Lecture 13 - Strongly Connected Components

Fall 2025, Korea University

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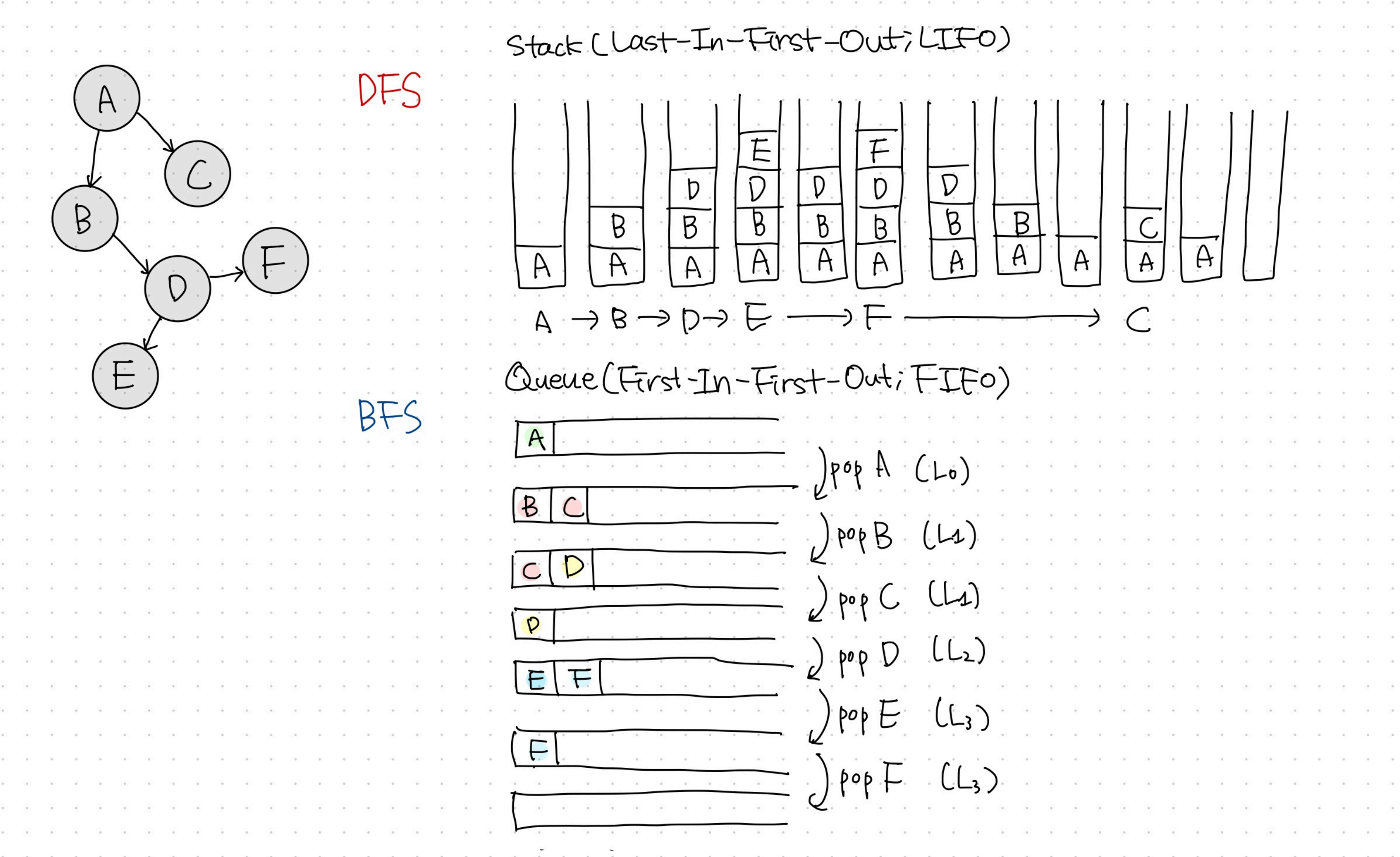
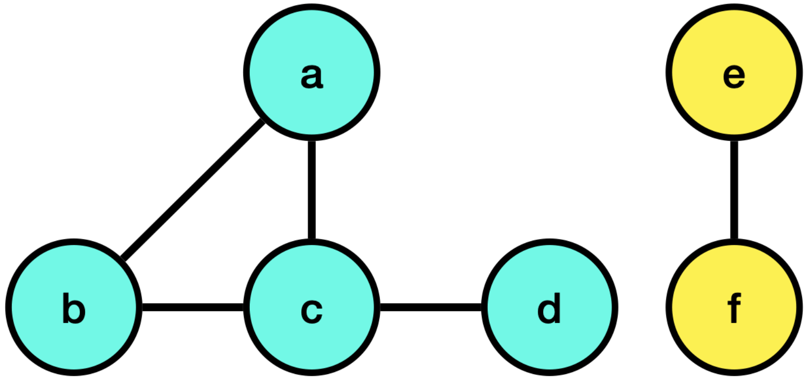
Recap: Graphs, DFS, and BFS



Connected Components (Undirected Graphs)

- A **connected component** of an undirected graph $G = (V, E)$ is:
  A **maximal** set $S \subseteq V$ such that
  $\forall u, v \in S$, **there exists a path from** u **to** v.
  - Maximal: If S' is connected and $S \subset S'$, then $S = S'$

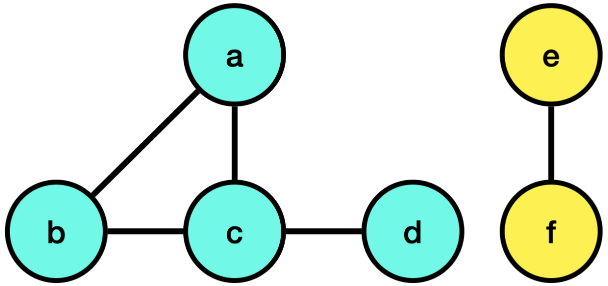
Example



- $\{a, b, c\}$ is connected but expandable → ❌ not maximal.
- $\{a, b, c, d\}$ is connected and not expandable → ✅ maximal → connected component.
- $\{e, f\}$ is connected & maximal → ✅ connected component.

