Lecture 13 – Parse Trees and Ambiguity

COSE215: Theory of Computation

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PLRG

2024 Spring

Recall

PLRG

- A **context-free grammar (CFG)**:

$$G = (V, \Sigma, S, R)$$

- The **language** of a CFG G:

$$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

- A language L is a **context-free language (CFL)**:

$$\exists \text{ CFG } G.\ L(G) = L$$

- For a given word $w \in L(G)$, a **derivation** for w is $S \Rightarrow^* w$
- A sequence $\alpha \in (V \cup \Sigma)^*$ is a **sentential form** if $S \Rightarrow^* \alpha$.

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Parse Trees

Consider the following CFG for balanced parentheses:

$$S \to \epsilon \mid (S) \mid SS$$

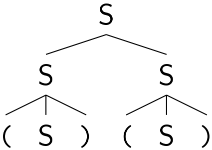
There are two different derivations for the sentential form $(S)(S)$:

$$① \quad S \Rightarrow_L SS \Rightarrow_L (S)S \Rightarrow_L (S)(S)$$

$$② \quad S \Rightarrow_R SS \Rightarrow_R S(S) \Rightarrow_R (S)(S)$$

However, **parse trees** focus on the structure of the derivations instead of considering the order of the derivation steps.

For example, the above two derivations have the same parse tree:



Parse Trees

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Definition (Parse Trees)

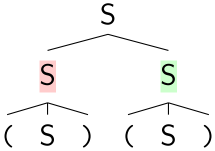
For a given CFG $G = (V, \Sigma, S, R)$, **parse trees** are trees satisfying:

1. The **root node** is labeled with the **start variable** S.
2. Each **internal node** is labeled with a **variable** $A \in V$.
   If its children are labeled with:
   $$X_1, X_2, \cdots, X_k$$
   from the left to the right, then $A \to X_1 X_2 \cdots X_k \in R$.
3. Each **leaf node** is labeled with a variable, symbol, or ϵ. However, if a leaf node is labeled with ϵ, it must be the only child of its parent.

Parse Trees – Example 1: Balanced Parentheses

$$S \to \epsilon \mid (S) \mid SS$$

A parse tree for $(S)(S)$:



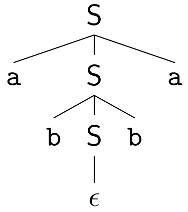
① $S \Rightarrow_L SS \Rightarrow_L (S)S \Rightarrow_L (S)(S)$

② $S \Rightarrow_R SS \Rightarrow_R S(S) \Rightarrow_R (S)(S)$

Parse Trees – Example 2: Even Palindromes

$$S \rightarrow \epsilon \mid \mathtt{a}S\mathtt{a} \mid \mathtt{b}S\mathtt{b}$$

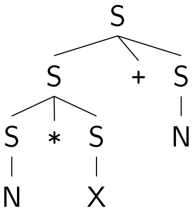
A parse tree for `abba`:



Parse Trees – Example 3: Arithmetic Expressions

$$
\begin{aligned}
S &\rightarrow N \mid X \mid S\texttt{+}S \mid S\texttt{*}S \mid \texttt{(}S\texttt{)} \\
N &\rightarrow \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\
X &\rightarrow \texttt{a} \mid \cdots \mid \texttt{z}
\end{aligned}
$$

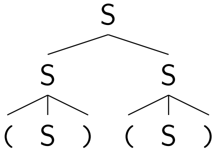
A parse tree for $N\texttt{*}X\texttt{+}N$:



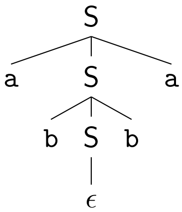
Yields

Definition (Yields)

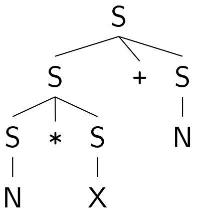
The sequence obtained by concatenating the labels (without ϵ) of the leaf nodes of a parse tree from left to right is called the **yield** of the parse tree.



Its yield is $(S)(S)$.



Its yield is abba.



