

Lecture 16 – Equivalence of Pushdown Automata and Context-Free Grammars

COSE215: Theory of Computation

Jihyeok Park



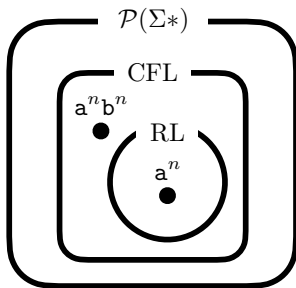
2025 Spring

A **context-free grammar** is a 4-tuple:

$$G = (V, \Sigma, S, R)$$

A **pushdown automaton (PDA)** is a finite automaton with a **stack**.

- Acceptance by **final states**
- Acceptance by **empty stacks**



PDA_{FS}
(by final states)

|| ?

PDA_{ES}
(by empty stacks)

|| ?

CFG

1. Equivalence of PDA by Final States and Empty Stacks

PDA_{FS} to PDA_{ES}

PDA_{ES} to PDA_{FS}

2. Equivalence of PDA and CFGs

CFGs to PDA_{ES}

PDA_{ES} to CFGs

$$\begin{array}{ccccc} \text{PDA}_{\text{FS}} & \longleftrightarrow & \text{PDA}_{\text{ES}} & \longleftrightarrow & \text{CFG} \\ \text{(by final states)} & & \text{(by empty stacks)} & & \end{array}$$

1. Equivalence of PDA by Final States and Empty Stacks

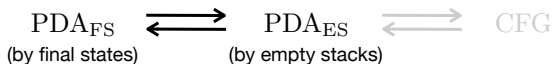
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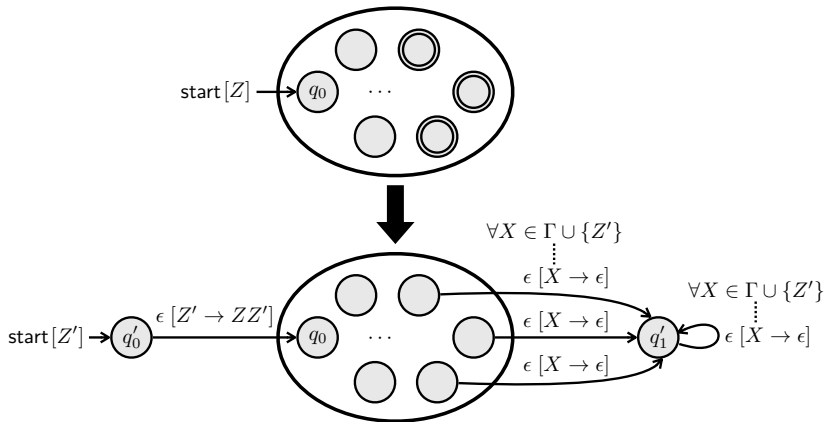


Theorem (PDA_{FS} to PDA_{ES})

For a given PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, \exists PDA P' . $L_F(P) = L_E(P')$.

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Define a PDA

$$P' = (Q \cup \{q'_0, q'_1\}, \Sigma, \Gamma \cup \{Z'\}, \delta', q'_0, Z', \emptyset)$$

where

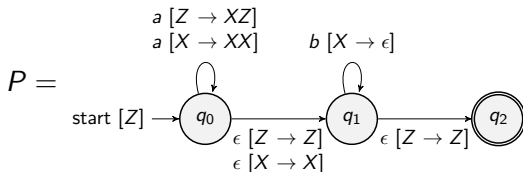
$$\delta'(q'_0, \epsilon, Z') = \{(q_0, ZZ')\}$$

