Lecture 14 - Single Source Shortest Path Problem and Dijkstra's Algorithm

Fall 2025, Korea University

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Course Outline (After Midterm)

- Part 3: Data Structures
  - Graphs, Graph Search (DFS, BFS) and Applications (**Finding SSCs w/ DFS**) 👈
- Part 4: Dynamic Programming
  - **Shortest-Path: Dijkstra** 👈, Bellman-Ford, Floyd-Warshall Algorithms
  - More General DP: Longest Common Subsequence, Knapsack Problem
- Part 5: Greedy Algorithms and Others
  - Scheduling Problem, Optimal Codes
  - Minimum Spanning Trees
  - Max Flow, Min Cut and Ford-Fulkerson Algorithms
  - Stable Matching, Gale-Shapley Algorithm

Review: Algorithm to Find SCCs

Kosaraju's Algorithm: A linear time algorithm to find the strongly connected components of a directed graph.

Step 1. Reverse the graph $G \rightarrow G^{rev}$.

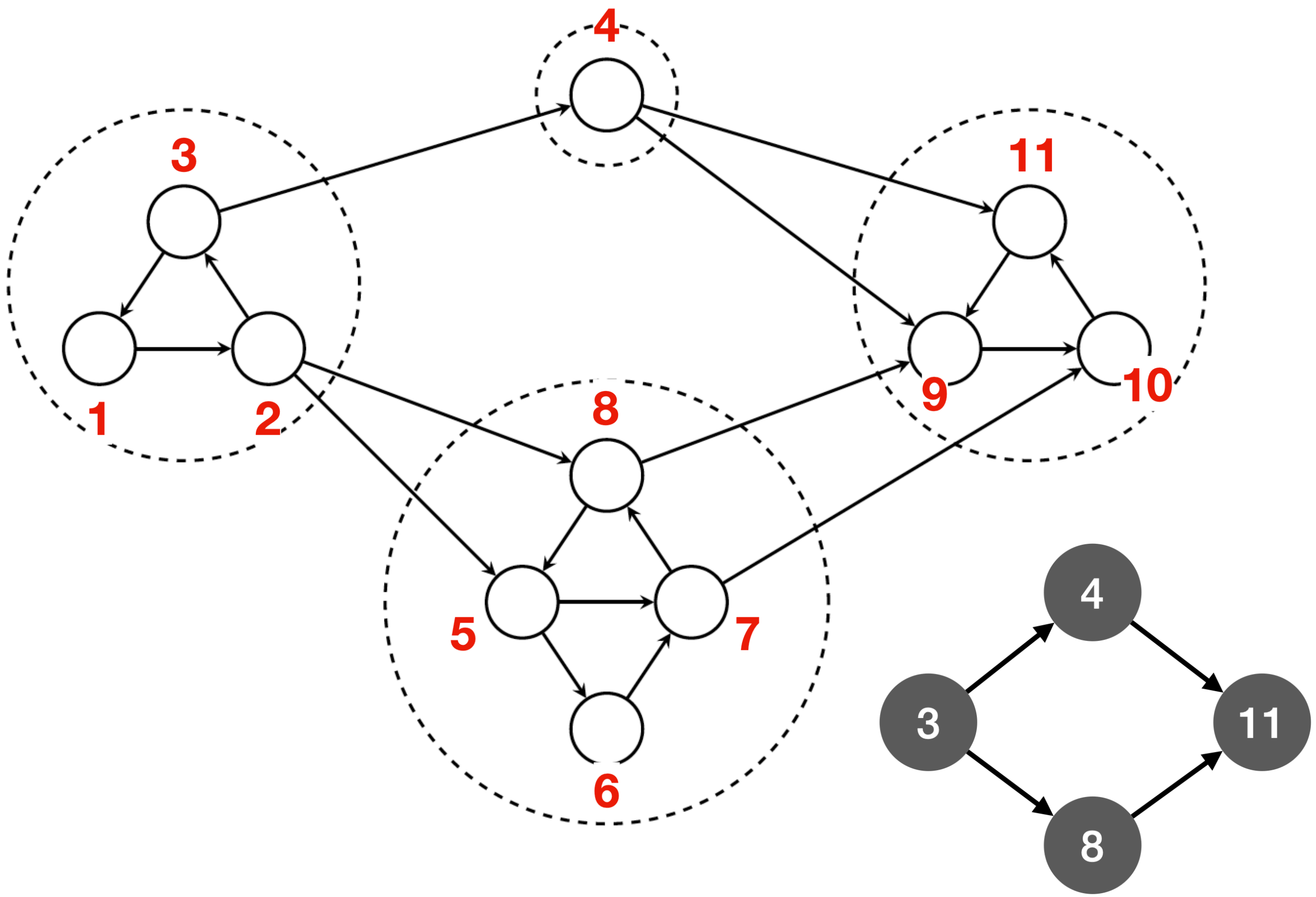
Step 2. Run DFS on G^{rev} (any order), compute finishing times $f(v)$.

- If there is an edge $C_1 \rightarrow C_2$ (between SCCs), then $\max_{v \in C_1} f(v) < \max_{v \in C_2} f(v)$.

Step 3. Run DFS on G, processing vertices in **decreasing order of** $f(v)$ and assigning a "leader" to each vertex (i.e., the source vertex that the DFS started from).

After Step 1 and 2...

> If there is an edge $C_1 \to C_2$ (between SCCs), then $\max_{v \in C_1} f(v) < \max_{v \in C_2} f(v)$



Applications of SCCs

🧑‍🤝‍🧑 Social Network Analysis

- Identify *communities* where users are mutually reachable

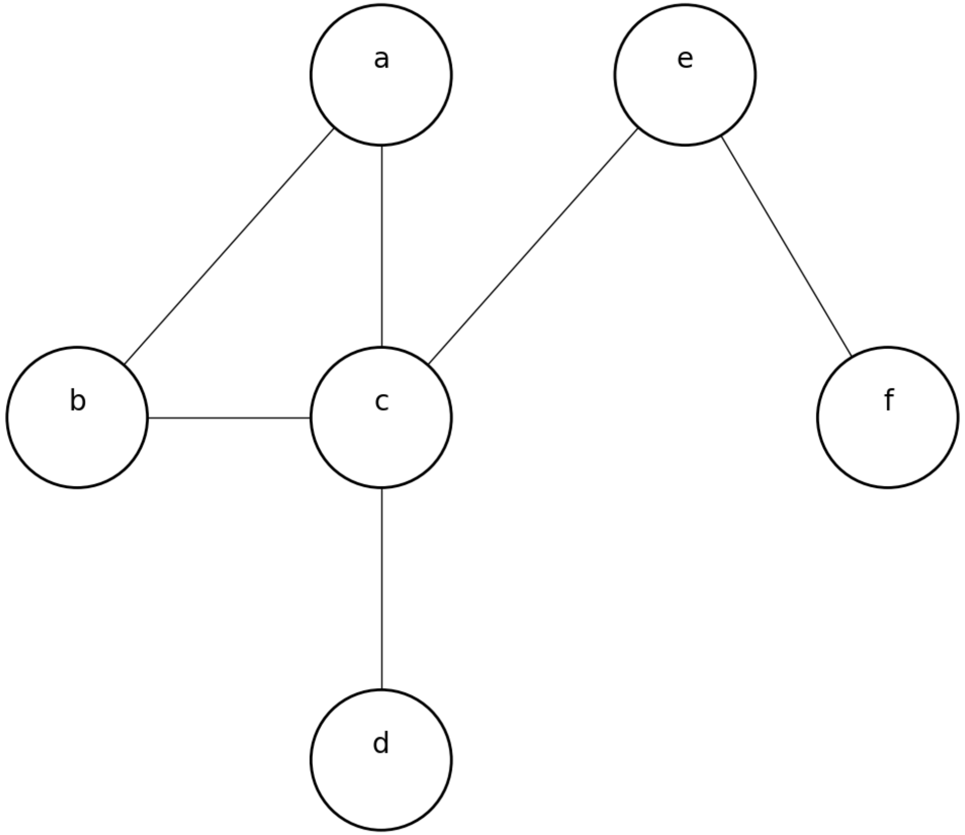
🔒 Deadlock Detection

- Represent processes/resources as a directed graph
- Cyclic SCCs indicate potential *deadlocks*

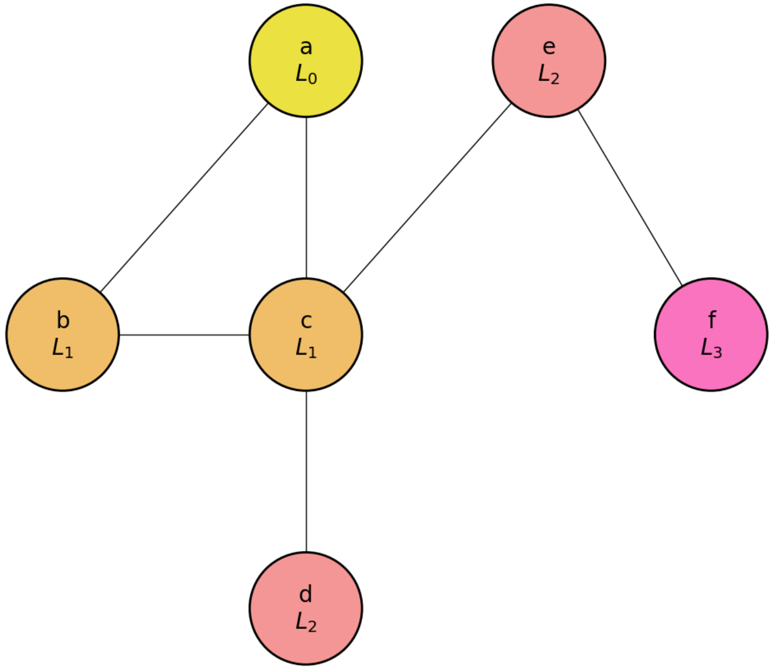
🧮 Simplifying Graph Problems

- Collapse SCCs into single nodes → *condensed DAG*
- Enables efficient algorithms (e.g., topological sort, reachability)

Today's Problem: How to Find the Shortest Path from `a` to every other node?



