Lecture 22 - Min Cut, Max Flow and Ford-Fulkerson - Part A

Fall 2025, Korea University

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Quiz #4

- Time limit: 20 minutes
- Start time: 13:40
- Materials: Lecture slides/notes may be used (open notes, not open internet)
- Access code:

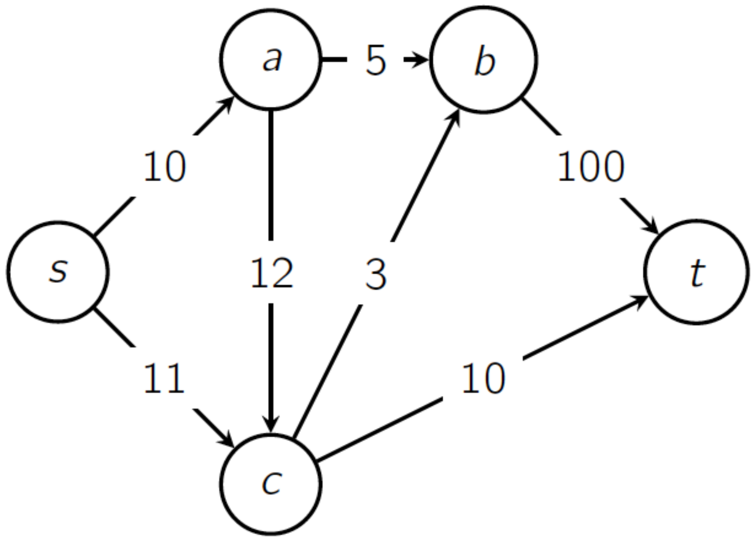
Roadmap of Min Cut, Max Flow and Ford-Fulkerson

1. **Minimum Cut Problem** ⭐
2. **Maximum Flow Problem** ⭐
3. **Max-Flow = Min-Cut Theorem** ⭐
4. Ford–Fulkerson Algorithm
5. Runtime Analysis
6. Applications of Max Flow

Setup

Given a **directed** graph $G = (V, E)$ with:

- **Source** node s and **target** node t
- Each edge (u, v) has capacity $c(u, v) \geq 0$

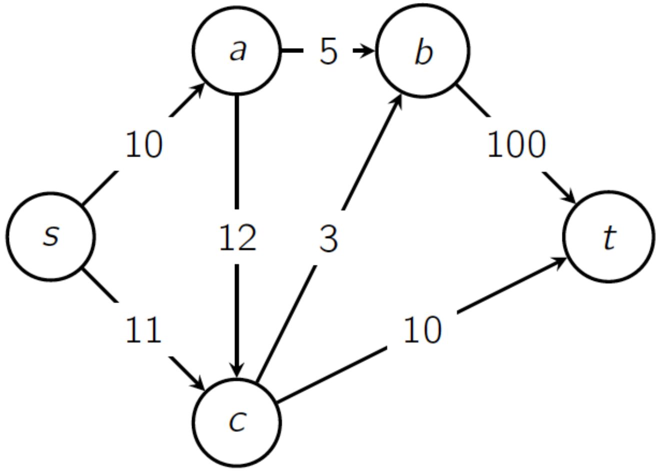


Minimum Cut Problem

Partition $V = S \cup T$ with $s \in S, t \in T, S \cap T = \emptyset$ (disjoint)

