Lecture 2 - Divide and Conquer (Karatsuba Integer Multiplication)

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Welcome Back!

In today's class, we will learn

- the **Integer Multiplication** problem
- the **Karatsuba Multiplication** method (or algorithm)
  - An *optimized* recursive approach to multiplying two numbers
  - A great example of **Divide-and-Conquer** algorithms

What's the Integer Multiplication Problem?

> Input: Two n-digit nonnegative integers, x and y.
>
> Output: The product $x \cdot y$.

- Examples:

    i. 1-digit integers

$$9 \cdot 8 = 72$$

    ii. 4-digit integers

$$5678 \cdot 1234 = 7006652$$

    i. n-digit integers

$$3141592653589793238462643383279502884197 \cdot 2718281828459045235360287471352662497757 = ?$$

The Grade-School Algorithms

Suppose you want to multiply two numbers $x = 5678$ and $y = 1234$ (so $n = 4$).

You already know a way to do this — the **grade-school algorithm** — which involves multiplying every digit of the first number with every digit of the second.

```
     5678
x    1234
---------
    22712 (= 5678 * 4)
   17034- (= 5678 * 3)
  11356-- (= 5678 * 2)
  5678--- (= 5678 * 1)
---------
  7006652
```

Analysis of the Number of the Operations

For now, let's think of a primitive operation as any of the following:

1. multipying two single-digit numbers
2. adding two single-digit numbers

For the first partial product, `5678 * 4`

- multiplying `4` times each of the digits `5`, `6`, `7`, and `8`
  - one per digit = n
- handling carries (add a single-digit number to double-digit one), e.g., `3` + `28`
  - at most two per carry = $2n$

Because there are n partial products, the computation requires at most $3n^2$ operations.

Analysis of the Number of the Operations - Continued

- We still have to add up the partial products to compute the final answer (at most another $3n^2$).

```
     5678
x    1234
---------
    22712 (= 5678 * 4) # partial product: roughly 3n operations ------|
   17034- (= 5678 * 3) # partial product: roughly 3n operations       n
  11356-- (= 5678 * 2) # partial product: roughly 3n operations    rows
  5678--- (= 5678 * 1) # partial product: roughly 3n operations ------|
---------
  7006652              #  final addition: roughly constant * n^2 operations
```

- # of operations overall ~ constant * n^2

Tiny Bit of Asymptotic Analysis

Instead of measuring a program's running time in seconds (which can vary based on hardware), we count the number of **primitive operations** (like additions or multiplications) it performs. This helps us understand how the **running time grows with the input size** n.

This is known as **Asymptotic Analysis**:

- It focuses on the **rate of growth** of an algorithm's running time as n becomes large.
- For example, the **grade-school multiplication algorithm** runs in $O(n^2)$ time.

We'll explore this notation in more detail next week!

Can We Do Better?

Have you ever taken the grade-school multiplication algorithm as the one and only—or even optimal—way to multiply numbers?

- But what if there's a *faster* way?
- What if we can design an algorithm that runs in **less than** $O(n^2)$ time?

> *"Perhaps the most important principle for the good algorithm designer is to refuse to be content."*
> — Aho, Hopcroft, and Ullman, *The Design and Analysis of Computer Algorithms* (1974)

Let's take that advice and see where it leads.

Divide and Conquer

The "Divide and Conquer" algorithm paradigm is a very useful and widely applicable technique. The high-level idea is just to split a given problem into smaller pieces and then solve the smaller ones, often recursively.

- Divide: Break the problem into smaller subproblems (often of equal size).
- Conquer: Solve each subproblem recursively.
- Combine: Merge the subproblem solutions into the final answer.

💡 A Recursive Algorithm (Divide and Conquer, Take 1)

Break each number into two halves

- Write $x = 10^{n/2}a + b$ and $y = 10^{n/2}c + d$.
- a, b, c, and d are $n/2$-digit numbers. For example,
  - $x = 5678 = 56 * 10^2 + 78$ ($a = 56, b = 78$)
  - $y = 1234 = 12 * 10^2 + 34$ ($c = 12, d = 34$)

Then,

$$\begin{aligned} x \cdot y &= (10^{n/2}a + b)(10^{n/2}c + d) \\ &= 10^n ac + 10^{n/2}(ad + bc) + bd \end{aligned}$$

It's **4 multiplications on numbers of size n/2**.

Is this Recursive Algorithm Better?

$$\begin{aligned} x \cdot y &= (10^{n/2}a + b)(10^{n/2}c + d) \\ &= 10^n ac + 10^{n/2}(ad + bc) + bd \end{aligned}$$

- Interestingly enough, this algorithm isn't actually better!
- Intuitively, this is because if we expand the recursion, we still have to multiply every pair of digits, i.e., when $x = 5678$ and $y = 1234$, we still need to compute $56 \cdot 12$, $56 \cdot 34$, $78 \cdot 12$, and $78 \cdot 34$.

✨ Karatsuba's Insight in 1960 (Divide and Conquer, Take 2)

An ingenious trick **reduces the number of multiplications from 4 to 3**.

$$\begin{aligned} x \cdot y &= (10^{n/2}a + b)(10^{n/2}c + d) \\ &= 10^n ac + 10^{n/2}(ad + bc) + bd \end{aligned}$$

- We don't care about ad or bc, but their sum $ad + bc$.
- **Gauss' Trick**: Instead of recursively computing ad and bc, we compute:

  $$t = (a+b)(c+d) = ac + bd + ad + bc$$

  because $ad + bc = t - ac - bd$ (we have to compute ac and bd, anyway.)
- So now we only compute ac, bd, and $(a+b)(c+d)$ (*just 3 multiplications*)!

