

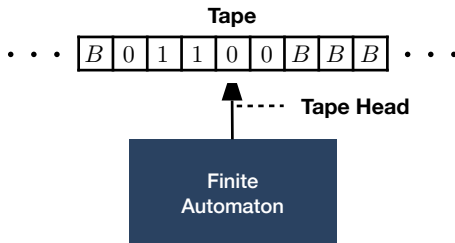
Lecture 22 – Examples of Turing Machines

COSE215: Theory of Computation

Jihyeok Park



2025 Spring



- A **Turing machine (TM)** is a **deterministic** FA with a **tape**.
 - ① A **tape** is an infinite sequence of cells containing **tape symbols**.
(The **blank symbol** B is a special symbol representing an empty cell.)
 - ② A **tape head** points to the current cell.
 - ③ A **transition** performs the following operations depending on the current 1) **state** and 2) **tape symbol** pointed by the tape head:
 - **Change** the current **state**.
 - **Replace** the current **tape symbol** pointed by the tape head.
 - **Move** the **tape head** left or right.
- We can use Turing machines as **computing machines**.

1. Turing Machines as Word Recognizers

Example 1: $L = \{a^n b^n c^n \mid n \geq 0\}$

Example 2: $L = \{ww \mid w \in \{a, b\}^*\}$

Example 3: $L = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$

2. Turing Machines as Computing Machines

Example 4: Flip Bits – $f(w \in \{0, 1\}^*) = (\text{flip of } w)$

Example 5: Unary Addition – $f(1^n + 1^m) = 1^{n+m}$

Example 6: Binary Increment – $f(w \in \{0, 1\}^*) = w + 1$

Example 7: Data Copy – $f(w \in \{a, b\}^*) = ww$

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Example 1: $L = \{a^n b^n c^n \mid n \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^n b^n c^n \mid n \geq 0\}$$

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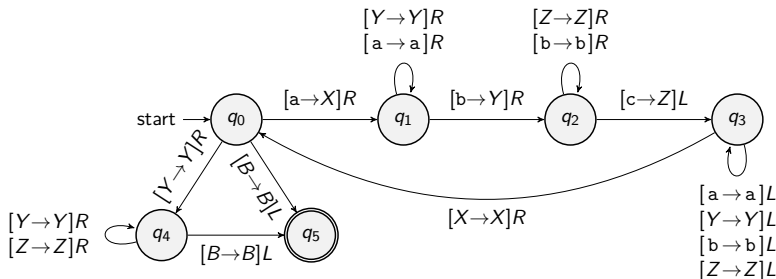
- 1: **while** there are a's **do**
- 2: Find and Replace a with X
- 3: Find and Replace b with Y
- 4: Find and Replace c with Z
- 5: Check if only Y's and Z's are left

Example 1: $L = \{a^n b^n c^n \mid n \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^n b^n c^n \mid n \geq 0\}$$

See the example for $aabbcc \in L(M)$.¹



¹<https://plrg.korea.ac.kr/courses/cose215/materials/tm-an-bn-cn.pdf>

Example 2: $L = \{ww \mid w \in \{a, b\}^*\}$

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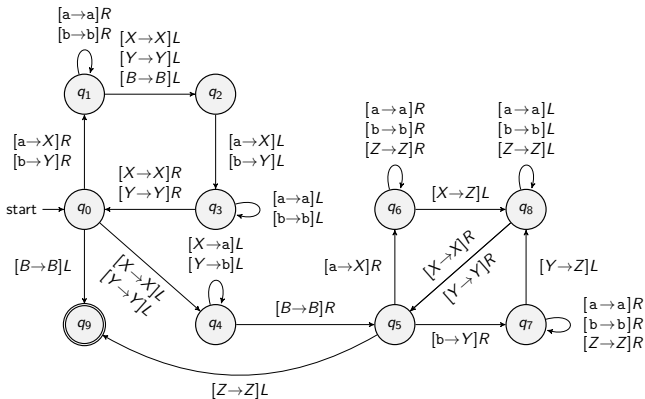
- 1: Find the middle of the input by repeatedly replacing leftmost and rightmost a's (or b's) with X's (or Y's)
- 2: Replace all X's (or Y's) with a's (or b's) in the first half
- 3: **while** there are input symbols in the first half **do**
- 4: Replace a (or b) with X (or Y) in the first half
- 5: Find and Replace matched X (or Y) with Z in the second half

Example 2: $L = \{ww \mid w \in \{a, b\}^*\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{ww \mid w \in \{a, b\}^*\}$$

See the example for $abbabb \in L(M)$.²



²<https://plrg.korea.ac.kr/courses/cose215/materials/tm-w-w.pdf>

Example 3: $L = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

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Construct a **Turing machine** that **accepts** the language:

