Lecture 8 – Closure Properties of Regular Languages

COSE215: Theory of Computation

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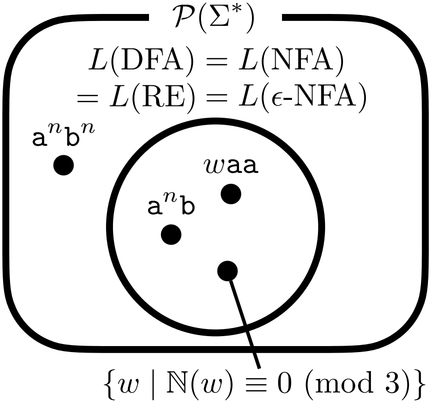
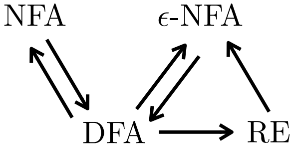
PLRG

2024 Spring

Recall

- Regular Languages





Contents

Closure Properties of Regular Languages

Let's consider a **regular** language. For example,

$$L = \{w\texttt{aa} \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$$

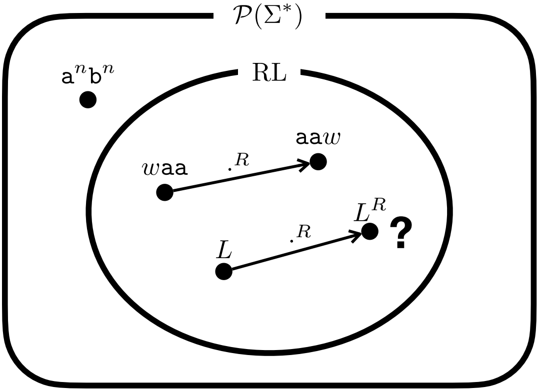
Then, is its **reverse** language L^R also **regular**?

$$L^R = \{\texttt{aa}w \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$$

Yes! We can construct a regular expression whose language is L^R as:

$$L(\texttt{aa(a|b)}^*) = L^R = \{\texttt{aa}w \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$$

Closure Properties of Regular Languages



Then, for any regular language L, is L^R always regular? **Yes!**

The class of regular languages is **closed** under the **reversal** operator.

In this lecture, we will discuss and prove the **closure properties** of regular languages for various language operators.

Closure Properties of Regular Languages

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Definition (Closure Properties)

The class of regular languages is **closed** under an n-ary operator op if and only if $\text{op}(L_1, \cdots, L_n)$ is regular for any regular languages $L_1, \cdots, L_n$. We say that such properties are **closure properties** of regular languages.

To prove the closure properties for the n-ary operator op, we need to provide a way to do the following for any regular languages $L_1, \cdots, L_n$:

1. Construct a regular expression R whose language is $\text{op}(L_1, \cdots, L_n)$ using the regular expressions $R_1, \cdots, R_n$ such that $L(R) = \text{op}(L(R_1), \cdots, L(R_n))$.

2. Construct a finite automaton A whose language is $\text{op}(L_1, \cdots, L_n)$ using the finite automata $A_1, \cdots, A_n$ such that $L(A) = \text{op}(L(A_1), \cdots, L(A_n))$.

Closure Properties of Regular Languages

In this lecture, we will prove the closure properties of regular languages for the following operators:

- Union
- Concatenation
- Kleene Star
- Complement
- Intersection
- Difference
- Reversal
- Homomorphism
- Inverse Homomorphism

Theorem (Closure under Union, Concatenation, and Kleene Star)

If L_1 and L_2 are regular languages, then so is $L_1 \cup L_2$, $L_1 L_2$, and L_1^*.

Proof) Let R_1 and R_2 be the regular expressions such that $L(R_1) = L_1$ and $L(R_2) = L_2$, respectively.

