Lecture 11b - Balanced Binary Search Tree (Red Black Tree) - Part 2

Fall 2025, Korea University

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Course Outline (Before Midterm) - Recap

- Part 1: Basics
  - ~~Divide and Conquer~~
  - ~~Basic Sorting Algorithms (Insertion Sort & Merge Sort)~~
  - ~~Asymptotic Analysis (Big-O, Big-Theta, Big-Omega)~~
  - ~~Solving Recurrences Using Master Method~~
- Part 2: Advanced Selection and Sorting
  - ~~Median and Selection Algorithm~~
  - ~~Solving Recurrences Using Substitution Method~~
  - ~~Quicksort, Counting Sort, Radix Sort~~
- Part 3: Data Structures
  - ~~Heaps, Binary Search Trees~~, **Balanced BSTs** - *Now we are here!* 📍

Today's Agenda

- Learn Red-Black Trees
  - Quick Review: Definition & Properties
  - Operations:
    - Tree Rotation (Left / Right)
    - Insertion (with Rotation & Recoloring)
    - Deletion (with Rotation & Recoloring)

Midterm Exam Information (Offline & In-Person)

- Date & Time:
  - Thursday, October 23
  - 1:30 – 2:45 PM
- Location:
  - Room 610 – Students with odd student IDs (33 students)
  - Room 616 – Students with even student IDs (32 students)

> ⚠️ Please sit with one empty seat between each student.
> 📚 Format: Closed book — but one A4 cheat sheet (both sides) is allowed.

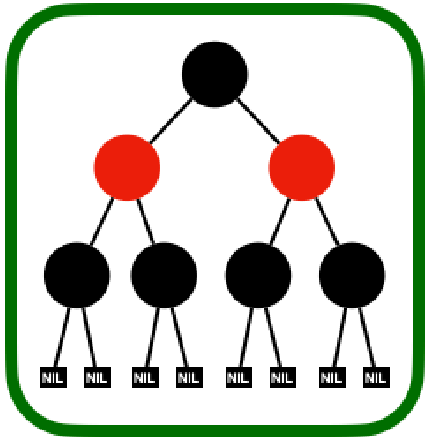
Red–Black Tree: Definition

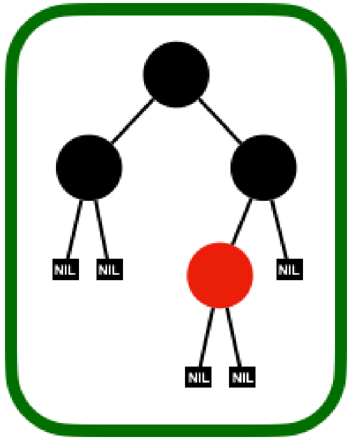
A Red–Black Tree is a BST with an extra bit of storage (color) per node.

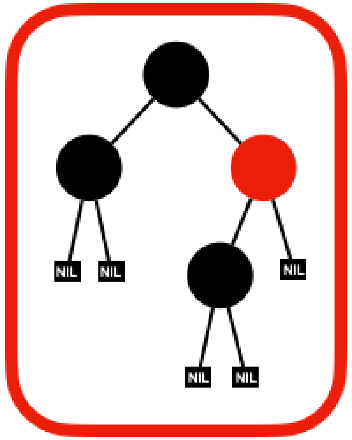
Properties:

1. Every node is red or black.
2. The root is black.
3. NILs are black.
4. The children of a red node are black.
5. For every node x, all x to NIL paths have the same number of black nodes on them.

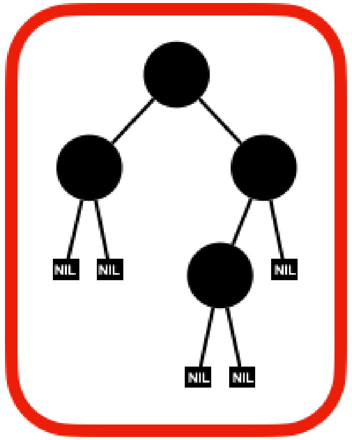
Examples



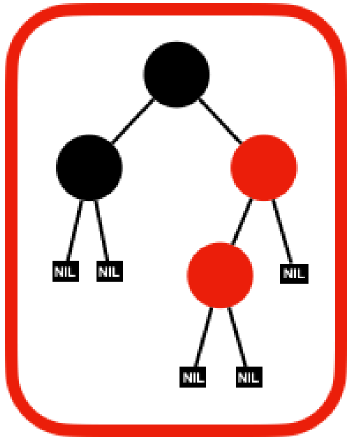




Violating 5



Violating 5



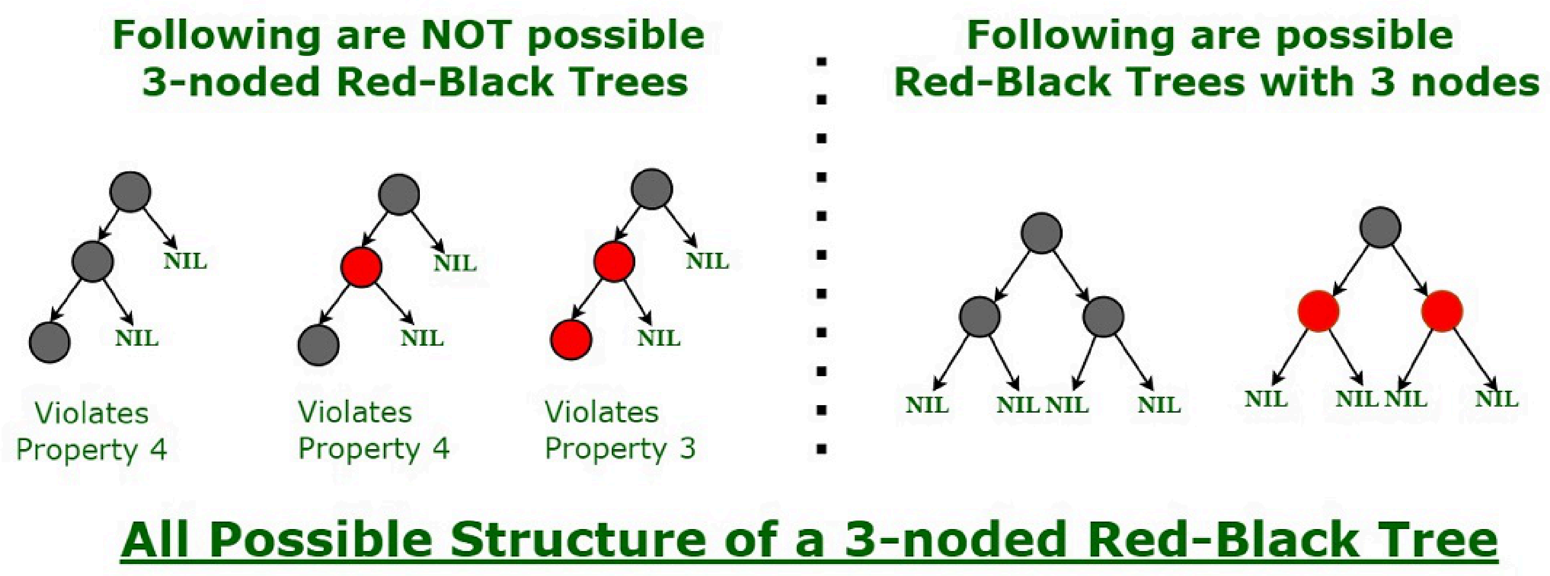
Violating 4

1. Every node is red or black.
2. The root is black.
3. NILs are black.
4. The children of a red node are black.
5. For every node x, all x to NIL paths have the same number of black nodes on them.

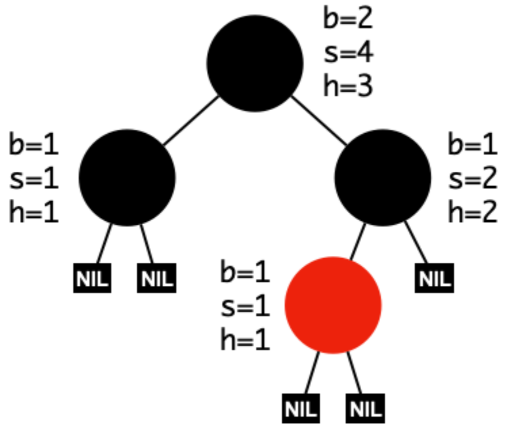
Why These Properties Work

- Property 4 limits consecutive reds → **prevents skewed long chains of red nodes**.
- Property 5 forces black nodes to be evenly distributed → **keeps height small**.

> A chain of 3 nodes is not possible in the Red-Black tree.