Formal Definition

- Define a relation $u \sim v$ if G has a path $u \to v$.
- This is an **equivalence relation** (symmetric, reflexive, transitive).
  i. **Reflexive:** $\forall u \in V, u \sim u$
  ii. **Symmetric:** $u \sim v \implies v \sim u$
  iii. **Transitive:** $u \sim v \wedge v \sim w \implies u \sim w$
- Each **equivalence class** under $\sim$ is a **connected component** — that is, a maximal set of vertices all connected to each other.



Finding Connected Components in Undirected Graphs

Algorithm

- Pick a vertex, run **BFS/DFS** → all reached vertices = one component.
- Repeat until all vertices are visited.

Runtime:

- Each vertex/edge is explored once.
- Total = $O(|V| + |E|)$.

Directed Graphs: Connectivity

- Undirected connectivity is straightforward.
- In **directed graphs**, Connectivity is **no longer symmetric**.
- We distinguish two notions:

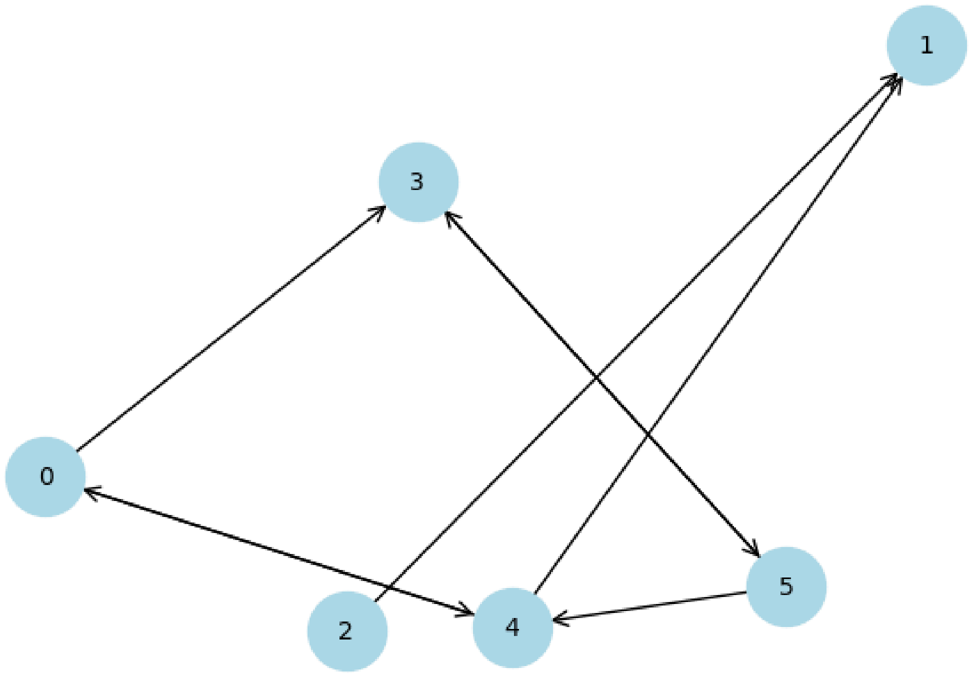
Weak Connectivity

- Ignore edge directions → compute undirected components.

Strong Connectivity

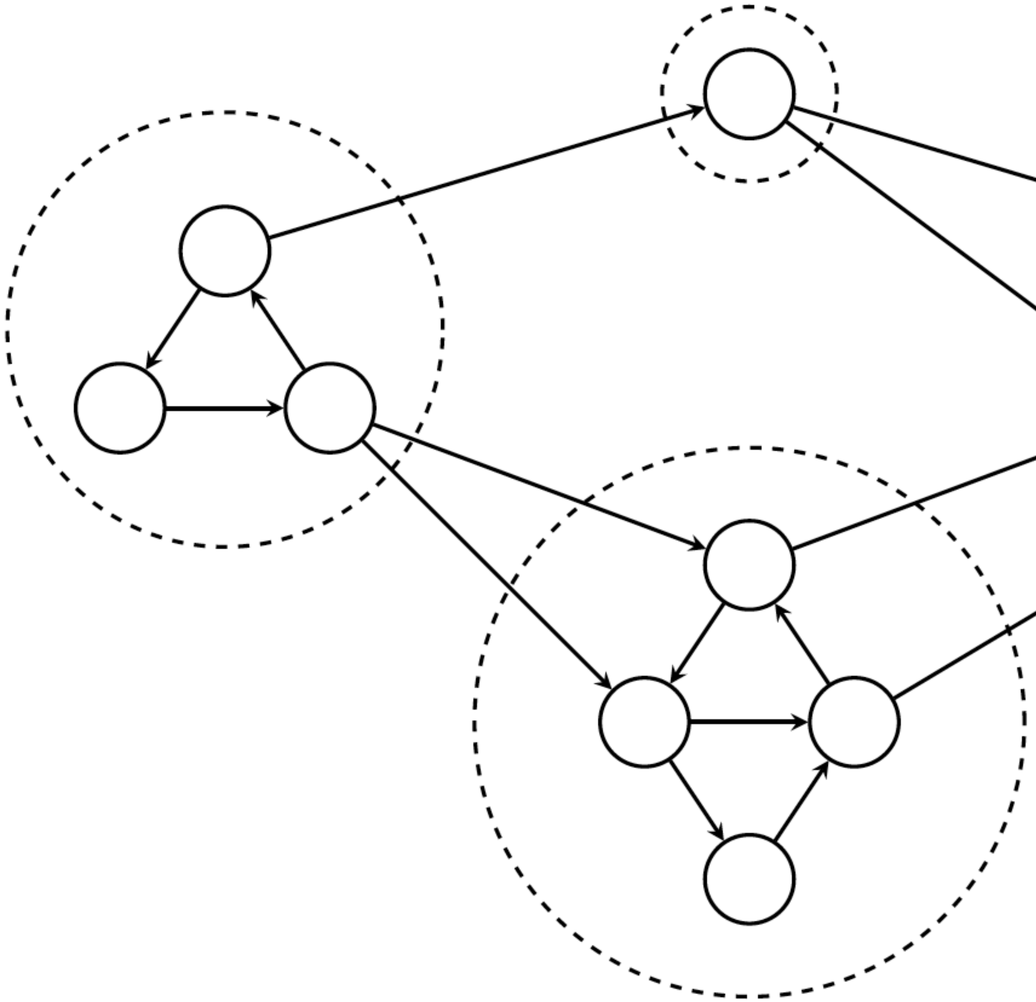
- u and v are in the same **Strongly Connected Component (SCC)** if: $u \to v$ AND $v \to u$ (*mutual reachability*).

