Lecture 16 - Dynamic Programming, All Pairs Shortest Path Problem and Floyd-Warshall Algorithm

Fall 2025, Korea University

Instructor: Gabin An (gabin_an@korea.ac.kr)

Course Outline (After Midterm)

- Part 3: Data Structures
  - Graphs, Graph Search (DFS, BFS) and Applications (Finding SSCs w/ DFS)
- Part 4: Dynamic Programming
  - Shortest-Path: Dijkstra, **Bellman-Ford, Floyd-Warshall Algorithms** 👈
  - More General DP: Longest Common Subsequence, Knapsack Problem
- Part 5: Greedy Algorithms and Others
  - Scheduling Problem, Optimal Codes
  - Minimum Spanning Trees
  - Max Flow, Min Cut and Ford-Fulkerson Algorithms
  - Stable Matching, Gale-Shapley Algorithm

Recap: Bellman-Ford Algorithm

1. Initialize $d[s] = 0$, others ∞.
2. **Repeat** $|V| - 1$ **times:**
   Relax **every edge** $(u, v) \in E$: $d[v] \leftarrow \min(d[v], d[u] + w(u, v))$
3. Extra pass:
   If any edge (u, v) can still be relaxed, i.e., $d[v] > d[u] + w(u, v)$
   → **negative cycle detected** 🔁.

Bellman-Ford as Dynamic Programming

Recall:

$d_k(v) = d_k[v] =$ shortest distance from s to v using at most k edges.

Recurrence:

$$d_k[v] = \min\left\{d_{k-1}[v],\ \min_{(u,v)\in E}\left(d_{k-1}[u] + w(u,v)\right)\right\}$$

- Build solutions iteratively: from paths with ≤ $k-1$ edges → paths with ≤ k edges.
- **Runtime**: $O(mn)$; for n iterations, we loop through all the edges.
- **Space**: can reuse arrays (only keep last iteration).
    - To compute d_k, we only need d_{k-1} (not all of $d_{k-1}, \ldots, d_0$).

What's Really Happening?

- Shortest path with ≤ k edges can be built from a shortest path with ≤ $k - 1$ edges.
- **DP Principle:** Store and reuse intermediate results instead of recomputing.
- Similar to divide-and-conquer, but with **memoization / tabulation** to avoid repeated work.

Dynamic Programming Recipe

When to use DP:

- **Optimal Substructure** → problem can be broken into smaller independent subproblems.
- **Overlapping Subproblems** → subproblems repeat, so store solutions in a table.

Optimal Substructure (Bellman-Ford Example)

> The optimal solution to the problem is composed of optimal solutions to smaller *independent* subproblems.

- For shortest paths:

$$d(s,t) = \min_{k \in V}\{d(s,k) + d(k,t)\}$$

- Any shortest $s \to t$ path is composed of:
  - A shortest $s \to k$ path, and
  - A shortest $k \to t$ path.

✅ Optimal solution is built from optimal solutions to subproblems.

Overlapping Subproblems (Bellman-Ford Example)

> The optimal solutions of subproblems can be reused multiple times to compute the optimal solutions of larger problems.

- $d(s, k)$ can be used to compute $d(s, t)$ **for any** t where the shortest $s - t$ path contains k.
- If (u, v) and (u, v') are both in E, relaxing (u, v) and (u, v') both need $d_{k-1}[u]$.
- Instead of recomputing many times, **store the computed valude** and reuse.

DP Implementations

Two common styles:

1. **Bottom-Up (Tabulation)**
    - Start with smallest subproblems.
    - Iteratively fill table → build up to full problem.
    - Example: Bellman-Ford.
2. **Top-Down (Memoization)**
    - Write recursion for the problem.
    - Store results of subproblems in a table and reuse stored results when needed.

💡 Both approaches are equivalent in power.

Today's Main Topic

So far, we solved the **Single-Source Shortest Paths (SSSP)** problem.

- Input: one source s, find shortest paths to all $v \in V$.

But what if we want the **shortest path between *every pair* of vertices**?

This leads us to the **All-Pairs Shortest Paths (APSP)** 🕸 problem.