Efficiency Analysis of Karatsuba Multiplication

- Upshot: To solve n-digit multiplication problem, we only need **three** recursive $n/2$-digit recursive multiplications (and some additions).
    - Saving a recursive call should save on the overall running time, but by how much?
    - Is the Karatsuba algorithm *faster* than the grade-school multiplication algorithm?

Let's Analyze the Runtime using Recurrence Relations!

Let $T(n)$ be the runtime of the algorithm, given an input of size n.

- The initial recursive algorithm: $T(n) = 4T(\frac{n}{2}) + O(n)$

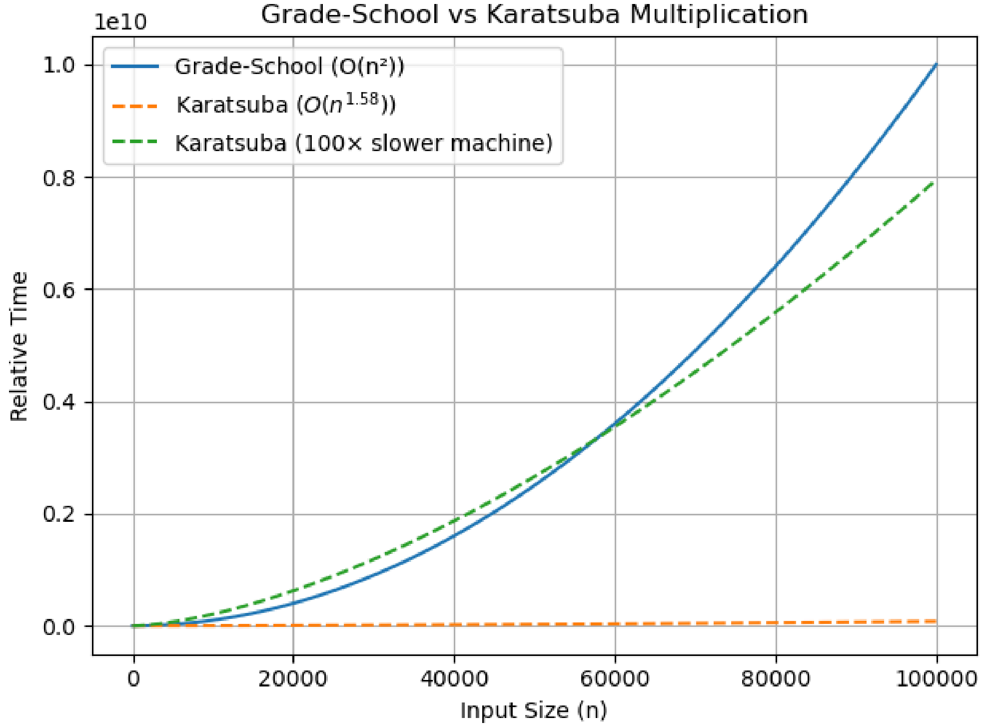
$$T(n) = 4T(\frac{n}{2}) = 4^2T(\frac{n}{2^2}) = \ldots = 4^tT(\frac{n}{2^t}) = n^2T(1) = O(n^2)$$

- The Karatsuba Multiplication algorithm: $T(n) = 3T(\frac{n}{2}) + O(n)$

$$T(n) = 3T(\frac{n}{2}) = \ldots = 3^t T(\frac{n}{2^t}) = 3^{\log_2 n} T(1) = 2^{(\log_2 3)(\log_2 n)} = n^{1.58} T(1) = O(n^{1.58})$$

> since $\log_2 3 \approx 1.58$

$O(n^2)$ vs $O(n^{1.58})$



Karatsuba is much faster than the n^2 algorithm that we learned in grade school!

Can We Do Better?

Algorithm	Recursive Calls	Time Complexity
Karatsuba (1962)	3T(n/2)	$O(n^{1.58})$
Toom and Cook (1963)	5T(n/3)	$O(n^{1.465})$
...	...	...
Harvey–van der Hoeven (2019)	Theoretical best	$O(n\log(n))$

- It is quite amazing that the seemingly simple question of multiplying two numbers has proved to be so mysterious and has seen new research advances as recently as 2019.
- This is what makes the study of algorithms so exciting!

Bonus: Where Does “Algorithm” Come From?

The word “algorithm” comes from **al-Khwarizmi**, a 9th-century Persian mathematician.

- He wrote about methods for arithmetic using **Arabic-Hindu numerals**.
    - i.e., a set of 10 symbols - 1,2,3,4,5,6,7,8,9,0
- His works were translated into Latin in the 1100s and introduced to Europe which had previously relied on Roman numerals such as I, V, X, L, C, and M.

  > Roman numerals were cumbersome for arithmetic and lacked the concept of zero. The Arabic-Hindu numeral system **revolutionized** computation by introducing place value and zero.

- The old French word "Algorisme" meant "the Arabic numerals system". Over time, the term evolved to mean **any systematic method for solving computational problems**.

Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/
- For more on the efficiency analysis of the Karatsuba algorithm, see: https://en.wikipedia.org/wiki/Karatsuba_algorithm#Efficiency_analysis