Lecture 11a - Balanced Binary Search Tree (Red Black Tree) - Part A

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Hope you had a great Chuseok! 🌾

In today's class, we'll

- Take a Quiz#2
- Review Binary Search Tree (BST)
- Learn Self-Balancing BSTs
- Learn Red-Black Trees
    - Definition & Properties
    - Operations:
        - Tree Rotation (Left / Right)
        - Insertion (with Rotation & Recoloring) - *next class*
        - Deletion (with Rotation & Recoloring) - *next class*

Quiz #2 ⭐

- Time limit: 15 minutes
- Start time: 13:40
- Materials: Lecture slides/notes may be used (open notes, not open internet)

Binary Search Tree (BST) 🌲

A binary search tree is a binary tree where all keys in a node's left subtree are less than the node's key, and all keys in its right subtree are greater.

```
      4                 4                 9                 1
    /   \             /   \             /   \             /   \
   2     5           2     6           2     6           2     6
  / \     \         / \   / \         / \   / \         / \
 1   3     6       1   3 5   7       1   4 7   8       3   4

      BST               BST            Not a BST         Not a BST
```

`search(i)`

```
      4
    /   \        search(3)
   2     5       -> 4 (large! search in the left tree)
  / \     \      -> 2 (small! search in the right tree)
 1   3     6     -> 3 (found!!)
```

- **What if the element does not exist in our BST?**: We simply return the node that would be the parent of this node if we inserted it into our tree.

```
      4
    /   \       search(3.5)
   2     5      -> 4 (large! search in the left tree)
  / \     \     -> 2 (small! search in the right tree)
 1   3     6    -> 3 (small! but the right child is NIL; return node 3)
      \
     (3.5) <- does not exist
```

`insert(i)`

- Find the parent node `x` where `i` should be inserted using `search(i)`.
- Create a new node `y` with `key(y)=i` and no children and attach `y` to `x`.

Example: `insert(3)`

```
      4
    /   \          -> search(3)
   2     5         -> 4 (large! search in the left tree)
  /       \        -> 2 (small! search in the right tree)
 1         6       -> NIL (parent node found)
```

```
      4
    /   \
   2     5
  / \     \
 1   3     6
```

`delete(i)`

- Delection is a bit more complicated.
- To delete a node `x` that exists in our tree, we consider several cases:
    - Case 1: `x` has no childeren
        - We simply remove it.
    - Case 2: `x` has only one child `c`
        - We **elevate** `c` to take `x`'s position in the tree.
    - Case 3: `x` has two childern, a left child `c1` and right child `c2`
        - We find `x`'s **immediate successor** `z` and have `z` take `x`'s position in the tree. Then, we remove the original succesor (Case 1 or Case 2).

```
        4                                   4
      /   \      --- delete(5) --->       /
     2     5                             2
```

```
        4                                   4
      /   \                               /   \
     2     5     --- delete(5) --->      2     6
    / \     \                           / \
   1   3     6                         1   3
```

```
      2                                             3               3
    /   \                                         /   \           /   \
   1     5  -- delete(2) --> successor = 3 -->   1     5   -->   1     5
        /                                             /               /
       3                                             3*              4
        \                                             \
         4                                             4
```

Time Complexity of `search(i)`, `insert(i)`, and `delete(i)`

- **Search** runs in $O(\text{height of tree})$ in the worst case.
- **Insert** and **delete** call `search` a constant number of times and do $O(1)$ extra work, so their runtimes are also $O(\text{height of tree})$.
- Height of tree:
  - **Best case** (completely balanced): $O(\log n)$, **Worst case** (long chain): $O(n)$

```
    2                    3                       1
   / \                  /                         \
  1   3                2                           2
                      /                             \
                     1                               3
 Balanced        Long Lefttward Path    Long Rightward Path
```

It's Time to Learn Self-Balancing BSTs.

