

# Lecture 14 – Pushdown Automata (PDA)

COSE215: Theory of Computation

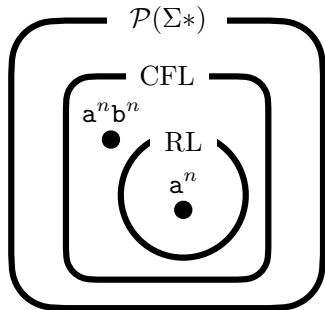
Jihyeok Park



2024 Spring

- A context-free grammar (CFG):

$$G = (V, \Sigma, S, R)$$



| Languages                   | Automata             | Grammars                   |
|-----------------------------|----------------------|----------------------------|
| Context-Free Language (CFL) | ???                  | Context-Free Grammar (CFG) |
| Regular Language (RL)       | Finite Automata (FA) | Regular Expression (RE)    |

## 1. Pushdown Automata

- Definition

- Transition Diagram

- Pushdown Automata in Scala

- Configurations and One-Step Moves

- Acceptance by Final States

- Acceptance by Empty Stacks

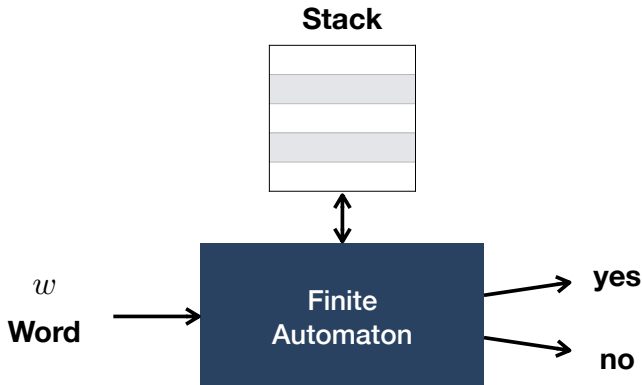
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- In FA, the next state is determined by the current **state** and **symbol**.
- In PDA, the next state is determined by the current **state**, **symbol**, and the **top element of the stack**.



## Definition (Pushdown Automata)

A **pushdown automaton (PDA)** is a 7-tuple:

$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$$

where

- $Q$  is a finite set of **states**
- $\Sigma$  is a finite set of **symbols**
- $\Gamma$  is a finite set of **stack alphabets**
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma^*)$  is a **transition function**
- $q_0 \in Q$  is the **initial state**
- $Z \in \Gamma$  is the **initial stack alphabet** (the stack is initially  $Z$ )
- $F \subseteq Q$  is a set of **final states**

$$P_1 = (\{q_0, q_1, q_2\}, \{a, b\}, \{X, Z\}, \delta, q_0, Z, \{q_2\})$$

where

$$\delta(q_0, a, Z) = \{(q_0, XZ)\}$$

$$\delta(q_0, b, Z) = \emptyset$$

$$\delta(q_0, \epsilon, Z) = \{(q_1, Z)\}$$

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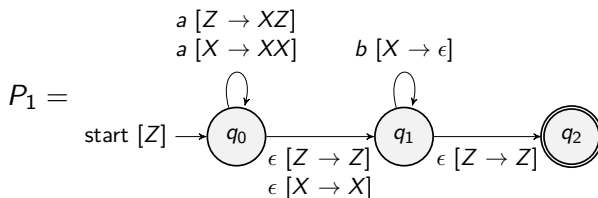
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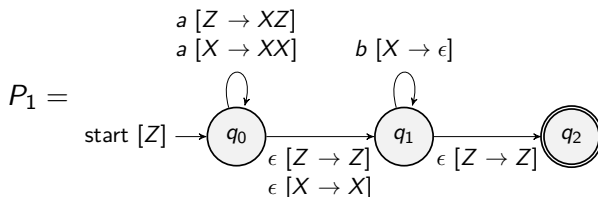
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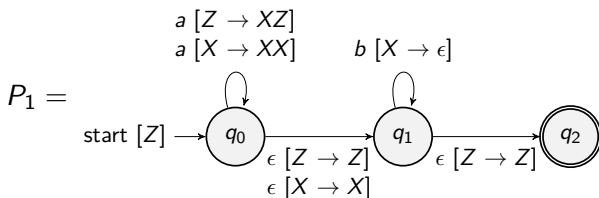
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For example,

