Lecture 25 – Type Inference (1)

COSE212: Programming Languages

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PLRG

2025 Fall

Recall

- **Polymorphism** is to use a single entity as **multiple types**, and there are various kinds of polymorphism:
  - **Parametric polymorphism**
  - **Subtype polymorphism**
  - **Ad-hoc polymorphism**
  - ...
- **PTFAE** – TFAE with **parametric polymorphism**.
- **STFAE** – TFAE with **subtype polymorphism**.
- In this lecture, we will learn **type inference**.

Type Inference

Definition (Type Inference)

Type inference is the process of automatically inferring the types of expressions.

The goal of **type inference algorithm** is to infer the type of an expression without **explicit type annotations** given by programmers.

Let's consider the following RFAE expression:

```
/* RFAE */
def sum(x) = if (x < 1) 0 else x + sum(x - 1)
sum
```

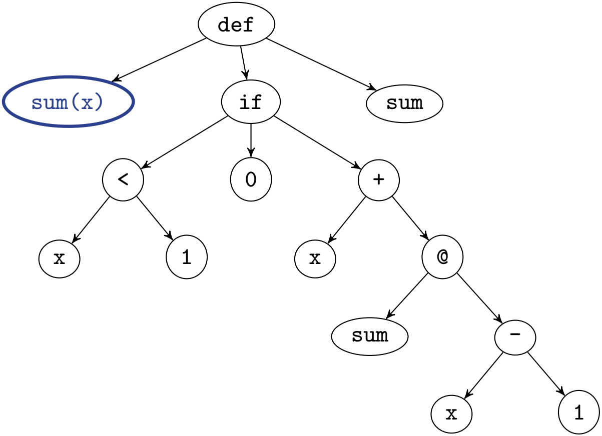
How can we **automatically infer** the type of `sum`?

1. Introduce **type variables** to denote unknown types
2. Collect the **type constraints** on the types
3. Find a **solution** (substitution of type variables) to the constraints

Contents

Contents

Example 1 – sum



Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	???
sum	???

Example 1 – sum

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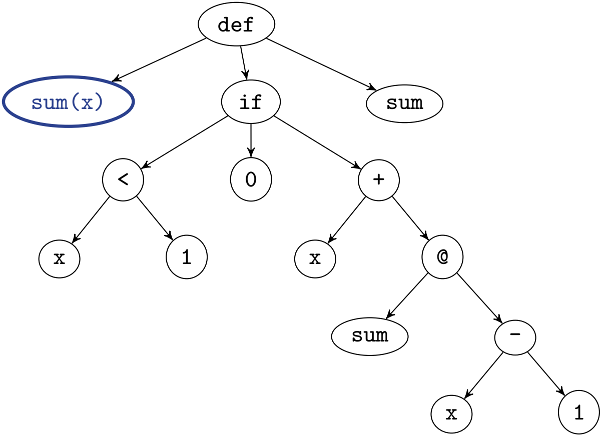


Type Environment

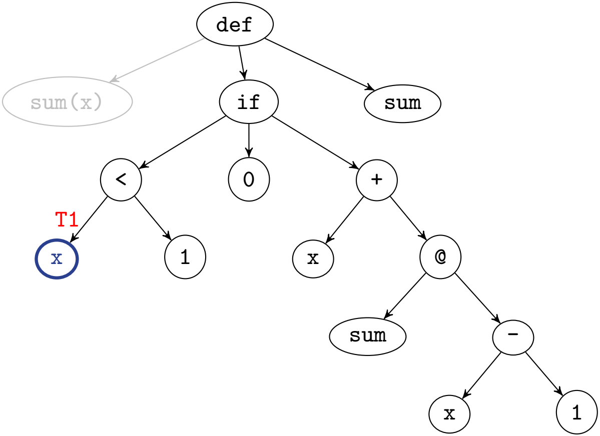
$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	-

Let's define **type variables** for unknown types.

Example 1 – sum



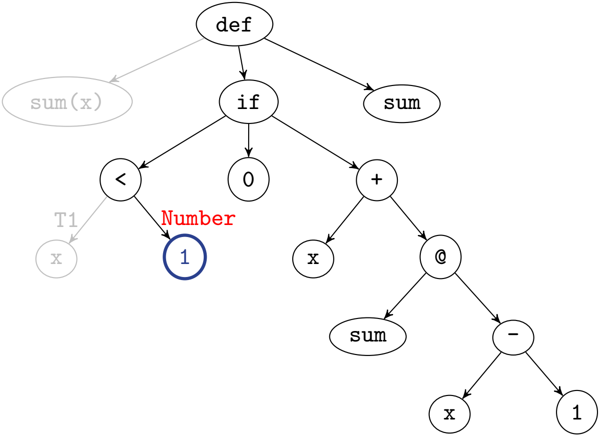
Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	-

Example 1 – sum



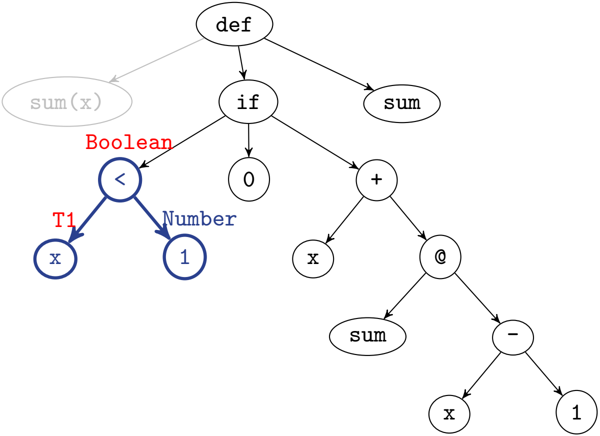
Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	-

Example 1 – sum



Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	-

The **operands** of < must be of type **Number**.
So, we collected a **type constraint**: T1 == Number.