Operation	Sorted Array	BST (height h)	Balanced BST
Search	$\Theta(\log n)$	$\Theta(h)$	$\Theta(\log n)$
Insert	$\Theta(n)$ 🤔	$\Theta(h)$	$\Theta(\log n)$
Delete	$\Theta(n)$ 🤔	$\Theta(h)$	$\Theta(\log n)$

- Sorted arrays: fast search, slow updates.
- Unbalanced BST: updates are fast **only if** $h(\text{height of tree}) = O(\log n)$.
  - Worst case: $h = O(n)$ → linked-list-like performance.
- Solution: **Self-balancing BSTs** (AVL tree, red–black tree, etc.).

Idea of Balancing ⚖️

- Keep tree height close to $\log n$ after every insertion/deletion.
- **Self-balancing BSTs**:
    - Detect when the tree is becoming too tall or skewed.
    - Apply **rotations** and **recoloring/restructuring** to restore balance.
- Many variants exist such as AVL trees (strict height balance) and Red–Black trees (color-based, less strict, easier insert/delete), Splay trees, Treaps, etc.
- In this class, we focus on **Red–Black Trees**.
    - Widely used in libraries (`std::map` , `TreeMap`).

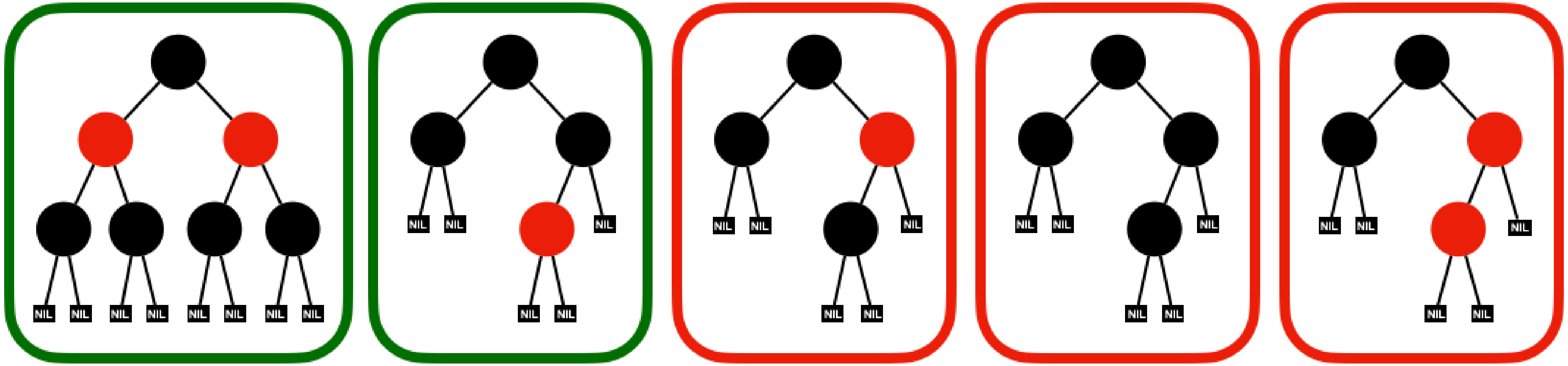
Red–Black Tree: Definition

A Red–Black Tree is a BST with an extra bit of storage (color) per node.

Properties:

1. Every node is red or black.
2. The root is black.
3. NILs are black.
4. The children of a red node are black.
5. For every node x, all x to NIL paths have the same number of black nodes on them.

