Lecture 26 – Type Inference (2)

COSE212: Programming Languages

Jihyeok Park

PLRG

2025 Fall

Recall

- **Type inference** is the process of automatically inferring the types of expressions.

Recall

- **Type inference** is the process of automatically inferring the types of expressions.
- We have seen three examples to learn how the type inference works.

```
/* RFAE */ def sum(x) = if (x < 1) 0 else x + sum(x - 1); sum
```

```
/* FAE */ val app = n => f => f(n); app(42)(x => x)
```

```
/* FAE */ val id = x => x; val n = id(42); val b = id(true); b
```

Recall

- **Type inference** is the process of automatically inferring the types of expressions.
- We have seen three examples to learn how the type inference works.

```
/* RFAE */ def sum(x) = if (x < 1) 0 else x + sum(x - 1); sum
```

```
/* FAE */ val app = n => f => f(n); app(42)(x => x)
```

```
/* FAE */ val id = x => x; val n = id(42); val b = id(true); b
```

- In this lecture, let's learn the details of the type inference algorithm.

Recall

- **Type inference** is the process of automatically inferring the types of expressions.
- We have seen three examples to learn how the type inference works.

```
/* RFAE */ def sum(x) = if (x < 1) 0 else x + sum(x - 1); sum
```

```
/* FAE */ val app = n => f => f(n); app(42)(x => x)
```

```
/* FAE */ val id = x => x; val n = id(42); val b = id(true); b
```

- In this lecture, let's learn the details of the type inference algorithm.
- **TIFAE** – TRFAE with **type inference**.
  - **Type Checker** and **Typing Rules** with **Type Inference**
  - Interpreter and Natural Semantics

Contents

Contents

Type Checker and Typing Rules

Let's ❶ design **typing rules** of TIFAE to define when an expression is well-typed in the form of:

$$\boxed{\Gamma \vdash e : \tau}$$

and ❷ implement a **type checker** in Scala according to typing rules:

```scala
def typeCheck(expr: Expr, tenv: TypeEnv): Type = ???
```

The type checker returns the **type** of e if it is well-typed, or rejects it and throws a **type error** otherwise.

We will keep track of the **variable types** using a **type environment** Γ as a mapping from variable names to their types.

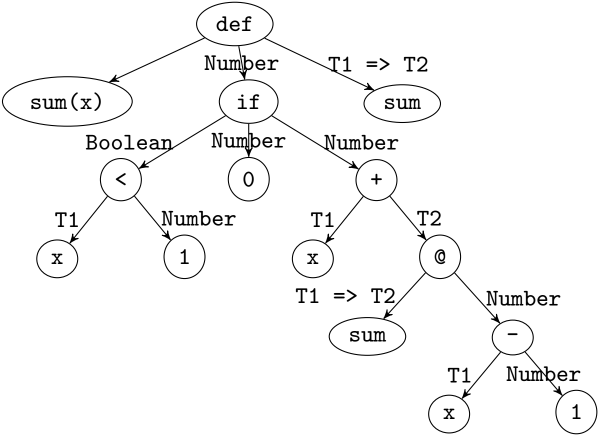
$$\text{Type Environments} \qquad \Gamma \;\in \mathbb{\Gamma} = \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T} \quad (\texttt{TypeEnv})$$

```scala
type TypeEnv = Map[String, Type]
```

Recall: Example 1 – sum

In addition, we need to keep track of the **solution** for **type constraints** over **type variables** to infer the types of expressions.

```
/* RFAE */ def sum(x) = if (x < 1) 0 else x + sum(x - 1); sum
```



Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	Number

Solutions for Type Constraints

A **solution** is a mapping from **type variables** to **types** or $\bullet$.

Types	$\mathbb{T} \ni \tau ::= \texttt{num} \mid \texttt{bool} \mid \tau \to \tau \mid \alpha$	($\texttt{Type}$)
Solutions	$\psi \in \Psi = \mathbb{X}_\alpha \xrightarrow{\text{fin}} (\mathbb{T} \uplus \{\bullet\})$	($\texttt{Solution}$)
Type Variables	$\alpha \in \mathbb{X}_\alpha$	($\texttt{Int}$)

```scala
type Solution = Map[Int, Option[Type]]
```

Note that $\bullet$ ($\texttt{None}$) represents a **not yet solved** (**free**) type variable.

Solutions for Type Constraints

A **solution** is a mapping from **type variables** to **types** or $\bullet$.

Types	$\mathbb{T} \ni \tau ::= \texttt{num} \mid \texttt{bool} \mid \tau \to \tau \mid \alpha$	$(\texttt{Type})$
Solutions	$\psi \in \Psi = \mathbb{X}_\alpha \xrightarrow{\text{fin}} (\mathbb{T} \uplus \{\bullet\})$	$(\texttt{Solution})$
Type Variables	$\alpha \in \mathbb{X}_\alpha$	$(\texttt{Int})$

```
type Solution = Map[Int, Option[Type]]
```

Note that $\bullet$ $(\texttt{None})$ represents a **not yet solved** (**free**) type variable.