Claim: A red–black tree with n nodes has height $\leq 2\log_2(n+1) = O(\log n)$



Let's show $s(x) \geq 2^{b(x)} - 1$ via induction on the height of x (Base case: NIL node ✅)
Using the *IH* $s(y) \geq 2^{b(y)} - 1$ for children y of x, i.e., $b(y) \geq b(x) - 1$

$$s(x) = 1 + s(\text{left}(x)) + s(\text{right}(x)) \geq 1 + \left(2^{b(x)-1} - 1\right) + \left(2^{b(x)-1} - 1\right) = 2^{b(x)} - 1$$

Finally, when r is a root,

$$n = s(r) \geq 2^{b(r)} - 1 \geq 2^{h(r)/2} - 1 \quad \rightarrow \quad h = h(r) \leq 2\log_2(n+1) \quad \blacksquare$$

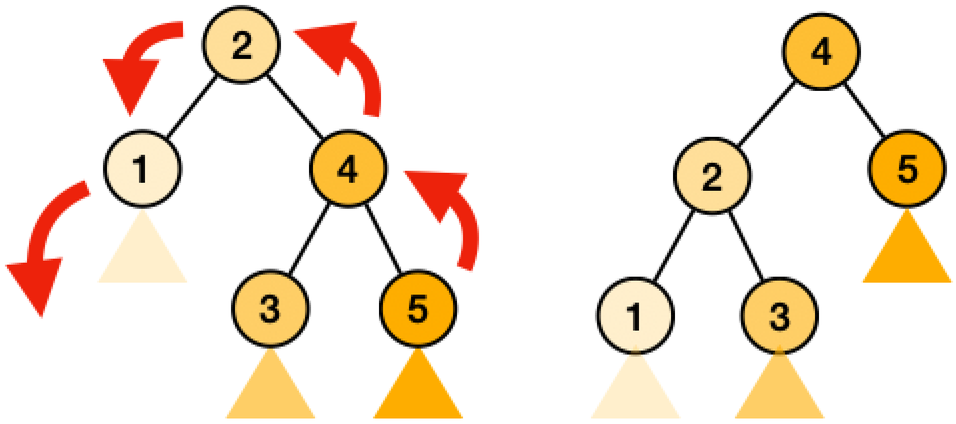
Basic Operations on a Red-Black Tree

1. **Search**: Same as in a normal BST (binary search based on key ordering).
2. **Insertion**: Insert as in a normal BST, then fix any violations of red-black properties via **rotations and recoloring**.
3. **Deletion**: Delete as in a normal BST, then restore red-black properties through **rotations and recoloring.**

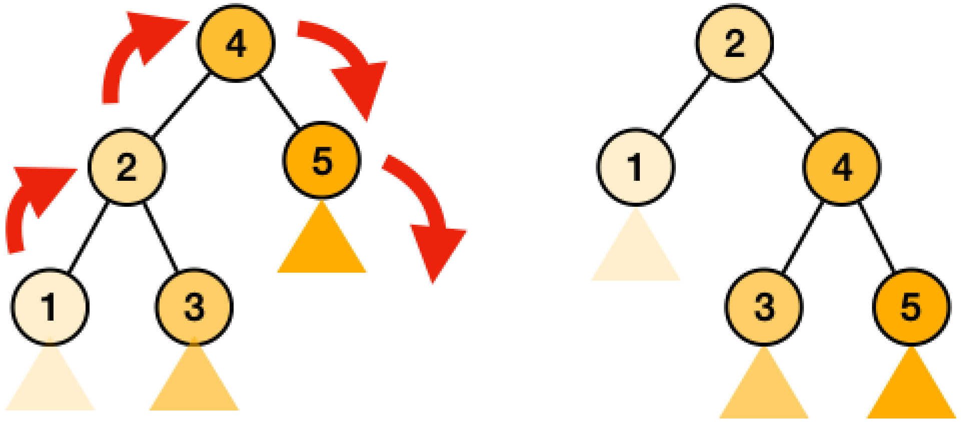
Rotation

- **Purpose**: Maintain balance after insertion/deletion
  - Tree rotations can be performed in $O(1)$ time.
- **Types**:
  i. Left Rotation
  ii. Right Rotation

Left Rotation



Right Rotation



Insertion

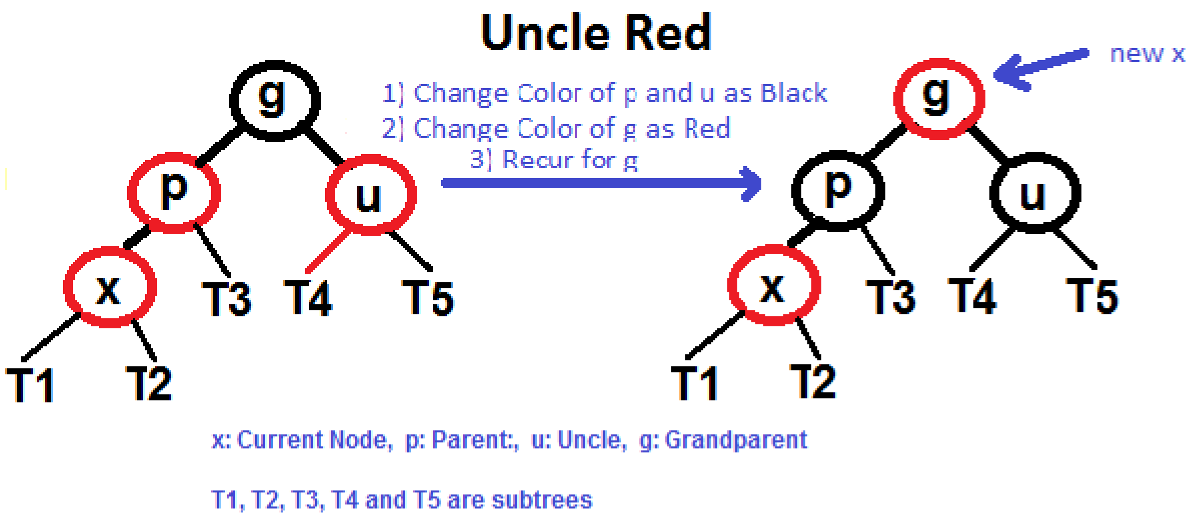
- Inserting a new node in a Red-Black Tree involves a two-step process:
    1. Performing a standard BST insertion
    2. Fixing any violations of Red-Black properties
- Rule: **A newly inserted node is always RED 🔴** !!
    - If new node is root, change color of new node as black.
    - If the parent of the new node is black, no properties are violated.
    - **If the parent is red, the tree might violate the Red Property, requiring fixes.**

Fixing Violations During Insertion

When inserting a **red** node under a **red** parent, we must fix violations of the Red-Black properties. We handle cases based on the color of the **uncle** (parent's sibling).

Case 1: Uncle is Red (Fix via Only Recoloring)

> Recolor the parent and uncle to black, and the grandparent to red. Then move up the tree to check for further violations.



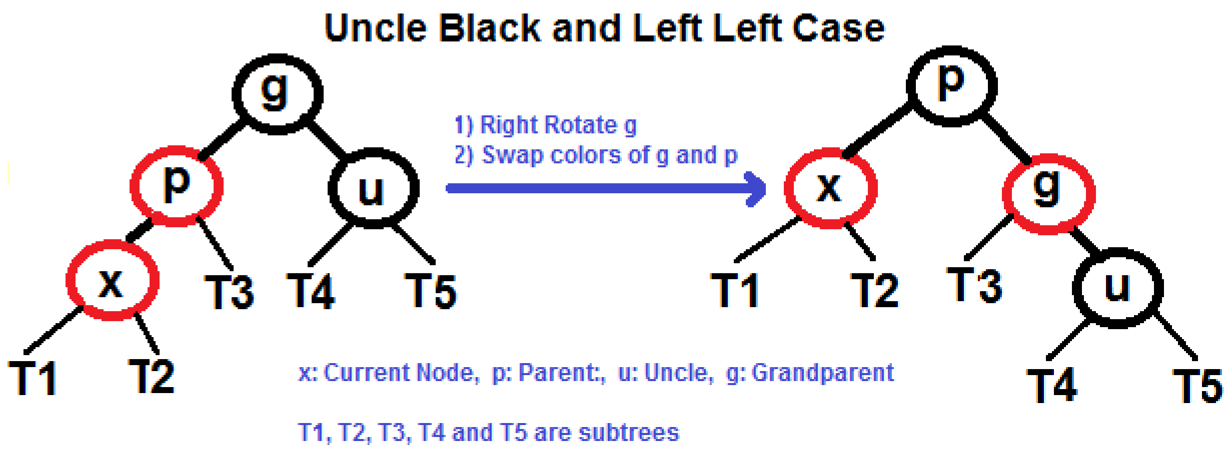
Case 2: Uncle is Black (Fix via Rotation Followed by Recoloring)

> Note that grandparent must have been black since parent is red.