Example 1 – sum

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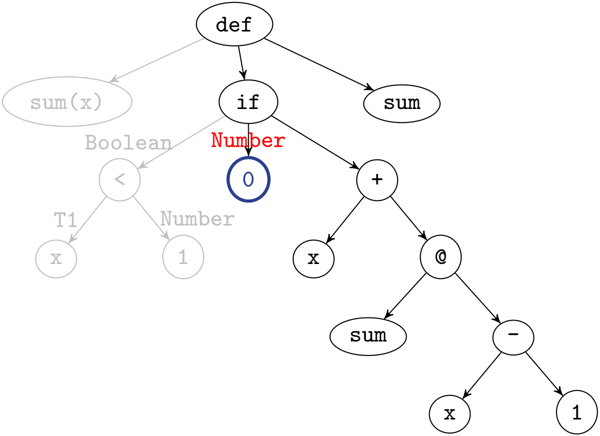


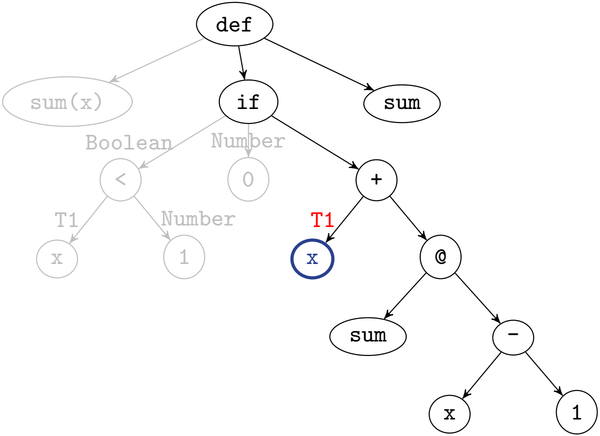
Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	-

Example 1 – sum



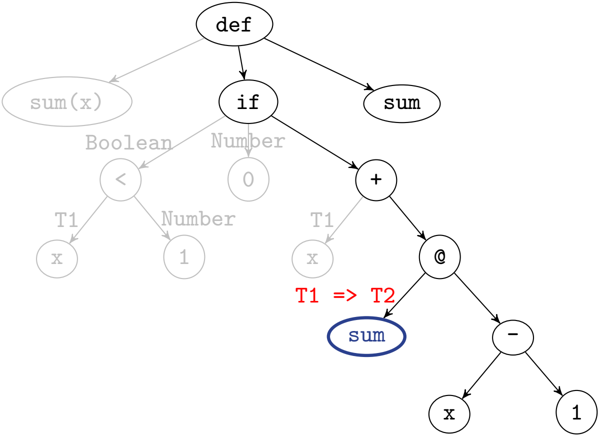
Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	-

Example 1 – sum



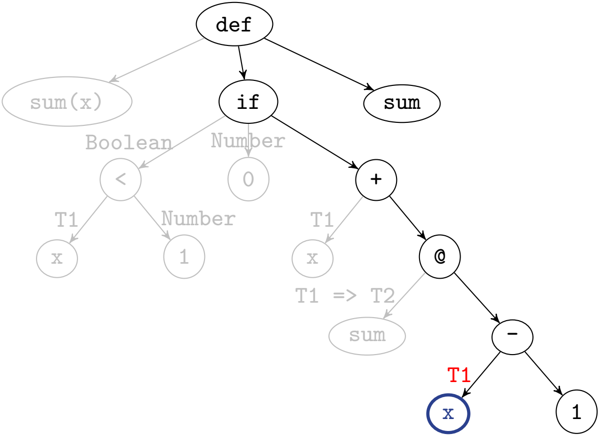
Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	-

Example 1 – sum



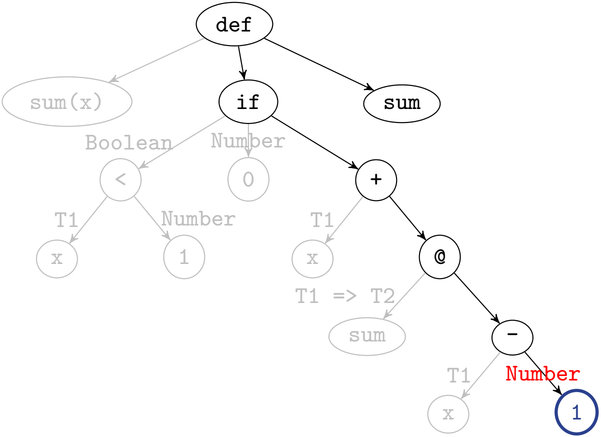
Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	-

