Lecture 8 – Lambda Calculus

COSE212: Programming Languages

Jihyeok Park

2024 Fall

Recall

- FVAE – VAE with First-Class Functions
  - First-Class Functions
  - Concrete and Abstract Syntax
  - Interpreter and Natural Semantics with Closures
  - Static and Dynamic Scoping

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- FVAE – VAE with First-Class Functions
  - First-Class Functions
  - Concrete and Abstract Syntax
  - Interpreter and Natural Semantics with Closures
  - Static and Dynamic Scoping
- In this lecture, we will learn **syntactic sugar** and **lambda calculus**

Contents

1. Syntactic Sugar
   - No More `val`
   - FAE – Removing `val` from FVAE
   - Syntactic Sugar and Desugaring

2. Lambda Calculus
   - Definition
   - Church Encodings
   - Church-Turing Thesis

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No More `val`

```
/* FVAE */
val x = 1; x + 2
```

It assigns a **value** 1 to the **variable** `x`, and then evaluates the **body expression** `x + 2` with the environment $[\texttt{x} \mapsto 1]$.

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val x = 1; x + 2
```

It assigns a **value** 1 to the **variable** `x`, and then evaluates the **body expression** `x + 2` with the environment $[\mathrm{x} \mapsto 1]$.

It is same as:

```
/* FVAE */
(x => x + 2)(1)
```

It assigns a **value** (argument) 1 to the **parameter** `x`, and then evaluates the **body expression** `x + 2` with the environment $[\mathrm{x} \mapsto 1]$.

No More `val`

In general, the following two expressions are equivalent:

$$\texttt{val}\ x\ \texttt{=}\ e_1;\ e_2 \qquad \text{is equivalent to} \qquad (\lambda x.e_2)(e_1)$$

Why?

No More `val`

In general, the following two expressions are equivalent:

$$\texttt{val}\ x\ \texttt{=}\ e_1;\ e_2 \qquad \text{is equivalent to} \qquad (\lambda x.e_2)(e_1)$$

Why?

The following inference rule for the semantics of $\texttt{val}\ x\ \texttt{=}\ e_1;\ e_2$:

$$\textsc{Val}\ \frac{\sigma \vdash e_1 \Rightarrow v_1 \qquad \sigma[x \mapsto v_1] \vdash e_2 \Rightarrow v_2}{\sigma \vdash \texttt{val}\ x = e_1;\ e_2 \Rightarrow v_2}$$

is equivalent to the following inference rule for the semantics of the combination $(\lambda x.e_2)(e_1)$ of a anonymous function and an application:

$$\texttt{App}\ \frac{\texttt{Fun}\ \dfrac{}{\sigma \vdash \lambda x.e_2 \Rightarrow \langle \lambda x.e_2, \sigma \rangle} \qquad \sigma \vdash e_1 \Rightarrow v_1 \qquad \sigma[x \mapsto v_1] \vdash e_2 \Rightarrow v_2}{\sigma \vdash (\lambda x.e_2)(e_1) \Rightarrow v_2}$$

FAE – Removing `val` from FVAE

Then, we can define a smaller language FAE

$$
\begin{array}{llll}
\text{Expressions} & e ::= & n & (\texttt{Num}) \\
& \mid & e \texttt{ + } e & (\texttt{Add}) \\
& \mid & e \texttt{ * } e & (\texttt{Mul}) \\
& \mid & x & (\texttt{Id}) \\
& \mid & \lambda x.e & (\texttt{Fun}) \\
& \mid & e(e) & (\texttt{App})
\end{array}
$$

by removing `val` from FVAE using the following equivalence:

$$\texttt{val } x \texttt{ = } e_1\texttt{;}\ e_2 \qquad \text{is equivalent to} \qquad (\lambda x.e_2)(e_1)$$

Syntactic Sugar and Desugaring

Definition (Syntactic Sugar)

Syntactic elements that can be expressed in terms of other syntactic elements are called **syntactic sugar**.

Syntactic Sugar and Desugaring

PLRG

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Definition (Desugaring)

Desugaring is a translation for removing syntactic sugar.

$$\mathcal{D}[\![-]\!] : \mathbb{E} \to \mathbb{E}$$

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Desugaring is a translation for removing syntactic sugar.

$$\mathcal{D}[\![-]\!] : \mathbb{E} \to \mathbb{E}$$

For example, we can define the desugaring of `val` as follows:

$$\mathcal{D}[\![\texttt{val}\ x\ \texttt{=}\ e_1;\ e_2]\!] = (\lambda x.\mathcal{D}[\![e_2]\!])(\mathcal{D}[\![e]\!])$$

Is it enough?

Syntactic Sugar and Desugaring

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Is it enough? **No!** Why?

Syntactic Sugar and Desugaring

$$\mathcal{D}[\![\texttt{val}\ x\ \texttt{=}\ e_1;\ e_2]\!] = (\lambda x.\mathcal{D}[\![e_2]\!])(\mathcal{D}[\![e]\!])$$

We need to **recursively desugar** sub-expressions of the given expression even if they are not syntactic sugars.

Syntactic Sugar and Desugaring

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$$\begin{array}{lcllcl}
\mathcal{D}[\![n]\!] & = & n & \mathcal{D}[\![x]\!] & = & x \\
\mathcal{D}[\![e_1\ \texttt{+}\ e_2]\!] & = & \mathcal{D}[\![e_1]\!]\ \texttt{+}\ \mathcal{D}[\![e_2]\!] & \mathcal{D}[\![\lambda x.e]\!] & = & \lambda x.\mathcal{D}[\![e]\!] \\
\mathcal{D}[\![e_1\ \texttt{*}\ e_2]\!] & = & \mathcal{D}[\![e_1]\!]\ \texttt{*}\ \mathcal{D}[\![e_2]\!] & \mathcal{D}[\![e_1(e_2)]\!] & = & \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_2]\!])
\end{array}$$

Syntactic Sugar and Desugaring

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$$\begin{array}{lcl@{\qquad}lcl}
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\mathcal{D}[\![e_1 \texttt{ * } e_2]\!] & = & \mathcal{D}[\![e_1]\!] \texttt{ * } \mathcal{D}[\![e_2]\!] & \mathcal{D}[\![e_1(e_2)]\!] & = & \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_2]\!])
\end{array}$$

For example,

$$\mathcal{D}[\![\texttt{val x = }1;\ 2 \texttt{ + } (\texttt{val y = }3;\ \texttt{x * y})]\!] = \lambda\texttt{x}.2 \texttt{ + } (\lambda\texttt{y}.\texttt{x * y})(3)(1)$$

Without desugaring rule for addition, the expression $(\texttt{val y = }3;\ \texttt{x * y})$ in the right-hand side of the addition cannot be desugared.

Syntactic Sugar and Desugaring

$$\mathcal{D}[\![\texttt{val}\ x\ \texttt{=}\ e_1;\ e_2]\!] = (\lambda x.\mathcal{D}[\![e_2]\!])(\mathcal{D}[\![e]\!])$$

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\end{array}$$

We can also implement **desugaring** in Scala:

```scala
def desugar(expr: Expr): Expr = expr match
  case Val(x, i, b) => App(Fun(x, desugar(b)), desugar(i))
  case Num(n)       => Num(n)
  case Add(l, r)    => Add(desugar(l), desugar(r))
  case Mul(l, r)    => Mul(desugar(l), desugar(r))
  case Id(x)        => Id(x)
  case Fun(p, b)    => Fun(p, desugar(b))
  case App(f, e)    => App(desugar(f), desugar(e))
```

Syntactic Sugar and Desugaring

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  case App(f, e)    => App(desugar(f), desugar(e))
```

Then, we can desugar the example FVAE expression as follows:

```
val expr: Expr = Expr("val x = 1; 2 + (val y = 3; x * y)")
desugar(expr) == Expr("(x => 2 + (y => x * y)(3))(1)")
```

Syntactic Sugar and Desugaring

Most programming languages have **syntactic sugar**:

- Scala

`for (x <- list) yield x * 2`	$\equiv$	`list.map(x => x * 2)`

- C++

`arr[i] + obj->field`	$\equiv$	`*(arr + i) + (*obj).field`

- JavaScript[1]

`x ||= y; x &&= y;`	$\equiv$	`x || (x = y); x && (x = y);`

- Haskell

`do x <- f; g x`	$\equiv$	`f >>= (\x -> g x)`

- ...

[1] https://babeljs.io/repl

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    Church Encodings
    Church-Turing Thesis

What is the minimal language that can express all the syntactic elements of FVAE?

Lambda Calculus

What is the minimal language that can express all the syntactic elements of FVAE? **Lambda calculus (LC)**!

The **lambda calculus (LC)** is a language only consisting of 1) **variables**, 2) **functions**, and 3) **applications**:

$$
\begin{array}{lrcl}
\text{Expressions} & e & ::= & x \\
 & & | & \lambda x.e \\
 & & | & e(e)
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We already showed that the **variable definition** can be desugared to a combination of a **function definition** and an **application**:

$$\mathcal{D}[\![\texttt{val}\ x\ \texttt{=}\ e_1;\ e_2]\!] = (\lambda x.\mathcal{D}[\![e_2]\!])(\mathcal{D}[\![e]\!])$$

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Then, how can we desugar other syntactic elements of FVAE?

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Then, how can we desugar other syntactic elements of FVAE?

Let's learn the **Church encodings**!

Church Encodings – Church Numerals

Church encodings are ways to encode **data** and **operations** in the **lambda calculus (LC)**.

