Lecture 23 - Stable Matching and Gale-Shapley Algorithm

Fall 2025, Korea University

Instructor: Gabin An (gabin_an@korea.ac.kr)

Course Outline (After Midterm)

- Part 3: Data Structures
  - Graphs, Graph Search (DFS, BFS) and Applications (Finding SSCs w/ DFS)
- Part 4: Dynamic Programming
  - Shortest-Path: Dijkstra, Bellman-Ford, Floyd-Warshall Algorithms
  - More General DP: Longest Common Subsequence, Knapsack Problem
- Part 5: Greedy Algorithms and Others
  - Activity Selection, Scheduling, Optimal Codes
  - Minimum Spanning Trees
  - Max Flow, Min Cut and Ford-Fulkerson Algorithms
  - **Stable Matching, Gale-Shapley Algorithm** 🎅🌲

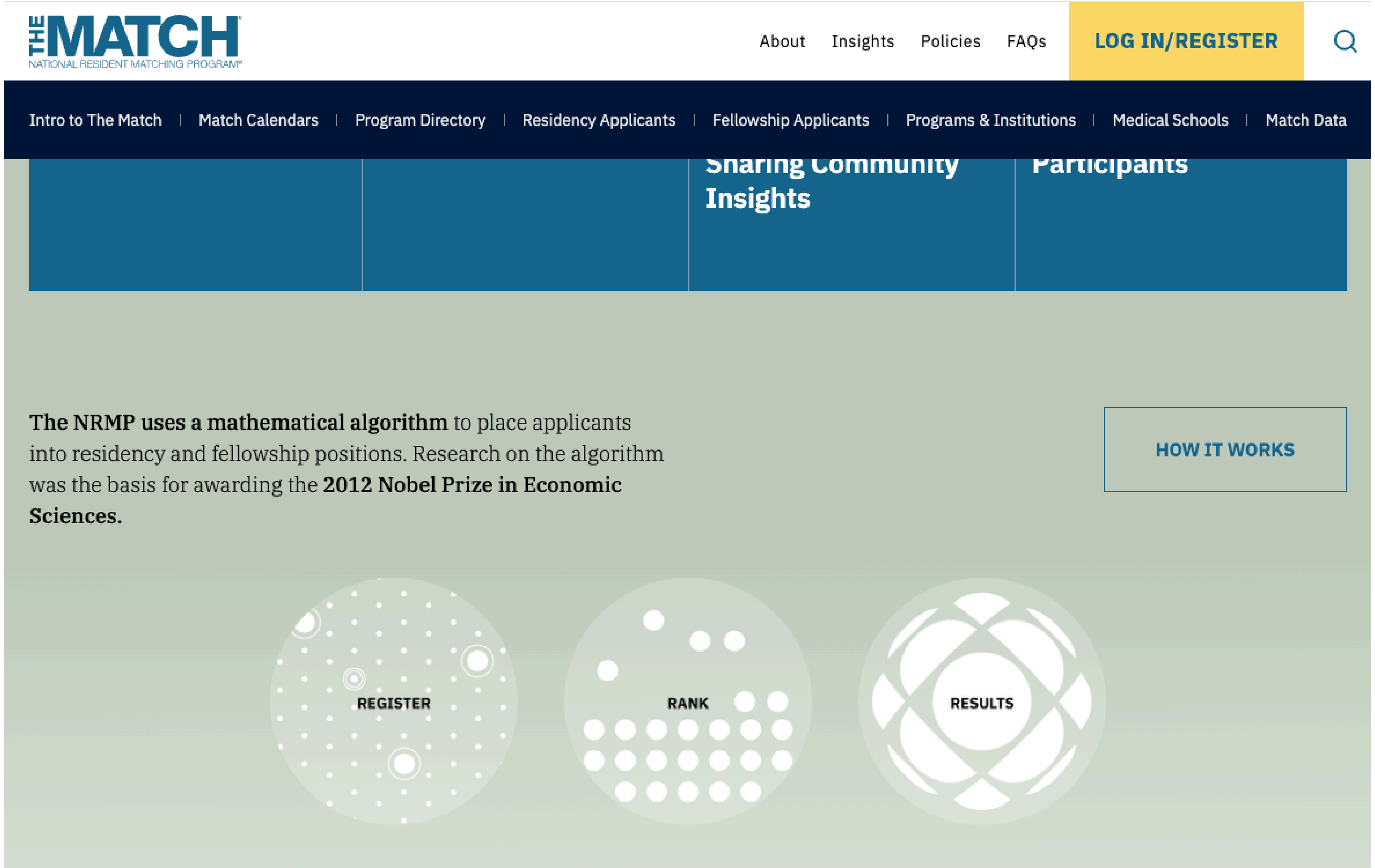
Motivation 🏥

In the US, each year, thousands of doctors are matched to hospitals through the **National Resident Matching Program (NRMP)** - https://www.nrmp.org/.

- Both **doctors** and **hospitals** have preferences.

Doctor	1st	2nd	3rd
Alice	Y	X	Z
Bob	X	Y	Z
Charlie	X	Y	Z

Hospital	1st	2nd	3rd
X	Alice	Charlie	Bob
Y	Charlie	Alice	Bob
Z	Bob	Charlie	Alice


THE MATCH
NATIONAL RESIDENT MATCHING PROGRAM®
About
Insights
Policies
FAQs
LOG IN/REGISTER
Intro to The Match
Match Calendars
Program Directory
Residency Applicants
Fellowship Applicants
Programs & Institutions
Medical Schools
Match Data
Sharing Community Insights
Participants
The NRMP uses a mathematical algorithm to place applicants into residency and fellowship positions. Research on the algorithm was the basis for awarding the 2012 Nobel Prize in Economic Sciences.
HOW IT WORKS
REGISTER
RANK
RESULTS

Motivation 🏥

In the US, each year, thousands of doctors are matched to hospitals through the **National Resident Matching Program (NRMP)**.

