

# TIFAE – TRFAE with Type Inference

## 1 INTRODUCTION

TIFAE is a toy language for the [COSE212](#) course at Korea University. TIFAE stands for an extension of the [RFAE](#) language with **type system**, and it supports the following features:

- **number (integer) values**
- **boolean values** (true and false)
- **arithmetic operators**: negation (-), addition (+), subtraction (-), multiplication (\*), division (/), and modulo (%)
- **arithmetic comparison operators**: equality (== and !=) and relational (<, >, <=, and >=)
- **first-class functions**
- **recursive functions** (def)
- **conditionals** (if-else)
- **logical operators**: conjunction (&&), disjunction (||), and negation (!)
- **static type checking with type inference**

This document is the specification of TIFAE. First, Section 2 describes the concrete syntax, and Section 3 describes the abstract syntax with the desugaring rules. Then, Section 4 describes the type system. Finally, Section 5 describes the big-step operational (natural) semantics of TIFAE.

## 2 CONCRETE SYNTAX

The concrete syntax of TIFAE is written in a variant of the extended Backus–Naur form (EBNF). The notation `<nt>` denotes a nonterminal, and `"t"` denotes a terminal. We use `?` to denote an optional element and `+` (or `*`) to denote one or more (or zero or more) repetitions of the preceding element. We use **butnot** to denote a set difference to exclude some strings from a producible set of strings. We omit some obvious terminals using the ellipsis (`...`) notation.

```
// basic elements
<digit>    ::= "0" | "1" | "2" | ... | "9"
<number>   ::= "-"? <digit>+
<alphabet> ::= "A" | "B" | "C" | ... | "Z" | "a" | "b" | "c" | ... | "z"
<idstart>  ::= <alphabet> | "_"
<idcont>   ::= <alphabet> | "_" | <digit>
<keyword>  ::= "true" | "false" | "def" | "if" | "else" | "val"
<id>       ::= <idstart> <idcont>* butnot <keyword>

// expressions
<expr> ::= <number> | "true" | "false" | <uop> <expr> | <expr> <bop> <expr>
        | "(" <expr> ")" | "{" <expr> "}"
        | "val" <id> "=" <expr> ";"? <expr> | <id>
        | "(" <id> ")" "=" <expr> | <expr> "(" <expr> ")"
        | "def" <id> "(" <id> ")" "=" <expr> ";"? <expr>
        | "if" "(" <expr> ")" <expr> "else" <expr>

// operators
<uop> ::= "-" | "!"
<bop>  ::= "+" | "-" | "*" | "/" | "%" | "&&" | "||"
        | "==" | "!=" | "<" | "<=" | ">" | ">="
```

For types, the arrow ( $\Rightarrow$ ) operator is right-associative. For expressions, the precedence and associativity of operators are defined as follows:

Description	Operator	Precedence	Associativity
Unary	$-, !$	1	right
Multiplicative	$*, /, \%$	2	left
Additive	$+, -$	3	
Relational	$<, <=, >, >=$	4	
Equality	$==, !=$	5	
Logical Conjunction	$\&\&$	6	
Logical Disjunction	$  $	7	

### 3 ABSTRACT SYNTAX

The abstract syntax of TIFAE is defined as follows:

Expressions	$\mathbb{E} \ni e ::= n$	(Num)	$  \text{val } x = e; e$	(Val)
	$  b$	(Bool)	$  x$	(Id)
	$  e + e$	(Add)	$  \lambda x. e$	(Fun)
	$  e * e$	(Mul)	$  \text{def } x(x) = e; e$	(Rec)
	$  e / e$	(Div)	$  e(e)$	(App)
	$  e \% e$	(Mod)	$  \text{if } (e) e \text{ else } e$	(If)
	$  e = e$	(Eq)		
	$  e < e$	(Lt)		

Numbers	$n \in \mathbb{Z}$	(BigInt)
Identifiers	$x \in \mathbb{X}$	(String)
Booleans	$b \in \mathbb{B} = \{\text{true}, \text{false}\}$	(Boolean)

The types or semantics of the remaining cases are defined with the following desugaring rules:

