

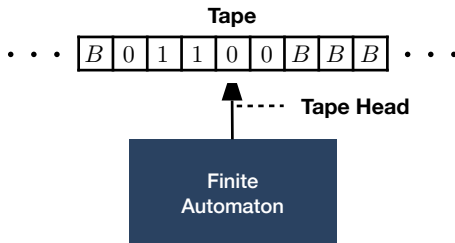
# Lecture 22 – Examples of Turing Machines

## COSE215: Theory of Computation

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2024 Spring



- A **Turing machine (TM)** is a **deterministic** FA with a **tape**.
  - ① A **tape** is an infinite sequence of cells containing **tape symbols**.  
(The **blank symbol**  $B$  is a special symbol representing an empty cell.)
  - ② A **tape head** points to the current cell.
  - ③ A **transition** performs the following operations depending on the current 1) **state** and 2) **tape symbol** pointed by the tape head:
    - **Change** the current **state**.
    - **Replace** the current **tape symbol** pointed by the tape head.
    - **Move** the **tape head** left or right.
- We can use Turing machines as **computing machines**.

## 1. Turing Machines as Word Recognizers

Example 1:  $L = \{a^n b^n c^n \mid n \geq 0\}$

Example 2:  $L = \{ww \mid w \in \{a, b\}^*\}$

Example 3:  $L = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$

## 2. Turing Machines as Computing Machines

Example 4: Flip Bits –  $f(w \in \{0, 1\}^*) = (\text{flip of } w)$

Example 5: Unary Addition –  $f(1^n + 1^m) = 1^{n+m}$

Example 6: Binary Increment –  $f(w \in \{0, 1\}^*) = w + 1$

Example 7: Data Copy –  $f(w \in \{a, b\}^*) = ww$

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Example 1:  $L = \{a^n b^n c^n \mid n \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^n b^n c^n \mid n \geq 0\}$$

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...	B	a	a	b	b	c	c	B	...
-----	---	---	---	---	---	---	---	---	-----

## Example 1: $L = \{a^n b^n c^n \mid n \geq 0\}$

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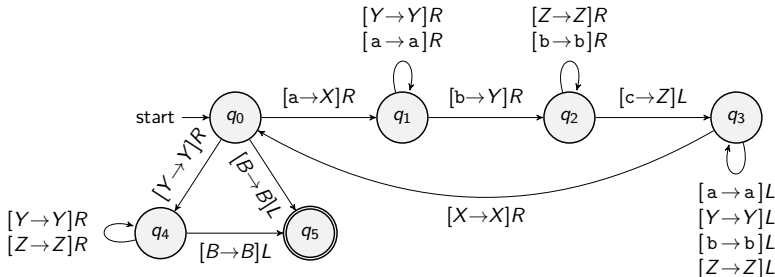
- 1: **while** there are a's **do**
- 2:     Find and Replace a with X
- 3:     Find and Replace b with Y
- 4:     Find and Replace c with Z
- 5: Check if only Y's and Z's are left

# Example 1: $L = \{a^n b^n c^n \mid n \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^n b^n c^n \mid n \geq 0\}$$

See the example for  $aabbcc \in L(M)$ .<sup>1</sup>



<sup>1</sup><https://plrg.korea.ac.kr/courses/cose215/materials/tm-an-bn-cn.pdf>



Example 2:  $L = \{ww \mid w \in \{a, b\}^*\}$

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Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{ww \mid w \in \{a, b\}^*\}$$

...	<i>B</i>	a	b	b	a	b	b	<i>B</i>	...
-----	----------	---	---	---	---	---	---	----------	-----

## Example 2: $L = \{ww \mid w \in \{a, b\}^*\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{ww \mid w \in \{a, b\}^*\}$$