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The key idea is to encode a **natural number** n as a **function** that takes another function f and an argument x and applies f to x n times:

$$\begin{array}{rcl}
\mathcal{D}[\![0]\!] & = & \lambda f.\lambda x.x \\
\mathcal{D}[\![1]\!] & = & \lambda f.\lambda x.f(x) \\
\mathcal{D}[\![2]\!] & = & \lambda f.\lambda x.f(f(x)) \\
\mathcal{D}[\![3]\!] & = & \lambda f.\lambda x.f(f(f(x))) \\
 & \vdots &
\end{array}$$

$$\begin{array}{rcl}
\mathcal{D}[\![e_1 \texttt{ + } e_2]\!] & = & \lambda f.\lambda x.\mathcal{D}[\![e_1]\!](f)(\mathcal{D}[\![e_2]\!](f)(x)) \\
\mathcal{D}[\![e_1 \texttt{ * } e_2]\!] & = & \lambda f.\lambda x.\mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_2]\!](f))(x)
\end{array}$$

Church Encodings – Church Numerals

For example,

$$\begin{array}{rcl} \mathcal{D}[\![1 + 1]\!] & = & \lambda f.\lambda x.\mathcal{D}[\![1]\!](f)(\mathcal{D}[\![1]\!](f)(x)) \\ & = & \lambda f.\lambda x.(\lambda f.\lambda x.f(x))(f)((\lambda f.\lambda x.f(x))(f)(x)) \\ & = & \lambda f.\lambda x.f((\lambda f.\lambda x.f(x))(f)(x)) \\ & = & \lambda f.\lambda x.f(f(x)) \\ & = & \mathcal{D}[\![2]\!] \end{array}$$

[2] https://en.wikipedia.org/wiki/Church_encoding

Church Encodings – Church Numerals

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$$
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\mathcal{D}[\![1 + 1]\!] & = & \lambda f.\lambda x.\mathcal{D}[\![1]\!](f)(\mathcal{D}[\![1]\!](f)(x)) \\
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& = & \lambda f.\lambda x.f((\lambda f.\lambda x.f(x))(f)(x)) \\
& = & \lambda f.\lambda x.f(f(x)) \\
& = & \mathcal{D}[\![2]\!]
\end{array}
$$

We can represent other data or operations in the **LC** using **Church encodings**, such as **integers**, **booleans**, **pairs**, **lists**, and so on.[2]

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We can represent other data or operations in the **LC** using **Church encodings**, such as **integers**, **booleans**, **pairs**, **lists**, and so on.[2]

Let's see one more example of **Church encoding** for **booleans** and **logical operations** (i.e., **Church booleans**).

[2] https://en.wikipedia.org/wiki/Church_encoding

Church Encodings – Church Booleans

The key idea is to encode a **boolean** b as a **function** that takes two arguments t and f and applies t if b is `true` or f if b is `false`:

$$
\begin{aligned}
\mathcal{D}[\![\texttt{true}]\!] &= \lambda t.\lambda f.t \\
\mathcal{D}[\![\texttt{false}]\!] &= \lambda t.\lambda f.f
\end{aligned}
\qquad
\begin{aligned}
\mathcal{D}[\![\texttt{if}(e_1)\ e_2\ \texttt{else}\ e_3]\!] &= \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_2]\!])(\mathcal{D}[\![e_3]\!]) \\
\mathcal{D}[\![e_1\ \texttt{\&\&}\ e_2]\!] &= \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_2]\!])(\mathcal{D}[\![e_1]\!]) \\
\mathcal{D}[\![e_1\ \texttt{||}\ e_2]\!] &= \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_1]\!])(\mathcal{D}[\![e_2]\!]) \\
\mathcal{D}[\![!\ e]\!] &= \lambda t.\lambda f.\mathcal{D}[\![e]\!](f)(t)
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$$

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$$

For example,

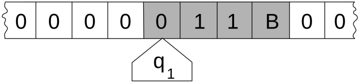
$$
\begin{aligned}
\mathcal{D}[\![\texttt{true \&\& false}]\!] &= \mathcal{D}[\![\texttt{true}]\!](\mathcal{D}[\![\texttt{false}]\!])(\mathcal{D}[\![\texttt{true}]\!]) \\
&= (\lambda t.\lambda f.t)(\mathcal{D}[\![\texttt{false}]\!])(\mathcal{D}[\![\texttt{true}]\!]) \\
&= \mathcal{D}[\![\texttt{false}]\!]
\end{aligned}
$$

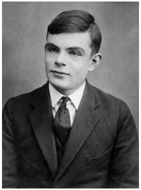
Church-Turing Thesis

Alonzo Church invented **lambda calculus** in 1930s, and it became the foundation of **programming languages**:

$$e ::= e \mid \lambda x.e \mid e(e)$$

Alan Turing invented **Turing machines (TM)** in 1936, and it became the foundation of **computers**:





Church-Turing Thesis: **Lambda Calculus** is **Turing complete**.

> *Any* **real-world computation** *can be translated into an equivalent computation involving a* **Turing machine** *or can be done using* **lambda calculus**.

Homework #2

PLRG

https://github.com/ku-plrg-classroom/docs/tree/main/cose212/cobalt

- Please see above document on GitHub:
  1. Implement `interp` function.
  2. Implement `subExpr1` and `subExpr2` functions.

- The due date is 23:59 on Oct. 14 (Mon.).

- Please only submit **`Implementation.scala`** file to **Blackboard**.

Summary

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Next Lecture

- Recursive Functions

Jihyeok Park
jihyeok_park@korea.ac.kr
https://plrg.korea.ac.kr