Lecture 3 – Deterministic Finite Automata (DFA)

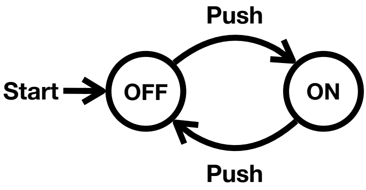
COSE215: Theory of Computation

Jihyeok Park

PLRG

2024 Spring

Recall



1. Mathematical Preliminaries
   - Mathematical Notations
   - Inductive Proofs
   - Notations in Languages
2. Basic Introduction of Scala
   - Basic Features
   - Object-Oriented Programming (OOP)
   - Functional Programming (FP)
   - Immutable Collections (Data Structures)

Contents

Definition of DFA

Definition (Deterministic Finite Automata (DFA))

A **deterministic finite automaton** (DFA) is a 5-tuple:

$$D = (Q, \Sigma, \delta, q_0, F)$$

- Q is a finite set of **states**
- Σ is a finite set of **symbols**
- $\delta : Q \times \Sigma \to Q$ is the **transition function**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

Definition of DFA

Definition (Deterministic Finite Automata (DFA))

A **deterministic finite automaton** (DFA) is a 5-tuple:

$$D = (Q, \Sigma, \delta, q_0, F)$$

- Q is a finite set of **states**
- Σ is a finite set of **symbols**
- $\delta : Q \times \Sigma \to Q$ is the **transition function**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

$$D_1 = (\{q_0, q_1, q_2\}, \{\mathtt{a}, \mathtt{b}\}, \delta, q_0, \{q_2\})$$

$$\begin{array}{lll} \delta(q_0, \mathtt{a}) = q_1 & \delta(q_1, \mathtt{a}) = q_2 & \delta(q_2, \mathtt{a}) = q_2 \\ \delta(q_0, \mathtt{b}) = q_0 & \delta(q_1, \mathtt{b}) = q_0 & \delta(q_2, \mathtt{b}) = q_0 \end{array}$$

Definition of DFA

```scala
// The type definitions of states and symbols
type State = Int
type Symbol = Char
// The definition of DFA
case class DFA(
  states: Set[State],
  symbols: Set[Symbol],
  trans: Map[(State, Symbol), State],
  initState: State,
  finalStates: Set[State],
)
```

Definition of DFA

```scala
// The type definitions of states and symbols
type State = Int
type Symbol = Char
// The definition of DFA
case class DFA(
  states: Set[State],
  symbols: Set[Symbol],
  trans: Map[(State, Symbol), State],
  initState: State,
  finalStates: Set[State],
)
```

```scala
// An example of DFA
val dfa1: DFA = DFA(
  states      = Set(0, 1, 2),
  symbols     = Set('a', 'b'),
  trans       = Map(
    (0, 'a') -> 1, (1, 'a') -> 2, (2, 'a') -> 2,
    (0, 'b') -> 0, (1, 'b') -> 0, (2, 'b') -> 0,
  ),
  initState   = 0,
  finalStates = Set(2),
)
```

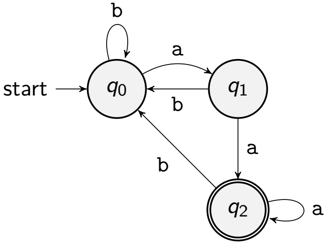
Transition Diagram and Transition Table

$$D_1 = (\{q_0, q_1, q_2\}, \{\mathtt{a}, \mathtt{b}\}, \delta, q_0, \{q_2\})$$

$$\delta(q_0, \mathtt{a}) = q_1 \qquad \delta(q_1, \mathtt{a}) = q_2 \qquad \delta(q_2, \mathtt{a}) = q_2$$

$$\delta(q_0, \mathtt{b}) = q_0 \qquad \delta(q_1, \mathtt{b}) = q_0 \qquad \delta(q_2, \mathtt{b}) = q_0$$

Transition Diagram



Transition Table

q	a	b
$\to q_0$	q_1	q_0
q_1	q_2	q_0
$*q_2$	q_2	q_0

Extended Transition Function

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

Extended Transition Function

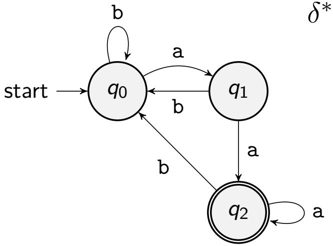
PLRG

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

$D_1 =$

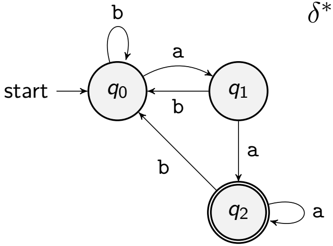


$\delta^*(q_0, \texttt{baa})$

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

$D_1 =$ 

$$\delta^*(q_0, \texttt{baa}) = \delta^*(\delta(q_0, \texttt{b}), \texttt{aa}) = \delta^*(q_0, \texttt{aa})$$