$\mathcal{D}[-e]$	$= \mathcal{D}[e] * (-1)$	$\mathcal{D}[e_1 != e_2]$	$= \mathcal{D}[\neg (e_1 == e_2)]$
$\mathcal{D}[\neg e]$	$= \text{if } (\mathcal{D}[e]) \text{ false else true}$	$\mathcal{D}[e_1 < e_2]$	$= \mathcal{D}[(e_1 < e_2)    (e_1 == e_2)]$
$\mathcal{D}[e_1 - e_2]$	$= \mathcal{D}[e_1] + \mathcal{D}[-e_2]$	$\mathcal{D}[e_1 > e_2]$	$= \mathcal{D}[\neg (e_1 < e_2)]$
$\mathcal{D}[e_1 \&\& e_2]$	$= \text{if } (\mathcal{D}[e_1]) \mathcal{D}[e_2] \text{ else false}$	$\mathcal{D}[e_1 >= e_2]$	$= \mathcal{D}[\neg (e_1 < e_2)]$
$\mathcal{D}[e_1    e_2]$	$= \text{if } (\mathcal{D}[e_1]) \text{ true else } \mathcal{D}[e_2]$		

The omitted cases recursively apply the desugaring rule to sub-expressions.

#### 4 TYPE SYSTEM

This section explains type system of TIFAE, and we use the following notations:

Type Environments	$\Gamma \in \mathbb{T} = \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T}^\vee$	(TypeEnv)
Solution	$\psi \in \Psi = \mathbb{X}_\alpha \xrightarrow{\text{fin}} (\mathbb{T} \uplus \{\perp\})$	(Solution)
Types	$\mathbb{T} \ni$ $\begin{array}{l} \tau ::= \text{num} \\ \quad   \text{bool} \\ \quad   \tau \rightarrow \tau \\ \quad   \alpha \end{array}$	(NumT) (BoolT) (ArrowT) (VarT)
Polymorphic Types	$\tau^\vee = \forall \alpha^*. \tau \in \mathbb{T}^\vee = \mathbb{X}_\alpha^* \times \mathbb{T}$	(TypeEnv)
Type Variables	$\alpha \in \mathbb{X}_\alpha$	(Int)

We skip the  $\forall$ -quantifier in polymorphic types if they have zero type variables. In the type system, type checking and type inference is defined with the following typing rules:

	$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$	
	$\tau\text{-Num} \frac{}{\Gamma, \psi \vdash n : \text{num}, \psi} \quad \tau\text{-Bool} \frac{}{\Gamma, \psi \vdash b : \text{bool}, \psi}$	
$\tau\text{-Add}$	$\frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \quad \Gamma, \psi_1 \vdash e_2 : \tau_2, \psi_2 \quad \text{unify}(\tau_1, \text{num}, \psi_2) = \psi_3 \quad \text{unify}(\tau_2, \text{num}, \psi_3) = \psi_4}{\Gamma, \psi_0 \vdash e_1 + e_2 : \text{num}, \psi_4}$	
$\tau\text{-Mul}$	$\frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \quad \Gamma, \psi_1 \vdash e_2 : \tau_2, \psi_2 \quad \text{unify}(\tau_1, \text{num}, \psi_2) = \psi_3 \quad \text{unify}(\tau_2, \text{num}, \psi_3) = \psi_4}{\Gamma, \psi_0 \vdash e_1 * e_2 : \text{num}, \psi_4}$	
$\tau\text{-Div}$	$\frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \quad \Gamma, \psi_1 \vdash e_2 : \tau_2, \psi_2 \quad \text{unify}(\tau_1, \text{num}, \psi_2) = \psi_3 \quad \text{unify}(\tau_2, \text{num}, \psi_3) = \psi_4}{\Gamma, \psi_0 \vdash e_1 / e_2 : \text{num}, \psi_4}$	
$\tau\text{-Mod}$	$\frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \quad \Gamma, \psi_1 \vdash e_2 : \tau_2, \psi_2 \quad \text{unify}(\tau_1, \text{num}, \psi_2) = \psi_3 \quad \text{unify}(\tau_2, \text{num}, \psi_3) = \psi_4}{\Gamma, \psi_0 \vdash e_1 \% e_2 : \text{num}, \psi_4}$	
$\tau\text{-Eq}$	$\frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \quad \Gamma, \psi_1 \vdash e_2 : \tau_2, \psi_2 \quad \text{unify}(\tau_1, \text{num}, \psi_2) = \psi_3 \quad \text{unify}(\tau_2, \text{num}, \psi_3) = \psi_4}{\Gamma, \psi_0 \vdash e_1 = e_2 : \text{bool}, \psi_4}$	
$\tau\text{-Lt}$	$\frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \quad \Gamma, \psi_1 \vdash e_2 : \tau_2, \psi_2 \quad \text{unify}(\tau_1, \text{num}, \psi_2) = \psi_3 \quad \text{unify}(\tau_2, \text{num}, \psi_3) = \psi_4}{\Gamma, \psi_0 \vdash e_1 < e_2 : \text{bool}, \psi_4}$	

