Lecture 3 - Sorting Algorithms (InsertionSort & MergeSort)

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Recap: Course Outline (Before Midterm)

- Part 1: Basics
  - ~~Divide and Conquer (w/ Integer Multiplication)~~ ✅
  - **Basic Sorting Algorithms (Insertion Sort & Merge Sort)** 👈
  - Asymptotic Analysis (Big-O, Big-Theta, Big-Omega)
  - Solving Recurrences Using Master Method
- Part 2: Advanced Selection and Sorting
  - Median and Selection Algorithm
  - Solving Recurrences Using Substitution Method
  - Quick Sort, Counting Sort, Radix Sort
- Part 3: Data Structures
  - Heaps, Binary Search Trees, Balanced BSTs

We're already at Week 2! 👀

In today's class, we'll dive into:

- the **Sorting** problem
- SelectionSort and BubbleSort (the simplest sorting algorithms)
- InsertionSort (as a warm-up for formal analysis)
- MergeSort (a classic Divide-and-Conquer example)

As we explore these algorithms, we'll focus on two key questions:

- **Correctness** — Does it actually work?
- **Efficiency** — How fast is it?

The Sorting Problem

> Input: An array of n numbers, in arbitrary order.
>
> Output: An array of the same numbers, sorted from smallest to largest.

For example, given the input array

```
[5,4,1,8,7,2,6,3]
```

the desired output array is

```
[1,2,3,4,5,6,7,8]
```

Simplest Sorting Algorithms

If you don't care about optimizing the running time, it's not too difficult to come up with a correct sorting algorithm.

- SelectionSort
- BubbleSort
- InsertionSort

SelectionSort

- Idea: Repeatedly find the smallest element from the unsorted part and move it to the front.

```
# [ sorted part | unsorted part ]

[|5, 4, 1, 8, 7, 2, 6, 3] # 1 is the smallest, swap with 5
[1 | 4, 5, 8, 7, 2, 6, 3] # 2 is next smallest, swap with 4
[1, 2 | 5, 8, 7, 4, 6, 3] # 3 is next, swap with 5
[1, 2, 3 | 8, 7, 4, 6, 5] # 4 is next, swap with 8
[1, 2, 3, 4 | 7, 8, 6, 5] # 5 is next, swap with 7
[1, 2, 3, 4, 5 | 8, 6, 7] # 6 is next, swap with 8
[1, 2, 3, 4, 5, 6 | 8, 7] # 7 is next, swap with 8
[1, 2, 3, 4, 5, 6, 7 | 8] # Done!
```

- SelectionSort makes $n - 1$ swaps, but always scans the rest of the array.
  - # Scan: $n + (n-1) + \ldots + 2 + 1 = n(n-1)/2 = O(n^2)$

BubbleSort 🫧

- Idea: Repeatedly swap adjacent elements if they're in the wrong order. Largest values “bubble up” to the end.

```
# Pass 1
# [ unsorted part | sorted part ]
[5, 4, 1, 8, 7, 2, 6, 3|] # compare 5, 4 (swap) -> 4, 5
[4, 5, 1, 8, 7, 2, 6, 3|] # compare 5, 1 (swap) -> 1, 5
[4, 1, 5, 8, 7, 2, 6, 3|] # compare 5, 8 (do not swap) -> 5, 8
[4, 1, 5, 8, 7, 2, 6, 3|] # compare 8, 7 (swap) -> 7, 8
[4, 1, 5, 7, 8, 2, 6, 3|] # compare 8, 2 (swap) -> 2, 8
[4, 1, 5, 7, 2, 8, 6, 3|] # compare 8, 6 (swap) -> 6, 8
[4, 1, 5, 7, 2, 6, 8, 3|] # compare 8, 3 (swap) -> 3, 8
[4, 1, 5, 7, 2, 6, 3 | 8] # the largest element 8 'bubble up' to the end!
```

BubbleSort - Let's Pratice Together ✏️

```
# Pass 2
[4, 1, 5, 7, 2, 6, 3 | 8]
```

BubbleSort - Continued

```
[4, 1, 5, 7, 2, 6, 3 | 8] # After Pass 1
[1, 4, 5, 2, 6, 3 | 7, 8] # After Pass 2
[1, 4, 2, 5, 3 | 6, 7, 8] # After Pass 3
[1, 2, 4, 3 | 5, 6, 7, 8] # After Pass 4
[1, 2, 3 | 4, 5, 6, 7, 8] # After Pass 5
[1, 2 | 3, 4, 5, 6, 7, 8] # After Pass 6
[1 | 2, 3, 4, 5, 6, 7, 8] # After Pass 7
```

- at most $(n-1)+(n-2)+...+1$ swaps = $O(n^2)$
- Both SelectionSort and Buble Sort have quadratic running times, meaning that the number of operations performed on arrays of length n scales with n^2, i.e., $O(n^2)$.