- There are four cases:
    - Left-Left (LL), Right-Right (RR), Left-Right (LR), and Right-Left (RL)

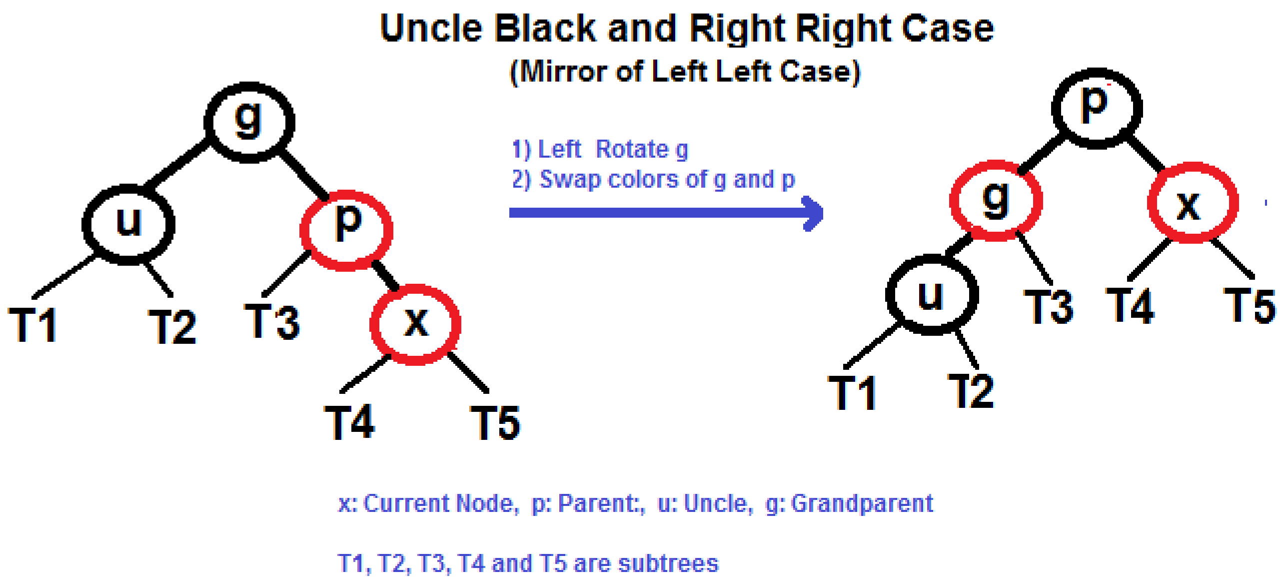
Sub-Case 2.1: Left-Left Case (`g.left == p` and `p.left == x`)

> Right rotate `g` and swap colors of `p` and `g`



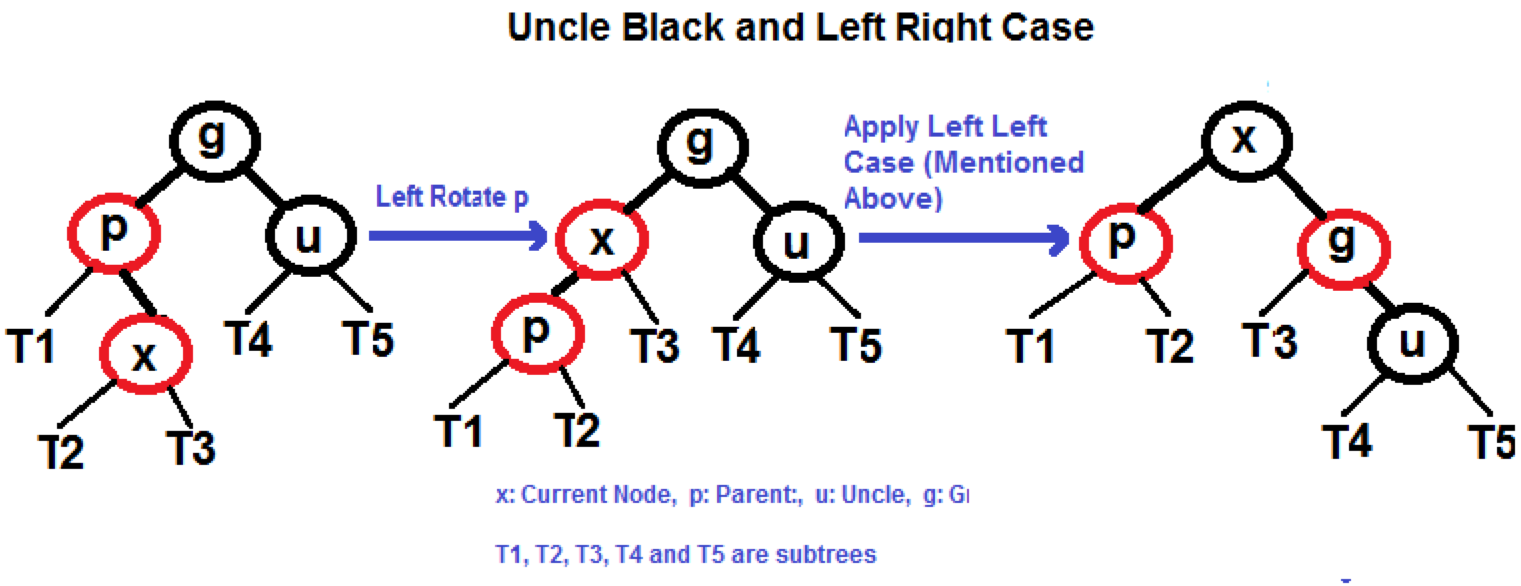
Sub-Case 2.2: Right-Right Case (`g.right == p` and `p.right == x`)

> Left rotate `g` and swap colors of `p` and `g`



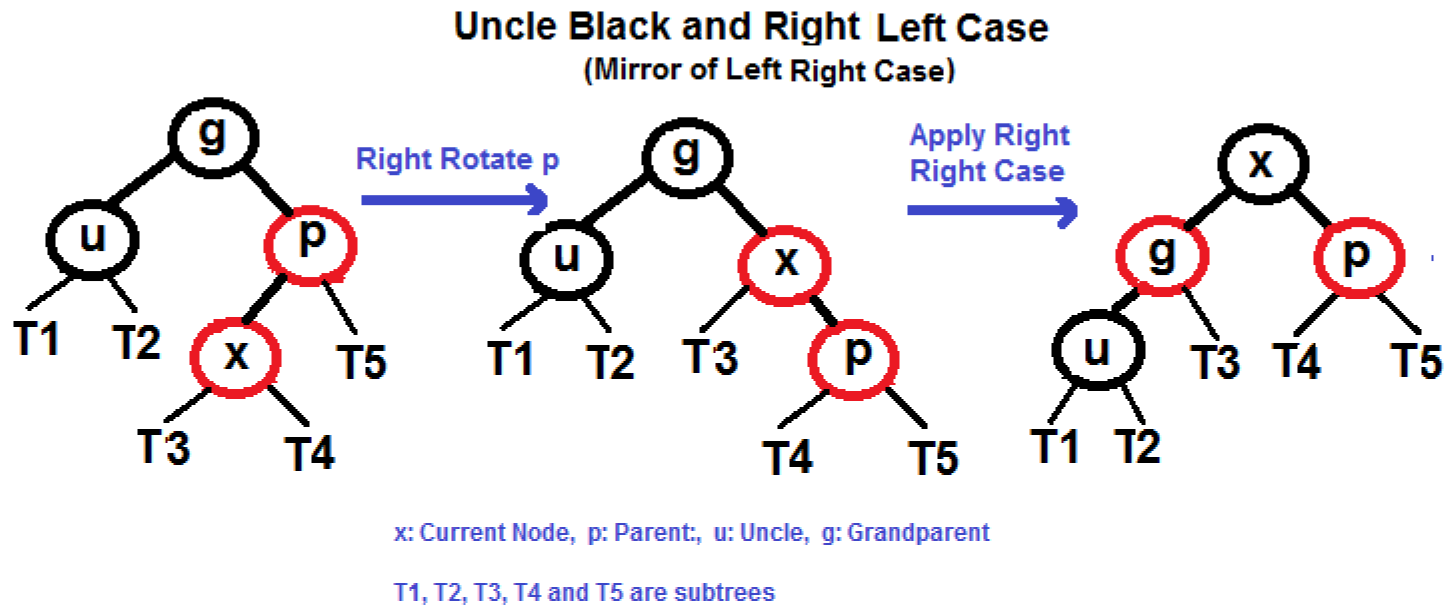
Sub-Case 2.3: Left-Right Case (`g.left == p` and `p.right == x`)

> Left rotate `p` and apply left-left case!