Example 1 – sum



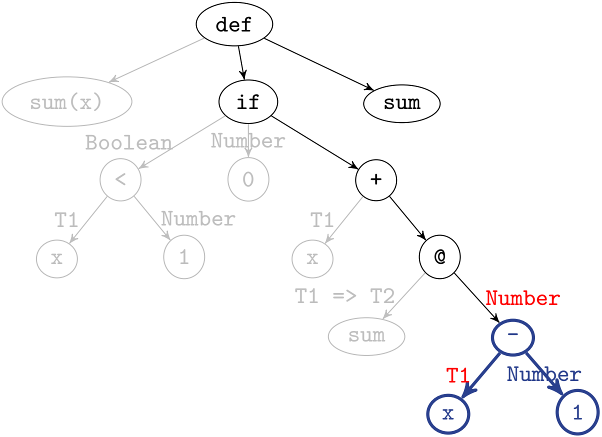
Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	-

Example 1 – sum



Type Environment

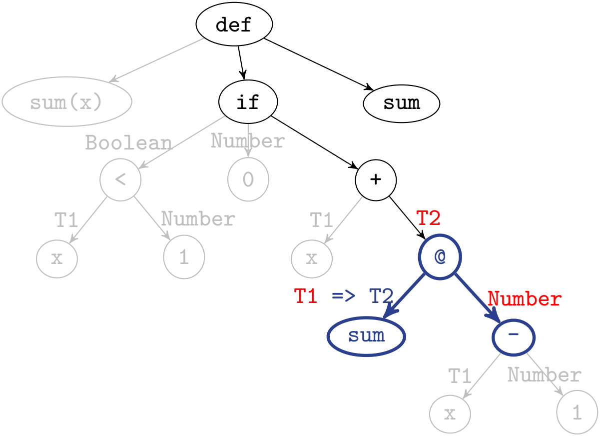
$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	-

The **operands** of - must be of type **Number**.
We collected a **type constraint**: `T1 == Number`.
But, it is not a new constraint.

Example 1 – sum



Type Environment

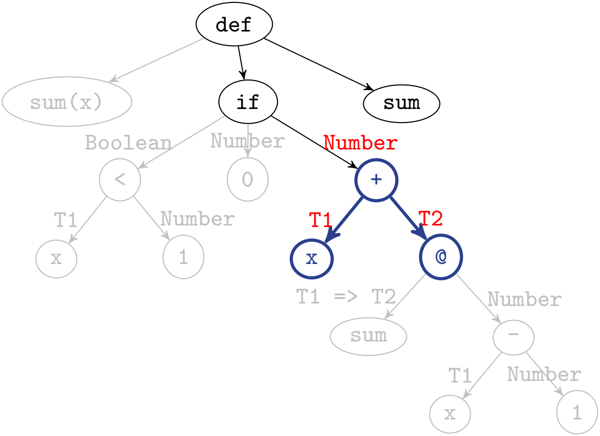
$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	-

The **argument type** should be equal to the **parameter type**.
We collected a **type constraint**: `T1 == Number`.
Again, it is not a new constraint.

Example 1 – sum



Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

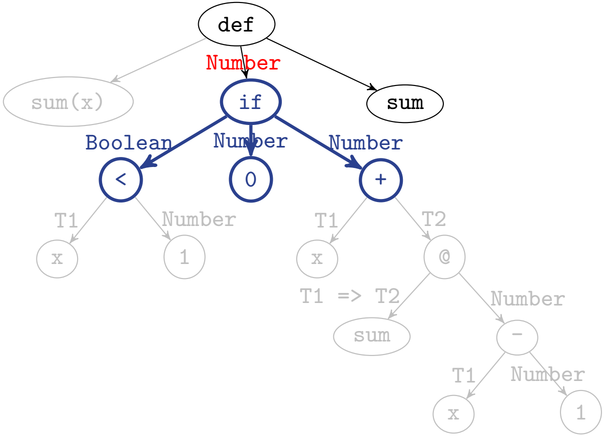
$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	Number

The **operands** of + must be of type **Number**.

We collected **type constraints**: `T1 == Number` and `T2 == Number`.

The second one is a new constraint!

Example 1 – sum



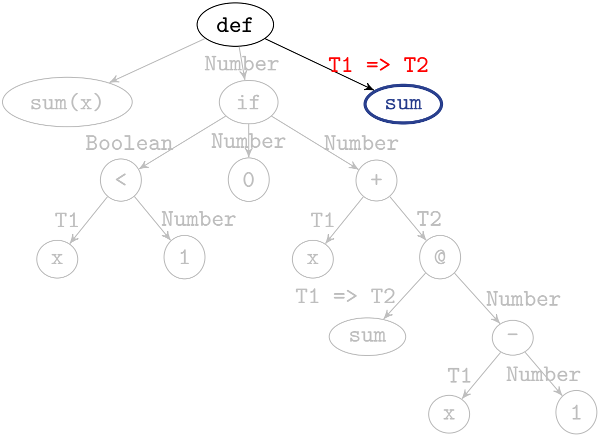
Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	Number

Example 1 – `sum`



Type Environment

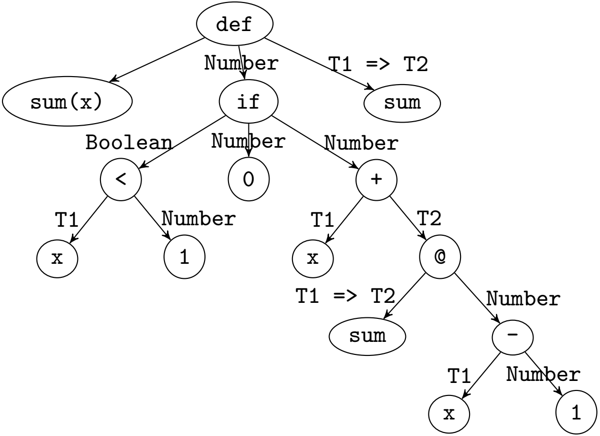
$\mathbb{X}$	$\mathbb{T}$
`sum`	`T1 => T2`

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
`T1`	`Number`
`T2`	`Number`

The type of `sum` is `T1 => T2`. Using the solution inferred by the collected constraints, we can instantiate it to `Number => Number`.