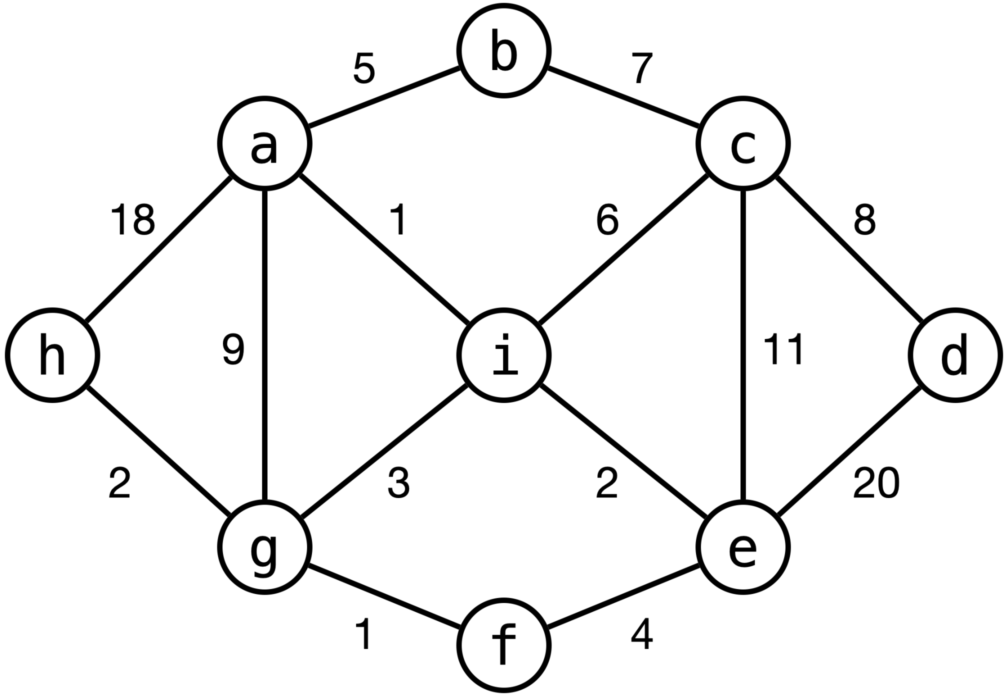
Use BFS!



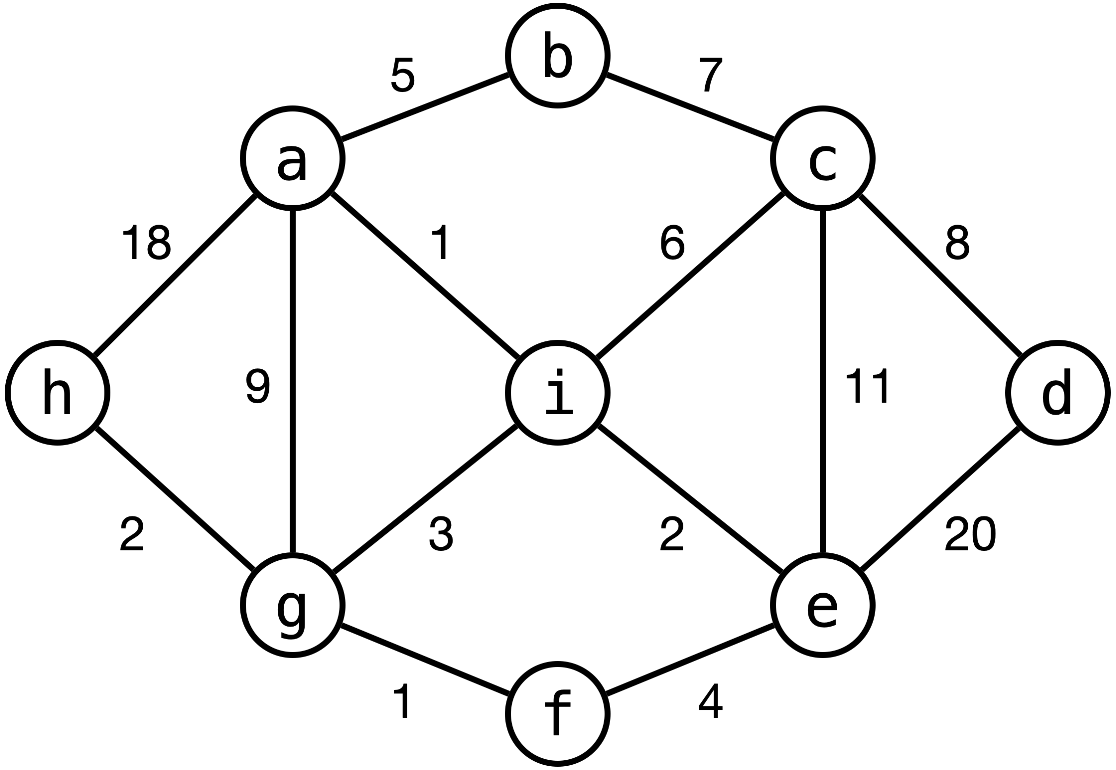
- In an **unweighted graph**, BFS can find the shortest path between two vertices.
- But what happens when the graph has **edge weights**?

The Single-Source Shortest Paths (SSSP) Problem

- **Input**: a graph $G = (V, E)$ with **nonnegative edge weights**, and a source vertex s
- **Task**: Compute the minimum-weight (shortest) path from s to every other vertex in G

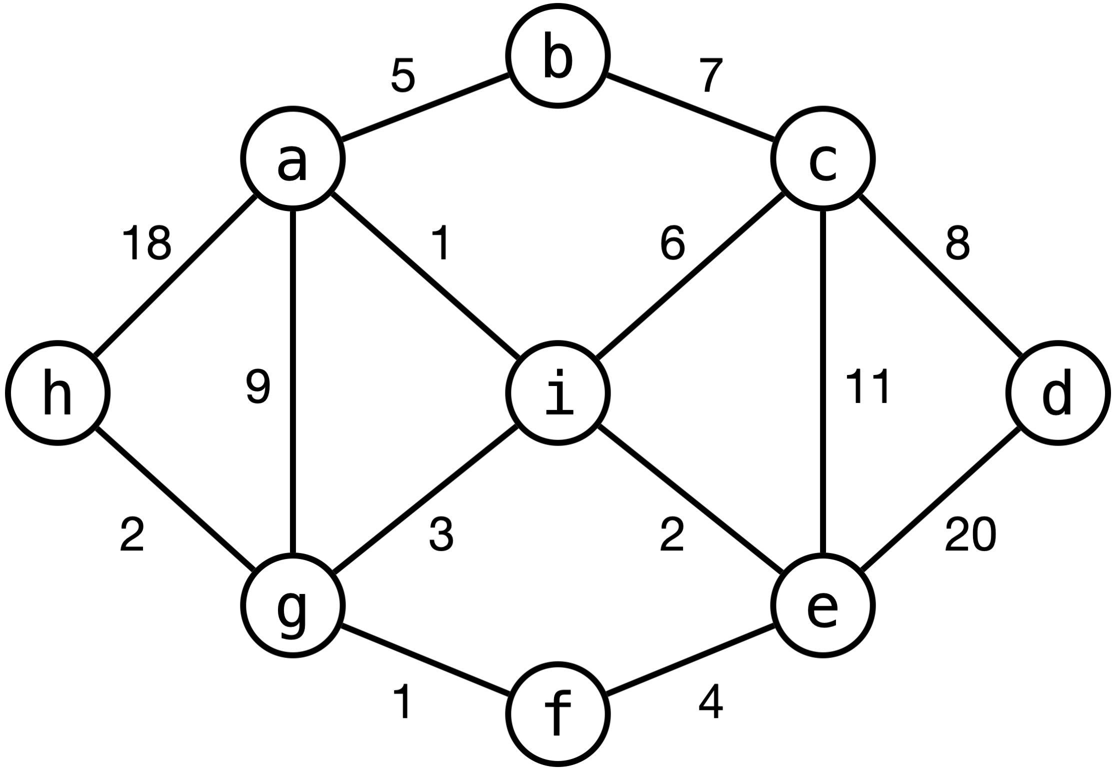


Example



- Starting from source node `a`, the goal is to find the shortest path from `a` to every other node (`b`, `c`, `d`, `e`, `f`, `g`, `h`, and `i`)
- For instance, the shortest path from `a` to `h` is not `a → h` (cost=18), but `a → i → g → h` (cost=6).