- We'll learn the **Floyd–Warshall** algorithm, a dynamic programming approach to solving APSP.

APSP: All-Pairs Shortest Paths

- **Input**: Graph $G = (V, E)$ with edge weights $w(u, v)$.
- **Goal**: Find shortest path distances $d(u, v)$ for **all pairs** $u, v \in V$.

Examples of applications:

- Routing in communication networks
- Social network analysis
- Circuit design

First Attempt 🤔

- Run **Dijkstra** from each node $u \in V$.
  - Runtime: $O(n \cdot (m + n \log n)) = O(mn + n^2 \log n)$
- Or run **Bellman–Ford** from each node.
  - Runtime: $O(n \cdot mn) = O(mn^2)$

Instead of running SSSP n times, **can we leverage Dynamic Programming to solve APSP directly?**

DP Perspective

Recall: DP works when we have

- **Optimal Substructure**
- **Overlapping Subproblems**

For shortest paths:

- If k is an intermediate vertex on a shortest path $u \to v$:

$$d(u, v) = d(u, k) + d(k, v)$$

Key Idea: Intermediate Nodes

- Label vertices as $1, 2, \ldots, n$.
- Define $d^{(k)}(u, v)$ = shortest path distance from u to v using **only** nodes from $\{1, \ldots, k\}$ as possible **intermediates** nodes
- Recurrence:

$$d^{(k)}(u, v) = \min\left(d^{(k-1)}(u, v),\ d^{(k-1)}(u, k) + d^{(k-1)}(k, v)\right)$$

- If P is a shortest path from u to v using nodes $\{1, \ldots, k\}$ internally, there are two cases:
  - 1️⃣ **P does not pass through** k: $d^{(k-1)}(u, v)$
  - 2️⃣ **P passes through** k: $d^{(k-1)}(u, k) + d^{(k-1)}(k, v)$

Key Idea: Intermediate Nodes

So:

- $d^{(0)}(u,v)$: shortest path using no intermediate nodes (i.e., just direct edges).
- $d^{(1)}(u,v)$: shortest path possibly passing through vertex 1.
- $d^{(2)}(u,v)$: shortest path possibly passing through vertics $\{1,2\}$.
- $d^{(3)}(u,v)$: shortest path possibly passing through vertics $\{1,2,3\}$.

...and so on, until

- $d^{(n)}(u,v)$: shortest path possibly passing through vertics $\{1,2,3,\ldots,n\}$.
    - This gives the shortest path over all vertices.

Solving APSP: Floyd–Warshall Algorithm

Initialization

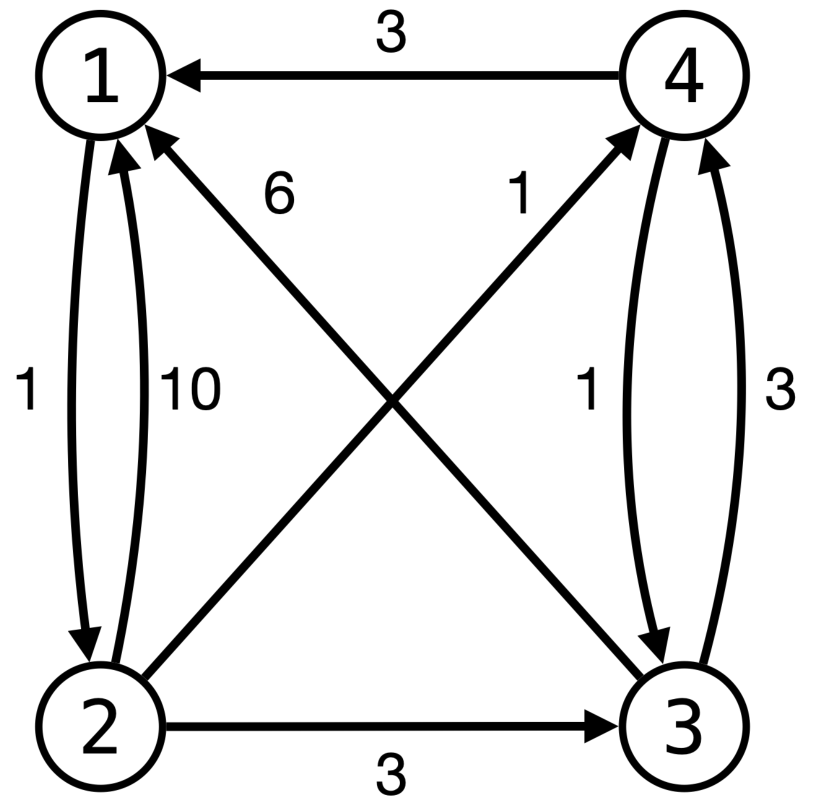
- $d^{(0)}(u, v) = w(u, v)$ if $(u, v) \in E$, else ∞.
- $d^{(0)}(u, u) = 0$.