...	B	a	b	b	a	b	b	B	...
-----	---	---	---	---	---	---	---	---	-----

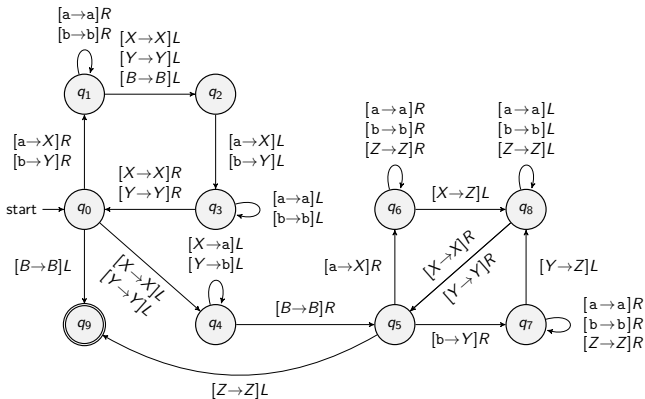
- 1: Find the middle of the input
- 2: Replace all  $X$ 's (or  $Y$ 's) with  $a$ 's (or  $b$ 's) in the first half
- 3: **while** there are input symbols in the first half **do**
- 4:     Replace  $a$  (or  $b$ ) with  $X$  (or  $Y$ ) in the first half
- 5:     Find and Replace matched  $X$  (or  $Y$ ) with  $Z$  in the second half

## Example 2: $L = \{ww \mid w \in \{a, b\}^*\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{ww \mid w \in \{a, b\}^*\}$$

See the example for  $abbabb \in L(M)$ .<sup>2</sup>



<sup>2</sup><https://plrg.korea.ac.kr/courses/cose215/materials/tm-w-w.pdf>

Example 3:  $L = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$$

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Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$$

...	B	a	a	b	b	b	c	c	c	c	c	c	B	...
-----	---	---	---	---	---	---	---	---	---	---	---	---	---	-----

### Example 3: $L = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$$

...	B	a	a	b	b	b	c	c	c	c	c	c	B	...
-----	---	---	---	---	---	---	---	---	---	---	---	---	---	-----

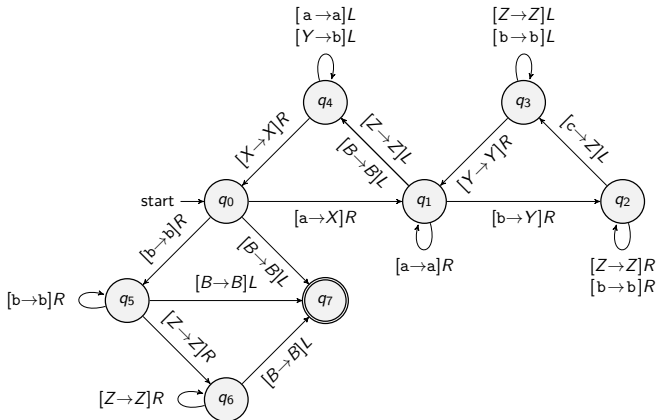
- 1: **while** there are a's **do**
- 2:     Find and Replace a with X
- 3:     **while** there are b's **do**
- 4:         Find and Replace b with Y
- 5:         Find and Replace c with Z
- 6:     Roll back all Y's to b's
- 7: Check if only b's and Z's are left

### Example 3: $L = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$$

See the example for  $aabb bcccccc \in L(M)$ .<sup>3</sup>



<sup>3</sup><https://plrg.korea.ac.kr/courses/cose215/materials/tm-ai-bj-cij.pdf>



## 1. Turing Machines as Word Recognizers

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Example 4: Flip Bits –  $f(w \in \{0, 1\}^*) = (\text{flip of } w)$

Example 5: Unary Addition –  $f(1^n + 1^m) = 1^{n+m}$

Example 6: Binary Increment –  $f(w \in \{0, 1\}^*) = w + 1$

Example 7: Data Copy –  $f(w \in \{a, b\}^*) = ww$

## Example 4: Flip Bits – $f(w \in \{0, 1\}^*) = (\text{flip of } w)$



Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = (\text{the flip of each bit in } w)$$

## Example 4: Flip Bits – $f(w \in \{0, 1\}^*) = (\text{flip of } w)$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = (\text{the flip of each bit in } w)$$

$$w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline \dots & B & 1 & 0 & 1 & 1 & 1 & 0 & 0 & B & \dots \\ \hline \end{array}$$

$$f(w) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline \dots & B & 0 & 1 & 0 & 0 & 0 & 1 & 1 & B & \dots \\ \hline \end{array}$$

