

Lecture 14 – Pushdown Automata (PDA)

COSE215: Theory of Computation

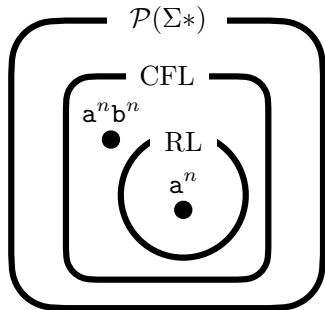
Jihyeok Park



2024 Spring

- A context-free grammar (CFG):

$$G = (V, \Sigma, S, R)$$



Languages	Automata	Grammars
Context-Free Language (CFL)	???	Context-Free Grammar (CFG)
Regular Language (RL)	Finite Automata (FA)	Regular Expression (RE)

1. Pushdown Automata

- Definition

- Transition Diagram

- Pushdown Automata in Scala

- Configurations and One-Step Moves

- Acceptance by Final States

- Acceptance by Empty Stacks

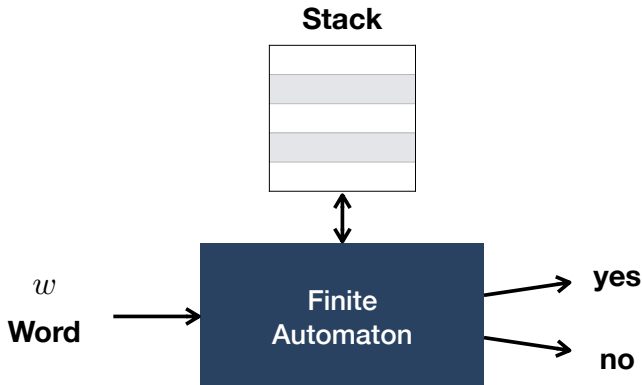
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- In FA, the next state is determined by the current **state** and **symbol**.
- In PDA, the next state is determined by the current **state**, **symbol**, and the **top element of the stack**.



Definition (Pushdown Automata)

A **pushdown automaton (PDA)** is a 7-tuple:

$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$$

where

- Q is a finite set of **states**
- Σ is a finite set of **symbols**
- Γ is a finite set of **stack alphabets**
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma^*)$ is a **transition function**
- $q_0 \in Q$ is the **initial state**
- $Z \in \Gamma$ is the **initial stack alphabet** (the stack is initially Z)
- $F \subseteq Q$ is a set of **final states**

$$P_1 = (\{q_0, q_1, q_2\}, \{a, b\}, \{X, Z\}, \delta, q_0, Z, \{q_2\})$$

where

$$\delta(q_0, a, Z) = \{(q_0, XZ)\}$$

$$\delta(q_0, b, Z) = \emptyset$$

$$\delta(q_0, \epsilon, Z) = \{(q_1, Z)\}$$

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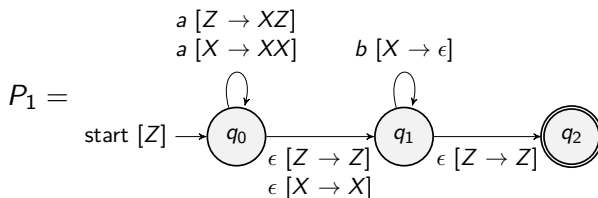
$$\delta(q_1, \epsilon, X) = \emptyset$$

$$\delta(q_2, a, X) = \emptyset$$

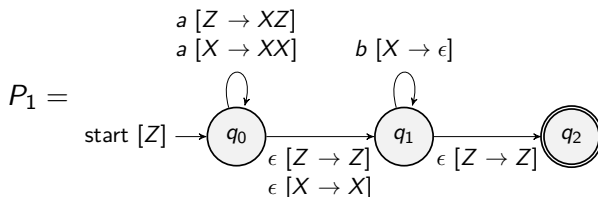
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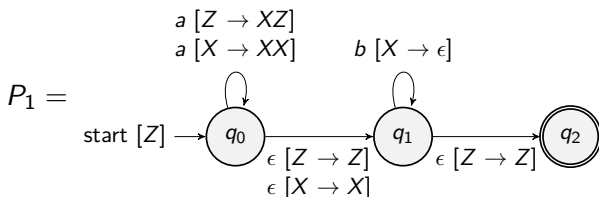
$$P_1 = (\{q_0, q_1, q_2\}, \{a, b\}, \{X, Z\}, \delta, q_0, Z, \{q_2\})$$



For example,

$$\delta(q_0, a, Z) = \{(q_0, XZ)\}$$

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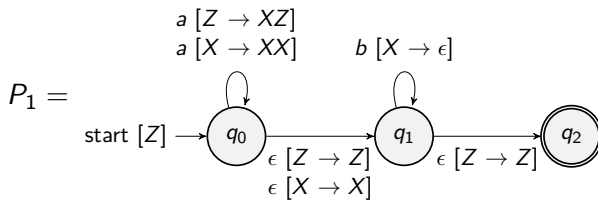


