Lecture 6 – First-Order Functions
COSE212: Programming Languages

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Recall

- **VAE – AE with variables**
    - Evaluation with Environments
    - Interpreter and Natural Semantics

- In this lecture, we will learn **first-order functions**.

- **F1VAE – VAE with first-order functions**
    - Concrete and Abstract Syntax
    - Interpreter and Natural Semantics

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First-Order Functions

Let's calculate the square of several numbers in Scala.

```
3 * 3       // 9
42 * 42     // 1764
2434 * 2434 // 5925796
```

With a **first-order function**, we can avoid the repetition of the code.

```
// A `square` function that takes an integer `n` and returns its square.
def square(n: Int): Int = n * n

square(3)    // 9
square(42)   // 1764
square(2434) // 5924356
```

A **first-order function** consists of 1) a function name, 2) a list of
parameters, and 3) a function body, and it can be **invoked** with
arguments.

First-Order Functions

Most programming languages support **first-order functions**.

- Scala

  ```
  def square(n: Int): Int = n * n
  ```
  ```
  square(3) // 9
  ```

- Python

  ```
  def square(n): return n * n
  ```
  ```
  square(3) # 9
  ```

- C++

  ```
  int square(int n) { return n * n; }
  ```
  ```
  square(3) // 9
  ```

- Rust

  ```
  fn square(n: i32) -> i32 { return n * n; }
  ```
  ```
  square(3) // 9
  ```

- ...

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F1VAE – VAE with First-Order Functions

Now, we want to extend VAE into F1VAE with **first-order functions**.

```
/* F1VAE */
def square(n) = n * n;
square(3) + 2 // 11
```

```
/* F1VAE */
def add3(n) = n + 3;
def mul2(m) = m * 2;
mul2(add3(4)) // 14
```

- An F1VAE **program** is a pair of
  1. a list of **function definitions**
  2. an **expression**

- A **function definition** in F1VAE only has a **single parameter**.

- We extend **expressions** with **function applications**.

Concrete Syntax

Let's define the **concrete syntax** of F1VAE in BNF:

- An F1VAE **program** is a pair of
  1 a list of **function definitions**
  2 an **expression**

- A **function definition** in F1VAE only has a **single parameter**.

- We extend **expressions** with **function applications**.

```
// programs
<program> ::= <fdef>* <expr>
// function definitions
<fdef>    ::= "def" <id> "(" <id> ")" "=" <expr> ";"
// expressions
<expr>    ::= ...
          | "{" <expr> "}"                      // block
          | "val" <id> "=" <expr> ";" <expr>    // variable definition
          | <id>                                // identifier lookup
          | <id> "(" <expr> ")"                 // function application
```

Let's define the **abstract syntax** of F1VAE in BNF and its **ADT** in Scala:

Programs	p	$::=$	$f^*\ e$	(Program)
Function Definitions	$\mathbb{F} \ni f$	$::=$	$\text{def } x(x) = e$	(FunDef)
Expressions	e	$::=$	$\dots$	
		$\mid$	$\text{val } x = e;\ e$	(Val)
		$\mid$	x	(Id)
		$\mid$	$x(e)$	(App)

```scala
// programs
case class Program(fdefs: List[FunDef], expr: Expr)
// function definitions
case class FunDef(name: String, param: String, body: Expr)
// expressions
enum Expr:
  ...
  case Val(name: String, init: Expr, body: Expr)
  case Id(name: String)
  case App(fname: String, arg: Expr) // function application
```

Abstract Syntax

For example, let's **parse** the following F1VAE program:

```
/* F1VAE */
def add3(n) = n + 3;
def mul2(m) = m * 2;
mul2(add3(4))
```

Then, the following **abstract syntax tree (AST)** is produced:

```
Program(
  List(
    FunDef("add3", "n", Add(Id("n"), Num(3))),
    FunDef("mul2", "m", Mul(Id("m"), Num(2)))
  ),
  App("mul2", App("add3", Num(4)))
)
```

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Evaluation with Function Environments

Let's evaluate the following F1VAE program:

```
/* F1VAE */
def add3(n) = n + 3;
def mul2(m) = m * 2;
mul2(add3(4)) // 14
```

How to find the function definition of add3 or mul2?