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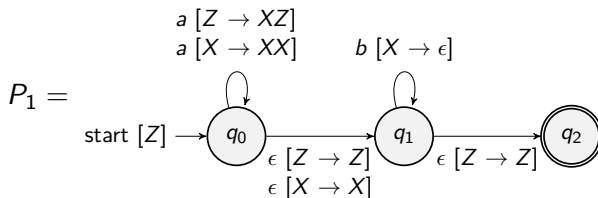


For example,

$$\begin{aligned}\delta(q_0, a, Z) &= \{(q_0, XZ)\} \\ \delta(q_0, \epsilon, X) &= \{(q_1, X)\}\end{aligned}$$

```
// The type definitions of states, symbols, words, and stack alphabets
type State = Int
type Symbol = Char
type Word = String
type Alphabet = String

// The definition of PDA
case class PDA(
  states: Set[State],
  symbols: Set[Symbol],
  alphabets: Set[Alphabet],
  trans: Map[
    (State, Option[Symbol], Alphabet),
    Set[(State, List[Alphabet])]
  ],
  initState: State,
  initAlphabet: Alphabet,
  finalStates: Set[State],
)
```



```

val pda1: PDA = PDA(
  states = Set(0, 1, 2),      symbols = Set('a', 'b'),
  alphabets = Set("X", "Z"),
  trans = Map(
    (0, Some('a'), "Z") -> Set((0, List("X", "Z"))),
    (0, Some('a'), "X") -> Set((0, List("X", "X"))),
    (0, None, "Z") -> Set((1, List("Z"))),
    (0, None, "X") -> Set((1, List("X"))),
    (1, Some('b'), "X") -> Set((1, List())),
    (1, None, "Z") -> Set((2, List("Z"))),
  ).withDefaultValue(Set()),
  initState = 0, initAlphabet = "Z", finalStates = Set(2),
)
    
```

## Definition (Configurations of PDA)

A **configuration** of a PDA  $P$  represents the current status of  $P$ . It is defined as a triple  $(q, w, \alpha)$  where

- $q \in Q$ : the current state
- $w \in \Sigma^*$ : the remaining word
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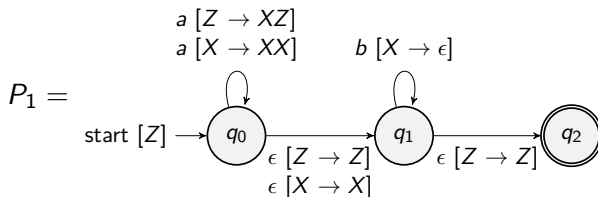
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## Definition (One-Step Moves of PDA)

A **one-step move** ( $\vdash$ ) of a PDA  $P$  is a transition from a configuration to another configuration:

$$\begin{aligned}(q, aw, X\beta) &\vdash (p, w, \alpha\beta) && \text{if } (p, \alpha) \in \delta(q, a, X) \\ (q, w, X\beta) &\vdash (p, w, \alpha\beta) && \text{if } (p, \alpha) \in \delta(q, \epsilon, X)\end{aligned}$$

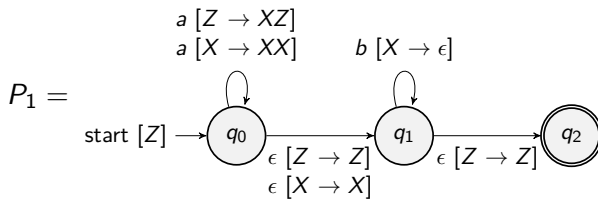
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$(q_0, ab, Z)$

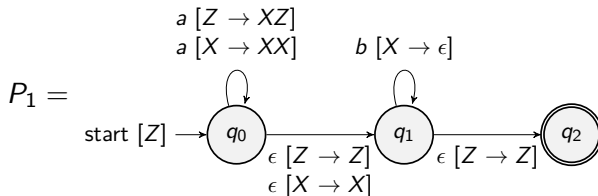


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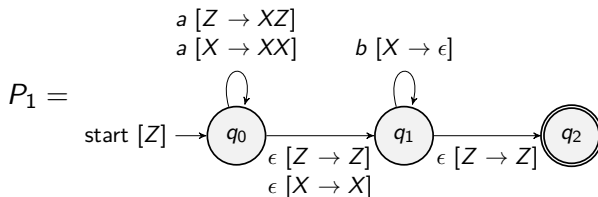
$$(q_0, ab, Z) \vdash (q_0, b, XZ) \quad (\because (q_0, XZ) \in \delta(q_0, a, Z))$$

$$\begin{aligned} (q, aw, X\beta) \vdash (p, w, \alpha\beta) & \text{ if } (p, \alpha) \in \delta(q, a, X) \\ (q, w, X\beta) \vdash (p, w, \alpha\beta) & \text{ if } (p, \alpha) \in \delta(q, \epsilon, X) \end{aligned}$$



