# Lecture 25 – Type Inference (1)

COSE212: Programming Languages

Jihyeok Park

PLRG

2025 Fall

# Recall

- **Polymorphism** is to use a single entity as **multiple types**, and there are various kinds of polymorphism:
  - **Parametric polymorphism**
  - **Subtype polymorphism**
  - **Ad-hoc polymorphism**
  - ...
- **PTFAE** – TFAE with **parametric polymorphism**.
- **STFAE** – TFAE with **subtype polymorphism**.

# Recall

- **Polymorphism** is to use a single entity as **multiple types**, and there are various kinds of polymorphism:
  - **Parametric polymorphism**
  - **Subtype polymorphism**
  - **Ad-hoc polymorphism**
  - ...
- **PTFAE** – TFAE with **parametric polymorphism**.
- **STFAE** – TFAE with **subtype polymorphism**.
- In this lecture, we will learn **type inference**.

# Type Inference

**Definition (Type Inference)**

**Type inference** is the process of automatically inferring the types of expressions.

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The goal of **type inference algorithm** is to infer the type of an expression without **explicit type annotations** given by programmers.

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Let's consider the following RFAE expression:

```
/* RFAE */
def sum(x) = if (x < 1) 0 else x + sum(x - 1)
sum
```

How can we **automatically infer** the type of `sum`?

# Type Inference

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The goal of **type inference algorithm** is to infer the type of an expression without **explicit type annotations** given by programmers.

Let's consider the following RFAE expression:

```
/* RFAE */
def sum(x) = if (x < 1) 0 else x + sum(x - 1)
sum
```

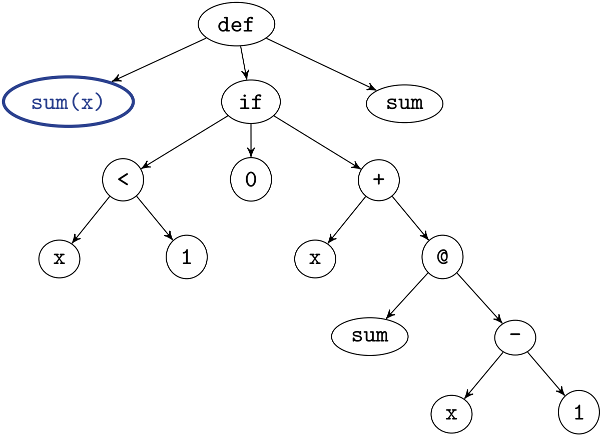
How can we **automatically infer** the type of `sum`?

1. Introduce **type variables** to denote unknown types
2. Collect the **type constraints** on the types
3. Find a **solution** (substitution of type variables) to the constraints

# Contents

# Contents

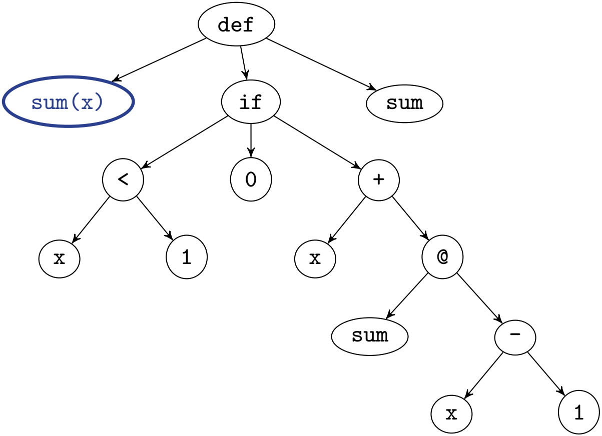
# Example 1 – sum




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | ??? |
| sum | ??? |

# Example 1 – sum




Type Environment

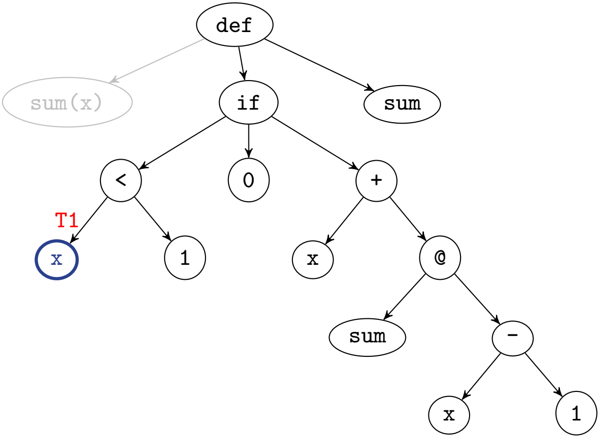
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |

Let's define **type variables** for unknown types.

