Lecture 17 – Deterministic Pushdown Automata (DPDA)

COSE215: Theory of Computation

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PLRG

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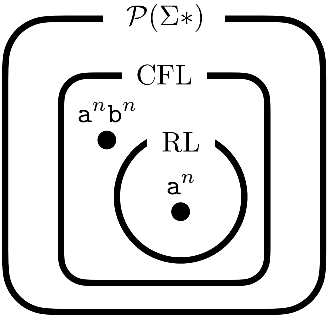
Recall

- A **pushdown automaton (PDA)** is an extension of ϵ-NFA with a **stack**. Thus, PDA is **non-deterministic**.
  - Acceptance by **final states**
  - Acceptance by **empty stacks**
- Then, how about **deterministic PDA (DPDA)**?
- What is the **language class** of DPDA? Still, CFL?



PDA_{FS}
(by final states)

$\|$

PDA_{ES}
(by empty stacks)

$\|$

CFG

Contents

1. Deterministic Pushdown Automata (DPDA)

2. Deterministic Context-Free Languages (DCFLs)
   - Fact 1: DCFL $\subsetneq$ CFL
   - Fact 2: RL $\subsetneq$ DCFL

3. Languages Accepted by Empty Stacks of DPDA ($\mathsf{DCFL}_{\mathsf{ES}}$)
   - Fact 3: $\mathsf{DCFL}_{\mathsf{ES}} \subsetneq$ DCFL
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Definition of Deterministic Pushdown Automata

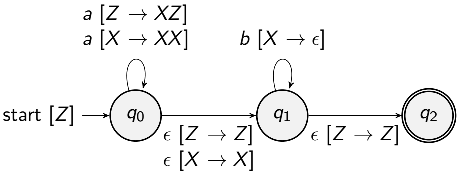
Definition (Deterministic Pushdown Automata (DPDA))

A **deterministic pushdown automaton (DPDA)** is a pushdown automaton having **at most one** one-step move ($\vdash$) from any configuration.

We can check it with the following conditions:

1. $|\delta(q, a, X)| \leq 1$ for all $q \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$.
2. If $\delta(q, \epsilon, X) \neq \varnothing$, then $\delta(q, a, X) = \varnothing$ for all $a \in \Sigma$.

For example, is the following PDA deterministic?



No, because it has multiple transitions for $(q_0, \mathtt{ab}, Z)$.

Definition of Deterministic Pushdown Automata

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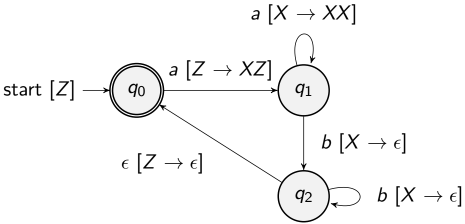
2. If $\delta(q, \epsilon, X) \neq \varnothing$, then $\delta(q, a, X) = \varnothing$ for all $a \in \Sigma$.

However, the following PDA is **deterministic**:



$$
\begin{aligned}
(q_0, \texttt{aabb}, Z) &\vdash (q_1, \texttt{abb}, XZ) \\
&\vdash (q_1, \texttt{bb}, XXZ) \\
&\vdash (q_2, \texttt{b}, XZ) \\
&\vdash (q_2, \epsilon, Z) \\
&\vdash (q_0, \epsilon, \epsilon)
\end{aligned}
$$

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Deterministic Context-Free Languages (DCFLs)

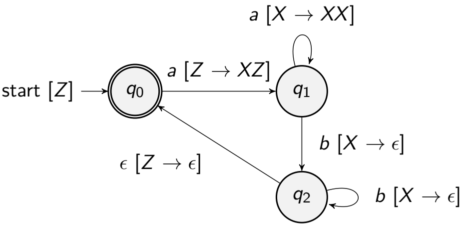
Definition (Deterministic Context-Free Languages (DCFLs))

A language L is a **deterministic context-free language (DCFL)** if and only if there exists a DPDA P such that $L = L_F(P)$ where $L_F(P)$ is the language accepted by **final states** of P.

