Lecture 20 - Greedy Algorithms (Optimal Codes)

Fall 2025, Korea University

Instructor: Gabin An (gabin_an@korea.ac.kr)

Agenda

- Recap: What is a **Greedy Algorithm**?
- New problem: **Optimal Codes**

Recap: What is a Greedy Algorithm?

- A greedy algorithm is an approach for solving a problem by selecting **the best option available at each step**.
- It builds up a solution piece by piece, always choosing the next piece that offers the most obvious and immediate benefit.
- **It assumes that local optimality will lead to global optimality**.

Property of Problems Suitable for Greedy Algorithms

- **Greedy Choice Property**
  - Choosing the best local option must be part of some optimal solution.
- **Optimal Substructure**
  - Optimal solutions to the full problem are composed of optimal solutions to subproblems.

Problems with Nice (Correct) Greedy Algorithms

- ~~Activity Selection~~ ✅
- ~~Scheduling~~ ✅
- **Optimal Codes** 👈
- Minimum Spanning Trees
- ...

"*Codes*"? "*Coding*"?

- Codes = binary representations of symbols
- Coding = the process of assigning these representations.

Representing Symbols (ASCII Example)

- **ASCII (American Standard Code for Information Interchange)**
- In ASCII, every character uses fixed 7 bits.

Example: `'a' = 1100001`, `'b' = 1100010`

Bits				$b_7 b_6 b_5$ →	0 0 0	0 0 1	0 1 0	0 1 1	1 0 0	1 0 1	1 1 0	1 1 1
b_4 ↓	b_3 ↓	b_2 ↓	b_1 ↓	Column → Row ↓	0	1	2	3	4	5	6	7
0	0	0	0	0	NUL	DLE	SP	0	@	P	`	p
0	0	0	1	1	SOH	DC1	!	1	A	Q	a	q
0	0	1	0	2	STX	DC2	"	2	B	R	b	r
0	0	1	1	3	ETX	DC3	#	3	C	S	c	s
0	1	0	0	4	EOT	DC4	$	4	D	T	d	t
0	1	0	1	5	ENQ	NAK	%	5	E	U	e	u
0	1	1	0	6	ACK	SYN	&	6	F	V	f	v
0	1	1	1	7	BEL	ETB	´	7	G	W	g	w
1	0	0	0	8	BS	CAN	(	8	H	X	h	x
1	0	0	1	9	HT	EM	)	9	I	Y	i	y
1	0	1	0	10	LF	SUB	*	:	J	Z	j	z
1	0	1	1	11	VT	ESC	+	;	K	[	k	{
1	1	0	0	12	FF	FS	,	<	L	\	l	¦
1	1	0	1	13	CR	GS	—	=	M	]	m	}
1	1	1	0	14	SO	RS	.	>	N	^	n	~
1	1	1	1	15	SI	US	/	?	O	_	o	DEL

Representing Symbols (Unicode Example)

ASCII

- 7-bit character set (128 characters) covering basic English letters, digits, punctuation, and control characters

Unicode

- A **much larger** character encoding standard (over 1.1 million code points possible)
- Covers characters for **virtually all languages**, emoji, symbols, etc.
- Designed to unify many old encodings

> 📝 **Unicode includes ASCII as its first 128 code points.** But Unicode is **not** just ASCII + extra characters. Unicode has **different architecture**, multiple planes, multiple encodings (UTF-8 / UTF-16 / UTF-32), and normalization rules.

In python, you can use `ord()` to see the Unicode representations of a character.