# Example 1 – sum



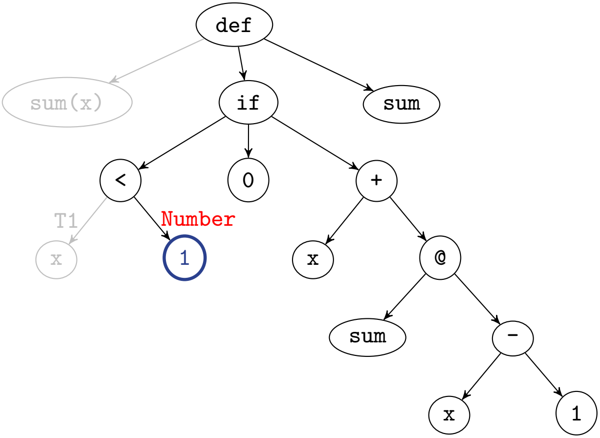

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
| --- | --- |
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
| --- | --- |
| T1 | - |
| T2 | - |

# Example 1 – sum



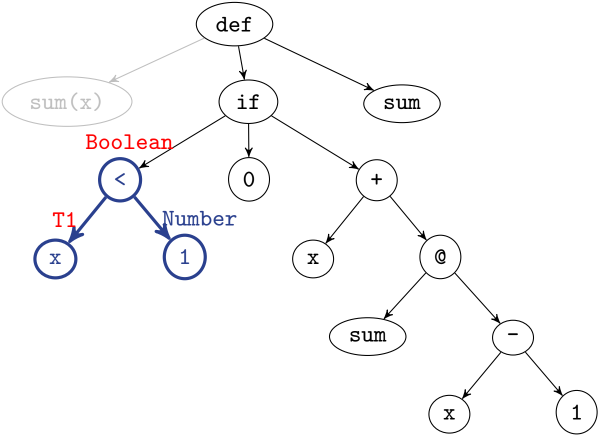

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |

# Example 1 – sum




Type Environment

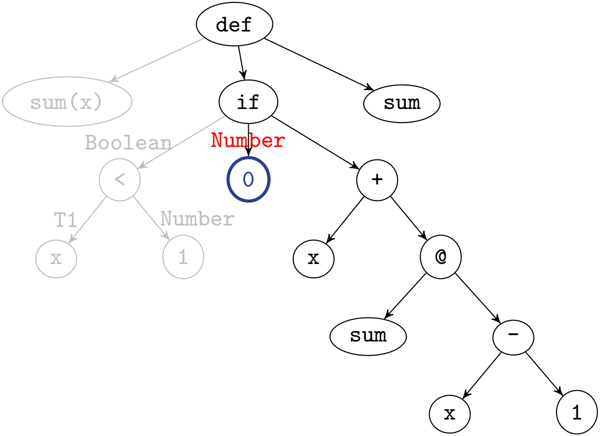
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

The **operands** of < must be of type **Number**.
So, we collected a **type constraint**: `T1 == Number`.