Examples



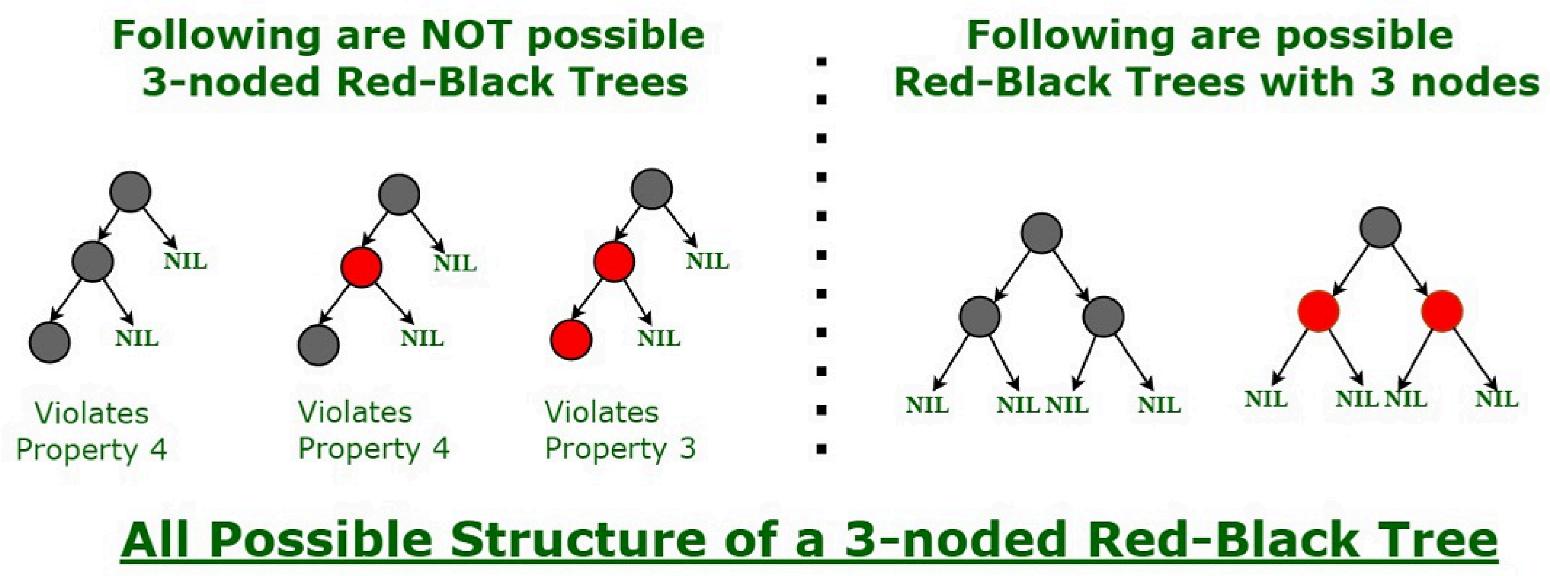
Violating 5 Violating 5 Violating 4

1. Every node is red or black.
2. The root is black.
3. NILs are black.
4. The children of a red node are black.
5. For every node x, all x to NIL paths have the same number of black nodes on them.

Why These Properties Work

- Property 4 limits consecutive reds → **prevents skewed long chains of red nodes**.
- Property 5 forces black nodes to be evenly distributed → **keeps height small**.

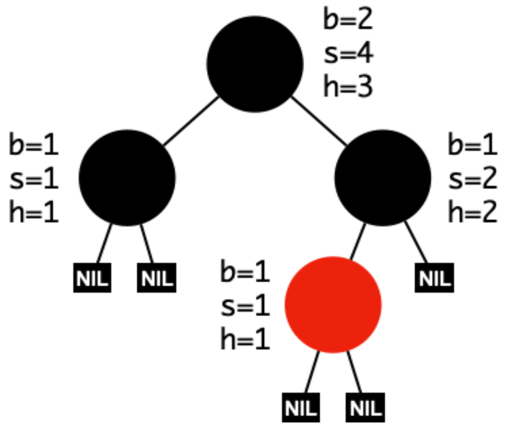
> A chain of 3 nodes is not possible in the Red-Black tree.



Claim: A red–black tree with n nodes has height $\leq 2\log_2(n+1) = O(\log n)$

Proof. For a node x,

- let $b(x)$ be its *black height* (i.e., # black nodes on any $x \to \mathrm{NIL}$ path excluding x)
- let $s(x)$ be the subtree size (including x).



- Notice that the black height is at least half of the height because there are **no two consecutive red nodes** on any root-to-NIL path.
  - $b(x) \geq \frac{h(x)}{2}$ (where $h(x)$ is the height of x)

Claim: A red–black tree with n nodes has height $\leq 2\log_2(n+1) = O(\log n)$

Proof (Continued).

