

# Lecture 8 – Closure Properties of Regular Languages

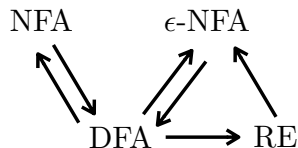
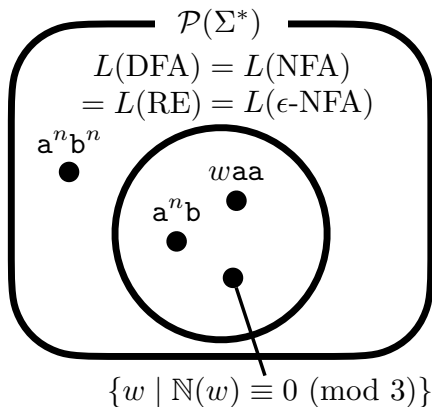
## COSE215: Theory of Computation

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2025 Spring

- Regular Languages



## 1. Closure Properties of Regular Languages

Union, Concatenation, and Kleene Star

Complement

Intersection

Difference

Reversal

Homomorphism

Inverse Homomorphism

Let's consider a **regular** language. For example,

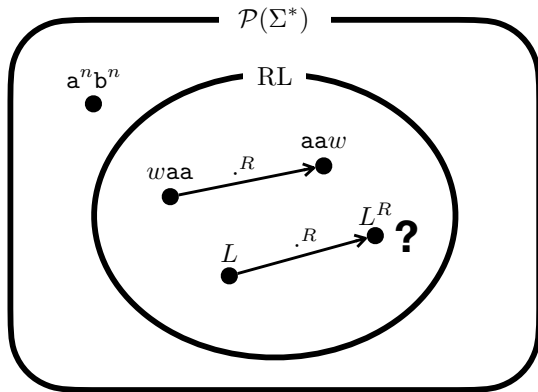
$$L = \{waa \mid w \in \{a, b\}^*\}$$

Then, is its **reverse** language  $L^R$  also **regular**?

$$L^R = \{aaw \mid w \in \{a, b\}^*\}$$

Yes! We can construct a regular expression whose language is  $L^R$  as:

$$L(aa(a|b)^*) = L^R = \{aaw \mid w \in \{a, b\}^*\}$$



Then, for any regular language  $L$ , is  $L^R$  always regular? **Yes!**

The class of regular languages is **closed** under the **reversal** operator.

In this lecture, we will discuss and prove the **closure properties** of regular languages for various language operators.

## Definition (Closure Properties)

The class of regular languages is **closed** under an  $n$ -ary operator  $\text{op}$  if and only if  $\text{op}(L_1, \dots, L_n)$  is regular for any regular languages  $L_1, \dots, L_n$ . We say that such properties are **closure properties** of regular languages.

To prove the closure properties for the  $n$ -ary operator  $\text{op}$ , we need to provide a way to do the following for any regular languages  $L_1, \dots, L_n$ :

- 1 Construct a regular expression  $R$  whose language is  $L(R) = \text{op}(L(R_1), \dots, L(R_n))$  for any regular expressions  $R_1, \dots, R_n$ .
- 2 Construct a finite automaton  $A$  whose language is  $L(A) = \text{op}(L(A_1), \dots, L(A_n))$  for any finite automata  $A_1, \dots, A_n$ .

In this lecture, we will prove the closure properties of regular languages for the following operators:

- Union
- Concatenation
- Kleene Star
- Complement
- Intersection
- Difference
- Reversal
- Homomorphism
- Inverse Homomorphism

**Theorem (Closure under Union, Concatenation, and Kleene Star)**

*If  $L_1$  and  $L_2$  are regular languages, then so is  $L_1 \cup L_2$ ,  $L_1 L_2$ , and  $L_1^*$ .*

**Proof)** Let  $R_1$  and  $R_2$  be the regular expressions such that  $L(R_1) = L_1$  and  $L(R_2) = L_2$ , respectively.