Example 1 – sum



Type Environment

$\mathbb{X}$	$\mathbb{T}$
sum	T1 => T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	Number

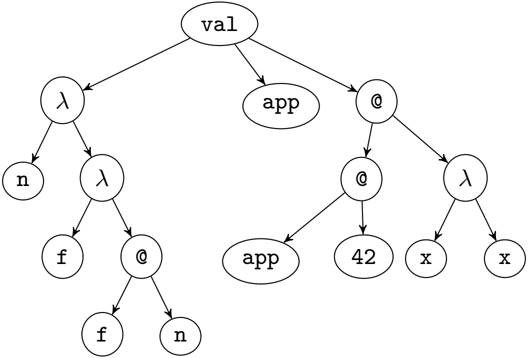
```
/* TRFAE */
def sum(x: Number): Number = if (x < 1) 0 else x + sum(x - 1)
sum
```

Contents

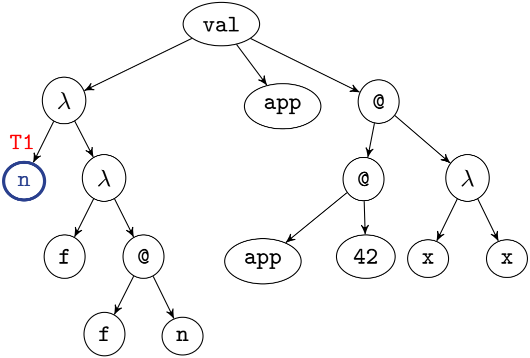
Example 2 – app

Let's infer the type of the following FAE expression:

```
/* FAE */
val app = n => f => f(n)
app(42)(x => x)
```



Example 2 – app



Type Environment

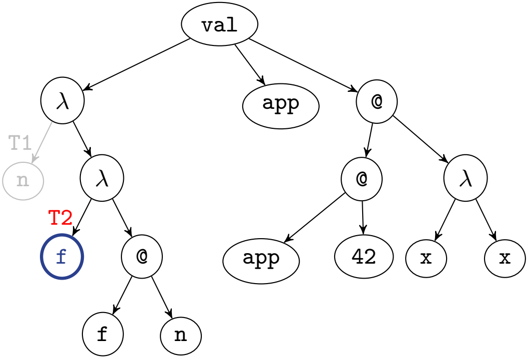
$\mathbb{X}$	$\mathbb{T}$
n	T1

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-

Let's define a new **type variable T1** for the parameter n.

Example 2 – app



Type Environment

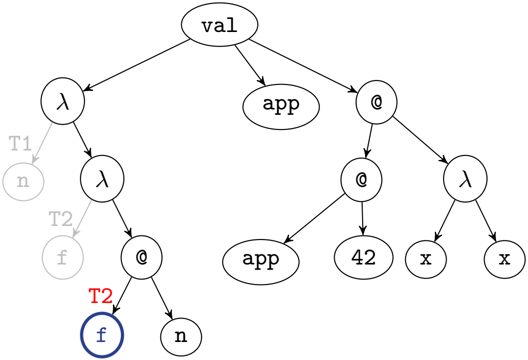
$\mathbb{X}$	$\mathbb{T}$
n	T1
f	T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	-

Let's define a new **type variable T2** for the parameter `f`.

Example 2 – app



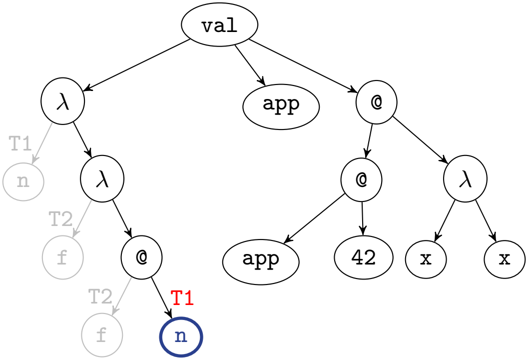
Type Environment

$\mathbb{X}$	$\mathbb{T}$
n	T1
f	T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	-

Example 2 – app



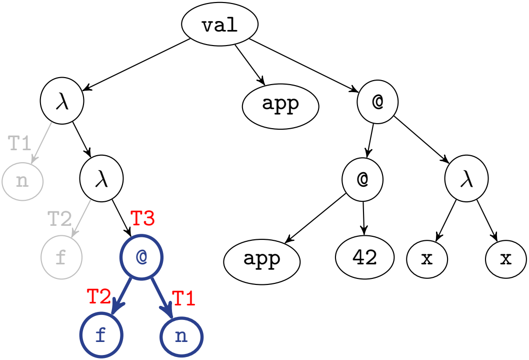
Type Environment

$\mathbb{X}$	$\mathbb{T}$
n	T1
f	T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	-