$$\begin{array}{c}
\tau\text{-Val} \frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \quad \text{gen}(\tau_1, \Gamma, \psi_1) = \tau_1^\forall \quad \Gamma[x : \tau_1^\forall] \vdash e_2 : \tau_2, \psi_2}{\Gamma, \psi_0 \vdash \text{val } x = e_1; e_2 : \tau_2, \psi_2} \\
\\
\tau\text{-Id} \frac{\Gamma(x) = \tau^\forall \quad \text{inst}(\tau^\forall, \psi) = (\tau, \psi')}{\Gamma, \psi \vdash x : \tau, \psi'} \quad \tau\text{-Fun} \frac{\alpha_p \notin \psi \quad \Gamma[x : \alpha_p], \psi[\alpha_p \mapsto \perp] \vdash e : \tau, \psi'}{\Gamma, \psi \vdash \lambda x. e : \alpha_p \rightarrow \tau, \psi'} \\
\\
\tau\text{-App} \frac{\Gamma, \psi \vdash e_f : \tau_f, \psi_f \quad \Gamma, \psi_f \vdash e_a : \tau_a, \psi_a \quad \alpha_r \notin \psi_a \quad \text{unify}(\tau_a \rightarrow \alpha_r, \tau_f, \psi_a[\alpha_r \mapsto \perp]) = \psi'}{\Gamma, \psi \vdash e_1(e_2) : \alpha_r, \psi'} \\
\\
\tau\text{-Rec} \frac{\alpha_p, \alpha_r \notin \psi \quad \alpha_p \neq \alpha_r \quad \Gamma_1 = \Gamma[x_f \mapsto (\alpha_p \rightarrow \alpha_r)] \quad \Gamma_2 = \Gamma_1[x_p \mapsto \alpha_p] \quad \Gamma_2, \psi[\alpha \mapsto \perp, \alpha' \mapsto \perp] \vdash e_b : \tau_b, \psi_b \quad \text{unify}(\tau_b, \alpha_r, \psi_b) = \psi_r \quad \Gamma_1, \psi_r \vdash e_s : \tau_s, \psi_s}{\Gamma, \psi \vdash \text{def } x_f(x_p) = e_b; e_s : \tau_s, \psi_s} \\
\\
\tau\text{-If} \frac{\Gamma, \psi_t \vdash e_e : \tau_e, \psi_e \quad \Gamma, \psi \vdash e_c : \text{bool}, \psi_c \quad \Gamma, \psi_c \vdash e_t : \tau_t, \psi_t \quad \text{unify}(\tau_c, \text{bool}, \psi_e) = \psi' \quad \text{unify}(\tau_t, \tau_e, \psi') = \psi''}{\Gamma, \psi \vdash \text{if } (e_c) e_t \text{ else } e_e : \tau_t, \psi''}
\end{array}$$

type unification is defined as a partial function:

$$\boxed{\text{unify} : (\mathbb{T} \times \mathbb{T} \times \Psi) \rightarrow \Psi}$$