For example, the following language is a DCFL:

$$L = \{a^n b^n \mid n \geq 0\}$$

because it is accepted by **final states** of the following **DPDA**:



Fact 1: DCFL $\subsetneq$ CFL

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① All DCFLs are CFLs **BUT** ② there exists a CFL that is not a DCFL.

1. $\boxed{\text{DCFL} \subseteq \text{CFL}}$: By definition of DCFL, we can easily prove it.
2. $\boxed{\text{CFL} \setminus \text{DCFL} \neq \varnothing}$: What is an example of a CFL that is not a DCFL?

The following language is a CFL but not a DCFL:

$$L = \{ww^R \mid w \in \{a, b\}^*\} \in \text{CFL} \setminus \text{DCFL}$$

The formal proof is complex, but we can intuitively understand it with the following two example words in L:

- $ww^R = \texttt{abba} \in L$ where $w = \texttt{ab}$
- $ww^R = \texttt{abbbba} \in L$ where $w = \texttt{abb}$

When we read `b` after `ab`, we need to consider two possible actions: ① pop Y for `b` (for `abba`) or ② push Y for `b` (for `abbbba`).

Fact 2: RL $\subsetneq$ DCFL

① All RLs are DCFLs **BUT** ② there exists a DCFL that is not an RL.

1. $\boxed{\text{RL} \subseteq \text{DCFL}}$: For a given RL L, consider its corresponding DFA D:

$$D = (Q, \Sigma, \delta, q_0, F)$$

Then, we can construct a DPDA P that accepts L as follows:

$$P = (Q, \Sigma, \{Z\}, \delta_P, q_0, Z, F)$$

where $\forall q \in Q.\ \forall a \in \Sigma.\ \delta_P(q, a, Z) = \{(\delta(q, a), Z)\}$ because

$$(q_0, w, Z) \vdash^* (q, \epsilon, Z) \iff \delta^*(q_0, w) = q \qquad \square$$

2. $\boxed{\text{DCFL} \setminus \text{RL} \neq \varnothing}$: We already know that the following language is a DCFL but not an RL:

$$L = \{a^n b^n \mid n \geq 0\} \in \text{DCFL} \setminus \text{RL}$$

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Languages Accepted by Empty Stacks of DPDA ($\mathrm{DCFL_{ES}}$)

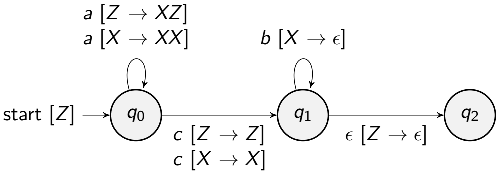
Definition ($\mathrm{DCFL_{ES}}$)

A language L is a **deterministic context-free language by empty stacks ($\mathbf{DCFL_{ES}}$)** if and only if there exists a DPDA P such that $L = L_E(P)$ where $L_E(P)$ is the language accepted by **empty stacks** of P.

For example, the following language is a $\mathrm{DCFL_{ES}}$:

$$L = \{a^n cb^n \mid n \geq 0\}$$

because it is accepted by empty stacks of the following DPDA:



Fact 3: $\text{DCFL}_{\text{ES}} \subsetneq \text{DCFL}$

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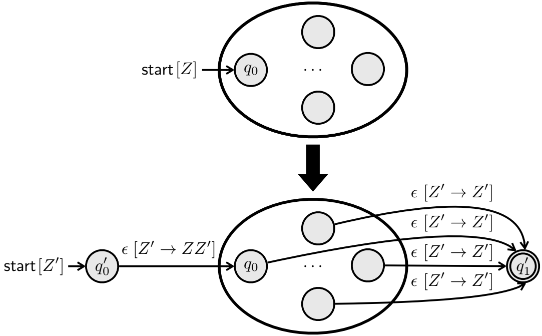
① All DCFL_{ES}s are DCFLs **BUT** ② there is a DCFL but not a DCFL_{ES}.

1. $\boxed{\text{DCFL}_{\text{ES}} \subseteq \text{DCFL}}$: For a given DCFL_{ES} L, consider its corresponding DPDA P that accepts L by **empty stacks**.