Let's show $s(x) \geq 2^{b(x)} - 1$ via induction on the height of x:

Base case ($h = 0$): NIL node has $b(x) = 0$ and $2^0 - 1 = 0$ non-NIL descendants. ✅

Inductive step: Using the *IH* $s(y) \geq 2^{b(y)} - 1$ for children y of x, i.e., $b(y) \geq b(x) - 1$

$$s(x) = 1 + s(\text{left}(x)) + s(\text{right}(x)) \geq 1 + \left(2^{b(x)-1} - 1\right) + \left(2^{b(x)-1} - 1\right) = 2^{b(x)} - 1$$

Therefore, $s(x) \geq 2^{b(x)} - 1$ for every x in a red-black tree. ✅

Finally, when r is a root,

$$n = s(r) \geq 2^{b(r)} - 1 \geq 2^{h(r)/2} - 1 \quad \rightarrow \quad h = h(r) \leq 2\log_2(n+1) \quad \blacksquare$$

Live Visualization: Red–Black Tree

🕹️ **Try it:** https://www.cs.usfca.edu/~galles/visualization/RedBlack.html

What to observe as you insert/delete:

- **No two reds in a row** (Property 4)
- **Uniform black-height** on all root→leaf paths (Property 5)
- **Rotations** (left/right) and **recoloring** after each update

Suggested demo sequence

- Insert: `41, 38, 31, 12, 19, 8`
- Delete: `8, 12, 19, 38`

Basic Operations on a Red-Black Tree

1. **Search**: Same as in a normal BST (binary search based on key ordering).
2. **Insertion**: Insert as in a normal BST, then fix any violations of red-black properties via **rotations and recoloring**.
3. **Deletion**: Delete as in a normal BST, then restore red-black properties through **rotations and recoloring.**

Rotation

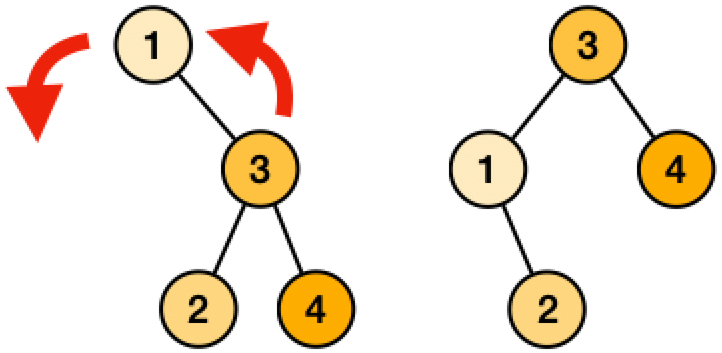
- **Purpose**: Maintain balance after insertion/deletion
  - Tree rotations can be performed in $O(1)$ time.
- **Types**:
  1. Left Rotation
  2. Right Rotation

Recall: BST's Subtree Property

```
1)         x                 2)      x
         /   \                     /   \
        y     ...               ...     y
      / \                              / \
   ...   z                            z   ...
        / \                          / \
```

1. If `left(x)` is `y` and `right(y)` is `z`, then all keys in `z`'s subtree satisfy:
   - `y` < keys < `x`
2. If `right(x)` is `y` and `left(y)` is `z`, then all keys in `z`'s subtree satisfy:
   - `x` < keys < `y`

Left Rotation

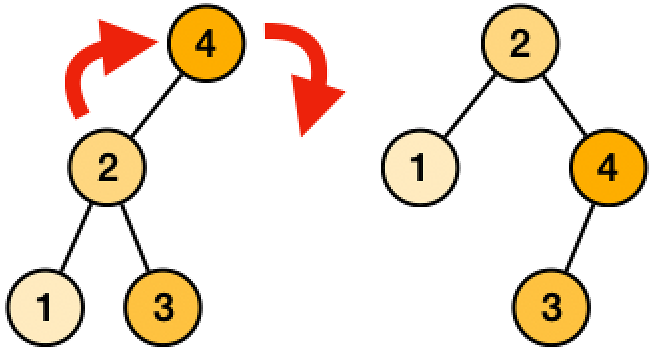


- **Goal**: Move a node down to the left, bring its right child up.

```
Before Rotation | After Rotation
      x         |         y
       \        |        / \
        y       |       x   b
       / \      |        \
      a   b     |         a
```