Example from Assignment #2



- $\{0, 3, 4, 5\}$ is a strongly connected component.
- $\{0, 1, 2, 3, 4, 5\}$ is a weakly connected component.

Example: Strongly Connected Components of a Directed Graph

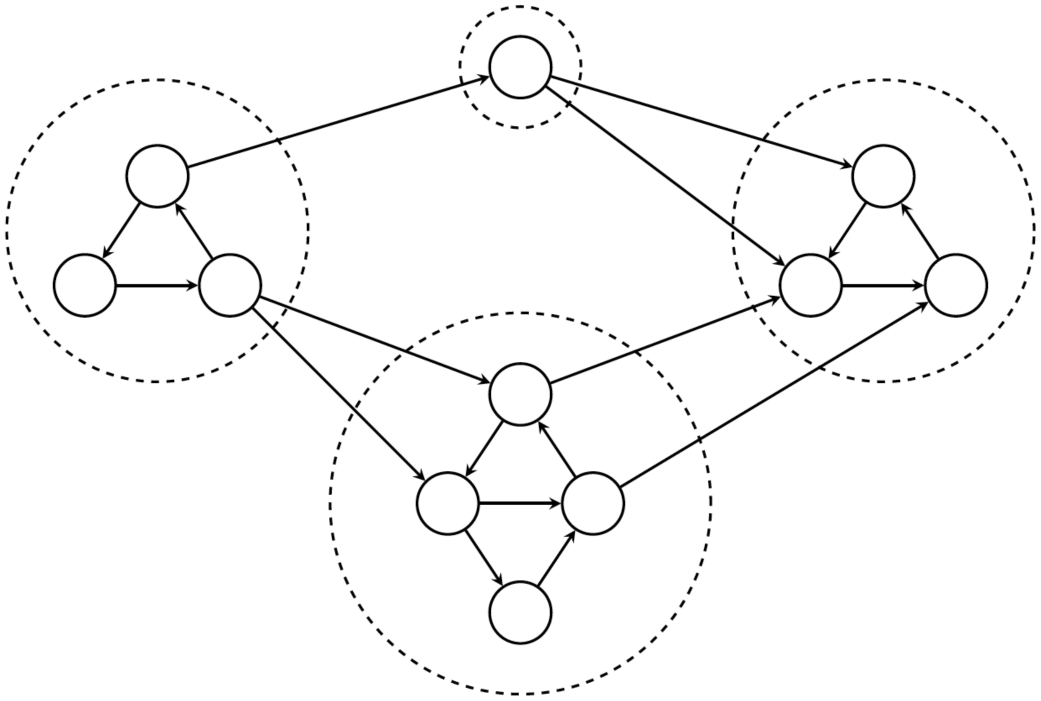


Strongly Connected Components (SCCs) - Today's Topic ✨

Definition:

A **strongly connected component (SCC)** is a maximal set $S \subseteq V$ such that every vertex has a path to every other vertex in S.

- SCCs partition the vertex set: Each vertex belongs to exactly one SCC.
- Within SCC: cycles possible.
- Between SCCs: no cycles — forms a **DAG** (Directed Acylic Graph).

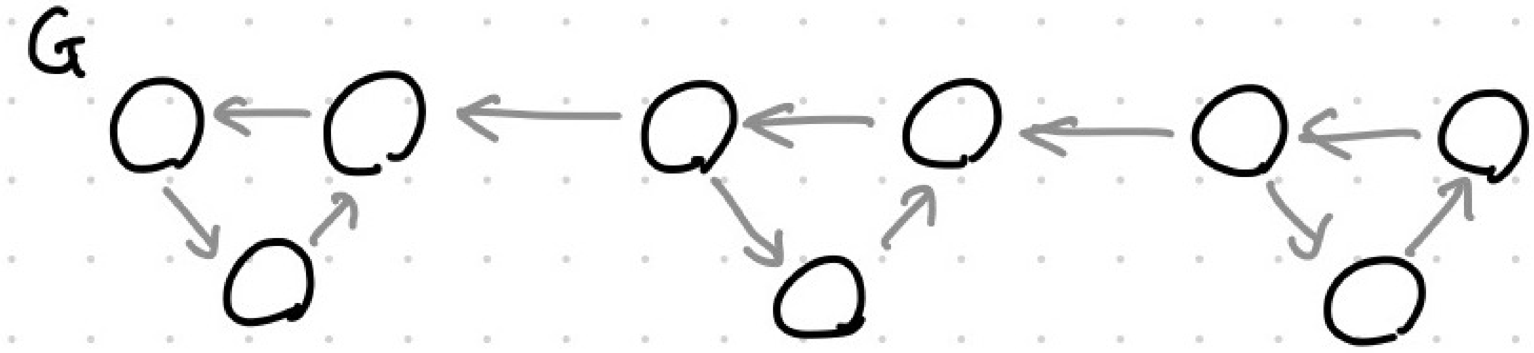


Algorithm to Find SCCs

Kosaraju's Algorithm: A linear time algorithm to find the strongly connected components of a directed graph.

Step 1. Reverse the graph $G \to G^{rev}$.
Step 2. Run DFS on G^{rev} (any order), compute finishing times $f(v)$.
Step 3. Run DFS on G, processing vertices in **decreasing order of** $f(v)$ and assigning a "leader" to each vertex (i.e., the source vertex that the DFS started from).