We need to construct a **function environment** that maps function names to function definitions from the **list of function definitions** in a program.

$$[\texttt{add3} \mapsto f_0, \texttt{mul2} \mapsto f_1]$$

where

$$
\begin{aligned}
f_0 &= \texttt{def add3}(n) = n + 3 \\
f_1 &= \texttt{def mul2}(m) = m * 2
\end{aligned}
$$

For VAE, the interpreter takes an **expression** e with an **environment** σ and returns a number n as the result.

```scala
type Value = BigInt                      // values
type Env = Map[String, Value]            // environments
def interp(expr: Expr, env: Env): Value = ... // interpreter
```

$$\sigma \vdash e \Rightarrow n$$

Now, we extend it to take a **function environment** Λ, a mapping from function names to function definitions, as well:

```scala
type Value = BigInt                      // values
type Env = Map[String, Value]            // environments
type FEnv = Map[String, FunDef]          // function environments
def interp(expr: Expr, env: Env, fenv: FEnv): Value = ... // interpreter
```

$$\sigma, \Lambda \vdash e \Rightarrow n$$

Constructing Function Environments

We have a **list of function definitions** in an F1VAE program, not a **function environment**, a mapping from function names to definitions.

```scala
case class Program(fdefs: List[FunDef], expr: Expr)
type FEnv = Map[String, FunDef]
```

How to construct a FEnv from a List[FunDef]?

```scala
def createFEnv(fdefs: List[FunDef]): FEnv =
  fdefs.foldLeft(Map.empty)((fenv: FEnv, fdef: FunDef) => {
    val fname: String = fdef.name
    if (fenv.contains(fname)) error(s"duplicate function: $fname")
    else fenv + (fname -> fdef)
  })
```

It will throw an error if there are **duplicate function names**:

```scala
createFEvn(List(
  FunDef("add3", "n", Add(Id("n"), Num(3))),
  FunDef("add3", "n", Add(Num(3), Id("n"))),
)) // error: duplicate function: add3
```

For F1VAE, we need to 1) implement the **interpreter**:

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = ???
```

and 2) define the **natural semantics** with environments and **function environments**:

$$\boxed{\sigma, \Lambda \vdash e \Rightarrow n}$$

Programs	p	$::=$	$f^* \; e$	(Program)
Function Definitions	$\mathbb{F} \ni f$	$::=$	def $x(x) = e$	(FunDef)
Expressions	e	$::=$	...	
		$\mid$	$x(e)$	(App)

where

Numbers	$n \in \mathbb{Z}$	(BigInt)
Identifiers	$x \in \mathbb{X}$	(String)
Environments	$\sigma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{Z}$	(Env)
Function Environments	$\Lambda \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{F}$	(FEnv)

Function Application

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
  ...
  case App(f, e) => ???
```

$$\boxed{\sigma, \Lambda \vdash e \Rightarrow n}$$

$$\text{App } \frac{???}{\sigma, \Lambda \vdash x_0(e_1) \Rightarrow \text{???}}$$

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
  ...
  case App(f, e) =>
    val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
    ...
```

$$\sigma, \Lambda \vdash e \Rightarrow n$$

$$x_0 \in \mathsf{Domain}(\Lambda) \qquad \Lambda(x_0) = \mathtt{def}\ x_0(x_1) = e_2$$

$$\text{App}\ \frac{\cdots}{\sigma, \Lambda \vdash x_0(e_1) \Rightarrow \text{???}}$$

Function Application

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
  ...
  case App(f, e) =>
    val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
    ... interp(e, env, fenv) ...
```

$$\sigma, \Lambda \vdash e \Rightarrow n$$

$$\text{App} \; \frac{\begin{array}{cc} x_0 \in \mathsf{Domain}(\Lambda) & \Lambda(x_0) = \mathtt{def} \; x_0(x_1) = e_2 \\ \sigma, \Lambda \vdash e_1 \Rightarrow n_1 & \ldots \end{array}}{\sigma, \Lambda \vdash x_0(e_1) \Rightarrow \mathtt{???}}$$