Example: Shortest Paths from `a`



target	shortest path	total cost
`b`	`a → b`	5
`c`	`a → i → c`	7
`d`	`a → i → c → d`	15
`e`	`a → i → e`	3
`f`	`a → i → g → f`	5
`g`	`a → i → g`	4
`h`	`a → i → g → h`	6
`i`	`a → i`	1

Applications of SSSP

Finding the shortest path from a source vertex to every other vertex has countless real-world applications, including:

- 🕸️ **Networking**: sending a message from one computer to all others as quickly as possible
- 🚛 **Logistics**: routing shipments from a central distribution center along the most efficient paths
- 🦠 **Epidemiology**: modeling how infectious diseases spread through social networks

And these are only a few examples!

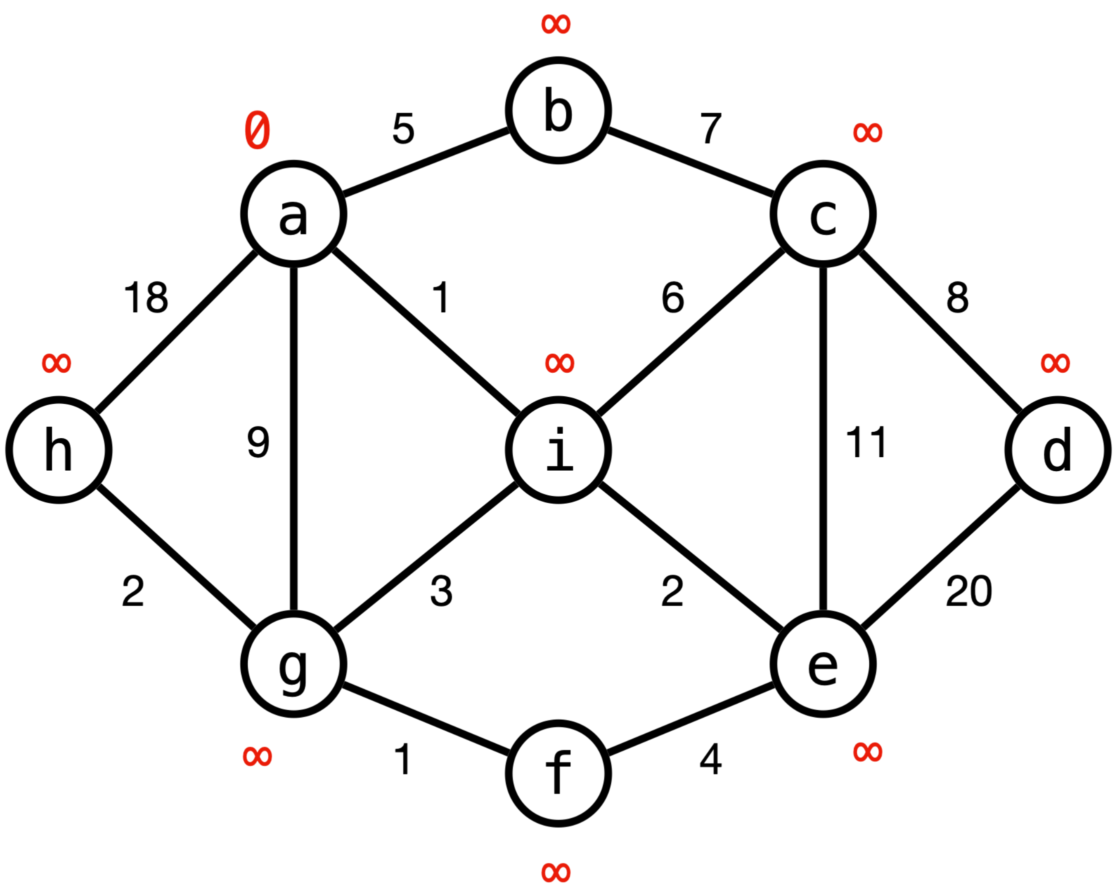
Solving SSSP: Dijkstra's Algorithm ✨

Key Idea:

Maintain distance estimates $d[t]$ for all nodes $t \in V$.

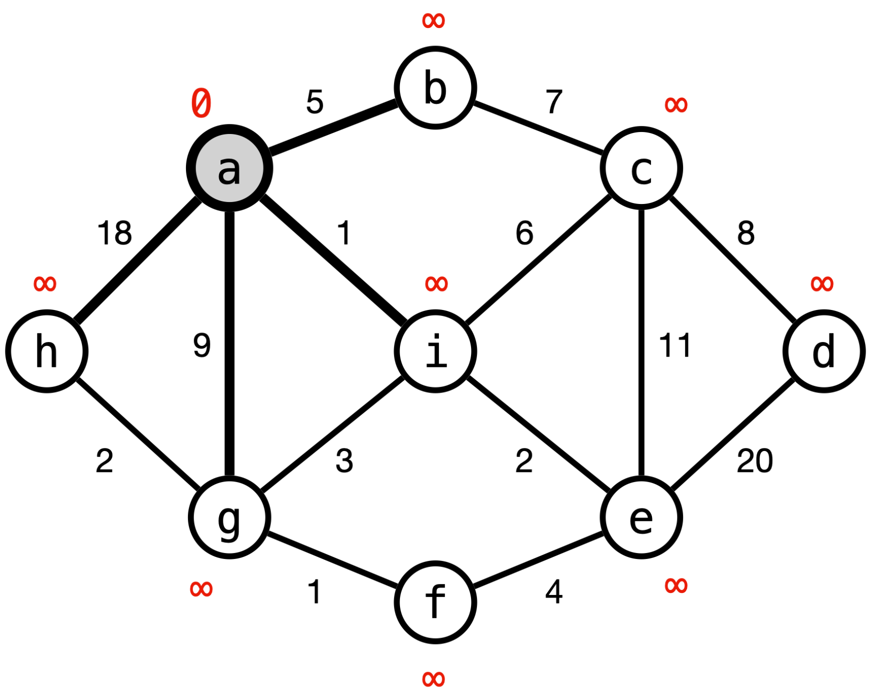
Initialization:

- $d[s] \leftarrow 0$, and $d[t] \leftarrow \infty$ for all $t \neq s$
- $F \leftarrow V$ (unfinalized nodes; white)
- $D \leftarrow \emptyset$ (finalized nodes; black)

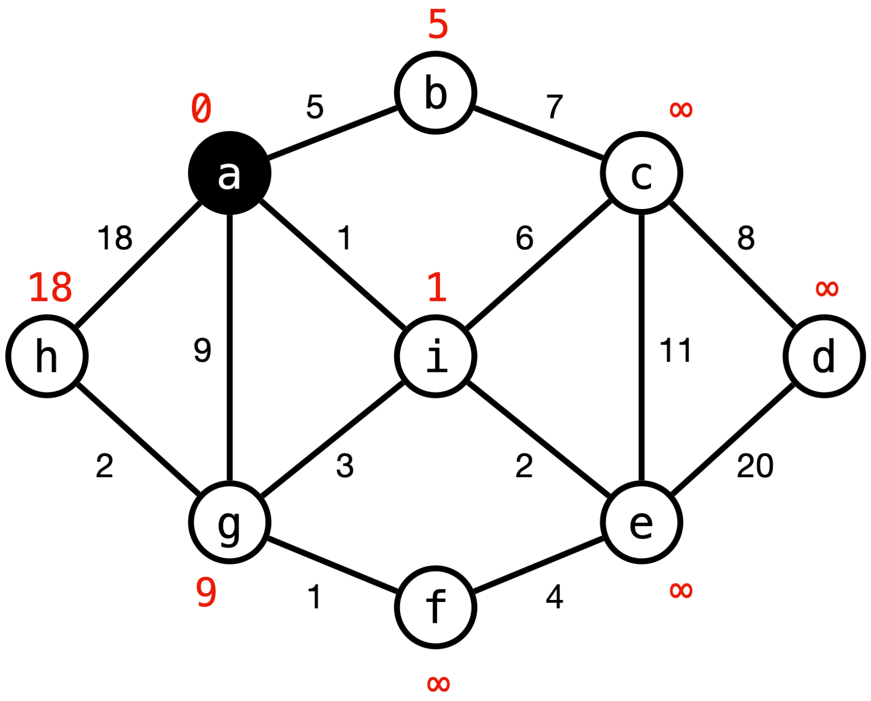


Main Loop (while $F \neq \emptyset$):