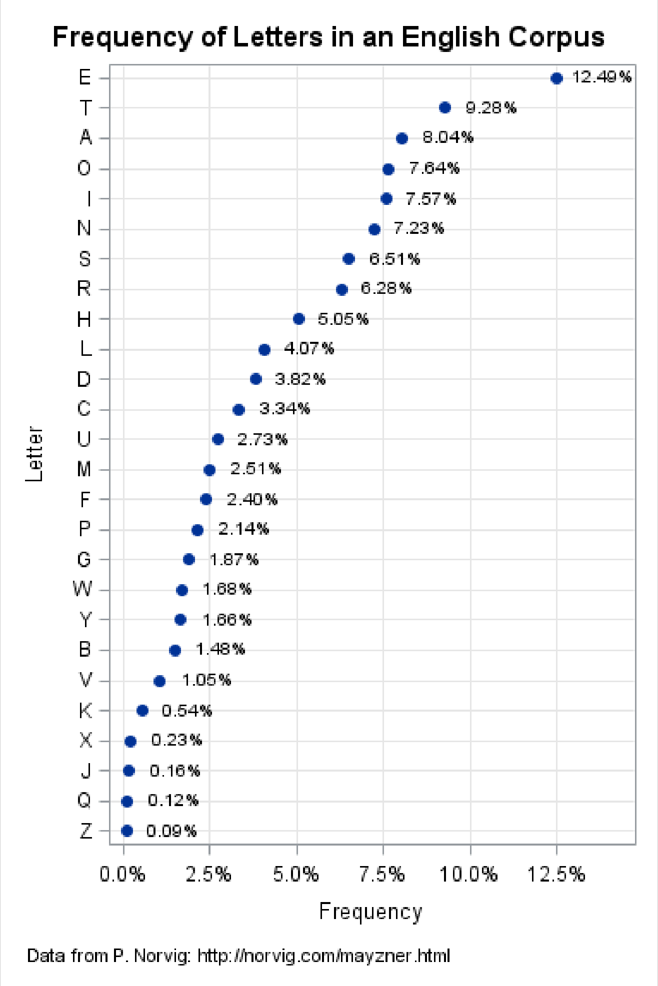
> **ord**(*character*, /)
>
> Return the ordinal value of a character.
>
> If the argument is a one-character string, return the Unicode code point of that character. For example, `ord('a')` returns the integer `97` and `ord('€')` (Euro sign) returns `8364`. This is the inverse of `chr()`.
>
> If the argument is a `bytes` or `bytearray` object of length 1, return its single byte value. For example, `ord(b'a')` returns the integer `97`.

```python
# character -> code (decimal)
ord('a'), ord('b')   # 97, 98

# character -> 8-bit binary representation
format(ord('a'), '08b'), format(ord('b'), '08b')
# '01100001', '01100010'
```

Problem: Fixed-length Encoding is Wasteful

- Not all characters appear equally often!
  - In English: `'e'` ≈ 13%, `'z'` ≈ 0.07%
- If frequencies of characters are very skewed, fixed bits wastes bandwidth.



Idea: Creating Optimal Codes

- Use *variable-length* codes so the expected bits per symbol is minimized.
- By assigning **shorter codes for frequent symbols, longer for rare**, we can save space on average.

Example: Compressing characters with different frequencies

Suppose you want to encode four symbols with the following frequencies:

Symbol	Frequency
A	50
B	30
C	15
D	5

1️⃣ Fixed-length code

```
A = 00
B = 01
C = 10
D = 11
```

Expected length

= 2 bits/symbol

2️⃣ Variable-length code

Assign shorter codes to frequent symbols:

```
A = 0
B = 10
C = 110
D = 111
```

Expected length

= 0.5 * 1 + 0.3 * 2 + 0.15 * 3 + 0.05 * 3

= 1.7 bits/symbol (🔋 savings of **15%**)

Key Question

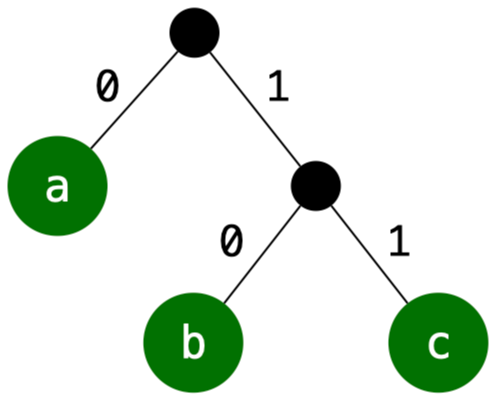
- How can we design a coding scheme that is **space-efficient** yet **decodable without ambiguity?**