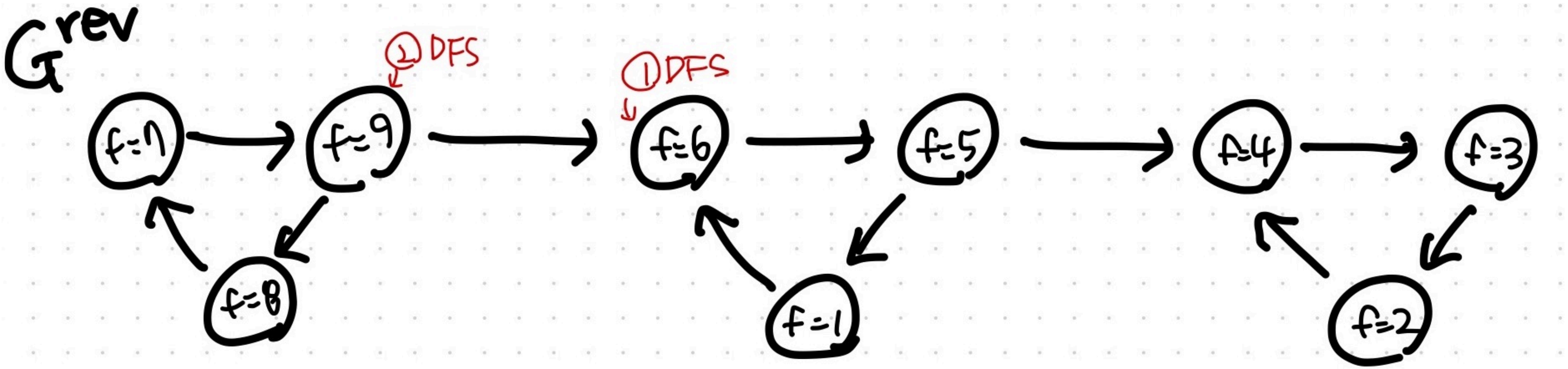
Step 1. Reverse the graph $G \rightarrow G^{rev}$.



G^{rev}

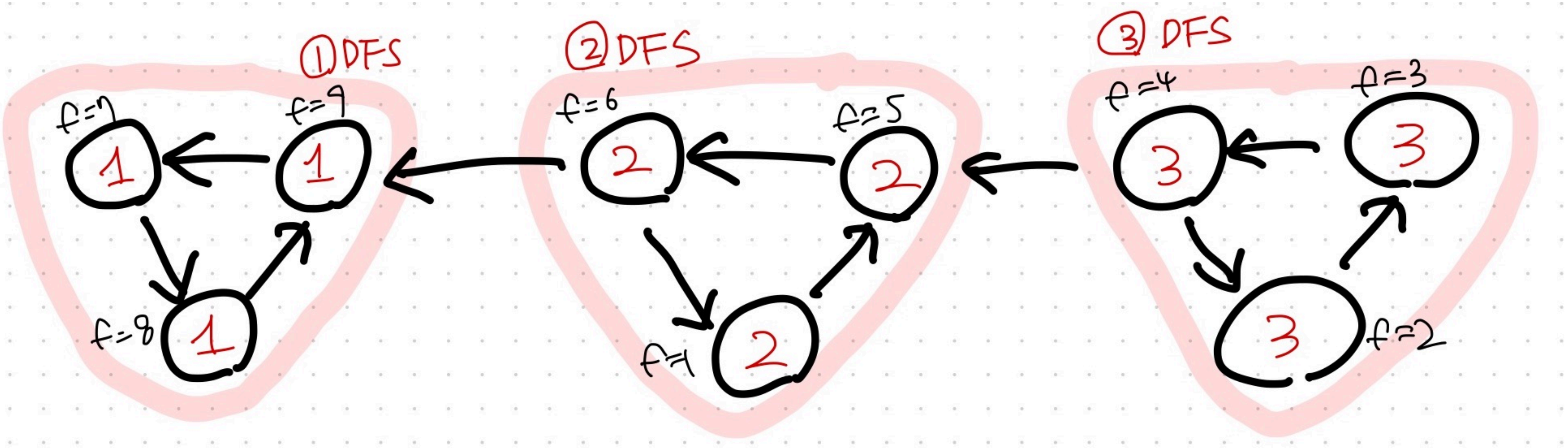
Step 2. Run DFS on G^{rev} (any order), compute finishing times $f(v)$.

```
Algorithm DFS(s, t):
    foreach v in N(s) do
        if vis(v) == false then
            t ← DFS(v, t)
            t ← t + 1
    f(s) ← t
    return f(s)
```



> Different DFS choices → different $f(v)$ assignments, but correctness still holds.

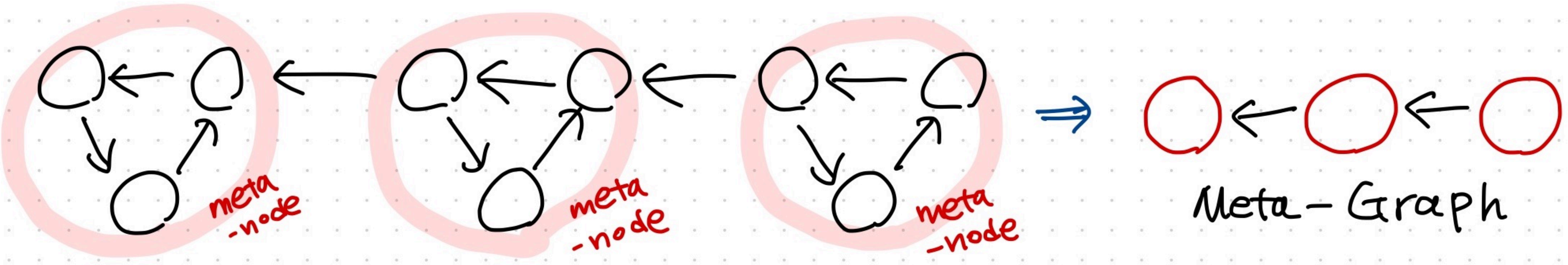
Step 3. Run DFS on G, processing vertices in **decreasing order of** $f(v)$ and assigning a "leader" to each vertex (i.e., the source vertex that the DFS started from).



- Each DFS call discovers one SCC.

The Meta-Graph of SCCs

- Contract each SCC into a "meta-node."
- Add edges between SCCs if original graph has them.



Fact: The SCC meta-graph is always a **Directed Acyclic Graph (DAG)**.

