

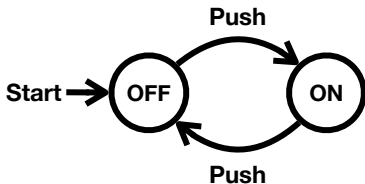
Lecture 1 – Mathematical Preliminaries

COSE215: Theory of Computation

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Theorem

*The current state is OFF if and only if the button is pushed **even** times.*

- Is it possible to prove it?

Let's learn **mathematical background and notation**.

1. Mathematical Notations

Notations in Logics

Notations in Set Theory

2. Inductive Proofs

Inductions on Integers

Structural Inductions

Mutual Inductions

3. Notations in Languages

Symbols & Words

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Notation	Description
A, B	arbitrary statements .
$P(x)$	a predicate that involves a variable x .
$A \wedge B$	the conjunction of A and B . (i.e., A and B).
$A \vee B$	the disjunction of A and B . (i.e., A or B).
$\neg A$	the negation of A .

Notation	Description
$A \Rightarrow B$	the implication of A and B (i.e., if A then B).
$A \Leftrightarrow B$	A if and only if (iff) B (i.e., $A \Rightarrow B \wedge B \Rightarrow A$).
$\forall x \in X. P(x)$	the universal quantifier (i.e., for all x in X , $P(x)$ holds).
$\exists x \in X. P(x)$	the existential quantifier (i.e., there exists x in X such that $P(x)$ holds).

- A **set** is a collection of elements, e.g.,
 - $\mathbb{N} = \{0, 1, 2, \dots\}$
 - $\{x \in \mathbb{N} \mid x \text{ is even}\} = \{0, 2, 4, 6, 8, 10, 12, \dots\}$
 - $\{x \in \mathbb{N} \mid x^2\} = \{0, 1, 4, 9, 16, 25, 36, \dots\}$
- The **empty set** is denoted by \emptyset .
- The **cardinality** of a set X is denoted by $|X|$.
- A **subset** X of a set Y is denoted by $X \subseteq Y$.

$$X \subseteq Y \iff \forall x \in X. x \in Y$$

- A **proper subset** X of a set Y is denoted by $X \subset Y$.

$$X \subset Y \iff X \subseteq Y \wedge X \neq Y$$

- The **union** of sets

$$\begin{aligned} X \cup Y &= \{x \mid x \in X \vee x \in Y\} \\ \bigcup \mathcal{C} &= X_1 \cup X_2 \cup \cdots \cup X_n = \{x \mid \exists X \in \mathcal{C}. x \in X\} \end{aligned}$$

where $\mathcal{C} = \{X_1, X_2, \dots, X_n\}$.

- The **intersection** of sets

$$\begin{aligned} X \cap Y &= \{x \mid x \in X \wedge x \in Y\} \\ \bigcap \mathcal{C} &= X_1 \cap X_2 \cap \cdots \cap X_n = \{x \mid \forall X \in \mathcal{C}. x \in X\} \end{aligned}$$

where $\mathcal{C} = \{X_1, X_2, \dots, X_n\}$.

- The **difference** of sets

$$X \setminus Y = \{x \mid x \in X \wedge x \notin Y\}$$

- The **complement** of a set X is denoted by \overline{X} .

$$\overline{X} = \{x \mid x \in U \wedge x \notin X\}$$

where U is the **universal set**.

- The **power set** of a set X is denoted by 2^X or $\mathcal{P}(X)$.

$$2^X = \mathcal{P}(X) = \{Y \mid Y \subseteq X\}$$

- The **Cartesian product** of sets X and Y is denoted by $X \times Y$.

$$X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$$

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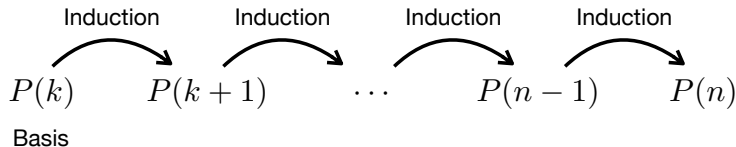
Definition (Inductions on Integers)

Let $P(n)$ be a predicate on integers, and if

- **(Basis Case)** $P(k)$ holds where k is an integer, and
- **(Induction Case)** for all $n \geq k$, $P(n) \Rightarrow P(n + 1)$,

then $P(i)$ holds for all $i \geq k$.