   Then, we can construct a DPDA P' that accepts L by **final states** as:



Fact 3: $\text{DCFL}_{\text{ES}} \subsetneq \text{DCFL}$

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① All DCFL_{ES}s are DCFLs **BUT** ② there is a DCFL but not a DCFL_{ES}.

❷ $\boxed{\text{DCFL} \setminus \text{DCFL}_{\text{ES}} \neq \varnothing}$: The following is a DCFL but not a DCFL_{ES}:

$$L = \{a^n b^n \mid n \geq 0\} \in \text{DCFL} \setminus \text{DCFL}_{\text{ES}}$$

Why?

The DPDA needs to accept the following two words by empty stacks:

- $w = \epsilon \in L$
- $w = \texttt{ab} \in L$

However, if a DPDA accepts the ϵ by empty stacks, then the stack must become empty without reading any input symbols.

Thus, the PDA **cannot accept** $\texttt{ab}$ by empty stacks.

We can generalize it as **prefix property** of DCFL_{ES}.

Fact 4: $\text{DCFL}_{\text{ES}} = \text{DCFL}$ having Prefix Property

Definition (Prefix Property)

A language L has the **prefix property** if and only if for any word $w \in L$, any proper prefix of w is not in L:

$$\forall x, y \in \Sigma^*. \ ((xy \in L \land y \neq \epsilon) \implies x \notin L)$$

Fact 4: $\text{DCFL}_{\text{ES}} = \text{DCFL}$ having Prefix Property

A language L is a DCFL_{ES} if and only if the language L is a DCFL having the **prefix property**.

For example, the following language is a **DCFL** but does **NOT** have the **prefix property** because $\epsilon \in L$ is a proper prefix of

$$L = \{a^n b^n \mid n \geq 0\}$$

Thus, L is a **DCFL** but **NOT** a $\mathbf{DCFL_{ES}}$.

Fact 5: RL $\not\subseteq$ DCFL$_{\text{ES}}$

Fact 5: RL $\not\subseteq$ DCFL$_{\text{ES}}$

There exists a RL that is not a DCFL$_{\text{ES}}$.

- $\boxed{\text{RL} \setminus \text{DCFL}_{\text{ES}} \neq \varnothing}$: For example, the following language is a **RL** but does **NOT** have the **prefix property**:

$$L = \{a^n \mid n \geq 0\} \in \text{RL} \setminus \text{DCFL}_{\text{ES}}$$

because aa $\in L$ is a proper prefix of aaaa $\in L$.

Thus, L is a **RL** but **NOT** a **DCFL$_{\text{ES}}$**.

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Inherent Ambiguity of DCFLs

Definition (Inherent Ambiguity)

A language L is **inherently ambiguous** if all CFGs whose languages are L are ambiguous. (i.e. there is no unambiguous grammar for L)

What is the relationship of **inherently ambiguous languages** and DCFLs?

It satisfies the following fact:

$$\text{DCFL} \subsetneq \text{Non Inherently Ambiguous Languages}$$

We prove this fact by the following three steps:

1. $\boxed{\text{DCFL}_{\text{ES}} \subseteq \text{Non Inherently Ambiguous Languages}}$
2. $\boxed{\text{DCFL} \subseteq \text{Non Inherently Ambiguous Languages}}$ (using 1)
3. $\boxed{\text{Non Inherently Ambiguous Languages} \setminus \text{DCFL} \neq \varnothing}$

Fact 6: DCFL $\subsetneq$ Non Inherently Ambiguous Languages

All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

1. $\boxed{\text{A DCFL}_{\text{ES}} \text{ has an unambiguous grammar}}$: For a given DCFL$_{\text{ES}}$ L and its corresponding DPDA P, we can define a CFG for P as follows:
   - For all $0 \leq j < n$,
   $$S \to A^Z_{0,j}$$
   - For all transition $(q_j, X_1 \cdots X_m) \in \delta(q_i, a, X)$ where $q_i \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$ and combinations $0 \leq k_1, \cdots, k_m < n$:
   $$A^X_{i,k_m} \to a\ A^{X_1}_{j,k_1}\ A^{X_2}_{k_1,k_2} \cdots A^{X_m}_{k_{m-1},k_m}$$

   For any word $w \in L$, w has a unique moves from the initial configuration to the final configuration in P. And, we know that:
   $$A^X_{i,j} \Rightarrow^* w \qquad \text{if and only if} \qquad (q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$$
   Thus, the above CFG is **unambiguous**.

