

Lecture 19 – Typed Languages

COSE212: Programming Languages

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Recall

- Safe Language Systems
 - Dynamic vs Static Analysis for Detecting Run-Time Errors
 - Soundness vs Completeness of Analysis
- Type Systems
 - Types
 - Type Errors
 - Type Checking
 - Type Soundness
- In this lecture, we will define our first **typed language**.
- **TFAE** – FAE with **type system**.
 - **Type Checker** and **Typing Rules**
 - Interpreter and Natural Semantics

Contents

1. TFAE – FAE with Type System

Concrete Syntax

Abstract Syntax

2. Type Checker and Typing Rules

Recall: Type Checking

Type Environment

Numbers

Addition and Multiplication

Immutable Variable Definition and Identifier Lookup

Function Definition and Application

Examples

3. Interpreter with Type Checker

Interpreter and Natural Semantics

Operational Semantics vs Typing Rules

Contents

1. TFAE – FAE with Type System

Concrete Syntax

Abstract Syntax

2. Type Checker and Typing Rules

Recall: Type Checking

Type Environment

Numbers

Addition and Multiplication

Immutable Variable Definition and Identifier Lookup

Function Definition and Application

Examples

3. Interpreter with Type Checker

Interpreter and Natural Semantics

Operational Semantics vs Typing Rules

Before defining TFAE, guess the types of the following FAE expressions:

```
/* FAE */ 42
```

Since it produces a **number**, let's say its type is **Number**.

```
/* FAE */ x => x + 1
```

It produces a function value, but can we say more about its type? **Yes!**

It should take a **number** type argument and return a **number**.

Let's say its type is **Number => Number** called **arrow type**.

```
/* FAE */ x => x
```

How about this? There is no information on the parameter *x*.

One simple solution is to explicitly add **type annotations!**

TFAE – FAE with Type System

Let's extend FAE into TFAE with **type annotations** to specify the types of function parameters:

```
/* TFAE */  
(x: Number) => x           // x is `Number` type  
(f: Number => Number) => f // f is `Number => Number` type
```

If we define immutable variable definitions as **syntactic sugar**, it requires the type annotations: $\mathcal{D}[\text{val } x:\tau = e; e'] = (\lambda x:\tau. \mathcal{D}[e'])(\mathcal{D}[e])$

```
/* TFAE */  
val x: Number = 42; x + 1    // == `((x: Number) => x + 1)(42)`
```

However, if we **explicitly define** them rather than syntactic sugar, we can guess variable types from their initial values:

```
/* TFAE */  
val x = 42; x + 1           // x is `Number` type because of `42`
```

Concrete Syntax

For TFAE, we need to extend **expressions** of FAE with

- ① **function definitions** with **type annotations**
- ② **immutable variable definitions** without **type annotations**
- ③ **types**

We can extend the **concrete syntax** of FAE as follows:

```
// expressions
<expr> ::= ...
    | "(" <id> ":" <type> ")"  ">" <expr>
    | "val" <id> "=" <expr> ";" <expr>

// types
<type> ::= "(" <type> ")"
            // only for precedence
    | "Number"
            // number type
    | <type> ">" <type>
            // arrow type
```

Since functions are first-class values, the parameter and return types could be recursively arrow types. And, $=>$ is **right-associative**.

We can extend the **abstract syntax** of FAE for TFAE as follows:

Expressions $\mathbb{E} \ni e ::= \dots$

| $\lambda x : \tau. e$ (Fun)
| $\text{val } x = e; e$ (Val)

Types $\mathbb{T} \ni \tau ::= \text{num}$ (NumT)

| $\tau \rightarrow \tau$ (ArrowT)

We can define the abstract syntax of TFAE in Scala as follows:

```
enum Expr:  
  ...  
  case Fun(param: String, ty: Type, body: Expr)  
  case Val(name: String, init: Expr, body: Expr)
```

```
enum Type:  
  case NumT  
  case ArrowT(paramTy: Type, retTy: Type)
```

Contents

1. TFAE – FAE with Type System

Concrete Syntax

Abstract Syntax

2. Type Checker and Typing Rules

Recall: Type Checking

Type Environment

Numbers

Addition and Multiplication

Immutable Variable Definition and Identifier Lookup

Function Definition and Application

Examples

3. Interpreter with Type Checker

Interpreter and Natural Semantics

Operational Semantics vs Typing Rules

Recall: Type Checking

If the following conditions hold, we say “**the expression e has type τ** ”:

- e does not cause any type error, and
- e evaluates to a value of type τ or does not terminate.

If so, we use the following notation and say that e is **well-typed**:

$$\vdash e : \tau$$

Definition (Type Checking)

Type checking is a kind of static analysis checking whether a given expression e is **well-typed**. A **type checker** returns the **type** of e if it is well-typed, or rejects it and reports the detected **type error** otherwise.

