Lecture 6 – Regular Expressions and Languages

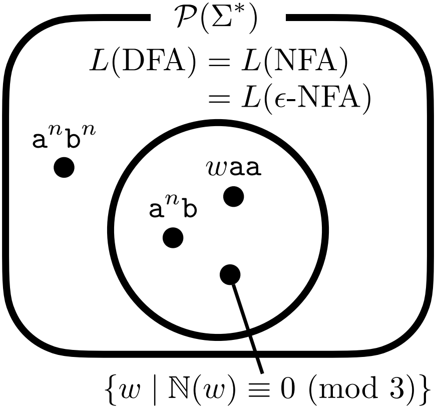
COSE215: Theory of Computation

Jihyeok Park

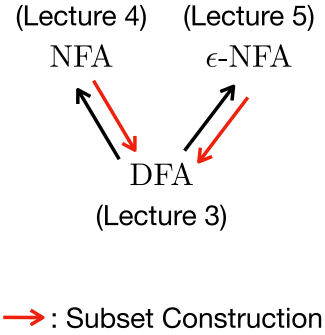
PLRG

2024 Spring

Recall



$\mathcal{P}(\Sigma^*)$
$L(\text{DFA}) = L(\text{NFA})$
$= L(\epsilon\text{-NFA})$
$\mathtt{a}^n\mathtt{b}^n$
$w\mathtt{aa}$
$\mathtt{a}^n\mathtt{b}$
$\{w \mid \mathbb{N}(w) \equiv 0 \pmod 3\}$



(Lecture 4)
NFA
(Lecture 5)
ϵ-NFA
DFA
(Lecture 3)
: Subset Construction

Contents

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Recall: Operations in Languages

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- **Union** of languages: $L_1 \cup L_2$
- **Concatenation** of languages: $L_1 L_2 = \{w_1 w_2 \mid w_1 \in L_1 \land w_2 \in L_2\}$
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Regular expressions (REs) provide a new way to define languages with above **operations** without using finite automata!

Definition of Regular Expressions

Definition (Regular Expressions)

A **regular expression** over a set of symbols Σ is inductively defined as:

- **(Basis Case)** $\varnothing$, ϵ, and $x \in \Sigma$ are regular expressions.
- **(Induction Case)** If R_1 and R_2 are regular expressions, then so are $R_1 \mid R_2$, $R_1 R_2$, R^*, and (R).

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The following is the **syntax** of regular expressions and examples:

$$
\begin{array}{rclll}
R & ::= & \varnothing & \text{(Empty)} & \mid \; R \mid R \quad \text{(Union)} \\
 & \mid & \epsilon & \text{(Epsilon)} & \mid \; R\;R \quad \text{(Concatenation)} \\
 & \mid & x & \text{(Symbol)} & \mid \; R^* \quad \text{(Kleene Star)} \\
 & & & & \mid \; (R) \quad \text{(Parentheses)}
\end{array}
$$

$\varnothing$	ϵ	a	a|b	ab
a^*	$a(\varnothing \mid c)^*$	$(a\epsilon) \mid b^*$	$(a(bc^*b)^*)^*$	$(a\varnothing a) \mid b^*$

Precedence Order

Arithmetic expressions have the following **precedence order**:

$$\times \qquad > \qquad +$$

It means that multiplication ($\times$) has higher precedence than addition ($+$). For example,

$$1 + 2 \times 3 \qquad \text{means} \qquad 1 + (2 \times 3)$$

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$$* \qquad > \qquad \cdot \qquad > \qquad |$$

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Similarly, **regular expressions** have the following **precedence order**:

$$^* \quad > \quad \cdot \quad > \quad \mid$$

For example,

$$\texttt{a} \mid \epsilon \texttt{b}^* \qquad \text{means} \qquad \texttt{a} \mid (\epsilon (\texttt{b}^*))$$

$$(\texttt{a} \mid \epsilon) \texttt{b}^* \qquad \text{means} \qquad (\texttt{a} \mid \epsilon) (\texttt{b}^*)$$

Definition of Regular Expressions

```
// The definition of regular expressions
enum RE:
  case Emp                                      // ∅
  case Eps                                      // ϵ
  case Sym(symbol: Symbol)                      // x
  case Union(left: RE, right: RE)               // R1 | R2
  case Concat(left: RE, right: RE)              // R1 R2
  case Star(re: RE)                             // R*
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```

In the algebraic data type (ADT) of regular expressions, we do **not need to explicitly define the parentheses** because it is already handled by the structure of the ADT.

