique.

3. Evaluate the following limits (also prove it using the definition of limit):

(a) $\lim_{n \rightarrow \infty} \frac{3n}{2n+1}$

(b) $\lim_{n \rightarrow \infty} \frac{n}{n^2+1}$

(c) $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$

4. Let x_n, y_n be two sequences such that $x_n \rightarrow x$ and $y_n \rightarrow y$. Prove that:

(a) $x_n + y_n \rightarrow x + y$

(b) $x_n - y_n \rightarrow x - y$

(c) $x_n y_n \rightarrow xy$

(d) $x_n/y_n \rightarrow x/y$ if $y \neq 0$

5. Prove that if $x_n \rightarrow x$ then $|x_n| \rightarrow |x|$ and $\sqrt{|x_n|} \rightarrow \sqrt{|x|}$.

6. Prove squeeze theroerm: If $x_n \leq y_n \leq z_n$ for all n and $x_n \rightarrow L$, $z_n \rightarrow L$, then $y_n \rightarrow L$.

7. Prove that for any real number x , there exists a sequence of rational numbers (x_n) such that $x_n \rightarrow x$.

3 Problem set 3(Sequences)

1. Give an example of an unbounded sequence that has a convergent subsequence.
2. Suppose that $x_n \geq 0$ and $(-1)^n x_n$ converges. Does it imply that x_n converges.
3. Let (x_n) be a bounded sequence and let $s := \sup\{x_n : n \in \mathbb{N}\}$. Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to s .
4. Show directly that bounded monotone sequence is a cauchy sequence.
5. Let $Y = (y_n)$ be defined inductively by $y_1 := 1$, $y_{n+1} := \frac{1}{4}(2y_n + 3)$ for $n \geq 1$. Show that $\lim Y = 3/2$.