Now, we have new forms of **type checker** and **typing rules**.

```
def typeCheck(expr: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = ???
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

Similar to the memory passing in MFAE for mutation, we will pass the solution ψ and update it during type checking.

Numbers

```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case Num(n) => (NumT, sol)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{-}\mathtt{Num}\ \frac{}{\Gamma, \psi \vdash n : \mathtt{num}, \psi}$$

Additions

```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case Add(l, r) =>
    val (lty, sol1) = typeCheck(l, tenv, sol)
    val (rty, sol2) = typeCheck(r, tenv, sol1)
    val sol3 = unify(lty, NumT, sol2)
    val sol4 = unify(rty, NumT, sol3)
    (NumT, sol4)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{-}\texttt{Add}\ \frac{\begin{array}{cc}\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 & \Gamma, \psi_1 \vdash e_2 : \tau_2, \psi_2 \\ \texttt{unify}(\tau_1, \texttt{num}, \psi_2) = \psi_3 & \texttt{unify}(\tau_2, \texttt{num}, \psi_3) = \psi_4\end{array}}{\Gamma, \psi_0 \vdash e_1 \texttt{ + } e_2 : \texttt{num}, \psi_4}$$

The `unify` function that takes two types must be the same and updates the given solution. We will see how it works later.

Conditionals

```scala
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case If(c, t, e) =>
    val (cty, sol1) = typeCheck(c, tenv, sol)
    val (tty, sol2) = typeCheck(t, tenv, sol1)
    val (ety, sol3) = typeCheck(e, tenv, sol2)
    val sol4 = unify(cty, BoolT, sol3)
    val sol5 = unify(tty, ety, sol4)
    (tty, sol5)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{-}\texttt{If}\ \frac{\begin{array}{ccc}\Gamma, \psi \vdash e_c : \tau_c, \psi_c & \Gamma, \psi_c \vdash e_t : \tau_t, \psi_t & \Gamma, \psi_t \vdash e_e : \tau_e, \psi_e \\ \multicolumn{3}{c}{\texttt{unify}(\tau_c, \texttt{bool}, \psi_e) = \psi' \qquad \texttt{unify}(\tau_t, \tau_e, \psi') = \psi''}\end{array}}{\Gamma, \psi \vdash \texttt{if}\ (e_c)\ e_t\ \texttt{else}\ e_e : \tau_t, \psi''}$$

Immutable Variable Defs. and Identifier Lookup

```scala
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...

  case Val(x, e, b) =>
    val (ety, sol1) = typeCheck(e, tenv, sol)
    typeCheck(b, tenv + (x -> ety), sol1)

  case Id(x) =>
    val ty = tenv.getOrElse(x, error(s"free identifier: $x"))
    (ty, sol)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{−}\texttt{Val}\ \frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \qquad \Gamma[x : \tau_1], \psi_1 \vdash e_2 : \tau_2, \psi_2}{\Gamma, \psi_0 \vdash \texttt{val}\ x\ \texttt{=}\ e_1;\ e_2 : \tau_2, \psi_2}$$

$$\tau\text{−}\texttt{Id}\ \frac{x \in \mathsf{Domain}(\Gamma)}{\Gamma, \psi \vdash x : \Gamma(x), \psi}$$

Function Definitions

```scala
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case Fun(p, b) =>
    val (pty, sol1) = newTypeVar(sol)
    val (rty, sol2) = typeCheck(b, tenv + (p -> pty), sol1)
    (ArrowT(pty, rty), sol2)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{–}\texttt{Fun}\ \frac{\alpha_p \notin \psi \qquad \Gamma[x : \alpha_p], \psi[\alpha_p \mapsto \bullet] \vdash e : \tau, \psi'}{\Gamma, \psi \vdash \lambda x.e : \alpha_p \to \tau, \psi'}$$

We need to introduce a **new type variable** α_p for the parameter x.

Recursive Function Definitions

```scala
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case Rec(f, p, b, s) =>
    val (pty, sol1) = newTypeVar(sol)
    val (rty, sol2) = newTypeVar(sol1)
    val fty = ArrowT(pty, rty)
    val tenv1 = tenv + (f -> fty)
    val tenv2 = tenv1 + (p -> pty)
    val (bty, sol3) = typeCheck(b, tenv2, sol2)
    val sol4 = unify(bty, rty, sol3)
    typeCheck(s, tenv1, sol4)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{–}\mathtt{Rec}\ \frac{\begin{array}{c}\alpha_p, \alpha_r \notin \psi \qquad \alpha_p \neq \alpha_r \qquad \Gamma_1 = \Gamma[x_f : (\alpha_p \to \alpha_r)] \\ \Gamma_2 = \Gamma_1[x_p : \alpha_p] \qquad \Gamma_2, \psi[\alpha_p \mapsto \bullet, \alpha_r \mapsto \bullet] \vdash e_b : \tau_b, \psi_b \\ \mathtt{unify}(\tau_b, \alpha_r, \psi_b) = \psi_r \qquad \Gamma_1, \psi_r \vdash e_s : \tau_s, \psi_s\end{array}}{\Gamma, \psi \vdash \mathtt{def}\ x_f(x_p)\ \mathtt{=}\ e_b;\ e_s : \tau_s, \psi_s}$$

Function Applications