```
// import all constructors (Emp, Eps, Sym, Union, Concat, Star) of RE
import RE.*

// a | ϵ b*
val re1: RE = Union(Sym('a'), Concat(Eps, Star(Sym('b'))))

// (a | ϵ) b*
val re2: RE = Concat(Union(Sym('a'), Eps), Star(Sym('b')))
```

Language of Regular Expressions

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For a given regular expression R on a set of symbols Σ, the **language** $L(R)$ of R is inductively defined as follows:

$$
\begin{array}{rcl}
L(\varnothing) & = & \varnothing \\
L(\epsilon) & = & \{\epsilon\} \\
L(x) & = & \{x\}
\end{array}
\qquad
\begin{array}{rcl}
L(R_1 \mid R_2) & = & L(R_1) \cup L(R_2) \\
L(R_1 R_2) & = & L(R_1)L(R_2) \\
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L(\mathtt{a} \mid \epsilon\mathtt{b}^*) & = & L(\mathtt{a}) \cup L(\epsilon\mathtt{b}^*) & = & \{\mathtt{a}\} \cup L(\epsilon)L(\mathtt{b}^*) \\
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& = & (L(\texttt{a}) \cup L(\epsilon))L(\texttt{b})^* & = & (\{\texttt{a}\} \cup \{\epsilon\})\{\texttt{b}\}^* \\
& = & \{\texttt{ab}^n \text{ or } \texttt{b}^n \mid n \geq 0\} & &
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Extended Regular Expressions

More operators can be added to regular expressions:

$$
\begin{array}{rcll}
R & ::= & \cdots & \\
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For examples,

$$\begin{array}{rcl} L(\texttt{(ab)}^{+}) & = & L(\texttt{ab(ab)}^{*}) = \{(\texttt{ab})^{n} \mid n \geq 1\} \\ L(\texttt{a}^{?}b) & = & L(\texttt{(a}\mid\epsilon\texttt{)b}) = \{\texttt{ab}, \texttt{b}\} \end{array}$$

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Examples

PLRG

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- $L = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$ – IMPOSSIBLE ($\nexists$ RE R. $L(R) = L$)

Equivalence of Regular Expressions

We say two regular expressions R_1 and R_2 are **equivalent** ($R_1 \equiv R_2$) if their languages are the same: $L(R_1) = L(R_2)$.

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- Left and right **distributive laws**:

$$(R_1 \mid R_2) R_3 \equiv R_1 R_3 \mid R_2 R_3 \qquad \text{and} \qquad R_1(R_2 \mid R_3) \equiv R_1 R_2 \mid R_1 R_3$$

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- Laws involving Kleene star:

$$(R^*)^* \equiv R^* \qquad \text{and} \qquad \varnothing^* \equiv \epsilon \qquad \text{and} \qquad \epsilon^* \equiv \epsilon$$

$$\epsilon \mid R^* \equiv R^* \mid \epsilon \equiv R^* \qquad \text{and} \qquad R \mid R^* \equiv R^* \mid R \equiv R^*$$

Simplifying Regular Expressions

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Contents

Regular Expressions in Practice

Most **programming languages** support **regular expressions**:

- **Scala** – `scala.util.matching.Regex` class
- **Python** – `re` module
- **JavaScript** – `RegExp` object
- **Rust** – `regex` crate
- ...

Regular Expressions in Practice

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For example, we can convert a string to a regular expression (`Regex`) object by using the `r` method in Scala:

```scala
import scala.util.matching.Regex

val re: Regex = "(a|b)c*".r
re.matches("a")      // true
re.matches("b")      // true
re.matches("accc")   // true
re.matches("bccccc") // true
re.matches("ba")     // false
re.matches("cba")    // false
re.matches("aacc")   // false
re.matches("cccccc") // false
```

Regular Expressions in Practice

In practice, regular expressions support more syntactic sugar:

Syntax	Description
`^`	start of the line
`$`	start and end of the line
`.`	any character
`[]`	any character in the set
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`"ci[dait]*".r` `"\\w+$".r` `"\\d+".r`

For example, above Scala regular expressions find patterns in each string:

```
Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor
incididunt ut 53 et dolore magna aliqua. Ut enim ad minim veniam, quis
nostrud exercitation 42 laboris nisi ut aliquip ex ea commodo consequat.
Duis aute irure dolor in reprehenderit in voluptate 129 esse cillum dolore eu
fugiat nulla 5323. Excepteur sint occaecat cupidatat non proident, sunt in
culpa qui officia deserunt mollit anim id est laborum.
```

Summary

Next Lecture

PLRG

- Equivalence of Regular Expressions and Finite Automata

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