InsertionSort

- Idea: In each step, larger elements are shifted right until the correct position for the current value is found.

Suppose we want to sort `5`, `4`, `2`, `3`

```
[_, _, _, _]
------------
[5, _, _, _] # Insert 5
------------
[_, 5, _, _] # 5 > 4. Shift 5 to the right
[4, 5, _, _] # Insert 4
------------
[4, _, 5, _] # 5 > 2. Shift 5 to the right
[_, 4, 5, _] # 4 > 2. Shift 5 to the right
[2, 4, 5, _] # Insert 2
```

InsertionSort - Let's Pratice Together ✏️

Insert `3` to the array!

```
[_, _, _, _]
------------
[5, _, _, _] # Insert 5
------------
[_, 5, _, _] # 5 > 4. Shift 5 to the right
[4, 5, _, _] # Insert 4
------------
[4, _, 5, _] # 5 > 2. Shift 5 to the right
[_, 4, 5, _] # 4 > 2. Shift 5 to the right
[2, 4, 5, _] # Insert 2
------------
```

InsertionSort - Algorithm

```python
def insertion_sort(A):
    for i in range(1, len(A)):
        current = A[i]    # save the current element
        j = i - 1         # start scanning
        while j >= 0 and A[j] > current:
            A[j+1] = A[j] # shift larger elements to the right
            j -= 1
        A[j+1] = current
```

Example:

```
# 1st iteration of the outer loop
[5, 4, 2, 3] # i = 1, current = 4, j = 0
[5, 5, 2, 3] # i = 1, current = 4, j = -1 (the inner loop ends because j < 0)
[4, 5, 2, 3] # A[0] = 4
```

Correctness of InsertionSort

- Goal: We want to prove that after `insertion_sort(A)` finishes, **the array A is sorted in non-decreasing order**.
- We'll do the proof by maintaining a **Loop Invariant**:

> After iteration i of the outer loop, the subarray $A[:i+1]$ is sorted.

```
[5, 4, 2, 3]
→ [4, 5, 2, 3] # After iteration 1, A[:2] is sorted.
→ [2, 4, 5, 3] # After iteration 2, A[:3] is sorted.
→ [2, 3, 4, 5] # After iteration 3, A[:4] (= A) is sorted.
```

Correctness of InsertionSort - Proof by Induction

- **Inductive Hypothesis**

  After iteration i, $A[:i+1]$ is sorted.

- **Base Case** ($i = 0$)

  $A[:1]$ contains one element, so it is trivially sorted. ✅

- **Inductive Step**

  Assume the hypothesis holds for iteration $i-1$:

  > **After iteration $i-1$, $A[:i]$ is sorted.**

- Now consider iteration i:

- Now consider iteration i:
  - At iteration i, let j be the largest index such that $0 \le j < i$ and $A[j] \le A[i]$.
  - Then, $A[i]$ is inserted between $A[j]$ and $A[j+1]$

$$A[j] \le A[i] \le A[j+1]$$

  - Since $A[: i]$ was already sorted:

$$A[0] \le A[1] \le \cdots \le A[j] \le A[i] \le A[j+1] \le \cdots \le A[i-1]$$

    - $A[: i+1]$ **is sorted**!! ✅

Conclusion

- By induction, the invariant holds for all i.
- After the final iteration ($i = n - 1$), $A[: n]$ (the entire array) is sorted. ✅

Running Time of InsertionSort

- The running time of InsertionSort is about n^2 operations.
- At iteration i, the algorithm may have to look through and move i elements, so that's about $1 + 2 + ... + n - 1 = \frac{n(n-1)}{2}$ operations.
- Can we do *asymptotically better* than n^2?
  - Can we come up with an algorithm that sorts an arbitrary list of n intergers in time that scales less than n^2? For example, like $n^{1.5}$, or $n \log(n)$, or even n.

