

Lecture 14 – Continuations (1)

COSE212: Programming Languages

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2025 Fall

- Lazy Evaluation
 - Call-by-Name (CBN)
 - Call-by-Need (CBN')
- LFAE – FAE with Lazy Evaluation
- We will learn about **continuations** with the following topics:
 - **Continuations** (Lecture 14 & 15)
 - **First-Class Continuations** (Lecture 16)
 - **Compiling with continuations** (Lecture 17)
- In this lecture, we will focus on the meaning of **continuations**.

1. Continuations

2. Continuation-Passing Style (CPS)

3. Interpreter of FAE in CPS

Addition and Multiplication

Function Application

1. Continuations

2. Continuation-Passing Style (CPS)

3. Interpreter of FAE in CPS

- Addition and Multiplication
- Function Application

Many real-world programming languages support **control statements** to change the **control-flow** of a program.

For example, C++ supports `break`, `continue`, and `return` statements:

```
int sumEvenUntilZero(int xs[], int len) {  
    if (len <= 0) return 0;           // directly return 0 if len <= 0  
    int sum = 0;  
    for (int i = 0; i < len; i++) {  
        if (xs[i] == 0) break;        // stop the loop if xs[i] == 0  
        if (xs[i] % 2 == 1) continue; // skip the rest if xs[i] is odd  
        sum += xs[i];  
    }  
    return sum;                       // finally return the sum  
}  
  
int xs[] = {4, 1, 3, 2, 0, 6, 5, 8};  
sumEvenUntilZero(xs, 8);             // 4 + 2 = 6
```

How can we represent them in functional languages? **Continuations!**

Intuitively, a **continuation** represents the **rest of the computation**.

For example, consider the following FAE expression:

```
/* FAE */  
(1 + 3) * 5
```

It **implicitly** represents the following computation:

- | | |
|---|-------------------------|
| ① Evaluate 1. | (Result: 1) |
| ② Evaluate 3. | (Result: 3) |
| ③ Add the results of step ① and ②. | (Result: $1 + 3 = 4$) |
| ④ Evaluate 5. | (Result: 5) |
| ⑤ Multiply the results of step ③ and ④. | (Result: $4 * 5 = 20$) |

The **continuation** of k -th step is the steps from $(k + 1)$ -th to the last one.

For instance, the **continuation** of the 3rd step is the 4th and 5th steps.

Can we **explicitly** represent the **continuations** in the expression?

Yes! Let's represent the **continuation** of the k -th step as a **function** that

- **takes** the result of the k -th step as an argument and
- **performs** the $(k + 1)$ -th to the last steps.

If e' is the **current evaluation part** in the expression e :

$$e = (\dots e' \dots)$$

we can revise it as:

$$(\lambda x. (\dots x \dots))(e')$$

where $\lambda x. (\dots x \dots)$ is the **continuation** of e' .

Let's explicitly represent the **continuations** of the previous example:

```
/* FAE */  
(1 + 3) * 5
```

- | | |
|---|-------------------------|
| ① Evaluate 1. | (Result: 1) |
| ② Evaluate 3. | (Result: 3) |
| ③ Add the results of step ① and ②. | (Result: $1 + 3 = 4$) |
| ④ Evaluate 5. | (Result: 5) |
| ⑤ Multiply the results of step ③ and ④. | (Result: $4 * 5 = 20$) |

```
/* FAE */  
(1 + 3) * 5
```


- | | |
|---|-------------------------|
| ① Evaluate 1. | (Result: 1) |
| ② Evaluate 3. | (Result: 3) |
| ③ Add the results of step ① and ②. | (Result: $1 + 3 = 4$) |
| ④ Evaluate 5. | (Result: 5) |
| ⑤ Multiply the results of step ③ and ④. | (Result: $4 * 5 = 20$) |

```
/* FAE */  
{  
  x1 => (x1 + 3) * 5      // step 2-5 (continuation of step 1)  
}(1)                    // step 1
```

- | | |
|---|-------------------------|
| ① Evaluate 1. | (Result: 1) |
| ② Evaluate 3. | (Result: 3) |
| ③ Add the results of step ① and ②. | (Result: $1 + 3 = 4$) |
| ④ Evaluate 5. | (Result: 5) |
| ⑤ Multiply the results of step ③ and ④. | (Result: $4 * 5 = 20$) |

```
/* FAE */
{
  x1 => {
    x2 => (x1 + x2) * 5    // step 3-5 (continuation of step 2)
  }(3)                  // step 2
}(1)                    // step 1
```