We need to

- ① design **typing rules** to define when an expression is well-typed
- ② implement a **type checker** in Scala according to typing rules

Type Environment

Let's ① design **typing rules** of TFAE to define when an expression is well-typed in the form of:

$$\vdash e : \tau$$

and ② implement a **type checker** in Scala according to typing rules:

```
def typeCheck(expr: Expr): Type = ???
```

The type checker returns the **type** of e if it is well-typed, or rejects it and throws a **type error** otherwise.

In addition, we need to keep track of the **variable types**.

Type Environment

Let's ① design **typing rules** of TFAE to define when an expression is well-typed in the form of:

$$\boxed{\Gamma \vdash e : \tau}$$

and ② implement a **type checker** in Scala according to typing rules:

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = ???
```

The type checker returns the **type** of e if it is well-typed, or rejects it and throws a **type error** otherwise.

In addition, we need to keep track of the **variable types**.

Let's define a **type environment** Γ as a mapping from variable names to their types and pass it to the type checker.

Type Environments $\Gamma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T}$ (TypeEnv)

```
type TypeEnv = Map[String, Type]
```

Numbers

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  case Num(_) => ???
  ...
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-}\text{Num} \quad \frac{}{\Gamma \vdash n : \textcolor{red}{???}}$$

Numbers

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  case Num(_) => NumT
  ...
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{}{\tau-\text{Num} \quad \frac{}{\Gamma \vdash n : \text{num}}}$$

The number literal n has `num` type in any type environment Γ .

Addition

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Add(left, right) =>
    ???
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Add} \quad \frac{\text{???}}{\Gamma \vdash e_1 + e_2 : \text{???}}$$

Addition

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Add(left, right) =>
    typeCheck(left, tenv)
    ???
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Add} \quad \frac{\Gamma \vdash e_1 : \tau \quad ???}{\Gamma \vdash e_1 + e_2 : ???}$$

Type checker should do

- ① get the type of e_1 in Γ

Addition

```

def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Add(left, right) =>
    mustSame(typeCheck(left, tenv), NumT)
    ???
  ...

def mustSame(lty: Type, rty: Type): Unit =
  if (lty != rty) error(s"type mismatch: ${lty.str} != ${rty.str}")

```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Add} \frac{\Gamma \vdash e_1 : \text{num} \quad ???}{\Gamma \vdash e_1 + e_2 : ???}$$

Type checker should do

- ① check the type of e_1 is num in Γ

Addition

```

def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Add(left, right) =>
    mustSame(typeCheck(left, tenv), NumT)
    mustSame(typeCheck(right, tenv), NumT)
    ???  

def mustSame(lty: Type, rty: Type): Unit =
  if (lty != rty) error(s"type mismatch: ${lty.str} != ${rty.str}")
  
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Add} \frac{\Gamma \vdash e_1 : \text{num} \quad \Gamma \vdash e_2 : \text{num}}{\Gamma \vdash e_1 + e_2 : ???}$$

Type checker should do

- ① check the types of e_1 and e_2 are num in Γ

Addition

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Add(left, right) =>
    mustSame(typeCheck(left, tenv), NumT)
    mustSame(typeCheck(right, tenv), NumT)
    NumT

def mustSame(lty: Type, rty: Type): Unit =
  if (lty != rty) error(s"type mismatch: ${lty.str} != ${rty.str}")
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Add} \frac{\Gamma \vdash e_1 : \text{num} \quad \Gamma \vdash e_2 : \text{num}}{\Gamma \vdash e_1 + e_2 : \text{num}}$$

Type checker should do

- ① check the types of e_1 and e_2 are num in Γ
- ② return num as the type of $e_1 + e_2$

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Mul(left, right) =>
    mustSame(typeCheck(left, tenv), NumT)
    mustSame(typeCheck(right, tenv), NumT)
    NumT

def mustSame(lty: Type, rty: Type): Unit =
  if (lty != rty) error(s"type mismatch: ${lty.str} != ${rty.str}")
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Mul} \quad \frac{\Gamma \vdash e_1 : \text{num} \quad \Gamma \vdash e_2 : \text{num}}{\Gamma \vdash e_1 * e_2 : \text{num}}$$

Type checker should do

- ① check the types of e_1 and e_2 are num in Γ
- ② return num as the type of $e_1 * e_2$

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Val(x, init, body) =>
    val initTy = typeCheck(init, tenv)
    typeCheck(body, tenv + (x -> initTy))
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Val} \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma[x : \tau_1] \vdash e_2 : \tau_2}{\Gamma \vdash \text{val } x = e_1; e_2 : \tau_2}$$

This rule stores the type of x in Γ inferred from the initial value.

```
/* TFAE */ val x = 1; x + 2      // `x: Number` in `tenv` <- `1: Number`
```

Identifier Lookup

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Id(x) =>
    tenv.getOrElse(x, error(s"free identifier: $x"))
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-}\text{Id} \frac{x \in \text{Domain}(\Gamma)}{\Gamma \vdash x : \Gamma(x)}$$

This rule looks up the type of x in Γ .

```
/* TFAE */ val x = 1; x + 2      // `x: Number` in `tenv` <- `1: Number`
```