> 💡 This is the foundation of **data compression** (e.g., ZIP, MP3, JPEG)

Prefix-Free Codes

- To decode correctly, we need to avoid ambiguity
  - Example: `a` ← `0`, `b` ← `1`, `c` ← `01`
    - Stream `01` could mean `ab` or `c` (❌ ambiguous!)
- Solution: Use ⭐**prefix-free codes**⭐
  - No codeword is a prefix of another.
  - Example: `a` ← `0`, `b` ← `10`, `c` ← `11`
    - Stream `010110` → decode uniquely: `abca` ✅
- Prefix-free codes ensure instant (unambiguous) decoding left-to-right.
- Non-prefix codes can be more compact per symbol but break unique decodability.

🌲 Binary Tree View



- Prefix-free codes are equivalent to placing characters only at **leaves** of a binary tree
- Build a binary tree where:
  - **Left edge = 0**, **Right edge = 1**
  - Leaves = characters (symbols), Codeword = path label
  - `a` ← `left` = `0`, `b` ← `right&left` = `10`, `c` ← `right&right` = `11`

Measuing Code Efficiency

- Suppose we have characters C and use the coding scheme represented by a tree T
- For each character $c \in C$,
  - $f(c)$: the frequencies of symbol c (probabilities of occurrence)
  - $d_T(c)$: the code length, i.e., depth of c in the tree T
- **Expected (average) code length**:

$$B(T) = \sum_{c \in C} f(c) \cdot d_T(c)$$

- Goal: Find a tree T that **minimizes** $B(T)$ → Shorter average code length

> A tree T is optimal if this expected cost B(T) is minimized among all prefix-free coding trees.

Huffman Coding (1951)

- Invented by **David Huffman** as a student project 👨🏻‍💻
  (instead of taking a final exam!)

History [edit]

In 1951, David A. Huffman and his MIT information theory classmates were given the choice of a term paper or a final exam. The professor, Robert M. Fano, assigned a term paper on the problem of finding the most efficient binary code. Huffman, unable to prove any codes were the most efficient, was about to give up and start studying for the final when he hit upon the idea of using a frequency-sorted binary tree and quickly proved this method the most efficient.[5]

Building Huffman Codes

- The basic idea 💡
  - Build subtrees for subsets of characters and merge them from the bottom up, combining the two trees with the characters of minimum total frequency.
- **Greedy algorithm**:
  i. Start with each character as a leaf node (weighted by frequency)
  ii. Repeatedly merge the two least frequent nodes into a new parent
  iii. Assign left=0, right=1 at each merge
  iv. Continue until one tree remains

Example: Huffman Codes

Characters + frequencies:

`a` : `0.45` , `b` : `0.13` , `c` : `0.12` , `d` : `0.16` , `e` : `0.05` , `f` : `0.09`

Step-by-step merges:

1. `e(0.05)` + `f(0.09)` → `node(0.14)`
2. `node(0.14)` + `c(0.12)` → `node(0.26)`
3. `b(0.13)` + `d(0.16)` → `node(0.29)`
4. `node(0.26)` + `node(0.29)` → `node(0.55)`
5. `a(0.45)` + `node(0.55)` → `root(1.00)`

Example: Huffman Codes (Results)

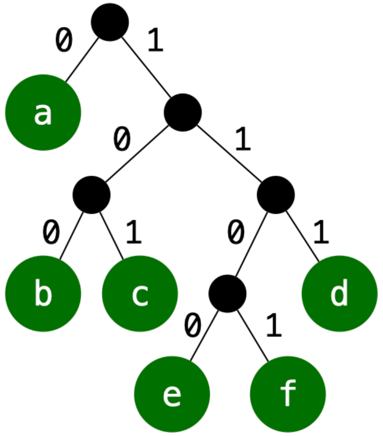
```
- a = 0
- b = 101
- c = 100
- d = 111
- e = 1100
- f = 1101
```

- Expected Length

$$E[L] = 0.45 \cdot 1 + 0.13 \cdot 3 + 0.12 \cdot 3 + 0.16 \cdot 3 + 0.05 \cdot 4 + 0.09 \cdot 4 = 2.24 \text{ bits/symbol}$$

- Compare 8 bits/symbol → > 72% reduction 🔋

Prefix-Free 👉 Instant Decoding



Let's decode `110011011111000101`!