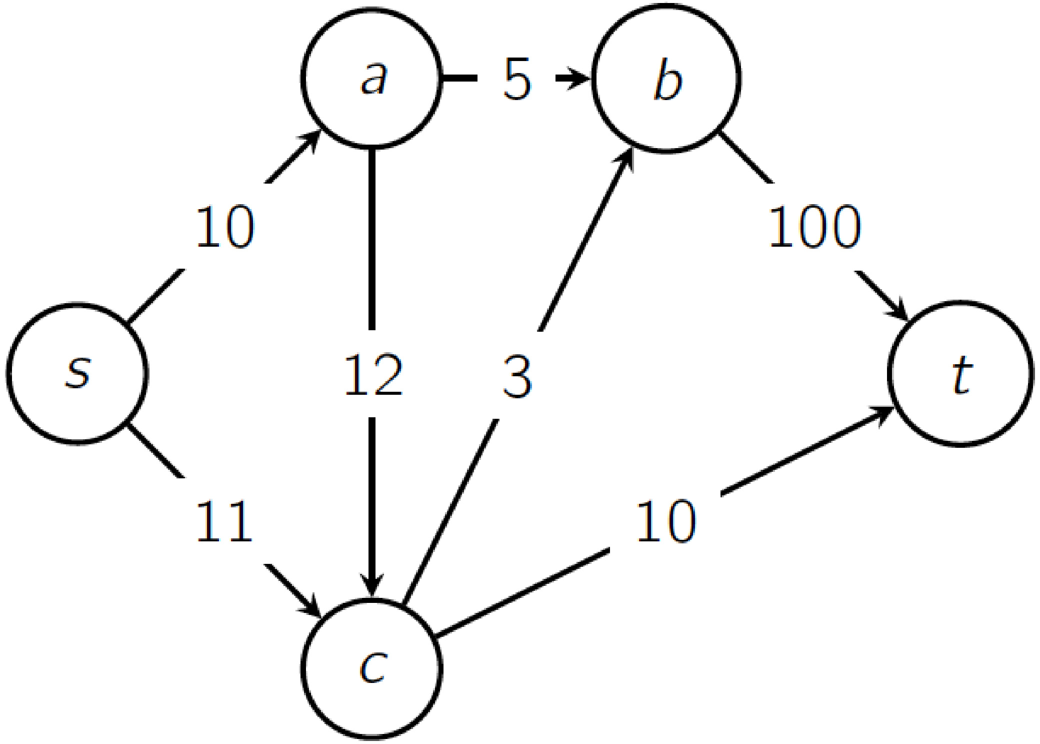
The **cost of a cut**: $c(S,T) = \sum_{x \in S,\, y \in T} c(x,y)$ (*does not count the edge from T to S*)

Minimum cut = a cut with the minimum cost



$$S = \{s, a, c\}, T = \{b, t\} \quad \rightarrow \quad c(S,T) = c(a,b) + c(c,b) + c(c,t) = 5 + 3 + 10 = 18$$

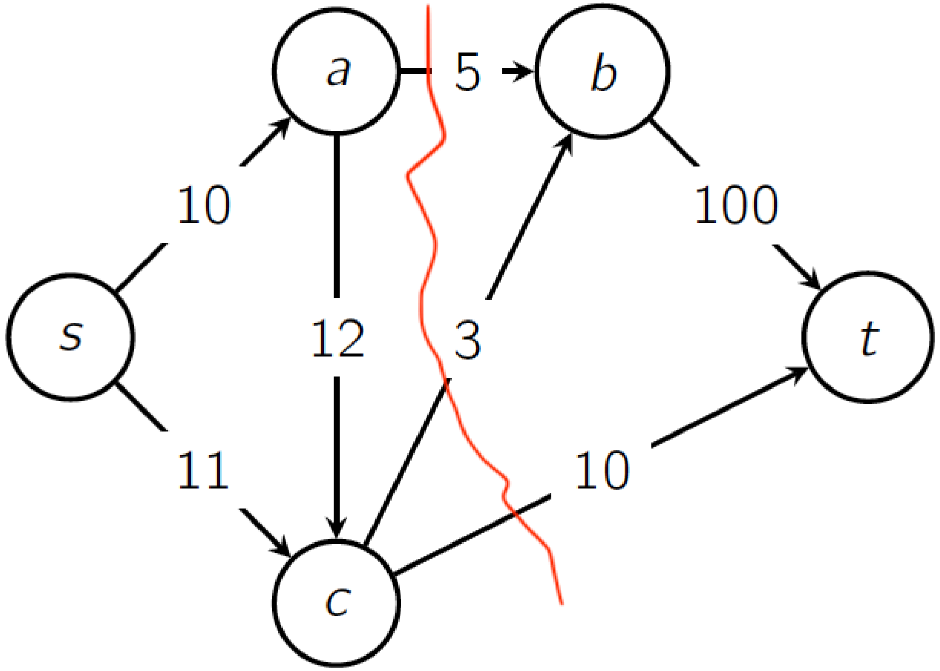
How to find the minimum cut?