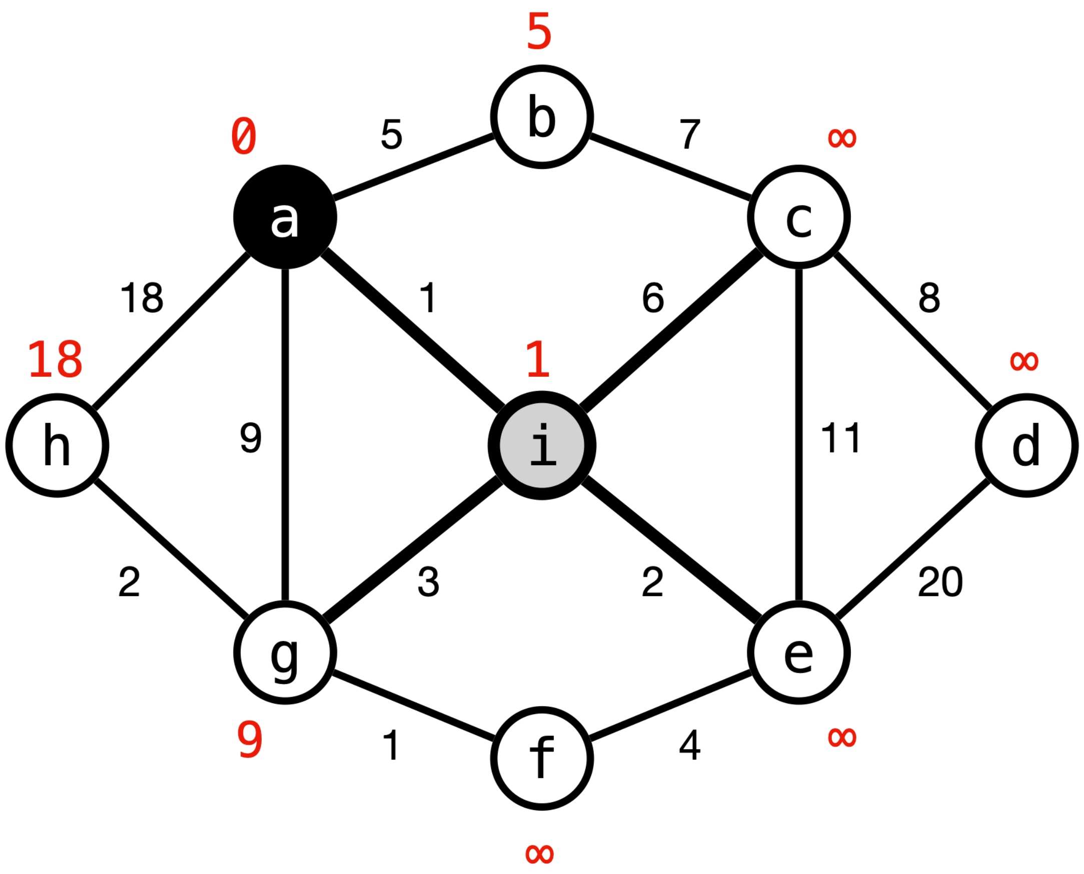
1. Select $x \in F$ with the smallest $d[x]$
2. Relax every edge (x, y): $d[y] \leftarrow \min(d[y], d[x] + w(x, y))$
3. Move x from F to D (finalized set)