$$\text{unify}(\tau_1, \tau_2, \psi) = \begin{cases} \psi & \text{if } \tau'_1 = \text{num} \wedge \tau'_2 = \text{num} \\ \psi & \text{if } \tau'_1 = \text{bool} \wedge \tau'_2 = \text{bool} \\ \text{unify}(\tau_{1,r}, \tau_{2,r}, \text{unify}(\tau_{1,p}, \tau_{2,p}, \psi)) & \text{if } \tau'_1 = (\tau_{1,p} \rightarrow \tau_{1,r}) \wedge \tau'_2 = (\tau_{2,p} \rightarrow \tau_{2,r}) \\ \psi & \text{if } \tau'_1 = \alpha = \tau'_2 \\ \psi[\alpha \mapsto \tau'_2] & \text{if } \tau'_1 = \alpha \wedge \neg \text{occur}(\alpha, \tau'_2, \psi) \\ \psi[\alpha \mapsto \tau'_1] & \text{if } \tau'_2 = \alpha \wedge \neg \text{occur}(\alpha, \tau'_1, \psi) \end{cases}$$

where  $\tau'_1 = \text{resolve}(\tau_1, \psi)$  and  $\tau'_2 = \text{resolve}(\tau_2, \psi)$ .

type resolving and occurrence checking are defined as following functions:

$$\boxed{\text{resolve} : (\mathbb{T} \times \Psi) \rightarrow \mathbb{T}}$$

$$\text{resolve}(\tau, \psi) = \begin{cases} \text{resolve}(\tau', \psi) & \text{if } \tau = \alpha \wedge \psi(\alpha) = \tau' \\ \tau & \text{otherwise} \end{cases}$$

$$\boxed{\text{occur} : (\mathbb{X}_\alpha \times \mathbb{T} \times \Psi) \rightarrow \text{bool}}$$

$$\text{occur}(\alpha, \tau, \psi) = \begin{cases} \text{true} & \text{if } \tau = \alpha \\ \text{occur}(\alpha, \tau_p, \psi) \vee \text{occur}(\alpha, \tau_r, \psi) & \text{if } \tau = (\tau_p \rightarrow \tau_r) \\ \text{false} & \text{otherwise} \end{cases}$$

type generalization and instantiation are defined as following functions:

$$\boxed{\text{gen} : (\mathbb{T} \times \Gamma \times \Psi) \rightarrow \mathbb{T}^\forall}$$

$$\text{gen}(\tau, \Gamma, \psi) = \forall \alpha_1, \dots, \alpha_m. \tau \quad \text{where} \quad \text{free}_\tau(\tau, \psi) \setminus \text{free}_\Gamma(\Gamma, \psi) = \{\alpha_1, \dots, \alpha_m\}$$

$$\boxed{\text{inst} : (\mathbb{T}^\forall \times \Psi) \rightarrow (\mathbb{T} \times \Psi)}$$

$$\text{inst}(\forall \alpha_1, \dots, \alpha_m. \tau, \psi) = (\text{subst}(\tau, \psi[\alpha_1 \mapsto \alpha'_1, \dots, \alpha_m \mapsto \alpha'_m]), \psi[\alpha'_1 \mapsto \perp, \dots, \alpha'_m \mapsto \perp])$$

$$\text{where} \quad \alpha'_1, \dots, \alpha'_m \notin \psi \wedge \forall 1 \leq i < j \leq m. \alpha'_i \neq \alpha'_j$$

free type variables and substitution are defined as following functions:

$$\boxed{\text{free}_\tau : (\mathbb{T} \times \Psi) \rightarrow \mathcal{P}(\mathbb{X}_\alpha)}$$

$$\text{free}_\tau(\tau, \psi) = \begin{cases} \text{free}_{\tau'}(\tau', \psi) & \text{if } \tau = \alpha \wedge \psi(\alpha) = \tau' \\ \{\alpha\} & \text{if } \tau = \alpha \wedge \psi(\alpha) = \perp \\ \text{free}_{\tau_p}(\tau_p, \psi) \cup \text{free}_{\tau_r}(\tau_r, \psi) & \text{if } \tau = (\tau_p \rightarrow \tau_r) \\ \emptyset & \text{otherwise} \end{cases}$$

$$\boxed{\text{free}_{\tau^\forall} : (\mathbb{T}^\forall \times \Psi) \rightarrow \mathcal{P}(\mathbb{X}_\alpha)}$$

$$\text{free}_{\tau^\forall}(\forall \alpha_1, \dots, \alpha_m. \tau, \psi) = \text{free}_\tau(\tau, \psi) \setminus \{\alpha_1, \dots, \alpha_m\}$$