$$\delta'(q \in Q, a \in \Sigma, X \in \Gamma) = \delta(q, a, X)$$

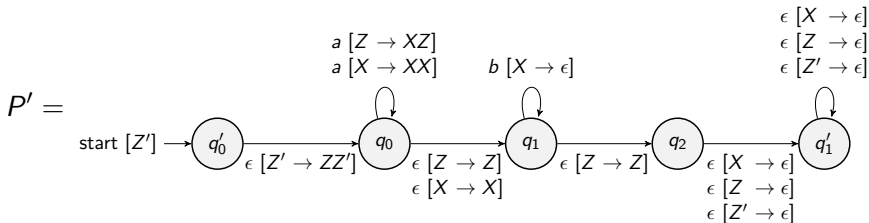
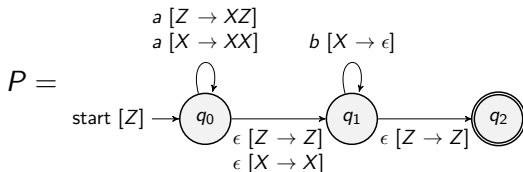
$$\delta'(q \in Q, \epsilon, X \in \Gamma \cup \{Z'\}) = \begin{cases} \delta(q, \epsilon, X) \cup \{(q'_1, \epsilon)\} & \text{if } q \in F \\ \delta(q, \epsilon, X) & \text{otherwise} \end{cases}$$

$$\delta'(q'_1, \epsilon, X \in \Gamma \cup \{Z'\}) = \{(q'_1, \epsilon)\}$$

$$L_F(P) = L_E(P') = \{a^n b^n \mid n \geq 0\}$$



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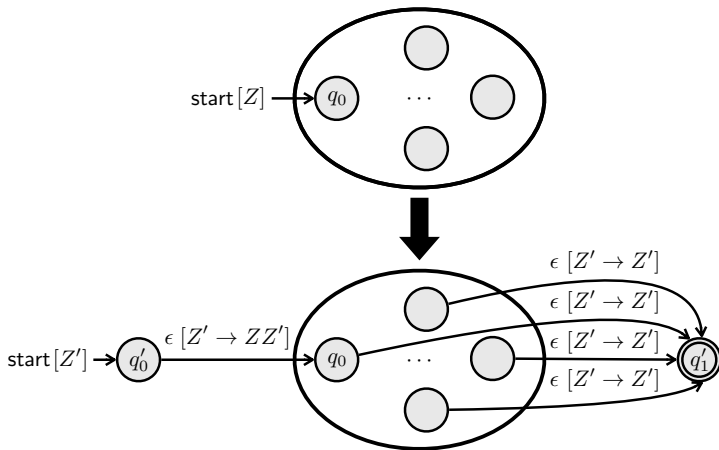


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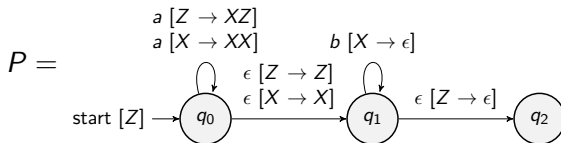
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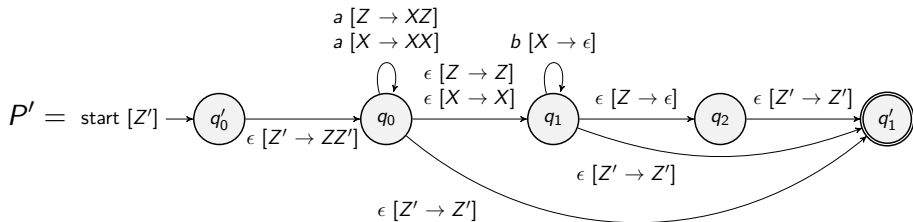
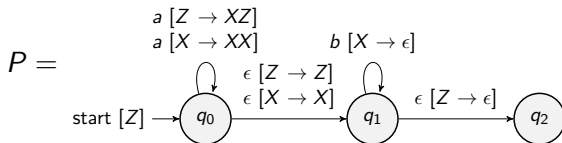
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2. Equivalence of PDA and CFGs

CFGs to PDA_{ES}

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Theorem (CFGs to PDA_{ES})

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Define a PDA

$$P = (\{q\}, \Sigma, V \cup \Sigma, \delta, q, S, \emptyset)$$

where

$$\delta(q, \epsilon, A \in V) = \{(q, \alpha) \mid A \rightarrow \alpha \in R\}$$

$$\delta(q, a \in \Sigma, a \in \Sigma) = \{(q, \epsilon)\}$$

$$\begin{aligned}\delta(q, \epsilon, A \in V) &= \{(q, \alpha) \mid A \rightarrow \alpha \in R\} \\ \delta(q, a \in \Sigma, a \in \Sigma) &= \{(q, \epsilon)\}\end{aligned}$$