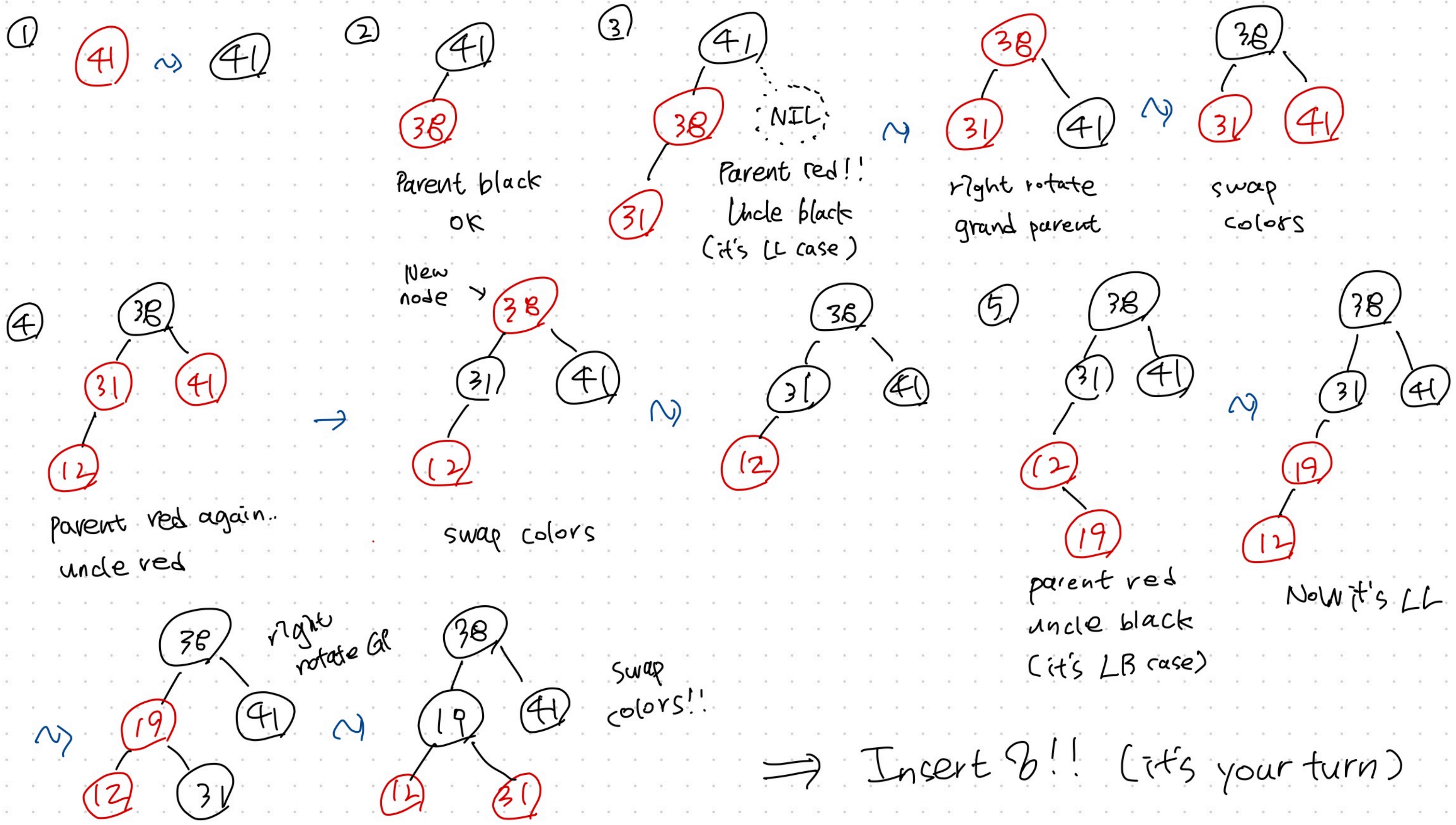


Sub-Case 2.4: Right-Left Case (`g.right == p` and `p.left == x`)

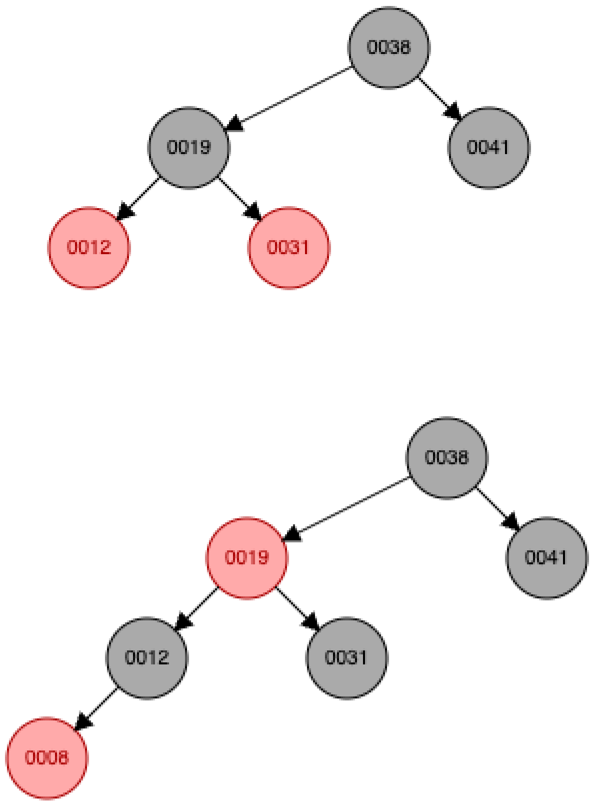
> Right rotate `p` and apply right-right case!



41 → 38 → 31 → 12 → 19 → 8



⇒ Insert 8!! (it's your turn)



Recoloring the grandparent to black helps preserve the black-height property.

Deletion

- Deletion operations in a Red-Black Tree (RBT) can be quite complex due to the balancing rules that need to be maintained.
- Deleting a new node in a Red-Black Tree involves a two-step process:

    i. Performing a standard BST deletion

    ii. Fixing any violations of Red-Black properties

> The main property that violates after insertion is two consecutive reds (i.e., Property 4). In delete, the main violated property is, **change of black height in subtrees (i.e., Property 5**) as deletion of a black node may cause reduced black height in one root to leaf path.

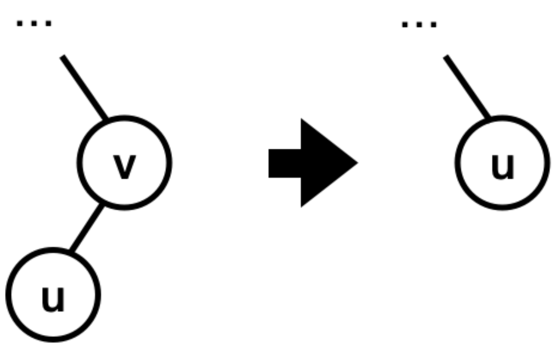
Recall: Standard BST Deletion

- When we perform standard delete operation in BST, we always end up deleting a node which is an either leaf or has only one child.
- For an internal node, we copy the successor and then recursively call delete for successor, successor is always a leaf node or a node with one child.

```
     2                                          3               3
   /   \                                      /   \           /   \
  1     5  -- delete(2) --> successor = 3 -->  1     5   -->  1     5
       /                                          /               /
      3                                          3*              4
       \                                          \
        4                                          4
```

Fixing Violations During Deletion

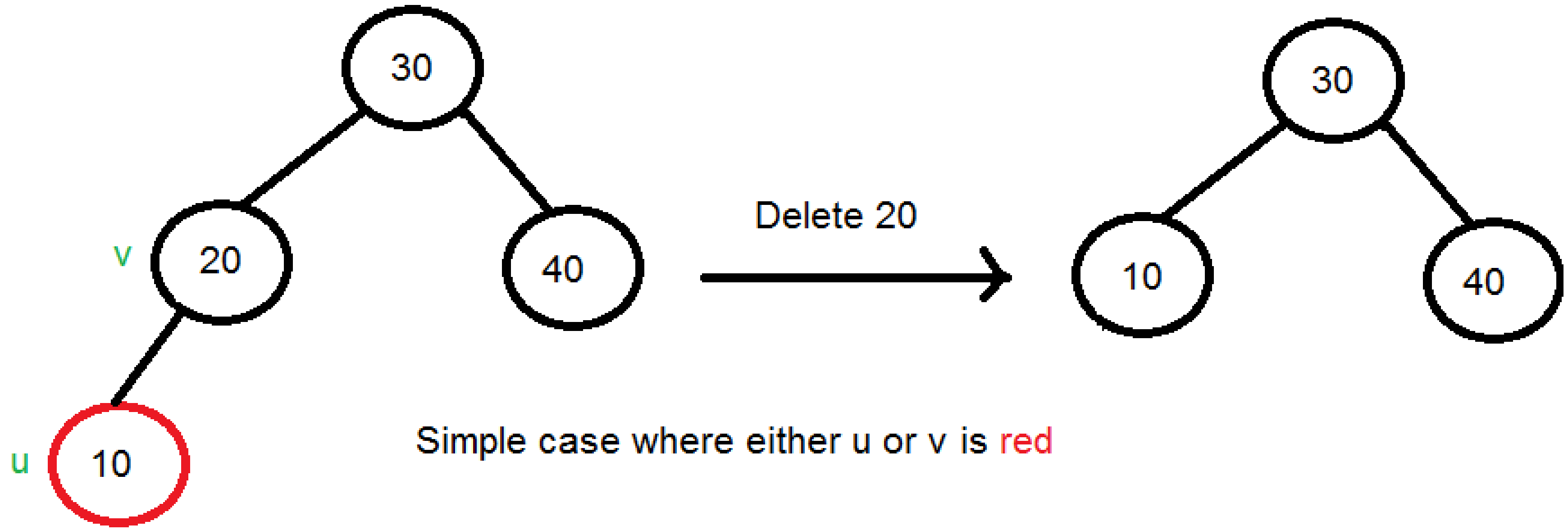
- We only need to handle cases **where a node is leaf or has one child**.
- Let `v` be the node to be deleted and `u` be the child that replaces `v`.
  - Note that `u` is NIL (black) when `v` is a leaf.
  - Note that both `u` and `v` cannot be red as `v` is parent of `u` (due to Property 4).



- There are two cases:
  - Case 1: Either `u` or `v` is red.
  - Case 2: Both `u` and `v` are black.