Example 2 – app



Type Environment

$\mathbb{X}$	$\mathbb{T}$
n	T1
f	T2

Solution

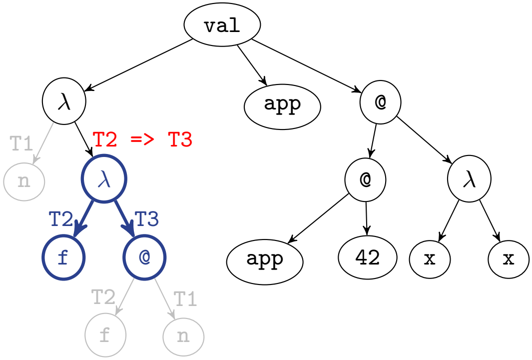
$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	T1 => T3
T3	-

The type T2 of f should be in the form of T1 => ???.

Let's define a new **type variable** T3 for ??? (the return type of f).

So, we collected a **type constraint**: T2 == T1 => T3.

Example 2 – app



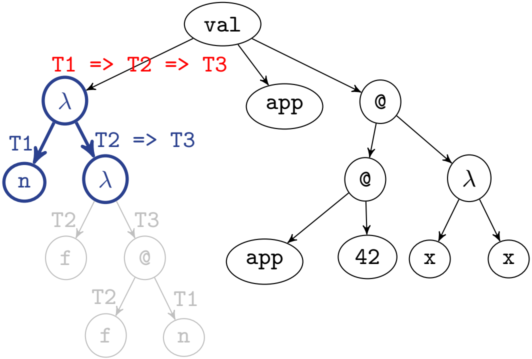
Type Environment

$\mathbb{X}$	$\mathbb{T}$
n	T1

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	T1 => T3
T3	-

Example 2 – app



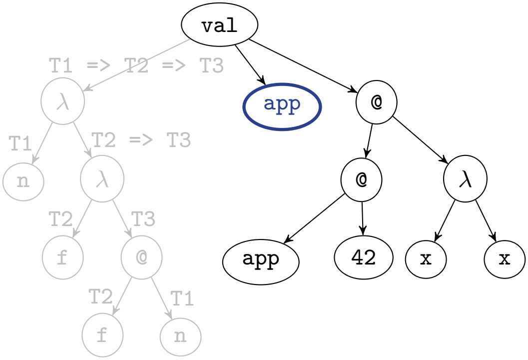
Type Environment

$\mathbb{X}$	$\mathbb{T}$

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	T1 => T3
T3	-

Example 2 – app



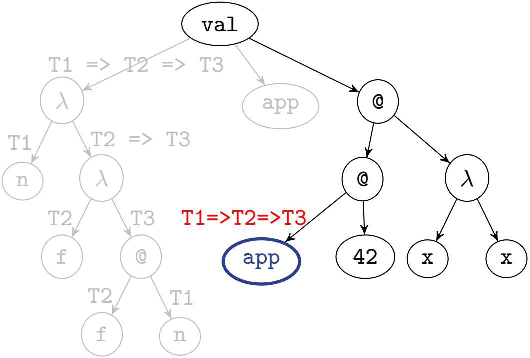
Type Environment

$\mathbb{X}$	$\mathbb{T}$
app	T1 => T2 => T3

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	T1 => T3
T3	-

Example 2 – app



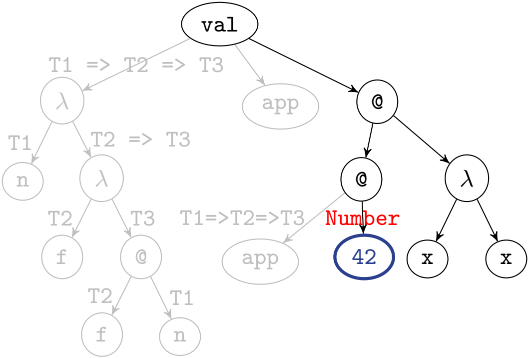
Type Environment

$\mathbb{X}$	$\mathbb{T}$
app	T1 => T2 => T3

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	–
T2	T1 => T3
T3	–

Example 2 – app



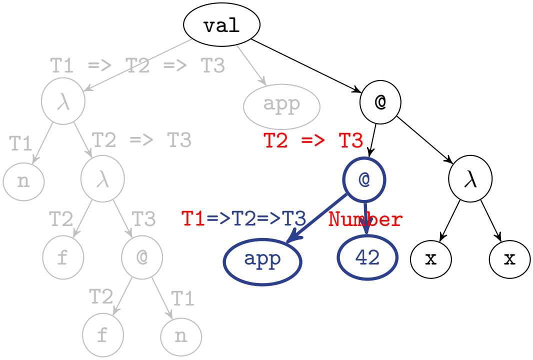
Type Environment

$\mathbb{X}$	$\mathbb{T}$
app	T1 => T2 => T3

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	T1 => T3
T3	-

Example 2 – app



Type Environment

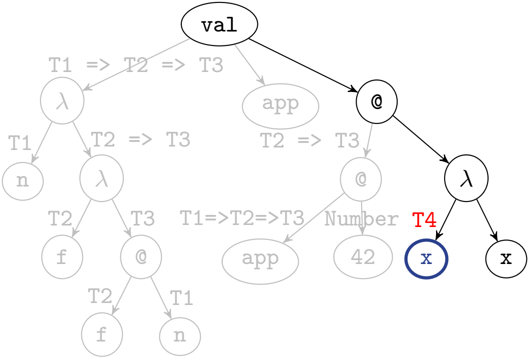
$\mathbb{X}$	$\mathbb{T}$
app	T1 => T2 => T3

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	T1 => T3
T3	-

The **parameter type** T1 should be equal to the **argument type** Number.
So, we collected a **type constraint**: T1 == Number.