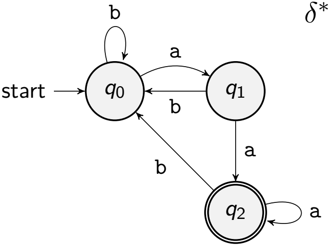
Extended Transition Function

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

$D_1 =$



$$\delta^*(q_0, \mathtt{baa}) = \delta^*(\delta(q_0, \mathtt{b}), \mathtt{aa}) = \delta^*(q_0, \mathtt{aa})$$

$$= \delta^*(\delta(q_0, \mathtt{a}), \mathtt{a}) = \delta^*(q_1, \mathtt{a})$$

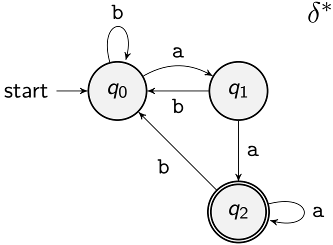
Extended Transition Function

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

$D_1 =$



$$
\begin{aligned}
\delta^*(q_0, \texttt{baa}) &= \delta^*(\delta(q_0, \texttt{b}), \texttt{aa}) = \delta^*(q_0, \texttt{aa}) \\
&= \delta^*(\delta(q_0, \texttt{a}), \texttt{a}) \quad = \delta^*(q_1, \texttt{a}) \\
&= \delta^*(\delta(q_1, \texttt{a}), \epsilon) \quad = \delta^*(q_2, \epsilon)
\end{aligned}
$$

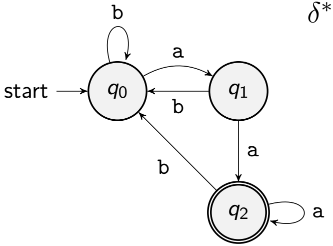
Extended Transition Function

PLRG

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

$D_1 =$ 

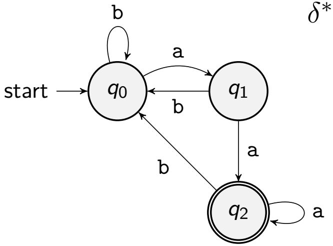
$$
\begin{aligned}
\delta^*(q_0, \mathtt{baa}) &= \delta^*(\delta(q_0, \mathtt{b}), \mathtt{aa}) = \delta^*(q_0, \mathtt{aa}) \\
&= \delta^*(\delta(q_0, \mathtt{a}), \mathtt{a}) \quad = \delta^*(q_1, \mathtt{a}) \\
&= \delta^*(\delta(q_1, \mathtt{a}), \epsilon) \quad = \delta^*(q_2, \epsilon) \\
&= q_2
\end{aligned}
$$

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

$D_1 =$

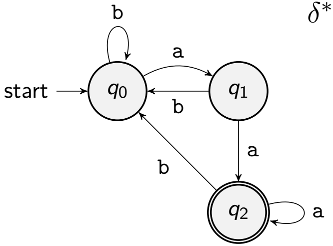


$\delta^*(q_0, \mathtt{aba})$

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

$D_1 =$ 

$$\delta^*(q_0, \texttt{aba}) = \delta^*(\delta(q_0, \texttt{a}), \texttt{ba}) = \delta^*(q_1, \texttt{ba})$$

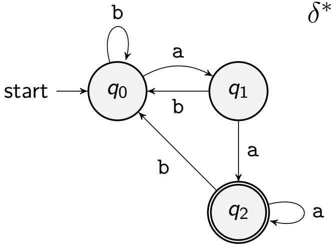
Extended Transition Function

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

$D_1 =$



$$\delta^*(q_0, \mathtt{aba}) = \delta^*(\delta(q_0, \mathtt{a}), \mathtt{ba}) = \delta^*(q_1, \mathtt{ba})$$
$$= \delta^*(\delta(q_1, \mathtt{b}), \mathtt{a}) = \delta^*(q_0, \mathtt{a})$$