- Both **doctors** and **hospitals** have preferences.
- A centralized algorithm must produce a **fair and stable** outcome.
- We will study the **Gale–Shapley (Deferred Acceptance)** algorithm.
    - the foundation of this real-world matching process!
    - invented by *David Gale* and *Lloyed Shapley*

Similar Scenarios

Stable matching problems appear in many real-world contexts:

- Students ↔️ Labs / Professors (graduate admissions)
- Employers ↔️ Teams (HR allocation)
- ...

Problem Setup: Stable Matching

We have:

- n **doctors** 🧑‍⚕️ and n **hospitals** 🏥 (each hospital fills one position)
- Each doctor ranks all hospitals.
- Each hospital ranks all doctors.

Goal: find a **stable matching** between doctors and hospitals.

A matching M is **stable** if there is **no blocking pair** (d, h) such that:

1. Doctor d prefers hospital h to her current match in M, **and**
2. Hospital h prefers doctor d to its current match in M.

Example

Doctor	1st	2nd	3rd
Alice	Y	X	Z
Bob	X	Y	Z
Charlie	X	Y	Z

Hospital	1st	2nd	3rd
X	Alice	Charlie	Bob
Y	Charlie	Alice	Bob
Z	Bob	Charlie	Alice

Example 1: (Alice–X), (Bob–Z), (Charlie–Y)

Doctor	1st	2nd	3rd
Alice	Y	**X**	Z
Bob	X	Y	**Z**
Charlie	X	**Y**	Z

Hospital	1st	2nd	3rd
X	**Alice**	Charlie	Bob
Y	**Charlie**	Alice	Bob
Z	**Bob**	Charlie	Alice

- Stable matching ✅
- Every hospital gets its top choice. -> No blocking pair can exist.

Example 2: (Alice–Y), (Bob–Z), (Charlie–X)

Doctor	1st	2nd	3rd
Alice	**Y**	X	Z
Bob	X	Y	**Z**
Charlie	**X**	Y	Z

Hospital	1st	2nd	3rd
X	Alice	**Charlie**	Bob
Y	Charlie	**Alice**	Bob
Z	**Bob**	Charlie	Alice

- Stable matching ✅
- Bob prefers X and Y over Z, but X and Y prefer other doctors over Bob.