# Example 1 – sum



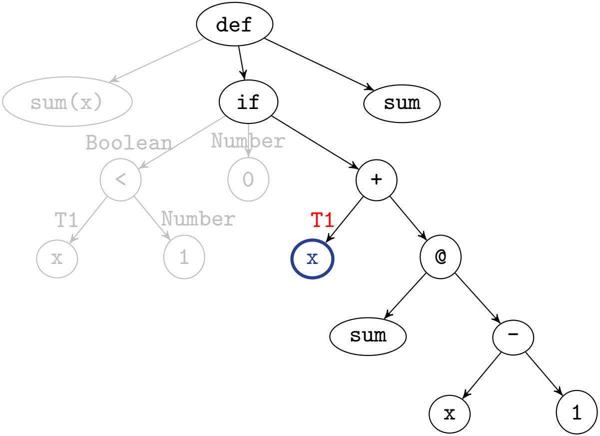

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

# Example 1 – sum



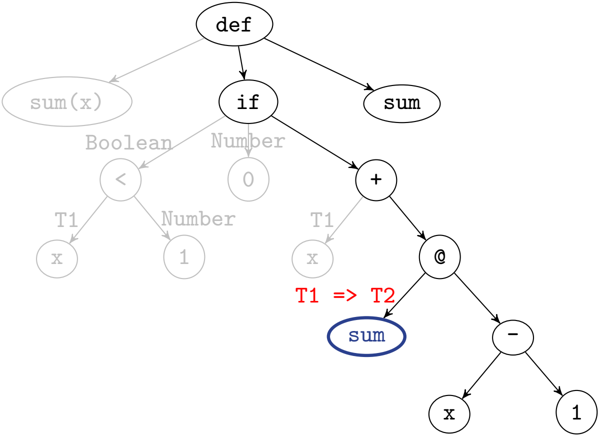

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

# Example 1 – sum



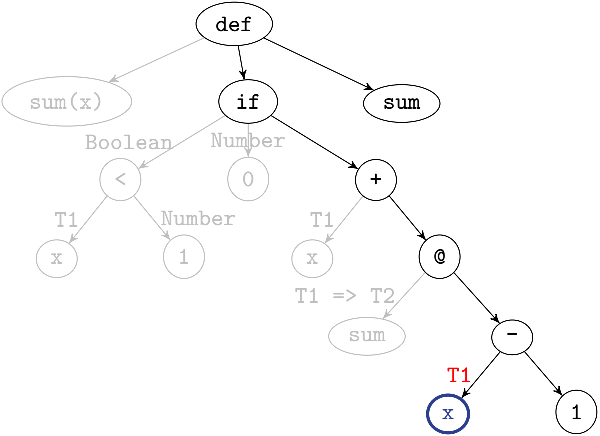

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

# Example 1 – sum



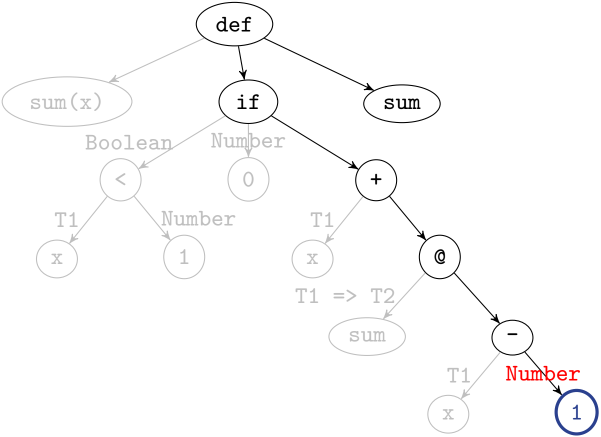

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

# Example 1 – sum



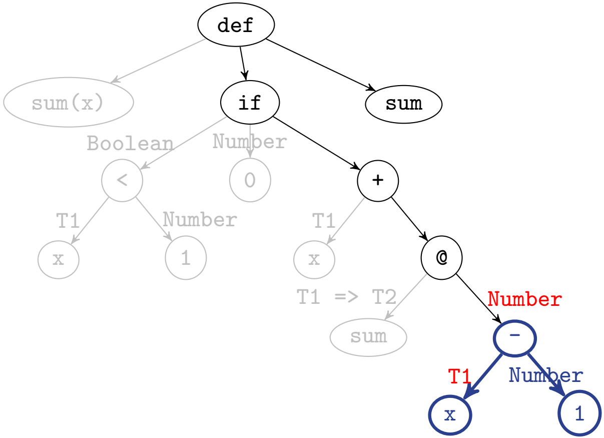

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

# Example 1 – sum




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

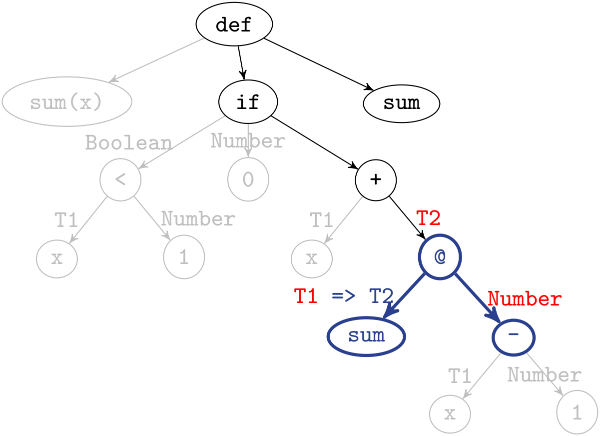
| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

The **operands** of - must be of type **Number**.

We collected a **type constraint**: `T1 == Number`.

But, it is not a new constraint.

# Example 1 – sum




Type Environment

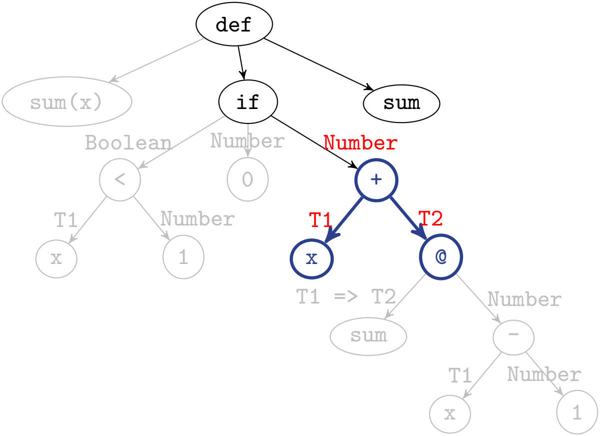
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

The **argument type** should be equal to the **parameter type**.
We collected a **type constraint**: T1 == Number.
Again, it is not a new constraint.

# Example 1 – sum




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

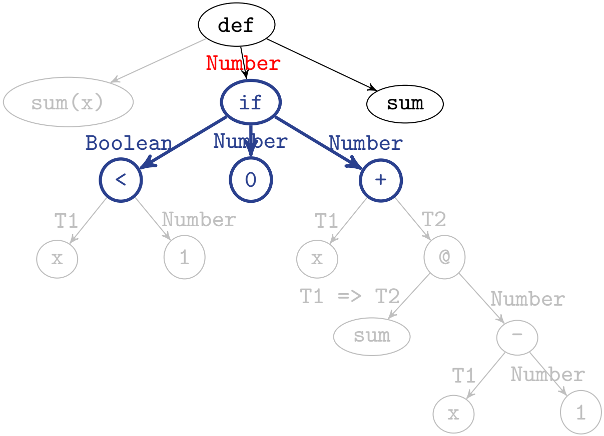
| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | Number |

The **operands** of + must be of type **Number**.

We collected **type constraints**: `T1 == Number` and `T2 == Number`.

The second one is a new constraint!