For example,

$$\begin{aligned}\delta(q_0, a, Z) &= \{(q_0, XZ)\} \\ \delta(q_0, \epsilon, X) &= \{(q_1, X)\}\end{aligned}$$

```
// The type definitions of states, symbols, words, and stack alphabets
type State = Int
type Symbol = Char
type Word = String
type Alphabet = String

// The definition of PDA
case class PDA(
  states: Set[State],
  symbols: Set[Symbol],
  alphabets: Set[Alphabet],
  trans: Map[
    (State, Option[Symbol], Alphabet),
    Set[(State, List[Alphabet])]
  ],
  initState: State,
  initAlphabet: Alphabet,
  finalStates: Set[State],
)
```



```

val pda1: PDA = PDA(
  states = Set(0, 1, 2),      symbols = Set('a', 'b'),
  alphabets = Set("X", "Z"),
  trans = Map(
    (0, Some('a'), "Z") -> Set((0, List("X", "Z"))),
    (0, Some('a'), "X") -> Set((0, List("X", "X"))),
    (0, None, "Z") -> Set((1, List("Z"))),
    (0, None, "X") -> Set((1, List("X"))),
    (1, Some('b'), "X") -> Set((1, List())),
    (1, None, "Z") -> Set((2, List("Z"))),
  ).withDefaultValue(Set()),
  initState = 0, initAlphabet = "Z", finalStates = Set(2),
)
    
```

Definition (Configurations of PDA)

A **configuration** of a PDA P represents the current status of P . It is defined as a triple (q, w, α) where

- $q \in Q$: the current state
- $w \in \Sigma^*$: the remaining word
- $\alpha \in \Gamma^*$: the current status of the stack

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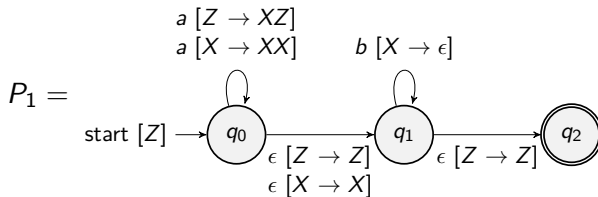
- $q \in Q$: the current state
- $w \in \Sigma^*$: the remaining word
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Definition (One-Step Moves of PDA)

A **one-step move** (\vdash) of a PDA P is a transition from a configuration to another configuration:

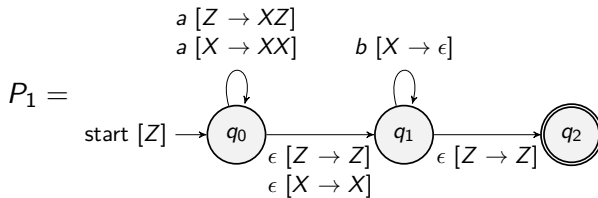
$$\begin{aligned}(q, aw, X\beta) &\vdash (p, w, \alpha\beta) && \text{if } (p, \alpha) \in \delta(q, a, X) \\ (q, w, X\beta) &\vdash (p, w, \alpha\beta) && \text{if } (p, \alpha) \in \delta(q, \epsilon, X)\end{aligned}$$

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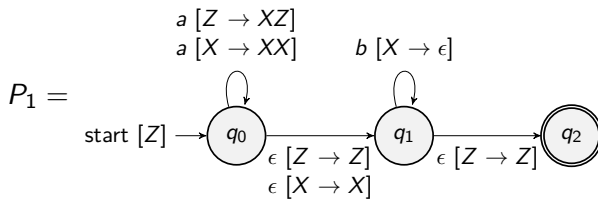
(q_0, ab, Z)

$$\begin{aligned} (q, aw, X\beta) \vdash (p, w, \alpha\beta) & \text{ if } (p, \alpha) \in \delta(q, a, X) \\ (q, w, X\beta) \vdash (p, w, \alpha\beta) & \text{ if } (p, \alpha) \in \delta(q, \epsilon, X) \end{aligned}$$



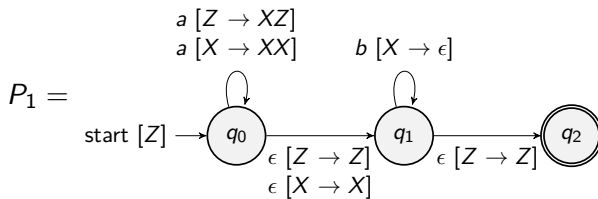
$$(q_0, ab, Z) \vdash (q_0, b, XZ) \quad (\because (q_0, XZ) \in \delta(q_0, a, Z))$$

$$\begin{aligned} (q, aw, X\beta) \vdash (p, w, \alpha\beta) & \text{ if } (p, \alpha) \in \delta(q, a, X) \\ (q, w, X\beta) \vdash (p, w, \alpha\beta) & \text{ if } (p, \alpha) \in \delta(q, \epsilon, X) \end{aligned}$$