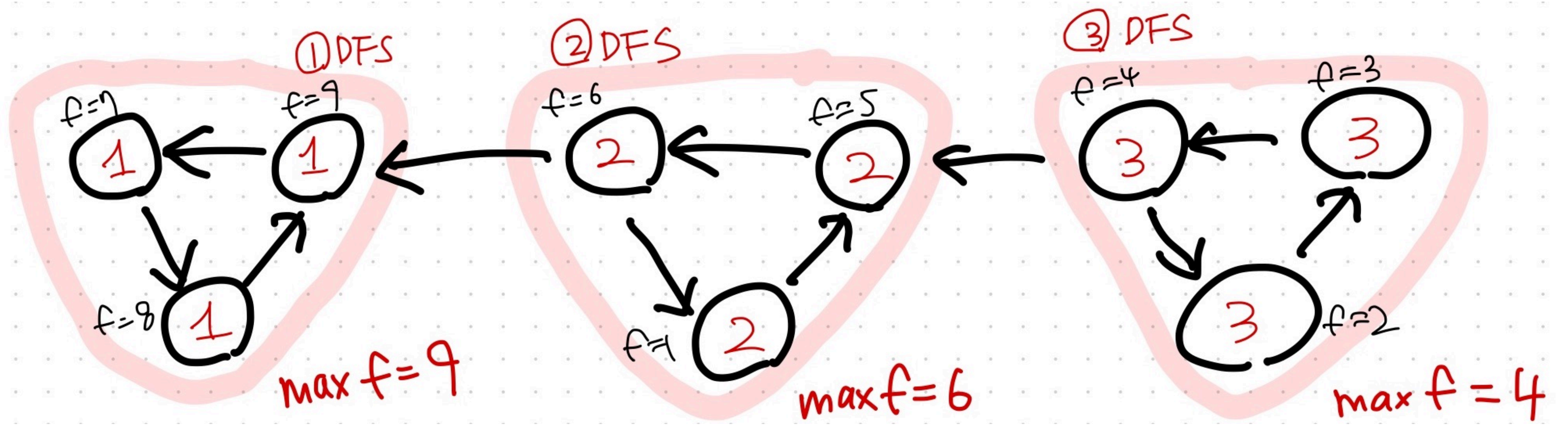
- Cycles within SCCs collapse into single nodes.
- No cycle can exist across SCCs.

The Key Lemma

Key Lemma

If there is an edge $C_1 \to C_2$ (between SCCs), then $\max_{v \in C_1} f(v) < \max_{v \in C_2} f(v)$.

Example



Proof

> If there is an edge $C_1 \to C_2$ (between SCCs), then $\max_{v \in C_1} f(v) < \max_{v \in C_2} f(v)$

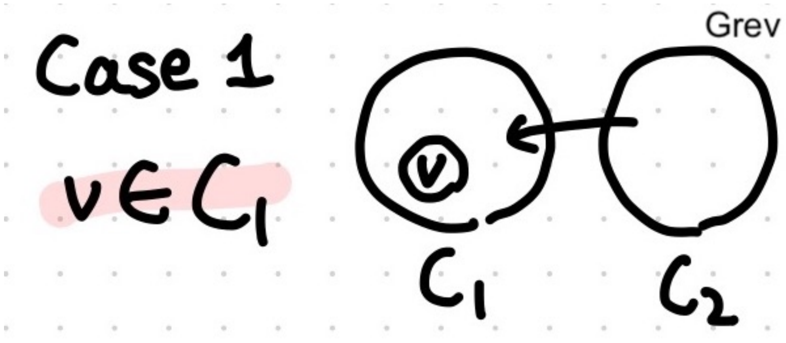
Let v denote the first vertex of $C_1 \cup C_2$ visited by DFS.

Then, there are two cases:

1. Case 1: $v \in C_1$
2. Case 2: $v \in C_2$

> If there is an edge $C_1 \to C_2$ (between SCCs), then $\max_{v \in C_1} f(v) < \max_{v \in C_2} f(v)$

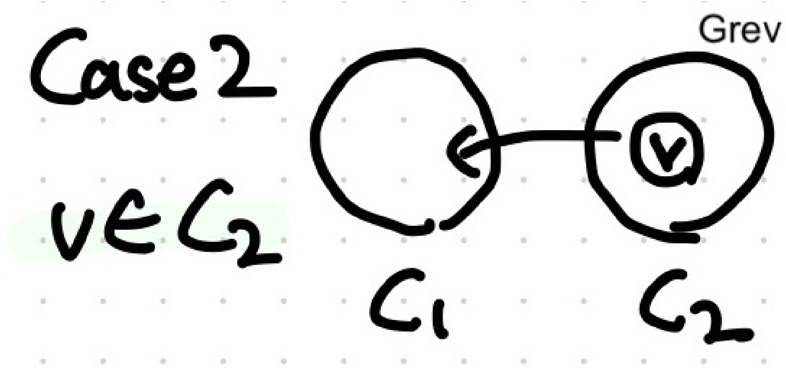
1. Case 1: $v \in C_1$



DFS will finish exploring all vertices in C_1 without reaching any vertices in C_2.

$$\forall v_1 \in C_1, \forall v_2 \in C_2, f(v_1) < f(v_2)$$
$$\to \quad \max_{v \in C_1} f(v) < \max_{v \in C_2} f(v)$$

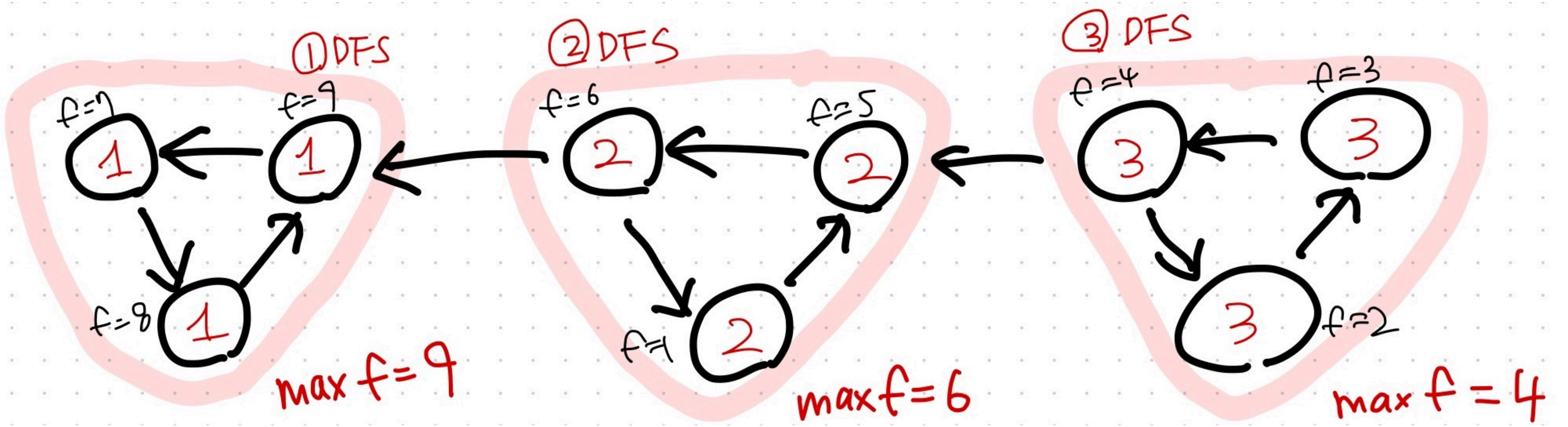
2. Case 2: $v \in C_2$



DFS will finish exploring all the vertices in C_1 and C_2 before ending (at v). Therefore, the fishing time of v is the largest among the vertices in $C_1 \cup C_2$.