Consider the following regular expression:

$$R_1 | R_2$$

$$R_1 R_2$$

$$R^*$$

Then, by the definition of the union ( $\cup$ ), concatenation ( $\cdot$ ), and Kleene star ( $*$ ) operators for regular expressions,

$$L(R_1 | R_2) = L_1 \cup L_2$$

$$L(R_1 R_2) = L_1 L_2$$

$$L(R^*) = L^*$$

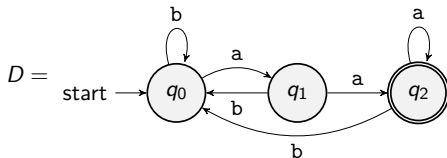
So, we proved that the class of regular languages are **closed** under the **union**, **concatenation**, and **Kleene star** operators. □



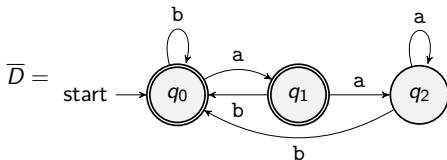
# Closure under Complement

Consider a regular language  $L = \{waa \mid w \in \{a, b\}^*\}$ .

Is its **complement**  $\bar{L} = \Sigma^* \setminus L$  also regular? **Yes!**



First, consider the above DFA  $D$  accepting the language  $L$ .



The key idea is to construct a new DFA  $\bar{D}$  by **swapping** the **final** and **non-final** states of the original DFA:

## Theorem (Closure under Complement)

*If  $L$  is a regular language, then so is  $\bar{L}$ .*

**Proof)** Let  $D = (Q, \Sigma, \delta, q_0, F)$  be the DFA such that  $L(D) = L$ . Consider the following DFA:

$$\bar{D} = (Q, \Sigma, \delta, q_0, Q \setminus F).$$

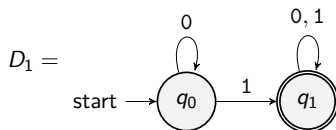
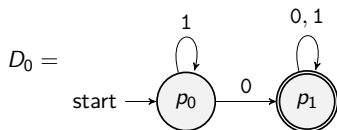
Then,

$$\begin{aligned} \forall w \in \Sigma^*, w \in L(\bar{D}) &\iff \delta^*(q_0, w) \in Q \setminus F \\ &\iff \delta^*(q_0, w) \notin F \\ &\iff w \notin L(D) \\ &\iff w \notin L \\ &\iff w \in \bar{L} \end{aligned}$$

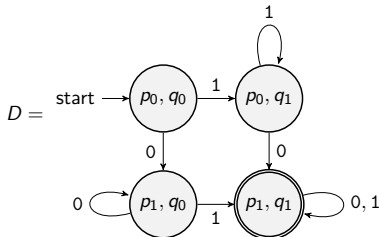


$$L_1 = \{w \in \{0, 1\}^* \mid w \text{ has } 0\} \quad L_2 = \{w \in \{0, 1\}^* \mid w \text{ has } 1\}$$

Is the **intersection** of two regular languages  $L_1 \cap L_2$  also regular? **Yes!**



First, consider the above DFAs  $D_0$  and  $D_1$  accepting the languages  $L_1$  and  $L_2$ , respectively.



The key idea is to construct a new DFA  $D$  by **combining** them with their **pair of states** as its states.

## Theorem (Closure under Intersection)

*If  $L_0$  and  $L_1$  are regular languages, then so is  $L_0 \cap L_1$ .*

**Proof)** Let  $D_0 = (Q_0, \Sigma, \delta_0, q_0, F_0)$  and  $D_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$  be the DFA such that  $L(D_0) = L_0$  and  $L(D_1) = L_1$ . Consider the following DFA:

$$D = (Q_0 \times Q_1, \Sigma, \delta, (q_0, q_1), F_0 \times F_1).$$

where  $\forall q \in Q_0, q' \in Q_1, a \in \Sigma. \delta((q, q'), a) = (\delta_0(q, a), \delta_1(q', a))$ . Then,

$$\begin{aligned} \forall w \in \Sigma^*, w \in L(D) &\iff \delta^*((q_0, q_1), w) \in F_0 \times F_1 \\ &\iff \delta^*(q_0, w) \in F_0 \text{ and } \delta^*(q_1, w) \in F_1 \\ &\iff w \in L(D_0) \text{ and } w \in L(D_1) \\ &\iff w \in L(D_0) \cap L(D_1) \\ &\iff w \in L_0 \cap L_1 \end{aligned}$$