Recap: Divide-and-Conquer

Recall the Divide-and-conquer paradigm from the previous lecture. In this paradigm, we use the following strategy:

- Break the problem into sub-problems.
- Solve the sub-problems (often recursively)
- Combine the results of the sub-problems to solve the big problem

At some point, the sub-problems become small enough that they are easy to solve, and then we can stop recursing.

> Sorting is a perfect candidate for this strategy!

New Sorting Algorithm: MergeSort

```
[6 5 12 10 9 1]
```

```
                [6 5 12 10 9 1]
               /              \
       [6 5 12] divide [10 9 1]
        /     \           /     \
    [6]     [5 12]    [10]     [9 1]
     |       /   \      |       /   \
    [6]    [5]   [12] [10]     [9]  [1]   # small enough
     |       \   /      |       \   /
    [6]     [5 12]    [10]      [1 9]
      \     /           \       /
     [5 6 12] combine   [1 9 10]
             \           /
              [1 5 6 9 10 12]
```

MergeSort - Implementation

```python
def merge_sort(A):
    n = len(A)
    if n <= 1:
        return A
    L = merge_sort(A[:n//2]) # left half
    R = merge_sort(A[n//2:]) # right half
    return merge(L, R) # key subroutine! ⭐
```

- `merge` is a procedure that take two sorted arrays and merge them into a sorted array that contains both of their elements.

```python
merge([1,3,5],[4,6,8]) # Output: [1,3,4,5,6,8]
```

Merging Two Arrays

```
def merge(L, R):
    i = j = 0
    S = []
    while i < len(L) and j < len(R):
        if L[i] < R[j]:
            S.append(L[i])
            i += 1
        else:
            S.append(R[j])
            j += 1
    while i < len(L):
        S.append(L[i])
        i += 1
    while j < len(R):
        S.append(L[k])
        j += 1
    return S
```

Correctness of MergeSort

```python
def merge_sort(A):
    n = len(A)
    if n <= 1:
        return A
    L = merge_sort(A[:n//2]) # solving left half
    R = merge_sort(A[n//2:]) # solving right half
    return merge(L, R)       # combine
```

- Goal: We want to prove that after `merge_sort(A)` finishes, **the array A is sorted in non-decreasing order**.
- This time, we'll maintain a **Recursion Invariant** that any time MergeSort returns, it returns a sorted array.

  > Whenever MergeSort returns an array of size $\leq n$, that array is sorted.

Correctness of MergeSort - Proof by Induction

- **Inductive Hypothesis**: Whenever MergeSort returns an array of size $\leq n$, that array is sorted.
- **Base Case** ($i = 0$ or $i = 1$): Whenever MergeSort returns an array of length 0 or length 1, that array is sorted. ✅ (this is trivial!)

- **Inductive Step**: Suppose the inductive hypothesis holds for $i-1$.
  - Suppose that MergeSort has an input of length i (≥ 2). Then `L` and `R` are both of length $\leq i-1$, so by induction, `L` and `R` are both sorted.
  - Therefore, we only need to show that *"When `merge` takes as inputs two sorted arrays `L` and `R`, then it returns a sorted array containing all of the elements of `L`, along with all of the elements of `R`."* (intuitively true; another proof needed!)
- **Conclusion**: By induction, the inductive hypothesis holds for all i. ✅