Fact 6: DCFL $\subsetneq$ Non Inherently Ambiguous Languages

All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

2. $\boxed{\text{A DCFL has an unambiguous grammar}}$: For a given DCFL L, we can define another DCFL L' with a special symbol \$ as follows:

$$L' = L\$ = \{w\$ \mid w \in L\}$$

Then, L' is a DCFL_{ES} because it has the **prefix property**, and L' has an **unambiguous grammar** G'. Now, we can define an **unambiguous grammar** G for L by treating \$ as a variable with a rule $\$ \rightarrow \epsilon$.

For example, $L = \{a^n b^n \mid n \geq 0\}$ is DCFL, then $L' = \{a^n b^n \$ \mid n \geq 0\}$ is a DCFL_{ES} and its **unambiguous grammar** G' is:

$$S \rightarrow X\$ \qquad X \rightarrow \mathtt{a}X\mathtt{b} \mid \epsilon$$

Then, the **unambiguous grammar** G for L is:

$$S \rightarrow X\$ \qquad X \rightarrow \mathtt{a}X\mathtt{b} \mid \epsilon \qquad \$ \rightarrow \epsilon$$

Fact 6: DCFL $\subsetneq$ Non Inherently Ambiguous Languages

All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

3. $\boxed{\text{Non Inherently Ambiguous Languages} \setminus \text{DCFL} \neq \varnothing}$: The following language is a **non inherently ambiguous language** but **not** a **DCFL**:

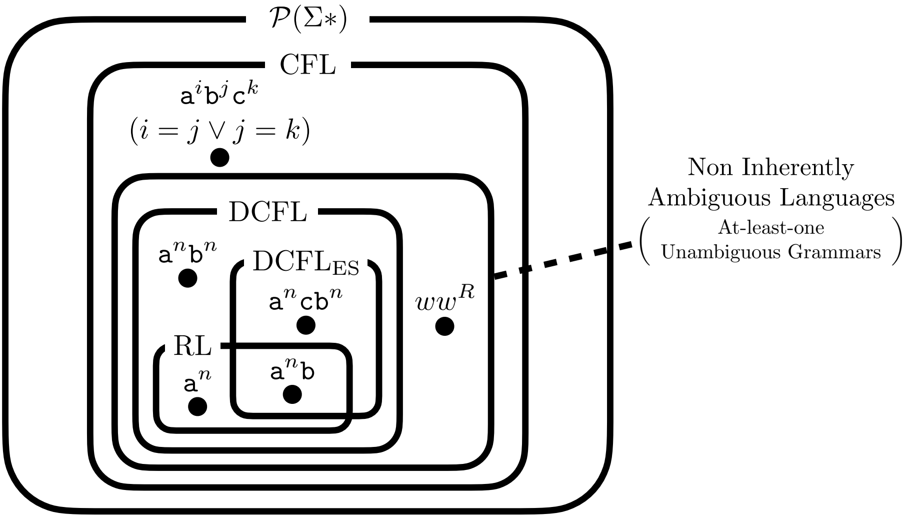
$$L = \{ww^R \mid w \in \{a, b\}^*\}$$

because the following **unambiguous grammar** G represents L:

$$S \to aSa \mid bSb \mid \epsilon$$

but we already know that L is **not** a **DCFL**.

Summary



$\mathcal{P}(\Sigma*)$
CFL
$\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k$
$(i = j \lor j = k)$
DCFL
$\mathtt{a}^n\mathtt{b}^n$
$\mathrm{DCFL_{ES}}$
$\mathtt{a}^n\mathtt{c}\mathtt{b}^n$
ww^R
RL
$\mathtt{a}^n$
$\mathtt{a}^n\mathtt{b}$
Non Inherently
Ambiguous Languages
At-least-one
Unambiguous Grammars

Next Lecture

- Normal Forms of Context-Free Grammars

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