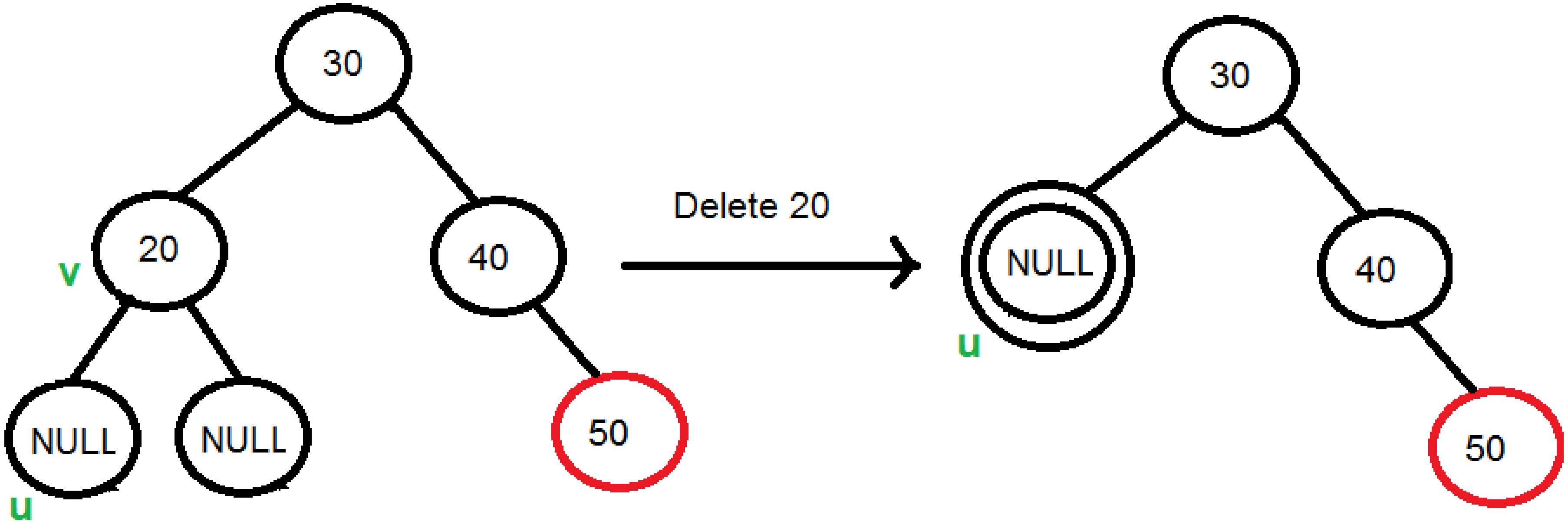
Case 1: Either `u` or `v` is Red (Simple!!)

> Mark the replaced child as black (no change in black height).



Case 2: Both `u` and `v` are black.

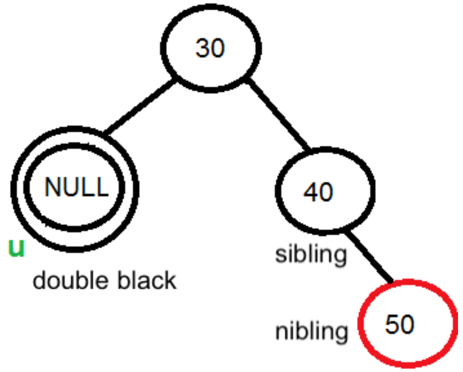
Step 1. Color `u` as double black. Now our task reduces to **convert this double black to single black.**



When 20 is deleted, it is replaced by a NULL, so the NULL becomes double black.
Note that deletion is not done yet, this double black must become single black

Step 2. Do following while the current node `u` is double black, and it is not the root. Let sibling of node be `s`. There are three cases:

- 2(a) **Black Sibling + Red Nibling Case** ⬅️
  - `s` is black, and *at least* one of `s`'s children is red.

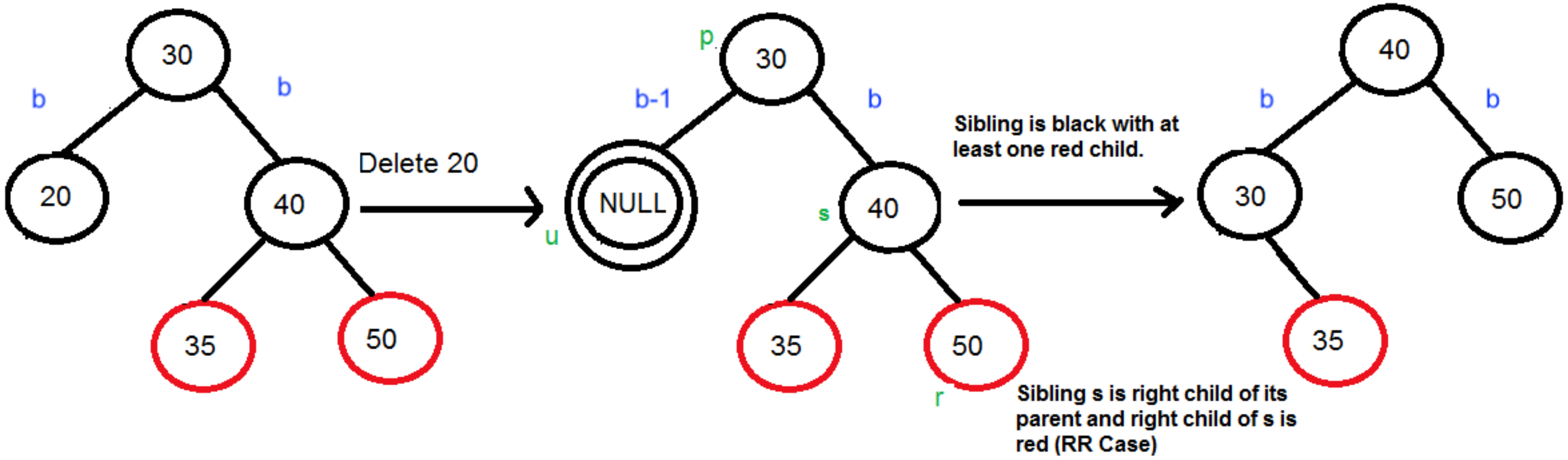


- 2(b) Black Sibling + Black Niblings Case
  - `s` is black, and both children of `s` are black.
- 2(c) Red Sibling Case
  - `s` is red.

- **2(a) Black Sibling + Red Nibling Case**
  - If `s` is black and at least one of `s`'s children is red, let the red child of `s` be `r`. There are four cases depending upon positions of `s` and `r`: Left-Left, Right-Right, Left-Right, Right-Left.

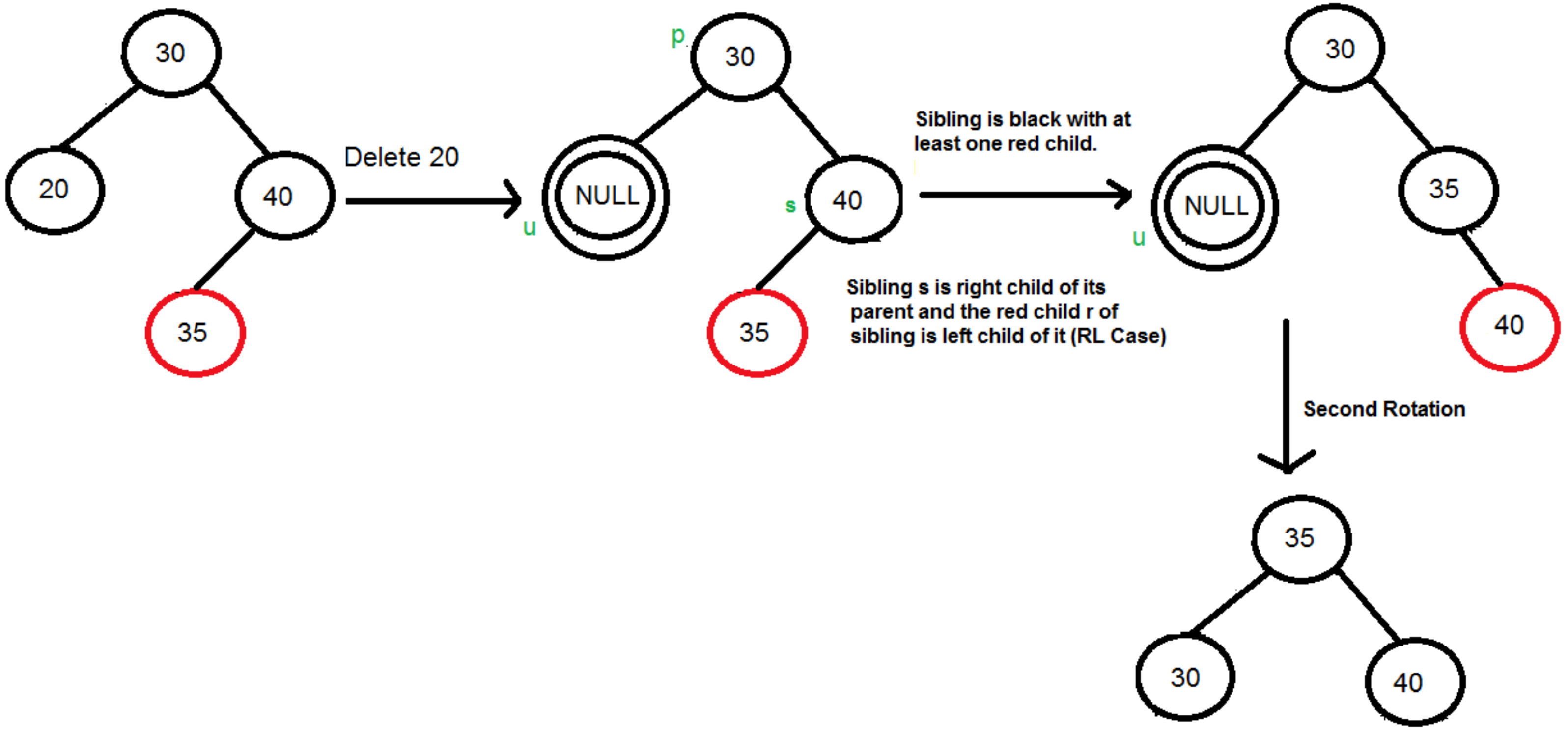
▲ Black Sibling + Red Nibling + **Right-Right Case** (`p.right == s`, and `s.right` red)

> Left rotate `p` and change the color of `r` to black.



