Lecture 5 - Solving Recurrences (Master Method)

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Welcome Back! 👀

In today's class, we'll

- Take a Quiz#1
- Revisit Divide and Conquer
- Learn **how to solve recurrences** using **Master Method**
  - i.e,, compute the running time of divide and conquer algorithms

Quiz #1 ⭐

- Time limit: 15 minutes
- Start time: 13:40
- Materials: Lecture slides/notes may be used (open notes, not open internet)
- Please prepare a sheet of paper and a pen/pencil for small calculations.
- Password:

Recap: Divide and Conquer

1. *Divide* the input into smaller subproblems
2. *Conquer* the subproblems recursively
3. *Combine* the solutions for the subproblems into a solution for the original problem.

Examples

- Karatsuba's Integer Multiplication Algorithm
- MergeSort

> Divide and Conquer algorithms are **recursive algorithms**!

Recurrence Examples

Let $T(n)$ be the runtime of the algorithm, given an input of size n.

- The simple recursive multiplication algorithm: $T(n) = 4T(\frac{n}{2}) + O(n)$
- The Karatsuba multiplication algorithm: $T(n) = 3T(\frac{n}{2}) + O(n)$
- MergeSort: $T(n) = 2T(\frac{n}{2}) + O(n)$

> The running time of divide and conquer algorithms can be naturally expressed in terms of the running time of smaller inputs.

Recurrence Examples - Running Time of Algorithms ✍️

- The simple recursive multiplication algorithm: $T(n) = 4T(\frac{n}{2}) + O(n)$
  - $O(n^2)$ - *See Lecture 2*
- The Karatsuba multiplication algorithm: $T(n) = 3T(\frac{n}{2}) + O(n)$
  - $O(n^{1.58})$ - *See Lecture 2*
- MergeSort: $T(n) = 2T(\frac{n}{2}) + O(n)$
  - $O(n \log_2 n)$ - *See Lecture 3*

> Is there easier way to compute the running time of these recursive algorithms? 🤔

Solving Recurrernces

There are techniques to solve these recurrences.

- **Master Method**: This method can only be used when the size of all the subproblems is the same (as was the case in the examples) - 📍 *Covered in Today's Class*
- **Substitution Method**: This method is used to solve more complex recurrences.

What is Master Method?

- A general method for solving recurrences
- Assumption: **All the subproblems have equal size**.
  - Karatsuba ($n = 4$) will be split into three subproblems of size 2
    - Karatsuba ($n = 2$), Karatsuba ($n = 2$), and Karatsuba ($n = 2$)
  - MergeSort ($n = 8$) will be split into two subproblems of size 4
    - MergeSort ($n = 4$) and MergeSort ($n = 4$)

Standard Recurrence Format

1. Base Case: For a sufficiently small n (say, when $n = 1$), the worst-case runtime of the algorithm is constant, namely $T(n) = O(1)$.
2. For all larger n:

$$T(n) \leq a \cdot T(\frac{n}{b}) + O(n^d)$$

   - a is the number of subproblems that we create from one problem (i.e., $a \geq 1$)
   - b is the factor by which the input size shrinks (i.e., $b > 1$).
   - d is the exponent of n in the time it takes to generate the subproblems and combine their solutions.

Master Theorem

If $T(n)$ is defined by a standard recurrence, i.e., $T(n) \leq a \cdot T(\frac{n}{b}) + O(n^d)$ where $a \geq 1$, $b > 1$ and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \quad \text{(Case 1)} \\ O(n^d) & \text{if } a < b^d \quad \text{(Case 2)} \\ O(n^{\log_b a}) & \text{if } a > b^d \quad \text{(Case 3)} \end{cases}$$

Master Theorem - Example for Case 1

If $T(n)$ is defined by a standard recurrence, i.e., $T(n) \leq a \cdot T(\frac{n}{b}) + O(n^d)$ where $a \geq 1$, $b > 1$ and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d & \textbf{(Case 1)} \\ O(n^d) & \text{if } a < b^d & \text{(Case 2)} \\ O(n^{\log_b a}) & \text{if } a > b^d & \text{(Case 3)} \end{cases}$$

Example:

MergeSort: $T(n) = 2T(\frac{n}{2}) + O(n)$

- $a = 2, b = 2, d = 1$, so $a = b^d$, hence $T(n) = O(n \log n)$.

