Lecture 4 - Asymptotic Notation (Big-O, Big-Theta, Big-Omega)

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Course Outline (Before Midterm)

- Part 1: Basics
  - ~~Divide and Conquer (w/ Integer Multiplication)~~ ✅
  - ~~Basic Sorting Algorithms (Insertion Sort & Merge Sort)~~ ✅
  - **Asymptotic Analysis (Big-O, Big-Theta, Big-Omega)** 👈
  - Solving Recurrences Using Master Method
- Part 2: Advanced Selection and Sorting
  - Median and Selection Algorithm
  - Solving Recurrences Using Substitution Method
  - Quick Sort, Counting Sort, Radix Sort
- Part 3: Data Structures
  - Heaps, Binary Search Trees, Balanced BSTs

👀 Review: MergeSort Implementation

```python
def merge_sort(A):
    n = len(A)

    if n <= 1:
        return A

    L = merge_sort(A[:n//2]) # left half
    R = merge_sort(A[n//2:]) # right half

    return merge(L, R) # key subroutine! ⭐
```

✏️ Correction: Running Time of ~~`merge`~~ -> `merge_sort`

Suppose the input array A has length m.

1. Base Case Check: **2 operations**
    - Retrieving length: 1 operation
    - Comparing if $m = 1$: 1 operation
2. Recursive Call Setup: $m + 2$ **operations**
    - Copying elements into `L` and `R`: m operations
    - Storing into variables `L` and `R`: 2 operations
3. Merging Two Halves: the merge step takes $3 + 3m \leq 6m$ **operations**
    - Why? 3 assignments + m scans + m appending + m cursor increase

- Total number of operations $\leq 2 + (m + 2) + 6m \leq 11m$

👀 Review: Running Time of `merge_sort`

```
level 0:                          n
                                /   \
level 1:                    n/2      n/2
                           /   \    /   \
level 2:                n/4  n/4  n/4  n/4
                         .               .
                         .               .
                   |  |  |  |       |  |  |  |
level log_2(n):    1  1  1  1  ...  1  1  1  1
```

- At each level $j = 0, 1, \ldots, \ldots, \log_2 n$, there are 2^j subproblems, each of size $n/2^j$.
- Work at each level j = (#subproblems * Work per subproblem) $\leq 2^j \cdot 11(\frac{n}{2^j}) = 11n$

Total Work = (Work per level * #levels) $\leq 11n \cdot (1 + \log_2(n)) = 11n \log_2 n + 11n$. ✅

We've Roughly Analyzed the Running Time So Far...

- O(n^2) algorithms: SelectionSort, BubbleSort, InsertionSort
- O($n \log n$) algorithm: MergeSort (divide-and-conquer)

1. The algorithm never "looks at" the input.
   - MergeSort runs through all $\log_2 n + 1$ levels of recursion for an input $[1, 2, 3, 4]$.
2. In our analysis, we've given a very loose upper bound on the time required of `merge` and dropped some constant factors and lower-order terms.

> Is this a problem? Have we been too sloppy? 🧐

We'll argue that these are *features*, not bugs, in the design and analysis of the algorithm.

Today's Objectives

> Big Question!! 🤓
>
> **What do we really mean by a “fast” algorithm?**

- Adopt three foundational principles for reasoning about performance:
    1. **Worst-Case Analysis** – focus on the most demanding inputs
    2. **Big-Picture Analysis** – ignore low-level machine details
    3. **Asymptotic Analysis** – emphasize scalability for large inputs
- Learn **Asymptotic Notation**: $O(n), \Omega(n), \Theta(n)$.

Principle #1: Worst-Case Analysis

- We want our **performance guarantees to hold for *every* possible input**.
- Think of it as a game against an **adversary**:
  - You choose the algorithm and claim it runs in time $T(n)$.
  - The adversary picks the **worst-case input** of size n.
  - You win if the algorithm still runs in $\leq T(n)$ time.
- Why Worst-Case?
  - Provides a **strong, robust guarantee**
  - No assumptions made about input distribution
  - Ensures the algorithm behaves well **even in the most challenging cases**