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...	<i>B</i>	a	a	b	b	b	c	c	c	c	c	c	<i>B</i>	...
-----	----------	---	---	---	---	---	---	---	---	---	---	---	----------	-----

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-----	---	---	---	---	---	---	---	---	---	---	---	---	---	-----

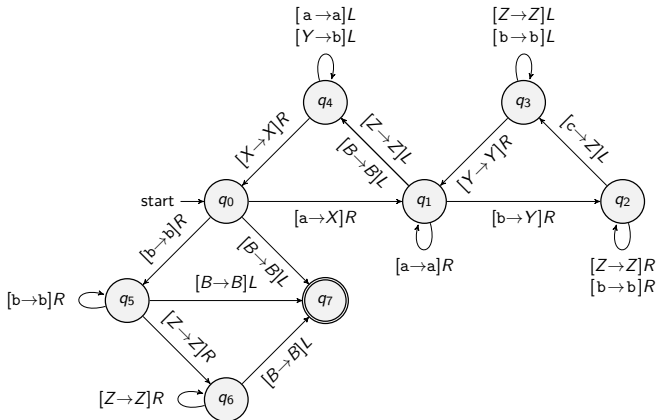
- 1: **while** there are a's **do**
- 2: Find and Replace a with X
- 3: **while** there are b's **do**
- 4: Find and Replace b with Y
- 5: Find and Replace c with Z
- 6: Roll back all Y's to b's
- 7: Check if only b's and Z's are left

Example 3: $L = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$$

See the example for $aabb bcccccc \in L(M)$.³



³<https://plrg.korea.ac.kr/courses/cose215/materials/tm-ai-bj-cij.pdf>

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2. Turing Machines as Computing Machines

Example 4: Flip Bits – $f(w \in \{0, 1\}^*) = (\text{flip of } w)$

Example 5: Unary Addition – $f(1^n + 1^m) = 1^{n+m}$

Example 6: Binary Increment – $f(w \in \{0, 1\}^*) = w + 1$

Example 7: Data Copy – $f(w \in \{a, b\}^*) = ww$

Example 4: Flip Bits – $f(w \in \{0, 1\}^*) = (\text{flip of } w)$



Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = (\text{the flip of each bit in } w)$$

Example 4: Flip Bits – $f(w \in \{0, 1\}^*) = (\text{flip of } w)$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = (\text{the flip of each bit in } w)$$

$$w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline \dots & B & 1 & 0 & 1 & 1 & 1 & 0 & 0 & B & \dots \\ \hline \end{array}$$

$$f(w) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline \dots & B & 0 & 1 & 0 & 0 & 0 & 1 & 1 & B & \dots \\ \hline \end{array}$$

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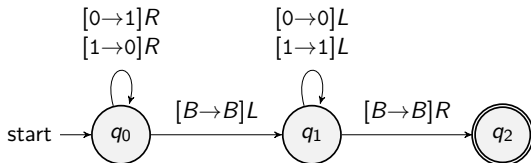
- 1: Flip each bit of the input: $1 \rightarrow 0$ and $0 \rightarrow 1$
- 2: Go to the first input symbol

Example 4: Flip Bits – $f(w \in \{0, 1\}^*) = (\text{flip of } w)$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = (\text{the flip of each bit in } w)$$

See the example for $f(1011100) = 0100011$.⁴



⁴<https://plrg.korea.ac.kr/courses/cose215/materials/tm-flip.pdf>

Example 5: Unary Addition – $f(1^n 1^m) = 1^{n+m}$

Construct a **Turing machine** that **computes** the function:

$$f(1^n 1^m) = 1^{n+m} \quad \text{where} \quad n, m \geq 0$$

Example 5: Unary Addition – $f(1^n+1^m) = 1^{n+m}$

Construct a **Turing machine** that **computes** the function:

$$f(1^n+1^m) = 1^{n+m} \quad \text{where} \quad n, m \geq 0$$

$$w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \dots & B & 1 & 1 & 1 & + & 1 & 1 & B & \dots \\ \hline \end{array}$$

$$f(w) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \dots & B & 1 & 1 & 1 & 1 & 1 & B & B & \dots \\ \hline \end{array}$$

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$$w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \dots & B & 1 & 1 & 1 & + & 1 & 1 & B & \dots \\ \hline \end{array}$$

$$f(w) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \dots & B & 1 & 1 & 1 & 1 & 1 & B & B & \dots \\ \hline \end{array}$$