# Example 1 – sum



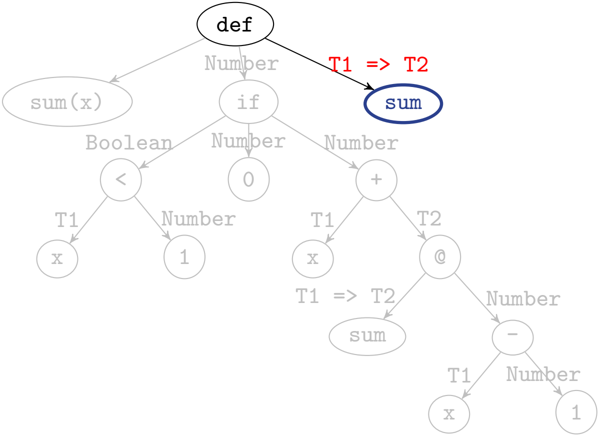

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | Number |

# Example 1 – sum




Type Environment

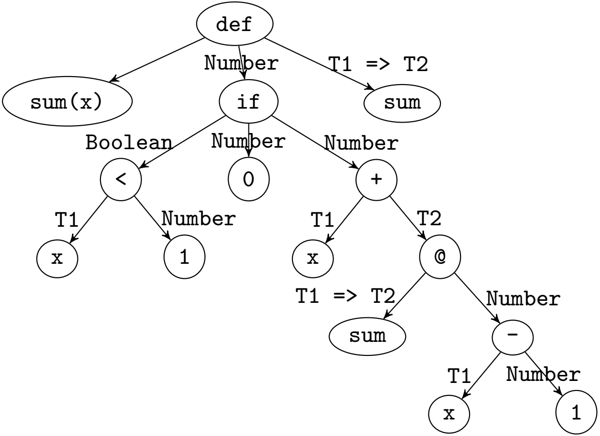
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| sum | T1 => T2 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | Number |

The type of sum is T1 => T2. Using the solution inferred by the collected constraints, we can instantiate it to Number => Number.

# Example 1 – sum




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| sum | T1 => T2 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | Number |

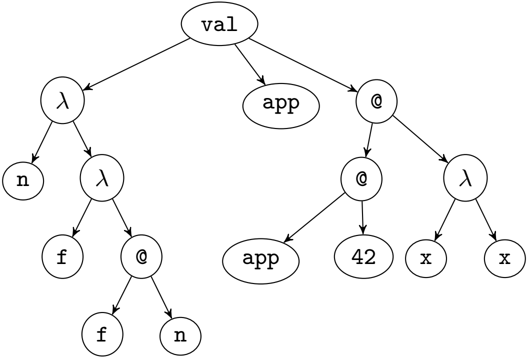
```
/* TRFAE */
def sum(x: Number): Number = if (x < 1) 0 else x + sum(x - 1)
sum
```

# Contents

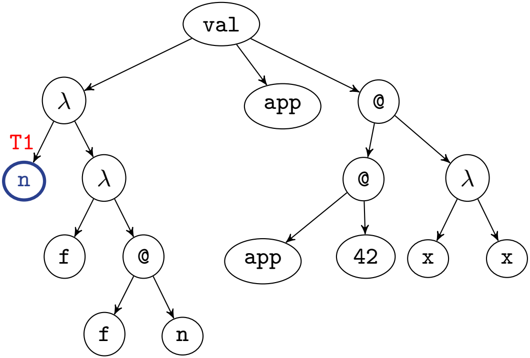
# Example 2 – app

Let's infer the type of the following FAE expression:

```
/* FAE */
val app = n => f => f(n)
app(42)(x => x)
```



# Example 2 – app




Type Environment

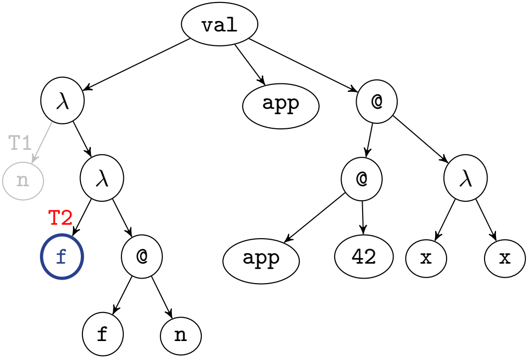
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| | |
| | |
| | |

Let's define a new **type variable T1** for the parameter n.

# Example 2 – app




Type Environment

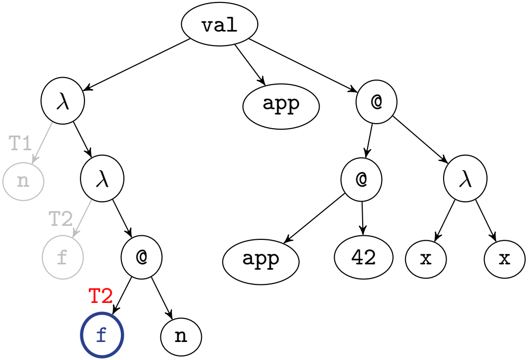
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| f | T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |
| | |
| | |

Let's define a new **type variable T2** for the parameter `f`.