Principle #2: Big-Picture Analysis

- We won't pay much attention to **constants** and **lower-order terms**.
  - e.g., treat $5n$, $n + 100$, and $0.01n$ all as $O(n)$
  - e.g., treat $11n \log_2 n + 11n$ as $O(n \log_2 n)$, or just $O(n \log n)$ 🧐
- Why?
  - **Simplifies** analysis and comparisons
  - Constants vary with **hardware, compilers, coding style**
  - We lose **very little predictive power** (as we'll soon see)

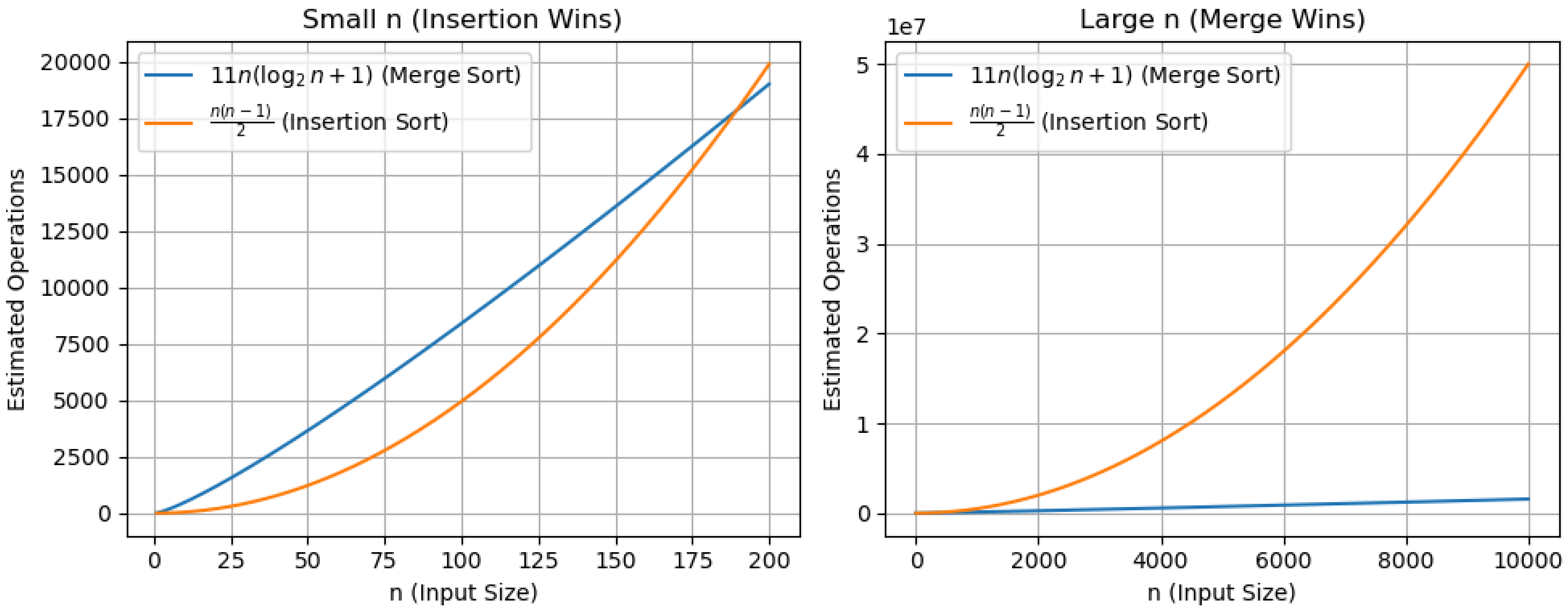
Principle #3: Asymptotic Analysis

- We focus on running time for large input sizes ($n \to \infty$)
- Small inputs can be solved even with brute force.
- However, for large inputs, efficiency **matters** — brute force breaks down.

> Only big problems are interesting!

Principle #3: Asymptotic Analysis - Continued

- e.g., $11n(\log_2 n + 1)$ (MergeSort) is better than $\frac{n(n-1)}{2}$ (InsertionSort)



🚀 What Counts as "Fast"?