Running Time of `merge_sort`

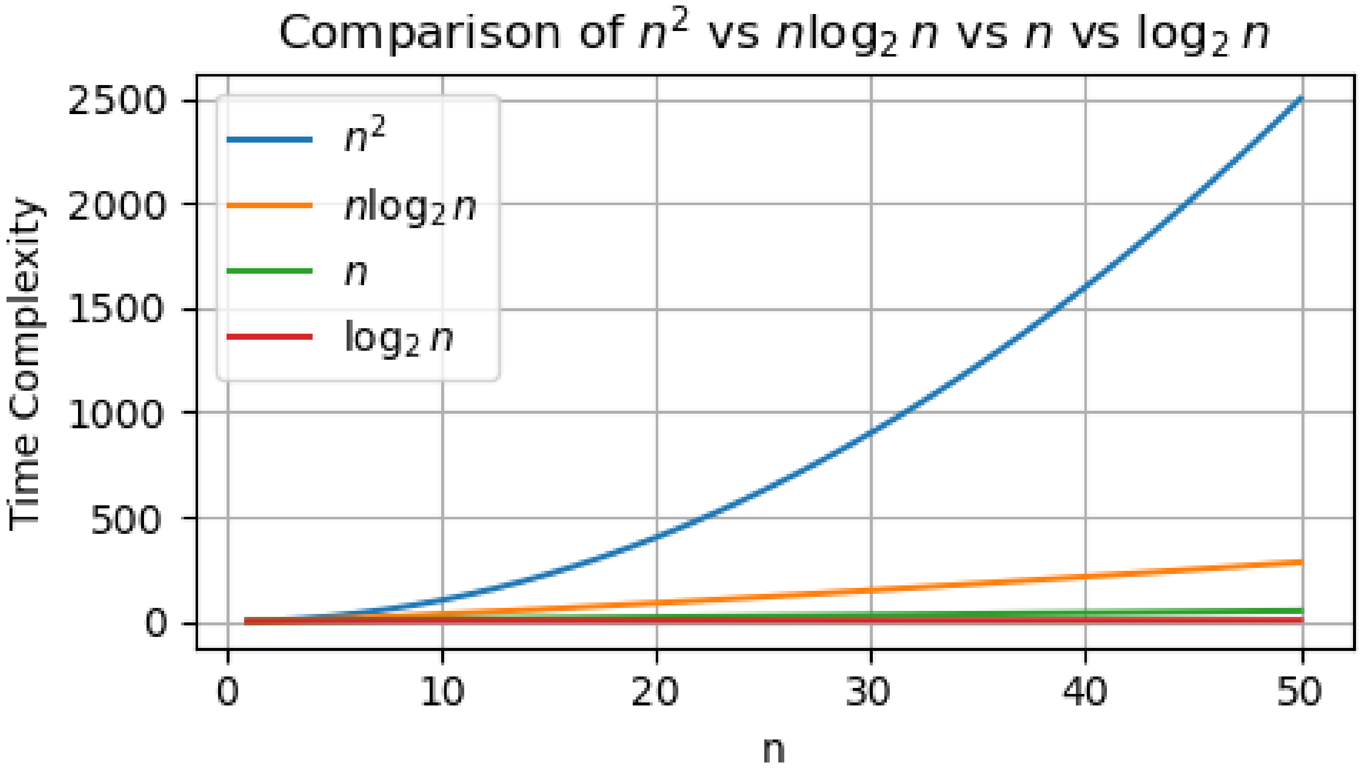
Suppose the input array A has length m.

1. Base Case Check: **2 operations**
   - Retrieving length: 1 operation
   - Comparing if $m = 1$: 1 operation
2. Recursive Call Setup: $m + 2$ **operations**
   - Copying elements into `L` and `R`: m operations
   - Storing into variables `L` and `R`: 2 operations
3. Merging Two Halves: the merge step takes $3 + 3m \leq 6m$ **operations**
   - Why? 3 assignments + m scans + m appending + m cursor increase

- Total number of operations $\leq 2 + (m + 2) + 6m \leq 11m$

Running Time of MergeSort

- **Claim**: For every input array of n numbers, Merge Sort produces a sorted output array and uses at most $11n\log_2 n + 11n$ operations. i.e., $O(n\log_2 n)$.
- MergeSort typically runs much faster than the simpler sorting algorithms with $O(n^2)$.



Proof of Claim (assuming n = power of 2)

```
level 0:                    4
                          /   \
level 1:                 2     2
                        / \   / \
level 2:               1   1 1   1
```

```
level 0:                   n
                         /   \
level 1:              n/2     n/2
                     /   \   /   \
level 2:          n/4  n/4  n/4  n/4
                   .              .
                   .              .
                   .              .
               |  |  |  |     |  |  |  |
level log_2(n):   1  1  1  1  ...  1  1  1  1
```

Proof of Claim - Finding the Pattern

```
level 0:                       n
                             /   \
level 1:                 n/2       n/2
                        /   \     /   \
level 2:             n/4   n/4   n/4   n/4
                      .                 .
                      .                 .
                 |  |  |  |         |  |  |  |
level log_2(n):  1  1  1  1  ...    1  1  1  1
```

- At each level $j = 0, 1, \ldots, \ldots, \log_2 n$, there are 2^j subproblems, each of size $n/2^j$.
- Work at each level j = (#subproblems * Work per subproblem) $\leq 2^j \cdot 11(\frac{n}{2^j}) = 11n$

Total Work = (Work per level * #levels) $\leq 11n \cdot (1 + \log_2(n)) = 11n \log_2 n + 11n$. ✅

Next Time

- Asymptotic Notation (Big-O, Big-Theta, Big-Omega)

Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/