Update Rule (for $k = 1 \to n$):

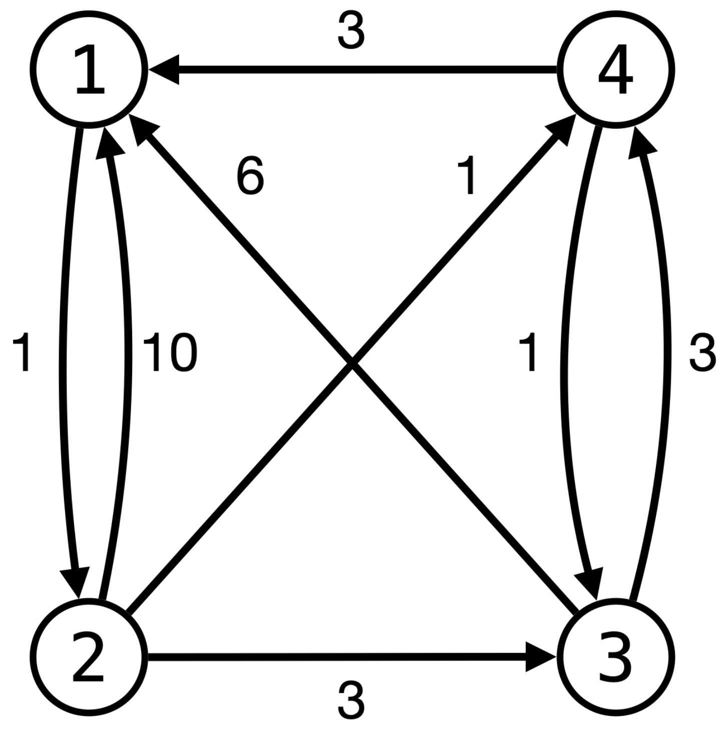
$$d^{(k)}(u, v) = \min\left(d^{(k-1)}(u, v),\ d^{(k-1)}(u, k) + d^{(k-1)}(k, v)\right)$$

Final Output: $d^{(n)}(u, v)$ = shortest distance from u to v.