Consider the following regular expression:

$$R_1 \mid R_2 \qquad\qquad R_1 R_2 \qquad\qquad R^*$$

Then, by the definition of the union ($\cup$), concatenation ($\cdot$), and Kleene star (*) operators for regular expressions,

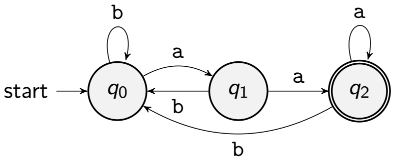
$$L(R_1 \mid R_2) = L_1 \cup L_2 \qquad\qquad L(R_1 R_2) = L_1 L_2 \qquad\qquad L(R^*) = L^*$$

So, we proved that the class of regular languages are **closed** under the **union**, **concatenation**, and **Kleene star** operators. $\square$

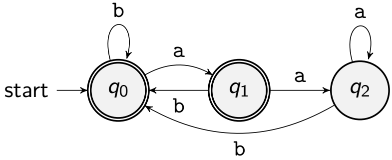
Closure under Complement

Consider a regular language $L = \{w\mathtt{aa} \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$.

Is its **complement** $\overline{L} = \Sigma^* \setminus L$ also regular? **Yes!**

$D =$ 

First, consider the above DFA D accepting the language L.

$\overline{D} =$ 

The key idea is to construct a new DFA $\overline{D}$ by **swapping** the **final** and **non-final** states of the original DFA:

Closure under Complement

Theorem (Closure under Complement)

If L is a regular language, then so is $\overline{L}$.

Proof) Let $D = (Q, \Sigma, \delta, q_0, F)$ be the DFA such that $L(D) = L$. Consider the following DFA:

$$\overline{D} = (Q, \Sigma, \delta, q_0, Q \setminus F).$$

Then,

$$\begin{aligned}
\forall w \in \Sigma^*, \;\; w \in L(\overline{D}) \quad &\iff \quad \delta^*(q_0, w) \in Q \setminus F \\
&\iff \quad \delta^*(q_0, w) \notin F \\
&\iff \quad w \notin L(D) \\
&\iff \quad w \notin L \\
&\iff \quad w \in \overline{L}
\end{aligned}$$

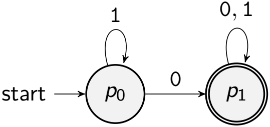
$\square$

Closure under Intersection

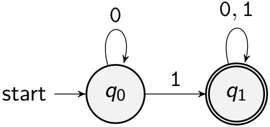
$$L_1 = \{w \in \{0,1\}^* \mid w \text{ has } 0\} \qquad L_2 = \{w \in \{0,1\}^* \mid w \text{ has } 1\}$$

Is the **intersection** of two regular languages $L_1 \cap L_2$ also regular? **Yes!**

$D_0 =$

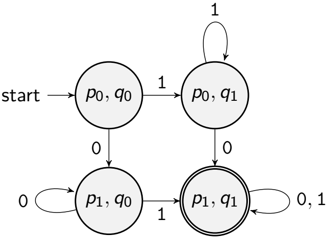


$D_1 =$



First, consider the above DFAs D_0 and D_1 accepting the languages L_1 and L_2, respectively.

$D =$



The key idea is to construct a new DFA D by **combining** them with their **pair of states** as its states.

Theorem (Closure under Intersection)

If L_0 and L_1 are regular languages, then so is $L_0 \cap L_1$.