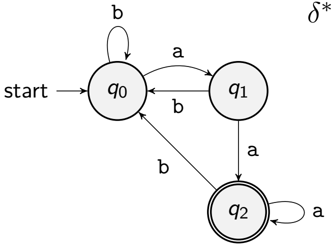
Extended Transition Function

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

$D_1 =$



$$
\begin{aligned}
\delta^*(q_0, \mathtt{aba}) &= \delta^*(\delta(q_0, \mathtt{a}), \mathtt{ba}) &&= \delta^*(q_1, \mathtt{ba}) \\
&= \delta^*(\delta(q_1, \mathtt{b}), \mathtt{a}) &&= \delta^*(q_0, \mathtt{a}) \\
&= \delta^*(\delta(q_0, \mathtt{a}), \epsilon) &&= \delta^*(q_1, \epsilon)
\end{aligned}
$$

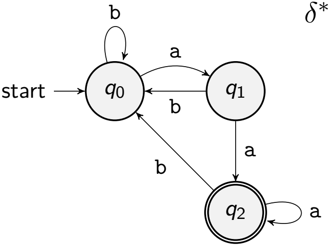
Extended Transition Function

PLRG

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, xw) = \delta^*(\delta(q, x), w)$ where $x \in \Sigma$, $w \in \Sigma^*$

$D_1 =$ 

$$
\begin{aligned}
\delta^*(q_0, \mathtt{aba}) &= \delta^*(\delta(q_0, \mathtt{a}), \mathtt{ba}) = \delta^*(q_1, \mathtt{ba}) \\
&= \delta^*(\delta(q_1, \mathtt{b}), \mathtt{a}) = \delta^*(q_0, \mathtt{a}) \\
&= \delta^*(\delta(q_0, \mathtt{a}), \epsilon) = \delta^*(q_1, \epsilon) \\
&= q_1
\end{aligned}
$$

Extended Transition Function

```
// The type definition of words
type Word = String
// The extended transition function of DFA
case class DFA(...):
  ...
  def extTrans(q: State, w: Word): State = w match
    case ""     => q
    case x <| w => extTrans(trans(q, x), w)

// An example transition for a word "baa"
dfa1.extTrans(0, "baa") // 2
// An example transition for a word "aba"
dfa1.extTrans(0, "aba") // 1
```

where <| is a helper function to extract the first symbol and the rest of the word but you do not need to understand the details of how it works.

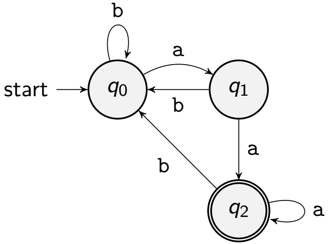
```
// A helper function to extract first symbol and rest of word
object `<|` { def unapply(w: Word) = w.headOption.map((_, w.drop(1))) }
```

Acceptance of a Word

Definition (Acceptance of a Word)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, we say that D **accepts** a word $w \in \Sigma^*$ if and only if $\delta^*(q_0, w) \in F$

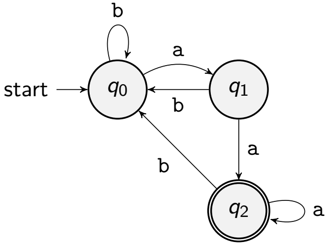
$D_1 =$



Definition (Acceptance of a Word)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, we say that D **accepts** a word $w \in \Sigma^*$ if and only if $\delta^*(q_0, w) \in F$

$D_1 =$



$$\delta^*(q_0, \mathtt{baa}) = q_2 \in F$$

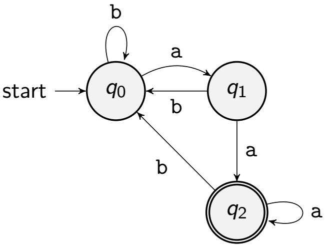
It means that D **accepts** baa.

Acceptance of a Word

Definition (Acceptance of a Word)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, we say that D **accepts** a word $w \in \Sigma^*$ if and only if $\delta^*(q_0, w) \in F$

$D_1 =$



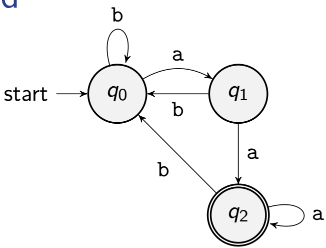
$$\delta^*(q_0, \texttt{baa}) = q_2 \in F$$

It means that D **accepts** baa.

$$\delta^*(q_0, \texttt{aba}) = q_1 \notin F$$

It means that D does **not accept** aba.