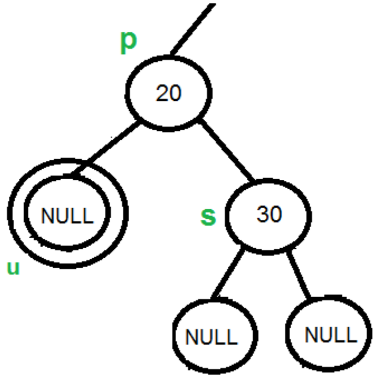
🔺 Black Sibling + Red Nibling + **Right-Left Case** (`p.right == s` , and `s.right` not red)

> Right rotate `s` , and swap colors and `s` and `r` . Now it's Right-Right Case!



Step 2. Do following while the current node `u` is double black, and it is not the root. Let sibling of node be `s`. There are three cases:

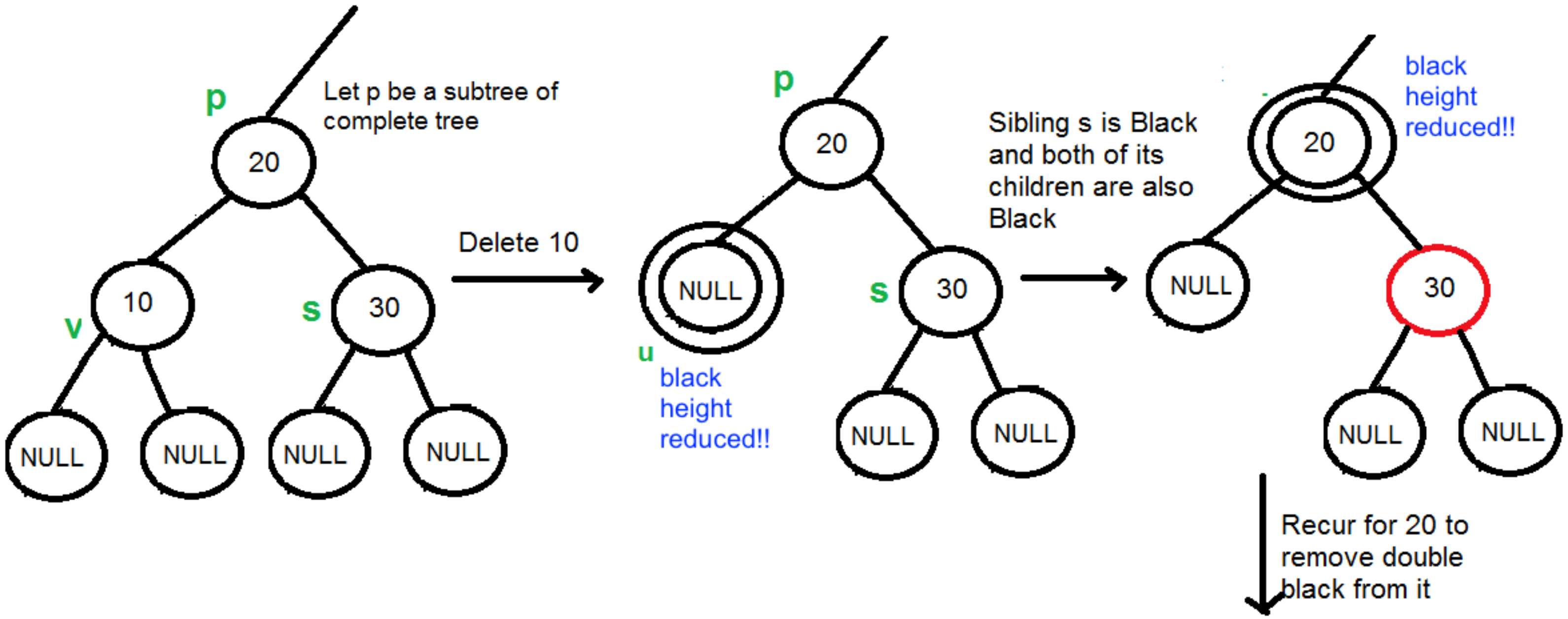
- 2(a) ~~Black Sibling + Red Nibling Case~~
  - `s` is black, and *at least* one of `s`'s children is red.
- 2(b) **Black Sibling + Black Niblings Case** ⬅️
  - `s` is black, and both children of `s` are black.



- 2(c) Red Sibling Case
  - `s` is red.

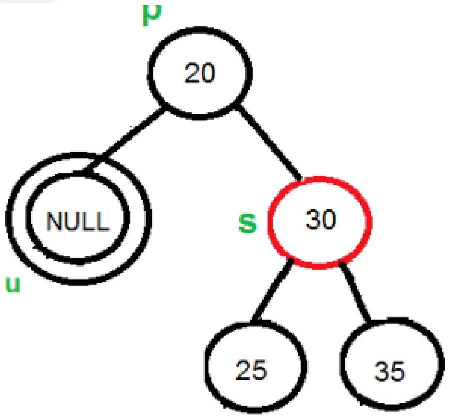
- **2(b) Black Sibling + Black Niblings Case**
  - If `s` is black and `s`'s both children is black,

    > Recolor `s` red. If `p` (the parent) is red, recolor `p` black and stop. If `p` is black, the “double black” moves up to `p`; recurse at `p`.



Step 2. Do following while the current node `u` is double black, and it is not the root. Let sibling of node be `s`. There are three cases:

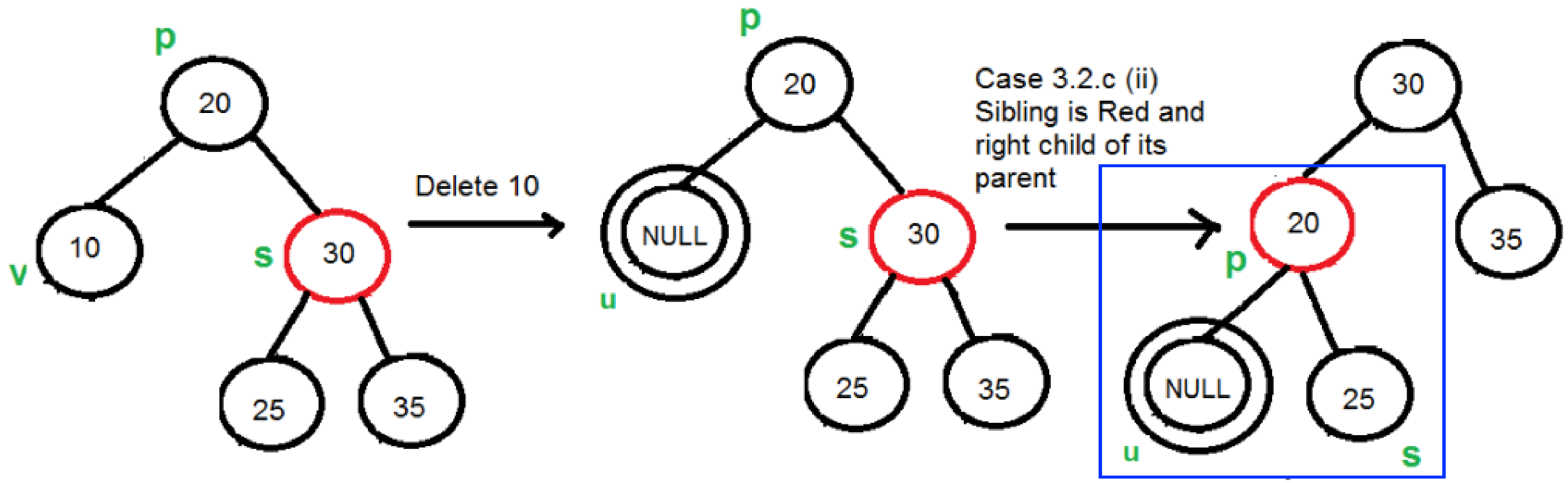
- 2(a) ~~Black Sibling + Red Nibling Case~~
  - `s` is black, and *at least* one of `s`'s children is red.
- 2(b) ~~Black Sibling + Black Niblings Case~~
  - `s` is black, and both children of `s` are black.
- 2(c) **Red Sibling Case** ⬅️
  - `s` is red.



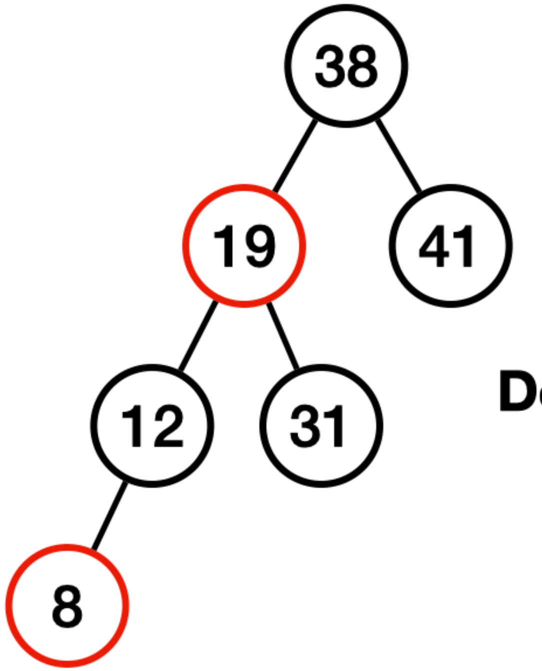
- **2(c) Red Sibling Case**

  If `s` is red, the childrens of `s` are black.