2^3 possibilities

- $S = \{s\}, T = \{a, b, c, t\}$
- $S = \{s, a\}, T = \{b, c, t\}$
- $S = \{s, b\}, T = \{a, c, t\}$
- $S = \{s, c\}, T = \{a, b, t\}$
- $S = \{s, a, b\}, T = \{c, t\}$
- $S = \{s, a, c\}, T = \{b, t\}$
- $S = \{s, b, c\}, T = \{a, t\}$
- $S = \{s, a, b, c\}, T = \{t\}$

Quantifying the Bottleneck using Min-Cut Capacity



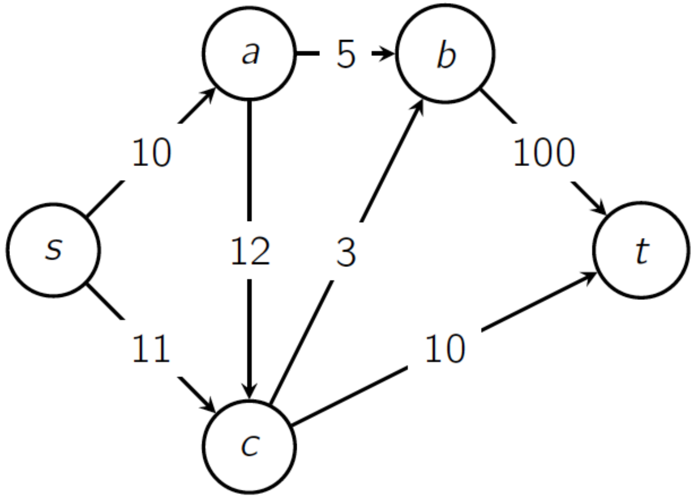
- Min-cut capacity precisely quantifies the network's **most critical bottleneck**.
  - If the min-cut capacity is 18, the network cannot, under any circumstances, handle a flow exceeding 18, regardless of the route taken.

How Do We Compute a Min-Cut?

Maximum Flow Problem

Given a **directed** graph $G = (V, E)$ with:

- **Source** s and **sink** t
- Each edge (u, v) has capacity $c(u, v) \geq 0$

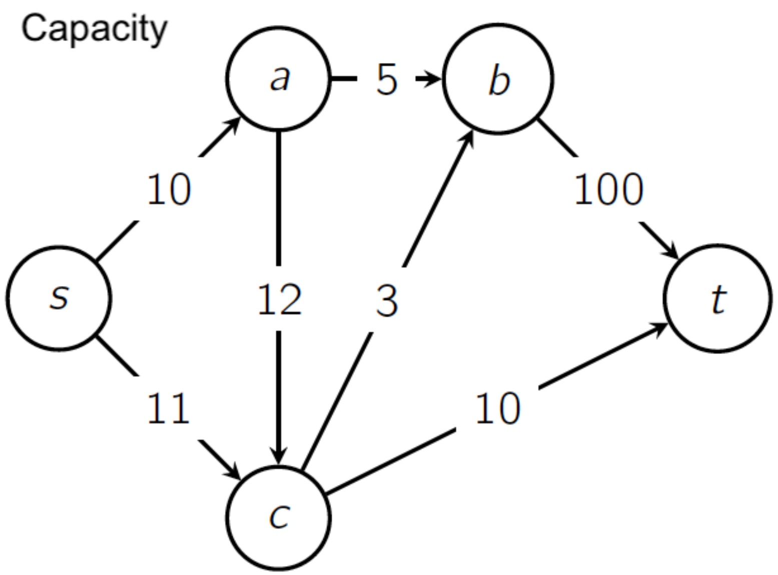


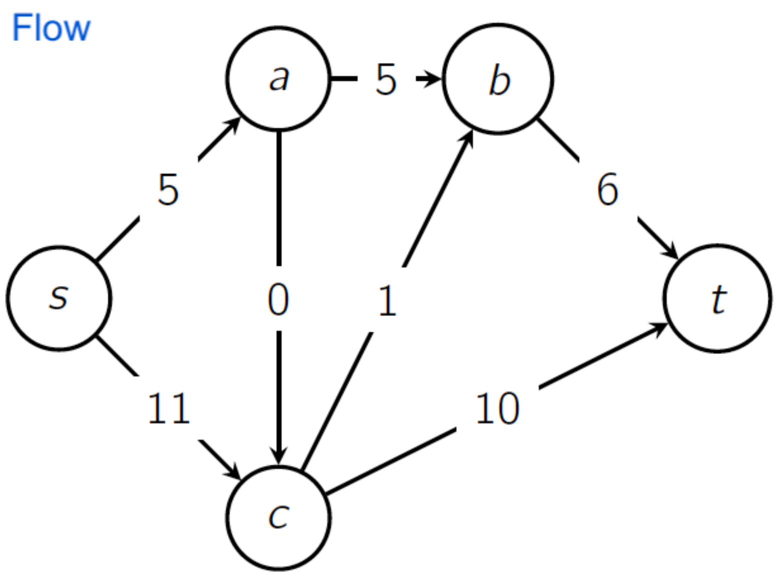
Max-Flow problem:

We seek a **flow** $f : E \to \mathbb{R}_{\geq 0}$ that obeys two constraints (*Capacity constraint* and *Flow conservation*) and **maximizes** the total amount sent from s to t.

Flow Constraint 1. Capacity Constraint

$$0 \le f(u,v) \le c(u,v)$$

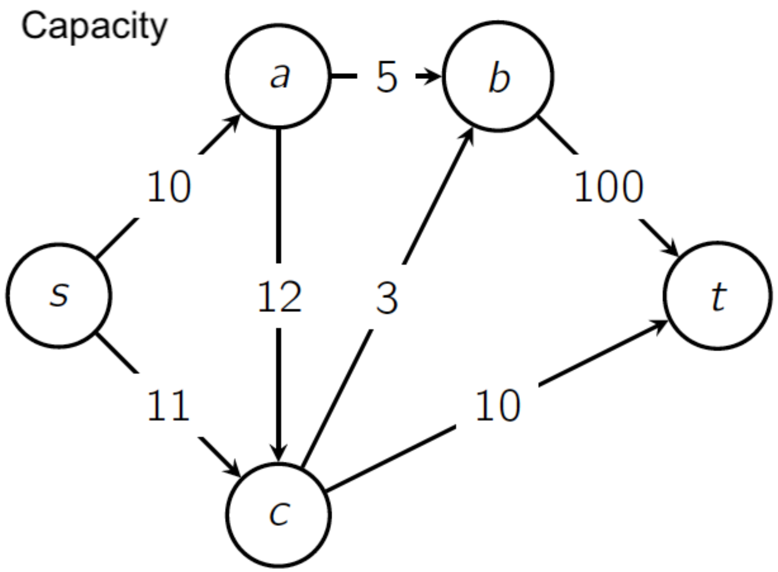


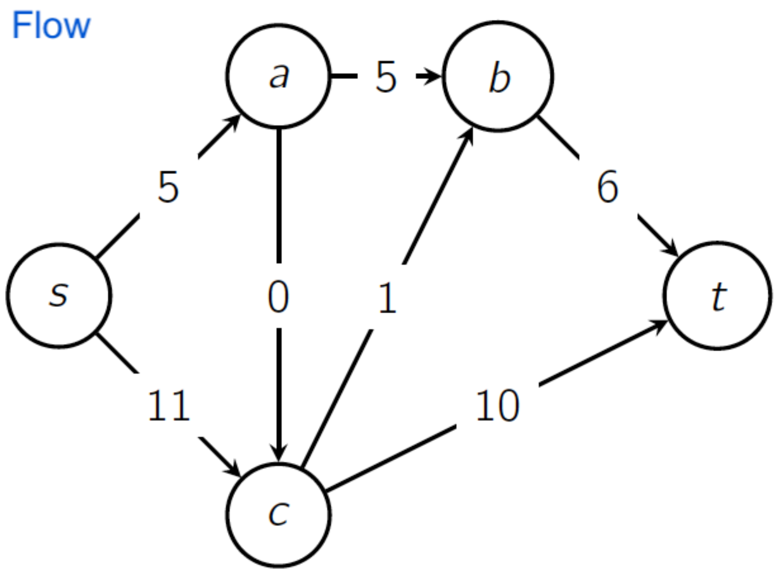


Flow Constraint 2. Flow Conservation

$$\sum_{x \in N_{in}(v)} f(x,v) = \sum_{y \in N_{out}(v)} f(v,y), \quad \forall v \in V - \{s,t\}$$