Example 2 – app



Type Environment

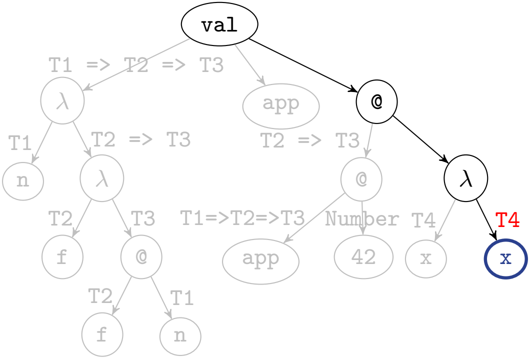
$\mathbb{X}$	$\mathbb{T}$
app	T1 => T2 => T3
x	T4

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	T1 => T3
T3	-
T4	-

Let's define a new **type variable T4** for the parameter x.

Example 2 – app



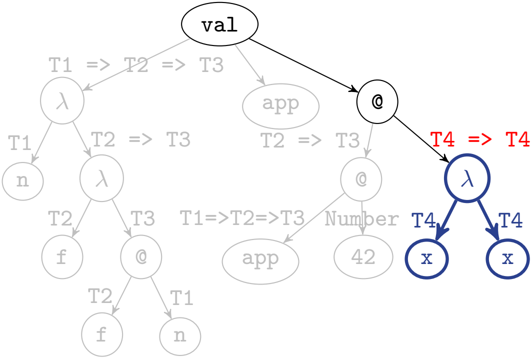
Type Environment

$\mathbb{X}$	$\mathbb{T}$
app	T1 => T2 => T3
x	T4

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	T1 => T3
T3	-
T4	-

Example 2 – app



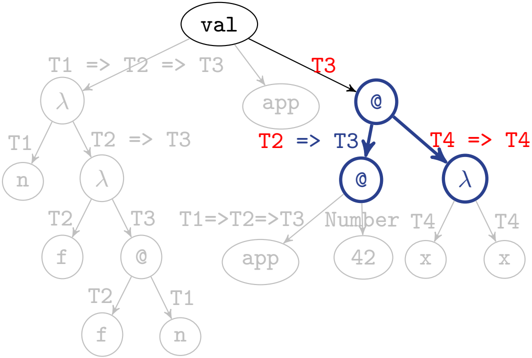
Type Environment

$\mathbb{X}$	$\mathbb{T}$
app	T1 => T2 => T3
x	T4

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	T1 => T3
T3	-
T4	-

Example 2 – app



Type Environment

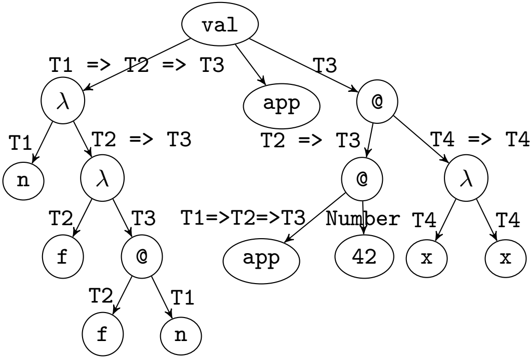
$\mathbb{X}$	$\mathbb{T}$
app	T1 => T2 => T3
x	T4

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	T1 => T3
T3	Number
T4	Number

The **parameter type** T2 should be equal to **argument type** T4 => T4.
We collected **type constraints**: T3 == Number and T4 == Number.
Finally, the entire expression has type T3 (= Number).

Example 2 – app



Type Environment

$\mathbb{X}$	$\mathbb{T}$
app	T1 => T2 => T3
x	T4

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	Number
T2	T1 => T3
T3	Number
T4	Number

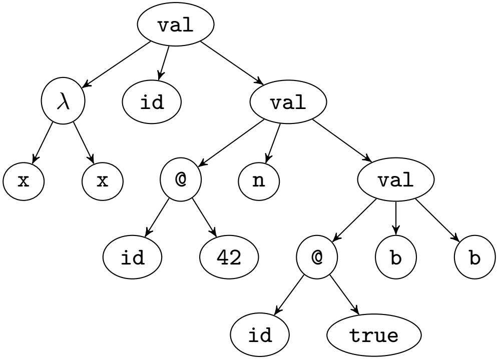
```
/* TFAE */
val app = (n: Number) => (f: Number => Number) => f(n)
app(42)((x: Number) => x)
```