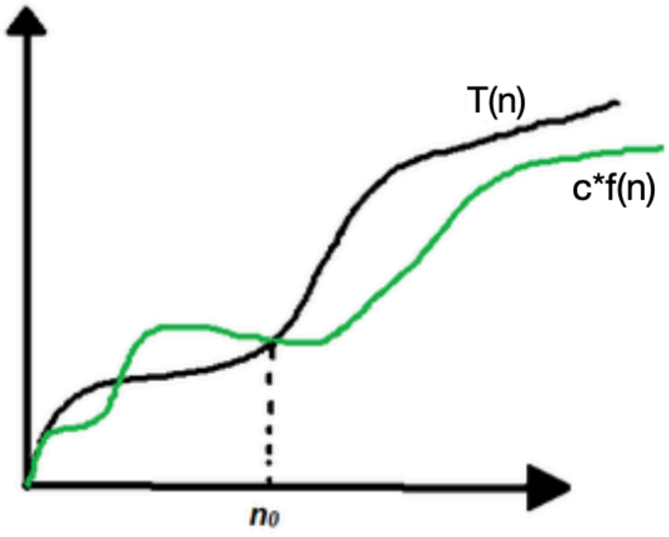
- A "fast" algorithm is one whose running time grows **slowly** with n.
- Ideally: as close to **linear** as possible
- This motivates a **formal system** for comparing algorithm growth rates.
  - 🔧 **Asymptotic Notation** 🔧

Asymptotic Notation: $O(n), \Omega(n), \Theta(n), \ldots$

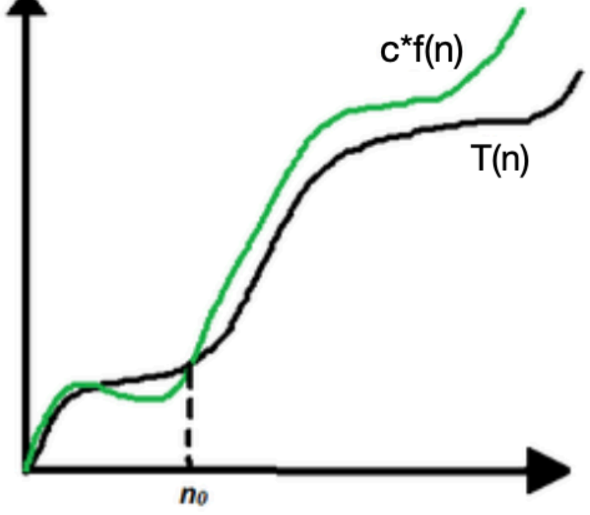
- We use **asymptotic notation** to describe algorithm performance.
  - Each describes a different kind of bound.
- Let $T(n)$ be the running time of an algorithm on input of size n.
- Example: Karatsuba Multiplication (from Lecture 2)

$$T(n) = 3T(\frac{n}{2}) = \ldots = 3^t T(\frac{n}{2^t}) = 3^{\log_2 n} T(1) \approx n^{1.58} T(1)$$

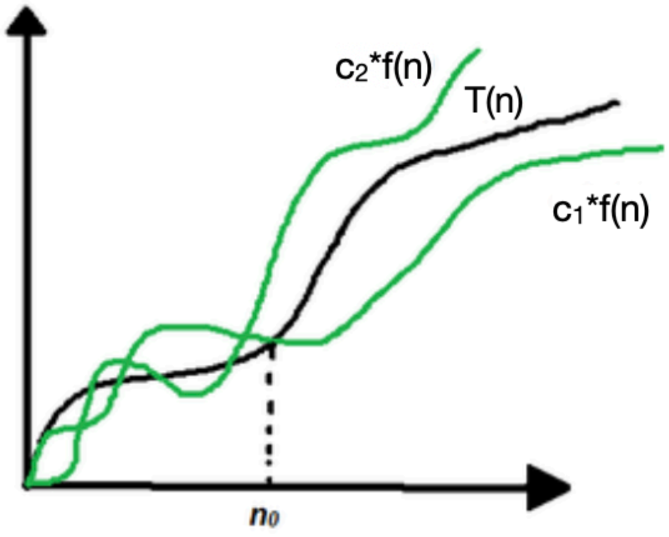
Big-Omega



Big-O



Big-Theta



1. Big-O Notation: $O(f(n))$

- Describes an **upper bound** on the running time
- Used to express the **worst-case** performance
- Interpretation: "$T(n)$ grows *no faster than* $f(n)$ up to constant factors"