- $N_{in}(v)$: the set of nodes with an edge that points to v
- $N_{out}(v)$: the set of nodes that v points to





Flow Value

If no edges enter s and none leave t:

$$|f| = \sum_{x \in N_{out}(s)} f(s,x) = \sum_{y \in N_{in}(t)} f(y,t)$$

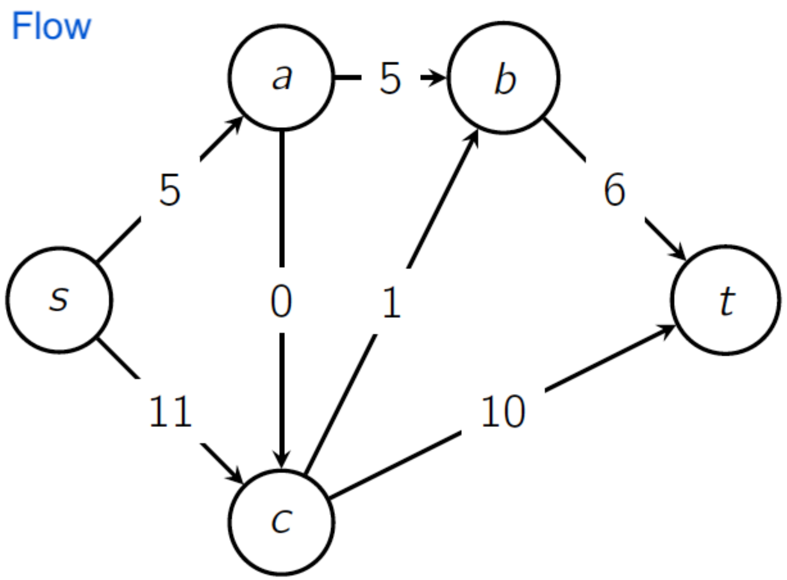
Otherwise:

$$|f| = \sum_{x \in N_{out}(s)} f(s,x) - \sum_{y \in N_{in}(s)} f(y,s)$$

Maximum Flow Problem

Goal: Find a flow f that **maximizes** $|f|$.