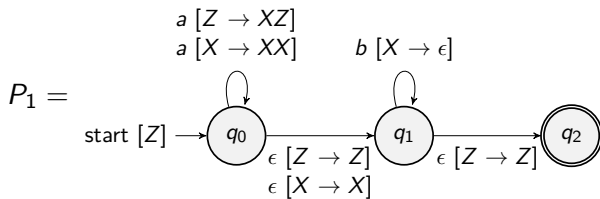
$$\begin{aligned} (q_0, ab, Z) & \vdash (q_0, b, XZ) & (\because (q_0, XZ) \in \delta(q_0, a, Z)) \\ & \vdash (q_1, b, XZ) & (\because (q_1, X) \in \delta(q_0, \epsilon, X)) \end{aligned}$$

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## Definition (Acceptance by Final States)

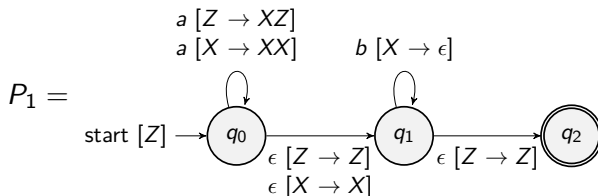
For a given PDA  $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$ , the language accepted by **final states** is defined as:

$$L_F(P) = \{w \in \Sigma^* \mid (q_0, w, Z) \vdash^* (q, \epsilon, \alpha) \text{ for some } q \in F, \alpha \in \Gamma^*\}$$

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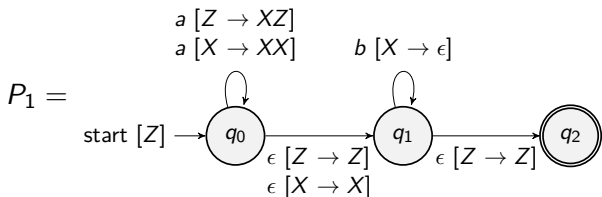
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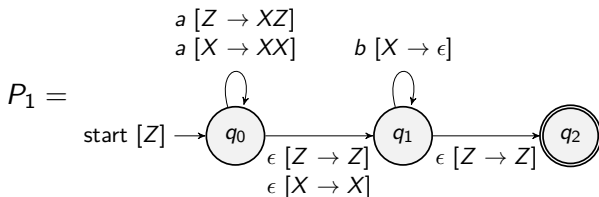


$$(q_0, ab, Z) \vdash^* (q_2, \epsilon, Z) \implies ab \in L_F(P)$$

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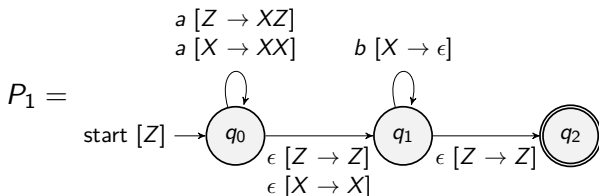
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 (q_0, ab, Z) \vdash^* (q_2, \epsilon, Z) &\implies ab \in L_F(P) \\
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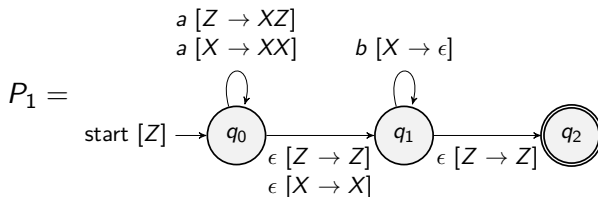
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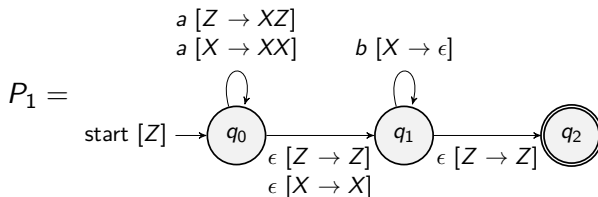
$$\begin{aligned} (q_0, ab, Z) &\vdash^* (q_2, \epsilon, Z) &\implies & ab \in L_F(P) \\ (q_0, aabb, Z) &\vdash^* (q_2, \epsilon, Z) &\implies & aabb \in L_F(P) \end{aligned}$$