Master Theorem - Example for Case 2

If $T(n)$ is defined by a standard recurrence, i.e., $T(n) \leq a \cdot T(\frac{n}{b}) + O(n^d)$ where $a \geq 1$, $b > 1$ and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d & \text{(Case 1)} \\ O(n^d) & \text{if } a < b^d & \textbf{(Case 2)} \\ O(n^{\log_b a}) & \text{if } a > b^d & \text{(Case 3)} \end{cases}$$

Example:

If an algorithm's running time is: $T(n) = 3T(\frac{n}{2}) + O(n^2)$

- $a = 3, b = 2, d = 2$, so $a < b^d$, hence $T(n) = O(n^2)$

Master Theorem - Example for Case 3

If $T(n)$ is defined by a standard recurrence, i.e., $T(n) \leq a \cdot T(\frac{n}{b}) + O(n^d)$ where $a \geq 1$, $b > 1$ and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \quad (\text{Case 1}) \\ O(n^d) & \text{if } a < b^d \quad (\text{Case 2}) \\ O(n^{\log_b a}) & \text{if } a > b^d \quad (\textbf{Case 3}) \end{cases}$$

Example:

The simple recursive multiplication algorithm: $T(n) = 4T(\frac{n}{2}) + O(n)$

- $a = 4, b = 2, d = 1$, so $a > b^d$, hence $T(n) = O(n^{\log_2 4}) = O(n^2)$

Master Theorem - Example for Case 3

If $T(n)$ is defined by a standard recurrence, i.e., $T(n) \le a \cdot T(\frac{n}{b}) + O(n^d)$ where $a \ge 1$, $b > 1$ and $d \ge 0$, then

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d & \text{(Case 1)} \\ O(n^d) & \text{if } a < b^d & \text{(Case 2)} \\ O(n^{\log_b a}) & \text{if } a > b^d & \textbf{(Case 3)} \end{cases}$$

Example:

The Karatsuba multiplication algorithm: $T(n) = 3T(\frac{n}{2}) + O(n)$

- $a = 3, b = 2, d = 1$, so $a > b^d$, hence $T(n) = O(n^{\log_2 3}) = O(n^{1.58})$

Proof of the Master Theorem - Preamble

Assumption

A recurrence is

1. $T(1) \leq c$ (For some constant c)
2. $T(n) \leq aT(\frac{n}{b}) + cn^d$
   and n is a power of b.

Idea: Generalize MergeSort analysis (i.e., use a **recursion tree**)

Recap: MergeSort Running Time Analysis

```
level 0:                        n
                              /   \
level 1:                  n/2       n/2
                         /   \     /   \
level 2:              n/4   n/4  n/4   n/4
                       .               .
                       .               .
                  |   |   |   |       |   |   |   |
level log_2(n):   1   1   1   1  ...  1   1   1   1
```

- At each level $j = 0, 1, \ldots, \ldots, \log_2 n$, there are 2^j subproblems, each of size $n/2^j$.
- Work at each level j = (#subproblems * Work per subproblem) $\leq 2^j \cdot 11(\frac{n}{2^j}) = 11n$