Key Lemma

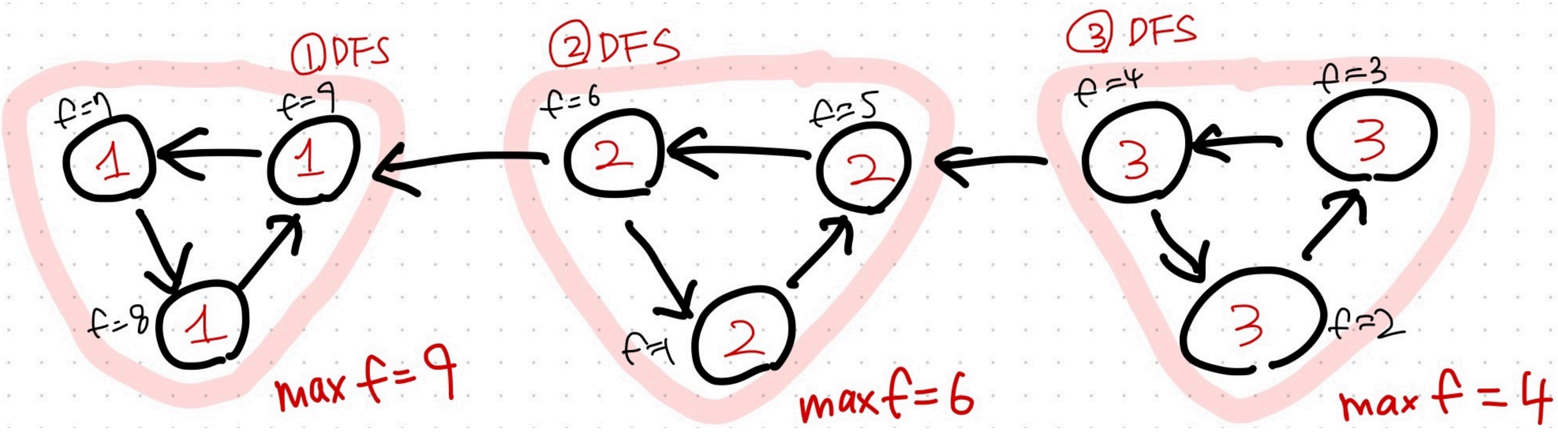
If there is an edge $C_1 \rightarrow C_2$ (between SCCs), then $\max_{v \in C_1} f(v) < \max_{v \in C_2} f(v)$.