Proof) Let $D_0 = (Q_0, \Sigma, \delta_0, q_0, F_0)$ and $D_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ be the DFA such that $L(D_0) = L_0$ and $L(D_1) = L_1$. Consider the following DFA:

$$D = (Q_0 \times Q_1, \Sigma, \delta, (q_0, q_1), F_0 \times F_1).$$

where $\forall q \in Q_0, q' \in Q_1, \mathrm{a} \in \Sigma.\ \delta((q, q'), \mathrm{a}) = (\delta_0(q, \mathrm{a}), \delta_1(q', \mathrm{a}))$. Then,

$$\begin{array}{rcl} \forall w \in \Sigma^*,\ \ w \in L(D) & \iff & \delta^*((q_0, q_1), w) \in F_0 \times F_1 \\ & \iff & \delta^*(q_0, w) \in F_0 \text{ and } \delta^*(q_1, w) \in F_1 \\ & \iff & w \in L(D_0) \text{ and } w \in L(D_1) \\ & \iff & w \in L(D_0) \cap L(D_1) \\ & \iff & w \in L_0 \cap L_1 \end{array}$$

$\square$

Closure under Intersection

Theorem (Closure under Intersection)

If L_0 and L_1 are regular languages, then so is $L_0 \cap L_1$.

Proof) Another proof is to use De Morgan's law:

$$L_0 \cap L_1 = \overline{\overline{L_0} \cup \overline{L_1}}$$

Since we already know that the regular languages are closed under complement and union, we are done. □

Closure under Difference

Theorem (Closure under Difference)

If L_0 and L_1 are regular languages, then so is $L_0 \setminus L_1$.

Proof) Similarly, we can use the following fact:

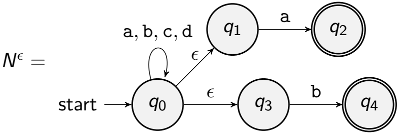
$$L_0 \setminus L_1 = L_0 \cap \overline{L_1}$$

Since we already know that the regular languages are closed under complement and intersection, we are done. □

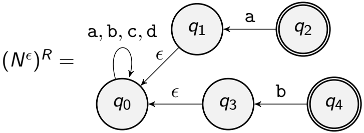
Closure under Reversal

$$L = \{w\mathtt{a} \text{ or } w\mathtt{b} \mid w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}^*\}$$

Is the **reversal** L^R of the above regular language L also regular? **Yes!**



The above ϵ-NFA N^ϵ accepts the language L.



The key idea is to construct a new ϵ-NFA $(N^\epsilon)^R$ by

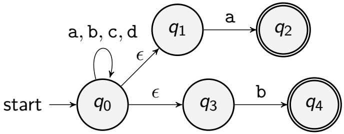
1. **reversing** the direction of the transitions
2.
3.

Closure under Reversal

$$L = \{wa \text{ or } wb \mid w \in \{a, b, c, d\}^*\}$$

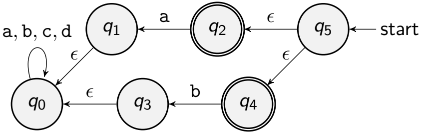
Is the **reversal** L^R of the above regular language L also regular? **Yes!**

$N^\epsilon =$



The above ϵ-NFA N^ϵ accepts the language L.

$(N^\epsilon)^R =$



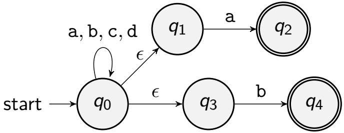
The key idea is to construct a new ϵ-NFA $(N^\epsilon)^R$ by

1. **reversing** the direction of the transitions
2. **adding** new initial state having ϵ-transitions to the original final states
3.

$$L = \{w\mathtt{a} \text{ or } w\mathtt{b} \mid w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}^*\}$$

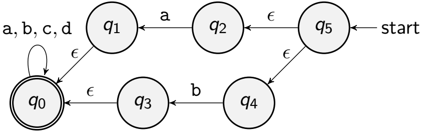
Is the **reversal** L^R of the above regular language L also regular? **Yes!**

$N^\epsilon =$



The above ϵ-NFA N^ϵ accepts the language L.

$(N^\epsilon)^R =$



The key idea is to construct a new ϵ-NFA $(N^\epsilon)^R$ by

1. **reversing** the direction of the transitions
2. **adding** new initial state having ϵ-transitions to the original final states
3. **change** original initial state to the unique new final state

Closure under Reversal

Theorem (Closure under Reversal)

If L is a regular language, then so is L^R.

