Lecture 4 – Nondeterministic Finite Automata (NFA)
COSE215: Theory of Computation

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PLRG

2024 Spring

Recall

PLRG

1. Deterministic Finite Automata (DFA)
   - Definition
   - Transition Diagram and Transition Table
   - Extended Transition Function
   - Acceptance of a Word
   - Language of DFA (Regular Language)
   - Examples

Contents

Contents

Definition (Nondeterministic Finite Automaton (NFA))

A **nondeterministic finite automaton** is a 5-tuple:

$$N = (Q, \Sigma, \delta, q_0, F)$$

- Q is a finite set of **states**
- Σ is a finite set of **symbols**
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is the **transition function**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

$$N = (\{q_0, q_1, q_2\}, \{\mathtt{a}, \mathtt{b}\}, \delta, q_0, \{q_2\})$$

$$\delta(q_0, \mathtt{a}) = \{q_0, q_1\} \qquad \delta(q_1, \mathtt{a}) = \{q_2\} \qquad \delta(q_2, \mathtt{a}) = \varnothing$$
$$\delta(q_0, \mathtt{b}) = \{q_0\} \qquad \delta(q_1, \mathtt{b}) = \varnothing \qquad \delta(q_2, \mathtt{b}) = \varnothing$$

Definition of NFA

```scala
// The definition of NFA
case class NFA(
  states: Set[State],
  symbols: Set[Symbol],
  trans: Map[(State, Symbol), Set[State]],
  initState: State,
  finalStates: Set[State],
)
```