# Example 2 – app



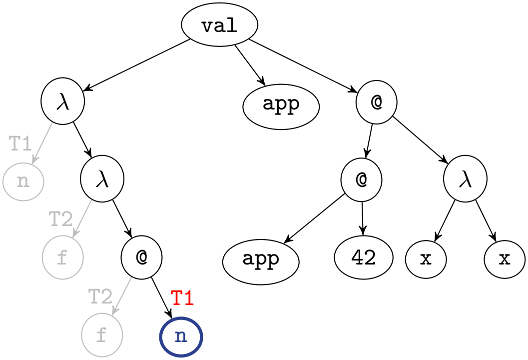

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| f | T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |
| | |
| | |

# Example 2 – app



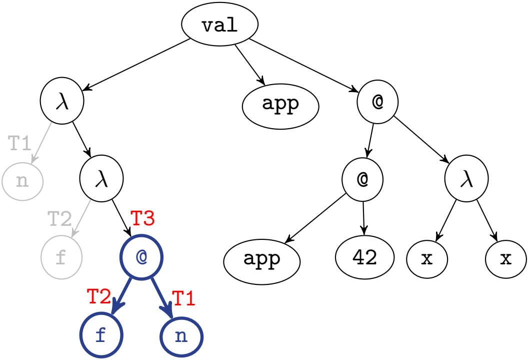

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| f | T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |
| | |
| | |

# Example 2 – app




Type Environment

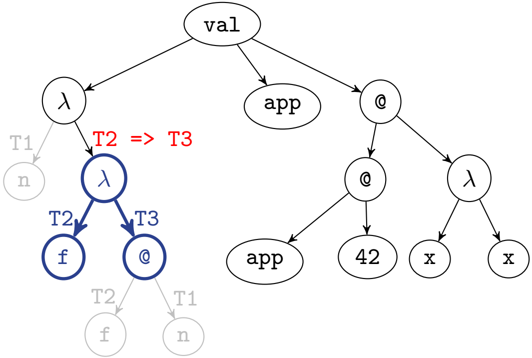
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| f | T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

The type T2 of f should be in the form of T1 => ???.
Let's define a new **type variable** T3 for ??? (the return type of f).
So, we collected a **type constraint**: T2 == T1 => T3.

# Example 2 – app



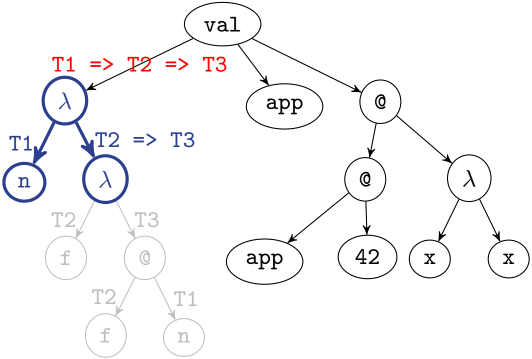

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

# Example 2 – app



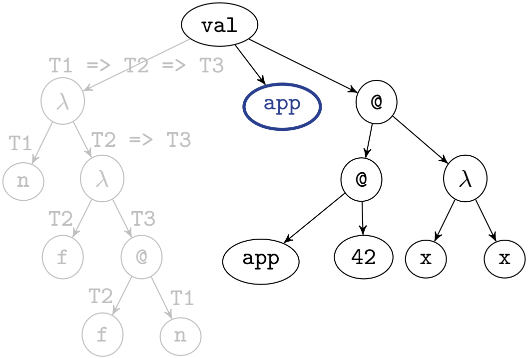

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

# Example 2 – app



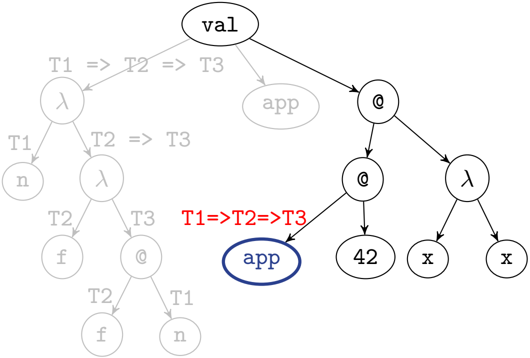

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

# Example 2 – app



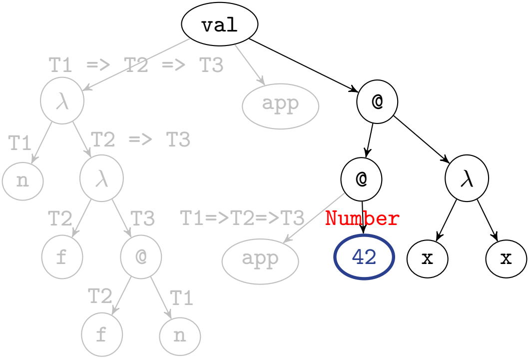

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

# Example 2 – app



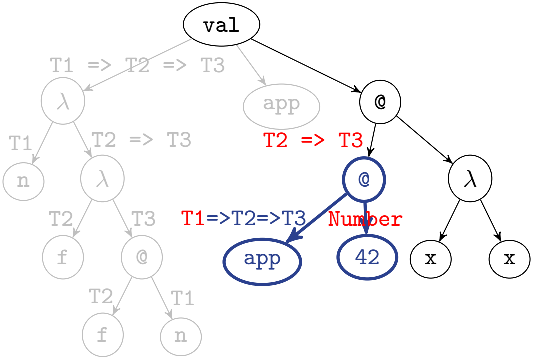

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

# Example 2 – app




Type Environment

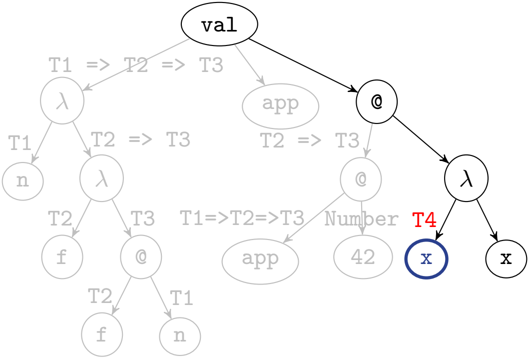
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | T1 => T3 |
| T3 | - |
| | |

The **parameter type** T1 should be equal to the **argument type** Number. So, we collected a **type constraint**: T1 == Number.