- Start at root, consume bits left-to-right
- Whenever you **hit a leaf, output the symbol and reset to root**
- No look-ahead, no backtracking required

> This is why prefix-free is the dominant practical constraint.

Quick Quiz

Build Huffman Codes for the following characters + frequencies:

Symbol	**Frequency**
A	50
B	20
C	15
D	10
E	5

Q. What is the expected length per symbol?

Huffman Code: Correctness Proof

Claim: Huffman's algorithm always produces a prefix-free code that minimizes the expected codeword length.

Proof Strategy:

We use **induction** on the number of symbols n.

1. **Base case ($n = 2$):**
   Only one possible prefix code — trivially optimal ✅

2. **Inductive step ($n > 2$):**
   Show that Huffman's greedy choice and reduction step preserve optimality.

Before we finish the induction, we need the two key tools.

1️⃣ Greedy step is safe

The two least frequent symbols should be merged — at least, there exists an optimal tree where they are siblings.

2️⃣ Reduction step preserves optimality:

If we combine the two least frequent symbols and solve a smaller problem, expanding them back yields an optimal solution for the original problem.

1️⃣ Greedy step is safe

> The two least frequent symbols should be merged — at least, there exists an optimal tree where they are siblings.

Let x and y be the **two least frequent symbols**.

The key statement is:

> In at least one optimal code tree, x and y appear as siblings at the maximum depth of the tree.

Justification: If not, we can swap them with two deepest siblings without increasing cost.

2️⃣ Reduction step preserves optimality:

> If we combine the two least frequent symbols and solve a smaller problem, expanding them back yields an optimal solution for the original problem.

Now suppose we merge x, y into a single combined symbol z, with: $f(z) = f(x) + f(y)$

This gives us a smaller alphabet of size $n - 1$.

The idea:

1. Solve the problem optimally for these $n - 1$ symbols.
2. Then expand z back into x and y, attaching 0 and 1.

Crucially:

- Expanding the tree **adds exactly 1** to the depth of x and y: $\Delta\mathrm{Cost} = f(x) + f(y)$
- So optimality for the smaller instance carries over to the larger one.

Inductive Proof

- **Inductive Hypothesis**: For a set of $n-1$ symbols, Huffman produces an optimal code.
- **Inductive Step** (From $n-1$ to n):
    1. Apply induction to the reduced problem with $n-1$ symbols (merging $x, y \to z$).
        - Huffman produces an optimal code C' for the reduced problem.
    2. Expand z into x, y and append `0` and `1` to their codes.
    3. The resulting code C is optimal for n symbols because x, y are the two least frequent symbols. ✅
        - Why? In some optimal prefix code, the two least frequent symbols must be siblings at the deepest level.

Huffman Codes Conclusion

- **Greedy choice** is always safe.
- **Reduction** preserves optimality.
- By **induction**, Huffman always yields an **optimal prefix code**.
- The technique of Huffman coding is the final stage in many compression methods, including JPEG, MP3, and zip.

Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/

Appendix: More Rigorous Proof

Outline

1. Setup & Notation
2. Lemma 0 (Full binary form)
3. Greedy-Choice Lemma (Deepest siblings)
4. Reduction Lemma (Merge two lightest)
5. Inductive Proof of Optimality
6. Remarks

Setup & Notation

- Alphabet (symbols) C; frequency/weight $f(c) > 0$ for each $c \in C$.
- A **prefix code** corresponds to a **binary tree** T with each symbol at a **leaf**.
- Let $d_T(c)$ be the depth (code length) of leaf c in T.
- **Cost (external path length)**:

$$B(T) \stackrel{\text{def}}{=} \sum_{c \in C} f(c)\, d_T(c).$$

  - Equivalently, expected codeword length is $\frac{B(T)}{\sum_{c \in C} f(c)}$.
- **Goal:** find T minimizing $B(T)$.