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)
```

```
// An example of NFA
val nfa1: NFA = NFA(
  states      = Set(0, 1, 2),
  symbols     = Set('a', 'b'),
  trans       = Map(
    (0, 'a') -> Set(0, 1), (1, 'a') -> Set(2), // (2, 'a') -> Set(),
    (0, 'b') -> Set(0), // (1, 'b') -> Set(),      (2, 'b') -> Set(),
  ).withDefaultValue(Set()),
  initState   = 0,
  finalStates = Set(2),
)
```

You can **skip empty transitions** using `withDefaultValue` method.

$$N_1 = (\{q_0, q_1, q_2\}, \{\mathtt{a}, \mathtt{b}\}, \delta, q_0, \{q_2\})$$

$$\delta(q_0, \mathtt{a}) = \{q_0, q_1\} \qquad \delta(q_1, \mathtt{a}) = \{q_2\} \qquad \delta(q_2, \mathtt{a}) = \varnothing$$
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Transition Diagram
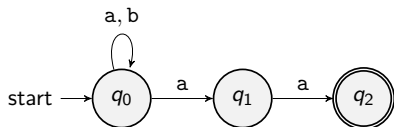
Transition Table

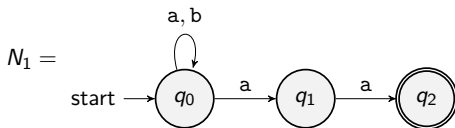| q	a	b
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\{q_2\}$	$\varnothing$
$*q_2$	$\varnothing$	$\varnothing$

Definition (Extended Transition Function)

For a given NFA $N = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function**
$\delta^* : \mathcal{P}(Q) \times \Sigma^* \to \mathcal{P}(Q)$ is defined as follows:

- **(Basis Case)** $\delta^*(S, \epsilon) = S$
- **(Induction Case)** $\delta^*(S, xw) = \delta^*(\bigcup_{q \in S} \delta(q, x), w)$

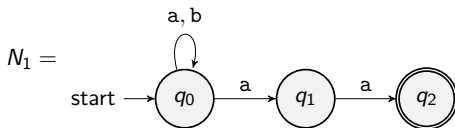where $S \subseteq Q$, $x \in \Sigma$, and $w \in \Sigma^*$

$$N_1 =$$

$\delta^*(\{q_0\}, \texttt{baa})$

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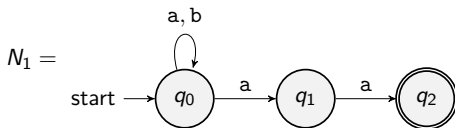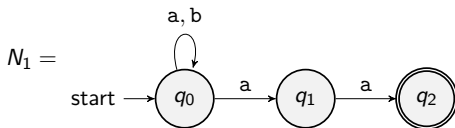where $S \subseteq Q$, $x \in \Sigma$, and $w \in \Sigma^*$

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$$\delta^*(\{q_0\}, \mathtt{baa}) \;=\; \delta^*(\delta(q_0, \mathtt{b}), \mathtt{aa}) \qquad\qquad =\; \delta^*(\{q_0\}, \mathtt{aa})$$

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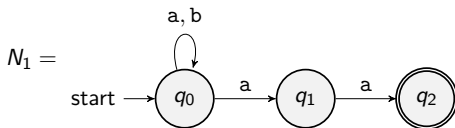where $S \subseteq Q$, $x \in \Sigma$, and $w \in \Sigma^*$

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\delta^*(\{q_0\}, \mathtt{baa}) &= \delta^*(\delta(q_0, \mathtt{b}), \mathtt{aa}) &&= \delta^*(\{q_0\}, \mathtt{aa}) \\
&= \delta^*(\delta(q_0, \mathtt{a}), \mathtt{a}) &&= \delta^*(\{q_0, q_1\}, \mathtt{a})
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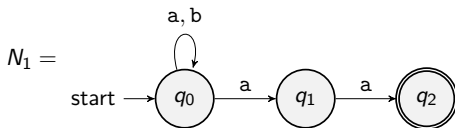where $S \subseteq Q$, $x \in \Sigma$, and $w \in \Sigma^*$

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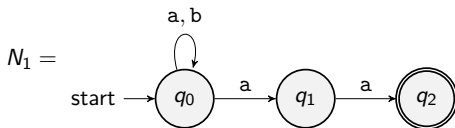where $S \subseteq Q$, $x \in \Sigma$, and $w \in \Sigma^*$

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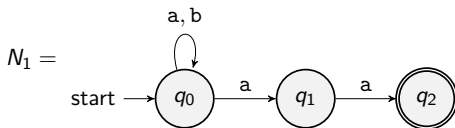where $S \subseteq Q$, $x \in \Sigma$, and $w \in \Sigma^*$

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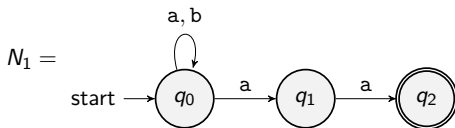$$\delta^*(\{q_0\}, \texttt{aba})$$

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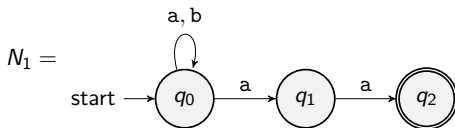where $S \subseteq Q$, $x \in \Sigma$, and $w \in \Sigma^*$

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Extended Transition Function

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&= \{q_0, q_1\}
\end{aligned}
$$

Extended Transition Function

```scala
// The type definition of words
type Word = String
case class NFA(...):
  ...
  // The extended transition function of NFA
  def extTrans(qs: Set[State], w: Word): Set[State] = w match
    case ""     => qs
    case x <| w => extTrans(qs.flatMap(q => trans(q, x)), w)

// An example transition for a word "baa"
nfa1.extTrans(Set(0), "baa") // Set(0, 1, 2)