$P(n)$ is called **induction hypothesis**.



Example

Prove that $\forall n \geq 0. \sum_{i=0}^n i = \frac{n(n+1)}{2}$.

Proof)

- (Basis Case): $0 = 0(0 + 1)/2$ \square
- (Induction Case): Assume that it holds for n (I.H.). Then,

$$\begin{aligned}\sum_{i=0}^{n+1} i &= (n+1) + \sum_{i=0}^n i \\ &= (n+1) + \frac{n(n+1)}{2} \quad (\because \text{I.H.}) \\ &= \frac{(n+1)(n+2)}{2} \quad \square\end{aligned}$$

Example

Prove that $\forall n \geq 0. \sum_{i=0}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof)

- (Basis Case): $0^2 = 0(0+1)(2 \cdot 0 + 1)/6$ \square
- (Induction Case): Assume that it holds for n (I.H.). Then,

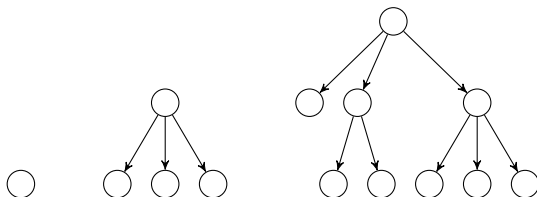
$$\begin{aligned}\sum_{i=0}^{n+1} i^2 &= (n+1)^2 + \sum_{i=0}^n i^2 \\ &= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \quad (\because \text{I.H.}) \\ &= \frac{(n+1)(n+2)(2(n+1)+1)}{6} \quad \square\end{aligned}$$

In CS, we often define somethings as **inductively-defined sets**.
For example, we can define **trees** as follows:

Example (Inductive Definition of Trees)

A **tree** is defined as follows:

- **(Basis Case)** A single **node** N is a tree.
- **(Induction Case)** If T_1, \dots, T_n are trees, then a graph defined with a new node N and edges from N to T_1, \dots, T_n is a tree as well.



Another example is a set of **arithmetic expressions**:

Example (Inductive Definition of Arithmetic Expressions)

An **arithmetic expression** is defined as follows:

- **(Basis Case)** A **number** or a **variable** is an arithmetic expression.
- **(Induction Case)** If E and F are arithmetic expressions, then so are $E+F$, $E * F$, and (E) .

42

x

$x + y$

$42 * x$

(x)

$(x * y) * z$

$(2 + x) * y$

$x * (x * y)$

$((((x))))$

Definition (Structural Inductions)

Let $P(x)$ be a predicate on a **inductively-defined set** X , and if

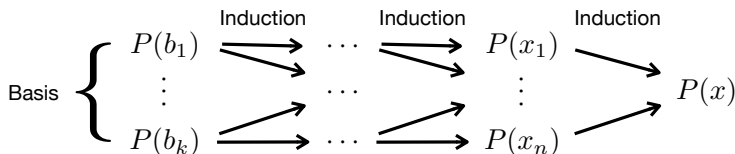
- **(Basis Case)** $P(b_1), \dots, P(b_k)$ hold for all basis cases b_1, \dots, b_k .
- **(Induction Case)** for all $x \in X$,

$$P(x_1) \wedge \dots \wedge P(x_n) \Rightarrow P(x)$$

where x_1, \dots, x_n are the **sub-structures** of x .

then $P(x)$ holds for all $x \in X$.

$P(x_1), \dots, P(x_n)$ are called **induction hypotheses**.