Acceptance of a Word

$D_1 =$ 


```
// The acceptance of a word by DFA
case class DFA(...):
  ...
  def accept(w: Word): Boolean =
    finalStates.contains(extTrans(initState, w))

// An example acceptance of a word "baa"
dfa1.accept("baa") // true

// An example non-acceptance of a word "aba"
dfa1.accept("aba") // false
```

Language of DFA (Regular Language)

Definition (Language of DFA)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **language** of D is defined as:

$$L(D) = \{w \in \Sigma^* \mid D \text{ accepts } w\}$$

Language of DFA (Regular Language)

Definition (Language of DFA)

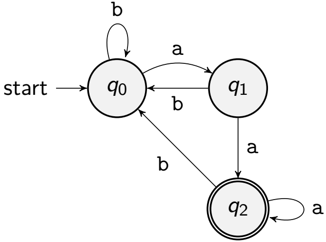
For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **language** of D is defined as:

$$L(D) = \{w \in \Sigma^* \mid D \text{ accepts } w\}$$

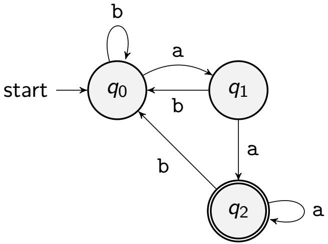
Definition (Regular Language)

A language L is **regular** if and only if there exists a DFA D such that $L(D) = L$

Example 1

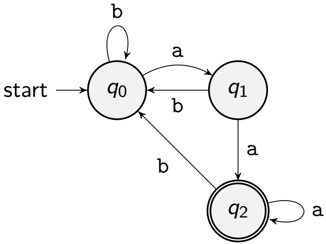
$D_1 =$ 

Example 1

$D_1 =$ 

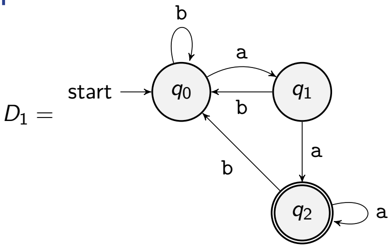
$$\delta^*(q_0, \texttt{baa}) = q_2 \in F$$

Example 1

$D_1 =$ 

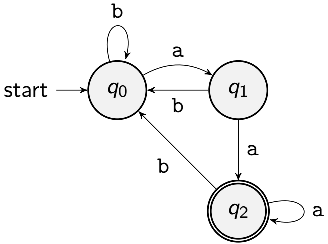
$$\Rightarrow \quad \begin{array}{l} \delta^*(q_0, \texttt{baa}) = q_2 \in F \\ D_1 \text{ accepts } \texttt{baa} \end{array}$$

Example 1



$$\delta^*(q_0, \texttt{baa}) = q_2 \in F$$
$$\Rightarrow \quad D_1 \text{ accepts } \texttt{baa}$$
$$\Rightarrow \quad \texttt{baa} \in L(D_1)$$

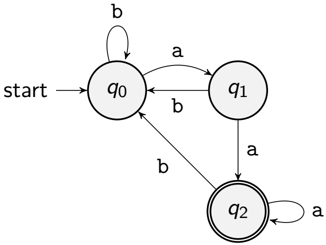
Example 1

$D_1 =$ 

$$\begin{aligned} & \delta^*(q_0, \texttt{baa}) = q_2 \in F \\ \Rightarrow \quad & D_1 \text{ accepts } \texttt{baa} \\ \Rightarrow \quad & \texttt{baa} \in L(D_1) \end{aligned}$$

$$\epsilon, \texttt{a}, \texttt{b}, \texttt{ab}, \texttt{ba}, \texttt{bb}, \texttt{aab}, \texttt{aba}, \texttt{abb}, \texttt{bab}, \cdots \notin L(D_1)$$