## Example 4: Flip Bits – $f(w \in \{0, 1\}^*) = (\text{flip of } w)$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = (\text{the flip of each bit in } w)$$

$w$	$=$	$\cdots$	$B$	$1$	$0$	$1$	$1$	$1$	$0$	$0$	$B$	$\cdots$
-----	-----	----------	-----	-----	-----	-----	-----	-----	-----	-----	-----	----------

$f(w)$	$=$	$\cdots$	$B$	$0$	$1$	$0$	$0$	$0$	$1$	$1$	$B$	$\cdots$
--------	-----	----------	-----	-----	-----	-----	-----	-----	-----	-----	-----	----------

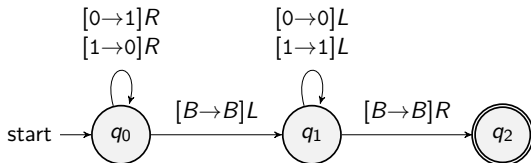
- 1: Flip each bit of the input:  $1 \rightarrow 0$  and  $0 \rightarrow 1$
- 2: Go to the first input symbol

## Example 4: Flip Bits – $f(w \in \{0, 1\}^*) = (\text{flip of } w)$ PLRG

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = (\text{the flip of each bit in } w)$$

See the example for  $f(1011100) = 0100011$ .<sup>4</sup>



<sup>4</sup><https://plrg.korea.ac.kr/courses/cose215/materials/tm-flip.pdf>

## Example 5: Unary Addition – $f(1^n 1^m) = 1^{n+m}$

Construct a **Turing machine** that **computes** the function:

$$f(1^n 1^m) = 1^{n+m} \quad \text{where} \quad n, m \geq 0$$

## Example 5: Unary Addition – $f(1^n+1^m) = 1^{n+m}$

Construct a **Turing machine** that **computes** the function:

$$f(1^n+1^m) = 1^{n+m} \quad \text{where} \quad n, m \geq 0$$

$$w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \dots & B & 1 & 1 & 1 & + & 1 & 1 & B & \dots \\ \hline \end{array}$$

$$f(w) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \dots & B & 1 & 1 & 1 & 1 & 1 & B & B & \dots \\ \hline \end{array}$$

## Example 5: Unary Addition – $f(1^n 1^m) = 1^{n+m}$

Construct a **Turing machine** that **computes** the function:

$$f(1^n 1^m) = 1^{n+m} \quad \text{where} \quad n, m \geq 0$$

$$w = \overline{\dots \mid B \mid 1 \mid 1 \mid 1 \mid + \mid 1 \mid 1 \mid B \mid \dots}$$

$$f(w) = \overline{\dots \mid B \mid 1 \mid 1 \mid 1 \mid 1 \mid 1 \mid B \mid B \mid \dots}$$

- 1: Find + after 1's
- 2: **if** the last symbol is 1 **then**
- 3:     Find and Remove the last 1
- 4:     Find and Replace the + with 1
- 5: **else**
- 6:     Remove the +
- 7: Go to the first input symbol

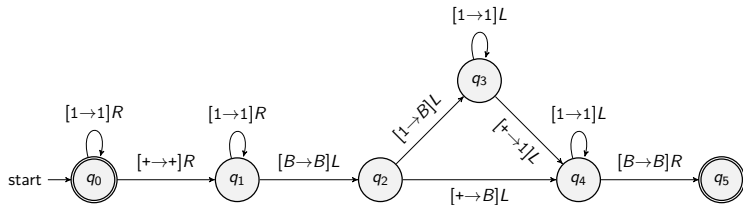


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Construct a **Turing machine** that **computes** the function:

$$f(1^n+1^m) = 1^{n+m} \quad \text{where} \quad n, m \geq 0$$

See the example for  $f(111+11) = 11111$ .<sup>5</sup>



<sup>5</sup><https://plrg.korea.ac.kr/courses/cose215/materials/tm-unary-add.pdf>