Lemma 0 (Full Binary Form)

> In some optimal tree T^*, every internal node has **exactly two** children (T^* is full).

Proof sketch. If an internal node had only one child, **contract** that edge; depths of all leaves do not increase, and at least one decreases, so $B(T)$ does not increase (strictly decreases unless already minimal). Hence we may assume an optimal tree is full. $\square$

Greedy-Choice Lemma (Deepest Siblings)

Let $x, y \in C$ be the two symbols with **smallest** frequencies (break ties arbitrarily).

> **Claim.** There exists an optimal tree T^* in which x and y are **siblings at the maximum depth** of T^*.

Proof. Take an optimal full tree T^*. Let a, b be two sibling leaves at **maximum depth**; w.l.o.g. order frequencies so that $f(x) \le f(y), \quad f(a) \le f(b)$

Because x, y are the two lightest symbols, we have $f(x) \le f(a)$ and $f(y) \le f(b)$. Construct $\tilde{T}$ by **swapping labels** so that the deepest sibling positions are occupied by x and y. Depths at those two positions are the same as before, say $d_a = d_b$. Then

$$\Delta \equiv B(\tilde{T}) - B(T^*) = (f(x) - f(a))\, d_a + (f(y) - f(b))\, d_b \le 0.$$

Thus $\tilde{T}$ is **also optimal** and has x, y as deepest siblings. $\square$

Reduction Lemma (Merge Two Lightest)

Form a reduced alphabet $C' = (C \setminus \{x, y\}) \cup \{z\}$ where the new symbol z has $f(z) = f(x) + f(y)$.

> **Claim A.** If T is a tree for C in which x, y are siblings, and T' is obtained by **contracting** their parent into leaf z, then $B(T) = B(T') + f(x) + f(y)$.

Reason. Only x and y change depths when expanding/contracting: each gains/loses exactly 1 in depth; others unchanged. Hence the cost difference is precisely $f(x) + f(y)$.

> **Claim B.** If T' is **optimal** for C' and we **expand** z into children x, y, the resulting T is optimal for C among trees where x, y are siblings. Moreover, if some optimal T for C has x, y siblings, then the contracted T' must be optimal for C'.

Proof. Suppose not: if there exists U' for C' with $B(U') < B(T')$, then expanding U' gives U with $B(U) = B(U') + f(x) + f(y) < B(T') + f(x) + f(y) = B(T)$, contradicting optimality of T. The converse is analogous. $\square$

Inductive Proof of Optimality (Huffman)

We prove by **induction on** $n = |C|$ that **Huffman** produces a tree minimizing $B(T)$.

Base ($n = 2$). There is only one full binary tree; Huffman returns it, trivially optimal.

Inductive Step. Assume true for all alphabets of size $n - 1$.

1. Let x, y be the two lightest symbols. By the **Greedy-Choice Lemma**, some optimal T^* has x, y as deepest siblings.
2. **Contract** x, y to z, obtaining C' and optimal T'^* by Claim B. By the inductive hypothesis, **Huffman** on C' returns an optimal tree T'_H with $B(T'_H) = B(T'^*)$.
3. **Expand** z back to x, y. By Claim A,

$$B(T_H) = B(T'_H) + f(x) + f(y) = B(T'^*) + f(x) + f(y) = B(T^*).$$

Thus T_H (Huffman's tree on C) is **optimal**. $\square$

Remarks

- The proof shows **Greedy-Choice** and **Optimal Substructure** explicitly.
- Ties among equal frequencies are harmless; any tie-breaking yields an optimal tree.
- The argument generalizes to **non-integer** positive weights.
- Implementation detail: use a min-heap to repeatedly extract the two lightest symbols; total time $O(n \log n)$.

Summary

- **Lemma 0:** assume an optimal **full** binary tree.
- **Greedy-Choice Lemma:** the two lightest symbols can be placed as **deepest siblings** in an optimal tree.
- **Reduction Lemma:** merging these yields a smaller **optimal** instance.
- **Induction:** applying Huffman recursively produces a globally **optimal** code.

Therefore, Huffman coding minimizes $B(T) = \sum_c f(c) d_T(c)$ over all prefix codes.