```scala
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case App(f, a) =>
    val (fty, sol1) = typeCheck(f, tenv, sol)
    val (aty, sol2) = typeCheck(a, tenv, sol1)
    val (rty, sol3) = newTypeVar(sol2)
    val sol4 = unify(ArrowT(aty, rty), fty, sol3)
    (rty, sol4)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{–}\texttt{App}\ \frac{\Gamma, \psi \vdash e_f : \tau_f, \psi_f \qquad \Gamma, \psi_f \vdash e_a : \tau_a, \psi_a \\ \alpha_r \notin \psi_a \qquad \texttt{unify}(\tau_a \to \alpha_r, \tau_f, \psi_a[\alpha_r \mapsto \bullet]) = \psi'}{\Gamma, \psi \vdash e_f(e_a) : \alpha_r, \psi'}$$

Contents

Type Unification

Definition (Type Unification)

Type unification is the process of updating a solution to make two types equal. If the types are not unifiable, then this process fails and throws an exception.

$$\boxed{\texttt{unify} : (\mathbb{T} \times \mathbb{T} \times \Psi) \rightharpoonup \Psi}$$

For example, if we unify a type variable α and the number type $\texttt{num}$, the solution $[\alpha \mapsto \bullet]$ is updated to $[\alpha \mapsto \texttt{num}]$.

$$\texttt{unify}(\alpha, \texttt{num}, [\alpha \mapsto \bullet]) = [\alpha \mapsto \texttt{num}]$$

Type Unification

Definition (Type Unification)

Type unification is the process of updating a solution to make two types equal. If the types are not unifiable, then this process fails and throws an exception.

$$\boxed{\texttt{unify} : (\mathbb{T} \times \mathbb{T} \times \Psi) \rightharpoonup \Psi}$$

For example, if we unify a type variable α and the number type $\texttt{num}$, the solution $[\alpha \mapsto \bullet]$ is updated to $[\alpha \mapsto \texttt{num}]$.

$$\texttt{unify}(\alpha, \texttt{num}, [\alpha \mapsto \bullet]) = [\alpha \mapsto \texttt{num}]$$

Before, we define the type unification, we need to define the **type resolving** and **occurrence checking** functions.

1. **Type resolving** is the process of recursively resolving a type variable to its representative type to deal with the **type aliasing**.
2. **Occurrence checking** is the process of checking whether a type variable occurs in a type to detect **recursive types**.

Type Resolving

To understand why we need the **type resolving** function, let's consider the following example:

$$\texttt{unify}(\alpha_1, \texttt{num}, \psi_1) = \psi_2$$

Solution

$\psi_1 =$

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	α_2
α_2	α_3
α_3	$\bullet$

$\psi_2 =$

Type Resolving

To understand why we need the **type resolving** function, let's consider the following example:

$$\texttt{unify}(\alpha_1, \texttt{num}, \psi_1) = \psi_2$$

$\psi_1 =$ Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	α_2
α_2	α_3
α_3	$\bullet$

$\psi_2 =$ Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	$\texttt{num}$
α_2	α_3
α_3	$\bullet$

Type Resolving

To understand why we need the **type resolving** function, let's consider the following example:

$$\texttt{unify}(\alpha_1, \texttt{num}, \psi_1) = \psi_2$$

$\psi_1 =$

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	α_2
α_2	α_3
α_3	$\bullet$

$\psi_2 =$

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	$\texttt{num}$
α_2	α_3
α_3	$\bullet$

If we directly update α_1 to $\texttt{num}$ in the solution ψ_2, it misses the information that α_2 and α_3 are also $\texttt{num}$.

Type Resolving

To understand why we need the **type resolving** function, let's consider the following example:

$$\texttt{unify}(\alpha_1, \texttt{num}, \psi_1) = \psi_2$$

$\psi_1 =$ Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	α_2
α_2	α_3
α_3	$\bullet$

$\psi_2 =$ Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	α_2
α_2	α_3
α_3	$\texttt{num}$

If we directly update α_1 to `num` in the solution ψ_2, it misses the information that α_2 and α_3 are also `num`.

Instead, we need to **resolve** the type variable α_1 to find its **representative type** (i.e., α_3) and unify it with `num` to deal with the **type aliasing**.

Type Resolving

We can define the **type resolving** function as follows:

$$\boxed{\texttt{resolve} : (\mathbb{T} \times \Psi) \to \mathbb{T}}$$

$$\texttt{resolve}(\tau, \psi) = \begin{cases} \texttt{resolve}(\tau', \psi) & \text{if } \tau = \alpha \wedge \psi(\alpha) = \tau' \\ \texttt{resolve}(\tau_p, \psi) \to \texttt{resolve}(\tau_r, \psi) & \text{if } \tau = (\tau_p \to \tau_r) \\ \tau & \text{otherwise} \end{cases}$$

and implement it in Scala as follows:

```scala
def resolve(ty: Type, sol: Solution): Type = ty match
  case VarT(k) => sol(k) match
    case Some(ty) => resolve(ty, sol)
    case None => ty
  case ArrowT(pty, rty) =>
    ArrowT(resolve(pty, sol), resolve(rty, sol))
  case _ => ty
```

Occurrence Checking

Let's understand why we need the **occurrence checking** function:

$$\texttt{unify}(\alpha_1, \texttt{num} \rightarrow \alpha_1, \psi) = \psi'$$

Can we unify α_1 and $\texttt{num} \rightarrow \alpha_1$?