Its yield is $N*X+N$.

Relationship between Parse Trees and Derivations

Theorem (Parse Trees and Derivations)

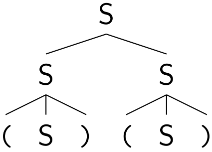
For a given CFG $G = (V, \Sigma, S, R)$, for any sequence $\alpha \in (V \cup \Sigma)^*$:

$$S \Rightarrow^* \alpha \iff \exists \text{ parse tree } T.\ \text{s.t. } T \text{ yields } \alpha$$

For example, consider the sequence $(S)(S)$:

$$S \Rightarrow SS \Rightarrow (S)S \Rightarrow (S)(S)$$

$$\Updownarrow$$



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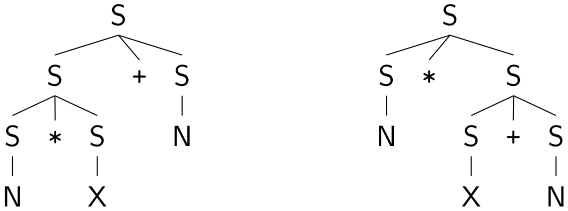
Ambiguous Grammars

Is there always a **unique** parse tree for a given sentential form?

$$
\begin{aligned}
S &\rightarrow N \mid X \mid S\texttt{+}S \mid S\texttt{*}S \mid \texttt{(}S\texttt{)} \\
N &\rightarrow \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\
X &\rightarrow \texttt{a} \mid \cdots \mid \texttt{z}
\end{aligned}
$$

For example, consider the sentential form $N\texttt{*}X\texttt{+}N$:



Actually, there are **two** parse trees for $N\texttt{*}X\texttt{+}N$.

Ambiguous Grammars

Definition (Ambiguous Grammar)

A context-free grammar $G = (V, \Sigma, S, R)$ is **ambiguous** if there exist a word $w \in \Sigma^*$ and two distinct parse trees for w. If not, G is **unambiguous**.

Theorem

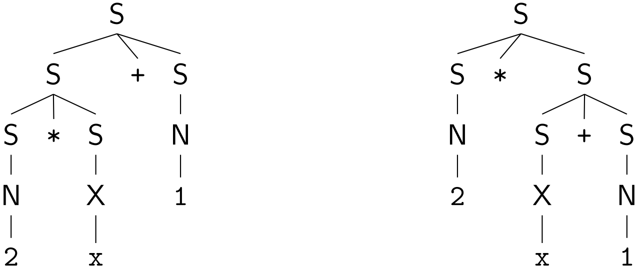
Let $G = (V, \Sigma, S, R)$ be a CFG. Then, the following numbers are equal for any sequence of variables or symbols $w \in (V \cup \Sigma)^*$:

1. *The number of parse trees whose yields are w.*
2. *The number of left-most derivations for w.*
3. *The number of right-most derivations for w.*

Ambiguous Grammars – Example

$$
\begin{aligned}
S &\rightarrow N \mid X \mid S\texttt{+}S \mid S\texttt{*}S \mid \texttt{(}S\texttt{)} \\
N &\rightarrow \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\
X &\rightarrow \texttt{a} \mid \cdots \mid \texttt{z}
\end{aligned}
$$

This grammar is **ambiguous** because there are **two** parse trees for the word `2 * x + 1`:



Note that it means that there are **two** left-most (or right-most) derivations for `2 * x + 1` by the previous theorem.

Ambiguous Grammars – Example

$$
\begin{aligned}
S &\rightarrow N \mid X \mid S\texttt{+}S \mid S\texttt{*}S \mid \texttt{(}S\texttt{)} \\
N &\rightarrow \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\
X &\rightarrow \texttt{a} \mid \cdots \mid \texttt{z}
\end{aligned}
$$

There are **two** left-most derivations for `2 * x + 1`:

1. Applying the production rule $S \rightarrow S\texttt{+}S$ first:

$$
\begin{array}{llllllllll}
S & \Rightarrow_L & S\texttt{+}S & \Rightarrow_L & S\texttt{*}S\texttt{+}S & \Rightarrow_L & N\texttt{*}S\texttt{+}S & \Rightarrow_L & \texttt{2*}S\texttt{+}S \\
 & \Rightarrow_L & \texttt{2*}X\texttt{+}S & \Rightarrow_L & \texttt{2*x+}S & \Rightarrow_L & \texttt{2*x+}N & \Rightarrow_L & \texttt{2*x+1}
\end{array}
$$

2. Applying the production rule $S \rightarrow S\texttt{*}S$ first:

$$
\begin{array}{llllllllll}
S & \Rightarrow_L & S\texttt{*}S & \Rightarrow_L & N\texttt{*}S & \Rightarrow_L & \texttt{2*}S & \Rightarrow_L & \texttt{2*}S\texttt{+}S \\
 & \Rightarrow_L & \texttt{2*}X\texttt{+}S & \Rightarrow_L & \texttt{2*x+}S & \Rightarrow_L & \texttt{2*x+}N & \Rightarrow_L & \texttt{2*x+1}
\end{array}
$$

Eliminating Ambiguity

Unfortunately,

- There is **NO** general algorithm to remove ambiguity from a CFG.
- There is even **NO** algorithm to determine a CFG is ambiguous.

Fortunately, there are well-known techniques to manually **eliminate** the ambiguity in a given grammar commonly used in programming languages.

$$
\begin{array}{rcl}
S & \rightarrow & N \mid X \mid S\texttt{+}S \mid S\texttt{*}S \mid \texttt{(}S\texttt{)} \\
N & \rightarrow & \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\
X & \rightarrow & \texttt{a} \mid \cdots \mid \texttt{z}
\end{array}
$$

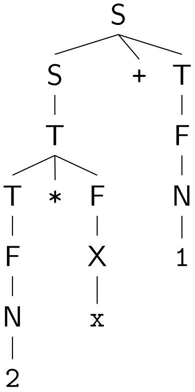
For example, an equivalent but unambiguous grammar is:

$$
\begin{array}{rcl}
S & \rightarrow & T \mid S\texttt{+}T \\
T & \rightarrow & F \mid T\texttt{*}F \\
F & \rightarrow & N \mid X \mid \texttt{(}S\texttt{)} \\
N & \rightarrow & \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\
X & \rightarrow & \texttt{a} \mid \cdots \mid \texttt{z}
\end{array}
$$

Eliminating Ambiguity

Now, the unique parse tree for `2 * x + 1` is:

$$
\begin{array}{lcl}
S & \rightarrow & T \mid S\texttt{+}T \\
T & \rightarrow & F \mid T\texttt{*}F \\
F & \rightarrow & N \mid X \mid \texttt{(}S\texttt{)} \\
N & \rightarrow & \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\
X & \rightarrow & \texttt{a} \mid \cdots \mid \texttt{z}
\end{array}
$$



Eliminating Ambiguity

First, analyze why the original grammar is ambiguous.

$$\begin{array}{lcl} S & \rightarrow & N \mid X \mid S\texttt{+}S \mid S\texttt{*}S \mid \texttt{(}S\texttt{)} \\ N & \rightarrow & \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\ X & \rightarrow & \texttt{a} \mid \cdots \mid \texttt{z} \end{array}$$

- The **precedence** of `+` and `*` is not specified.
  - For example, two parse trees for `1 * 2 + 3` interpreted as:

    `1 * (2 + 3)` and `(1 * 2) + 3`

  - Let's give `*` higher precedence than `+` to interpret it as `(1 * 2) + 3`.
- The **associativity** of `+` (or `*`) is not specified.
  - For example, two parse trees for `1 + 2 + 3` interpreted as:

    `1 + (2 + 3)` and `(1 + 2) + 3`

  - Let's give the left-associativity to `+` to interpret it as `(1 + 2) + 3`.