$$L_F(P_1) = \{a^n b^n \mid n \geq 0\}$$



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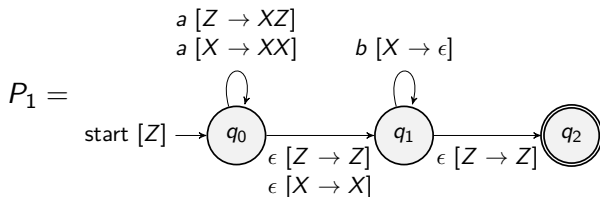


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The idea to construct a PDA accepting the language by final states is:

- ① Push alphabet  $X$  for each  $a$  to the stack.
- ② Pop alphabet  $X$  for each  $b$  from the stack.
- ③ Accept when all  $X$ 's are popped.

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See the additional material for the input string  $aaabbb$ .<sup>1</sup>

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```
// The type definition of configurations
case class Config(state: State, word: Word, stack: List[Alphabet])
case class PDA(...):
  // Configurations reachable from the initial configuration
  def reachableConfig(init: Config): Set[Config] = ... // See PDA.scala
  // The initial configuration
  def init(word: Word): Config =
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  // Acceptance by final states
  def acceptByFinalState(word: Word): Boolean =
    reachableConfig(init(word)).exists(config => {
      val Config(q, w, _) = config
      w.isEmpty && finalStates.contains(q)
    })

acceptByFinalState(pda1)("ab")    // true
acceptByFinalState(pda1)("aba")  // false
```

## Definition (Acceptance by Empty Stacks)

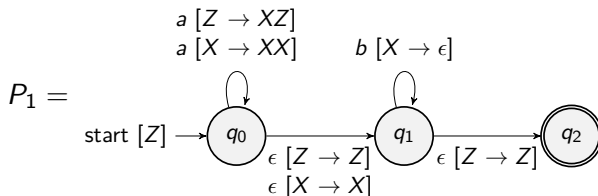
For a given PDA  $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$ , the language accepted by **empty stacks** is defined as:

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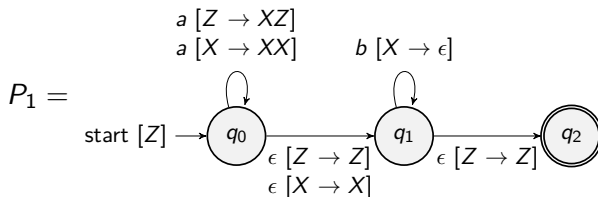
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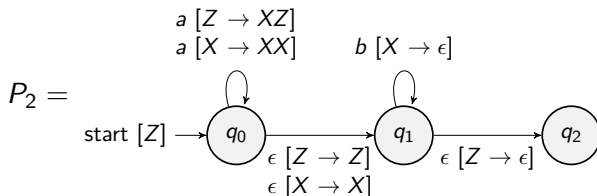
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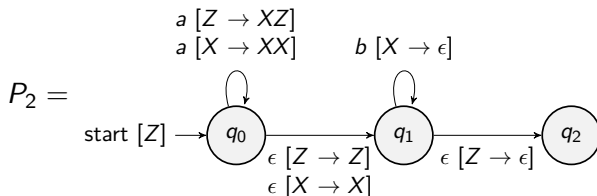
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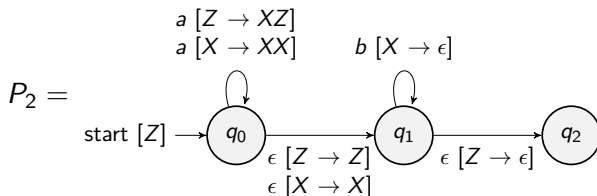


$$(q_0, ab, Z) \vdash^* (q_2, \epsilon, \epsilon) \implies ab \in L_E(P)$$

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  // Acceptance by empty stacks
  def acceptByEmptyStack(word: Word): Boolean =
    reachableConfig(init(word)).exists(config => {
      val Config(_, w, xs) = config
      w.isEmpty && xs.isEmpty
    })

acceptByEmptyStack(pda2)("ab")    // true
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```

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- Pushdown Automata in Scala

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- Examples of Pushdown Automata

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