Occurrence Checking

Let's understand why we need the **occurrence checking** function:

$$\texttt{unify}(\alpha_1, \texttt{num} \rightarrow \alpha_1, \psi) = \psi'$$

Can we unify α_1 and $\texttt{num} \rightarrow \alpha_1$? **No!** because it requires **recursive types** not supported in our type system.

Occurrence Checking

Let's understand why we need the **occurrence checking** function:

$$\mathtt{unify}(\alpha_1, \mathtt{num} \to \alpha_1, \psi) = \psi'$$

Can we unify α_1 and $\mathtt{num} \to \alpha_1$? **No!** because it requires **recursive types** not supported in our type system.

Let's define the **occurrence checking** function to detect type constraints that require recursive types

$$\boxed{\mathtt{occur} : (\mathbb{X}_\alpha \times \mathbb{T}) \to \mathtt{bool}}$$

$$\mathtt{occur}(\alpha, \tau) = \begin{cases} \mathtt{true} & \text{if } \tau = \alpha \\ \mathtt{occur}(\alpha, \tau_p) \vee \mathtt{occur}(\alpha, \tau_r) & \text{if } \tau = (\tau_p \to \tau_r) \\ \mathtt{false} & \text{otherwise} \end{cases}$$

Occurrence Checking

Let's understand why we need the **occurrence checking** function:

$$\mathtt{unify}(\alpha_1, \mathtt{num} \to \alpha_1, \psi) = \psi'$$

Can we unify α_1 and $\mathtt{num} \to \alpha_1$? **No!** because it requires **recursive types** not supported in our type system.

Let's define the **occurrence checking** function to detect type constraints that require recursive types

$$\boxed{\mathtt{occur} : (\mathbb{X}_\alpha \times \mathbb{T}) \to \mathtt{bool}}$$

$$\mathtt{occur}(\alpha, \tau) = \begin{cases} \mathtt{true} & \text{if } \tau = \alpha \\ \mathtt{occur}(\alpha, \tau_p) \lor \mathtt{occur}(\alpha, \tau_r) & \text{if } \tau = (\tau_p \to \tau_r) \\ \mathtt{false} & \text{otherwise} \end{cases}$$

and implement it in Scala as follows:

```
def occurs(k: Int, ty: Type): Boolean = ty match
  case VarT(l) => k == l
  case ArrowT(pty, rty) => occurs(k, pty) || occurs(k, rty)
  case _ => false
```

Type Unification

Using the **type resolving** and **occurrence checking** functions, we could define the **type unification** as a partial function:

$$\boxed{\texttt{unify} : (\mathbb{T} \times \mathbb{T} \times \Psi) \rightharpoonup \Psi}$$

$$\texttt{unify}(\tau_1, \tau_2, \psi) = \left\{ \right.$$

.

1

2

3

Type Unification

Using the **type resolving** and **occurrence checking** functions, we could define the **type unification** as a partial function:

$$\boxed{\texttt{unify} : (\mathbb{T} \times \mathbb{T} \times \Psi) \rightharpoonup \Psi}$$

$$\texttt{unify}(\tau_1, \tau_2, \psi) = \left\{ \right.$$

where $\tau_1' = \texttt{resolve}(\tau_1, \psi)$ and $\tau_2' = \texttt{resolve}(\tau_2, \psi)$.

1. First, it resolves the types τ_1 and τ_2 with the current solution ψ into τ_1' and τ_2' using the **type resolving** function `resolve`.
2.
3.

Type Unification

Using the **type resolving** and **occurrence checking** functions, we could define the **type unification** as a partial function:

$$\boxed{\texttt{unify} : (\mathbb{T} \times \mathbb{T} \times \Psi) \rightharpoonup \Psi}$$

$$\texttt{unify}(\tau_1, \tau_2, \psi) = \begin{cases} \psi[\alpha \mapsto \tau_2'] & \text{if } \tau_1' = \alpha \wedge \neg\texttt{occur}(\alpha, \tau_2') \\ \psi[\alpha \mapsto \tau_1'] & \text{if } \tau_2' = \alpha \wedge \neg\texttt{occur}(\alpha, \tau_1') \end{cases}$$

where $\tau_1' = \texttt{resolve}(\tau_1, \psi)$ and $\tau_2' = \texttt{resolve}(\tau_2, \psi)$.

1. First, it resolves the types τ_1 and τ_2 with the current solution ψ into τ_1' and τ_2' using the **type resolving** function `resolve`.
2. If one of τ_1' or τ_2' is a type variable and does **not occur** in the other type, it updates the solution of the type variable.
3.

Type Unification

PLRG

Using the **type resolving** and **occurrence checking** functions, we could define the **type unification** as a partial function:

$$\boxed{\texttt{unify} : (\mathbb{T} \times \mathbb{T} \times \Psi) \rightharpoonup \Psi}$$

$$\texttt{unify}(\tau_1, \tau_2, \psi) = \begin{cases} \psi & \text{if } \tau_1' = \texttt{num} \wedge \tau_2' = \texttt{num} \\ \psi & \text{if } \tau_1' = \texttt{bool} \wedge \tau_2' = \texttt{bool} \\ \texttt{unify}(\tau_{1,r}, \tau_{2,r}, \texttt{unify}(\tau_{1,p}, \tau_{2,p}, \psi)) & \text{if } \tau_1' = (\tau_{1,p} \to \tau_{1,r}) \wedge \tau_2' = (\tau_{2,p} \to \tau_{2,r}) \\ \psi & \text{if } \tau_1' = \alpha = \tau_2' \\ \psi[\alpha \mapsto \tau_2'] & \text{if } \tau_1' = \alpha \wedge \neg\texttt{occur}(\alpha, \tau_2') \\ \psi[\alpha \mapsto \tau_1'] & \text{if } \tau_2' = \alpha \wedge \neg\texttt{occur}(\alpha, \tau_1') \end{cases}$$

where $\tau_1' = \texttt{resolve}(\tau_1, \psi)$ and $\tau_2' = \texttt{resolve}(\tau_2, \psi)$.