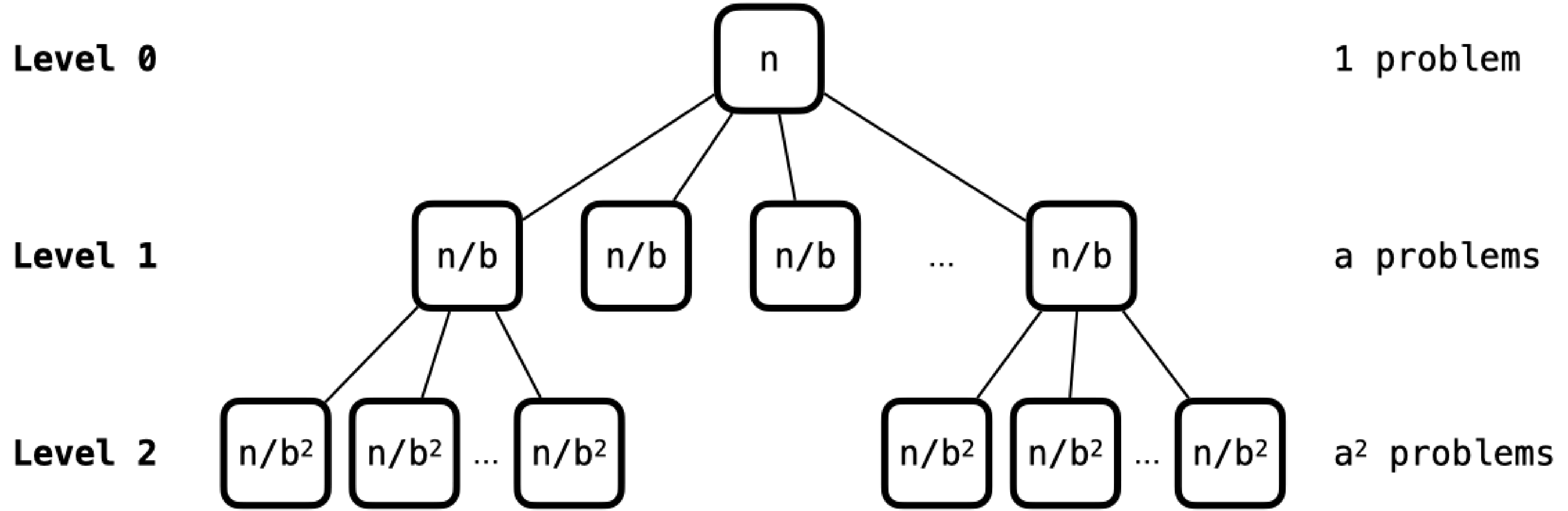
Total Work = (Work per level * #levels) $\leq 11n \cdot (1 + \log_2(n)) = O(n \log n)$. ✅

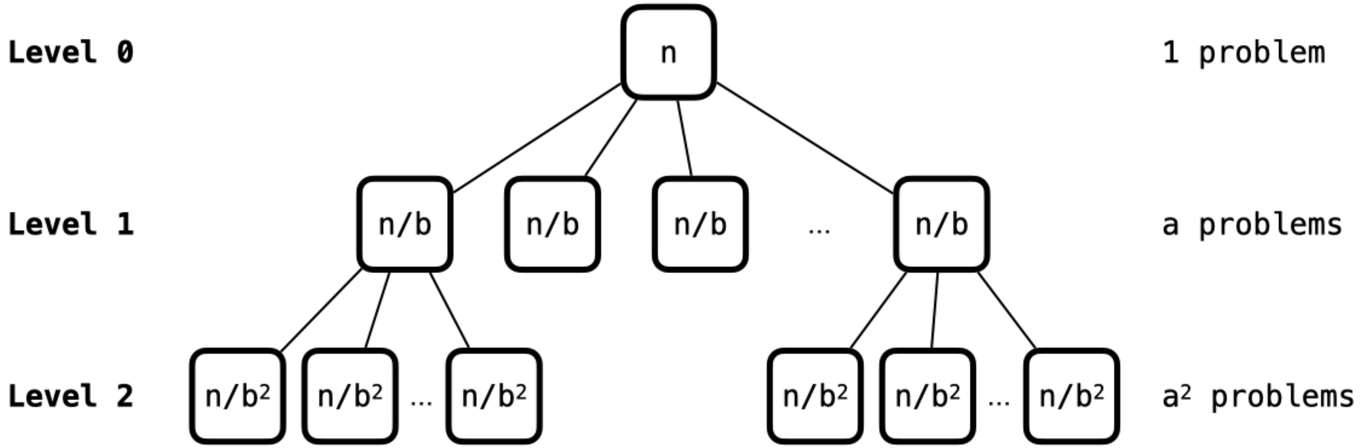
Quiz!

Let $T(n) = aT(\frac{n}{b}) + cn^d$, then in a recursion tree, at each level $j = 0, 1, 2, \ldots, \log_b n$, there are ____ subproblems, each of size ____.

1. a^j and n/a^j, respectively.
2. a^j and n/b^j, respectively.
3. b^j and n/a^j, respectively.
4. b^j and n/b^j, respectively.