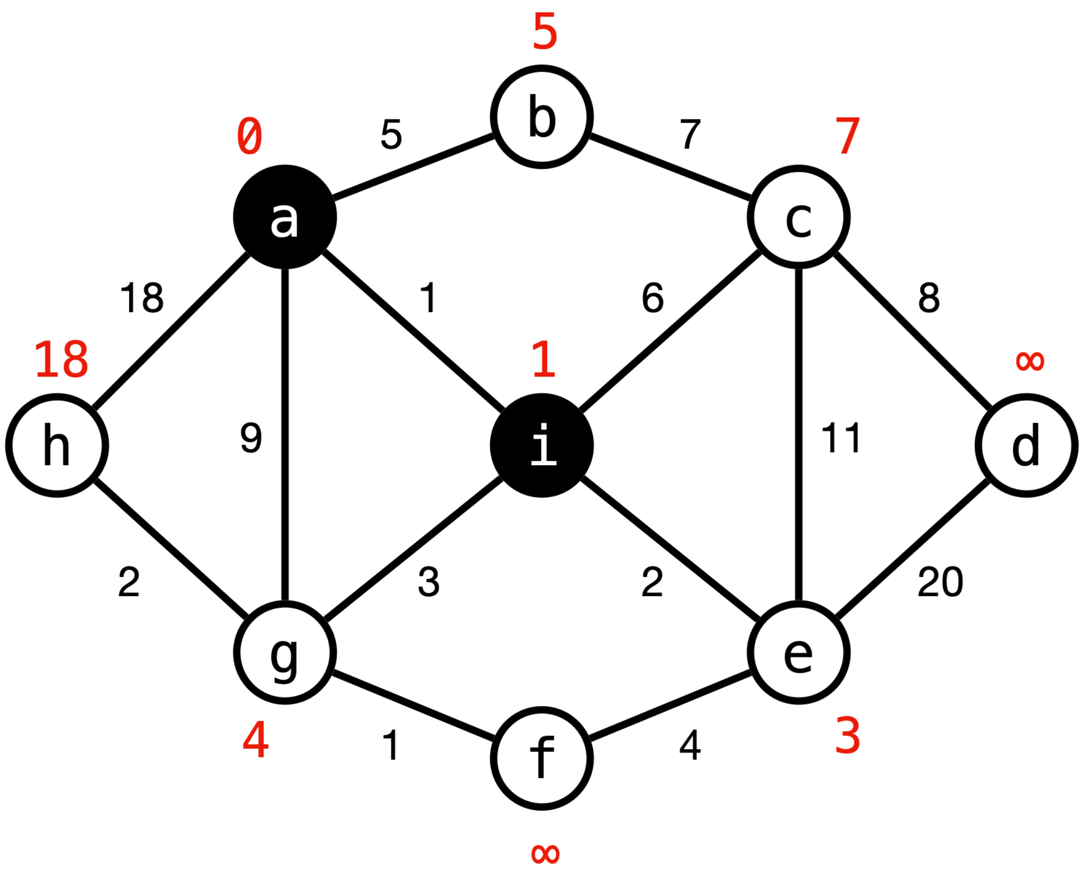
Pick a



Relax its edges & move a from F to D

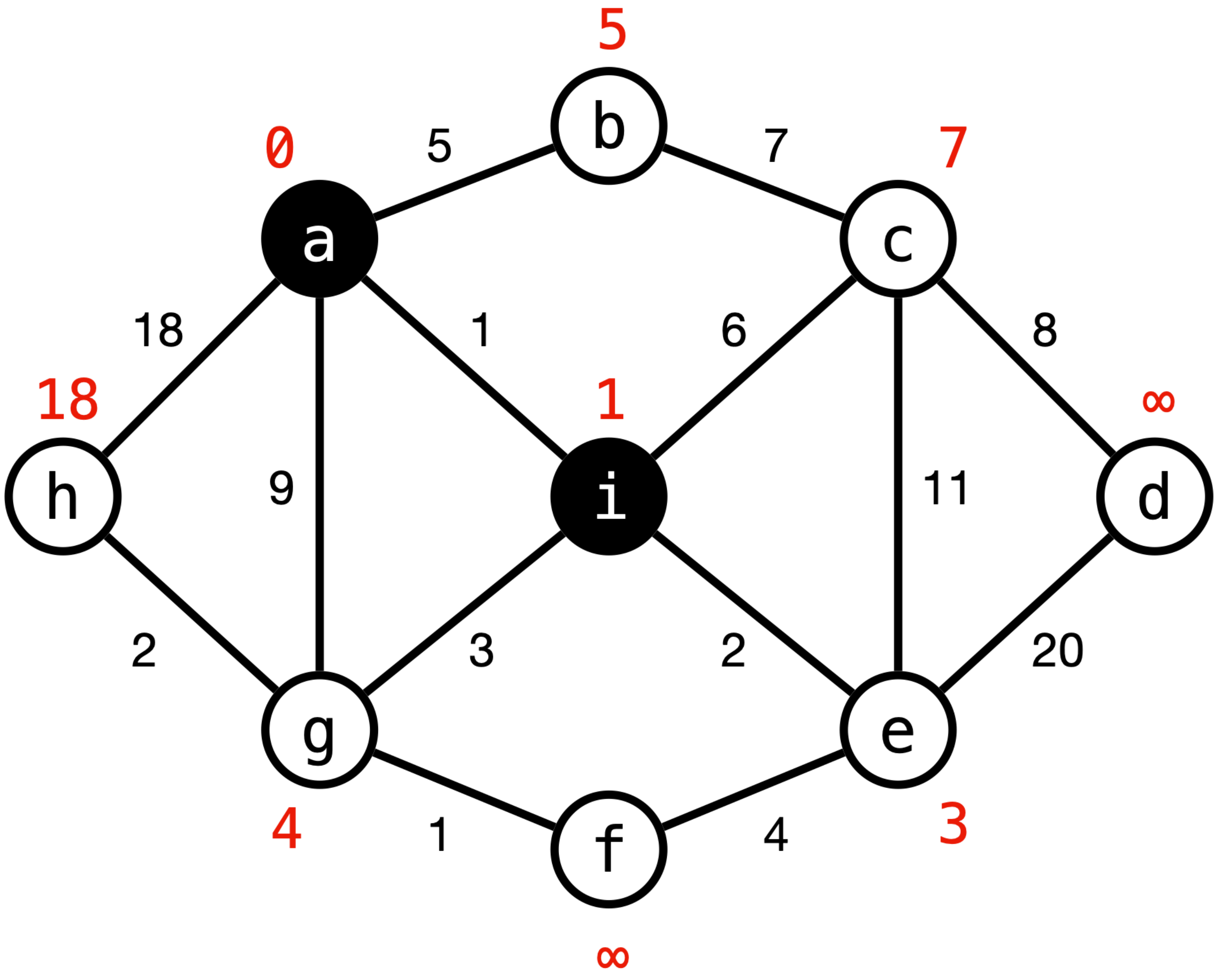


Pick i

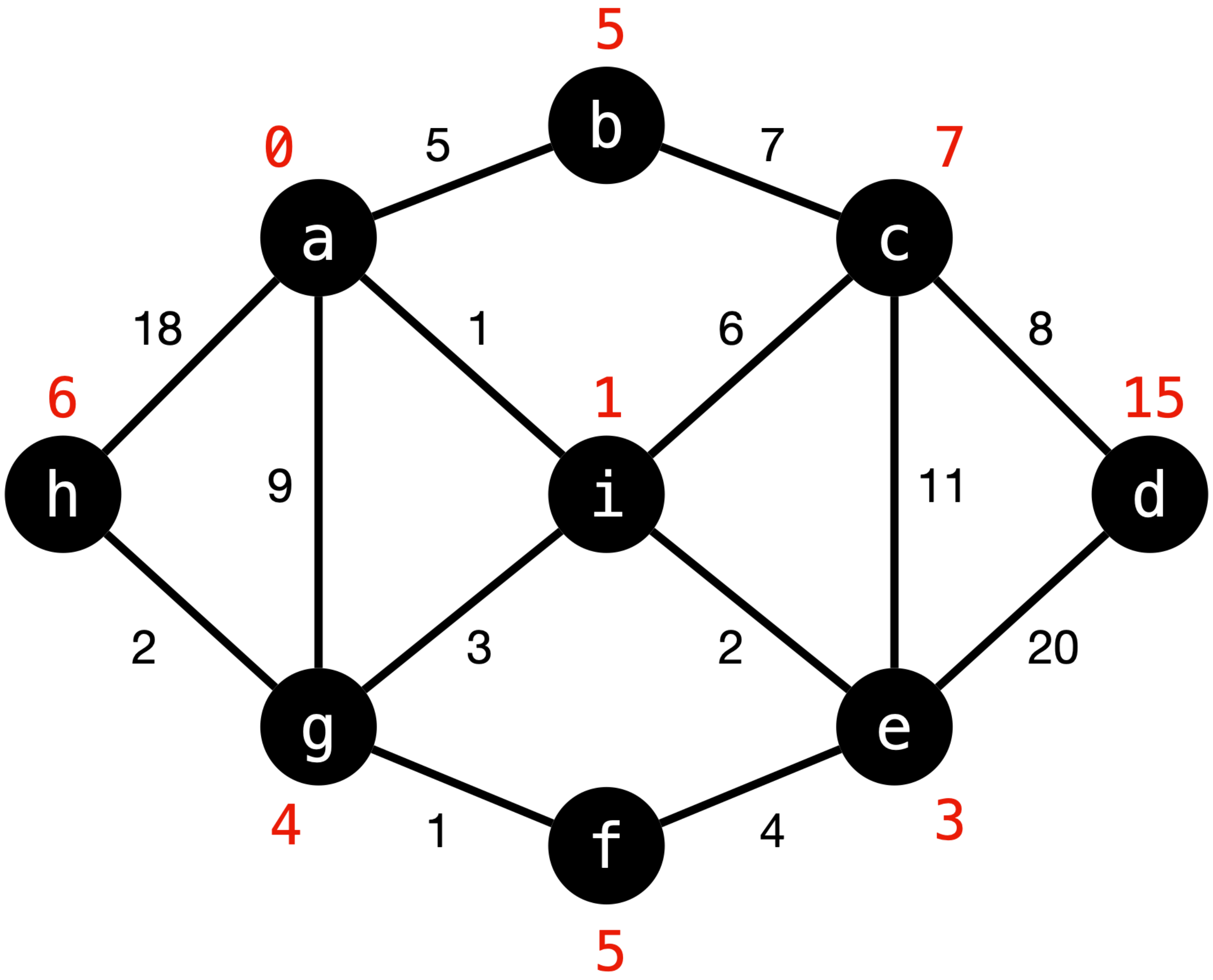


Relax its edges & move i from F to D

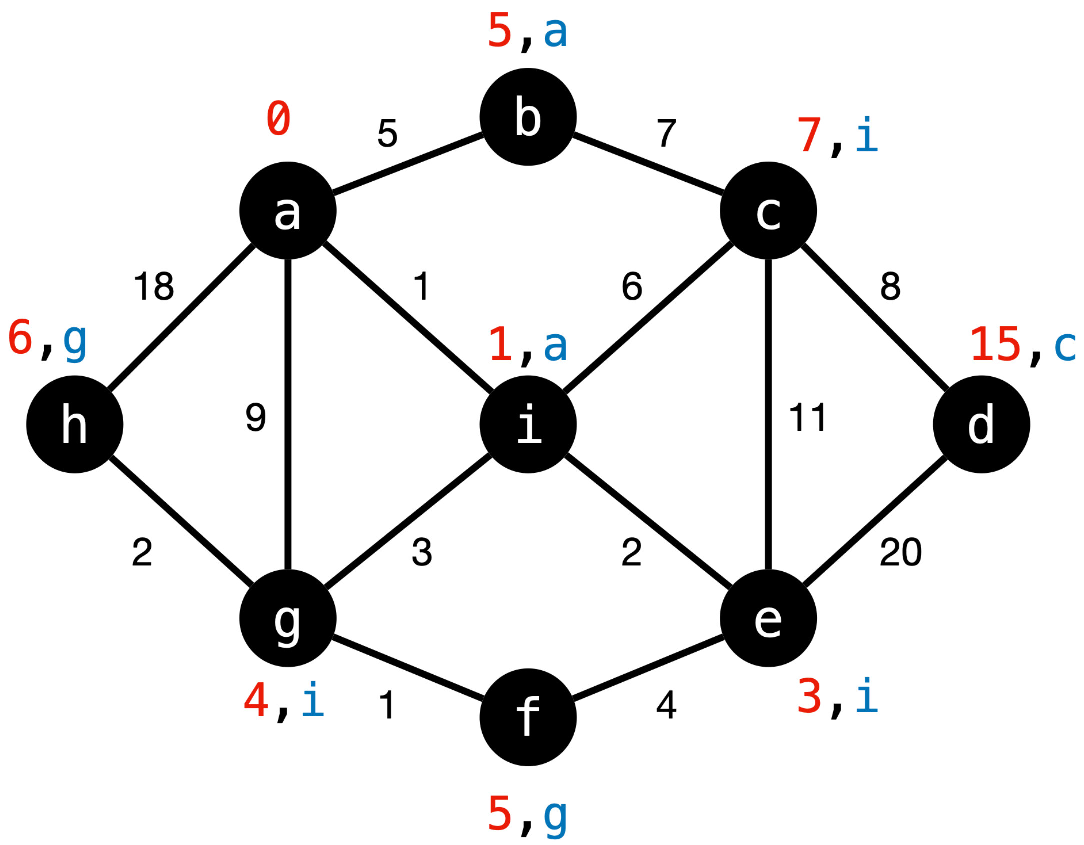
Let's Practice Together!



Fast Forward...