# Example 2 – app




Type Environment

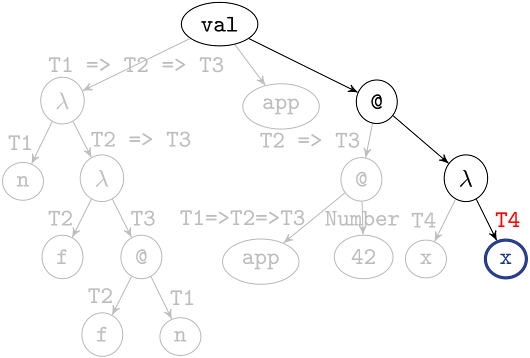
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| x | T4 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | T1 => T3 |
| T3 | - |
| T4 | - |

Let's define a new **type variable T4** for the parameter x.

# Example 2 – app



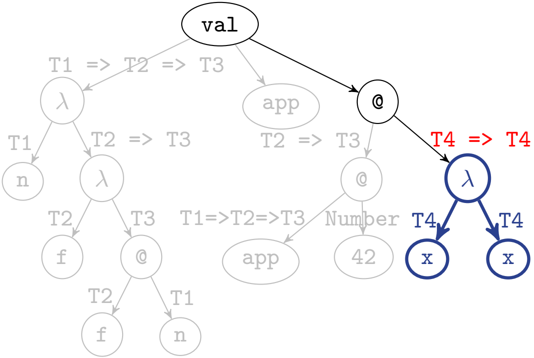

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| x | T4 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | T1 => T3 |
| T3 | - |
| T4 | - |

# Example 2 – app



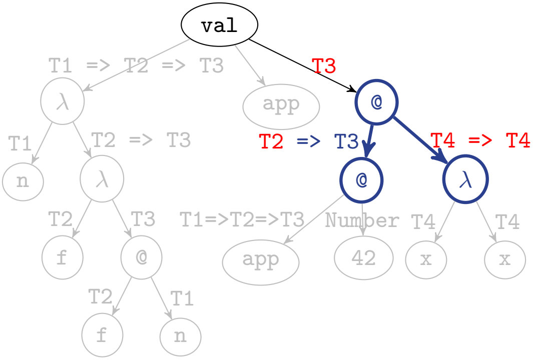

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| x | T4 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | T1 => T3 |
| T3 | - |
| T4 | - |

# Example 2 – app




Type Environment

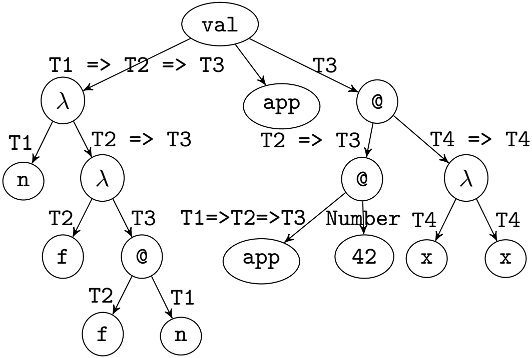
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| x | T4 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | T1 => T3 |
| T3 | Number |
| T4 | Number |

The **parameter type** T2 should be equal to **argument type** T4 => T4.
We collected **type constraints**: T3 == Number and T4 == Number.
Finally, the entire expression has type T3 (= Number).

# Example 2 – app




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| x | T4 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | T1 => T3 |
| T3 | Number |
| T4 | Number |

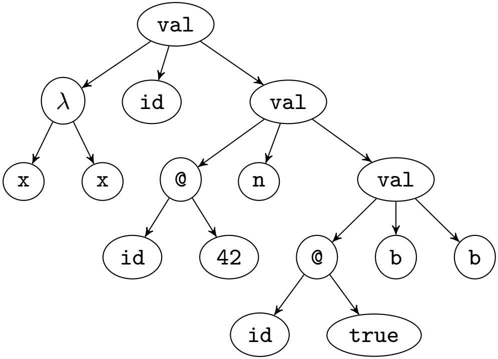
```
/* TFAE */
val app = (n: Number) => (f: Number => Number) => f(n)
app(42)((x: Number) => x)
```

# Contents

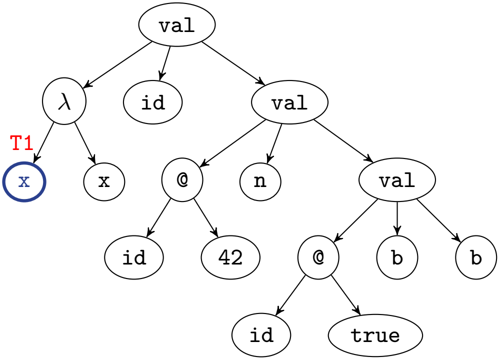
# Example 3 – id

Let's infer the type of the following FAE expression:

```
/* FAE */
val id = x => x
val n = id(42)
val b = id(true)
b
```