Example Walkthrough

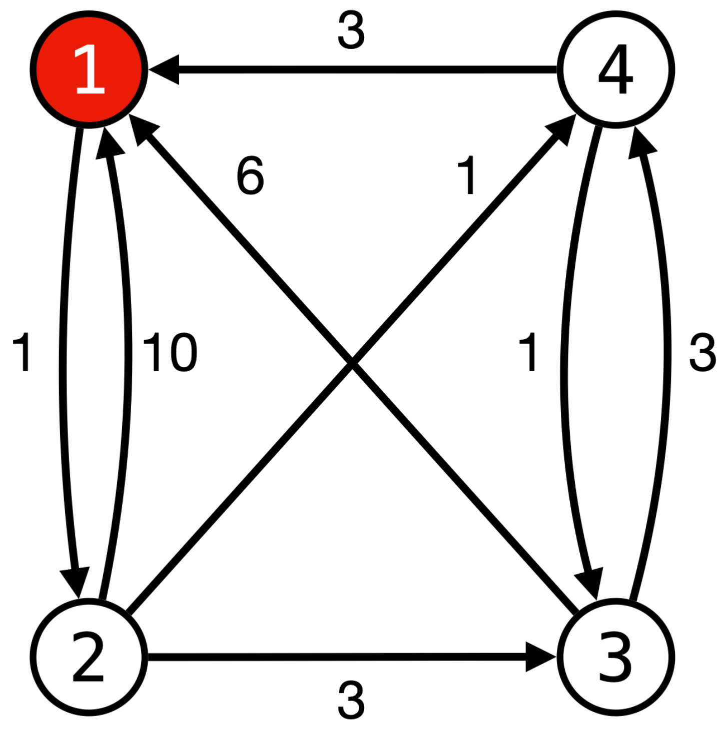


Example Walkthrough: $d^{(0)}$



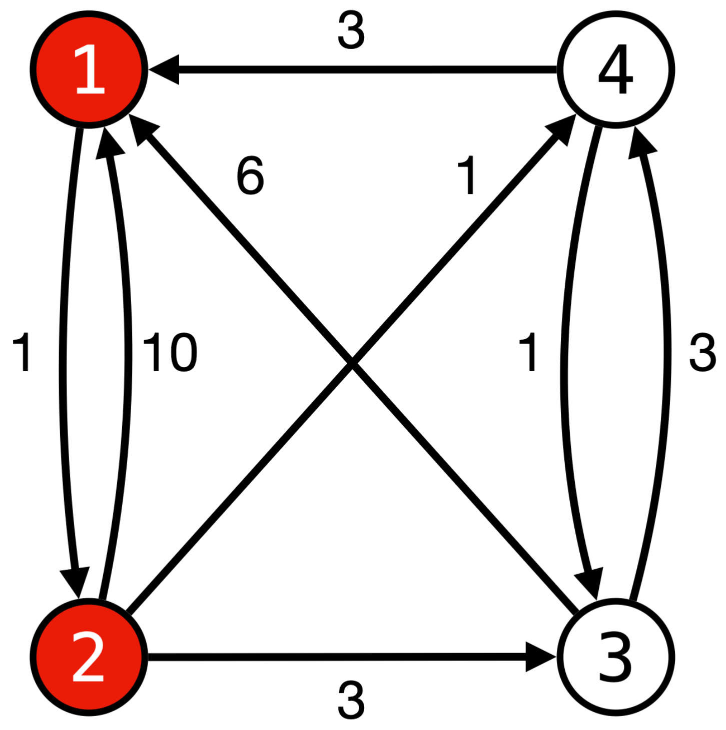
$d^{(0)}$	1	2	3	4
1	0	1	∞	∞
2	10	0	3	1
3	6	∞	0	3
4	3	∞	1	0

Example Walkthrough: $d^{(1)}$



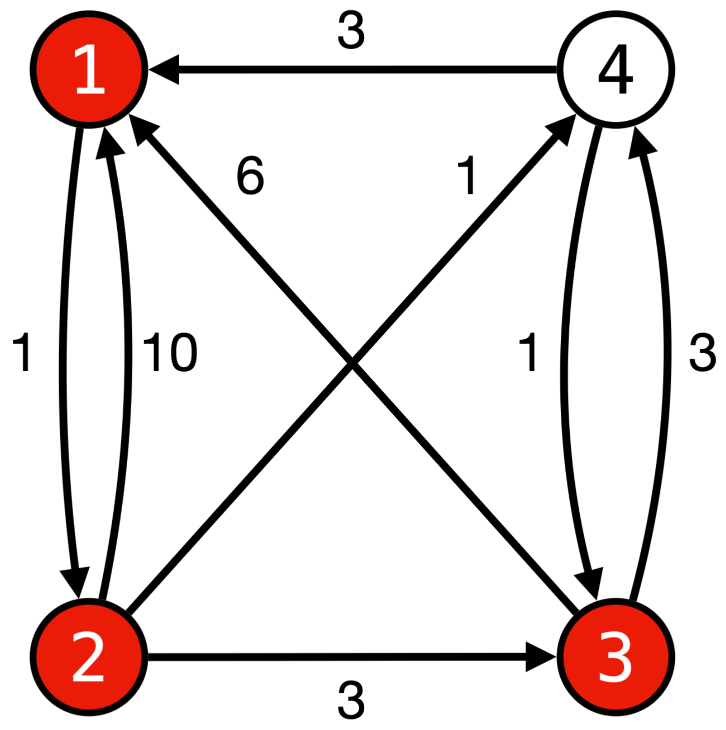
$d^{(1)}$	1	2	3	4
1	0	1	∞	∞
2	10	0	3	1
3	6	7	0	3
4	3	4	1	0