Function Definition

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Fun(param, paramTy, body) =>
    val retTy = typeCheck(body, tenv + (param -> paramTy))
    ArrowT(paramTy, retTy)
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-}\mathbf{Fun} \quad \frac{\Gamma[x : \tau] \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau. e : \tau \rightarrow \tau'}$$

We can check the body of a function with its parameter type.

```
/* TFAE */ (x: Number) => x      // Number => Number
```

Function Application

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case App(fun, arg) => typeCheck(fun, tenv) match
    case ArrowT(paramTy, retTy) =>
      mustSame(typeCheck(arg, tenv), paramTy)
      retTy
    case ty => error(s"not a function type: ${ty.str}")
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-App} \quad \frac{\Gamma \vdash e_0 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash e_0(e_1) : \tau_2}$$

We don't have to check the type of the function body because it is already checked when the function is defined.

```
/* TFAE */ ((x: Number) => x)(1) // Number
```

Examples

```
/* TFAE */ val x = 1; x + 2           // 3: Number
```

$$\frac{\frac{x \in \text{Domain}([x : \text{num}])}{[x : \text{num}] \vdash x : \text{num}} \quad [x : \text{num}] \vdash 2 : \text{num}}{[x : \text{num}] \vdash x + 2 : \text{num}}
 \frac{\emptyset \vdash 1 : \text{num}}{\emptyset \vdash \text{val } x = 1; x + 2 : \text{num}}$$

```
/* TFAE */ ((x: Number) => x)(2) * 3           // 6: Number
```

$$\frac{x \in \text{Domain}([x : \text{num}])}{[x : \text{num}] \vdash x : \text{num}}
 \frac{\emptyset \vdash \lambda x : \text{num}. x : \text{num} \rightarrow \text{num} \quad \emptyset \vdash 2 : \text{num}}{\emptyset \vdash (\lambda x : \text{num}. x)(2) : \text{num}}
 \frac{}{\emptyset \vdash 3 : \text{num}}$$

$$\frac{\emptyset \vdash (\lambda x : \text{num}. x)(2) : \text{num}}{\emptyset \vdash (\lambda x : \text{num}. x)(2) * 3 : \text{num}}$$

Contents

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Concrete Syntax

Abstract Syntax

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Numbers

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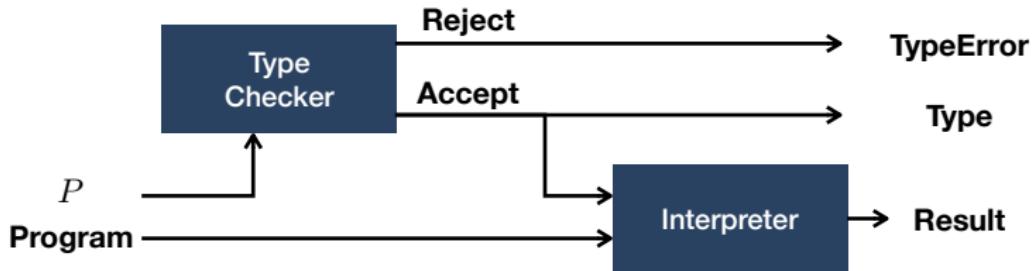
Examples

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Interpreter and Natural Semantics

Operational Semantics vs Typing Rules

Interpreter with Type Checker



```
def eval(str: String): String =  
  val expr = Expr(str)  
  val ty = typeCheck(expr, Map.empty)  
  val v = interp(expr, Map.empty)  
  s"${v.str}: ${ty.str}"
```

```
eval("val x = 1; x + 2")           // 3: Number  
eval("((x: Number) => x)(2) * 3") // 6: Number  
eval("1 + (x: Number) => x")      // a type error thrown by `typeCheck`
```

For interpreter and natural semantics for TFAE, it is just enough to extend those for function definitions in FAE.

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Fun(p, t, b) => CloV(p, b, env)
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\text{Fun} \quad \frac{}{\sigma \vdash \lambda x : \tau . e \Rightarrow \langle \lambda x . e, \sigma \rangle}$$

The type annotation is ignored in the interpreter and natural semantics.

What is the difference between **operational semantics** and **typing rules**?

$$\sigma \vdash e \Rightarrow v$$

vs

$$\Gamma \vdash e : \tau$$

See the table below for the comparison.

	Operational Semantics	Typing Rules
Mathematical Notation	$\sigma \vdash e \Rightarrow v$	$\Gamma \vdash e : \tau$
Dynamic/Static	Dynamic	Static
Concrete/Abstract	Concrete	Abstract
Purpose	Evaluation	Type Checking
Implementation	Interpreter	Type Checker
Result	Value	Type

Summary

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Examples

3. Interpreter with Type Checker

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Operational Semantics vs Typing Rules

Exercise #11

<https://github.com/ku-plrg-classroom/docs/tree/main/cose212/tfae>

- Please see above document on GitHub:
 - Implement typeCheck function.
 - Implement interp function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

Next Lecture

- Typing Recursive Functions

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