Lecture 9 - Sorting Lower Bounds, Counting Sort, Radix Sort

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Welcome Back! 👋

- Today, we'll prove that **every deterministic comparison-based sorting algorithm** requires **Ω(n log n)** time in the worst case.
- We'll also introduce **non-comparison-based** sorting algorithms, including **Counting Sort** and **Radix Sort**.

Comparison-Based Sorting Algorithms ⚖️

- The sorting algorithms we have learned in this course (i.e., SelectionSort, BubbleSort, InsertionSort, MergeSort, QuickSort) are all **COMPARISON-based** sorting algorithms!
    - i.e., they sort an array via asking *whether a given element is greater than, less than, or equal to some other element*.
- For such algorithms, there exists **theoretical lower bounds** for running time.

Lower Bounds for Sorting

> Any *comparison-based* deterministic sorting algorithm takes $\Omega(n \log n)$ time.

Lower Bounds for Sorting (Worst-Case Bound)

> Any **deterministic comparison-based** sorting algorithm must make $\Omega(n \log n)$ comparisons.

- Based on:
  - There are **n!** possible permutations for sorting outputs.
  - Sorting algorithm must **distinguish** between all of them.
- Think about:
  - Game of "20 Questions" (스무고개)
  - Each comparison at best halves the search space $\Rightarrow$ need $\log_2(n!)$ steps

Example: Sorting 3 Elements

- Input: three numbers x_1, x_2, x_3
- Possible orders: $3! = 6$ permutations
  - List: (123), (132), (213), (231), (312), (321)
  - (123) means $x_1 > x_2 > x_3$

Comparison as Information

- Each comparison is a **yes/no** question, cutting possibilities at most in half.
- Example: check $x_1 > x_2$
  - If true: possible → (123), (132), (312)
  - If false: possible → (213), (231), (321)
  - So the 6 possibilities reduce to 3.

Minimum number of comparisons

- After each comparison, at most half of the candidates remain.
- Need enough comparisons to distinguish among all $3! = 6$ orders.
  - $\#\text{comparisons} \geq \lceil \log_2(6) \rceil = \lceil 2.58 \rceil = 3$
- Hence, 3 comparisons are sufficient to fully determine the order of three elements.

Time Complexity Analysis

- Any deterministic comparison-based sorting algorithm must make at least $\log_2(n!)$ comparisons before it can halt.

$$\begin{aligned}\log_2(n!) &= \log_2(n \times (n-1) \times \cdots \times 1) \\ &= \log_2(n) + \log_2(n-1) + \cdots + \log_2(1) \\ &= \Omega(n \log n). \quad \blacksquare\end{aligned}$$

Randomized Algorithms

> Randomized comparison-based sorting also requires $\Omega(n \log n)$ comparisons on average.

- Treat randomized algorithm as a **distribution over deterministic ones**
- Expected runtime is weighted average over deterministic algorithms
- Since all have $\Omega(n \log n)$ average, so must the randomized algorithm

Non-Comparison-Based Sorting Algorithms?

> Comparison-based sorting has a known lower bound of $\Omega(n \log n)$.

- This bound applies to algorithms that only compare values (e.g., MergeSort, QuickSort).
- But what if we **don't use comparisons** to sort?

  ➡️ **Counting Sort** and **Radix Sort** escape this lower bound!

Counting Sort

- For a given input of n objects, each with a corresponding key (or value) in the range $\{0, 1, \ldots, r-1\}$, Counting Sort will sort the objects by their keys:
    i. Create an array A of r buckets where each bucket contains a linked list.
    ii. For each element in the input array with key k, concatenate the element to the end of the linked list $A[k]$.
    iii. Concatenate all the linked lists: $A[0], \ldots, A[r-1]$.

Counting Sort: Example

- Input: `[o1(k=4), o2(k=2), o3(k=2), o4(k=8), o5(k=3), o6(k=3), o7(k=1)]`
- Assume keys range from 0 to $r - 1 = 9$ (so $r = 10$)

1. Create 10 buckets

```
A[0]: [], ..., A[9]: []
```

2. Distribute elements into buckets

```
A[1]: [o7]
A[2]: [o2, o3]
A[3]: [o5, o6]
A[4]: [o1]
A[8]: [o4]
```

3. Sorted output: `[o7, o2, o3, o5, o6, o1, o4]` (Sorted without comparisons!)

Counting Sort: Why It Works

- Items are grouped by key.
- Bucket `i` (key = i) appears before bucket `j` (key = j) when $i < j$.
- Resulting order is **correct** and **stable**.
  - **Stability**: If x appears before y in input and both have same key, x appears before y in output.
    - Example: `Input = [o1(k=4), o2(k=2), o3(k=2), o4(k=8), o5(k=3), o6(k=3), o7(k=1)]`, `Output = [o7, o2, o3, o5, o6, o1, o4]`,
      - `o2` appears before `o3`, and `o5` appears before `o6` in output.

Counting Sort: Time and Space Complexity

- Time: $O(n + r)$
  (one pass over input + one pass over buckets)
- Space: $O(n + r)$

> ⚠️ Works best when r is **small** (e.g., constant or $O(n)$).