Example Walkthrough: $d^{(2)}$



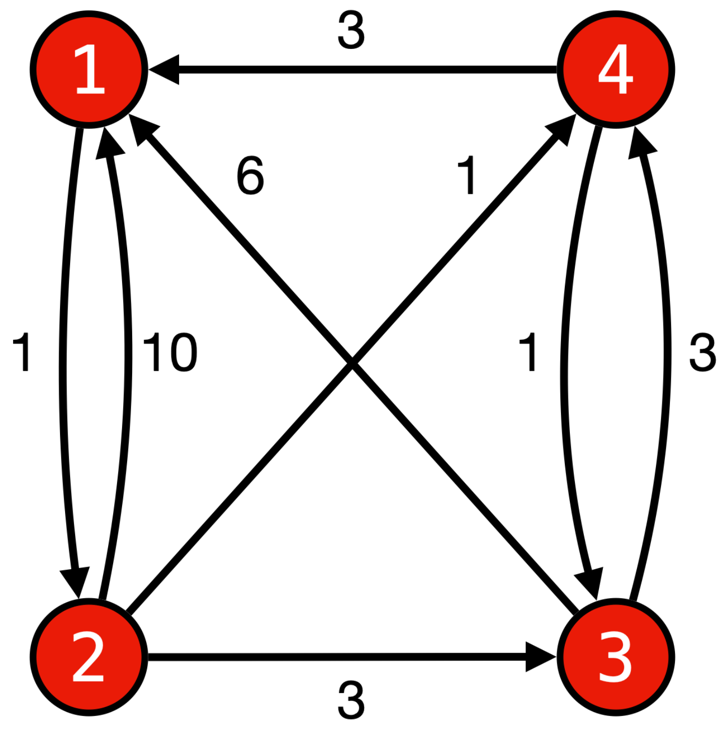
$d^{(2)}$	1	2	3	4
1	0	1	4	2
2	10	0	3	1
3	6	7	0	3
4	3	4	1	0

Example Walkthrough: $d^{(3)}$



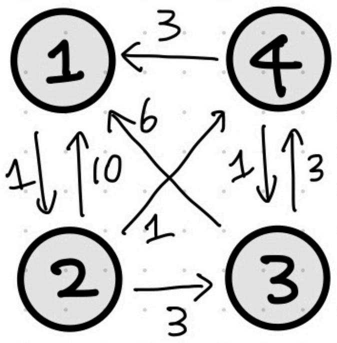
$d^{(3)}$	1	2	3	4
1	0	1	4	2
2	9	0	3	1
3	6	7	0	3
4	3	4	1	0