1. First, it resolves the types τ_1 and τ_2 with the current solution ψ into τ_1' and τ_2' using the **type resolving** function `resolve`.
2. If one of τ_1' or τ_2' is a type variable and does **not occur** in the other type, it updates the solution of the type variable.
3. Otherwise, it checks τ_1' and τ_2' are equal or recursively unifies them.

Type Unification

$$
\texttt{unify}(\tau_1, \tau_2, \psi) = \begin{cases}
\psi & \text{if } \tau_1' = \texttt{num} \land \tau_2' = \texttt{num} \\
\psi & \text{if } \tau_1' = \texttt{bool} \land \tau_2' = \texttt{bool} \\
\texttt{unify}(\tau_{1,r}, \tau_{2,r}, \texttt{unify}(\tau_{1,p}, \tau_{2,p}, \psi)) & \text{if } \tau_1' = (\tau_{1,p} \rightarrow \tau_{1,r}) \land \tau_2' = (\tau_{2,p} \rightarrow \tau_{2,r}) \\
\psi & \text{if } \tau_1' = \alpha = \tau_2' \\
\psi[\alpha \mapsto \tau_2'] & \text{if } \tau_1' = \alpha \land \neg\texttt{occur}(\alpha, \tau_2') \\
\psi[\alpha \mapsto \tau_1'] & \text{if } \tau_2' = \alpha \land \neg\texttt{occur}(\alpha, \tau_1')
\end{cases}
$$

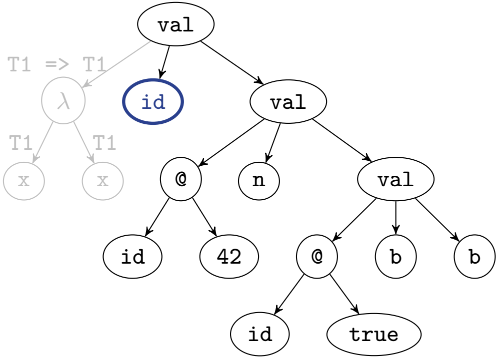
where $\tau_1' = \texttt{resolve}(\tau_1, \psi)$ and $\tau_2' = \texttt{resolve}(\tau_2, \psi)$.

And, we can implement the **type unification** function in Scala as follows:

```scala
def unify(lty: Type, rty: Type, sol: Solution): Solution =
  (resolve(lty, sol), resolve(rty, sol)) match
    case (NumT, NumT) => sol
    case (BoolT, BoolT) => sol
    case (ArrowT(lpty, lrty), ArrowT(rpty, rrty)) =>
      unify(lrty, rrty, unify(lpty, rpty, sol))
    case (VarT(k), VarT(l)) if k == l => sol
    case (VarT(k), rty) if !occurs(k, rty) => sol + (k -> Some(rty))
    case (lty, VarT(k)) if !occurs(k, lty) => sol + (k -> Some(lty))
    case _ => error(s"Cannot unify ${lty.str} and ${rty.str}")
```

Contents

Recall: Example 3 – `id`



Type Environment

$\mathbb{X}$	$\mathbb{T}$
`id`	`[T1] { T1 => T1 }`

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$

Let's **generalize** the type `T1 => T1` into a **polymorphic type** for `id` with **type variable** `T1` as a **type parameter**.

We call this **let-polymorphism** because it only introduces polymorphism for the let-binding (e.g., `val`).

Type Environment with Type Schemes

We need to extend the **type environment** with **type schemes**, restricted forms of polymorphic types.

Type Environments	Γ	$\in$	$\mathbb{\Gamma} = \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T}^{\forall}$
Type Schemes	$\forall(\alpha^*).\tau = \tau^{\forall}$	$\in$	$\mathbb{T}^{\forall} = \mathbb{X}_{\alpha}^{*} \times \mathbb{T}$
Types	$\mathbb{T} \ni$		$\tau ::= \texttt{num} \mid \texttt{bool} \mid \tau \to \tau \mid \alpha$

Type Environment with Type Schemes

We need to extend the **type environment** with **type schemes**, restricted forms of polymorphic types.

Type Environments	Γ	$\in$	$\mathbb{\Gamma} = \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T}^{\forall}$
Type Schemes	$\forall(\alpha^*).\tau = \tau^{\forall}$	$\in$	$\mathbb{T}^{\forall} = \mathbb{X}_{\alpha}^* \times \mathbb{T}$
Types	$\mathbb{T} \ni$	$\tau ::=$	$\texttt{num} \mid \texttt{bool} \mid \tau \to \tau \mid \alpha$

Note that polymorphic types are not types in TIFAE, and **type schemes** are restricted forms of polymorphic types used in type environments.