// An example transition for a word "aba"
nfa1.extTrans(Set(0), "aba") // Set(0, 1)
```
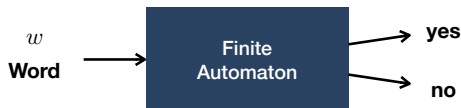
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w
Word → Finite Automaton → **yes** / **no**

Definition (Acceptance of a Word)

For a given NFA $N = (Q, \Sigma, \delta, q_0, F)$, we say that N **accepts** a word $w \in \Sigma^*$ if and only if $\delta^*(\{q_0\}, w) \cap F \neq \varnothing$

$N_1 =$

start $\rightarrow q_0 \xrightarrow{\ a\ } q_1 \xrightarrow{\ a\ } q_2$ with self-loop a, b on q_0

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$$N_1 =$$

$$\delta^*(\{q_0\}, \mathtt{baa}) \cap F = \{q_0, q_1, q_2\} \cap \{q_2\} = \{q_2\} \neq \varnothing$$

It means that N_1 **accepts** baa.
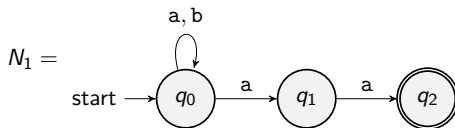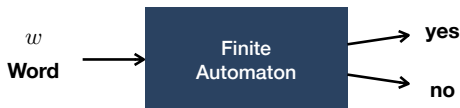
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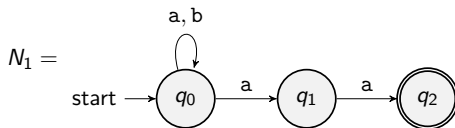
$$\delta^*(\{q_0\}, \texttt{baa}) \cap F = \{q_0, q_1, q_2\} \cap \{q_2\} = \{q_2\} \neq \varnothing$$

It means that N_1 **accepts** baa.

$$\delta^*(\{q_0\}, \texttt{aba}) \cap F = \{q_0, q_1\} \cap \{q_2\} = \varnothing$$

It means that N_1 does **not accept** aba.

Acceptance of a Word

$$N_1 =$$

a, b

start $\rightarrow$ q_0 $\xrightarrow{\text{a}}$ q_1 $\xrightarrow{\text{a}}$ q_2
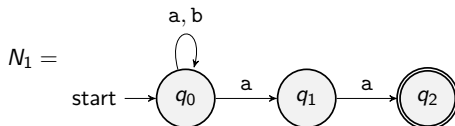
```scala
case class NFA(...):
  ...
  // The acceptance of a word by NFA
  def accept(w: Word): Boolean =
    extTrans(Set(initState), w).intersect(finalStates).nonEmpty