Function Application

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
  ...
  case App(f, e) =>
    val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
    ... Map(fdef.param -> interp(e, env, fenv)) ...
```

$$\sigma, \Lambda \vdash e \Rightarrow n$$

$$\text{App } \frac{\begin{array}{cc} x_0 \in \mathsf{Domain}(\Lambda) & \Lambda(x_0) = \mathtt{def} \ x_0(x_1) = e_2 \\ \sigma, \Lambda \vdash e_1 \Rightarrow n_1 & \dots [x_1 \mapsto n_1] \dots \end{array}}{\sigma, \Lambda \vdash x_0(e_1) \Rightarrow n_2}$$

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
  ...
  case App(f, e) =>
    val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
    interp(fdef.body, Map(fdef.param -> interp(e, env, fenv)), fenv)
```

$$\sigma, \Lambda \vdash e \Rightarrow n$$

$$\text{App } \frac{\begin{array}{cc} x_0 \in \text{Domain}(\Lambda) & \Lambda(x_0) = \text{def } x_0(x_1) = e_2 \\ \sigma, \Lambda \vdash e_1 \Rightarrow n_1 & [x_1 \mapsto n_1], \Lambda \vdash e_2 \Rightarrow n_2 \end{array}}{\sigma, \Lambda \vdash x_0(e_1) \Rightarrow n_2}$$

We skip the other cases because they are only augmented with passing function environments. If you are interested, please refer to this spec:

https://github.com/ku-plrg-classroom/docs/blob/main/cose212/f1vae/f1vae-spec.pdf

```
/* F1VAE */
def inc(x) = x + 1;
inc(2) // 3
```

Let's evaluate the above F1VAE program:

$$
\text{App} \ \dfrac{\Lambda(\texttt{inc}) = f \quad \text{Num} \ \dfrac{}{\varnothing, \Lambda \vdash 2 \Rightarrow 2} \quad \text{Add} \ \dfrac{\text{Id} \ \dfrac{\texttt{x} \in \mathsf{Domain}(\sigma)}{\sigma, \Lambda \vdash \texttt{x} \Rightarrow 2} \quad \text{Num} \ \dfrac{}{\sigma, \Lambda \vdash 2 \Rightarrow 2}}{\sigma, \Lambda \vdash \texttt{x + 1} \Rightarrow 3}}{\varnothing, \Lambda \vdash \texttt{inc(2)} \Rightarrow 3}
$$

where

$$
\begin{array}{rcl}
\Lambda & = & [\texttt{inc} \mapsto f] \\
f & = & \texttt{def inc(x) = x + 1} \\
\sigma & = & [\texttt{x} \mapsto 2]
\end{array}
$$

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4. Static Scoping vs Dynamic Scoping
Example

Static Scoping vs Dynamic Scoping

The current semantics is called **static scoping** (or **lexical scoping**) because a binding occurrence is determined statically without considering the function application but only the function definition.

```
/* F1VAE */
def f(x) = x + y;        // y is a free variable
{ val y = 2; f(1) } + { val y = 4; f(3) }
```

However, we can define the semantics of F1VAE in another way by using the **dynamic scoping** instead; a binding occurrence is determined dynamically when function application is executed:

```
/* F1VAE */
def f(x) = x + y;        // y = 2 or y = 4 depending on the call-site
{ val y = 2; f(1) } + { val y = 4; f(3) } // (1 + 2) + (3 + 4) = 10
```

Static Scoping vs Dynamic Scoping

We can design and implement the semantics of F1VAE with the **dynamic scoping** by changing the definition of the function application:

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
  ...
  case App(f, e) =>
    val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
    interp(fdef.body, env + (fdef.param -> interp(e, env, fenv)), fenv)
```

$$\boxed{\sigma, \Lambda \vdash e \Rightarrow n}$$

$$\text{App} \;\; \frac{x_0 \in \text{Domain}(\Lambda) \qquad \Lambda(x_0) = \texttt{def } x_0(x_1) \texttt{ = } e_2}{\sigma, \Lambda \vdash e_1 \Rightarrow n_1 \qquad \sigma[x_1 \mapsto n_1], \Lambda \vdash e_2 \Rightarrow n_2}$$
$$\sigma, \Lambda \vdash x_0(e_1) \Rightarrow n_2$$

However, we will use the **static scoping** by default in this course.

Example