Formal Definition:

$$T(n) = O(f(n)) \iff \exists\, c > 0,\ n_0 \text{ such that } \forall n \geq n_0,\ 0 \leq T(n) \leq c \cdot f(n)$$

Big-O Example #1: Karatsuba Multiplication is $O(n^{1.58})$

$$T(n) \approx n^{1.58} T(1) = O(n^{1.58})$$

This is true because:

$$\exists\, c > 0,\ n_0 \text{ such that } \forall n \geq n_0,\ T(n) \leq c \cdot n^{1.58}$$

- For **sufficiently large** n, the algorithm takes **no more than** $c \cdot n^{1.58}$ time
- This gives a **guaranteed upper bound** on the running time

> Karatsuba is **faster than** $O(n^2)$ multiplication.

- Technically, it is also $O(n^2)$, but $O(n^2)$ is **not tight** and gives less precise information.

Big-O Example #2: All degree-k polynomials are $O(n^k)$

Claim: If $T(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0$, then $T(n) = O(n^k)$

Proof:

- Choose: $n_0 = 1$ and $c = |a_k| + |a_{k-1}| + \ldots + |a_1| + |a_0|$
- We want to show: $\forall n \geq 1,\ T(n) \leq c \cdot n^k$ *(Definition of Big-O)*

> $T(n) = O(f(n)) \iff \exists\, c > 0,\ n_0 \text{ such that } \forall n \geq n_0,\ 0 \leq T(n) \leq c \cdot f(n)$

- $\forall n \geq 1$, we have:

$$\begin{aligned} T(n) &= a_k n^k + \ldots + a_1 n + a_0 \\ &\leq |a_k| n^k + \ldots + |a_1| n + |a_0| \\ &\leq (|a_k| + \ldots + |a_1| + |a_0|) \cdot n^k = c \cdot n^k \end{aligned}$$

- Hence, $T(n) = O(n^k)$ ✅

Big-O Example #3: For every $k \geq 1, n^k$ is not $O(n^{k-1})$

Proof (by contradiction):

Suppose, for contradiction, that: $T(n) = n^k = O(n^{k-1})$

Then, by the definition of Big-O, there exist constants $c > 0, n_0 \geq 1$ such that:

$$n^k \leq c \cdot n^{k-1} \quad \forall n \geq n_0$$

Divide both sides by n^{k-1}:

$$n \leq c \quad \forall n \geq n_0$$

But this is **clearly false**, since $n \to \infty$ as n increases. So we have a **contradiction**.
Hence, n^k is not $O(n^{k-1})$. ✅

> This shows that Big-O cannot *underestimate* growth rate.

2. Big-Omega Notation: $\Omega(f(n))$

- Describes a **lower bound** on the running time
- Indicates **best-case** or guaranteed performance
- Interpretation: “T(n) grows *at least as fast as* f(n)”

Formal Definition:

$$T(n) = \Omega(f(n)) \iff \exists\, c > 0,\ n_0 \text{ such that } \forall n \geq n_0,\ 0 \leq c \cdot f(n) \leq T(n)$$

Example:

Karatsuba's runtime, $T(n) \approx n^{1.58} T(1)$, grows **faster than linear**, i.e., $T(n) = \Omega(n)$.

3. Big-Theta Notation: $\Theta(f(n))$

- Describes a **tight bound**: both upper and lower
- Indicates that the algorithm grows at **exactly** the rate of $f(n)$ (up to constants)
- Interpretation: "T(n) grows *as fast as* f(n)"

Formal Definition:

$$T(n) = \Theta(f(n)) \iff \exists\, c_1, c_2 > 0,\ n_0 \text{ such that } \forall n \geq n_0,\ 0 \leq c_1 f(n) \leq T(n) \leq c_2 f(n)$$

Example:

- Karatsuba's runtime grows **asymptotically exactly like** $n^{1.58}$, i.e., $T(n) = \Theta(n^{1.58})$