> As you can see there can be multiple possible stable matching.

Example 3: (Alice–Z), (Bob–X), (Charlie–Y)

Doctor	1st	2nd	3rd
Alice	Y	X	**Z**
Bob	**X**	Y	Z
Charlie	X	**Y**	Z

Hospital	1st	2nd	3rd
X	Alice	Charlie	**Bob**
Y	**Charlie**	Alice	Bob
Z	Bob	Charlie	**Alice**

- Unstable matching ❌
- **(Alice, X)** form a **blocking pair**!
    1. Alice prefers X to her current match, Z. 🤕
    2. and.. X also prefers Alice to its current match, Bob. 🙁

The Gale–Shapley Algorithm 🤝

(also called **Deferred Acceptance Algorithm**)

is an algorithm designed to compute a stable matching based on the given preference lists!

The Gale–Shapley Algorithm (doctor-proposing version)

1. All doctors start free.
2. Each free doctor proposes to her/his most-preferred hospital not yet rejected.
3. Each hospital:
    - keeps the best proposal (tentatively engaged 💍),
    - rejects others.
4. Repeat until no free doctors remain.

Example: Doctor-Proposing version

Doctor	1st	2nd	3rd
Alice	**Y** 💍	X	Z
Bob	X	Y	Z
Charlie	X	Y	Z

Hospital	1st	2nd	3rd
X	Alice	Charlie	Bob
Y	Charlie	**Alice** 💍	Bob
Z	Bob	Charlie	Alice

1. **Alice → Y** → Y is free → ✅ accepts

Example: Doctor-Proposing version

Doctor	1st	2nd	3rd
Alice	**Y** 💍	X	Z
Bob	**X** 💍	Y	Z
Charlie	X	Y	Z

Hospital	1st	2nd	3rd
X	Alice	Charlie	**Bob** 💍
Y	Charlie	**Alice** 💍	Bob
Z	Bob	Charlie	Alice

1. **Alice → Y** → Y is free → ✅ accepts
2. **Bob → X** → X is free → ✅ accepts

Example: Doctor-Proposing version

Doctor	1st	2nd	3rd
Alice	Y 💍	X	Z
Bob	~~X~~ 😢	Y	Z
Charlie	X 💍	Y	Z

Hospital	1st	2nd	3rd
X	Alice	**Charlie** 💍	~~Bob~~ 👋
Y	Charlie	**Alice** 💍	Bob
Z	Bob	Charlie	Alice

1. **Alice → Y** → Y is free → ✅ accepts
2. **Bob → X** → X is free → ✅ accepts
3. **Charlie → X** → X prefers Charlie → ❌ rejects Bob → Bob becomes free

Example: Doctor-Proposing version

Doctor	1st	2nd	3rd
Alice	Y 💍	X	Z
Bob	~~X~~ 😢	~~Y~~ 😢	Z
Charlie	X 💍	Y	Z

Hospital	1st	2nd	3rd
X	Alice	**Charlie** 💍	~~Bob~~ 👋
Y	Charlie	**Alice** 💍	~~Bob~~ 👋
Z	Bob	Charlie	Alice

1. **Alice → Y** → Y is free → ✅ accepts
2. **Bob → X** → X is free → ✅ accepts
3. **Charlie → X** → X prefers Charlie → ❌ rejects Bob → Bob becomes free
4. **Bob → Y** → Y prefers Alice → ❌ rejects Bob

Example: Doctor-Proposing version

Doctor	1st	2nd	3rd
Alice	**Y** 💍	X	Z
Bob	~~X~~ 😢	~~Y~~ 😢	**Z** 💍
Charlie	**X** 💍	Y	Z

Hospital	1st	2nd	3rd
X	Alice	**Charlie** 💍	~~Bob~~ 👋
Y	Charlie	**Alice** 💍	~~Bob~~ 👋
Z	**Bob** 💍	Charlie	Alice