For Path Recovery, Keep Track of the Parents



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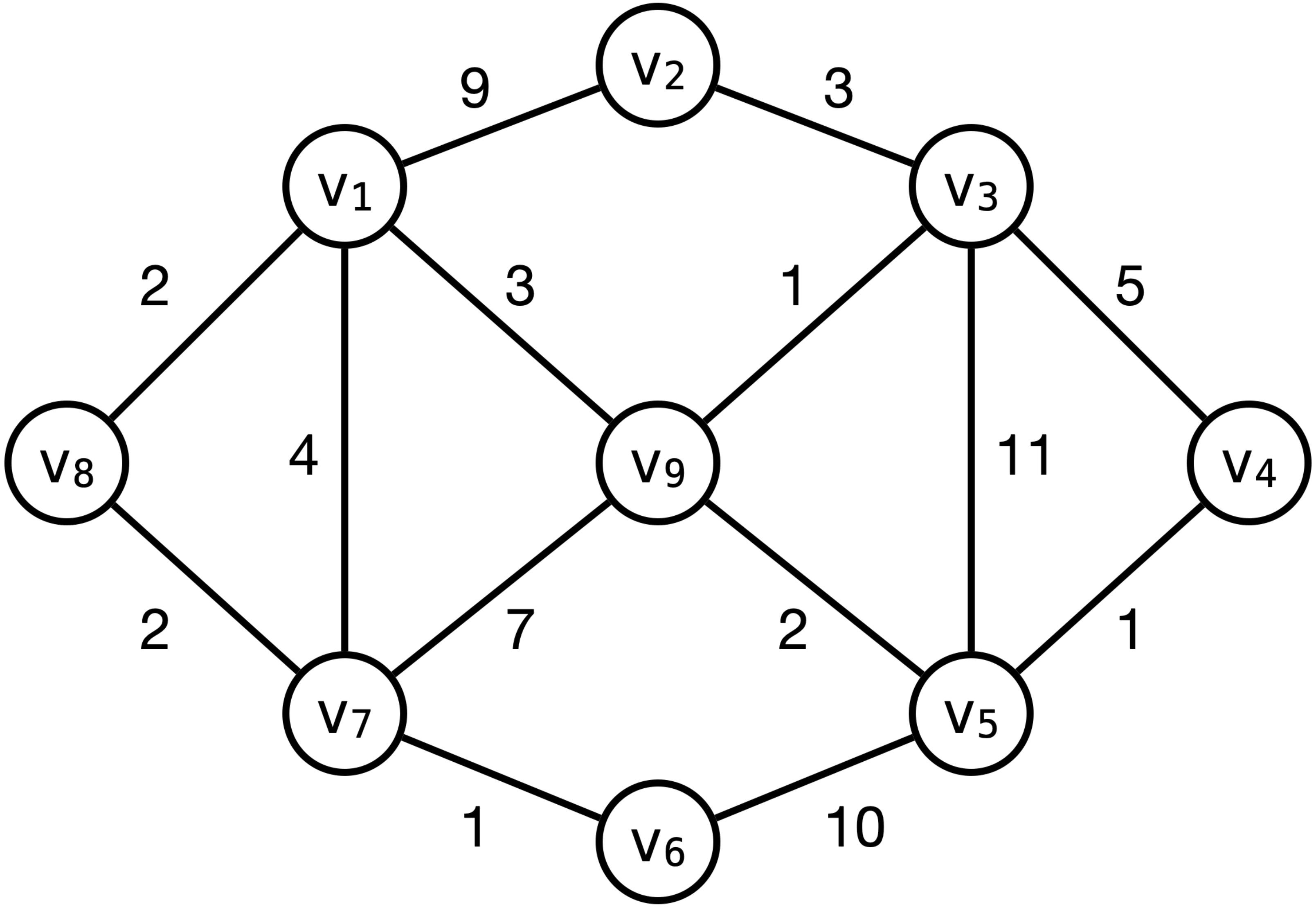
e	a → i → e	3
f	a → i → g → f	5
g	a → i → g	4
h	a → i → g → h	6
i	a → i	1



v1
v2
v3
v4
v5
v6
v7
v8
v9
9
3
2
3
1
5
4
11
2
7
2
1
1
10

Invariant of Dijkstra's Algorithm

- **Claim 1**. At any point in time, for every node t:

$$d[t] \geq d(s,t)$$

> Distance estimates, $d[t]$, never fall below the actual shortest path length.

- **Claim 2**. Once t is finalized (added to D),

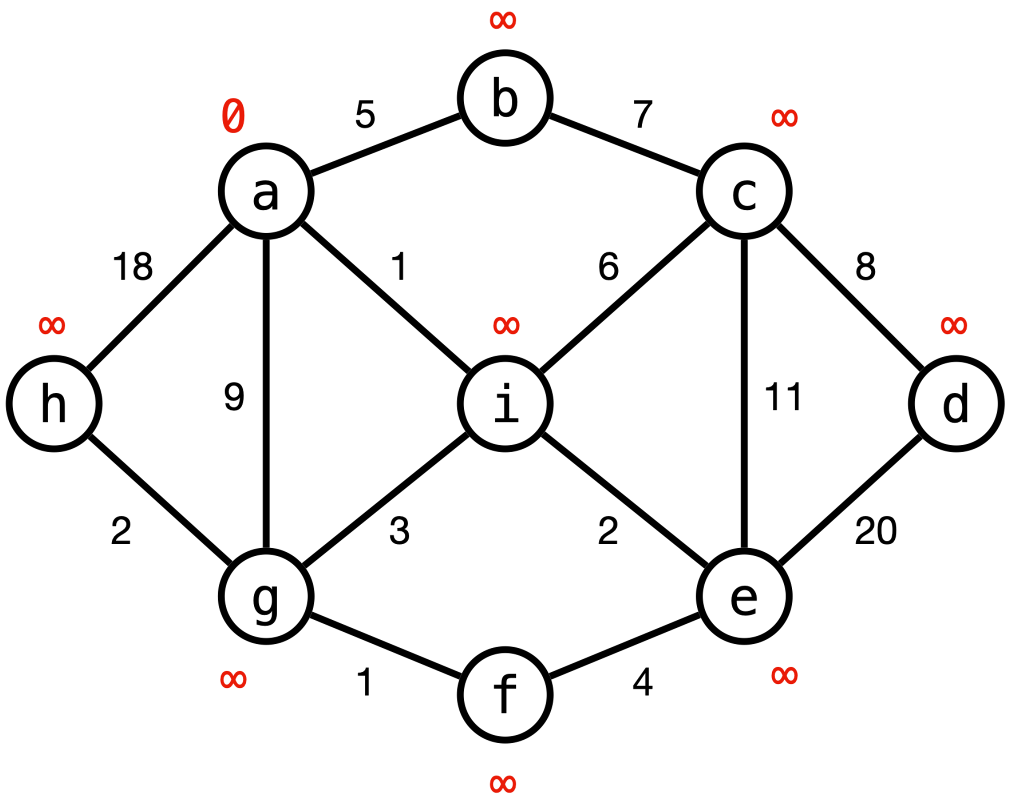
$$d[t] = d(s,t)$$

> A finalized node always has the correct shortest distance.

Proof of Claim 1: For every node t, at any point in time $d[t] \geq d(s,t)$

Inductive Hypothesis: At any point in time, if $d[t] < \infty$, then $d[t]$ is the weight of some path from s to t.

Base Case: At the start, $d[s] = 0 = d(s,s)$, and $d[t] = \infty$ for all $t \neq s$. Clearly the hypothesis holds. ✅



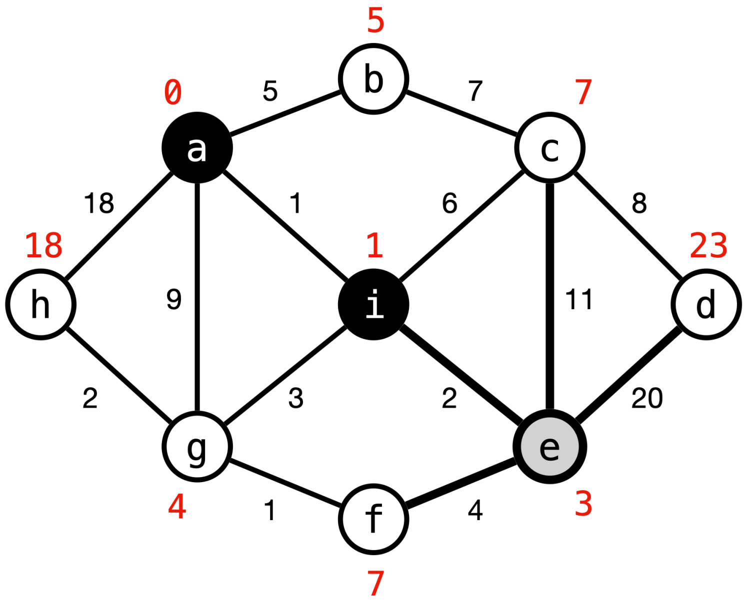
Inductive Step:

- Suppose $d[t]$ is updated via relaxation: $d[t] \leftarrow d[x] + w(x, t)$.
- This means that (by the induction hypothesis) there is a path from s to x of weight $d[x]$ and an edge (x, t) of weight $w(x, t)$.
  ➡️ There is a path from s to t of weight $d[x] + w(x, t)$!
- $d[t]$ is at least the weight of the shortest path, i.e., $d[t] \geq d(s, t)$. ✅

Example:

$$d[d] \leftarrow d[e] + w(e, d) = 3 + 20 = 23$$

- There is a path from `a` to `e` of weight 3 and an edge (`e`, `d`) of weight 20.
- Therefore, there is a path from `a` to `d` of weight 23.



Invariant of Dijkstra's Algorithm

- ✅ **Claim 1.** At any point in time, for every node t:

$$d[t] \geq d(s,t)$$

> Distance estimates, $d[t]$, never fall below the actual shortest path length.