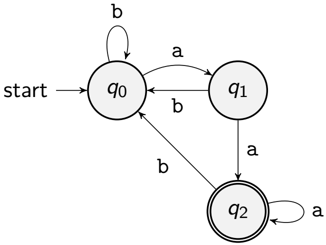
Example 1

$D_1 =$ 

$$\begin{aligned} & \delta^*(q_0, \texttt{baa}) = q_2 \in F \\ \Rightarrow \quad & D_1 \text{ accepts } \texttt{baa} \\ \Rightarrow \quad & \texttt{baa} \in L(D_1) \end{aligned}$$

$$\epsilon, \texttt{a}, \texttt{b}, \texttt{ab}, \texttt{ba}, \texttt{bb}, \texttt{aab}, \texttt{aba}, \texttt{abb}, \texttt{bab}, \cdots \notin L(D_1)$$
$$\texttt{aa}, \texttt{aaa}, \texttt{baa}, \texttt{aaaa}, \texttt{abaa}, \texttt{baaa}, \texttt{bbaa}, \cdots \in L(D_1)$$

Example 1

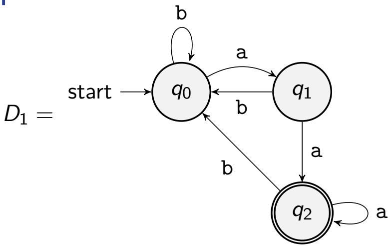
$D_1 =$ 

$$\delta^*(q_0, \texttt{baa}) = q_2 \in F$$
$$\Rightarrow \quad D_1 \text{ accepts } \texttt{baa}$$
$$\Rightarrow \quad \texttt{baa} \in L(D_1)$$

$$\epsilon, \texttt{a}, \texttt{b}, \texttt{ab}, \texttt{ba}, \texttt{bb}, \texttt{aab}, \texttt{aba}, \texttt{abb}, \texttt{bab}, \cdots \notin L(D_1)$$
$$\texttt{aa}, \texttt{aaa}, \texttt{baa}, \texttt{aaaa}, \texttt{abaa}, \texttt{baaa}, \texttt{bbaa}, \cdots \in L(D_1)$$

$$L(D_1) = \{w\texttt{aa} \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$$

Example 1



$$\begin{aligned} & \delta^*(q_0, \mathtt{baa}) = q_2 \in F \\ \Rightarrow\quad & D_1 \text{ accepts } \mathtt{baa} \\ \Rightarrow\quad & \mathtt{baa} \in L(D_1) \end{aligned}$$

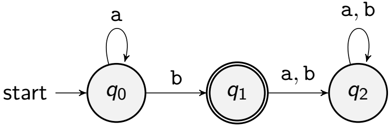
$$\epsilon, \mathtt{a}, \mathtt{b}, \mathtt{ab}, \mathtt{ba}, \mathtt{bb}, \mathtt{aab}, \mathtt{aba}, \mathtt{abb}, \mathtt{bab}, \cdots \notin L(D_1)$$
$$\mathtt{aa}, \mathtt{aaa}, \mathtt{baa}, \mathtt{aaaa}, \mathtt{abaa}, \mathtt{baaa}, \mathtt{bbaa}, \cdots \in L(D_1)$$

$$L(D_1) = \{w\mathtt{aa} \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$$

- q_0 represents ϵ or any word ending with b
- q_1 represents any word ending with exactly one a
- q_2 represents any word ending with at least two a's

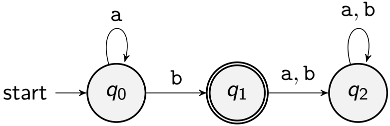
Example 2

$D_2 \ =$



Example 2

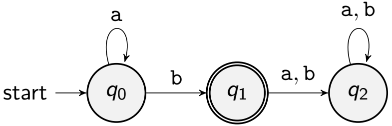
$D_2 \;=$



$$\epsilon, \texttt{a}, \texttt{aa}, \texttt{ba}, \texttt{bb}, \texttt{aaa}, \texttt{aba}, \texttt{abb}, \texttt{baa}, \texttt{bab}, \texttt{bba}, \cdots \notin L(D_2)$$

Example 2

$D_2 =$

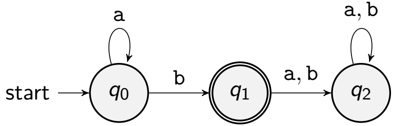


ϵ, a, aa, ba, bb, aaa, aba, abb, baa, bab, bba, $\cdots \notin L(D_2)$

b, ab, aab, aaab, aaaab, aaaaab, aaaaaab, $\cdots \in L(D_2)$

Example 2

$D_2 =$

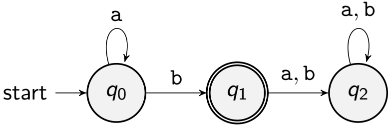


$$\epsilon, \texttt{a}, \texttt{aa}, \texttt{ba}, \texttt{bb}, \texttt{aaa}, \texttt{aba}, \texttt{abb}, \texttt{baa}, \texttt{bab}, \texttt{bba}, \cdots \notin L(D_2)$$
$$\texttt{b}, \texttt{ab}, \texttt{aab}, \texttt{aaab}, \texttt{aaaab}, \texttt{aaaaab}, \texttt{aaaaaab}, \cdots \in L(D_2)$$