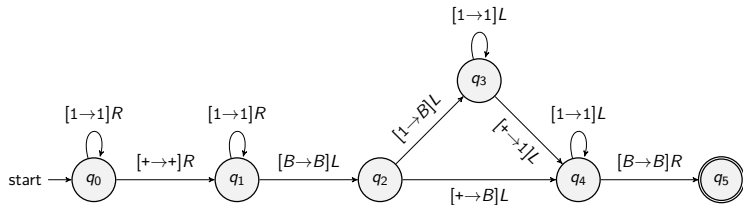
- 1: Find + after 1's
- 2: **if** the last symbol is 1 **then**
- 3: Find and Remove the last 1
- 4: Find and Replace the + with 1
- 5: **else**
- 6: Remove the +
- 7: Go to the first input symbol

Example 5: Unary Addition – $f(1^n+1^m) = 1^{n+m}$

Construct a **Turing machine** that **computes** the function:

$$f(1^n+1^m) = 1^{n+m} \quad \text{where} \quad n, m \geq 0$$

See the example for $f(111+11) = 11111$.⁵



⁵<https://plrg.korea.ac.kr/courses/cose215/materials/tm-unary-add.pdf>

Example 6: Binary Increment – $f(w \in \{0, 1\}^*) = w + 1$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = w + 1 \quad \text{where} \quad w \text{ starts with } 1$$

Example 6: Binary Increment – $f(w \in \{0, 1\}^*) = w + 1$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = w + 1 \quad \text{where} \quad w \text{ starts with } 1$$

$w =$

...	B	1	0	1	0	1	1	1	1	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

$f(w) =$

...	B	1	0	1	1	0	0	0	0	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

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$w =$

...	B	1	0	1	0	1	1	1	1	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

$f(w) =$

...	B	1	0	1	1	0	0	0	0	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

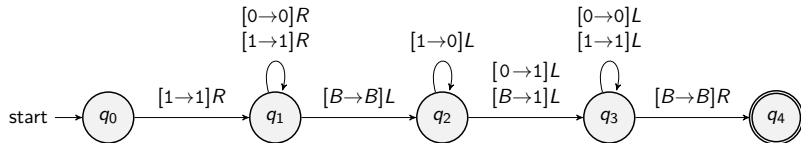
- 1: Check if the first bit is 1.
- 2: Move to the end of the input.
- 3: Repeatedly replace the rightmost 1 with 0.
- 4: Replace 0 (or B) with 1.
- 5: Go to the first input symbol.

Example 6: Binary Increment – $f(w \in \{0, 1\}^*) = w + 1$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0, 1\}^*) = w + 1 \quad \text{where} \quad w \text{ starts with } 1$$

See the example for $f(10101111) = 10110000$.⁶



⁶<https://plrg.korea.ac.kr/courses/cose215/materials/tm-inc.pdf>

Example 7: Data Copy – $f(w \in \{a, b\}^*) = ww$

Construct a **Turing machine** that **computes** the function:

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$w =$

...	B	a	b	b	B	B	B	B	...
-----	-----	---	---	---	-----	-----	-----	-----	-----

$f(w) =$

...	B	a	b	b	a	b	b	B	...
-----	-----	---	---	---	---	---	---	-----	-----

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-----	-----	---	---	---	-----	-----	-----	-----	-----

$f(w) =$

...	B	a	b	b	a	b	b	B	...
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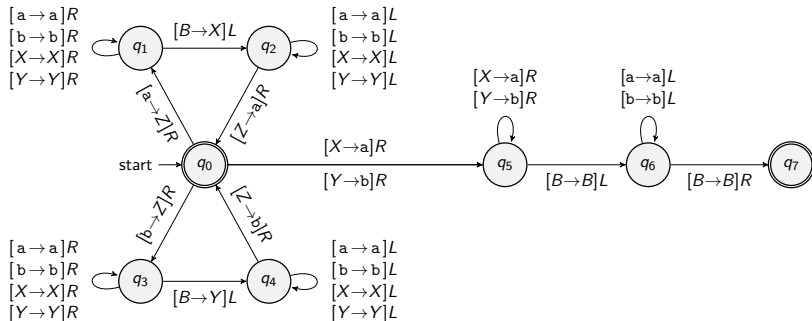
- 1: **while** there are input symbols **do**
- 2: Find and Replace a (or b) with Z
- 3: Find and Fill the first blank with X (or Y) for a (or b)
- 4: Roll back Z to the original a (or b)
- 5: Replace X 's and Y 's with a's and b's
- 6: Go to the first input symbol

Example 7: Data Copy – $f(w \in \{a, b\}^*) = ww$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{a, b\}^*) = ww$$

See the example for $f(abb) = abbabb$.⁷



⁷<https://plrg.korea.ac.kr/courses/cose215/materials/tm-data-copy.pdf>

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- Extensions of Turing Machines

Jihyeok Park

jihyeok_park@korea.ac.kr

<https://plrg.korea.ac.kr>