- 👉 **Claim 2.** Once t is finalized (added to D),

$$d[t] = d(s,t)$$

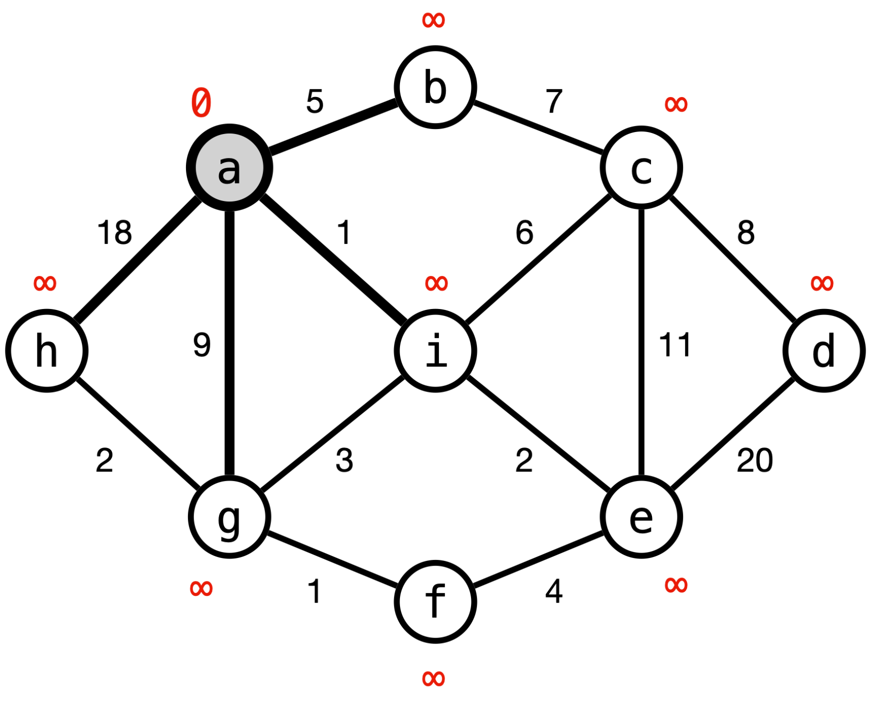
> **A finalized node always has the correct shortest distance.**

Here, $d(s,t)$ is the weight of the shortest path from source s to node t.

Proof of Claim 2: Once t is finalized (added to D), $d[t] = d(s, t)$

We prove this claim by induction on *the order of placement of nodes into* D.

Base Case: At the start, s is placed into D where $d[s] = d(s, s) = 0$. ✅



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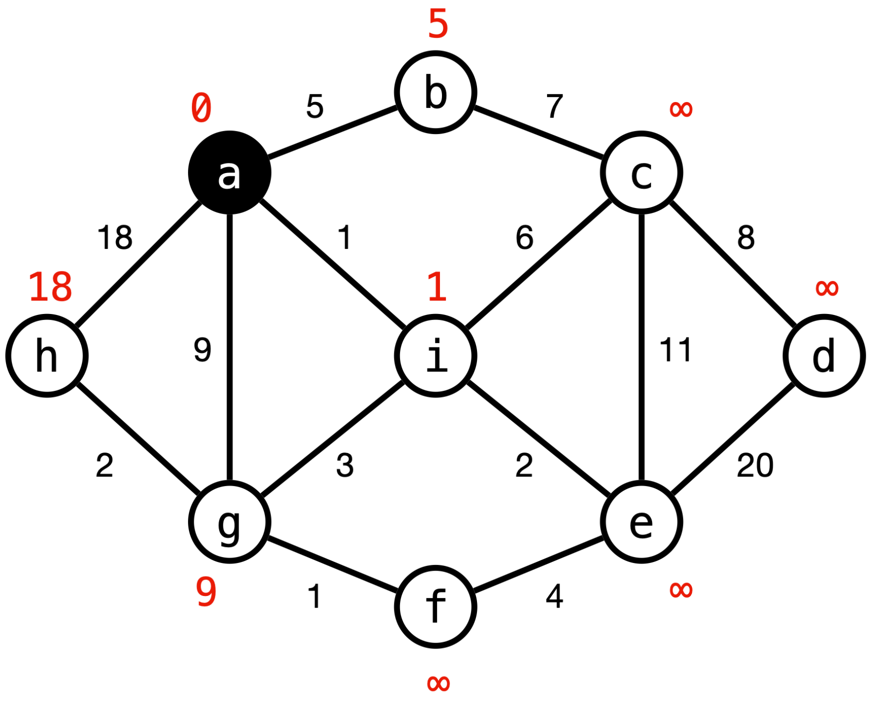
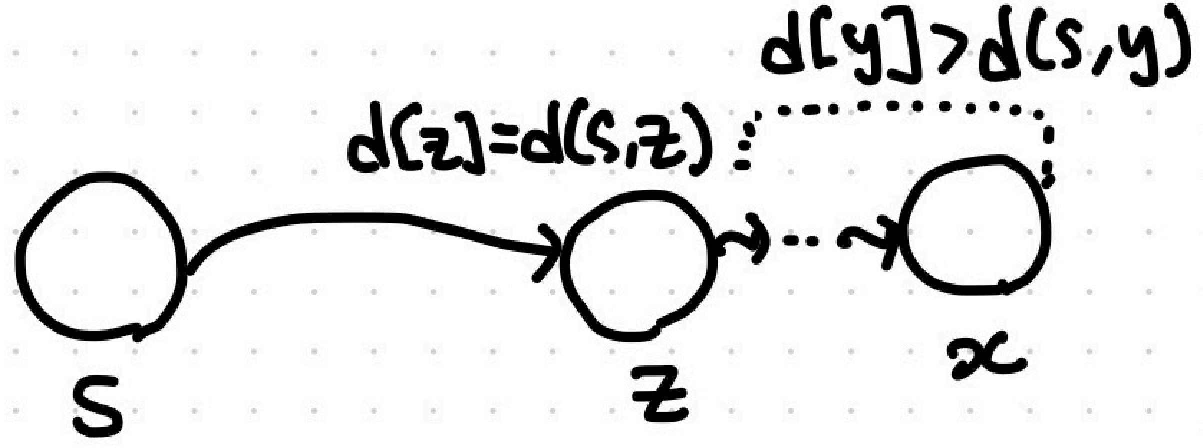
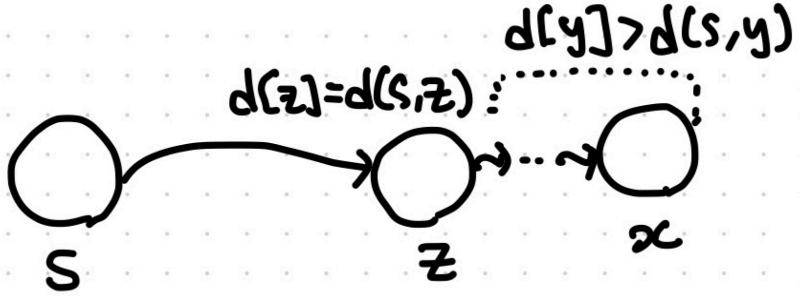


Relax its edges & move a from F to D

Inductive Step: We assume that for all nodes y currently in D, $d[y] = d(s, y)$.

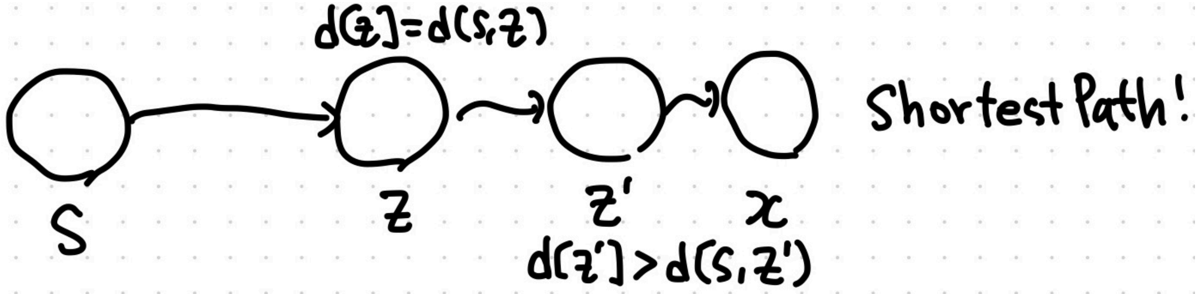
- Suppose x is the node chosen (with minimum $d[x]$ in F).
- Let p be a shortest path from s to x, and z be the *last node on* p (closest to x) for which $d[z] = d(s, z)$. z always exists since there is at least one such node, namely s.
- By the choice of z, for every node y on p between z (not inclusive) to x (inclusive), $d[y] > d(s, y)$.





Case 1. $z = x$: On the shortest path, x is the last node for which $d[z] = d(s, z)$

Case 2. $z \neq x$: There exists a node z' after z on the shortest path p (possibly $z' = x$).