# Example 3 – id




Type Environment

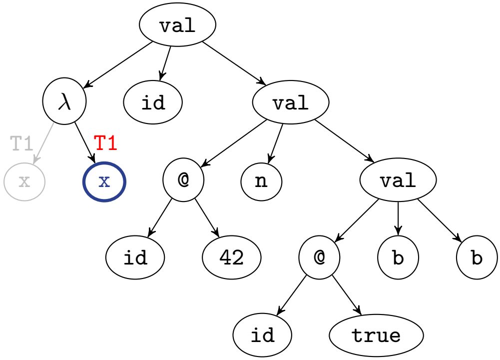
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| | |

Let's define a new **type variable T1** for the parameter x.

# Example 3 – id



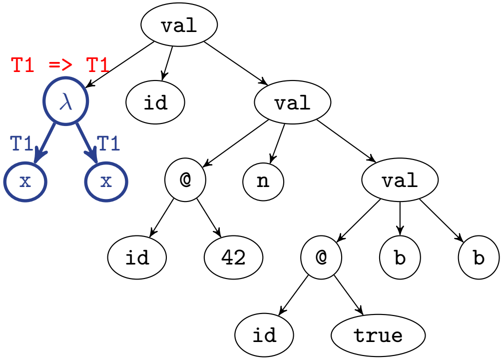

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| | |

# Example 3 – id



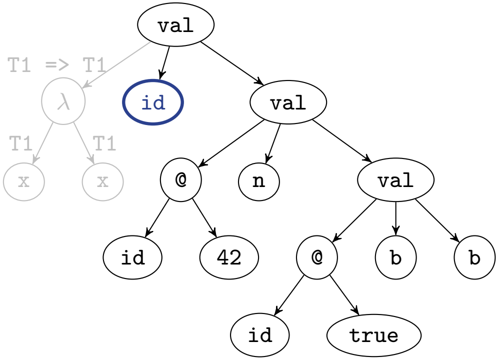

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| | |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| | |

# Example 3 – `id`




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| `id` | `[T1]` { `T1 => T1` } |
| | |
| | |

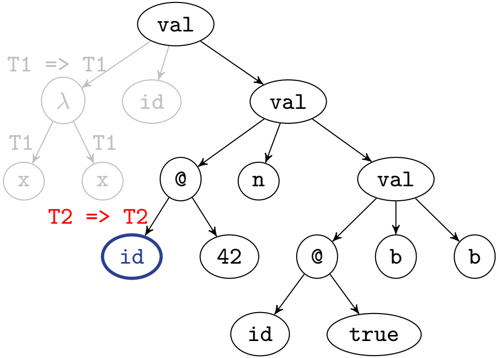
Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| | |
| | |

Let's **generalize** the type `T1 => T1` into a **polymorphic type** for `id` with **type variable** `T1` as a **type parameter**.

We call this **let-polymorphism** because it only introduces polymorphism for the let-binding (e.g., `val`).

# Example 3 – id




Type Environment

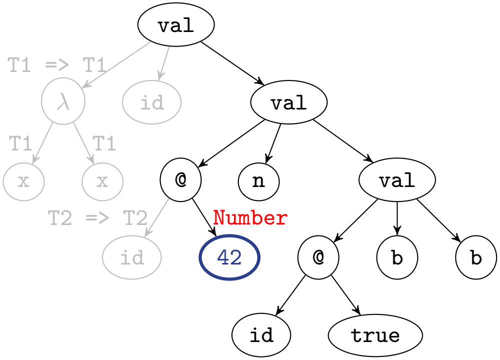
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T2 | – |
| | |

Let's define a new **type variable** T2 to **instantiate** the **type variable** T1.
And, **substitute** T1 with T2.

# Example 3 – id



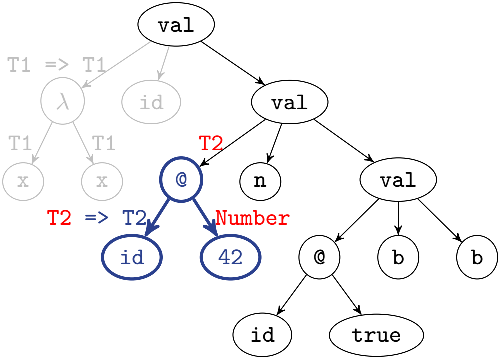

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T2 | - |
| | |

# Example 3 – `id`




Type Environment

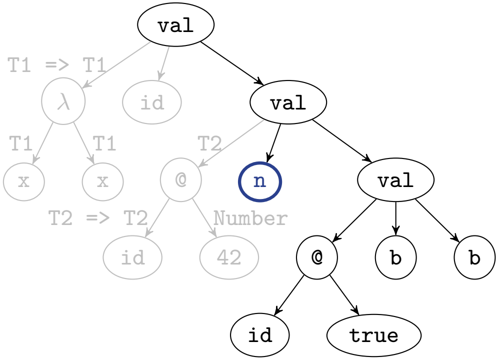
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T2 | Number |
| | |

The **parameter type** T2 should be equal to **argument type** Number.
We collected a **type constraint**: `T2 == Number`.