// An example acceptance of a word "baa"
nfa1.accept("baa") // true

// An example non-acceptance of a word "aba"
nfa1.accept("aba") // false
```

Definition (Language of NFA)

For a given NFA $N = (Q, \Sigma, \delta, q_0, F)$, the **language** of N is defined as:

$$L(N) = \{w \in \Sigma^* \mid N \text{ accepts } w\}$$

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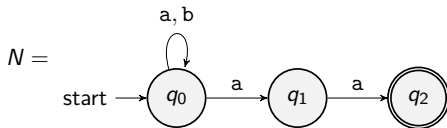
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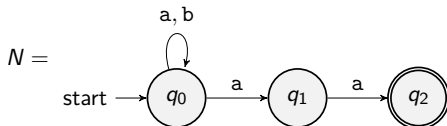
$$L(N) = \{w\text{aa} \mid w \in \{\text{a}, \text{b}\}^*\}$$

$L = \{\mathtt{a}^n\mathtt{b} \mid n \geq 0\}$

Examples

$L = \{a^n b \mid n \geq 0\}$

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$L = \{w \in \{0,1\}^* \mid w \text{ contains}$
exactly two 0's $\}$

Examples

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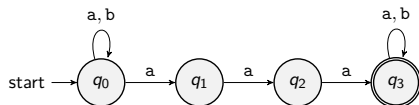$L = \{w \in \{0, 1\}^* \mid w$ contains exactly two 0's $\}$

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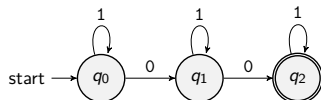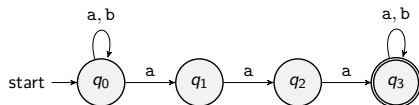$L = \{w \in \{0, 1\}^* \mid w \text{ contains exactly two 0's }\}$

$L = \{w \in \{a, b\}^* \mid w \text{ contains three consecutive a's }\}$

Examples

$L = \{a^n b \mid n \geq 0\}$

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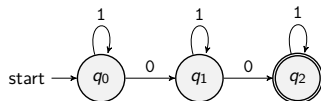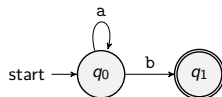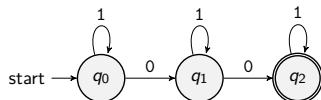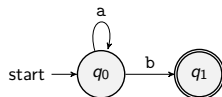
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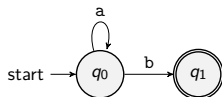$L = \{w \in \{0,1\}^* \mid w \text{ contains exactly two 0's }\}$

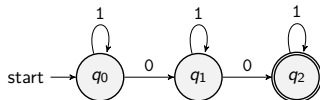$L = \{w \in \{a,b\}^* \mid w \text{ contains three consecutive a's }\}$

$L = \{w \in \{0,1\}^* \mid d(w) \equiv 0 \pmod{3}\}$ where $d(w)$ is the natural number represented by w in binary
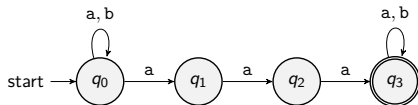
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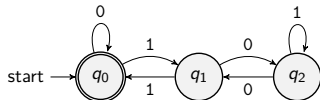$L = \{w \in \{0,1\}^* \mid w \text{ contains exactly two 0's }\}$

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Examples

$L = \{ a^n b \mid n \geq 0 \}$

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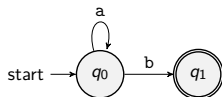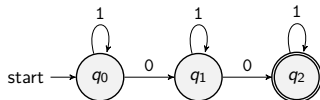$L = \{ w \in \{a, b\}^* \mid w \text{ contains three consecutive a's} \}$

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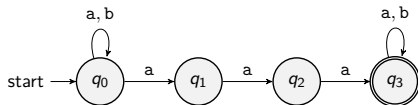

$L = \{ a^n b^n \mid n \geq 0 \}$
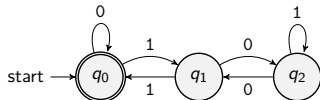
Examples

$L = \{a^n b \mid n \geq 0\}$

$L = \{w \in \{0, 1\}^* \mid w \text{ contains exactly two 0's }\}$

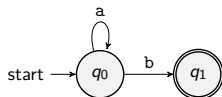$L = \{w \in \{a, b\}^* \mid w \text{ contains three consecutive a's }\}$

$L = \{w \in \{0, 1\}^* \mid d(w) \equiv 0 \pmod 3\}$ where $d(w)$ is the natural number represented by w in binary
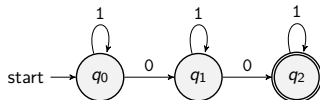

$L = \{a^n b^n \mid n \geq 0\}$

IMPOSSIBLE ($\nexists$ NFA N. $L(N) = L$)

Contents