Example

Prove that for all tree T , the number of nodes in T is equal to the number of edges in T plus one.

Proof) Let $N(T)$ be the number of node and $E(T)$ be the number of edges in T . Let's prove $\forall T. N(T) = E(T) + 1$.

- (Basis Case): $N(T) = 1$ and $E(T) = 0$. \square
- (Induction Case): Assume that it holds for T_1, \dots, T_n (I.H.). Then,

$$\begin{aligned} N(T) &= 1 + \sum_{i=1}^n N(T_i) \\ &= 1 + \sum_{i=1}^n (E(T_i) + 1) \quad (\because \text{I.H.}) \\ &= 1 + n + \sum_{i=1}^n E(T_i) \\ &= 1 + E(T) \quad \square \end{aligned}$$

Example

Prove that for all arithmetic expression E , the number of left parentheses in E is equal to the number of right parentheses in E .

Proof) Let $L(E)$ be the number of left parentheses and $R(E)$ be the number of right parentheses in E . Let's prove $\forall E. L(E) = R(E)$.

- (Basis Case): $L(E) = R(E) = 0$ for numbers and variables. \square
- (Induction Case): Assume that it holds for E and F (I.H.). Then,

$$\begin{aligned} L(E+F) &= L(E) + L(F) = R(E) + R(F) && (\because \text{I.H.}) \\ &= R(E+F) && \square \end{aligned}$$

$$\begin{aligned} L(E * F) &= L(E) + L(F) = R(E) + R(F) && (\because \text{I.H.}) \\ &= R(E * F) && \square \end{aligned}$$

$$\begin{aligned} L((E)) &= L(E) + 1 = R(E) + 1 && (\because \text{I.H.}) \\ &= R((E)) && \square \end{aligned}$$

Definition (Mutual Inductions)

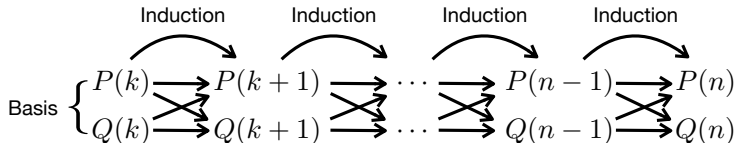
Let $P(x)$ and $Q(x)$ are predicates on integers, and if

- **(Basis Case)** $P(k)$ and $Q(k)$ hold where k is an integer, and
- **(Induction Case)** for all $n \geq k$,

$$P(n) \wedge Q(n) \Rightarrow P(n+1) \wedge Q(n+1)$$

then $P(i)$ and $Q(i)$ hold for all $i \geq k$.

$P(n)$ and $Q(n)$ are called **induction hypotheses**.



Theorem

*The current state is OFF if and only if the button is pushed **even** times.*

Proof)

Theorem

*The current state is OFF if and only if the button is pushed **even** times, and the current state is ON if and only if the button is pushed **odd** times.*

Proof) Let $S(i)$ be the current state after i times of pushing. Let's prove

$$\forall i \geq 0. S(i) = \text{OFF} \iff i \equiv 0 \pmod{2} \quad (P)$$

$$\forall i \geq 0. S(i) = \text{ON} \iff i \equiv 1 \pmod{2} \quad (Q)$$

- (Basis Case): Known facts: $S(0) = \text{OFF}$ and $0 \equiv 0 \pmod{2}$
 - (P, \Rightarrow) : $0 \equiv 0 \pmod{2} \implies S(0) = \text{OFF} \Rightarrow 0 \equiv 0 \pmod{2}$
 - (P, \Leftarrow) : $S(0) = \text{OFF} \implies S(0) = \text{OFF} \Leftarrow 0 \equiv 0 \pmod{2}$
 - (Q, \Rightarrow) : $\neg(S(0) = \text{ON}) \implies S(0) = \text{ON} \Rightarrow 0 \equiv 1 \pmod{2}$
 - (Q, \Leftarrow) : $\neg(0 \equiv 1 \pmod{2}) \implies S(0) = \text{ON} \Leftarrow 0 \equiv 1 \pmod{2}$