## Theorem (Closure under Intersection)

*If  $L_0$  and  $L_1$  are regular languages, then so is  $L_0 \cap L_1$ .*

**Proof)** Another proof is to use De Morgan's law:

$$L_0 \cap L_1 = \overline{\overline{L_0} \cup \overline{L_1}}$$

Since we already know that the regular languages are closed under complement and union, we are done. □

## Theorem (Closure under Difference)

*If  $L_0$  and  $L_1$  are regular languages, then so is  $L_0 \setminus L_1$ .*

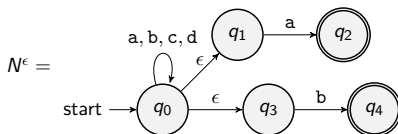
**Proof)** Similarly, we can use the following fact:

$$L_0 \setminus L_1 = L_0 \cap \overline{L_1}$$

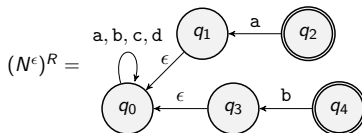
Since we already know that the regular languages are closed under complement and intersection, we are done. □

$$L = \{wa \text{ or } wb \mid w \in \{a, b, c, d\}^*\}$$

Is the **reversal**  $L^R$  of the above regular language  $L$  also regular? **Yes!**



The above  $\epsilon$ -NFA  $N^\epsilon$  accepts the language  $L$ .

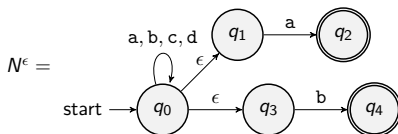


The key idea is to construct a new  $\epsilon$ -NFA  $(N^\epsilon)^R$  by

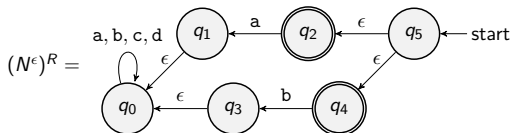
- 1 **reversing** the direction of the transitions
- 2
- 3

$$L = \{wa \text{ or } wb \mid w \in \{a, b, c, d\}^*\}$$

Is the **reversal**  $L^R$  of the above regular language  $L$  also regular? **Yes!**



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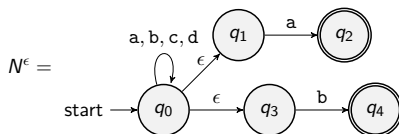
The key idea is to construct a new  $\epsilon$ -NFA  $(N^\epsilon)^R$  by

- 1 **reversing** the direction of the transitions
- 2 **adding** new initial state having  $\epsilon$ -transitions to the original final states
- 3

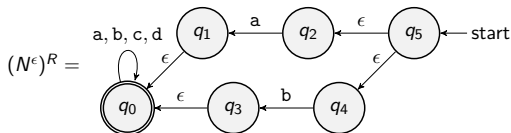


$$L = \{wa \text{ or } wb \mid w \in \{a, b, c, d\}^*\}$$

Is the **reversal**  $L^R$  of the above regular language  $L$  also regular? **Yes!**



The above  $\epsilon$ -NFA  $N^\epsilon$  accepts the language  $L$ .