  - x < a < y < b

Right Rotation

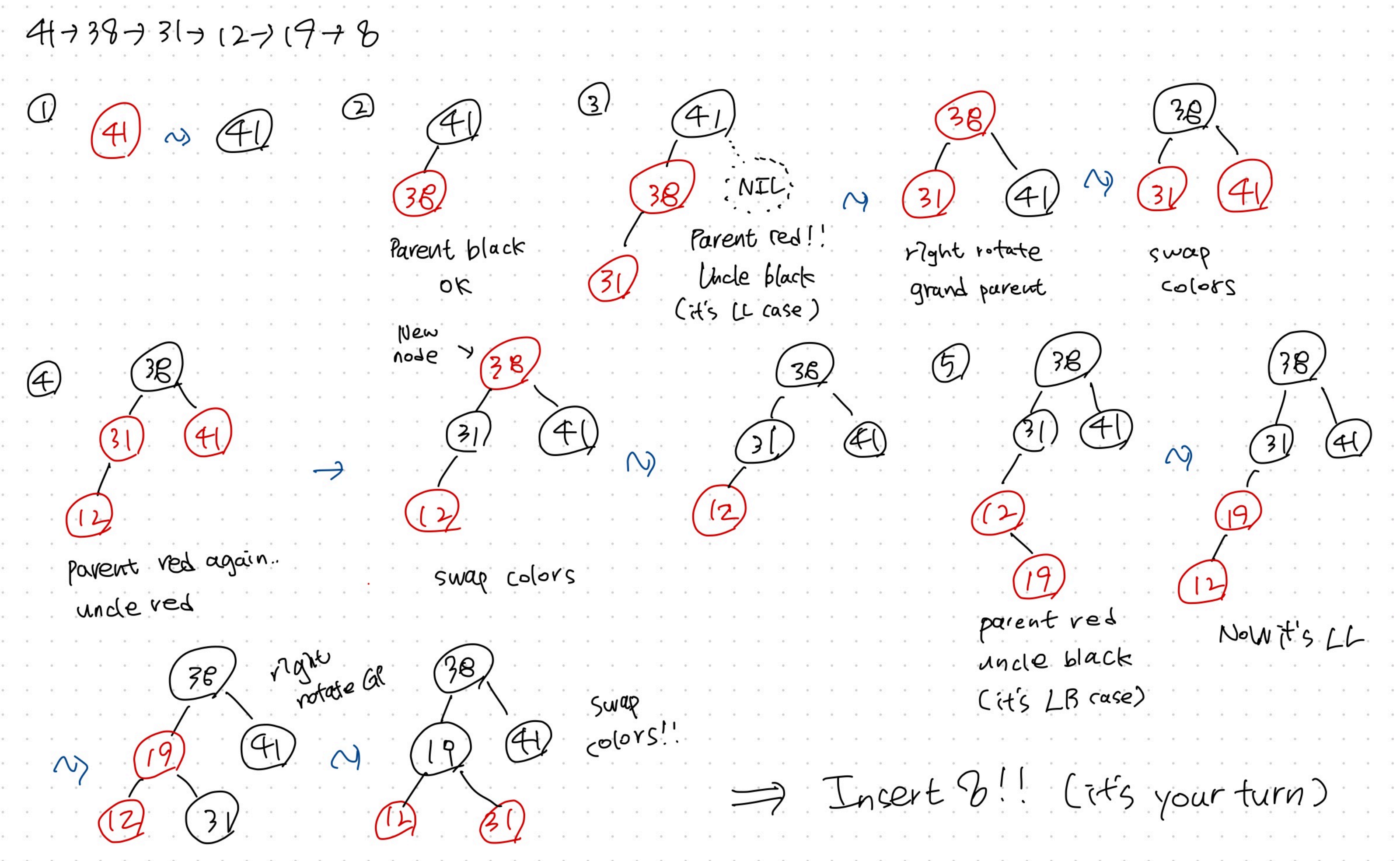


- **Goal**: Move a node down to the left, bring its right child up.