- (Induction Case): Assume that it holds for n (I.H.):

$$S(n) = \text{OFF} \iff n \equiv 0 \pmod{2} \quad (P - \text{I.H.})$$

$$S(n) = \text{ON} \iff n \equiv 1 \pmod{2} \quad (Q - \text{I.H.})$$

- (P, \iff) :

$$\begin{aligned} S(n+1) = \text{OFF} &\iff S(n) = \text{ON} \\ &\iff n \equiv 1 \pmod{2} \quad (\because Q - \text{I.H.}) \\ &\iff n+1 \equiv 0 \pmod{2} \end{aligned}$$

- (Q, \iff) :

$$\begin{aligned} S(n+1) = \text{ON} &\iff S(n) = \text{OFF} \\ &\iff n \equiv 0 \pmod{2} \quad (\because P - \text{I.H.}) \\ &\iff n+1 \equiv 1 \pmod{2} \end{aligned}$$

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- We first define a finite and non-empty set of **symbols** Σ .
- A **word** $w \in \Sigma^*$ is a sequence of symbols.
 - $\Sigma = \{0, 1\}$ – binary symbols.

$\epsilon, 0, 1, 00, 01, 10010, \dots \in \Sigma^*$

- $\Sigma = \{a, b, \dots, z\}$ – lowercase letters.

$\epsilon, a, b, abc, \text{hello}, \text{cs}, \text{students}, \dots \in \Sigma^*$

- $\Sigma = \{a \mid a \text{ is an Unicode character}\}$ – Unicode characters.

$\epsilon, \text{안녕하세요}, \text{こんにちは}, \star \blacksquare \blacktriangle \oplus, \dots \in \Sigma^*$

Notation	Description
ϵ	the empty word .
$w_1 w_2$	the concatenation of w_1 and w_2 . (w_1 is a prefix of $w_1 w_2$ and w_2 is a suffix of $w_1 w_2$)
w^R	the reverse of w .
$ w $	the length of w .
Σ^k	the set of all words of length k .
Σ^*	the set of all words (the Kleene star). (i.e., $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \dots = \bigcup_{k=0} \Sigma^k$)
Σ^+	the set of all words except ϵ (the Kleene plus). (i.e., $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \dots = \bigcup_{k=1} \Sigma^k$)

A **language** $L \subseteq \Sigma^*$ is a specific set of words.

When $\Sigma = \{0, 1\}$, we can define the following languages:

- $L = \{\epsilon, 0, 1\}$ – the empty word, zero, and one.
- $L = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \dots\}$ – all binary words.
- $L = \{0^n 1^n \mid n \geq 0\}$ – equal number of consecutive zeros and ones.
- $L = \{10, 11, 101, 111, 1011, \dots\}$ – ???

- The **union**, **intersection**, and **difference** of languages:

$$L_1 \cup L_2 \quad L_1 \cap L_2 \quad L_1 \setminus L_2$$

- The **reverse** of a language:

$$L^R = \{w^R \mid w \in L\}$$

- The **complement** of a language:

$$\bar{L} = \Sigma^* \setminus L$$

- The **concatenation** of languages:

$$L_1 L_2 = \{w_1 w_2 \mid w_1 \in L_1 \wedge w_2 \in L_2\}$$

- The **power** of a language:

$$\begin{aligned} L^0 &= \{\epsilon\} \\ L^n &= L^{n-1}L \quad (n \geq 1) \end{aligned}$$

- The **Kleene star** of a language:

$$L^* = L^0 \cup L^1 \cup L^2 \cup \dots = \bigcup_{n \geq 0} L^n$$

- The **Kleene plus** of a language:

$$L^+ = L^1 \cup L^2 \cup L^3 \cup \dots = \bigcup_{n \geq 1} L^n$$

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- Basic Introduction of Scala

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