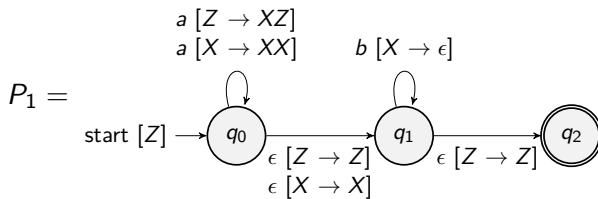
$$\begin{aligned} (q_0, ab, Z) & \vdash (q_0, b, XZ) & (\because (q_0, XZ) \in \delta(q_0, a, Z)) \\ & \vdash (q_1, b, XZ) & (\because (q_1, X) \in \delta(q_0, \epsilon, X)) \end{aligned}$$

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Definition (Acceptance by Final States)

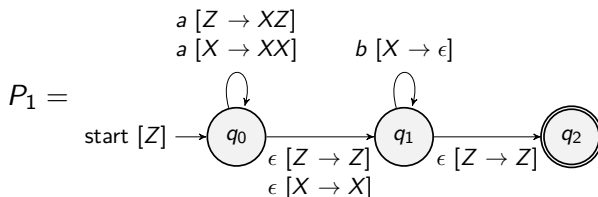
For a given PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, the language accepted by **final states** is defined as:

$$L_F(P) = \{w \in \Sigma^* \mid (q_0, w, Z) \vdash^* (q, \epsilon, \alpha) \text{ for some } q \in F, \alpha \in \Gamma^*\}$$

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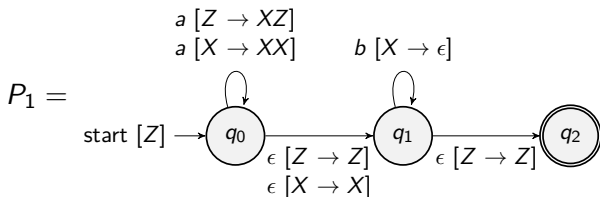
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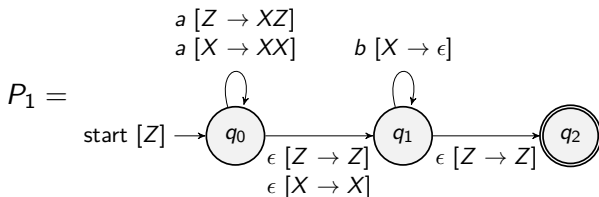


$$(q_0, ab, Z) \vdash^* (q_2, \epsilon, Z) \implies ab \in L_F(P)$$

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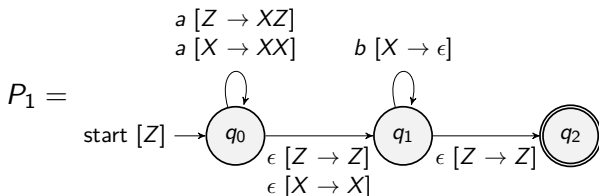


$$\begin{aligned} (q_0, ab, Z) &\vdash^* (q_2, \epsilon, Z) &\implies & ab \in L_F(P) \\ (q_0, aabb, Z) &\vdash^* (q_2, \epsilon, Z) &\implies & aabb \in L_F(P) \end{aligned}$$

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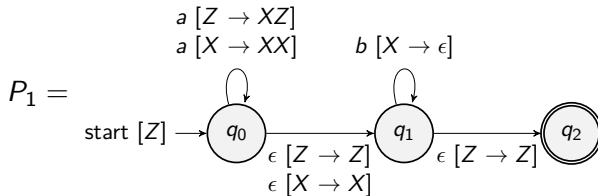
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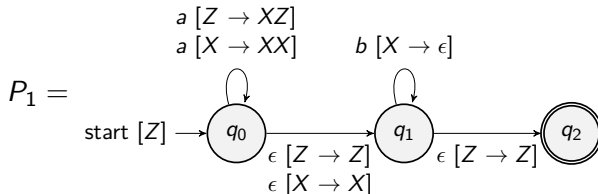
$$\begin{aligned} (q_0, ab, Z) \vdash^* (q_2, \epsilon, Z) &\implies ab \in L_F(P) \\ (q_0, aabb, Z) \vdash^* (q_2, \epsilon, Z) &\implies aabb \in L_F(P) \end{aligned}$$

$$L_F(P_1) = \{a^n b^n \mid n \geq 0\}$$



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¹<https://plrg.korea.ac.kr/courses/cose215/materials/pda-an-bn-final.pdf>

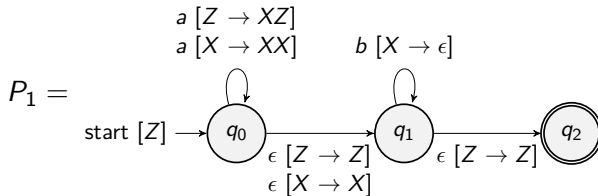


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The key idea is to **count** the number of a's using the stack.