Eliminating Ambiguity – Precedence

To enforce the **precedence**, define new variables F for factors and T for terms:

- A **factor** is a number, a variable, or a parenthesized expression:

  `42,   x,   (1 + 2),   ...`

  In the grammar, F is defined as:

  $$F \to N \mid X \mid (S)$$

- A **term** is the multiplication of one or more factors:

  `42,   2 * x,   2 * (1 + 2),   1 * (x * y) * z,   ...`

  In the grammar, T is defined as:

  $$T \to F \mid T * F$$

- An **expression** is the addition of one or more terms:

  `42,   1 + 2,   1 + 2 * 3,   (1 + 2) * 3 + 4),   ...`

  In the grammar, S is defined as:

  $$S \to T \mid S + T$$

Eliminating Ambiguity – Associativity

The unambiguous grammar is:

$$
\begin{array}{lcl}
S & \rightarrow & T \mid S\texttt{+}T \\
T & \rightarrow & F \mid T\texttt{*}F \\
F & \rightarrow & N \mid X \mid \texttt{(}S\texttt{)} \\
N & \rightarrow & \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\
X & \rightarrow & \texttt{a} \mid \cdots \mid \texttt{z}
\end{array}
$$

- This grammar supports the **left-associativity** of + and *. Why?

Eliminating Ambiguity – Associativity

The unambiguous grammar is:

$$
\begin{array}{rcl}
S & \rightarrow & T \mid S\texttt{+}T \\
T & \rightarrow & F \mid T\texttt{*}F \\
F & \rightarrow & N \mid X \mid \texttt{(}S\texttt{)} \\
N & \rightarrow & \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\
X & \rightarrow & \texttt{a} \mid \cdots \mid \texttt{z}
\end{array}
$$

- This grammar supports the **left-associativity** of + and ∗. Why?
  - $S \rightarrow S\texttt{+}T$ and $T \rightarrow T\texttt{*}F$ are **left-recursive**.
- Then, how to support the **right-associativity** of + and ∗?
  - Replace the **left-recursive** rules with **right-recursive** rules!

$$
\begin{array}{rcl}
S & \rightarrow & T \mid T\texttt{+}S \\
T & \rightarrow & F \mid F\texttt{*}T \\
\cdots & &
\end{array}
$$

Inherent Ambiguity

So far, we have discussed the **ambiguity** for grammars.
We will now discuss the **inherent ambiguity** for languages.

Definition (Inherent Ambiguity)

A language L is **inherently ambiguous** if all CFGs whose languages are L are ambiguous. (i.e. there is no unambiguous grammar for L)

For example, the following language is **inherently ambiguous**:

$$L = \{\mathrm{a}^i\mathrm{b}^j\mathrm{c}^k \mid i, j, k \geq 0 \wedge (i = j \vee j = k)\}$$

An example of ambiguous grammar for L is:

$$\begin{array}{lcl} S & \rightarrow & L \mid R \\ L & \rightarrow & A \mid L\mathtt{c} \\ A & \rightarrow & \epsilon \mid \mathtt{a}A\mathtt{b} \\ R & \rightarrow & B \mid \mathtt{a}R \\ B & \rightarrow & \epsilon \mid \mathtt{b}B\mathtt{c} \end{array}$$

Summary

1. Parse Trees
   - Definition
   - Yields
   - Relationship between Parse Trees and Derivations
2. Ambiguity
   - Ambiguous Grammars
   - Eliminating Ambiguity
   - Inherent Ambiguity

Midterm Exam

- The midterm exam will be given in class.
- **Date:** 13:30-14:45 (1 hour 15 minutes), April 24 (Wed.).
- **Location:** 604, Woojung Hall of Informatics (우정정보관 604호)
- **Coverage:** Lectures 1 – 13
- **Format:** 7–9 questions with closed book and closed notes
  - Filling blanks in some tables, sentences, or expressions.
  - Construction of automata or grammars for given languages.
  - Proofs of given statements related to automata or grammars.
  - Yes/No questions about concepts in the theory of computation.
  - etc.
- Note that there is **no class** on **April 22 (Mon.)**.
- Please refer to the **previous exams** in the course website:

https://plrg.korea.ac.kr/courses/cose215/

Next Lecture

- Pushdown Automata (PDA)

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