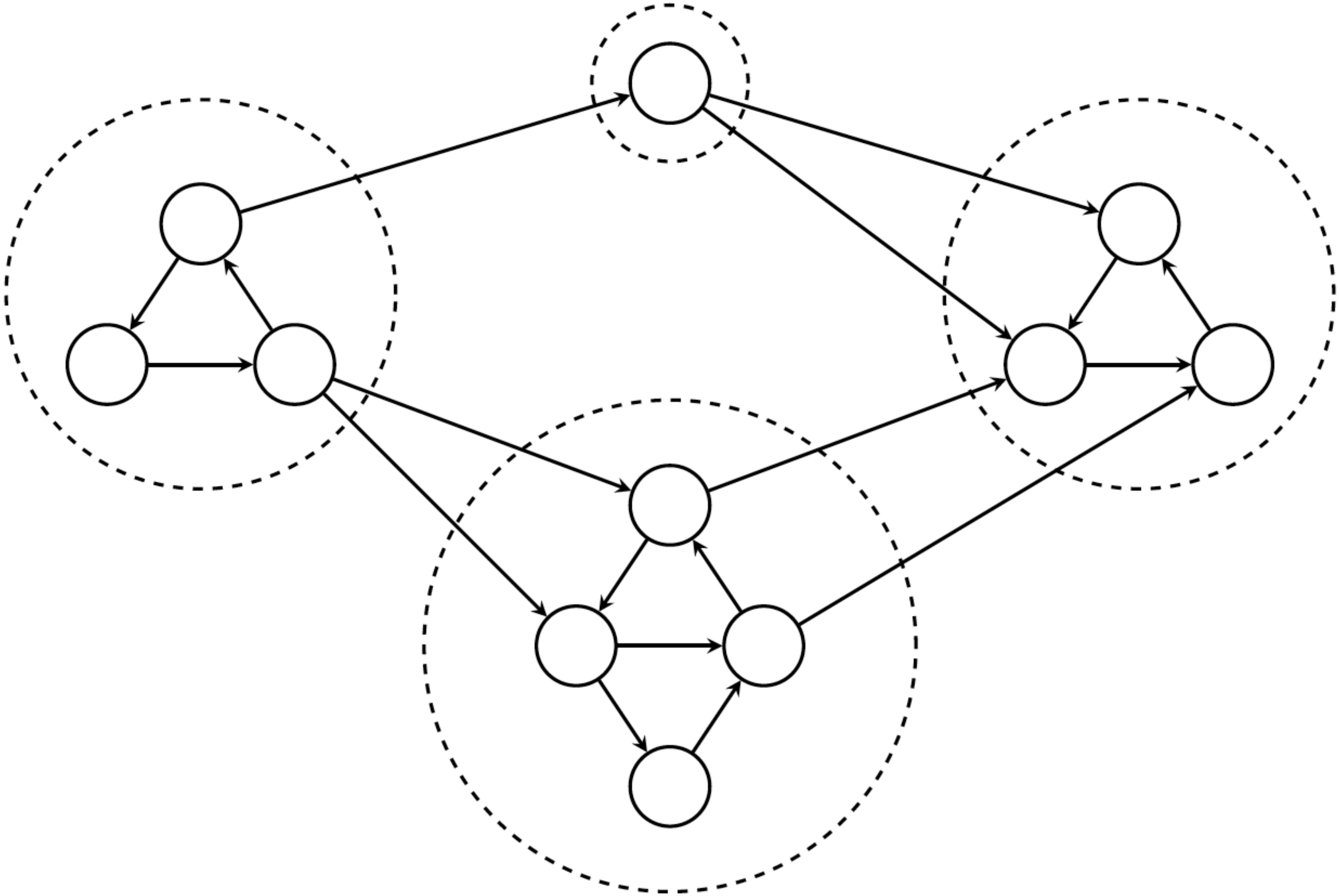
Consequence

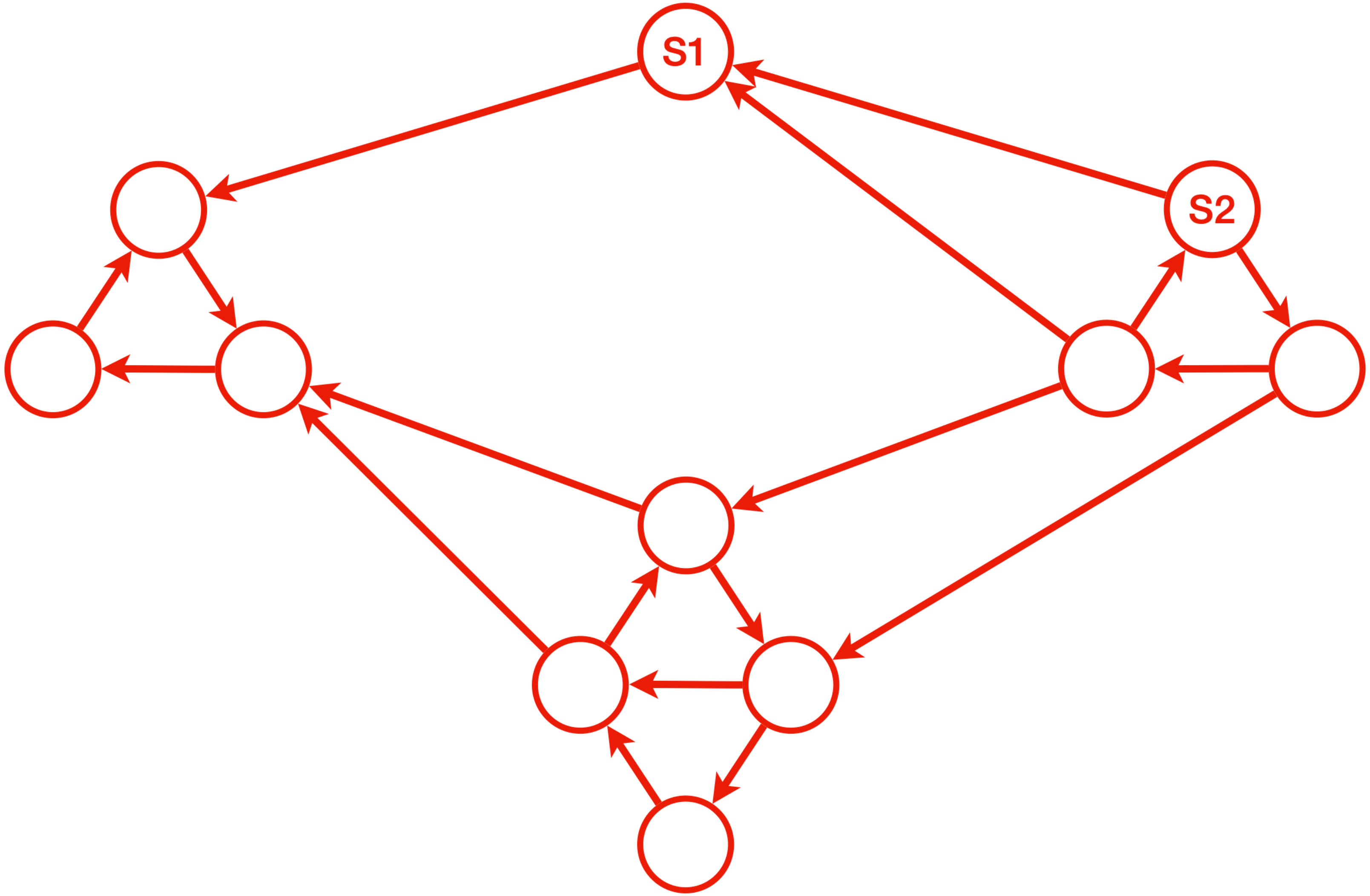
Processing vertices in **decreasing order of finishing times** ensures that each DFS call discovers **exactly one SCC**.

Correctness of Kosaraju's Algorithm (Intuition: "Onion Layers" 🧅 !)



- Intuitively, DFS peels off the graph **one SCC at a time**, like removing successive **layers of an onion**.
- All outgoing edges from the current SCC lead to SCCs that have **already been explored** (e.g., when exploring $C_3 \rightarrow C_2$, C_2 is guaranteed to be finished).