## Example 6: Binary Increment – $f(w \in \{0, 1\}^*) = w + 1$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = w + 1 \quad \text{where} \quad w \text{ starts with } 1$$

## Example 6: Binary Increment – $f(w \in \{0, 1\}^*) = w + 1$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = w + 1 \quad \text{where} \quad w \text{ starts with } 1$$

$w =$ 

...	B	1	0	1	0	1	1	1	1	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

$f(w) =$ 

...	B	1	0	1	1	0	0	0	0	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

## Example 6: Binary Increment – $f(w \in \{0,1\}^*) = w + 1$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0,1\}^*) = w + 1 \quad \text{where} \quad w \text{ starts with } 1$$

$w =$ 

...	B	1	0	1	0	1	1	1	1	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

$f(w) =$ 

...	B	1	0	1	1	0	0	0	0	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

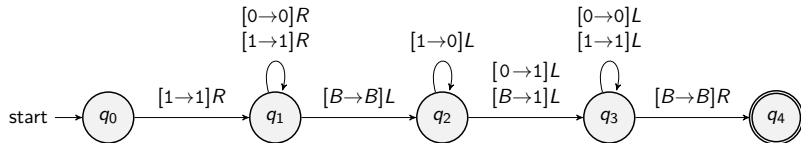
- 1: Check if the first bit is 1.
- 2: Move to the end of the input.
- 3: Repeatedly replace the rightmost 1 with 0.
- 4: Replace 0 (or B) with 1.
- 5: Go to the first input symbol.

## Example 6: Binary Increment – $f(w \in \{0, 1\}^*) = w \triangleleft \text{PLRG}$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = w + 1 \quad \text{where} \quad w \text{ starts with } 1$$

See the example for  $f(10101111) = 10110000$ .<sup>6</sup>



<sup>6</sup><https://plrg.korea.ac.kr/courses/cose215/materials/tm-inc.pdf>

## Example 7: Data Copy – $f(w \in \{a, b\}^*) = ww$

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$w =$ 

...	$B$	a	b	b	$B$	$B$	$B$	$B$	...
-----	-----	---	---	---	-----	-----	-----	-----	-----

$f(w) =$ 

...	$B$	a	b	b	a	b	b	$B$	...
-----	-----	---	---	---	---	---	---	-----	-----

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Construct a **Turing machine** that **computes** the function:

$$f(w \in \{a, b\}^*) = ww$$

$w =$ 

...	B	a	b	b	B	B	B	B	...
-----	---	---	---	---	---	---	---	---	-----

$f(w) =$ 

...	B	a	b	b	a	b	b	B	...
-----	---	---	---	---	---	---	---	---	-----

- 1: **while** there are input symbols **do**
- 2:     Find and Replace a (or b) with Z
- 3:     Find and Fill the first blank with X (or Y) for a (or b)
- 4:     Roll back Z to the original a (or b)
- 5: Replace X's and Y's with a's and b's
- 6: Go to the first input symbol

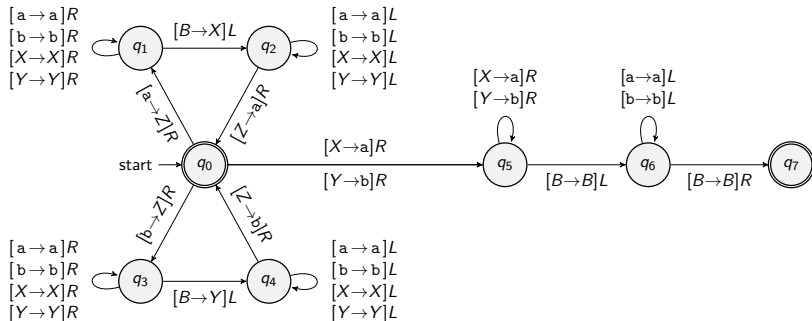


## Example 7: Data Copy – $f(w \in \{a, b\}^*) = ww$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{a, b\}^*) = ww$$

See the example for  $f(abb) = abbabb$ .<sup>7</sup>



<sup>7</sup><https://plrg.korea.ac.kr/courses/cose215/materials/tm-data-copy.pdf>

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- Extensions of Turing Machines

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