1. **Alice → Y** → Y is free → ✅ accepts
2. **Bob → X** → X is free → ✅ accepts
3. **Charlie → X** → X prefers Charlie → ❌ rejects Bob → Bob becomes free
4. **Bob → Y** → Y prefers Alice → ❌ rejects Bob
5. **Bob → Z** → Z is free → ✅ accepts

- **Final Matching:** (Alice–Y), (Bob–Z), (Charlie–X)

Example: Doctor-Proposing version

Doctor	1st	2nd	3rd
Alice	**Y** 💍	X	Z
Bob	~~X~~ 🥲	~~Y~~ 🥲	**Z** 💍
Charlie	**X** 💍	Y	Z

Hospital	1st	2nd	3rd
X	Alice	**Charlie** 💍	~~Bob~~ 👋
Y	Charlie	**Alice** 💍	~~Bob~~ 👋
Z	**Bob** 💍	Charlie	Alice

Hospitals only improve over time,

and doctors gradually move down their list.

Pseudocode

```
for each doctor d:
    d.i = 0  # index in preference list
for each hospital h:
    h.match = NIL

free_doctors = all doctors

while free_doctors:
    d = pick any free doctor
    d.i += 1
    h = d.pref[d.i]
    if h.match == NIL or h.prefers(d, h.match):
        if h.match != NIL:
            free_doctors.add(h.match)
        h.match = d
        free_doctors.remove(d)
```

Proposition 1. Once a hospital is matched, it never becomes unmatched again.

Proof.

- We only assign NIL values at initialization.
- So once a hospital has a *potential match* it can never run out of matches again; it will only reject a potential match for another match (a *more preferred* one).

```
for each doctor d:
    d.i = 0  # index in preference list
for each hospital h:
    h.match = NIL 👈

free_doctors = all doctors

while free_doctors:
    d = pick any free doctor
    d.i += 1
    h = d.pref[d.i]
    if h.match == NIL or h.prefers(d, h.match):
        if h.match != NIL:
            free_doctors.add(h.match)
        h.match = d 👈
        free_doctors.remove(d)
```

Proposition 2. Every doctor eventually gets matched (no one "runs out").

Proof. Suppose, for contradiction, some doctor d "runs out" of hospitals to propose to.

- Then d must have already proposed to **all** n **hospitals**.
- Every hospital that rejected d must have been **already matched** to some other doctor.
  - By **Proposition 1**, once a hospital is matched, it never becomes unmatched again.
- Therefore, at this point:
  - All n hospitals are matched,
  - ... but only to the other $n-1$ doctors (excluding d).
- ❌ Contradiction! It's impossible for n hospitals have a match if one doctor remains unmatched.

Proposition 3. Algorithm terminates in $\leq O(n^2)$ steps.

Proof.

- In every iteration, some $d.\,i$ gets incremented by 1.
- Since they start at 0 and can never reach $n+1$, in total the number of increments is at most $n \times O(n) = O(n^2)$.
- By **Proposition 2** (*"Every doctor eventually gets matched"*), when the algorithm terminates, we have a full matching.

Theorem 1. The resulting matching is STABLE.

This is the results from the doctor-proposing version of Gale-Shapley.

Doctor	1st	2nd	3rd
Alice	**Y**	X	Z
Bob	X	Y	**Z**
Charlie	**X**	Y	Z

Hospital	1st	2nd	3rd
X	Alice	**Charlie**	Bob
Y	Charlie	**Alice**	Bob
Z	**Bob**	Charlie	Alice

✅ This is stable!

❓ *Why does the algorithm produce the stable matching?*

Theorem 1. The resulting matching is STABLE.

- Note that hospitals' matches only **improve** over the course of the algorithm.
  - This is because h will only reject a potential match for a **better** one.
- Assume by contradition, that there is a blocking pair (d, h) in the final matching.
  - By definition, d prefers h to her match, and h prefers d to its match.
  - Let h' be the hospital d got matched to.
  - d must have proposed to h before her final match h' because h' comes later in the doctor d's preference list than h.
  - h **must have rejected** d **for someone it prefers more**.
- ❌ Contradiction because hospitals' matches only improve → no blocking pairs exist!