Contents

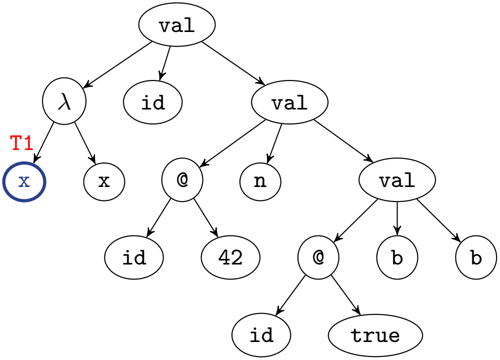
Example 3 – id

Let's infer the type of the following FAE expression:

```
/* FAE */
val id = x => x
val n = id(42)
val b = id(true)
b
```



Example 3 – `id`



Type Environment

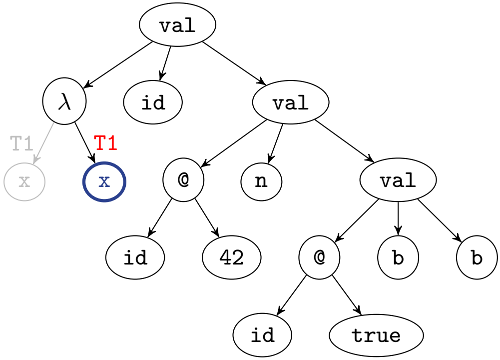
$\mathbb{X}$	$\mathbb{T}$
x	T1

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-

Let's define a new **type variable `T1`** for the parameter `x`.

Example 3 – id



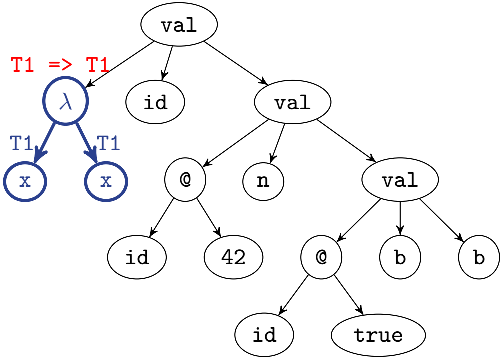
Type Environment

$\mathbb{X}$	$\mathbb{T}$
x	T1

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-

Example 3 – id



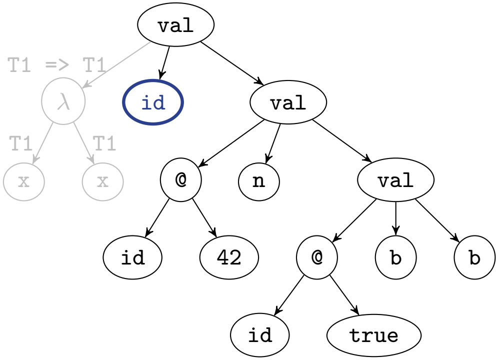
Type Environment

$\mathbb{X}$	$\mathbb{T}$

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-

Example 3 – id



Type Environment

$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }

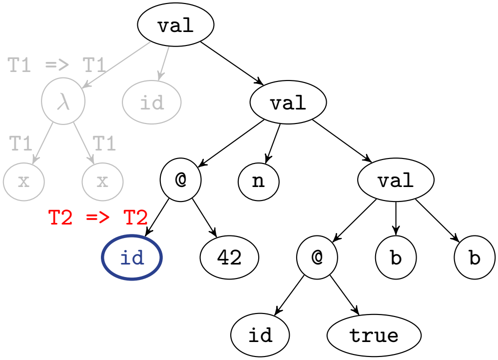
Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$

Let's **generalize** the type `T1 => T1` into a **polymorphic type** for `id` with **type variable** `T1` as a **type parameter**.

We call this **let-polymorphism** because it only introduces polymorphism for the let-binding (e.g., `val`).

Example 3 – id



Type Environment

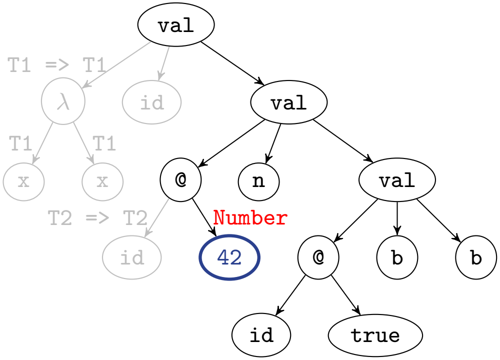
$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	-

Let's define a new **type variable** T2 to **instantiate** the **type variable** T1. And, **substitute** T1 with T2.

Example 3 – id



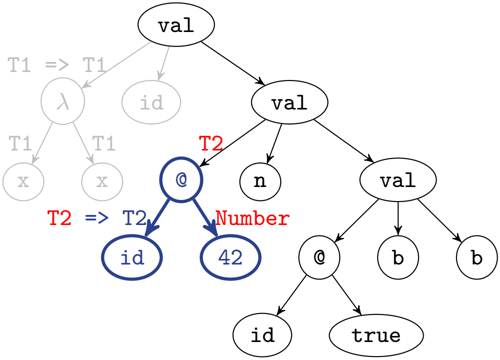
Type Environment

$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	-

Example 3 – id



Type Environment

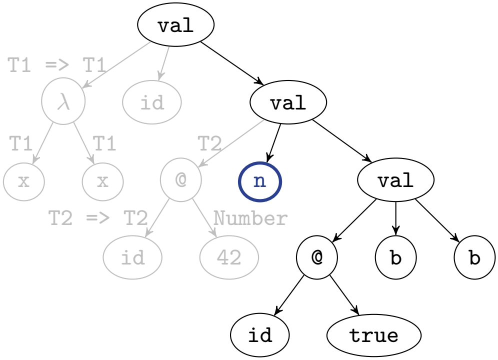
$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	Number

The **parameter type** T2 should be equal to **argument type** Number.
We collected a **type constraint**: T2 == Number.