- Counting Sort behaves poorly if the range of values r is very large.
  - Radix Sort builds on Counting Sort and fixes this issue!

Radix Sort

- Input: n numbers with d digits (each digit in $\{0, 1, \ldots, r-1\}$)
- **Idea:** Sort digit-by-digit from **least** to **most** significant

```
for j = 1 to d:
    CountingSort(A, key = j-th digit)
```

Radix Sort: Example

- Input: 7 numbers with **3 digits** `[329, 457, 657, 839, 436, 720, 355]`
  - Base: $r = 10$ (digits range from 0–9)

1. Sort by **1st digit (units)**
   - Digits: `[9, 7, 7, 9, 6, 0, 5]`
   - Counting Sort: `[720, 355, 436, 457, 657, 329, 839]`
2. Sort by **2nd digit (tens)**
   - Digits: `[2, 5, 3, 5, 5, 2, 3]`
   - Counting Sort: `[720, 329, 436, 839, 355, 457, 657]`
3. Sort by **3rd digit (hundreds)**
   - Digits: `[7, 3, 4, 8, 3, 4, 6]`
   - Counting Sort: `[329, 355, 436, 457, 657, 720, 839]`

Radix Sort: Why It Works

We prove correctness by **induction on digit position**:

- **Base case** ($j = 1$):
  After the 1st pass, the array is correctly sorted by the **least significant digit**.

- **Inductive step**:
  Suppose array is correctly sorted by the first $j - 1$ digits.
  When we sort by the j-th digit using a **stable** sort:
  - Elements with the same j-th digit **retain their order** from earlier digits.
  - So the array becomes sorted by the **first j digits**.

➡️ After the d-th pass, the array is sorted by all d digits → **Fully sorted!**

Radix Sort: Time Complexity

- Each digit uses Counting Sort: $O(n+r)$
- Total time: $O(d(n+r))$
  - If $r = O(n)$ and $d = O(1)$, then:
    - $O(n)$ time

Radix Sort: Varying the Base r

- The number of **passes** (digits d) depends on:
  - The **maximum value** in the input: M
  - The **base** r used for digit representation

$$d = \lfloor \log_r M \rfloor + 1$$

- So the **runtime** is:

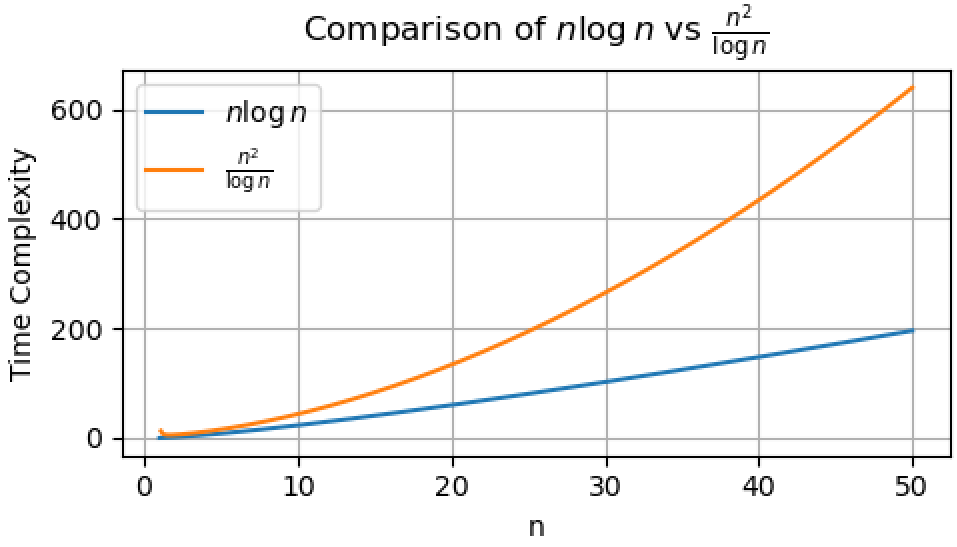
$$O(d(n+r)) \to O\left((\log_r M + 1)(n+r)\right)$$

- How should we choose r?
  - One smart strategy would be setting $r = n$ to balance the two terms in $(n+r)$
    - It leads to runtime: $O\left(n \cdot (\log_n M + 1)\right)$

When is Radix Sort Fast?

- When $r = n$, the runtime is $O(n \cdot (\log_n M + 1))$
  - If $M = n^c$ for constant c, $O(n)$ time ✅
  - If $M = 2^n$, $O(n^2 / \log n)$ ❌ *This is worse than* $O(n \log n)$

$$O(n \cdot (\log_n 2^n + 1)) = O(n \cdot (n \log_n 2 + 1)) = O(n \cdot (n \frac{\log_2 2}{\log_2 n} + 1)) = O(n \cdot (\frac{n}{\log_2 n} + 1)) = O(n^2 / \log n)$$



When is Radix Sort Fast?

- Radix Sort (with $r = n$) is efficient when M is **not too large** relative to n.
    - Use Radix Sort **when the sorting keys (numbers) are bounded**!
    - When M grows exponentially in n (e.g., $M = 2^n$), Radix Sort becomes inefficient and loses its edge over comparison-based sorts.

Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/