Example Walkthrough: $d^{(4)}$



$d^{(4)}$	1	2	3	4
1	0	1	3	2
2	4	0	2	1
3	6	7	0	3
4	3	4	1	0

Example Walkthrough: Summary



$d^{(0)}$

	1	2	3	4
1	0	1	∞	∞
2	10	0	3	1
3	6	∞	0	3
4	3	∞	1	0

$d^{(1)}$

	1	2	3	4
1	0	1	∞	∞
2	10	0	3	1
3	6	**7**	0	3
4	3	**4**	1	0

$d^{(0)}(u,v) / d^{(0)}(u,1)+d^{(0)}(1,v)$

3→2 vs 3→1 + 1→2
4→2 vs 4→1 + 1→2

$d^{(2)}$

	1	2	3	4
1	0	1	**4**	**2**
2	10	0	3	1
3	6	7	0	3
4	3	4	1	0

$d^{(1)}(u,v) / d^{(1)}(u,2)+d^{(1)}(2,v)$

1→3 vs 1→2 + 2→3
1→4 vs 1→2 + 2→4

$d^{(3)}$

	1	2	3	4
1	0	1	4	2
2	**9**	0	3	1
3	6	7	0	3
4	3	4	1	0

$d^{(2)}(u,v) / d^{(2)}(u,3)+d^{(2)}(3,v)$

2→1 vs 2→3 + 3→1

$d^{(4)}$

	1	2	3	4
1	0	1	**3**	2
2	**4**	0	**2**	1
3	6	7	0	3
4	3	4	1	0

$d^{(3)}(u,v) / d^{(3)}(u,4)+d^{(3)}(4,v)$

1→3 vs 1→4 + 4→3
2→1 vs 2→4 + 4→1
2→3 vs 2→4 + 4→3

Correctness of Floyd–Warshall (Induction Proof)

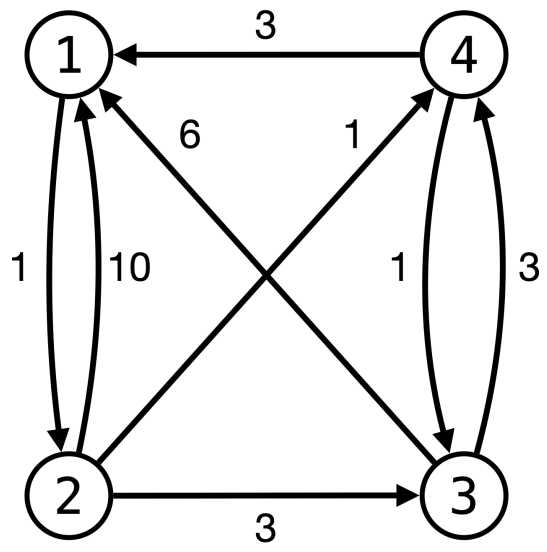
Inductive Claim: For every $k = 0, 1, \ldots, n$ and all u, v, the table entry $d^{(k)}(u, v)$ equals the shortest path distance from u to v whose **intermediate vertices** (if any) are drawn only from $\{1, \ldots, k\}$.

Thus, after n iterations:

$$d^{(n)}(u, v) = d(u, v)$$

Base Case ($k=0$):

- Allowed intermediates: none
- Paths are either the $u=v$ path (cost 0) or **single edge** (u,v) (cost $w(u,v)$), else ∞.
- This is exactly $d^{(0)}$ by initialization. ✅



$d^{(0)}$	1	2	3	4
1	0	1	∞	∞
2	10	0	3	1
3	6	∞	0	3
4	3	∞	1	0

Inductive Step ($k - 1 \to k$):

Inductive Hypothesis: $d^{(k-1)}(u, v)$ equals the shortest path distance from u to v whose **intermediate vertices** are drawn only from $\{1, \dots, k-1\}$.

Consider any shortest $u \to v$ path P whose intermediates lie in $\{1, \dots, k\}$.

Case 1️⃣. P does not pass through k

Then P's intermediates lie in $\{1, \dots, k-1\}$, so by inductive hypothesis,
$d^{(k)}(u, v) = d^{(k-1)}(u, v) = \text{cost}(P)$.

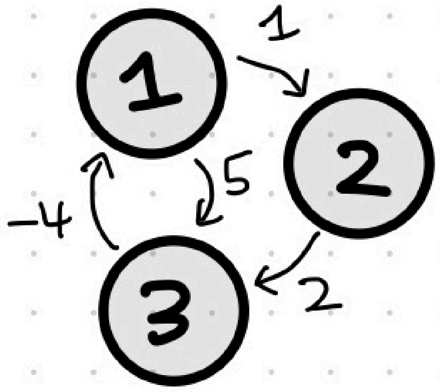
Case 2️⃣. P passes through k:
Let $P = (u \to k) \circ (k \to v)$. Each subpath's intermediates are in $\{1, \dots, k-1\}$, so by inductive hypothesis, $d^{(k-1)}(u, k)$ and $d^{(k-1)}(k, v)$ equal their true costs, hence

$$d^{(k)}(u, v) = d^{(k-1)}(u, k) + d^{(k-1)}(k, v) = \text{cost}(P).$$

Handling Negative Cycles

- Floyd–Warshall can also **detect negative cycles**.
- Check the diagonal:
  - If $d^{(n)}(u, u) < 0$ for some u, then there is a negative cycle reachable from u.
- Why?
  - If there is a simple path P from u to u of negative weight (i.e., a negative cycle containing u), then $d^{(n)}(u, u)$ will be at most its weight, and hence, will be negative.