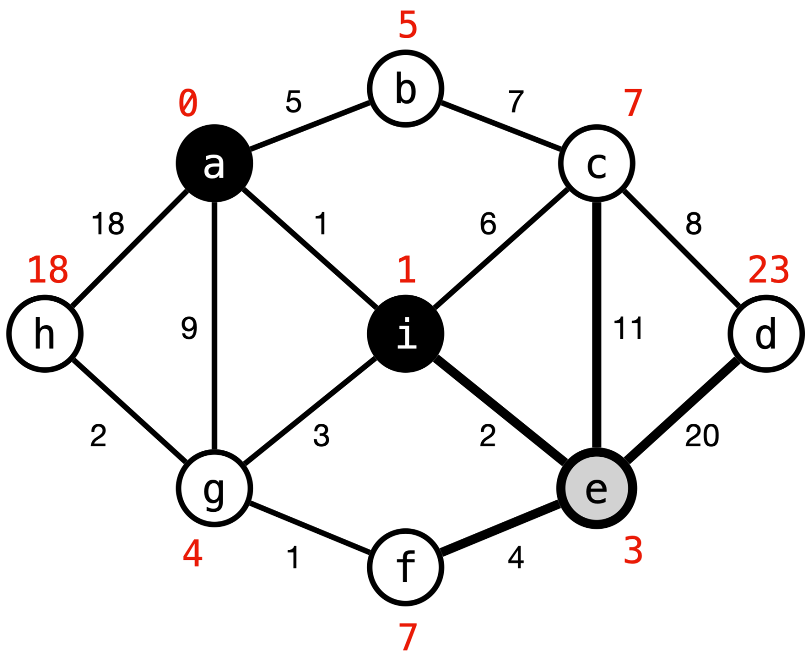


Case 1. $z = x$: Clearly $d[x] = d(s,x)$ (by the choice of z), and we are done.

Reasoning:

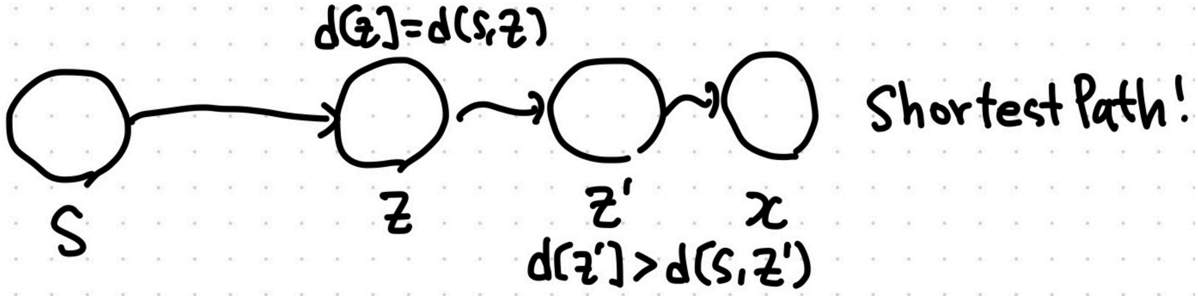
- After x is chosen, $d[x]$ never changes again (no relaxation can reduce it).
- Hence, when x is finalized (moved to D), its estimate is exact:

$$d[x] = d(s,x)$$



- `e` is the node chosen (with minimum $d[x]$ in F).
- Let p be a shortest path from `a` to `e` : `a → i → e`
- `e` is the *last node on* p for which $d[z] = d(s, z)$.
- Therefore, when `e` is added to D after relaxing its edges, $d[e] = d(a, e)$.

Case 2. $z \neq x$: There exists a node z' after z on the shortest path p (possibly $z' = x$).

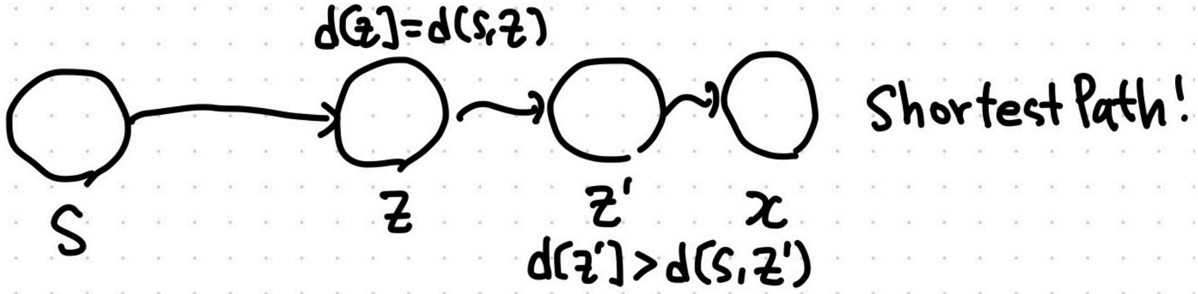


We know: $d[z] = d(s, z) \leq d(s, x) \leq d[x]$

Why?

- $d(s, z) \leq d(s, x)$ since prefixes of shortest paths are shortest paths.
- $d(s, x) \leq d[x]$ by Claim 1
    - Distance estimates never fall below the actual shortest path length.

Case 2. $z \neq x$: There exists a node z' after z on the shortest path p (possibly $z' = x$).



We know: $d[z] = d(s,z) \leq d(s,x) \leq d[x]$

- If $d[z] = d[x]$, then $d[x] = d(s,x)$, contradicting the choice of z 🚫.
- If $d[z] < d[x]$. Then z must already be in D (since x is the minimum in F), and (z, z') would have been relaxed earlier. So $d[z'] = d(s,z')$, contradicting the choice of z 🚫.
- Therefore, $z \neq x$ is false, $z = x \implies d[x] = d(s,x)$ ✅

Proof of Claim 2: **Once t is finalized (added to D), $d[t] = d(s, t)$**

Inductive Step: We assume that for all nodes y currently in D, $d[y] = d(s, y)$.

- Suppose x is the node chosen (with minimum $d[x]$ in F).
- Let p be a shortest path from s to x, and z be the *last node on* p (closest to x) for which $d[z] = d(s, z)$.
    - z is always x. 👉 $d[x] = d(s, x)$.
- Therefore, when x is added to D, $d[x] = d(s, x)$.

Runtime of Dijkstra's Algorithm

Consider implementing Dijkstra’s algorithm using a **priority queue** to store the set F, with distance estimates as the keys. The priority queue needs to support three operations:

- **FindMin**: return the vertex in F with the smallest distance estimate
- **DecreaseKey**: update a vertex’s key (distance estimate) when a shorter path is found during relaxation
- **DeleteMin**: remove the vertex with the smallest distance estimate from F

Suppose $n = |V|, m = |E|$, then the total runtime of the algorithm is:

$$n \cdot (T_{FindMin}(n) + T_{DeleteMin}(n)) + m \cdot T_{DecreaseKey}(n)$$

Runtime of Dijkstra's Algorithm Depends on Priority Queue Choice! (1/2)

$$n \cdot (T_{FindMin}(n) + T_{DeleteMin}(n)) + m \cdot T_{DecreaseKey}(n)$$

Using an **array**:

- FindMin = $O(n)$, DeleteMin = $O(n)$, DecreaseKey = $O(1)$
- Total runtime = $O(n^2 + m) = O(n^2)$

Using a **balanced BST** (e.g., Red-Black Tree):

- All operations: $O(\log n)$ (DecreaseKey = Delete + Insert)
- Total runtime = $O((n + m) \log n)$
- Note: efficient for sparse graphs, but slower on dense graphs ($m = \Theta(n^2)$)

Runtime of Dijkstra's Algorithm Depends on Priority Queue Choice! (2/2)

$$n \cdot (T_{FindMin}(n) + T_{DeleteMin}(n)) + m \cdot T_{DecreaseKey}(n)$$

Using a **Fibonacci Heap**:

Fibonacci Heaps are a complex data strcuture which is able to support the operations:

- FindMin = $O(1)$, DecreaseKey = $O(1)$, DeleteMin = $O(\log n)$ *amortized*
- Total runtime = $O(m + n \log n)$ 👍

> See the Stanford Lecture 11 notes for details on the meaning of amortized time.

A new breakthrough!

This year, a single-source shortest path (SSSP) algorithm has been introduced that asymptotically outperforms Dijkstra's algorithm: [Breaking the Sorting Barrier for Directed Single-Source Shortest Paths]

> We give a deterministic $O(m \log^{2/3} n)$-time algorithm for single-source shortest paths (SSSP) on directed graphs with real non-negative edge weights in the comparison-addition model. This is the first result to break the $O(m + n \log n)$ time bound of Dijkstra's algorithm on sparse graphs, showing that Dijkstra's algorithm is not optimal for SSSP.

Dijkstra's Algorithm

- Finds the shortest paths from a single source node to all other nodes in $O(m + n \log n)$ time.
- Does not work with negative edge weights.
- Not robust to graph updates (e.g., when edge weights change).

Next time: we'll look at the **Bellman–Ford** algorithm, which can handle negative edges.

Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/
- Stanford CS106B slides
  - https://web.stanford.edu/class/archive/cs/cs106b/cs106b.1258/lectures/26-graph-algorithms/