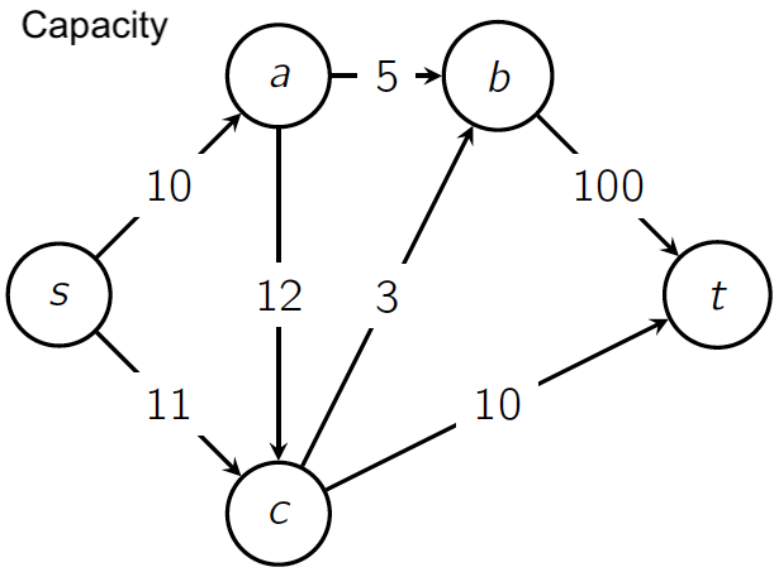
Example

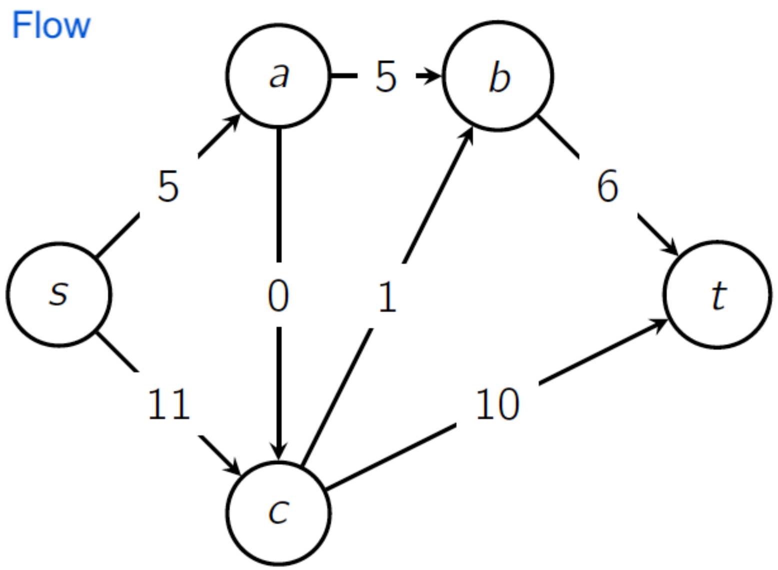


The current flow value is:

$$|f| = \sum_{x \in N_{out}(s)} f(s,x) = \sum_{y \in N_{in}(t)} f(y,t) = 16$$

The maximum flow is 18.

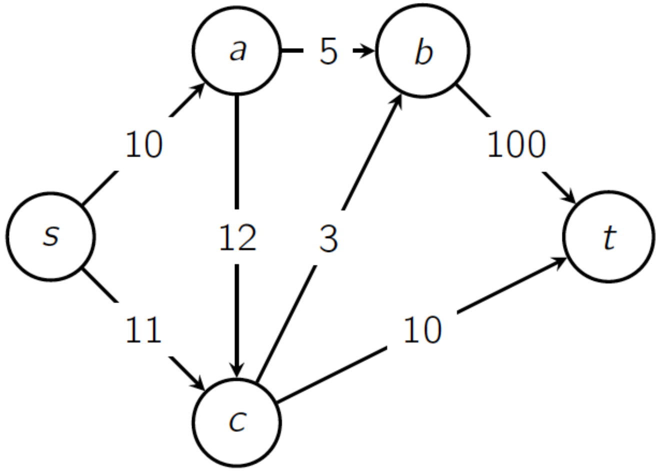




- You can still increase the flow of $s \to a$, $a \to c$, $c \to b$, and $b \to t$ by 2.

💡 Lemma: Max Flow $\leq$ Min Cut

For any flow f and cut (S, T), we have $|f| \leq c(S, T)$.



Intuition:

- Flow *out of* S cannot exceed the total **capacity** of edges *leaving* S.
- Every unit of flow from s to t must cross from S to T somewhere.

Lemma: Max Flow $\leq$ Min Cut (Proof)

$$
\begin{aligned}
|f| &= \sum_{x \in N_{out}(s)} f(s,x) - \sum_{y \in N_{in}(s)} f(y,s) + \sum_{v \in S \setminus \{s\}} \left(\sum_{x \in N_{out}(v)} f(v,x) - \sum_{y \in N_{in}(v)} f(y,v) \right) \\
&= \sum_{v \in S} \left(\sum_{x \in N_{out}(v)} f(v,x) - \sum_{y \in N_{in}(v)} f(y,v) \right) \\
&= \sum_{v \in S} \left(\sum_{x \in N_{out}(v) \cap S} f(v,x) - \sum_{y \in N_{in}(v) \cap S} f(y,v) \right) + \sum_{v \in S} \left(\sum_{x \in N_{out}(v) \cap T} f(v,x) - \sum_{y \in N_{in}(v) \cap T} f(y,v) \right) \\
&= \sum_{v \in S} \left(\sum_{x \in N_{out}(v) \cap T} f(v,x) - \sum_{y \in N_{in}(v) \cap T} f(y,v) \right) \\
&\leq \sum_{v \in S, x \in T, x \in N_{out}(v)} f(v,x) \leq \sum_{v \in S, x \in T, x \in N_{out}(v)} c(v,x) = c(S,T)
\end{aligned}
$$

Corollary

If there exists a cut (S, T) such that

$$|f| = c(S, T)$$

then the flow f is a **maximum flow** and (S, T) is a **minimum cut**.

Theorem (Max-Flow Min-Cut Theorem)

For any graph G,

$$\max_{f} |f| = \min_{(S,T)} c(S, T).$$

that is, the value of the maximum flow equals the capacity of the minimum cut.

Next Time: Ford–Fulkerson Algorithm

Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/