Type Environment with Type Schemes

We need to extend the **type environment** with **type schemes**, restricted forms of polymorphic types.

Type Environments	$\Gamma \in \mathbb{\Gamma} = \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T}^{\forall}$
Type Schemes	$\forall(\alpha^*).\tau = \tau^{\forall} \in \mathbb{T}^{\forall} = \mathbb{X}_{\alpha}^{*} \times \mathbb{T}$
Types	$\mathbb{T} \ni \tau ::= \texttt{num} \mid \texttt{bool} \mid \tau \to \tau \mid \alpha$

Note that polymorphic types are not types in TIFAE, and **type schemes** are restricted forms of polymorphic types used in type environments.

We can define the **type environment** and **type schemes** in Scala:

```
// type environments
type TypeEnv = Map[String, TypeScheme]
// type schemes
case class TypeScheme(ks: List[Int], ty: Type)
```

Immutable Variable Defs. with Type Generalization

```scala
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...

  case Val(x, e, b) =>
    val (ety, sol1) = typeCheck(e, tenv, sol)
    val polyty = gen(ety, tenv, sol1)
    typeCheck(b, tenv + (x -> polyty), sol1)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{-}\mathtt{Val}\ \frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \quad \mathtt{gen}(\tau_1, \Gamma, \psi_1) = \tau_1^{\forall} \qquad \Gamma[x : \tau_1^{\forall}], \psi_1 \vdash e_2 : \tau_2, \psi_2}{\Gamma, \psi_0 \vdash \mathtt{val}\ x\ \mathtt{=}\ e_1;\ e_2 : \tau_2, \psi_2}$$

We need to **generalize** the type τ_1 of the expression e_1 into a **type scheme** $\tau_1^{\forall}$ using the **type generalization** function gen.

Immutable Variable Defs. with Type Generalization

```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...

  case Val(x, e, b) =>
    val (ety, sol1) = typeCheck(e, tenv, sol)
    val polyty = gen(ety, tenv, sol1)
    typeCheck(b, tenv + (x -> polyty), sol1)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\texttt{-Val}\ \frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \qquad \texttt{gen}(\tau_1, \Gamma, \psi_1) = \tau_1^{\forall} \qquad \Gamma[x : \tau_1^{\forall}], \psi_1 \vdash e_2 : \tau_2, \psi_2}{\Gamma, \psi_0 \vdash \texttt{val}\ x\ \texttt{=}\ e_1;\ e_2 : \tau_2, \psi_2}$$

We need to **generalize** the type τ_1 of the expression e_1 into a **type scheme** $\tau_1^{\forall}$ using the **type generalization** function gen. For example,

$$\texttt{gen}(\alpha \to \alpha, \varnothing, [\alpha \mapsto \bullet]) = \forall \alpha.(\alpha \to \alpha)$$

Type Generalization

We can define the **type generalization** function $\texttt{gen}$ as follows:

$$\boxed{\texttt{gen} : (\mathbb{T} \times \Gamma \times \Psi) \to \mathbb{T}^{\forall}}$$

$$\texttt{gen}(\tau, \Gamma, \psi) = \forall(\alpha_1, \ldots, \alpha_m).\tau \quad \text{where} \quad \texttt{free}_{\tau}(\tau, \psi) \setminus \texttt{free}_{\Gamma}(\Gamma, \psi) = \{\alpha_1, \ldots, \alpha_m\}$$

Type Generalization

We can define the **type generalization** function gen as follows:

$$\boxed{\texttt{gen} : (\mathbb{T} \times \Gamma \times \Psi) \to \mathbb{T}^{\forall}}$$

$$\texttt{gen}(\tau, \Gamma, \psi) = \forall(\alpha_1, \ldots, \alpha_m).\tau \quad \text{where} \quad \texttt{free}_\tau(\tau, \psi) \setminus \texttt{free}_\Gamma(\Gamma, \psi) = \{\alpha_1, \ldots, \alpha_m\}$$

with the following definitions of **free type variables** in each component:

$$\boxed{\texttt{free}_\tau : (\mathbb{T} \times \Psi) \to \mathcal{P}(\mathbb{X}_\alpha)}$$

$$\texttt{free}_\tau(\tau, \psi) = \begin{cases} \{\alpha\} & \text{if } \tau = \alpha \land \psi(\alpha) = \bullet \\ \texttt{free}_\tau(\tau', \psi) & \text{if } \tau = \alpha \land \psi(\alpha) = \tau' \\ \texttt{free}_\tau(\tau_p, \psi) \cup \texttt{free}_\tau(\tau_r, \psi) & \text{if } \tau = (\tau_p \to \tau_r) \\ \varnothing & \text{otherwise} \end{cases}$$