Example 3 – `id`



Type Environment

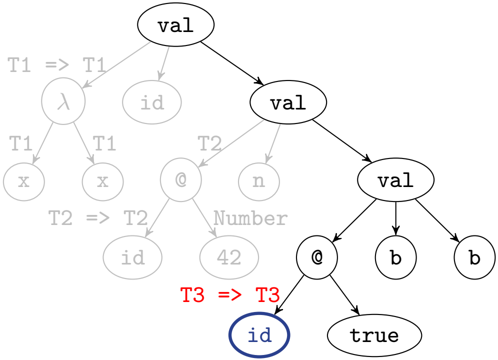
$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }
n	T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	Number

T2 is not a free type variable because it actually represents Number.
So, we will not introduce a polymorphic type in this case.

Example 3 – id



Type Environment

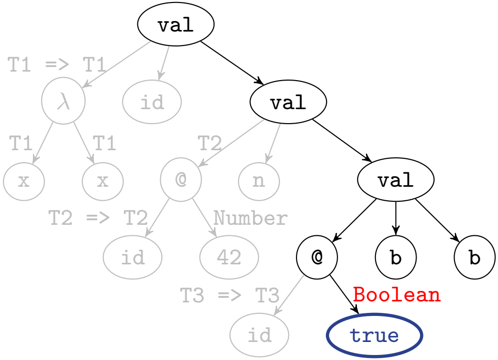
$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }
n	T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	Number
T3	-

Let's define a new **type variable T3** to **instantiate** the **type variable T1**. And, **substitute T1** with T3.

Example 3 – id



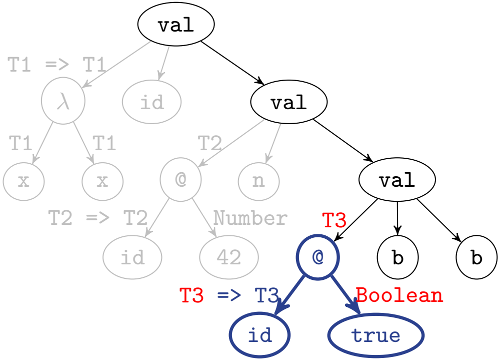
Type Environment

$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }
n	T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	Number
T3	-

Example 3 – id



Type Environment

$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }
n	T2

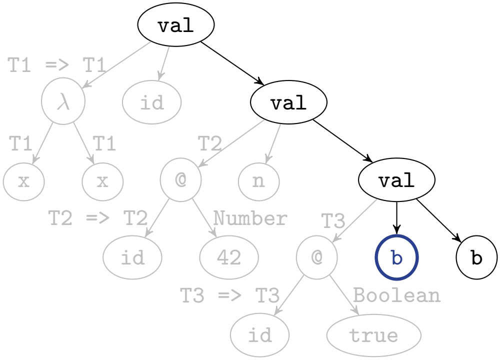
Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	Number
T3	Boolean

The **parameter type** T3 should be equal to **argument type** Boolean.
We collected a **type constraint**: T3 == Boolean.

Example 3 – id

PLRG



Type Environment

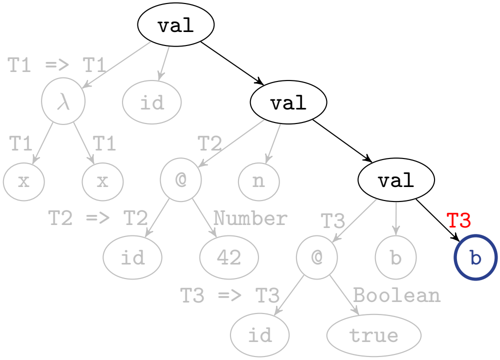
$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }
n	T2
b	T3

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	Number
T3	Boolean

T3 is not a free type variable because it actually represents Boolean.
So, we will not introduce a polymorphic type in this case.

Example 3 – id



Type Environment

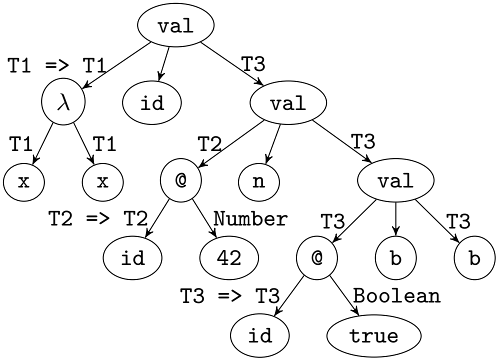
$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }
n	T2
b	T3

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	Number
T3	Boolean

Finally, the entire expression has type T3 (= Boolean).

Example 3 – id



Type Environment

$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }
n	T2
b	T3

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T2	Number
T3	Boolean

```
/* PTFAE */
val id = forall[T] { (x: T) => x }
val n = id[Number](42)
val b = id[Boolean](true)
b
```

Summary

1. Example 1 – `sum`
2. Example 2 – `app`
3. Example 3 – `id`

Next Lecture

- Type Inference (2)

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