$$L(D_2) = \{\texttt{a}^n\texttt{b} \mid n \geq 0\}$$

Example 2

$D_2 =$



$$\epsilon, \mathtt{a}, \mathtt{aa}, \mathtt{ba}, \mathtt{bb}, \mathtt{aaa}, \mathtt{aba}, \mathtt{abb}, \mathtt{baa}, \mathtt{bab}, \mathtt{bba}, \cdots \notin L(D_2)$$

$$\mathtt{b}, \mathtt{ab}, \mathtt{aab}, \mathtt{aaab}, \mathtt{aaaab}, \mathtt{aaaaab}, \mathtt{aaaaaab}, \cdots \in L(D_2)$$

$$L(D_2) = \{\mathtt{a}^n\mathtt{b} \mid n \geq 0\}$$

- q_0 represents zero or more a's
- q_1 represents zero or more a's followed by b
- q_2 represents any other words

Example 3

Theorem

The language $L = \{w \in \{0, 1\}^* \mid w$ is a binary format (allowing leading zeros) of natural numbers divisible by $3\}$ is regular.

Proof)

Example 3

Theorem

The language $L = \{w \in \{0, 1\}^* \mid w$ *is a binary format (allowing leading zeros) of natural numbers divisible by* $3\}$ *is regular.*

Proof) You need to construct a DFA D_2 such that $L(D_2) = L$.

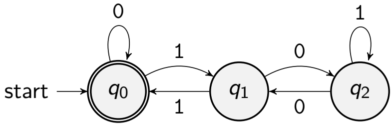
Example 3

Theorem

The language $L = \{w \in \{0, 1\}^* \mid w$ is a binary format (allowing leading zeros) of natural numbers divisible by $3\}$ is regular.

Proof) You need to construct a DFA D_2 such that $L(D_2) = L$. Consider the following DFA D_2:

$D_3 =$



□

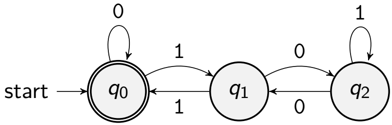
Example 3

Theorem

The language $L = \{w \in \{0, 1\}^* \mid w$ is a binary format (allowing leading zeros) of natural numbers divisible by $3\}$ is regular.

Proof) You need to construct a DFA D_2 such that $L(D_2) = L$. Consider the following DFA D_2:

$D_3 =$



$\square$

- q_0 represents binary format of an integer n s.t. $n \equiv 0 \pmod 3$
- q_1 represents binary format of an integer n s.t. $n \equiv 1 \pmod 3$
- q_2 represents binary format of an integer n s.t. $n \equiv 2 \pmod 3$

Example 4

Theorem

The language $L = \{a^n b^n \mid n \geq 0\}$ *is regular.*

You need to construct a DFA D such that $L(D) = L$.

Example 4

Theorem

The language $L = \{a^n b^n \mid n \geq 0\}$ *is regular.*

You need to construct a DFA D such that $L(D) = L$. However, it is impossible because L is actually not regular.

Example 4

Theorem

The language $L = \{a^n b^n \mid n \geq 0\}$ is regular.

You need to construct a DFA D such that $L(D) = L$. However, it is impossible because L is actually not regular.

Then, is it possible to prove that L is not regular?

Example 4

Theorem

The language $L = \{a^n b^n \mid n \geq 0\}$ is regular.

You need to construct a DFA D such that $L(D) = L$. However, it is impossible because L is actually not regular.

Then, is it possible to prove that L is not regular?

Yes, it is possible BUT you will learn how to prove it (using Pumping Lemma) later in this course.

Summary

1. Deterministic Finite Automata (DFA)
    - Definition
    - Transition Diagram and Transition Table
    - Extended Transition Function
    - Acceptance of a Word
    - Language of DFA (Regular Language)
    - Examples

Next Lecture

- Nondeterministic Finite Automata (NFA)

Jihyeok Park
jihyeok_park@korea.ac.kr
https://plrg.korea.ac.kr