$$L(D) = \{w\mathrm{aa} \mid w \in \{\mathrm{a}, \mathrm{b}\}^*\} = L(N)$$

Is there any language that is the language of a DFA but not the language of an NFA, or vice versa?

$$\{w \mid \mathbb{N}(w) \equiv 0 \pmod 3\}$$

Is there any language that is the language of a DFA but not the language of an NFA, or vice versa? No! DFA and NFA are **equivalent**.

$$\{w \mid \mathbb{N}(w) \equiv 0 \pmod 3\}$$

Equivalence of DFA and NFA

Theorem (Equivalence of DFA and NFA)

A language L is the language L(D) of a DFA D if and only if L is the language L(N) of an NFA N.

Equivalence of DFA and NFA

Theorem (Equivalence of DFA and NFA)

A language L is the language L(D) of a DFA D if and only if L is the language L(N) of an NFA N.

Proof) By the following two theorems.

Theorem (DFA to NFA)

For a given DFA $D = (Q, \Sigma, \delta, q, F)$, $\exists$ NFA N. $L(D) = L(N)$.

It means ① we can always construct an NFA equivalent to a given DFA.

Theorem (NFA to DFA – Subset Construction)

For a given NFA $N = (Q, \Sigma, \delta, q_0, F)$, $\exists$ DFA D. $L(D) = L(N)$.

It means ② we can always construct a DFA equivalent to a given NFA.

① **Let's learn how to construct an NFA equivalent to a given DFA.**

DFA D

q	a	b
$\rightarrow q_0$	q_1	q_0
q_1	q_2	q_0
$*q_2$	q_2	q_0

NFA N

q	a	b
$\rightarrow q_0$	$\{q_1\}$	$\{q_0\}$
q_1	$\{q_2\}$	$\{q_0\}$
$*q_2$	$\{q_2\}$	$\{q_0\}$

DFA → NFA
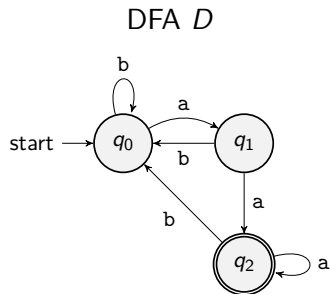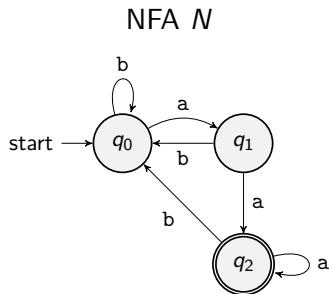
Theorem (DFA to NFA)

For a given DFA $D = (Q, \Sigma, \delta_D, q_0, F)$, $\exists$ NFA N. $L(D) = L(N)$.

Proof) Consider the following NFA:

$$N = (Q, \Sigma, \delta_N, q_0, F)$$

where $\forall q \in Q. \ \forall x \in \Sigma.$

$$\delta_N(q, x) = \{\delta_D(q, x)\}$$

Then,

$$
\begin{aligned}
w \in L(D) &\iff \delta_D^*(q_0, w) \in F && (\because \text{definition of } L(D)) \\
&\iff \{\delta_D^*(q_0, w)\} \cap F \neq \varnothing && (\because \text{set theory}) \\
&\iff \delta_N^*(\{q_0\}, w) \cap F \neq \varnothing && (\because \text{lemma in the next slide}) \\
&\iff w \in L(N) && (\because \text{definition of } L(N)) \quad \square
\end{aligned}
$$

DFA → NFA – Lemma

Lemma

$\forall q \in Q. \ \forall w \in \Sigma^*. \ \delta_N^*(\{q\}, w) = \{\delta_D^*(q, w)\}.$

Proof) By induction on the **length of word**.

- **(Base Case)** $\delta_N^*(\{q\}, \epsilon) = \{q\} = \{\delta_D^*(q, \epsilon)\}$.
- **(Inductive Case)** Assume it holds for w (I.H.).

$$\delta_N^*(\{q\}, xw) = \delta_N^*(\delta_N(q, x), w) \qquad (\because \text{definition of } \delta_N^*)$$

$$= \delta_N^*(\{\delta_D(q, x)\}, w) \quad (\because \text{definition of } \delta_N)$$

$$= \{\delta_D^*(\delta_D(q, x), w)\} \quad (\because \text{I.H.})$$

$$= \{\delta_D^*(q, xw)\} \qquad (\because \text{definition of } \delta_D^*) \quad \square$$
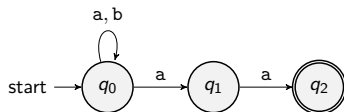
② **Let's learn how to construct a DFA equivalent to a given NFA.**
We will use **subsets of states** in the NFA as **states** in the DFA.
(This is called the **subset construction** approach.)

DFA D

q	a	b