# Example 3 – `id`




Type Environment

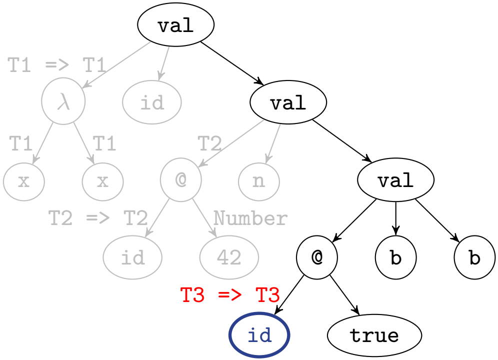
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T2 | Number |
| | |

T2 is not a free type variable because it actually represents Number.
So, we will not introduce a polymorphic type in this case.

# Example 3 – id




Type Environment

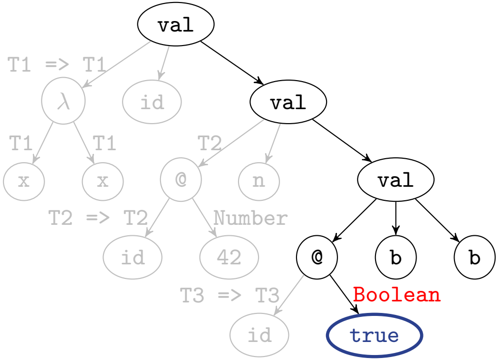
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T2 | Number |
| T3 | - |

Let's define a new **type variable T3** to **instantiate** the **type variable T1**. And, **substitute T1** with T3.

# Example 3 – `id`



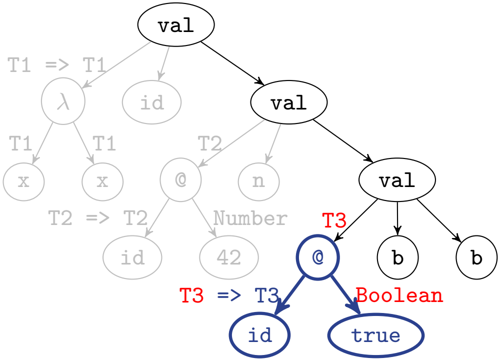

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T2 | Number |
| T3 | - |

# Example 3 – id




Type Environment

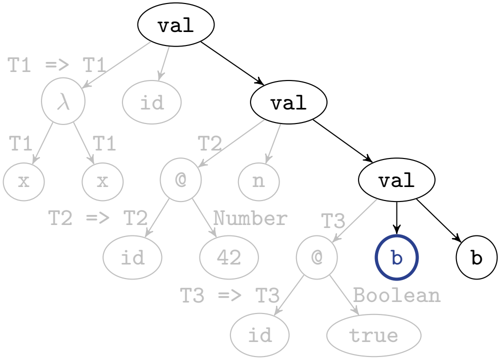
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T2 | Number |
| T3 | Boolean |

The **parameter type** T3 should be equal to **argument type** Boolean.
We collected a **type constraint**: T3 == Boolean.

# Example 3 – `id`




Type Environment

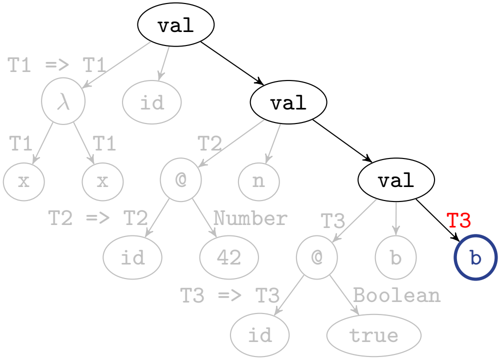
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| b | T3 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T2 | Number |
| T3 | Boolean |

T3 is not a free type variable because it actually represents Boolean.
So, we will not introduce a polymorphic type in this case.

# Example 3 – id




Type Environment

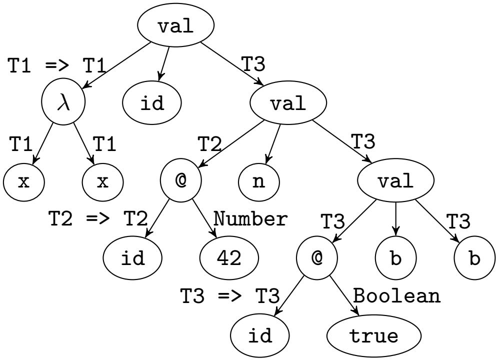
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| b | T3 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T2 | Number |
| T3 | Boolean |

Finally, the entire expression has type T3 (= Boolean).

# Example 3 – id




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| b | T3 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T2 | Number |
| T3 | Boolean |

```
/* PTFAE */
val id = forall[T] { (x: T) => x }
val n = id[Number](42)
val b = id[Boolean](true)
b
```

# Summary

1. Example 1 – `sum`

2. Example 2 – `app`

3. Example 3 – `id`

# Next Lecture

- Type Inference (2)

Jihyeok Park
jihyeok_park@korea.ac.kr
https://plrg.korea.ac.kr