Doctor-Optimality 👩‍⚕️

We say a stable matching σ is **doctor-optimal** if for any other stable matching σ' and any docter d, the doctor d weakly prefers her hospital in σ to her hospital in σ'.

✅ Stable matching from the doctor-proposing version of Gale-Sharpley

Doctor	1st	2nd	3rd
Alice	**Y**	X	Z
Bob	X	Y	**Z**
Charlie	**X**	Y	Z

> doctor-optimal

✅ Stable matching - but Alice prefers Y over X, and Charlie prefers X over Y.

Doctor	1st	2nd	3rd
Alice	Y	**X**	Z
Bob	X	Y	**Z**
Charlie	X	**Y**	Z

Theorem 2. The output of the doctor-proposing version of Gale-Sharpley is doctor-optimal.

Proof.

For each doctor d, let $h^*(d)$ be the **best feasible hospital** for d (the one she prefers most among all stable matchings).

Assume by contradiction that the algorithm is not doctor-optimal: some doctor d gets rejected by $h^*(d)$ at some point during the algorithm.

Consider the **first** time where this happens: $h^*(d)$ rejects d because of d'

- **Case 1:** If $h^*(d') = h^*(d)$, then $(d', h^*(d))$ would block any matching that pairs d with $h^*(d)$ → ❌ Contradiction. $h^*(d)$ cannot be feasible for d in a stable matching.
- **Case 2:** Otherwise, d' must have been rejected by her own $h^*(d')$ earlier, contradicting that d is the **first** doctor rejected by her best feasible hospital. ❌

Incentive Compatibility 🧩

- An algorithm is **incentive-compatible** if **no participant can benefit by lying** about their true preferences.
- In our context, a matching algorithm is **incentive-compatible for doctors** if no doctor can get a better hospital by **misreporting** her preferences.
- **Why it matters:** In real-world systems like *The Match* (NRMP), thousands of participants submit preferences. If someone could gain by lying, others would be forced to lie too, leading to chaos and unfair outcomes.

The **doctor-proposing Gale–Shapley algorithm** is:

> ✅ **Incentive-compatible for doctors** (proved by Dubins & Freedman, 1981)
> ❌ **Not incentive-compatible for hospitals**

The doctor-proposing Gale–Shapley algorithm is not incentive-compatible for hospitals. (*Hospitals can get better matches by lying🤥!*)

Doctor	1st	2nd	3rd
Alice	**Y**	X	Z
Bob	X	Y	**Z**
Charlie	**X**	Y	Z

Hospital	1st	2nd	3rd
X	Alice	**Charlie** 🫡	Bob 🫡
Y	Charlie	**Alice**	Bob
Z	**Bob**	Charlie	Alice

Doctor	1st	2nd	3rd
Alice	Y	**X**	Z
Bob	X	Y	**Z**
Charlie	X	**Y**	Z

Hospital	1st	2nd	3rd
X	**Alice**	Bob 🤥	Charlie 🤥
Y	**Charlie**	Alice	Bob
Z	**Bob**	Charlie	Alice

Summary

Property	Gale–Shapley Algorithm
Termination	Always halts in O(n^2)
Stability	No blocking pairs
Doctor-optimal	Best for proposers
Incentive-compatibility	For proposers only

Final Exam Information (Offline & In-Person)

- Date & Time:
  - Thursday, December 18
  - 1:30 – 2:45 PM
- Location:
  - Room 609 – Students with odd student IDs (33 students)
  - Room 610 – Students with even student IDs (32 students)

> ⚠️ Please sit with one empty seat between each student.
> 📚 Format: Closed book — but one A4 cheat sheet (both sides) is allowed.

Good luck with your exam! 🎅

Credits & Resources

Gale & Shapley, *College Admissions and the Stability of Marriage*, 1962.

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/