Consider the following CFG:

$$S \rightarrow \epsilon \mid aSb \mid bSa \mid SS$$

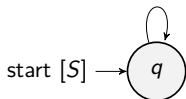
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Then, the equivalent PDA (by empty stacks) is:

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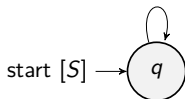
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| | |
|-------------------------------------|--------------------------------------|
| $\epsilon [S \rightarrow \epsilon]$ | $(q, abab, S) \vdash (q, abab, aSb)$ |
| $\epsilon [S \rightarrow aSb]$ | $\vdash (q, bab, Sb)$ |
| $\epsilon [S \rightarrow bSa]$ | $\vdash (q, bab, bSab)$ |
| $\epsilon [S \rightarrow SS]$ | $\vdash (q, ab, Sab)$ |
| $a [a \rightarrow \epsilon]$ | $\vdash (q, ab, ab)$ |
| $b [b \rightarrow \epsilon]$ | $\vdash (q, b, b)$ |
| | $\vdash (q, \epsilon, \epsilon)$ |



Theorem (PDA_{ES} to CFGs)

*For a given PDA $P = (Q = \{q_0, \dots, q_{n-1}\}, \Sigma, \Gamma, \delta, q_0, Z, F)$,
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The key idea is defining a variable $A_{i,j}^X$ for each $0 \leq i, j < n$ and $X \in \Gamma$ that generates all words causing the PDA to move from q_i to q_j by popping X :

$$A_{i,j}^X \Rightarrow^* w \quad \text{if and only if} \quad (q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$$

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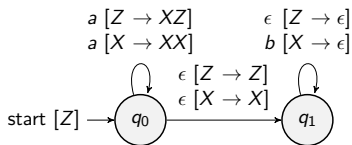
To cover all possible combinations of k_1, \cdots, k_m , we need to define a production rule for A_{i,k_m}^X as follows:

$$A_{i,k_m}^X \rightarrow a A_{j,k_1}^{X_1} A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m} \text{ for all } 0 \leq k_1, \cdots, k_m < n$$

$$S \rightarrow A_{0,j}^Z$$

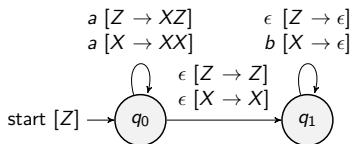
$$A_{i,k_m}^X \rightarrow a A_{j,k_1}^{X_1} A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m}$$

Consider the following PDA (by empty stacks):



$$S \rightarrow A_{0,j}^Z \qquad A_{i,k_m}^X \rightarrow a A_{j,k_1}^{X_1} A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m}$$

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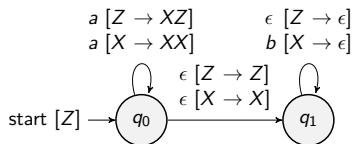


Then, the equivalent CFG is:

$$\begin{aligned}
 S &\rightarrow A_{0,0}^Z \mid A_{0,1}^Z \\
 A_{0,0}^Z &\rightarrow a A_{0,0}^X A_{0,0}^Z \mid a A_{0,1}^X A_{1,0}^Z \mid A_{1,0}^Z \\
 A_{0,1}^Z &\rightarrow a A_{0,0}^X A_{0,1}^Z \mid a A_{0,1}^X A_{1,1}^Z \mid A_{1,1}^Z \\
 A_{0,0}^X &\rightarrow a A_{0,0}^X A_{0,0}^X \mid a A_{0,1}^X A_{1,0}^X \mid A_{1,0}^X \\
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 A_{1,1}^Z &\rightarrow \epsilon \\
 A_{1,1}^X &\rightarrow b
 \end{aligned}$$

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$$\begin{aligned}
 S &\Rightarrow A_{0,1}^Z \\
 &\Rightarrow a A_{0,1}^X A_{1,1}^Z \\
 &\Rightarrow aa A_{0,1}^X A_{1,1}^X A_{1,1}^Z \\
 &\Rightarrow aa A_{1,1}^X A_{1,1}^X A_{1,1}^Z \\
 &\Rightarrow aab A_{1,1}^X A_{1,1}^Z \\
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- Deterministic Pushdown Automata (DPDA)

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