$$\boxed{\text{free}_\Gamma : (\Gamma \times \Psi) \rightarrow \mathcal{P}(\mathbb{X}_\alpha)}$$

$$\text{free}_\Gamma([x_1 : \tau_1^\forall, \dots, x_n : \tau_n^\forall], \psi) = \text{free}_{\tau_1^\forall}(\tau_1^\forall, \psi) \cup \dots \cup \text{free}_{\tau_n^\forall}(\tau_n^\forall, \psi)$$

$$\boxed{\text{subst} : (\mathbb{T} \times \Psi) \rightarrow \mathbb{T}}$$

$$\text{subst}(\tau, \psi) = \begin{cases} \text{subst}(\tau', \psi) & \text{if } \tau = \alpha \wedge \psi(\alpha) = \tau' \\ \text{subst}(\tau_p, \psi) \rightarrow \text{subst}(\tau_r, \psi) & \text{if } \tau = (\tau_p \rightarrow \tau_r) \\ \tau & \text{otherwise} \end{cases}$$

## 5 SEMANTICS

We use the following notations in the semantics:

Values	$\mathbb{V} \ni v ::= n$	(NumV)
	$  b$	(BoolV)
	$  \langle \lambda x. e, \sigma \rangle$	(CloV)
Environments	$\sigma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{V}$	(Env)

The big-step operational (natural) semantics of TIFAE is defined as follows:

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\begin{array}{c}
 \text{Num} \frac{}{\sigma \vdash n \Rightarrow n} \qquad \text{Bool} \frac{}{\sigma \vdash b \Rightarrow b} \\
 \\
 \text{Add} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 + e_2 \Rightarrow n_1 + num_2} \qquad \text{Mul} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 * e_2 \Rightarrow n_1 * n_2} \\
 \\
 \text{Div} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2 \quad n_2 \neq 0}{\sigma \vdash e_1 / e_2 \Rightarrow n_1 / n_2} \qquad \text{Mod} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2 \quad n_2 \neq 0}{\sigma \vdash e_1 \% e_2 \Rightarrow n_1 \% n_2} \\
 \\
 \text{Eq} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 == e_2 \Rightarrow n_1 = n_2} \qquad \text{Lt} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 < e_2 \Rightarrow n_1 < n_2} \\
 \\
 \text{Val} \frac{\sigma \vdash e_1 \Rightarrow v_1 \quad \sigma[x \mapsto v_1] \vdash e_2 \Rightarrow v_2}{\sigma \vdash \text{val } x = e_1; e_2 \Rightarrow v_2} \qquad \text{Id} \frac{x \in \text{Domain}(\sigma)}{\sigma \vdash x \Rightarrow \sigma(x)} \\
 \\
 \text{Fun} \frac{}{\sigma \vdash \lambda x. e \Rightarrow \langle \lambda x. e, \sigma \rangle} \qquad \text{Rec} \frac{\sigma' = \sigma[x_0 \mapsto \langle \lambda x_1. e_2, \sigma' \rangle] \quad \sigma' \vdash e_3 \Rightarrow v_3}{\sigma \vdash \text{def } x_0(x_1) = e_2; e_3 \Rightarrow v_3} \\
 \\
 \text{App} \frac{\sigma \vdash e_0 \Rightarrow \langle \lambda x. e_2, \sigma' \rangle \quad \sigma \vdash e_1 \Rightarrow v_1 \quad \sigma'[x \mapsto v_1] \vdash e_2 \Rightarrow v_2}{\sigma \vdash e_0(e_1) \Rightarrow v_2} \\
 \\
 \text{If}_T \frac{\sigma \vdash e_0 \Rightarrow \text{true} \quad \sigma \vdash e_1 \Rightarrow v_1}{\sigma \vdash \text{if } (e_0) e_1 \text{ else } e_2 \Rightarrow v_1} \qquad \text{If}_F \frac{\sigma \vdash e_0 \Rightarrow \text{false} \quad \sigma \vdash e_2 \Rightarrow v_2}{\sigma \vdash \text{if } (e_0) e_1 \text{ else } e_2 \Rightarrow v_2}
 \end{array}$$