- | | |
|---|-------------------------|
| ① Evaluate 1. | (Result: 1) |
| ② Evaluate 3. | (Result: 3) |
| ③ Add the results of step ① and ②. | (Result: $1 + 3 = 4$) |
| ④ Evaluate 5. | (Result: 5) |
| ⑤ Multiply the results of step ③ and ④. | (Result: $4 * 5 = 20$) |

```
/* FAE */  
{  
  x1 => {  
    x2 => {  
      x3 => x3 * 5           // step 4-5 (continuation of step 3)  
    }(x1 + x2)              // step 3  
  }(3)                      // step 2  
(1)                        // step 1
```

- | | |
|---|-------------------------|
| ① Evaluate 1. | (Result: 1) |
| ② Evaluate 3. | (Result: 3) |
| ③ Add the results of step ① and ②. | (Result: $1 + 3 = 4$) |
| ④ Evaluate 5. | (Result: 5) |
| ⑤ Multiply the results of step ③ and ④. | (Result: $4 * 5 = 20$) |

```
/* FAE */
{
  x1 => {
    x2 => {
      x3 => {
        x4 => x3 * x4    // step 5 (continuation of step 4)
      }(5)              // step 4
    }(x1 + x2)          // step 3
  }(3)                 // step 2
}(1)                   // step 1
```

- | | |
|---|-------------------------|
| ① Evaluate 1. | (Result: 1) |
| ② Evaluate 3. | (Result: 3) |
| ③ Add the results of step ① and ②. | (Result: $1 + 3 = 4$) |
| ④ Evaluate 5. | (Result: 5) |
| ⑤ Multiply the results of step ③ and ④. | (Result: $4 * 5 = 20$) |

```
/* FAE */
{
  x1 => {
    x2 => {
      x3 => {
        x4 => {
          x5 => x5           // no more steps (continuation of step 5)
        }(x3 * x4)         // step 5
      }(5)                 // step 4
    }(x1 + x2)             // step 3
  }(3)                     // step 2
}(1)                       // step 1
```

- | | |
|---|----------------------|
| ① Evaluate 1. | (Result: 1) |
| ② Evaluate 3. | (Result: 3) |
| ③ Add the results of step ① and ②. | (Result: 1 + 3 = 4) |
| ④ Evaluate 5. | (Result: 5) |
| ⑤ Multiply the results of step ③ and ④. | (Result: 4 * 5 = 20) |

```

/* FAE */
val x1 = 1;           // step 1
val x2 = 3;           // step 2
val x3 = x1 + x2;     // step 3
val x4 = 5;           // step 4
val x5 = x3 * x4;     // step 5
x5                   // no more steps (continuation of step 5)
    
```

by using the syntactic sugar for variable definitions (`val`).

$$\mathcal{D}[\text{val } x = e; e'] = (\lambda x. \mathcal{D}[e']) (\mathcal{D}[e])$$

1. Continuations

2. Continuation-Passing Style (CPS)

3. Interpreter of FAE in CPS

Addition and Multiplication

Function Application

So far, we implement functions in **direct style**, where the result of a function is **returned** to the caller.

For example, the following Scala `sum` function is written in **direct style**:

```
def sum(n: Int): Int =  
  if (n <= 1) 1  
  else sum(n - 1) + n  
  
sum(3) * 5           // (1 + 2 + 3) * 5 = 30
```

Continuation-passing style (CPS) is a style of programming that passes the continuation as an explicit parameter to a function and calls it to give the result to the continuation. Let's rewrite the `sum` function in CPS:

```
type Cont = Int => Int  
def sumCPS(n: Int, k: Cont): Int = ???  
  
sumCPS(3, x => x * 5)   // (1 + 2 + 3) * 5 = 30
```



```
def sum(n: Int): Int =  
  if (n <= 1) 1  
  else sum(n - 1) + n
```

Let's rewrite the sum function in CPS:

```
type Cont = Int => Int  
def sumCPS(n: Int, k: Cont): Int = ???
```

```
def sum(n: Int): Int =  
  if (n <= 1) 1  
  else sum(n - 1) + n
```

Let's rewrite the sum function in CPS:

```
type Cont = Int => Int  
def sumCPS(n: Int, k: Cont): Int = k(sum(n))
```

It is not the correct implementation of sum in CPS because it depends on the original sum function.