Proof) Let $N^\epsilon = (Q, \Sigma, \delta, q_0, F)$ be the ϵ-NFA such that $L(N^\epsilon) = L$. Consider the following

$$(N^\epsilon)^R = (Q \uplus \{q_s\}, \Sigma, \delta^R, q_s, \{q_0\})$$

where

$$\begin{aligned} \forall q \in Q.\ \forall a \in \Sigma.\ \delta^R(q, \mathrm{a}) &= \{q' \in Q \mid q \in \delta(q', \mathrm{a})\} \\ \forall q \in Q.\ \delta^R(q, \epsilon) &= \{q' \in Q \mid q \in \delta(q', \epsilon)\} \\ \forall a \in \Sigma.\ \delta^R(q_s, \mathrm{a}) &= \varnothing \\ \delta^R(q_s, \epsilon) &= F \end{aligned}$$

□

Closure under Reversal

Theorem (Closure under Reversal)

If L is a regular language, then so is L^R.

Proof) Another proof is to use the structural induction on the regular expressions. Let R be a regular expression. Then, we can define its reversed regular expression R^R as follows:

- If $R = \varnothing$, then $R^R = \varnothing$.
- If $R = \epsilon$, then $R^R = \epsilon$.
- If $R = \mathtt{a}$, then $R^R = \mathtt{a}$.
- If $R = R_0 \,|\, R_1$, then $R^R = R_0^R \,|\, R_1^R$.
- If $R = R_0 R_1$, then $R^R = R_1^R R_0^R$.
- If $R = R_0^*$, then $R^R = (R_0^R)^*$.
- If $R = (R_0)$, then $R^R = (R_0^R)$. $\square$

$$R = \mathtt{ab(cd)^*|ef}$$

$$R^R = \mathtt{(dc)^*ba|fe}$$

Closure under Homomorphism

Definition (Homomorphism)

Suppose Σ and Γ are two finite sets of symbols. Then, a function

$$h : \Sigma \to \Gamma^*$$

is called a **homomorphism**. For a given word $w = \mathtt{a}_1\mathtt{a}_2 \cdots \mathtt{a}_n \in \Sigma^*$,

$$h(w) = h(\mathtt{a}_1)h(\mathtt{a}_2) \cdots h(\mathtt{a}_n)$$

For a language $L \subseteq \Sigma^*$,

$$h(L) = \{h(w) \mid w \in L\} \subseteq \Gamma^*$$

Example

Let $\Sigma = \{0, 1\}$, $\Gamma = \{\mathtt{a}, \mathtt{b}\}$, and $h(0) = \mathtt{ab}$, $h(1) = \mathtt{a}$. Then,

$$h(10) = \mathtt{aab} \qquad h(010) = \mathtt{abaab} \qquad h(1100) = \mathtt{aaabab}$$

Closure under Homomorphism

Theorem (Closure under Homomorphism)

If h is a homomorphism and L is a regular language, then so is $h(L)$.

Proof) Let R be the regular expression such that $L(R) = L$. Then, we can define its homomorphic regular expression $h(R)$ as follows:

- If $R = \varnothing$, then $h(R) = \varnothing$.
- If $R = \epsilon$, then $h(R) = \epsilon$.
- If $R = \mathtt{a}$, then $h(R) = h(\mathtt{a})$.
- If $R = R_0 \,|\, R_1$, then $h(R) = h(R_0) \,|\, h(R_1)$.
- If $R = R_0 R_1$, then $h(R) = h(R_0) h(R_1)$.
- If $R = R_0^*$, then $h(R) = (h(R_0))^*$.
- If $R = (R_0)$, then $h(R) = (h(R_0))$. □

$$h(0) = \mathtt{ab}$$
$$h(1) = \mathtt{a}$$

$$R = 0(0|1)^*0^*$$

$$h(R) = \mathtt{ab}(\mathtt{ab}|\mathtt{a})^*(\mathtt{ab})^*$$

Closure under Inverse Homomorphism

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Definition (Inverse Homomorphism)