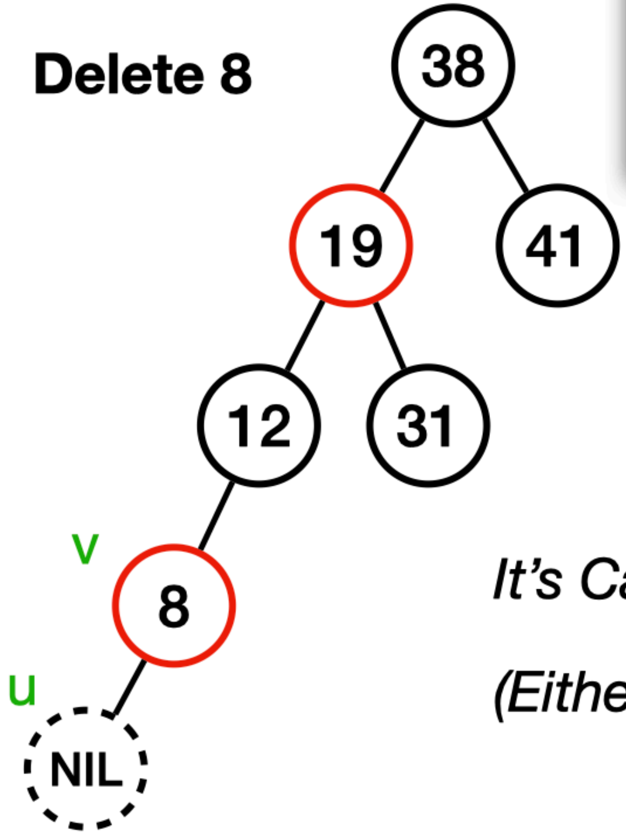
  > Perform a rotation to move old sibling up, and recolor the old sibling and parent.
  > Now it's Black Sibling Case.

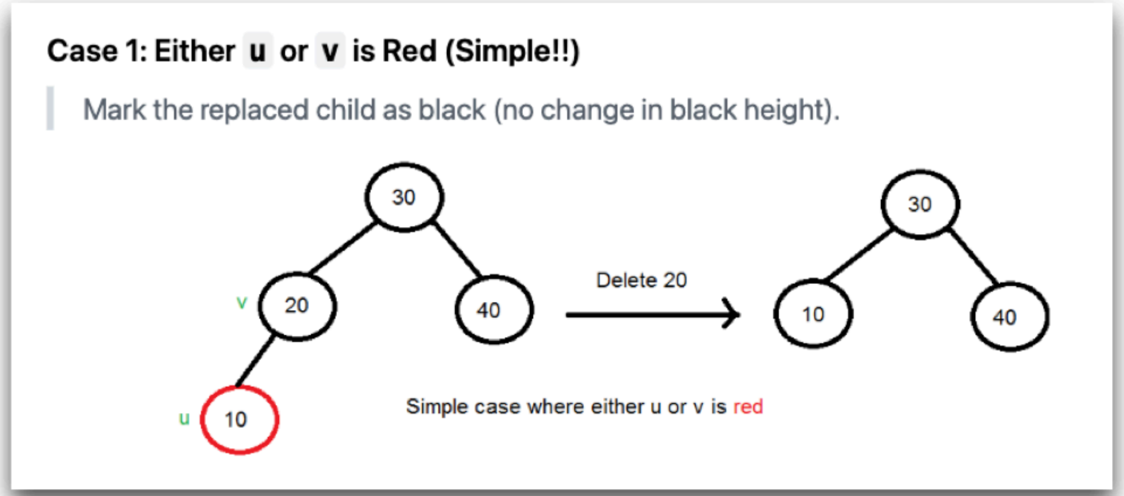


Step 3. If the current double black `u` is root, make it single black and return.



Delete 8



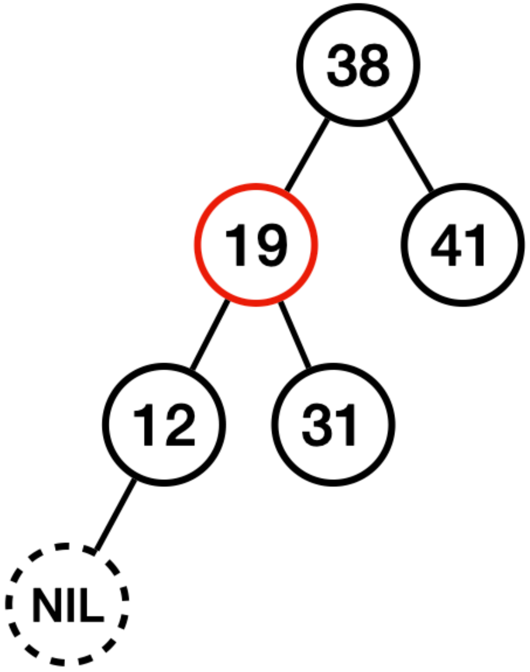


Case 1: Either u or v is Red (Simple!!)

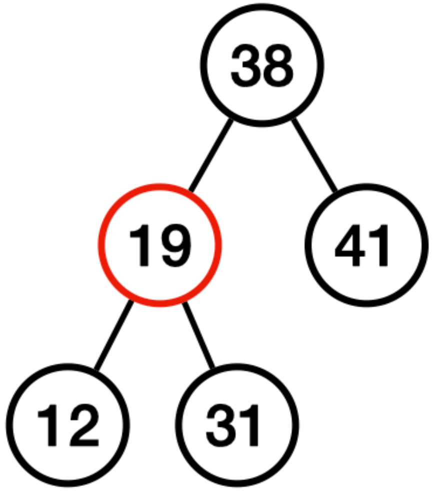
> Mark the replaced child as black (no change in black height).

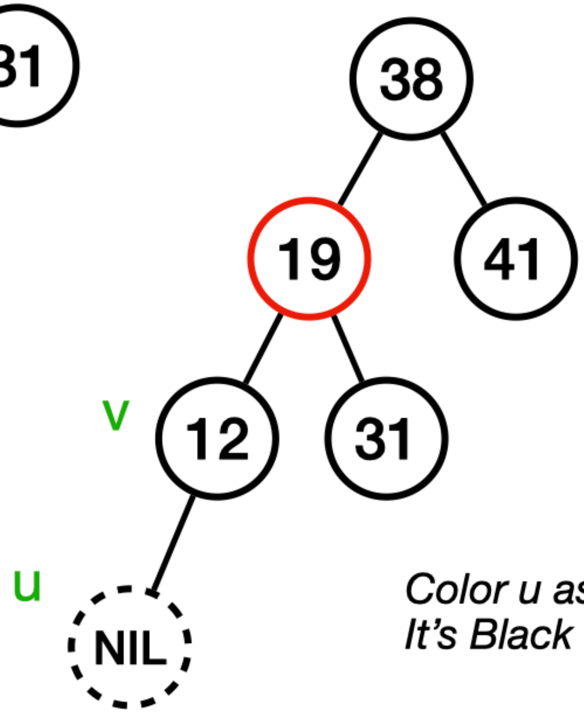
It's Case 1.

(Either v or u is red)

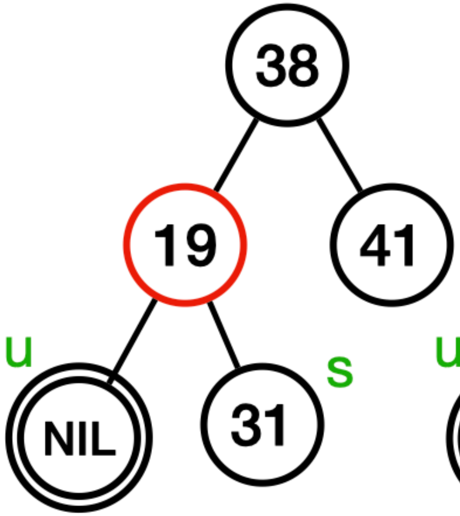


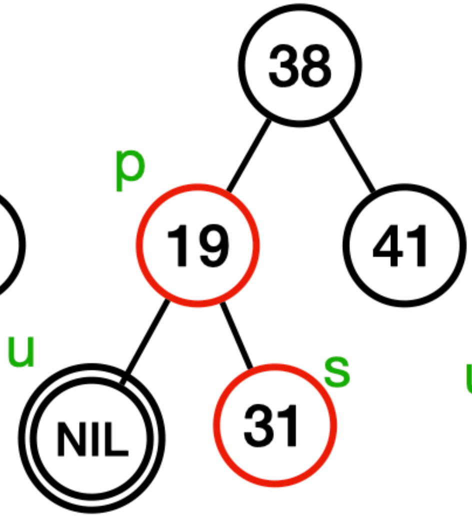
Delete 12



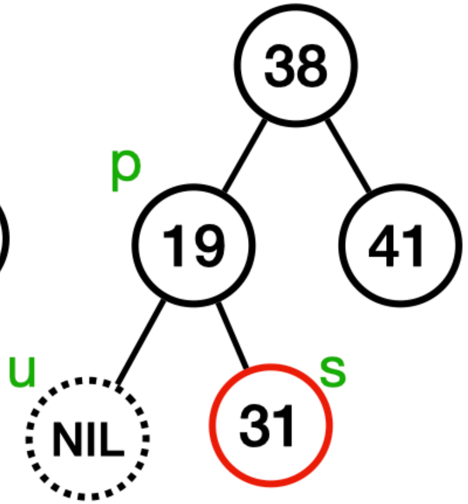


It's Case 2.
(Both u and v are Black)