Type Generalization

We can define the **type generalization** function $\texttt{gen}$ as follows:

$$\boxed{\texttt{gen} : (\mathbb{T} \times \Gamma \times \Psi) \to \mathbb{T}^\forall}$$

$$\texttt{gen}(\tau, \Gamma, \psi) = \forall(\alpha_1, \ldots, \alpha_m).\tau \quad \text{where} \quad \texttt{free}_\tau(\tau, \psi) \setminus \texttt{free}_\Gamma(\Gamma, \psi) = \{\alpha_1, \ldots, \alpha_m\}$$

with the following definitions of **free type variables** in each component:

$$\boxed{\texttt{free}_\tau : (\mathbb{T} \times \Psi) \to \mathcal{P}(\mathbb{X}_\alpha)}$$

$$\texttt{free}_\tau(\tau, \psi) = \begin{cases} \{\alpha\} & \text{if } \tau = \alpha \wedge \psi(\alpha) = \bullet \\ \texttt{free}_\tau(\tau', \psi) & \text{if } \tau = \alpha \wedge \psi(\alpha) = \tau' \\ \texttt{free}_\tau(\tau_p, \psi) \cup \texttt{free}_\tau(\tau_r, \psi) & \text{if } \tau = (\tau_p \to \tau_r) \\ \varnothing & \text{otherwise} \end{cases}$$

$$\boxed{\texttt{free}_{\tau^\forall} : (\mathbb{T}^\forall \times \Psi) \to \mathcal{P}(\mathbb{X}_\alpha)}$$

$$\texttt{free}_{\tau^\forall}(\forall(\alpha_1, \ldots, \alpha_m).\tau, \psi) = \texttt{free}_\tau(\tau, \psi) \setminus \{\alpha_1, \ldots, \alpha_m\}$$

Type Generalization

We can define the **type generalization** function $\texttt{gen}$ as follows:

$$\boxed{\texttt{gen} : (\mathbb{T} \times \Gamma \times \Psi) \to \mathbb{T}^{\forall}}$$

$$\texttt{gen}(\tau, \Gamma, \psi) = \forall(\alpha_1, \ldots, \alpha_m).\tau \quad \text{where} \quad \texttt{free}_{\tau}(\tau, \psi) \setminus \texttt{free}_{\Gamma}(\Gamma, \psi) = \{\alpha_1, \ldots, \alpha_m\}$$

with the following definitions of **free type variables** in each component:

$$\boxed{\texttt{free}_{\tau} : (\mathbb{T} \times \Psi) \to \mathcal{P}(\mathbb{X}_{\alpha})}$$

$$\texttt{free}_{\tau}(\tau, \psi) = \begin{cases} \{\alpha\} & \text{if } \tau = \alpha \wedge \psi(\alpha) = \bullet \\ \texttt{free}_{\tau}(\tau', \psi) & \text{if } \tau = \alpha \wedge \psi(\alpha) = \tau' \\ \texttt{free}_{\tau}(\tau_p, \psi) \cup \texttt{free}_{\tau}(\tau_r, \psi) & \text{if } \tau = (\tau_p \to \tau_r) \\ \varnothing & \text{otherwise} \end{cases}$$

$$\boxed{\texttt{free}_{\tau^{\forall}} : (\mathbb{T}^{\forall} \times \Psi) \to \mathcal{P}(\mathbb{X}_{\alpha})}$$

$$\texttt{free}_{\tau^{\forall}}(\forall(\alpha_1, \ldots, \alpha_m).\tau, \psi) = \texttt{free}_{\tau}(\tau, \psi) \setminus \{\alpha_1, \ldots, \alpha_m\}$$

$$\boxed{\texttt{free}_{\Gamma} : (\Gamma \times \Psi) \to \mathcal{P}(\mathbb{X}_{\alpha})}$$

$$\texttt{free}_{\Gamma}([x_1 : \tau_1^{\forall}, \ldots, x_n : \tau_n^{\forall}], \psi) = \texttt{free}_{\tau^{\forall}}(\tau_1^{\forall}, \psi) \cup \ldots \cup \texttt{free}_{\tau^{\forall}}(\tau_n^{\forall}, \psi)$$

Type Generalization

We can define the **type generalization** function $\texttt{gen}$ as follows:

$$\boxed{\texttt{gen} : (\mathbb{T} \times \mathbb{\Gamma} \times \Psi) \rightarrow \mathbb{T}^{\forall}}$$

$$\texttt{gen}(\tau, \Gamma, \psi) = \forall(\alpha_1, \ldots, \alpha_m).\tau \quad \text{where} \quad \texttt{free}_{\tau}(\tau, \psi) \setminus \texttt{free}_{\Gamma}(\Gamma, \psi) = \{\alpha_1, \ldots, \alpha_m\}$$

Why do we need to subtract the free type variables $\texttt{free}_{\Gamma}(\Gamma, \psi)$ in the type environment Γ when generalizing the type τ?

Type Generalization

We can define the **type generalization** function gen as follows:

$$\boxed{\texttt{gen} : (\mathbb{T} \times \Gamma \times \Psi) \to \mathbb{T}^{\forall}}$$

$$\texttt{gen}(\tau, \Gamma, \psi) = \forall(\alpha_1, \ldots, \alpha_m).\tau \quad \text{where} \quad \texttt{free}_{\tau}(\tau, \psi) \setminus \texttt{free}_{\Gamma}(\Gamma, \psi) = \{\alpha_1, \ldots, \alpha_m\}$$

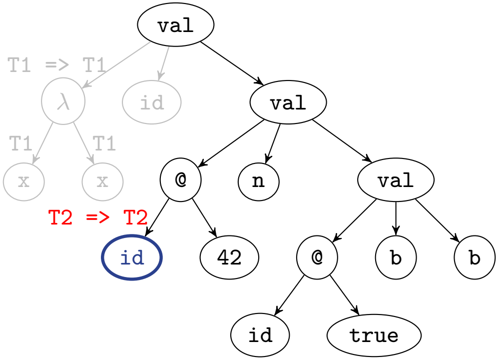
Why do we need to subtract the free type variables $\texttt{free}_{\Gamma}(\Gamma, \psi)$ in the type environment Γ when generalizing the type τ?