```
/* F1VAE */
def add(x) = x + y;
val y = 2; add(1)
```

If we evaluate the above F1VAE program with a **static scoping**:

$$
\text{Val} \cfrac{\text{Num} \cfrac{}{\varnothing, \Lambda \vdash 2 \Rightarrow 2} \quad \text{App} \cfrac{\Lambda(\text{add}) = f \quad \text{Num} \cfrac{}{\sigma_0, \Lambda \vdash 1 \Rightarrow 1} \quad \text{Add} \cfrac{\text{Id} \cfrac{\text{x} \in \text{Domain}(\sigma)}{\sigma_1, \Lambda \vdash \text{x} \Rightarrow 1} \quad \text{Id} \cfrac{\text{x} \notin \text{Domain}(\sigma)}{\sigma_1, \Lambda \vdash \text{y} \Rightarrow \text{FAIL}}}{\sigma_1, \Lambda \vdash \text{x + y} \Rightarrow \text{FAIL}}}{\sigma_0, \Lambda \vdash \text{add(1)} \Rightarrow \text{FAIL}}}{\varnothing, \Lambda \vdash \text{val y = 2; add(1)} \Rightarrow \text{FAIL}}
$$

where

$$
\begin{array}{rcl}
\Lambda &=& [\text{add} \mapsto f] \\
f &=& \text{def add(x) = x + y} \\
\sigma_0 &=& [\text{y} \mapsto 2] \\
\sigma_1 &=& [\text{x} \mapsto 1]
\end{array}
$$

it cannot be evaluated because y is a **free variable** in the add function.

```
/* F1VAE */
def add(x) = x + y;
val y = 2; add(1)
```

However, if we evaluate it with a **dynamic scoping**:

$$
\text{VAL} \frac{\text{NUM} \frac{}{\varnothing, \Lambda \vdash 2 \Rightarrow 2} \quad \text{APP} \frac{\Lambda(\text{add}) = f \quad \text{NUM} \frac{}{\sigma_0, \Lambda \vdash 1 \Rightarrow 1} \quad \text{ADD} \frac{\text{ID} \frac{x \in \text{Domain}(\sigma)}{\sigma_1, \Lambda \vdash x \Rightarrow 1} \quad \text{ID} \frac{x \in \text{Domain}(\sigma)}{\sigma_1, \Lambda \vdash y \Rightarrow 2}}{\sigma_1, \Lambda \vdash x + y \Rightarrow 3}}{\sigma_0, \Lambda \vdash \text{add}(1) \Rightarrow 3}}{\varnothing, \Lambda \vdash \text{val } y = 2; \ \text{add}(1) \Rightarrow 3}
$$

where

$$
\begin{aligned}
\Lambda &= [\text{add} \mapsto f] \\
f &= \text{def add}(x) = x + y \\
\sigma_0 &= [y \mapsto 2] \\
\sigma_1 &= \sigma_0[x \mapsto 1] = [y \mapsto 2, x \mapsto 1,]
\end{aligned}
$$

it evaluates to 3 because y is **dynamically bound** to 2.

Summary

$$
\begin{array}{llll}
\text{Programs} & p & ::= & f^* \ e & (\texttt{Program}) \\
\text{Function Definitions} & \mathbb{F} \ni f & ::= & \texttt{def} \ x(x) = e & (\texttt{FunDef}) \\
\text{Expressions} & e & ::= & \dots & \\
& & | & x(e) & (\texttt{App})
\end{array}
$$

```scala
type FEnv = Map[String, FunDef]
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
  ...
  case App(f, e) =>
    val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
    interp(fdef.body, Map(fdef.param -> interp(e, env, fenv)), fenv)
```

$$\boxed{\sigma, \Lambda \vdash e \Rightarrow n}$$

$$
\text{App} \ \frac{
\begin{array}{cc}
x_0 \in \mathsf{Domain}(\Lambda) & \Lambda(x_0) = \texttt{def} \ x_0(x_1) = e_2 \\
\sigma, \Lambda \vdash e_1 \Rightarrow n_1 & [x_1 \mapsto n_1], \Lambda \vdash e_2 \Rightarrow n_2
\end{array}
}{
\sigma, \Lambda \vdash x_0(e_1) \Rightarrow n_2
}
$$

https://github.com/ku-plrg-classroom/docs/tree/main/cose212/f1vae

- Please see above document on GitHub:
    - Implement interp function.
    - Implement interpDS function.

- It is just an exercise, and you **don't need to submit** anything.

- However, some exam questions might be related to this exercise.

- First-Class Functions

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