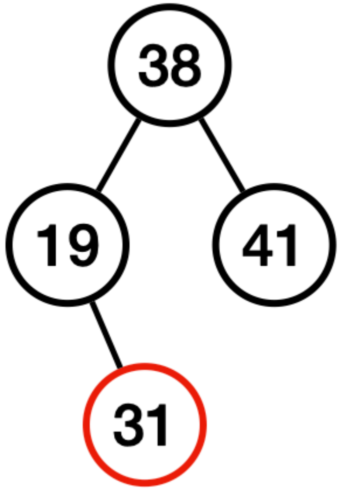




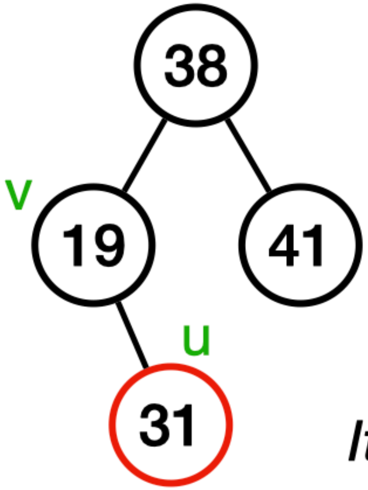
Color u as double black.
It's Black Sibling + Black Niblings Case.



Recolor s red.
Since p is red, recolor p black and stop.

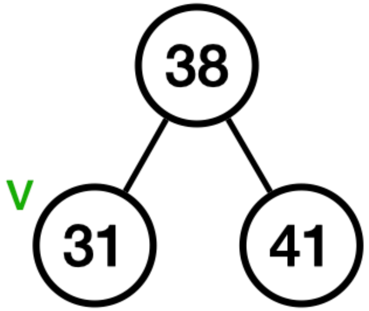


Delete 19

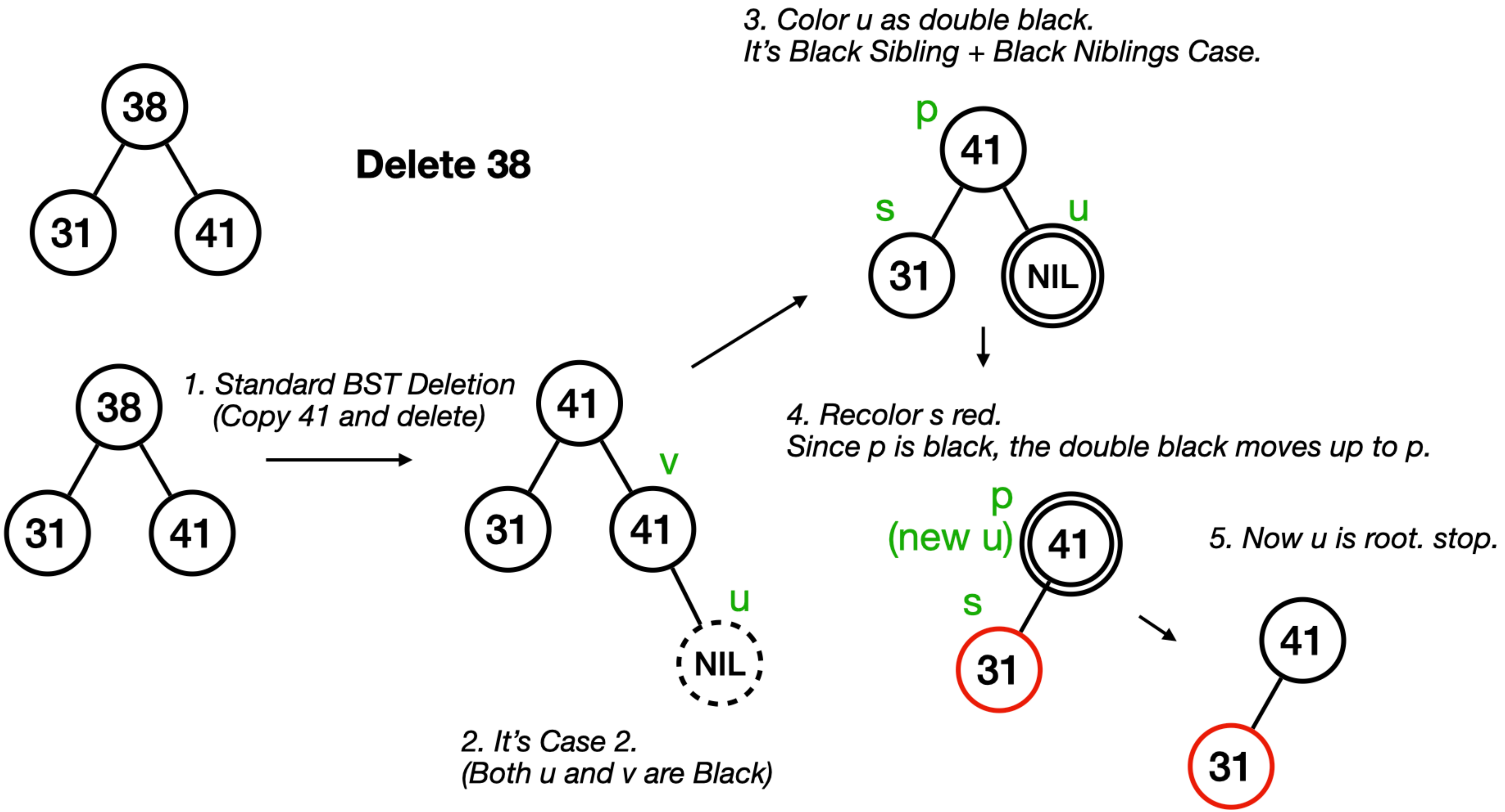


It's Case 1.

(Either v or u is red)



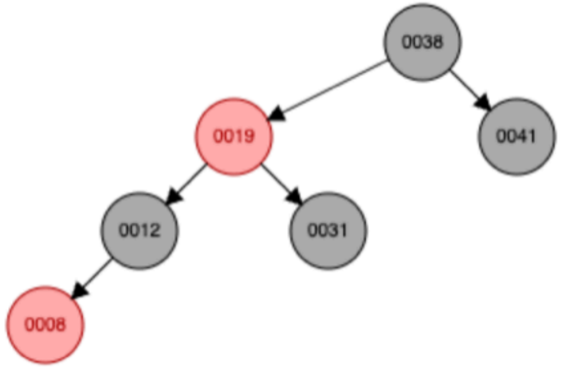
Mark the replaced child as black.



38
31
41
Delete 38
38
31
41
1. Standard BST Deletion
(Copy 41 and delete)
41
v
31
41
u
NIL
2. It's Case 2.
(Both u and v are Black)
3. Color u as double black.
It's Black Sibling + Black Niblings Case.
p
41
s
u
31
NIL
4. Recolor s red.
Since p is black, the double black moves up to p.
p
(new u)
41
s
31
5. Now u is root. stop.
41
31

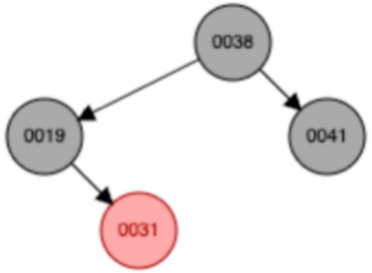
Visualizer Results

https://www.cs.usfca.edu/~galles/visualization/RedBlack.html



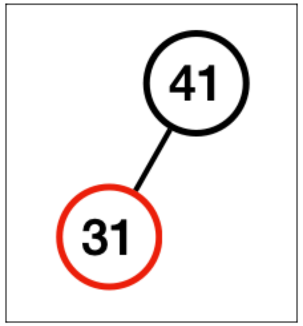
Delete 8

Delete 12



Delete 19

Delete 38



Compare this result with the the previous example. Think about why they are different.

Comparison with AVL Tree

- AVL trees are more strictly balanced than Red–Black trees.
  - This strict balance means **faster searches** in worst case.
  - However, AVL trees often require more rotations during insertions and deletions.
- Red–Black trees are less strictly balanced, so searches can be slightly slower.
  - But Red–Black trees typically require fewer rotations, making them better for frequent updates.
- Choose Red–Black if your application involves **frequent insertions/deletions**.
- Choose AVL if **searches are frequent** and updates are rare.

Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
    - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
    - https://algorithmsilluminated.com/

Reference:

- https://www.geeksforgeeks.org/dsa/introduction-to-red-black-tree
- https://www.geeksforgeeks.org/dsa/c-program-red-black-tree-insertion/
- https://studyglance.in/ds/display.php?tno=27&topic=Red-Black-Tree