The Recursion Tree





- Work at level 1 $\leq a \cdot c(\frac{n}{b})^d$, Work at level 2 $\leq a^2 \cdot c(\frac{n}{b^2})^d$, ...
- Work at each level $j \leq a^j \cdot c(\frac{n}{b^j})^d = cn^d \cdot (\frac{a}{b^d})^j$ (*ignoring work in recursive calls*)
  - a = rate of subproblem proliferation (RSP)
  - b^d = rate of work shrinkage (RWS) - per subproblem
- Total Work $\leq \sum_{j=0}^{\log_b n} \{cn^d \cdot (\frac{a}{b^d})^j\} = cn^d \cdot \sum_{j=0}^{\log_b n} (\frac{a}{b^d})^j$

Intuition for the 3 Cases 💡

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \quad (\text{Case 1}) \\ O(n^d) & \text{if } a < b^d \quad (\text{Case 2}) \\ O(n^{\log_b a}) & \text{if } a > b^d \quad (\text{Case 3}) \end{cases}$$

> Upper bound for level j: $cn^d \cdot (\frac{a}{b^d})^j$

- Case 1: $a = b^d$ (RSP = RWS), i.e., $\frac{a}{b^d} = 1$
  - 🟰 Same amount of work each level (like MergeSort)
- Case 2: $a < b^d$ (RSP < RWS), i.e., $\frac{a}{b^d} < 1$
  - ⬆️ Top-heavy work: most work is done near the root.
- Case 3: $a > b^d$ (RSP > RWS), i.e., $\frac{a}{b^d} > 1$
  - ⬇️ Bottom-heavy work: most work is at the leaves.

Proof of the Master Theorem - Case 1 ($a = b^d$)

- Total Work
  $\leq cn^d \cdot \sum_{j=0}^{\log_b n} (\frac{a}{b^d})^j$
  $= cn^d \cdot \sum_{j=0}^{\log_b n} 1$
  $= cn^d (\log_b n + 1)$
  $= O(n^d \log n)$

Proof of the Master Theorem - Case 2 ($a < b^d$)

- Total Work
  $\leq cn^d \cdot \sum_{j=0}^{\log_b n} (\frac{a}{b^d})^j$
  $= cn^d \cdot \sum_{j=0}^{\log_b n} r^j$ (where $r = \frac{a}{b^d} < 1$)
  $= cn^d \cdot (1 + r^1 + r^2 + \cdots + r^{\log_b n})$
  $\leq cn^d \cdot (\frac{1}{1-r})(\because r < 1)$
  $= O(n^d)$

- Basic Fact
  - If $|r| < 1$, $\sum_{j=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^k + \cdots = \frac{a}{1-r}$

Proof of the Master Theorem - Case 3 ($a > b^d$)

- Total Work

$\leq cn^d \cdot \sum_{j=0}^{\log_b n} (\frac{a}{b^d})^j$

$= cn^d \cdot \sum_{j=0}^{\log_b n} r^j$ (where $r = \frac{a}{b^d} > 1$)

$= cn^d \cdot (1 + r^1 + r^2 + \cdots + r^{\log_b n})$

$= cn^d \cdot \frac{r^{\log_b n} - 1}{r - 1}$

$= O(n^d \cdot r^{\log_b n})$

$= O(n^d \cdot (\frac{a}{b^d})^{\log_b n})$

$= O(a^{\log_b n} \cdot n^d \cdot b^{-d \log_b n})$

$= O(a^{\log_b n} \cdot n^d \cdot n^{-d})$

$= O(a^{\log_b n})$ (*this equals to* $\#\text{leaves}$ 🍃)

$= O(b^{\log_b a \log_b n}) = O((b^{\log_b n})^{\log_b a}) = O(n^{\log_b a})$

The Master Method 🧙‍♂️

If $T(n) \leq a \cdot T(\frac{n}{b}) + O(n^d)$ where $a \geq 1, b > 1$ and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \quad (\text{Case } 1) \\ O(n^d) & \text{if } a < b^d \quad (\text{Case } 2) \\ O(n^{\log_b a}) & \text{if } a > b^d \quad (\text{Case } 3) \end{cases}$$

Note that this method can only be used when the size of all the subproblems is the same!

Credits & Resources

Lecture materials adapted from:

- Stanford CS161 slides and lecture notes
  - https://stanford-cs161.github.io/winter2025/
- *Algorithms Illuminated* by Tim Roughgarden
  - https://algorithmsilluminated.com/