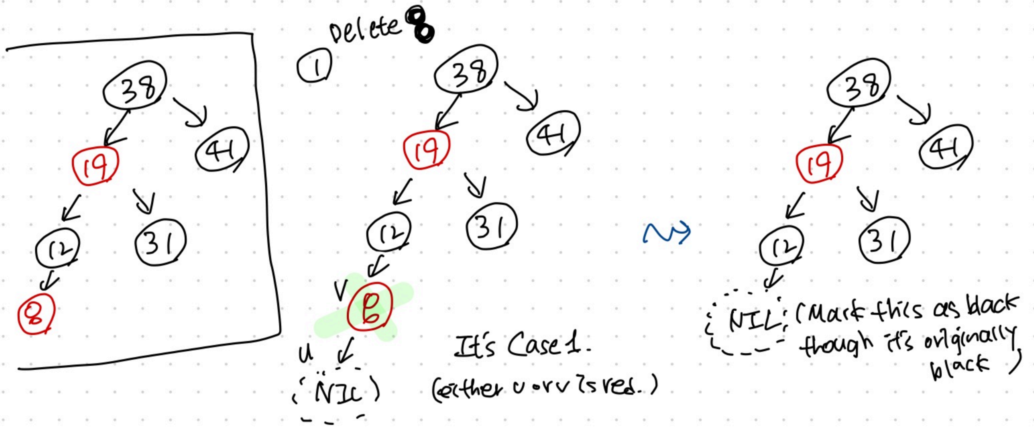
```
Before Rotation | After Rotation
        x       |        y
       /        |       / \
      y         |      a   x
     / \        |         /
    a   b       |        b
```

  - a < y < b < x

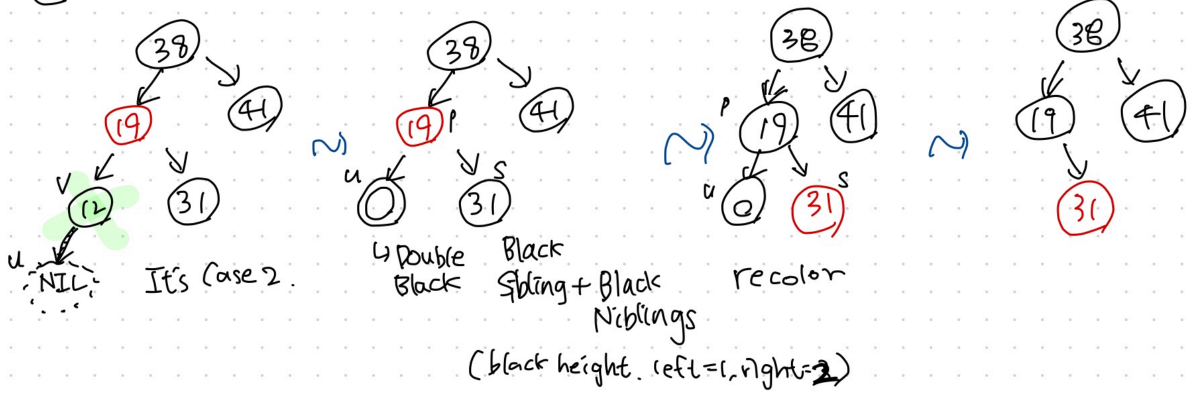
Next Class: Red-Black Tree Insertion / Deletion



Let's delete 8 → 12 → 19 → 38



② Delete 12



Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/

Reference:

- https://www.geeksforgeeks.org/dsa/introduction-to-red-black-tree
- https://www.geeksforgeeks.org/dsa/c-program-red-black-tree-insertion/
- https://studyglance.in/ds/display.php?tno=27&topic=Red-Black-Tree