The key idea is to construct a new  $\epsilon$ -NFA  $(N^\epsilon)^R$  by

- ① **reversing** the direction of the transitions
- ② **adding** new initial state having  $\epsilon$ -transitions to the original final states
- ③ **change** original initial state to the unique new final state

## Theorem (Closure under Reversal)

*If  $L$  is a regular language, then so is  $L^R$ .*

**Proof)** Let  $N^\epsilon = (Q, \Sigma, \delta, q_0, F)$  be the  $\epsilon$ -NFA such that  $L(N^\epsilon) = L$ . Consider the following

$$(N^\epsilon)^R = (Q \uplus \{q_s\}, \Sigma, \delta^R, q_s, \{q_0\})$$

where

$$\begin{aligned}\forall q \in Q. \forall a \in \Sigma. \delta^R(q, a) &= \{q' \in Q \mid q \in \delta(q', a)\} \\ \forall q \in Q. \delta^R(q, \epsilon) &= \{q' \in Q \mid q \in \delta(q', \epsilon)\} \\ \forall a \in \Sigma. \delta^R(q_s, a) &= \emptyset \\ \delta^R(q_s, \epsilon) &= F\end{aligned}$$



## Theorem (Closure under Reversal)

*If  $L$  is a regular language, then so is  $L^R$ .*

**Proof)** Another proof is to use the structural induction on the regular expressions. Let  $R$  be a regular expression. Then, we can define its reversed regular expression  $R^R$  as follows:

- If  $R = \emptyset$ , then  $R^R = \emptyset$ .
- If  $R = \epsilon$ , then  $R^R = \epsilon$ .
- If  $R = a$ , then  $R^R = a$ .
- If  $R = R_0 | R_1$ , then  $R^R = R_0^R | R_1^R$ .
- If  $R = R_0 R_1$ , then  $R^R = R_1^R R_0^R$ .
- If  $R = R_0^*$ , then  $R^R = (R_0^R)^*$ .
- If  $R = (R_0)$ , then  $R^R = (R_0^R)$ . □

$$R = ab(cd)^* | ef$$

$$R^R = (dc)^* ba | fe$$

## Definition (Homomorphism)

Suppose  $\Sigma_0$  and  $\Sigma_1$  are two finite sets of symbols. Then, a function

$$h : \Sigma_0 \rightarrow \Sigma_1^*$$

is called a **homomorphism**. For a given word  $w = a_1 a_2 \cdots a_n \in \Sigma_0^*$ ,

$$h(w) = h(a_1)h(a_2) \cdots h(a_n)$$

For a language  $L \subseteq \Sigma_0^*$ ,

$$h(L) = \{h(w) \mid w \in L\} \subseteq \Sigma_1^*$$

## Example

Let  $\Sigma_0 = \{0, 1\}$ ,  $\Sigma_1 = \{a, b\}$ , and  $h(0) = ab$ ,  $h(1) = a$ . Then,

$$h(10) = aab \quad h(010) = abaab \quad h(1100) = aaabab$$

## Theorem (Closure under Homomorphism)

*If  $h$  is a homomorphism and  $L$  is a regular language, then so is  $h(L)$ .*

**Proof)** Let  $R$  be the regular expression such that  $L(R) = L$ . Then, we can define its homomorphic regular expression  $h(R)$  as follows:

- If  $R = \emptyset$ , then  $h(R) = \emptyset$ .

- If  $R = \epsilon$ , then  $h(R) = \epsilon$ .

- If  $R = a$ , then  $h(R) = h(a)$ .

- If  $R = R_0 | R_1$ , then  $h(R) = h(R_0) | h(R_1)$ .

- If  $R = R_0 R_1$ , then  $h(R) = h(R_0) h(R_1)$ .

- If  $R = R_0^*$ , then  $h(R) = (h(R_0))^*$ .

- If  $R = (R_0)$ , then  $h(R) = (h(R_0))$ . □

$$h(0) = ab$$

$$h(1) = a$$

$$R = 0(0|1)^*0^*$$

$$h(R) = ab(ab|a)^*(ab)^*$$

## Definition (Inverse Homomorphism)

Suppose  $\Sigma_0$  and  $\Sigma_1$  are two finite sets of symbols. For a given language  $L \subseteq \Sigma_1^*$  and a homomorphism  $h : \Sigma_0 \rightarrow \Sigma_1^*$ ,