Suppose Σ and Γ are two finite sets of symbols. For a given language $L \subseteq \Gamma^*$ and a homomorphism $h : \Sigma \to \Gamma^*$,

$$h^{-1}(L) = \{w \in \Sigma^* \mid h(w) \in L\} \subseteq \Sigma^*$$

Example

Let $\Sigma = \{0, 1\}$, $\Gamma = \{\mathtt{a}, \mathtt{b}\}$, and $h(0) = \mathtt{ba}$, $h(1) = \mathtt{a}$. Consider the following language $L \subseteq \Gamma^*$:

$$L = \{w\mathtt{aa} \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$$

Then, $01 \in h^{-1}(L)$ because $h(01) = \mathtt{baa} \in L$.

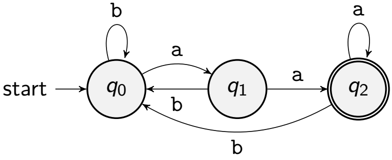
However, $10 \notin h^{-1}(L)$ because $h(10) = \mathtt{aba} \notin L$.

Closure under Inverse Homomorphism

$L = \{w\mathtt{aa} \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$ $\qquad h : \Sigma \to \Gamma^*.\ h(0) = \mathtt{ba} \land h(1) = \mathtt{a}$

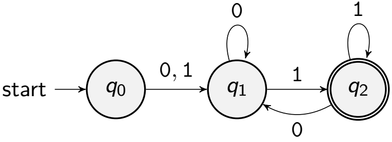
Is the **inverse homomorphism** $h^{-1}(L)$ of the above regular language L also regular ($\Sigma = \{0, 1\}$ and $\Gamma = \{\mathtt{a}, \mathtt{b}\}$)? **Yes!**

$D =$



The above DFA D accepts the language L.

$h^{-1}(D) =$



The key idea is to construct a new DFA $h^{-1}(D)$ by **reconstructing** the **transitions** by following the path $h(x)$ for each symbol in $x \in \Sigma$.

Closure under Inverse Homomorphism

Theorem (Closure under Inverse Homomorphism)

If $h : \Sigma \to \Gamma^*$ is a homomorphism and $L \subseteq \Gamma^*$ is a regular language, then so is $h^{-1}(L)$.

Proof) Let $D = (Q, \Gamma, \delta, q_0, F)$ be the DFA such that $L(D) = L$.

Consider the following DFA:

$$h^{-1}(D) = (Q, \Sigma, \delta', q_0, Q).$$

where $\forall q \in Q, x \in \Sigma.\ \delta'(q, x) = \delta^*(q, h(x))$. Then, $\forall w = x_1 \cdots x_n \in \Sigma^*$.

$$\begin{array}{rcl}
w \in L(h^{-1}(D)) & \iff & (\delta')^*(q_0, w) \in F \\
& \iff & \delta'(\ldots(\delta'(\delta'(q_0, x_1), x_2), \ldots, x_n)) \in F \\
& \iff & \delta(\ldots(\delta(\delta(q_0, h(x_1)), h(x_2)), \ldots, h(x_n)) \in F \\
& \iff & \delta^*(q_0, h(x_1) \cdots h(x_n)) \in F \\
& \iff & \delta^*(q_0, h(w)) \in F \\
& \iff & h(w) \in L(D) \\
& \iff & h(w) \in L
\end{array}$$

Summary

1. Closure Properties of Regular Languages
   - Union, Concatenation, and Kleene Star
   - Complement
   - Intersection
   - Difference
   - Reversal
   - Homomorphism
   - Inverse Homomorphism

Next Lecture

- The Pumping Lemma for Regular Languages

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