- Start with the stack only having the initial stack alphabet Z .
- Repeatedly **push** X onto the stack for each a .
- Repeatedly **pop** X from the stack for each b .
- Accept when the stack only contains Z .

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See the additional material for the input string $aaabbb$.¹

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```
// The type definition of configurations
case class Config(state: State, word: Word, stack: List[Alphabet])
case class PDA(...):
  // Configurations reachable from the initial configuration
  def reachableConfig(init: Config): Set[Config] = ... // See PDA.scala
  // The initial configuration
  def init(word: Word): Config =
    Config(initState, word, List(initAlphabet))
  // Acceptance by final states
  def acceptByFinalState(word: Word): Boolean =
    reachableConfig(init(word)).exists(config => {
      val Config(q, w, _) = config
      w.isEmpty && finalStates.contains(q)
    })

acceptByFinalState(pda1)("ab")    // true
acceptByFinalState(pda1)("aba")  // false
```

Definition (Acceptance by Empty Stacks)

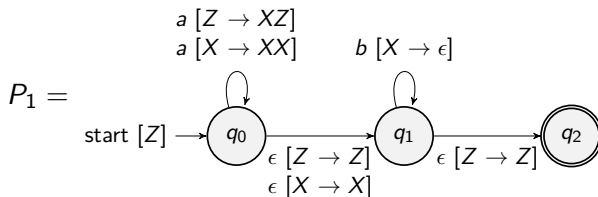
For a given PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, the language accepted by **empty stacks** is defined as:

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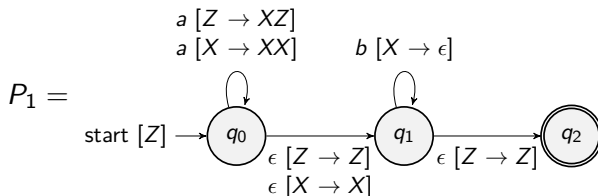
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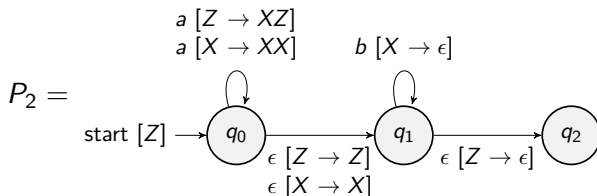


$$L_E(P_1) = \emptyset$$

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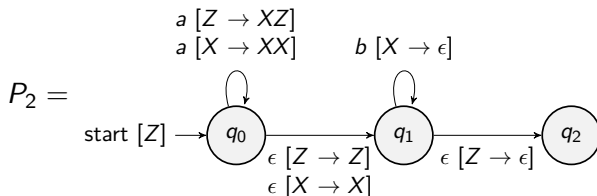
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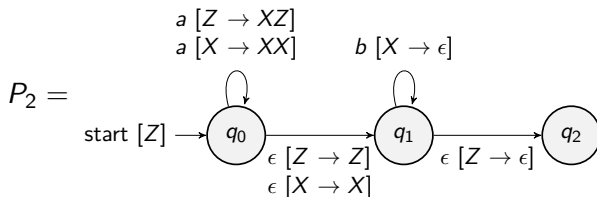


$$(q_0, ab, Z) \vdash^* (q_2, \epsilon, \epsilon) \implies ab \in L_E(P)$$

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$$(q_0, ab, Z) \vdash^* (q_2, \epsilon, \epsilon) \implies ab \in L_E(P)$$

$$L_E(P_2) = \{a^n b^n \mid n \geq 0\}$$

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  // The initial configuration
  def init(word: Word): Config =
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  // Acceptance by empty stacks
  def acceptByEmptyStack(word: Word): Boolean =
    reachableConfig(init(word)).exists(config => {
      val Config(_, w, xs) = config
      w.isEmpty && xs.isEmpty
    })

acceptByEmptyStack(pda2)("ab")    // true
acceptByEmptyStack(pda2)("aba")  // false
```

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- Transition Diagram

- Pushdown Automata in Scala

- Configurations and One-Step Moves

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- Examples of Pushdown Automata

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