Consider the following example:

```
/* TIFAE */
x => {
  // tyenv = [x: T1]  and  solution = [T1 -> _]    (T1 is free in tyenv)
  val z = x;        // z: T1 (O)    not     z: [T1] { T1 } (X)
  z                 // z: T1 (O)    not     z: T2          (X)
}
```

If we generalize the type `T1` to `[T1] { T1 => T1 }` for `z`, the types of `x` and `z` will be different even though they have exactly the same value.

Recall: Example 3 – `id`



Type Environment

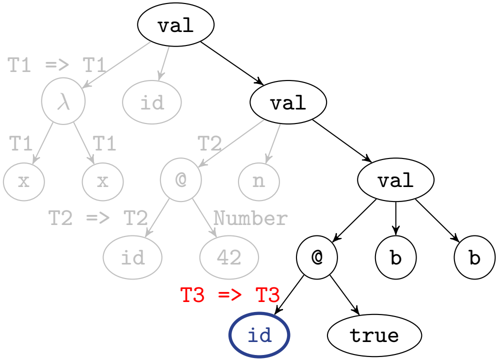
$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	-

Let's define a new **type variable** `T2` to **instantiate** the **type variable** `T1`. And, **substitute** `T1` with `T2`.

Recall: Example 3 – `id`



Type Environment

$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }
n	T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	Number
T3	-

Let's define a new **type variable `T3`** to **instantiate** the **type variable `T1`**. And, **substitute `T1`** with `T3`.

Identifier Lookup with Type Instantiation

```scala
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...

  case Id(x) =>
    val ty = tenv.getOrElse(x, error(s"free identifier: $x"))
    inst(ty, sol)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{–}\mathtt{Id}\ \frac{\Gamma(x) = \tau^{\forall} \qquad \mathtt{inst}(\tau^{\forall}, \psi) = (\tau, \psi')}{\Gamma, \psi \vdash x : \tau, \psi'}$$

We need to **instantiate** the type scheme $\tau^{\forall}$ with new type variables using the **type instantiation** function `inst`.

Identifier Lookup with Type Instantiation

```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...

  case Id(x) =>
    val ty = tenv.getOrElse(x, error(s"free identifier: $x"))
    inst(ty, sol)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{–}\mathtt{Id}\ \frac{\Gamma(x) = \tau^{\forall} \qquad \mathtt{inst}(\tau^{\forall}, \psi) = (\tau, \psi')}{\Gamma, \psi \vdash x : \tau, \psi'}$$

We need to **instantiate** the type scheme $\tau^{\forall}$ with new type variables using the **type instantiation** function `inst`. For example,

$$\mathtt{inst}(\forall\alpha.(\alpha \to \alpha), \varnothing) = (\beta \to \beta, [\beta \mapsto \bullet])$$

Type Instantiation

We can define the **type instantiation** function `inst` as follows:

$$\boxed{\texttt{inst} : (\mathbb{T}^{\forall} \times \Psi) \to (\mathbb{T} \times \Psi)}$$

$$\begin{array}{l}\texttt{inst}(\forall(\alpha_1, \ldots, \alpha_m).\tau, \psi) = (\\ \qquad \texttt{resolve}(\tau, \psi[\alpha_1 \mapsto \alpha'_1, \ldots, \alpha_m \mapsto \alpha'_m]), \\ \qquad \psi[\alpha'_1 \mapsto \bullet, \ldots, \alpha'_m \mapsto \bullet] \\ )\end{array}$$

where $\quad \alpha'_1, \ldots, \alpha'_m \notin \psi \wedge \forall 1 \leq i < j \leq m.\ \alpha'_i \neq \alpha'_j$

Summary

1. Type Checker and Typing Rules with Type Inference
   - Solutions for Type Constraints
   - Numbers
   - Additions
   - Conditionals
   - Immutable Variable Definitions and Identifier Lookup
   - Function Definitions
   - Recursive Function Definitions
   - Function Applications
2. Type Unification
   - Type Resolving
   - Occurrence Checking
   - Type Unification
3. Type Inference with Let-Polymorphism
   - Type Generalization
   - Type Instantiation

Exercise #16

https://github.com/ku-plrg-classroom/docs/tree/main/cose212/tifae

- Please see above document on GitHub:
  - Implement `typeCheck` function.
  - Implement `interp` function.

- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

Final Exam

- **Date:** 18:30 – 21:00 (150 min.), December 17 (Wed.).
- **Location:** B102, IT & General Education Center (정운오IT교양관)
- **Coverage:** Lectures 14 – 26
- **Format:** closed book and closed notes
    - Fill-in-the-blank questions about the PL concepts.
    - Write the evaluation results of given expressions.
    - Draw derivation trees of given expressions.
    - Define the syntax or semantics of extended language features.
    - Define typing rules for the given language features.
    - etc.
- Note that there is **no class** on **December 15 (Mon.)**.

Next Lecture

- Course Review

Jihyeok Park
jihyeok_park@korea.ac.kr
https://plrg.korea.ac.kr