Let's replace `sum(n)` with the body of `sum`.

```
def sum(n: Int): Int =  
  if (n <= 1) 1  
  else sum(n - 1) + n
```

Let's rewrite the sum function in CPS:

```
type Cont = Int => Int  
def sumCPS(n: Int, k: Cont): Int = k(  
  if (n <= 1) 1  
  else sum(n - 1) + n  
)
```

Let's utilize the following equivalence:

```
e0(if (e1) e2 else e3)    ==    if (e1) e0(e2) else e0(e3)
```

```
def sum(n: Int): Int =  
  if (n <= 1) 1  
  else sum(n - 1) + n
```

Let's rewrite the sum function in CPS:

```
type Cont = Int => Int  
def sumCPS(n: Int, k: Cont): Int =  
  if (n <= 1) k(1)  
  else k(sum(n - 1) + n)
```

But, it still depends on the original sum function.

Let's utilize the following equivalence:

$$\begin{aligned} k(\text{sum}(n - 1) + n) &== (x \Rightarrow k(x + n))(\text{sum}(n - 1)) \\ &== \text{sumCPS}(n - 1, x \Rightarrow k(x + n)) \end{aligned}$$

because $x \Rightarrow k(x + n)$ is the continuation of $\text{sum}(n - 1)$.

```
def sum(n: Int): Int =  
  if (n <= 1) 1  
  else sum(n - 1) + n
```

Let's rewrite the sum function in CPS:

```
type Cont = Int => Int  
def sumCPS(n: Int, k: Cont): Int =  
  if (n <= 1) k(1)  
  else sumCPS(n - 1, x => k(x + n))
```

If all functions are written in CPS, a program satisfies the properties:

- Every function takes a continuation as an explicit parameter.
- A continuation is used at most once in a function body.
- Every function call is in a tail position. (tail-call optimization)
- Every function ends with a function call.

1. Continuations

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Addition and Multiplication

Function Application

The original interpreter of FAE is written in **direct style**, and continuations of the evaluation of expressions are **implicitly** represented.

```
def interp(expr: Expr, env: Env): Value = expr match
  case Num(n)      => NumV(n)
  case Add(l, r)    => numAdd(interp(l, env), interp(r, env))
  case Mul(l, r)    => numMul(interp(l, env), interp(r, env))
  case Id(x)        => env.getOrElse(x, error(s"free identifier: $x"))
  case Fun(p, b)    => CloV(p, b, env)
  case App(f, a)    => interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
```

To **explicitly** represent continuations of the evaluation of each expression in the interpreter of FAE, we need to modify the interpreter in **CPS**:

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = ???
```

The original interpreter of FAE is written in **direct style**, and continuations of the evaluation of expressions are **implicitly** represented.

```
def interp(expr: Expr, env: Env): Value = expr match
  case Num(n)      => NumV(n)
  case Add(l, r)    => numAdd(interp(l, env), interp(r, env))
  case Mul(l, r)    => numMul(interp(l, env), interp(r, env))
  case Id(x)        => env.getOrElse(x, error(s"free identifier: $x"))
  case Fun(p, b)    => CloV(p, b, env)
  case App(f, a)    => interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
```

To **explicitly** represent continuations of the evaluation of each expression in the interpreter of FAE, we need to modify the interpreter in **CPS**:

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value =
  k(interp(expr, env))
```


The original interpreter of FAE is written in **direct style**, and continuations of the evaluation of expressions are **implicitly** represented.

To **explicitly** represent continuations of the evaluation of each expression in the interpreter of FAE, we need to modify the interpreter in **CPS**:

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = k(expr match
  case Num(n)      => NumV(n)
  case Add(l, r)   => numAdd(interp(l, env), interp(r, env))
  case Mul(l, r)   => numMul(interp(l, env), interp(r, env))
  case Id(x)       => env.getOrElse(x, error(s"free identifier: $x"))
  case Fun(p, b)   => CloV(p, b, env)
  case App(f, a)   => interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
)
```

The original interpreter of FAE is written in **direct style**, and continuations of the evaluation of expressions are **implicitly** represented.

To **explicitly** represent continuations of the evaluation of each expression in the interpreter of FAE, we need to modify the interpreter in **CPS**:

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  case Num(n)      => k(NumV(n))
  case Add(l, r)   => k(numAdd(interp(l, env), interp(r, env)))
  case Mul(l, r)   => k(numMul(interp(l, env), interp(r, env)))
  case Id(x)       => k(env.getOrElse(x, error(s"free identifier: $x")))
  case Fun(p, b)   => k(CloV(p, b, env))
  case App(f, a)   => k(interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
  )
)
```

The original interpreter of FAE is written in **direct style**, and continuations of the evaluation of expressions are **implicitly** represented.