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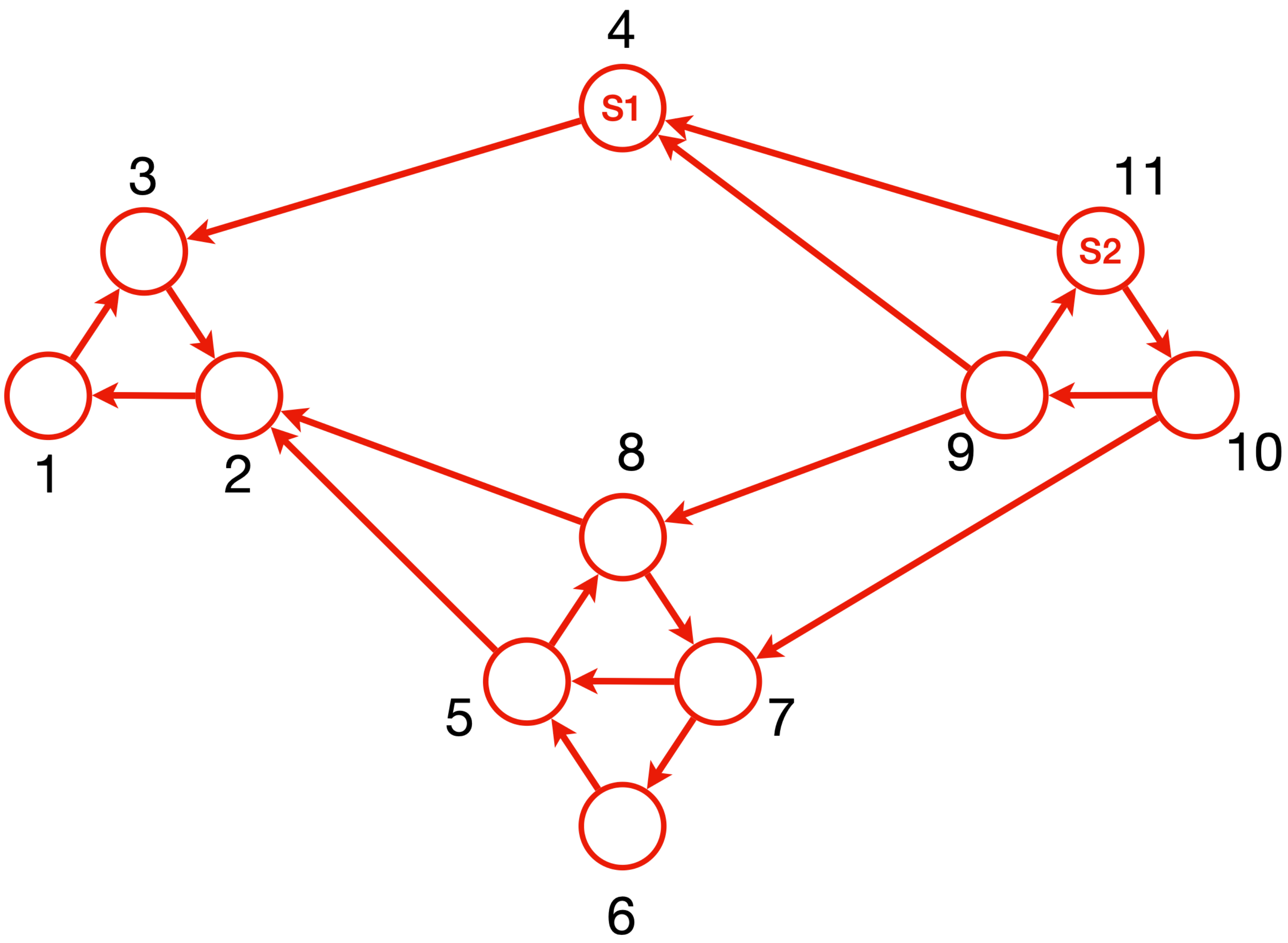
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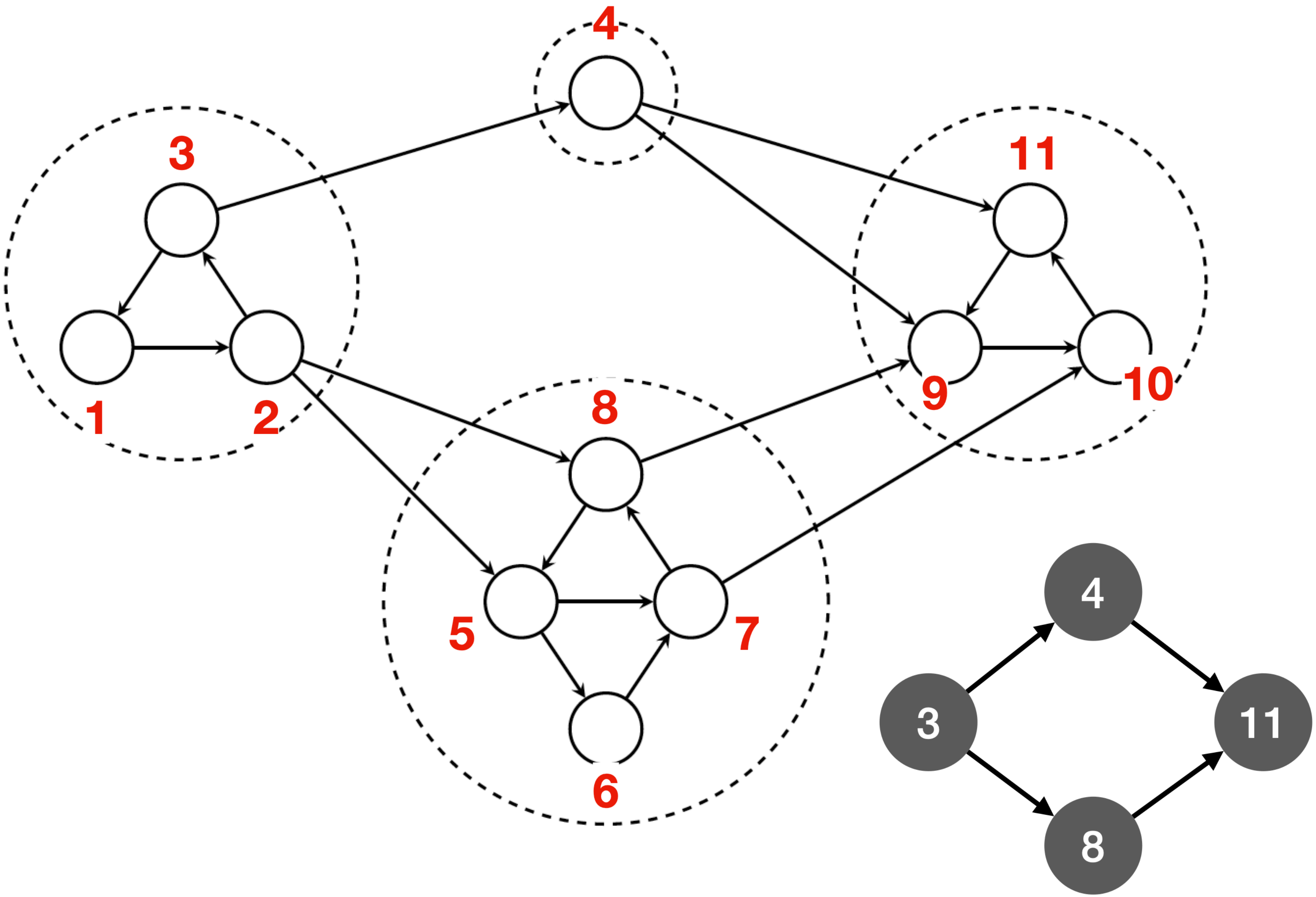
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Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/