Example: Nagative Cycle Detection



$$
d^{(0)} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 5 \\ 2 & \infty & 0 & 2 \\ 3 & -4 & \infty & 0 \end{array}
\qquad
d^{(1)} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 5 \\ 2 & \infty & 0 & 2 \\ 3 & -4 & \mathbf{-3} & 0 \end{array}
$$

$$
d^{(2)} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & \mathbf{3} \\ 2 & \infty & 0 & 2 \\ 3 & -4 & -3 & \mathbf{-1} \end{array}
\qquad
d^{(3)} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ 2 & \mathbf{-2} & \mathbf{-1} & \mathbf{1} \\ 3 & \mathbf{-5} & \mathbf{-4} & \mathbf{-2} \end{array}
$$

- There is a negative cycle reachable from all vertices!

Runtime and Space Complexity

- **Runtime**: Filling an n^2 table across n iterations, i.e., $O(n^3)$
- **Space**: $O(n^2)$ (store distance matrix).
  - Note: can optimize to use only 2 layers (current + previous k).

👉 Simple to implement, practical for dense graphs.

APSP: Dijkstra vs. Floyd–Warshall

Algorithm	Runtime	Works with Negative Weights?	When to Use
Floyd-Warshall	$O(n^3)$	✅ Yes	Good for **dense graphs**, simple to implement
Dijkstra $\times n$	$O(nm + n^2 \log n)$	❌ No	Best for **sparse graphs**, more efficient

- **Sparse graphs** ($m = O(n)$): Dijkstra n times $\approx O(n^2 \log n)$ → **much faster**
- **Dense graphs** ($m = \Theta(n^2)$): Both $\approx O(n^3)$ → Floyd–Warshall may be simpler and practical

Summary

- **APSP Problem**: compute shortest paths between every pair of vertices.
- **Naïve approach**: run SSSP n times.
- **Floyd–Warshall Algorithm**:
    - Elegant **DP formulation**
    - Works with negative edge weights
    - Detects negative cycles
    - Runtime: $O(n^3)$

Announcement

- The bonus assignment details will be uploaded to LMS today.
- The next quizzes will be on November 18 (Tuesday) and December 4 (Thursday).

Appendix: Why is it called Dynamic Programming?

Historical Context

- The term was coined by **Richard Bellman** in the 1950s.
- At the time, he was working at the RAND Corporation on mathematical optimization problems for the U.S. Air Force.
- The U.S. Secretary of Defense (Charles Wilson) had a strong dislike of the words **“research”** and **“mathematics.”**
    - If Bellman called his work “mathematical research,” it would not get funded. 💸😰
    - So he needed a safer, more appealing name.

Why “Programming”?

- “Programming” here means a **plan** or **schedule of actions** (like in “linear programming” or “planning a program of events”), not computer programming.
- DP is about constructing an optimal plan by combining solutions of smaller subproblems.

Why “Dynamic”?

- “Dynamic” emphasized that the method involved **multistage decision-making** and **updating over time**.
- Bellman also noted that “dynamic” has a positive, action-oriented connotation — it’s hard to use it negatively.
- This made it **politically safe** and even attractive to sponsors.

So… It’s called Dynamic Programming because:

- **Programming** = planning/optimization, not coding.
- **Dynamic** = problems solved stage by stage, with decisions unfolding over time.
- Bellman chose the term partly for **political reasons** so his work could be supported, while secretly he was doing deep mathematics 😛.

Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/
- Example: https://favtutor.com/blogs/floyd-warshall-algorithm