$\varnothing$	$\varnothing$	$\varnothing$
$\rightarrow \{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_1\}$	$\{q_2\}$	$\varnothing$
$*\{q_2\}$	$\varnothing$	$\varnothing$
$\{q_0, q_1\}$	$\{q_0, q_1, q_2\}$	$\{q_0\}$
$*\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$
$*\{q_1, q_2\}$	$\{q_2\}$	$\varnothing$
$*\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0\}$

NFA N

q	a	b
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\{q_2\}$	$\varnothing$
$*q_2$	$\varnothing$	$\varnothing$

② **Let's learn how to construct a DFA equivalent to a given NFA.**
We will use **subsets of states** in the NFA as **states** in the DFA.
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DFA D

NFA N

q	a	b
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\{q_2\}$	$\varnothing$
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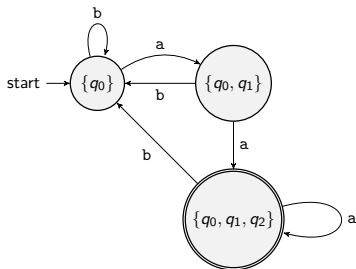
q	a	b
$\rightarrow \{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
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$*\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0\}$
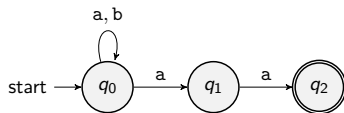
DFA ← NFA (Subset Construction)

Theorem (NFA to DFA – Subset Construction)

For a given NFA $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$, $\exists$ DFA D. $L(D) = L(N)$.

Proof) Define a DFA

$$D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$$

where

- $Q_D = \mathcal{P}(Q_N)$
- $\forall S \in Q_D. \ \forall x \in \Sigma.$
$$\delta_D(S, x) = \bigcup_{q \in S} \delta_N(q, x)$$

- $F_D = \{S \in Q_D \mid S \cap F_N \neq \varnothing\}$

Then,

$$
\begin{aligned}
w \in L(D) &\iff \delta_D^*(\{q_0\}, w) \in F_D && (\because \text{definition of } L(D)) \\
&\iff \delta_D^*(\{q_0\}, w) \cap F_N \neq \varnothing && (\because \text{definition of } F_D) \\
&\iff \delta_N^*(\{q_0\}, w) \cap F_N \neq \varnothing && (\because \text{lemma in the next slide}) \\
&\iff w \in L(N) && (\because \text{definition of } L(N)) \quad \square
\end{aligned}
$$

DFA ← NFA (Subset Construction) – Lemma ◆PLRG

Lemma

$\forall S \in Q_D. \forall w \in \Sigma^*. \delta_D^*(S, w) = \delta_N^*(S, w)$

Proof) By induction on the **length of word**.

- **(Base Case)** $\delta_D^*(S, \epsilon) = S = \delta_N^*(S, \epsilon)$.
- **(Inductive Case)** Assume it holds for w (I.H.).

$$\delta_D^*(S, xw) = \delta_D^*(\delta_D(S, x), w) \qquad (\because \text{definition of } \delta_D^*)$$

$$= \delta_D^*(\bigcup_{q \in S} \delta_N(q, x), w) \quad (\because \text{definition of } \delta_D)$$

$$= \delta_N^*(\bigcup_{q \in S} \delta_N(q, x), w) \quad (\because \text{I.H.})$$

$$= \delta_N^*(S, xw) \qquad (\because \text{definition of } \delta_N^*) \quad \square$$

Summary

1. Nondeterministic Finite Automata (NFA)
   Definition
   Transition Diagram and Transition Table
   Extended Transition Function
   Language of NFA
   Examples

2. Equivalence of DFA and NFA
   DFA $\to$ NFA
   DFA $\leftarrow$ NFA (Subset Construction)

PLRG

- ϵ-Nondeterministic Finite Automata (ϵ-NFA)

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