To **explicitly** represent continuations of the evaluation of each expression in the interpreter of FAE, we need to modify the interpreter in **CPS**:

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  case Num(n)      => k(NumV(n))
  case Add(l, r)   => k(numAdd(interp(l, env), interp(r, env)))
  case Mul(l, r)   => k(numMul(interp(l, env), interp(r, env)))
  case Id(x)       => k(env.getOrElse(x, error(s"free identifier: $x")))
  case Fun(p, b)   => k(CloV(p, b, env))
  case App(f, a)   => k(interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
  )
)
```

Let's modify the Add, Mul, and App cases because they still use the original interp function.

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    k(numAdd(interp(l, env), interp(r, env)))
  ...
```

The current evaluation part is `interp(l, env)`.

Its continuation is `lv => k(numAdd(lv, interp(r, env)))`.

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    {
      lv => k(numAdd(lv, interp(r, env))) // cont. of `interp(l, env)`
    }(interp(l, env))
  ...
```

Let's rewrite it by passing the continuation into `interpCPS`.

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    interpCPS(l, env, {
      lv => k(numAdd(lv, interp(r, env))) // cont. of `interp(l, env)`
    })
  ...
```

Similarly, the current evaluation part is `interp(r, env)`.

Its continuation is `rv => k(numAdd(lv, rv))`.

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    interpCPS(l, env, {
      lv => {
        rv => k(numAdd(lv, rv))           // cont. of `interp(r, env)`
      }(interp(r, env))
    })
  ...
```

Let's rewrite it by passing the continuation into interpCPS.

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    interpCPS(l, env, {
      lv => interpCPS(r, env, {
        rv => k(numAdd(lv, rv))           // cont. of `interp(r, env)`
      })
    })
  ...
```



```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    interpCPS(l, env, {
      lv => interpCPS(r, env, {
        rv => k(numAdd(lv, rv))
      })
    })
  case Mul(l, r) =>
    interpCPS(l, env, {
      lv => interpCPS(r, env, {
        rv => k(numMul(lv, rv))
      })
    })
  ...
```

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case App(f, a) => k(interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
  )
  ...
```

In a similar way, we can rewrite function application case.

The current evaluation part is `interp(f, env)`.

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
...
case App(f, a) => interpCPS(f, env, {
  // cont. of `interp(f, env)`
  fv => k(fv match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
  )
})
...
```

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case App(f, a) => interpCPS(f, env, fv => k(fv match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
  ))
  ...
```

Let's move the continuation invocation `k(...)` into the inside of the `match` expression.

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case App(f, a) => interpCPS(f, env, fv => fv match
    case CloV(p, b, fenv) => k(interp(b, fenv + (p -> interp(a, env))))
    case v                => error(s"not a function: ${v.str}")
  )
  ...
```

We do not need to wrap `error(...)` with `k` because it does not return a value but throws an exception.

Now, the current evaluation part is `interp(a, env)`.

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
...
case App(f, a) => interpCPS(f, env, fv => fv match
  case CloV(p, b, fenv) =>
    interpCPS(a, env, {
      // cont. of `interp(a, env)`
      av => k(interp(b, fenv + (p -> av)))
    })
  case v => error(s"not a function: ${v.str}")
)
...
```

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case App(f, a) => interpCPS(f, env, fv => fv match
    case CloV(p, b, fenv) =>
      interpCPS(a, env, av => k(interp(b, fenv + (p -> av))))
    case v => error(s"not a function: ${v.str}")
  )
  ...
```

Now, the current evaluation part is `interp(b, fenv + (p -> av))`.

```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
...
case App(f, a) => interpCPS(f, env, fv => fv match
  case CloV(p, b, fenv) =>
    interpCPS(a, env, av => interpCPS(b, fenv + (p -> av), {
      // cont. of `interp(b, fenv + (p -> av))`
      k
    })
  case v => error(s"not a function: ${v.str}")
)
...
```



```
type Cont = Value => Value
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case App(f, a) => interpCPS(f, env, fv => fv match
    case CloV(p, b, fenv) =>
      interpCPS(a, env, av => interpCPS(b, fenv + (p -> av), k)
    case v => error(s"not a function: ${v.str}")
  )
  ...
```

1. Continuations
2. Continuation-Passing Style (CPS)
3. Interpreter of FAE in CPS
 - Addition and Multiplication
 - Function Application

- The score for the midterm exam will be uploaded to [LMS](#) by tomorrow before noon.
- The claim hours are scheduled as follows:
 - **10/27 (Mon.)** 15:00-18:00
 - **10/28 (Tue.)** 15:00-18:00
 - **Place:** 507, Jung Woonoh IT & General Education Center

- Continuations (2)

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