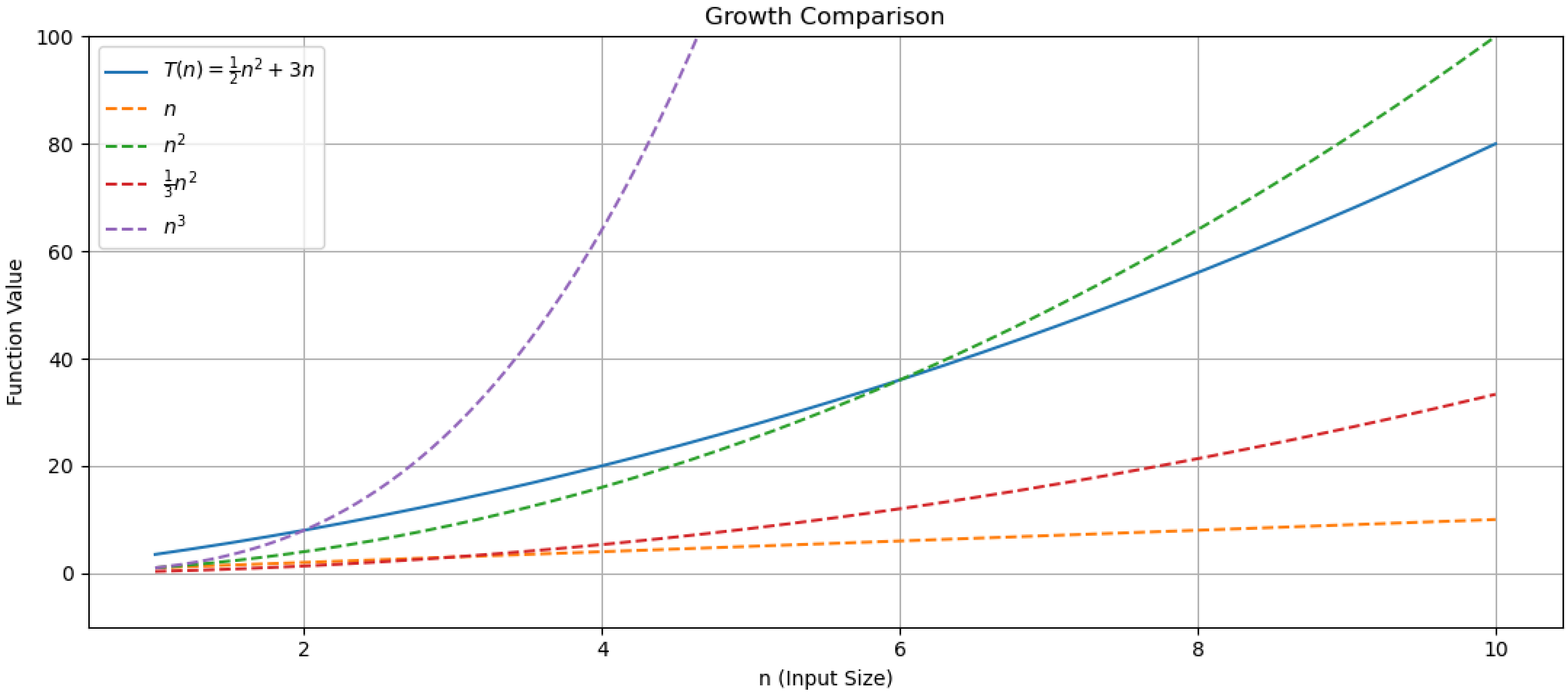
Check Your Understanding 🤓

Let

$$T(n) = \frac{1}{2}n^2 + 3n$$

Which of the following are **true**? Choose *all that apply*.

1. $T(n) = O(n)$
2. $T(n) = \Omega(n)$
3. $T(n) = \Theta(n^2)$
4. $T(n) = O(n^3)$


Growth Comparison
$T(n) = \frac{1}{2}n^2 + 3n$
n
n^2
$\frac{1}{3}n^2$
n^3
Function Value
n (Input Size)
100
80
60
40
20
0
2
4
6
8
10

Check Your Understanding 🤔 - 2nd try!

Let

$$T(n) = 7n \log n + 20n$$

Which of the following are **true**? Choose *all that apply*.

1. $T(n) = O(n)$
2. $T(n) = O(n \log n)$
3. $T(n) = \Omega(n \log n)$
4. $T(n) = \Theta(n^2)$

✨ Heads up! Quiz #1 is coming up next Tuesday.

- It will cover material from Lectures 2, 3, and 4.
- 5-6 questions :-)
- 15 minutes
- I recommend running through the Python code we covered in class at least once!

Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
    - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
    - https://algorithmsilluminated.com/