$$h^{-1}(L) = \{w \in \Sigma_0^* \mid h(w) \in L\} \subseteq \Sigma_0^*$$

## Example

Let  $\Sigma_0 = \{0, 1\}$ ,  $\Sigma_1 = \{a, b\}$ , and  $h(0) = ba$ ,  $h(1) = a$ . Consider the following language  $L \subseteq \Sigma_1^*$ :

$$L = \{waa \mid w \in \{a, b\}^*\}$$

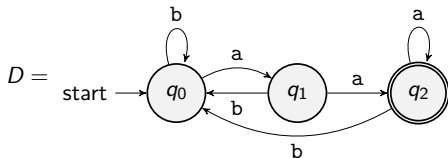
Then,  $01 \in h^{-1}(L)$  because  $h(01) = baa \in L$ .

However,  $10 \notin h^{-1}(L)$  because  $h(10) = aba \notin L$ .

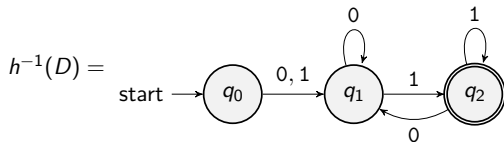
$$L = \{waa \mid w \in \{a, b\}^*\}$$

$$h : \Sigma_0 \rightarrow \Sigma_1^*. h(0) = ba \wedge h(1) = a$$

Is the **inverse homomorphism**  $h^{-1}(L)$  of the above regular language  $L$  also regular ( $\Sigma_0 = \{0, 1\}$  and  $\Sigma_1 = \{a, b\}$ )? **Yes!**



The above DFA  $D$  accepts the language  $L$ .



The key idea is to construct a new DFA  $h^{-1}(D)$  by **reconstructing** the **transitions** by following the path  $h(a)$  for each symbol in  $a \in \Sigma_0$ .

## Theorem (Closure under Inverse Homomorphism)

*If  $h : \Sigma_0 \rightarrow \Sigma_1^*$  is a homomorphism and  $L \subseteq \Sigma_1^*$  is a regular language, then so is  $h^{-1}(L)$ .*

**Proof)** Let  $D = (Q, \Sigma_1, \delta, q_0, F)$  be the DFA such that  $L(D) = L$ .

Consider the following DFA:

$$h^{-1}(D) = (Q, \Sigma_0, \delta', q_0, F).$$

where  $\forall q \in Q, a \in \Sigma_0$ .  $\delta'(q, a) = \delta^*(q, h(a))$ . Then,  $\forall w = a_1 \cdots a_n \in \Sigma_0^*$ .

$$\begin{aligned} w \in L(h^{-1}(D)) &\iff (\delta')^*(q_0, w) \in F \\ &\iff \delta'(\dots(\delta'(\delta'(q_0, a_1), a_2), \dots, a_n)) \in F \\ &\iff \delta(\dots(\delta(\delta(q_0, h(a_1)), h(a_2)), \dots, h(a_n)) \in F \\ &\iff \delta^*(q_0, h(a_1) \cdots h(a_n)) \in F \\ &\iff \delta^*(q_0, h(w)) \in F \\ &\iff h(w) \in L(D) \\ &\iff h(w) \in L \end{aligned}$$



## 1. Closure Properties of Regular Languages

Union, Concatenation, and Kleene Star

Complement

Intersection

Difference

Reversal

Homomorphism

Inverse Homomorphism

- Please see this document for the exercise.

<https://github.com/ku-plrg-classroom/docs/tree/main/cose215/r1-closure>

- Please implement the following functions in `Implementation.scala`.
  - `complementDFA` for the **complement** of a DFA.
  - `intersectDFA` for the **intersection** of two DFAs.
  - `reverseENFA` for the **reverse** of an  $\epsilon$ -NFA.
  - `reverseRE` for the **reverse** of a regular expression.
  - `homRE` for the **homomorphism** of a regular expression.
  - `ihomDFA` for the **inverse homomorphism** of a DFA.
- It is just an exercise, and you **don't need to submit** anything.

- The Pumping Lemma for Regular Languages

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